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Avoiding monochromatic solutions to 3-term equations

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Given an equation, the integers $[n] = \{1, 2, \ldots, n\}$ as inputs, and the colors red and blue, how can we color $[n]$ in order to minimize the number of monochromatic solutions to the equation, and what is the minimum? The answer is only known for a handful of equations, but much progress has been made on improving upper and lower bounds on minima for various equations. A well-studied characteristic of an equation, which has its roots in graph Ramsey theory, is to determine if the minimum number of monochromatic solutions can be achieved (asymptotically) by uniformly random colorings. Such equations are called *common*. We prove that no 3-term equations are common and provide a lower bound for a specific class of 3-term equations.

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1. Introduction

Given an equation

$$a_1x_1 + \cdots + a_kx_k = 0 \text{ with } a_i \in \mathbb{Z} \tag{1}$$

and the colors red and blue, how should we color the elements of $[n] = \{1, 2, \ldots, n\}$ in order to reduce the number of monochromatic solutions, with the ultimate goal being to find the asymptotic (as $n \to \infty$) minimum number? To be precise, by a coloring we mean a function $f : [n] \to \{-1, 1\}$ (where -1 represents blue and 1 represents red), by a solution we mean a vector $(x_1^*, \ldots, x_k^*) \in [n]^k$ that satisfies the equation, and by monochromatic we mean $f(x_1^*) = \cdots = f(x_k^*)$.

Before we proceed, we clarify the asymptotic notation used throughout. Let f, g be functions of n. If $f = O(g)$, there exist some constants C, N such that $|f(n)| \leq Cg(n)$ for all $n \geq N$. By $f = \Omega(g)$, we mean $g = O(f)$. If $f = o(g)$, this indicates $f/g \to 0$ as $n \to \infty$[1].

Asymptotic minima are difficult to come by, but much progress has been made on improving upper and lower bounds. The most comprehensive result on lower bounds is due to Frankl, Graham, and Rödl, who showed that as long there is a nonempty subset of coefficients which sum to 0, the equation will always have $\Omega(n^{k-1})$ monochromatic solutions [8]. In fact, their result is more general, considering systems of equations and an arbitrary number of colors.

For upper bounds, a well-studied problem is to determine if colorings can be found which yield fewer monochromatic solutions asymptotically than uniformly random colorings. This problem has its roots in graph Ramsey theory, where one can ask a similar question: given a fixed graph H, can the edges of K_n always be colored in such a way that produces asymptotically fewer monochromatic copies of H in K_n than what would be expected from uniformly random colorings? Graphs with this property are referred to as *uncommon*. In 1959, Goodman showed that K_3 was *common*, i.e. every coloring of K_n has asymptotically at least as many monochromatic copies of K_3 as one would expect from uniformly random colorings [9]. Three years later, Erdős conjectured that K_s was common for all $s \geq 2$ [6], and in 1980 Burr and Rosta were even bolder, conjecturing that all graphs were common [1]. However, in 1989 Thomason showed that K_4 was uncommon, disproving both conjectures [19].

The first result regarding equations came nearly a decade later, and we will highlight certain aspects of the original equation of study: $x + y = z$, known as *Schur's equation*. Each solution is generally represented as a *Schur triple* $(x, y, x + y)$. There are $\binom{n}{2}$ solutions over $[n]$ (when $(x, y, x + y)$ and $(y, x, x + y)$ are considered distinct). With a uniformly random coloring, a given solution with $x \neq y$ will be monochromatic with probability $1/4$ (and solutions with $x = y$ contribute only to lower order terms), so we would expect $n^2/8 + O(n)$ monochromatic solutions. In 1998, Robertson and Zeilberger found the asymptotic minimum number of monochromatic solutions: $n^2/11 + O(n)$ [14]. In particular, there is always a coloring of $[n]$ with fewer monochromatic solutions than what would be expected from a uniformly random coloring. To borrow the terminology from graph theory, $x + y = z$ is *uncommon*. A coloring that achieves this minimum is quite simple to describe:

[1]In all cases, the implicit constants are allowed to depend on the equation being analyzed.

(or as close to this as possible when n is not a multiple of 11).

One can ask the same questions about other equations or systems of equations. Generally the equations considered are linear with integer coefficients, which enables another variation on the problem: replace $[n]$ with an abelian group. Colorings of $\mathbb{Z}_n$ and $\mathbb{F}_p^n$ are frequently studied [12, 7, 15]. Over $[n]$, true (asymptotic) minima are only known for a handful of equations, such as $x+by = z$ with $b \in \mathbb{N} = \{1, 2, 3, \dots\}$ [14, 18] and $x+y = z+w$, where it turns out the $1/8$ fraction of monochromatic solutions from a random coloring is asymptotically optimal (see Appendix A for proof). Asymptotic minima are studied most often, but there are also some results on *exact* minima [11].

In this paper, we restrict our focus to 3-term equations and address both upper and lower bounds. For upper bounds, we show that all 3-term equations are *uncommon*, i.e. we can always color $[n]$ in such a way which produces asymptotically fewer monochromatic solutions than what is expected from uniformly random colorings. This result is of interest because all 3-term equations (in fact, all equations with an odd number of terms) are actually *common* over any abelian group whose order is coprime to each coefficient of the equation [4]. For lower bounds, we use a structure theorem (a robust version of Freiman's $3k-4$ Theorem [16]) to show equations of the form $ax + ay = cz$, $a, c \in \mathbb{N}$, always have $\Omega(n^2)$ monochromatic solutions.

Even though the main focus in this paper is 3-term equations, the notation below is kept more general in order to make connections with the result in [20], which we will discuss later. Let E be an equation $a_1x_1+\cdots+a_kx_k = 0$ with integer coefficients whose inputs are taken from a finite set A, and let $f : A \to \{-1, 1\}$ a coloring. Denote the set of all solutions

$$T_E(A) \coloneqq \{(x_1, \dots, x_k) \in A^k \mid a_1x_1 + \cdots + a_kx_k = 0\}, \tag{2}$$

and denote the set of all monochromatic solutions

$$M_E(f) \coloneqq \{(x_1, \dots, x_k) \in T_E(A) \mid f(x_1) = \cdots = f(x_k)\}. \tag{3}$$

Next, the proportion of monochromatic solutions under f is denoted

$$\mu_E(f) \coloneqq \frac{|M_E(f)|}{|T_E(A)|} \tag{4}$$

(with $\mu_E(f)$ defined to equal 0 if $|T_E(A)| = 0$). Finally, the value in question is the minimum monochromatic proportion:

$$\mu_E(A) \coloneqq \min_{f:A\to\{\pm 1\}} \mu_E(f). \tag{5}$$

2. Upper bounds

With the previous notation, an equation E is **uncommon over** $[n]$ if

$$\limsup_{n\to\infty} \mu_E([n]) < \frac{1}{2^{k-1}} \tag{6}$$

(asymptotically strictly less than 2^{1-k}). To reiterate, 2^{1-k} is the expected value of $\mu_E(f)$ when f is a uniformly random coloring, assuming that the equation has both positive and negative coefficients[2]. Note that what makes an equation uncommon over $[n]$ is a *sequence* of colorings in n, but will often refer to the sequence simply as a single coloring $f : [n] \to \{-1, 1\}$ defined in terms of n. With this, we can now state our main result formally.

Theorem 2.1. *All equations $ax+by+cz = 0$ with $a, b, c \in \mathbb{Z}$ are uncommon over $[n]$.*

Here we emphasize "over $[n]$" because of other results when $[n]$ is replaced by an abelian group [4, 12, 15, 7]. To show many equations are uncommon over $[n]$, we will color cyclic groups and extend them to colorings of $[n]$, an idea that has some similarities with techniques for solving related problems [12, 2]. We will actually define our colorings via probability distributions and use Fourier-analytic techniques like those in [4, 5, 7, 15, 20]. We do not make a distinction between the general cyclic group of order m and the integers modulo m, which we will denote $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

Below is the crucial lemma that allows us to work in $\mathbb{Z}_m$ rather than $[n]$. An analogous statement can be found in [12] regarding arithmetic progressions. While our results are centered around 3-term equations, we state this fact more generally for use in a later discussion.

Lemma 2.1. *Given an equation $E : a_1x_1 + \cdots + a_kx_k = 0$ and a positive integer m,*

$$\limsup_{n\to\infty} \mu_E([n]) \le \mu_E(\mathbb{Z}_m). \tag{7}$$

[2] Note that equations with all coefficients the same sign (e.g. $x + y + z = 0$) will have 0 solutions for any n, and therefore $\mu_E(f) := 0$. Such equations are always uncommon by our definition.

Proof. Let $f : \mathbb{Z}_m \to \{-1, 1\}$ be a coloring that achieves the minimum on the right-hand side. This coloring can be extended to a coloring $\tilde{f} : [n] \to \{-1, 1\}$ very naturally by composing f with the canonical projection map $[n] \to \mathbb{Z}_m$. By design, a vector in $[n]^k$ is monochromatic if and only if it is monochromatic when projected onto the corresponding vector in $\mathbb{Z}_m^k$. Let $Cn^{k-1} + O(n^{k-2})$ be the number of solutions to the equation over $[n]$, where C is some positive constant. Then each solution over $\mathbb{Z}_m$ corresponds to

$$C\left(\frac{n}{m}\right)^{k-1} + O(n^{k-2})$$

solutions over $[n]$. Using the fact that $|T_E(\mathbb{Z}_m)| = m^{k-1}$, we get

$$\begin{aligned}\mu_E([n])& \\ &\leq \frac{|M_E(\tilde{f})|}{|T_E([n])|} = \frac{\mu_E(\mathbb{Z}_m)m^{k-1}[C(n/m)^{k-1} + O(n^{k-2})]}{Cn^{k-1} + O(n^{k-2})} = \mu_E(\mathbb{Z}_m) + o(1),\end{aligned}$$

and the result follows. □

Lemma 2.1 is critical because it allows us to prove results (and use past results) over $\mathbb{Z}_m$ and apply them to scenarios over $[n]$. In practice, the "colorings" we use are actually defined probabilistically, and we invoke the probabilistic method to say that if there is a random coloring whose expected proportion of monochromatic solutions is at most some value K, then there must exist an actual coloring f such that $\mu_E(f) \leq K$.

Remark. In the graph theoretic setting, the proportion analogous to μ,

$$\frac{\text{min. \# of monochr. } H \text{ in } K_n}{\text{total \# of } H \text{ in } K_n},$$

has a limit as $n \to \infty$ (often referred to as the *Ramsey multiplicity constant*). The proof of this fact is straightforward, as the sequence is bounded and monotonic. However, for equations the corresponding sequence is not monotonic. We still expect the limit to exist, but a proof (or counterexample) has not yet been found.

We prove Theorem 2.1 in a series of steps, each of which handles some subset of 3-term equations $ax + by + cz = 0$, $a, b, c \in \mathbb{Z}$. First, we use Fourier-analytic techniques and Lemma 2.1 to deal with most equations. Next, we state and prove a modest proposition for nearly all the equations not covered in the first step and again utilize Lemma 2.1. Finally, the equations which remain are a small and rigid class of equations and one isolated equation, and for these we explicitly define colorings with asymptotically fewer than the critical $1/4$ monochromatic fraction expected from uniformly random colorings. Detailed computations for the equations in this step are provided in Appendix B. We always assume the equations are fully reduced, i.e. $\gcd(a, b, c) = 1$.

2.1. Fourier-analytic techniques

First, we will cover the standard notation for Fourier analysis in this setting. For a generalized and thorough introduction, we recommend [17, Chapter 4]. Let $f : \mathbb{Z}_m \to [0, 1]$, which we associate with a probabilistic coloring via

$$f(t) = \mathbf{P}[t \text{ is red}]. \tag{8}$$

When we identify elements in $\mathbb{Z}_m$ with the integers $0, 1, \ldots, m-1$, both

$$f(t) \quad \text{and} \quad e^{-2\pi i \xi t/m} \quad (t, \xi \in \mathbb{Z}_m)$$

are well-defined notions. The **Fourier transform of** f, denoted $\widehat{f}$, is the function from $\mathbb{Z}_m$ to $\mathbb{C}$ given by

$$\widehat{f}(\xi) \coloneqq \frac{1}{m} \sum_{t \in \mathbb{Z}_m} f(t) e^{-2\pi i \xi t/m}. \tag{9}$$

In the arguments of [4, 7, 20], it was crucial that the order of the group was relatively prime to each coefficient of the equation. We will use similar tools, but we will actually exploit the fact that these results do not always hold without this condition.

Without loss of generality we may assume $|c| = \max\{|a|, |b|, |c|\}$. Put $m = |c|$. In order to use Fourier transforms effectively, we need additional assumptions:

$$m > |a|, |b|, \tag{10}$$

$$\text{One of } \gcd(a, m), \ \gcd(b, m) \text{ is equal to } 1, \tag{11}$$

$$a + b \not\equiv 0 \mod m). \tag{12}$$

Our goal now is to show that all equations of this form are uncommon over $\mathbb{Z}_m = \mathbb{Z}_{|c|}$, as this combined with Lemma 2.1 implies they are also uncommon over $[n]$. We will then show, using various other techniques, that the equations not satisfying one of the above assumptions are still uncommon.

First, we can write the expected number of red solutions over $\mathbb{Z}_m$ in terms of Fourier transforms:

$$\mathbf{E}[\#\text{ of red solutions}] = m^2\widehat{f}(0)\sum_{t\in\mathbb{Z}_m}\widehat{f}(at)\widehat{f}(bt). \tag{13}$$

Note, this formula requires that m does not divide a or b, which is guaranteed by (10). Extending this idea, the expected proportion of monochromatic solutions is

$$\begin{aligned}&\mu_{\{ax+by+cz=0\}}(f)\\ &\quad= \widehat{f}(0)\sum_{t\in\mathbb{Z}_m}\widehat{f}(at)\widehat{f}(bt) + \widehat{(1-f)}(0)\sum_{t\in\mathbb{Z}_m}\widehat{(1-f)}(at)\widehat{(1-f)}(bt).\end{aligned} \tag{14}$$

Therefore, to show $ax+by+cz=0$ is uncommon over $\mathbb{Z}_m$, we simply need to find an f such that (14) is strictly less than $1/4$. In order to simplify calculations, we will impose the restriction $\widehat{f}(0)=1/2$, which is equivalent to requiring overall red and blue appear with equal probability. This gives us

$$\begin{aligned}\mu_{\{ax+by+cz=0\}}(f) &= \frac{1}{4}+\frac{1}{2}\sum_{t\in\mathbb{Z}_m-\{0\}}[\widehat{f}(at)\widehat{f}(bt)+\widehat{(1-f)}(at)\widehat{(1-f)}(bt)]\\ &= \frac{1}{4}+\sum_{\substack{t\in\mathbb{Z}_m\\ at,bt\neq 0}}\widehat{f}(at)\widehat{f}(bt).\end{aligned}$$

The last equality follows from the fact that $\widehat{(1-f)}(s) = -\widehat{f}(s)$ whenever $s\neq 0$, so any summand in the first sum with exactly one of at, bt equal to 0 will be 0 (and the case $at=bt=0$ will only occur when $t=0$ since the equation is fully reduced). Therefore, it suffices to find an f such that $\widehat{f}(0)=1/2$ and

$$\sum_{\substack{t\in\mathbb{Z}_m\\ at,bt\neq 0}}\widehat{f}(at)\widehat{f}(bt) < 0. \tag{15}$$

We will refer to the above sum as the **deviation**. By the Fourier inversion formula, we may define f by its Fourier coefficients, although some care must be taken to ensure $\mathrm{Range}(f)\subseteq[0,1]$. First, we will utilize the fact that f is real-valued if and only if $\widehat{f}$ is Hermitian: $\overline{\widehat{f}(s)}=\widehat{f}(-s)$ for all s. Next, we must find Fourier coefficients that guarantee f is between 0 and 1. To do this, we will use the **Fourier inversion formula**:

$$f(v) = \sum_{t \in \mathbb{Z}_m} \widehat{f}(t) e^{2\pi i t v/m}. \tag{16}$$

By requiring $\widehat{f}(0) = 1/2$ and using the triangle inequality with (16), we have

$$|f(v) - 1/2| \leq \sum_{t \in \mathbb{Z}_m - \{0\}} |\widehat{f}(t)|. \tag{17}$$

Regarding Assumption (11), without loss of generality we may assume $\gcd(a, m) = 1$. We split the work into two cases: $a \neq b$ and $a = b$. When $a \neq b$, we may set $\widehat{f}(\pm a) = -1/8$, $\widehat{f}(\pm b) = 1/9$, and $\widehat{f}(s) = 0$ for all other $s \neq 0$. Note that if we did not assume (12), these choices could not be made. With this $\widehat{f}$ is Hermitian, and by (17) $0 \leq f(v) \leq 1$ for all v. Now we argue the deviation is negative. Here, the deviation will have at least two negative terms and at most two positive terms. To see this, the negative terms are guaranteed by $t = \pm 1$, which are distinct since (10) and (12) together imply $m \geq 3$. Positive terms arise when

$$(at, bt) \in \{(\pm a, \pm a), (\pm a, \mp a), (\pm b, \pm b), (\pm b, \mp b)\}.$$

Since $\gcd(a, m) = 1$, $t \mapsto at$ is injective modulo m, so $(at, bt) \in \{(\pm a, \pm a),$ $(\pm a, \mp a)\}$ will only occur when $t = \pm 1$. In the other cases, $t = \pm b a^{-1} \notin \{\pm 1\}$ is possible. Therefore, the deviation is at most

$$-2 \cdot \frac{1}{8} \cdot \frac{1}{9} + 2 \cdot \frac{1}{9^2} < 0,$$

and hence the equation is uncommon over $\mathbb{Z}_m$.

If $a = b$, then we simply take $\widehat{f}(\pm a) = \pm i/4$ and $\widehat{f}(s) = 0$ for all other $s \neq 0$. Again, $\widehat{f}$ is Hermitian and f takes values within $[0, 1]$, and here the deviation $-1/8$. This covers all cases, proving any equation satisfying the initial assumptions (10), (11), and (12) is uncommon over $\mathbb{Z}_m$ and is therefore uncommon over $[n]$ by Lemma 2.1. Next we will cover equations that do not satisfy those assumptions.

2.2. Remaining equations

As discussed previously, the Fourier-analytic techniques do not cover every equation. Recall the assumptions we needed:

(10) $m > |a|, |b|$,
(11) One of $\gcd(a, m)$, $\gcd(b, m)$ is equal to 1,
(12) $a + b \not\equiv 0 \mod m$).

If (11) does not hold, then we may assume one of these gcds is at least 3, as they cannot both be 2 with the equation fully reduced. For these equations, we have the following proposition.

Proposition 2.1. *Every 3-term equation with two coefficients that have a common factor of at least 3 is uncommon over* $[n]$.

Proof. Without loss of generality, assume $m = \gcd(a, c) \geq 3$. As done previously, we will work in $\mathbb{Z}_m$. The coloring is quite simple: $f(0) = -1$, and $f(t) = 1$ otherwise. Note that since the equation is fully reduced, every solution will be of the form $(x, 0, z) \in \mathbb{Z}_m^3$, and x, z are unrestricted. Therefore, only one solution, namely $(0, 0, 0)$, will be monochromatic, and hence the monochromatic proportion is $1/m^2 \leq 1/9 < 1/4$. By Lemma 2.1 this extends to a coloring of $[n]$, and hence the equation is uncommon over $[n]$. □

Equations where (12) does not hold are equivalent to one of three types of equations:

$$ax + by = -(a + b)z, \quad ax + by = (a + b)z, \quad \text{and} \quad ax - ay + cz = 0.$$

For the first type, note that our assumption that z has the largest coefficient in magnitude implies that a and b have the same sign. Therefore, the first type has no solutions and is automatically uncommon. The second type is equivalent to a *constellation* shown to be uncommon in [3]. Equations of the third type with $|a| \geq 3$ can be eliminated by Proposition 2.1, which does not require that c is the largest coefficient. If $|a| = 1$, the equations are equivalent to ones of the form $x - y + cz = 0$, which were shown to be uncommon in [18] (in fact, the authors found asymptotic minima). If $|a| = 2$, we are left with equations of the form

$$2x - 2y + cz = 0. \tag{18}$$

If (10) does not hold but the largest coefficients are at least 3, then Proposition 2.1 ensures these equations are uncommon. Up to equivalence, the equations left in this case are

$$x + y - z = 0, \quad 2x - y + 2z = 0, \quad \text{and} \quad 2x + y - 2z = 0.$$

The first equation is Schur's equation, discussed previously. Therefore, the only equations not yet covered are, up to equivalence:

$$2x - 2y + cz = 0 \qquad \text{and} \qquad 2x - y + 2z = 0. \tag{19}$$

We now describe colorings for these equations that yield asymptotically fewer than a $1/4$ proportion of monochromatic solutions, and detailed computations can be found in Appendix B.

The colorings for equations of the form $2x-2y+cz = 0$ all have a similar construction: alternate between red and blue until some boundary point αn that depends on c, and then color from αn to n entirely red. Let the coloring $f : [n] \to \{-1, 1\}$ be defined as follows:

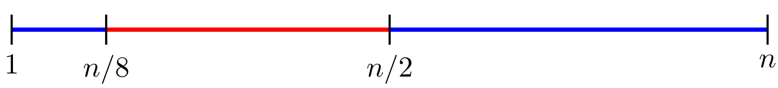
$$f(t) = \begin{cases} -1, & t \text{ even}, t \leq \alpha n, \\ 1, & \text{otherwise}, \end{cases} \qquad \text{where} \qquad \alpha = \begin{cases} 3/4, & c = 1, \\ 2/c, & c \geq 3 \end{cases}$$

(note that c is odd because our equations are fully reduced). With these colorings, we get monochromatic proportions of

$$\begin{cases} 5/24 + o(1), & c = 1, \\ 1/c^2 + o(1), & c \geq 3, \end{cases}$$

both of which are asymptotically less than $1/4$, proving these equations are uncommon.

For the final equation, $2x - y + 2z = 0$, we use the following coloring:



(or as close to this as possible if n is not a multiple of 8). With this coloring, the proportion of monochromatic solutions is $1/64 + o(1)$, far less than the $1/4$ threshold. This finally proves Theorem 2.1, i.e. all 3-term equations are uncommon over $[n]$. Next, we will calculate lower bounds for a specific class of 3-term equations.

3. Lower bounds

Every equation

$$a_1x_1 + \cdots + a_kx_k = 0 \quad (a_i \in \mathbb{Z}) \tag{20}$$

with at least one positive and one negative coefficient has $Cn^{k-1} + O(n^{k-2})$ solutions for some $C > 0$ depending on the coefficients, and we believe a positive fraction of these will always be monochromatic, as long as at least three coefficients are nonzero. Note that it follows from a result of Rado (see discussion following equation (3) in section 2 of [13]) that a 2-coloring of these equations has at least one monochromatic solution.

Conjecture 3.1. *Given an equation $a_1x_1 + \cdots + a_kx_k = 0$ with nonzero $a_i \in \mathbb{Z}$, $k \geq 3$, and at least one negative and one positive coefficient, every coloring has $\Omega(n^{k-1})$ monochromatic solutions over $[n]$.*

As stated previously, a result of Frankl, Graham, and Rödl confirms this conjecture for equations which have a subset of coefficients that sum to 0 [8]. And in fact, they showed this for systems of equations (with an analogous assumption on the coefficients) and colorings of an arbitrary number of colors. They also showed that this is not necessarily true for equations in general via the equation $x + y - 3z = 0$ using 5 colors. We expect this lower bound on the number of monochromatic solutions to still hold when only two colors are used. We make partial progress towards this conjecture.

Theorem 3.1. *Equations of the form $ax + ay - cz = 0$ ($a, c \in \mathbb{N}$) always have $\Omega(n^2)$ monochromatic solutions.*

We will prove this by using the structure theorem from Xuancheng Shao and Max Wenqiang [16]. We may assume a and c are relatively prime. Fix a coloring $f : [n] \to \{-1, 1\}$, and let $R = f^{-1}(\{1\})$ and $B = f^{-1}(\{-1\})$ denote the red and blue elements, respectively. We will actually show there are $\Omega(n^2)$ monochromatic solutions just among the multiples of c, so we denote $R' = R \cap c\mathbb{Z}$ and $B' = B \cap c\mathbb{Z}$. Let C_1 and C_2 be small, positive constants possibly depending on a and c to be determined later.

Claim 3.1. *If $|R'| \leq C_1 n$ for sufficiently small C_1, then there are $\Omega(n^2)$ blue solutions.*

Proof. There are $\Omega(n^2)$ total solutions involving only multiples of c. Since each number in R' is present in at most $3n$ solutions, by assumption there are at most $3C_1n^2$ solutions with an input from R'. If we make C_1 small enough, this still leaves $\Omega(n^2)$ solutions with inputs exclusively from B'. □

By this claim, we may assume $|R'|, |B'| \geq C_1 n$. Now we will cover some necessary notation. Let $X, Y \subseteq \mathbb{Z}$. The **sum set** of two sets X and Y, denoted $X + Y$, is

$$X + Y = \{x + y : x \in X, y \in Y\}. \tag{21}$$

The basic outline of our argument is as follows: if the sum sets $R' + R'$ and $B' + B'$ are both large, then they will have a nontrivial intersection, and if one of these sum sets is small, then $[n] \cap c\mathbb{Z}$ will contain large monochromatic arithmetic progressions, and both cases imply there will be $\Omega(n^2)$ monochromatic solutions. Rather than use sum sets, We will use the *robust* sum sets defined in [16]: given a subset $\Gamma \subseteq X \times Y$, let

$$X +_\Gamma Y := \{x + y : (x, y) \in \Gamma\}. \tag{22}$$

For $A \in \{R', B'\}$, let $\Gamma = \Gamma(A)$ be the set of all pairs in $A \times A$ whose sum has at least $C_2 n$ distinct representations as a sum of pairs, i.e.

$$\Gamma = \{(a_1, a_2) \in A \times A : |\{(b_1, b_2) \in A \times A : b_1 + b_2 = a_1 + a_2\}| \geq C_2 n\}. \tag{23}$$

We will first show that if the robust sum sets in question are large, then we will have $\Omega(n^2)$ monochromatic solutions. Let $\epsilon > 0$. We leave it arbitrary for now, but later we will pick a specific ϵ which depends on C_1 and C_2.

Claim 3.2. *If* $|A +_{\Gamma(A)} A| \geq (2 + \epsilon)|A|$ *for* $A = R', B'$, *then*

$$|(R' +_{\Gamma(R')} R') \cap (B' +_{\Gamma(B')} B')| = \Omega(n), \tag{24}$$

which implies there are $\Omega(n^2)$ *monochromatic solutions.*

Proof. We have

$$|R' +_{\Gamma(R')} R'| + |B' +_{\Gamma(B')} B'| \geq (2 + \epsilon)(|R'| + |B'|) = (2 + \epsilon) \left\lfloor \frac{n}{c} \right\rfloor,$$

and since $A +_{\Gamma(A)} A \subseteq c\mathbb{Z} \cap [2n]$ (which has only $\lfloor 2n/c \rfloor$ elements), (24) follows from the Inclusion-Exclusion Principle.

Note that by construction $v \in (R' +_{\Gamma(R')} R') \cap (B' +_{\Gamma(R')} B')$ corresponds to at least $C_2 n$ monochromatic solutions: if v is colored red, each distinct representation will correspond to a red solution, and similarly if v is colored blue. Since there are $\Omega(n)$ such v, we have $\Omega(n^2)$ monochromatic solutions. $\square$

Because of the above claim, we may now assume that one of the robust sum sets is not too large. Without loss of generality, suppose

$$|R' +_{\Gamma(R')} R'| < (2+\epsilon)|R'|. \tag{25}$$

We are now in a position to use the previously mentioned structure theorem [16]. Rather than state the theorem verbatim, we state only what we need for this scenario.

Theorem 3.2. *Let $\epsilon > 0$. Suppose $|R'| \geq \max\{3, 2\epsilon^{-1/2}\}$, and let $\Gamma \subseteq R' \times R'$ be a subset with $|\Gamma| \geq (1-\epsilon)|R'|^2$. If $|R' +_\Gamma R'| < (1+\theta-11\epsilon^{1/2})|R'|$, where $\theta = \frac{1+\sqrt{5}}{2}$, then there is an arithmetic progression P with $|P| \leq |R' +_\Gamma R'| - (1-5\epsilon^{1/2})|R'|$, $|R' \cap P| \geq (1-\epsilon^{1/2})|R'|$.*

With (25), $R' +_\Gamma R'$ is small enough to fit the corresponding assumption to the theorem. We also have an appropriate lower bound on Γ. To see this, note the following claim.

Claim 3.3. *There are at most $\epsilon|R'|^2$ pairs in $R' \times R' - \Gamma$, where $\epsilon = \frac{2C_2}{cC_1^2}$.*

Proof. Since $R' + R'$ contains only multiples of c and lies inside $[2n]$, $|R' + R'| \leq 2n/c$. By the definition of Γ, each of these elements leads to at most $C_2 n$ pairs that are not in Γ. Therefore, since $|R'| \geq C_1 n$,

$$|R' \times R' - \Gamma| \leq (C_2 n)(2n/c) \leq \frac{2C_2}{cC_1^2}|R'|^2. \qquad \square$$

By this claim,

$$|\Gamma| = |R' \times R'| - |R' \times R' - \Gamma| \geq |R'|^2 - \epsilon|R'|^2 = (1-\epsilon)|R'|^2,$$

as required.

Note that the line of reasoning from Claim 3.1 up to this point is valid for B' as well (with the same choice for ϵ), and that once C_1 is fixed (by Claim 3.1) this choice of ϵ can be made arbitrarily small by decreasing C_2.

By Theorem 3.2, we can now say that R' strongly resembles an arithmetic progression. That is,

$$|R' \cap P| = (1 - o_{\epsilon\to 0}(1))|R'|, \tag{26}$$

where the notation $o_{\epsilon\to 0}(1)$ emphasizes things being asymptotic as $\epsilon \to 0$, rather than $n \to \infty$. With so much information about the coloring (at least on the multiples of c), we can now find a specific progression which contains the desired amount of monochromatic solutions.

Claim 3.4. *There exists an arithmetic progression* $Q = \{dk : 1 \leq k \leq \lfloor n/d \rfloor\}$ *with* $|Q \cap A| = (1 - o_{\epsilon \to 0}(1))|Q|$ *for some* $A \in \{R, B\}$.

Proof. Let P be the progression guaranteed by Theorem 8, containing numbers of the form $a + dk$. If $a = 0$, then P satisfies the conclusions of the claim, so we may assume $0 < a < d$. In this case, the progression $Q = \{dk : 1 \leq k \leq \lfloor n/d \rfloor\}$ is contained almost entirely blue (since it's almost entirely disjoint from R' and everything is still a multiple of c). In other words, $|Q \cap R'| = o_{\epsilon \to 0}(1)$, and so $|Q \cap B| = (1 - o_{\epsilon \to 0}(1))|Q|$. □

Finally, if $x, y \in Q$, then

$$z = \frac{x + y}{c} = \frac{dk_1 + dk_2}{c} = d\left(\frac{k_1 + k_2}{c}\right),$$

which is in Q whenever $k_1 + k_2 \in c\mathbb{Z}$ and $k_1 + k_2 \leq n/d$. There are approximately

$$\frac{1}{2c}\left((1 - o_{\epsilon \to 0}(1))|Q|\right)^2$$

blue pairs $(x, y) \in Q^2$ satisfying these constraints. Since Q contains at most $o_{\epsilon \to 0}(1)|Q|$ red numbers, there are at most $o_\epsilon(1)|Q|^2$ solutions (x, y, z) in Q with x and y blue and z red. Subtracting, we are left with

$$\frac{1}{2c}\left((1 - o_{\epsilon \to 0}(1))|Q|\right)^2 = \Omega(n^2)$$

monochromatic blue solutions. Note that by Theorem 3.2 and the assumption following Claim 3.1, we have

$$|P| \geq |R' \cap P| \geq (1 - \epsilon^{1/2})|R'| \geq (1 - \epsilon^{1/2})C_1 n.$$

In particular, since $|P|$ is linear in size, d must be bounded above by a constant, and hence Q is also linear in size. Therefore, every equation of the form $ax + ay - cz = 0$ has $\Omega(n^2)$ monochromatic solutions regardless of how $[n]$ is colored, proving Theorem 3.1.

4. Conclusion and new directions

We have shown that all 3-term equations are uncommon over $[n]$. For any single equation $a_1x_1 + \cdots + a_kx_k = 0$ over an abelian group A whose order is relatively prime to each a_i, a full classification is known.

Theorem 4.1 ([20]). *An equation is uncommon over A if and only if k is even and has no canceling partition.*

A **canceling partition** of an equation is a partition of the coefficients into pairs $\{a_i, a_j\}$ such that $a_i + a_j = 0$. Over $[n]$, we expect the following to be true.

Conjecture 4.1. *An equation is common over $[n]$ if and only if k is even and has a canceling partition.*

Note that if this were true, equations with k even would behave the same over A and $[n]$, while equations with k odd would behave differently. With this paper, the conjecture is now confirmed for $k = 3$. Much is still unknown, but we can also definitively say that equations with k even and no canceling partition are uncommon over $[n]$. This is simply because they are uncommon over $\mathbb{Z}_p$ if p is a large enough prime ($p > \max\{|a_i|\}$) by Theorem 4.1, and Lemma 2.1 implies they are also uncommon over $[n]$. Appendix A provides a proof for the only type of equation known to be common over $[n]$:

$$x_1 + \cdots + x_{k/2} = x_{k/2+1} + \cdots + x_k \quad (k \text{ even}). \tag{27}$$

For instance, it is not even known if $x + 2y = z + 2w$ is common.

Aside from these types of classification problems (and ones which address systems of equations as in [10]), improving upper and lower bounds on minima remains widely open. Furthermore, all these questions and more can be asked about colorings of more than 2 colors.

Appendix A. Monochromatic solutions for additive tuples

Fix $k \in \mathbb{N}$ even. Here we prove all *additive tuples*, equations of the form

$$x_1 + \cdots + x_{k/2} = x_{k/2+1} + \cdots + x_k, \tag{28}$$

are common over $[n]$, i.e. the minimum fraction of monochromatic solutions the same as what is expected from uniformly random colorings: 2^{1-k}. Let p be a prime which is larger than $kn/2$. We identify $[n]$ with the subset $S = \{1, 2, \ldots, n\} \subseteq \mathbb{Z}_p$ and note that by our choice of p any solution to (28) over S is also a solution over the integers.

Let $\mathbb{1}_S$ be the indicator function of S, and recall the definition of the Fourier transform from Section 2.1:

$$\widehat{\mathbb{1}_S}(\xi) = \frac{1}{p} \sum_{t \in \mathbb{Z}_p} \mathbb{1}_S(t) e^{-2\pi i \xi t/p}.$$

It is standard (see, for example, Equation (4.14) in [17]) that the number of solutions to (28) in S is given by

$$p^{k-1}\sum_{t\in\mathbb{Z}_p}\left|\widehat{\mathbb{1}_S}(t)\right|^k.$$

Now suppose we have a partition of S into a red set R and a blue set B. The total number of monochromatic solutions to $x_1+\cdots+x_{k/2}=x_{k/2+1}+\cdots+x_k$ is then given by

$$p^{k-1}\sum_{j\in\mathbb{Z}_p}\left|\widehat{\mathbb{1}_R}(j)\right|^k+p^{k-1}\sum_{j\in\mathbb{Z}_p}\left|\widehat{\mathbb{1}_B}(j)\right|^k.$$

Using the inequality $x^k+y^k\geq 2^{1-k}(x+y)^k$, which is valid for all real x,y and $k\geq 2$ even (by Jensen's inequality), we get that the number of monochromatic solutions is at least

$$\begin{aligned}(p/2)^{k-1}\sum_{j\in\mathbb{Z}_p}\left(|\widehat{\mathbb{1}_R}(j)|+|\widehat{\mathbb{1}_B}(j)|\right)^k &\geq (p/2)^{k-1}\sum_{j\in\mathbb{Z}_p}\left|\widehat{\mathbb{1}_R}(j)+\widehat{\mathbb{1}_B}(j)\right|^k\\ &= 2^{1-k}\left(p^{k-1}\sum_{j\in\mathbb{Z}_p}\left|\widehat{\mathbb{1}_S}(j)\right|^k\right).\end{aligned}$$

In other words, the number of monochromatic solutions is always at least a 2^{1-k} fraction of the total number of solutions.

Appendix B. Computations

Below are detailed calculations for the equations remaining after using the Fourier-analytic techniques, Proposition 2.1, and past results [14, 3, 18].

B.1. $2x-2y+cz=0$

Recall the colorings $f:[n]\to\{-1,1\}$ (with -1 as blue and 1 as red) used for these equations:

$$f(t)=\begin{cases}-1, & t \text{ even}, t\leq\alpha n,\\ 1, & \text{otherwise},\end{cases}\qquad\text{where}\qquad \alpha=\begin{cases}3/4, & c=1,\\ 2/c, & c\geq 3\end{cases}$$

(note that c is odd because our equations are fully reduced). We address the case when $c \geq 3$ and $c = 1$ separately.

Let $c \geq 3$. Since in any solution z is even, this coloring forces z to be blue: if $z \in [2n/c, n]$, then

$$2(y - x) = cz \geq c(2n/c) = 2n,$$

so $y - x \geq n$, but this is not possible. Therefore, all monochromatic solutions are blue, and in particular $x, y, z \in [1, 2n/c]$. There are $2n^2/c^2 + O(n)$ ways to choose two numbers x and y in $[1, 2n/c]$ (note $y > x$ is required for a valid solution to the equation). Only $1/4 + O(n^{-1})$ of the pairs (x, y) are blue[3]. Furthermore, $2(y - x)$ must be divisible by c, and only $1/c + O(n^{-1})$ of the pairs (x, y) meet that requirement. Finally, once x and y are chosen, z is determined, and z is always in $[1, 2n/c]$: $z = 2(y - x)/c < 2n/c$. Therefore, there are

$$\frac{n^2}{2c^3} + O(n)$$

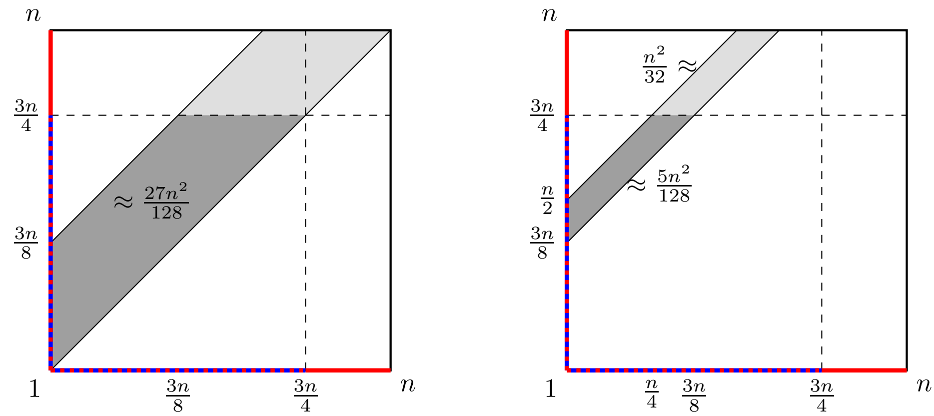
monochromatic solutions. The total number of solutions is $n^2/2c + O(n)$: there are $\binom{n}{2}$ ways to choose two numbers in $[n]$ and set them as x and y, and $1/c + O(n^{-1})$ of these pairs will have $z = 2(y - x)/c \in \mathbb{Z}$ (and z will always be in $[n]$). This gives us

$$\mu_{\{2x-2y+cz=0\}}(f) \leq \frac{n^2/2c^3 + O(n)}{n^2/2c + O(n)} = \frac{1}{c^2} + o(1) = \frac{1}{4} - \Omega(1) \quad \text{(for } c \geq 3\text{)}.$$

Now let $c = 1$. We will use the fact that for any solution z must be even and break the counting into two cases: (a) $z \in [1, 3n/4]$ (blue solutions) and (b) $z \in [3n/4, n]$ (red solutions). To count the number of monochromatic solutions, it helps to visualize solutions on an $n \times n$ grid. For our purposes here the horizontal axis will represent the x values, and the vertical axis will represent the y values. Once x and y are chosen, $z = 2(y - x)$ is determined, and valid solutions $(x, y, 2(y - x))$ in $[n]^3$ will lie within a certain area on the grid. Figure 1 is provided as a visual aid for the following computations.

(a) For a blue solution, we must have $x, y, z \in [1, 3n/4]$. There are $27n^2/128 + O(n)$ valid choices for x and y in $[1, 3n/4]$ that also lead to

[3]The $O(n^{-1})$ error term here is due to edge effects from the boundaries of the regions; the total number of pairs involved in such effects is $O(n)$. We use facts similar to this several more times throughout this Appendix.



(a) $z \in [1, 3n/4]$: The gray areas combined represent all pairs (x, y) with $1 \leq z = 2(y - x) \leq 3n/4$. Since here we are counting blue solutions, $x, y \in [1, 3n/4]$, as well, i.e. we only consider the dark gray area. The area of the dark gray trapezoid must be multiplied by $1/4$, since only about $1/4$ of the pairs (x, y) in that region are blue.

(b) $z \in [3n/4, n]$: The gray areas combined represent all pairs (x, y) with $3n/4 \leq z = 2(y - x) \leq n$. Since we are counting red solutions, the dark gray area must be multiplied by $1/4$, because only about $1/4$ of the pairs (x, y) in that trapezoid are red, and the light gray area must be multiplied by $1/2$, because only about half of the x values there are red (the y values in the light gray area are all red).

Figure 1: Two depictions of the $n \times n$ grid in the xy-plane. Labeled points on the axes correspond to boundary points of the colored regions.

$z \in [1, 3n/4]$. Note, however, that only $1/4 + O(n^{-1})$ of the pairs (x, y) will be blue. Therefore, there are

$$\frac{27}{512}n^2 + O(n)$$

blue solutions.

(b) For a red solution, note that since z must be even, $z \in [3n/4, n]$. For valid x and y, there are two possible cases here: (i) $x \in [1, 3n/4]$ and $y \in [3n/4, n]$, or (ii) $x, y \in [1, 3n/4]$. In (i) there are $n^2/32 + O(n)$ valid choices for x and y, but only $1/2 + O(n^{-1})$ of the x will be red. Therefore, the contribution from (i) is

$$\frac{1}{64}n^2 + O(n).$$

In (ii) there are $5n^2/128+O(n)$ solutions, but only a $1/4+O(n^{-1})$ proportion of the pairs (x,y) will be red, so the contribution from (ii) is

$$\frac{5}{512}n^2+O(n).$$

Adding up all the blue solutions and all the red solutions, we get

$$\left(\frac{27}{512}+\frac{1}{64}+\frac{5}{512}\right)n^2+O(n)=\frac{5}{64}n^2+O(n)$$

monochromatic solutions.

The total number of solutions is $3n^2/8+O(n)$, because for a solution we must have

$$1\le z=2(y-x)\le n,$$

or $0<y-x\le n/2$, and there are $3n^2/8+O(n)$ pairs (x,y) which satisfy this. Therefore,

$$\mu_{\{2x-2y+z=0\}}([n])\le\frac{5}{24}+o(1)=\frac{1}{4}-\Omega(1),$$

i.e. $2x-2y+z=0$ is uncommon over $[n]$.

B.2. $\boldsymbol{2x-y+2z=0}$

We will now cover a general technique to show an individual equation is uncommon, and then we will use it on the equation $2x-y+2z=0$. Fix the equation $ax+by+cz=0$, and let $f:[n]\to\{-1,1\}$ be a coloring. Consider the value

$$L=\sum_{ai+bj+ck=0} f(i)f(j)+f(i)f(k)+f(j)f(k). \tag{29}$$

Here and elsewhere in this section, the variables i,j,k are implicitly assumed to lie in $[n]$. L, in a sense, indirectly counts the number of monochromatic solutions: by direct computation, each summand is 3 if i,j,k are monochromatic and is -1 otherwise, so

$$L=3(\#\text{ monochr. solutions})-(\#\text{ non-monochr. solutions}).$$

With a straightforward manipulation, we get

$$\#\text{ monochr. solutions}=\frac{1}{4}(\#\text{ total solutions})+\frac{L}{4}, \tag{30}$$

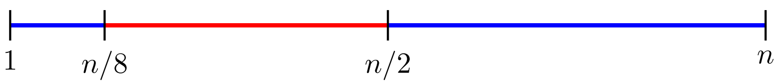
which means that to show $ax + by + cz = 0$ is uncommon, we only need to exhibit a family of colorings with $L = Cn^2 + O(n)$ for some $C < 0$.

Our next task is to find a way to actually compute L. For $i < j$, let $N(i, j)$ denote the number of times a solution contains i and j as two of the three values for x, y, z (in no particular order). Then we can rewrite (29) as

$$L = \sum_{i<j} N(i,j)f(i)f(j) + O(n). \tag{31}$$

Note the $O(n)$ term accounts for the possibility of solutions with $i = j$. We can view $N(i, j)$ as the sum of six indicator-like functions, each corresponding to where there exists a solution with (i, j) playing the role of some ordered pair from $\{x, y, z\}$. We will examine the total contribution of each of these functions separately in computing L, using areas in an $n \times n$ grid to aid the calculations.

The coloring

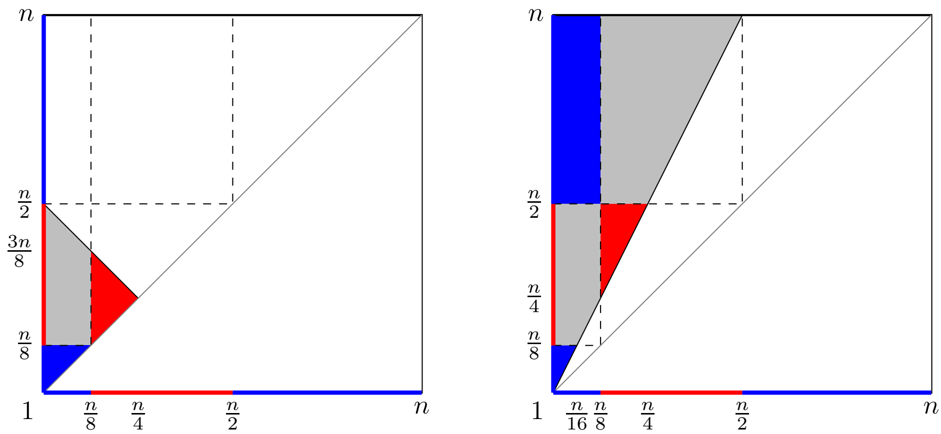


will be enough for our purposes[4]. To compute L, the cases to consider are

1. (i, j) plays the role of (x, z): $2i-y+2j = 0$; restriction: $1 \le 2i+2j \le n$.
2. (i, j) plays the role of (z, x): $2j-y+2i = 0$; restriction: $1 \le 2j+2i \le n$.
3. (i, j) plays the role of (x, y): $2i-j+2z = 0$; restrictions: $2 \le j-2i \le 2n$, j even.
4. (i, j) plays the role of (y, x): $2j-i+2z = 0$; restrictions: $2 \le i-2j \le 2n$, i even.
5. (i, j) plays the role of (y, z): $2x-i+2j = 0$; restrictions: $2 \le i-2j \le 2n$, i even.
6. (i, j) plays the role of (z, y): $2x-j+2i = 0$; restrictions: $2 \le j-2i \le 2n$, j even.

Note in each of these six cases one of the bounds holds trivially.

[4] This coloring was obtained by first running a basic version of the local optimization algorithm described in [3] for $n = 1000$. We then simplified the coloring by hand and blew it up to an arbitrary n. The hand-manipulation did increase the number of monochromatic solutions slightly, but it greatly simplified the following calculations.



(a) Visual for L_1.

(b) Visual for L_3.

Figure 2: In each case, the contributions to L are computed by subtracting the gray area (dichromatic pairs) from the red/blue area (monochromatic pairs). The lighter regions represent $i \geq j$ and are not a part of L. Labeled points on the axes correspond to boundary points of the colored regions.

Let us explore Case 1. We start by defining an "indicator" of sorts, which will help us rewrite (31):

$$I_1(i,j) = \begin{cases} f(i)f(j), & 1 \leq 2i + 2j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Aggregating, we define $L_1 = \sum_{i<j} I_1(i,j)$, which simply counts up Case 1's contribution to (31). Note Case 2 is identical to Case 1.

We can approach Case 3 in a similar manner, but there is an additional twist. If we define

$$I_3(i,j) = \begin{cases} f(i)f(j), & 2 \leq j - 2i \leq 2n, \\ 0, & \text{otherwise,} \end{cases}$$

then $L_3 = \sum_{i<j} I_3(i,j)$ includes the contributions from both even and odd j, so the contribution from Case 3 is actually $\frac{1}{2}L_3 + O(n)$. The rest of the L_r are defined similarly.

Each L_r can be computed by considering the pairs (i,j) in Case r with $i < j$ and subtracting the number of dichromatic pairs from the number of monochromatic pairs. Similar to (yet distinct from) the counting technique implemented for the equation $2x - 2y + z = 0$, to compute a given L_r we can consider areas within an $n \times n$ grid, as seen in Figure 2, now with i represented on the horizontal axis and j on the vertical axis.

Case 2 is identical to 1, Case 5 is identical to 3, and Cases 4 and 6 lie completely in $i \geq j$ and therefore will not contribute to L. This allows us to simplify the calculation:

$$L = L_1 + L_2 + \frac{1}{2}(L_3 + L_5) + O(n) = 2L_1 + L_3 + O(n) = -\frac{15}{128}n^2 + O(n). \tag{32}$$

The coefficient of n^2 is negative, so by (30) this coloring gives (asymptotically) fewer monochromatic solutions than what is expected from uniformly random colorings, i.e. $2x - y + 2z = 0$ is uncommon over $[n]$.

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