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On Mallows' variation of the Stern-Brocot tree

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Mallows (2011) showed that if we, starting with the initial sequence $\langle \frac{0}{1}, \frac{1}{1} \rangle$, successively insert two fractions between two adjacent fractions in a certain way, then every fraction from $\mathbb{Q} \cap (0,1)$ eventually appears. In this article, we first show that Mallows' variation MT_k $(k \geq 0)$ of the Stern-Brocot tree can be obtained from the left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree. We then present algorithms for the following two questions: Where is the j'th element of $\mathrm{MT}_{k'}$ placed in $\mathrm{SBT}^{\mathrm{L}}$? Conversely, where is the jth element of the kth level of $\mathrm{SBT}^{\mathrm{L}}$ placed in Mallows' variation?

We then do similar things for the tree R-DT obtained from the Ducci tree by reversing the paths. More precisely, inspired by the way that we obtained Mallows' variation from the left subtree of the Stern-Brocot tree, we introduce a variation VT_k $(k \geq 0)$ of the tree R-DT and study analogous questions concerning placement between VT_k $(k \geq 0)$ and the left subtree R-DT^{L} of the tree R-DT. We also provide an algorithm, which, given a $k \geq 2$, outputs the ordered set VT_k as a sequence.

Lastly, we explain how Mallows' variation of the Stern-Brocot tree and our variation of the tree R-DT are related to each other.

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1. Introduction

Starting with the initial sequence $\langle \frac{0}{1}, \frac{1}{0} \rangle$, successively update it by inserting fractions as follows: In the first step, between $\frac{0}{1}$ and $\frac{1}{0}$, insert their mediant $\frac{0+1}{1+0}$ to obtain $\langle \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \rangle$. In the second step, between each adjacent pair of fractions from $\langle \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \rangle$, insert their mediant $\frac{0+1}{1+1}$ and $\frac{1+1}{1+0}$ to obtain $\langle \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0} \rangle$. In the third step, between each adjacent pair of fractions from $\langle \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0} \rangle$, insert their mediant $\frac{0+1}{1+2}, \frac{1+1}{2+1}, \frac{1+2}{1+1}$ and $\frac{2+1}{1+0}$ to obtain $\langle \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0} \rangle$. And so on. The *Stern-Brocot tree* [1, 7] is the labeled binary tree whose kth level $(k \geq 0)$ is labeled by the fractions inserted at the $(k+1)$st step. It is known that this tree contains (as a label of some vertex) each positive fractions precisely once. For background information, we refer the reader to [2].

When studying the relationship between the Stern-Brocot tree and the Ducci map, we introduced an analogous tree, which we termed the *Ducci tree* [4]. By reversing the paths in the Ducci tree, we also defined the tree R-DT in [3]. Like the Stern-Brocot tree, both of these trees contain each positive fractions precisely once. Conversion algorithms between the kth levels of the Stern-Brocot tree and the tree R-DT as well as properties of the tree R-DT are presented in [3].

Another variation was introduced by Mallows who inserted not one but two fractions in the above construction. Explicitly, in the first step, between $\frac{0}{1}$ and $\frac{1}{1}$, insert two fractions $\frac{0+1}{1+1}$ and $\frac{0+2\cdot 1}{1+2\cdot 1}$ to obtain $\langle \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{1}{0} \rangle$. In the second step, between $\frac{0}{1}$ and $\frac{1}{2}$ (resp. $\frac{1}{2}$ and $\frac{2}{3}$, $\frac{2}{3}$ and $\frac{1}{1}$), insert $\frac{0+1}{1+2}$ and $\frac{0+2\cdot 1}{1+2\cdot 2}$ (resp. $\frac{2\cdot 1+2}{2\cdot 2+3}$ and $\frac{1+2}{2+3}$, $\frac{2+1}{3+1}$ and $\frac{2+2\cdot 1}{3+2\cdot 1}$) to obtain $\langle \frac{0}{1}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \rangle$. And so forth. Let us write MT_k $(k \geq 0)$ for the ordered set consisting of the fractions inserted at the $(k+1)$st step. Then each fractions from $\mathbb{Q} \cap (0,1)$ appears in one of MT_k $(k \geq 0)$ precisely once [6].

In Section 3, we shall first show that Mallows' variation MT_k $(k \geq 0)$ can be obtained from the left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree. We then present algorithms for the following two questions: Where is the j'th element of $\mathrm{MT}_{k'}$ placed in $\mathrm{SBT}^{\mathrm{L}}$? Conversely, where is the jth element of the kth level of $\mathrm{SBT}^{\mathrm{L}}$ placed in Mallows' variation? In Section 4, inspired by the way that we obtained Mallows' variation from the left subtree of the Stern-Brocot tree, we shall introduce a variation VT_k $(k \geq 0)$ of the tree R-DT and study similar questions concerning placement between VT_k $(k \geq 0)$ and the left subtree R-DT^{L} of the tree R-DT. We also provide an algorithm, which, given a $k \geq 2$, outputs the ordered set VT_k as a sequence. In the last section, we explain how Mallows' variation of the Stern-Brocot tree and our variation of the tree R-DT are related to each other.

2. Notation and terminology

For the later sections, we first prepare notation and terminology here.

Let T be a rooted binary tree. For any its vertices v, v' such that either $v = v'$ or one of them is an ancestor of the other, we recursively define a finite sequence $\mathrm{Path}_{\mathrm{T}}(v, v')$ of $\mathtt{L}$'s and $\mathtt{R}$'s by:

$$\mathrm{Path}_{\mathrm{T}}(v, v') = \begin{cases} \langle\rangle & \text{if } v = v' \\ \langle \mathtt{L} \rangle ^\frown \mathrm{Path}_{\mathrm{T}}(v'', v') & \text{if either (a) } v' \text{ is equal to or a descendant of the left child } v'' \text{ of } v \text{ or (b) } v'' \text{ is equal to or a descendant of } v' \text{ and } v \text{ is the left child of } v'' \\ \langle \mathtt{R} \rangle ^\frown \mathrm{Path}_{\mathrm{T}}(v'', v') & \text{if either (a) } v' \text{ is equal to or a descendant of the right child } v'' \text{ of } v \text{ or (b) } v'' \text{ is equal to or a descendant of } v' \text{ and } v \text{ is the right child of } v'' \end{cases}.$$

Here and in what follows, the symbol $^\frown$ represents the concatenation operator, e.g., $\langle \mathtt{L}, \mathtt{R} \rangle ^\frown \langle \mathtt{R}, \mathtt{R}, \mathtt{L} \rangle = \langle \mathtt{L}, \mathtt{R}, \mathtt{R}, \mathtt{R}, \mathtt{L} \rangle$.

Another fundamental concept is that of *level* in T, which is defined inductively as follows: The root is at level 0. If a vertex is at level k, then its children are at level $k+1$. We write T_k for the set of all vertices at level k. By ordering vertices from left to right, we shall often view T_k as an ordered set.

In this paper, we shall sometimes identify an ordered set with a sequence. Hence for example, if v' and v'' are the left and right children of the root of T, respectively, then two equations $\mathrm{T}_1 = \{v', v''\}$ and $\mathrm{T}_1 = \langle v', v'' \rangle$ are both correct for us.

We shall also use finite continued fraction. Write $[a_0; a_1, a_2, \ldots, a_\ell]$ for

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell}}}}.$$

$a_0, a_1, a_2, \ldots, a_\ell$ (called *elements* of the continued fraction) are such that $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}_{>0}$. In order to use the finite continued fractions as an apparatus for representing the rationals, we shall adopt the convention that the last element a_ℓ is larger than 1 if $\ell > 0$. Then every rational

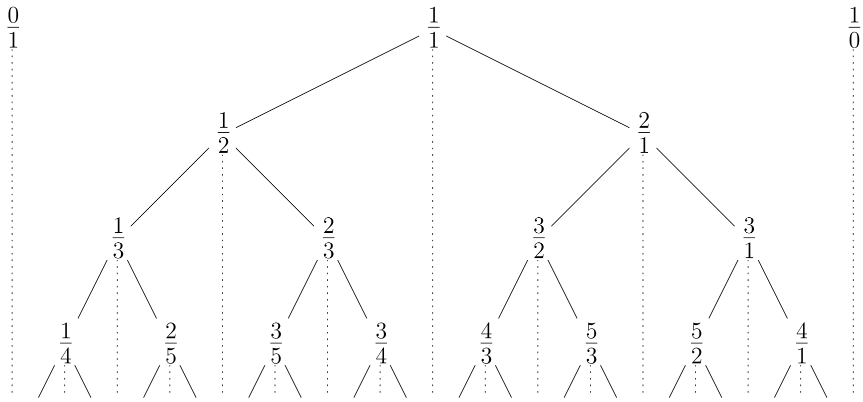

Figure 1: Top levels of the Stern-Brocot tree.

number admits a unique representation as a finite continued fraction. For more details on continued fraction, we refer the reader to [5].

3. The Stern-Brocot tree and its variation

Let us start with the definition of the Stern-Brocot tree [1, 7]:

Definition 1. Define (ordered) sets SBT_k ($k \geq 0$) of 2^k fractions by induction as follows: Let $\mathrm{SBT}_0 := \left\{\frac{1}{1}\right\}$. Suppose we have defined $\mathrm{SBT}_0, \mathrm{SBT}_1, \ldots, \mathrm{SBT}_k$ and let $\frac{n_1}{m_1} < \frac{n_2}{m_2} < \cdots < \frac{n_{2^{k+1}+1}}{m_{2^{k+1}+1}}$ be the elements of $\left\{\frac{0}{1}, \frac{1}{0}\right\} \cup \mathrm{SBT}_0 \cup \mathrm{SBT}_1 \cup \cdots \cup \mathrm{SBT}_k$. ($\frac{1}{0}$ is viewed as the largest element.) Then SBT_{k+1} is the (ordered) set $\left\{\frac{n_1+n_2}{m_1+m_2}, \frac{n_2+n_3}{m_2+m_3}, \ldots, \frac{n_{2^{k+1}}+n_{2^{k+1}+1}}{m_{2^{k+1}}+m_{2^{k+1}+1}}\right\}$.

The *Stern-Brocot tree* (SBT) is the labeled binary tree such that the labels of its kth level ($k \geq 0$), in left-to-right order, is SBT_k. (See Figure 1.)

Since the fraction $\frac{n_1+n_2}{m_1+m_2}$ is called the *mediant* of $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$, the above way of constructing SBT_{k+1} from $\left\{\frac{0}{1}, \frac{1}{0}\right\} \cup \mathrm{SBT}_0 \cup \mathrm{SBT}_1 \cup \cdots \cup \mathrm{SBT}_k$ is referred to as the *mediant construction*.

In what follows, we are concerned with the left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree. It is formally defined as the rooted binary tree whose kth level is the left half of the $(k+1)$st level of the Stern-Brocot tree. Observe that this definition is equivalent to the ensuing inductive one, which does not refer to the Stern-Brocot tree: Let $\mathrm{SBT}^{\mathrm{L}}_0 := \left\{\frac{1}{2}\right\}$. Suppose we have defined $\mathrm{SBT}^{\mathrm{L}}_0, \mathrm{SBT}^{\mathrm{L}}_1, \ldots, \mathrm{SBT}^{\mathrm{L}}_k$ and let $\frac{n_1}{m_1} < \frac{n_2}{m_2} < \cdots < \frac{n_{2^{k+1}+1}}{m_{2^{k+1}+1}}$ be the elements of $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_k^{\mathrm{L}}$. Then $\mathrm{SBT}_{k+1}^{\mathrm{L}}$ is the (ordered) set $\{\frac{n_1+n_2}{m_1+m_2}, \frac{n_2+n_3}{m_2+m_3}, \ldots, \frac{n_{2^{k+1}}+n_{2^{k+1}+1}}{m_{2^{k+1}}+m_{2^{k+1}+1}}\}$.

Let us list some known properties:

Proposition 1 ([2]). *Let $k \geq 0$.*

(i) *The left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree contains each fraction from $\mathbb{Q} \cap (0,1)$ precisely once;*

(ii) *The left-to-right order and the order defined by the magnitude relation coincide on $\mathrm{SBT}_k^{\mathrm{L}}$;*

(iii) *If $\frac{n_1}{m_1} < \frac{n_2}{m_2}$ are adjacent elements in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_k^{\mathrm{L}}$, then $n_1 m_2 - m_1 n_2 = -1$;*

(iv) *If $\frac{n}{m}$ appears in $\mathrm{SBT}_k^{\mathrm{L}}$ and if $\frac{n_L}{m_L}$ (resp. $\frac{n_R}{m_R}$) is its left (resp. right) adjacent element in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_k^{\mathrm{L}}$, then $\frac{n_L}{m_L}$ (resp. $\frac{n_R}{m_R}$) and the left (resp. right) child of $\frac{n}{m}$ are adjacent in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_{k+1}^{\mathrm{L}}$.* □

Since the first item above guarantees that different vertices get different labels, we shall hereafter freely identify a vertex with its label without explicit mention.

The following property will be well-known but, as it is important for us, we shall include its proof here:

Proposition 2 (folklore). *Let $\frac{n}{m} \in \mathbb{Q} \cap (0,1)$ and let $\frac{n'}{m'}, \frac{n''}{m''}$ be its left and right children in $\mathrm{SBT}^{\mathrm{L}}$, respectively.*

(i) *If n is odd, then precisely one of n' and n'' is even;*

(ii) *If n is even, then both n' and n'' are odd.*

Proof. Let $k \geq 0$ be such that the fraction $\frac{n}{m}$ is at level k of $\mathrm{SBT}^{\mathrm{L}}$.
(i): If $k = 0$, then the statement is evidently true. So suppose $k \geq 1$ and let $\frac{n_1}{m_1}$ be the parent of $\frac{n}{m}$ in $\mathrm{SBT}^{\mathrm{L}}$. Then it follows from the mediant construction that there exists a fraction $\frac{n_2}{m_2}$ adjacent to $\frac{n_1}{m_1}$ in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_{k-1}^{\mathrm{L}}$ such that $\frac{n_1+n_2}{m_1+m_2} = \frac{n}{m}$. The mediant construction also implies that children of $\frac{n}{m}$ are $\frac{n_1+n}{m_1+m}$ and $\frac{n_2+n}{m_2+m}$. Hence $n' + n'' = n_1 + n + n_2 + n = 3n$ is odd, which indicates that precisely one of n' and n'' is even.
(ii): Since $\frac{n}{m}$ and $\frac{n'}{m'}$ (resp. $\frac{n}{m}$ and $\frac{n''}{m''}$) are adjacent in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup \cdots \cup \mathrm{SBT}_{k+1}^{\mathrm{L}}$, which is immediate from the mediant construction, we have $|nm' - mn'| = |nm'' - mn''| = 1$ by Proposition 1 (iii). As n is even, these equations force n' and n'' to be odd. □

There are several trees closely related to the Stern-Brocot tree: In the left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree, the set $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathrm{L}} \cup \mathrm{SBT}_1^{\mathrm{L}} \cup$

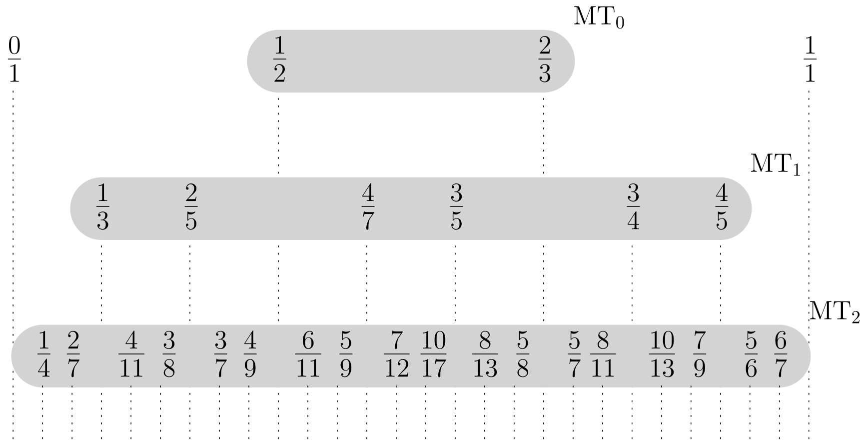

Figure 2: Top levels of MT_k $(k \geq 0)$.

$\cdots \cup \mathrm{SBT}^{\mathrm{L}}_{k+1}$ can be obtained from $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}^{\mathrm{L}}_0 \cup \mathrm{SBT}^{\mathrm{L}}_1 \cup \cdots \cup \mathrm{SBT}^{\mathrm{L}}_k$ by inserting one fraction between each pair of adjacent fractions. By inserting not one but two fractions, Mallows introduced the following variation:

Definition 2 ([6]). Define (ordered) sets MT_k $(k \geq 0)$ of $2 \cdot 3^k$ fractions inductively as follows: Let $\mathrm{MT}_0 := \{\frac{1}{2}, \frac{2}{3}\}$. Suppose we have defined $\mathrm{MT}_0, \mathrm{MT}_1, \ldots, \mathrm{MT}_k$ and let $\frac{n_1}{m_1} < \frac{n_2}{m_2} < \cdots < \frac{n_{3^{k+1}+1}}{m_{3^{k+1}+1}}$ be the elements of $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_k$. Then MT_{k+1} is the (ordered) set $\{\frac{n'_1}{m'_1}, \frac{n'_2}{m'_2}, \ldots, \frac{n'_{2\cdot 3^{k+1}}}{m'_{2\cdot 3^{k+1}}}\}$, where

$$\left\langle \frac{n'_{2j-1}}{m'_{2j-1}}, \frac{n'_{2j}}{m'_{2j}} \right\rangle = \begin{cases} \left\langle \frac{n_j+n_{j+1}}{m_j+m_{j+1}}, \frac{n_j+2n_{j+1}}{m_j+2m_{j+1}} \right\rangle & \text{if } n_j \text{ is even} \\ \left\langle \frac{2n_j+n_{j+1}}{2m_j+m_{j+1}}, \frac{n_j+n_{j+1}}{m_j+m_{j+1}} \right\rangle & \text{otherwise} \end{cases}$$

for each $j = 1, 2, \ldots, 3^{k+1}$. (See Figure 2.)

A simple calculation shows the inequality $\frac{n_j}{m_j} < \frac{n'_{2j-1}}{m'_{2j-1}} < \frac{n'_{2j}}{m'_{2j}} < \frac{n_{j+1}}{m_{j+1}}$, which indicates that the set $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_{k+1}$ can indeed be obtained from $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_k$ by inserting two fractions between each pair of adjacent fractions and also that three pairs $\frac{n_j}{m_j} < \frac{n'_{2j-1}}{m'_{2j-1}}$, $\frac{n'_{2j-1}}{m'_{2j-1}} < \frac{n'_{2j}}{m'_{2j}}$, $\frac{n'_{2j}}{m'_{2j}} < \frac{n_{j+1}}{m_{j+1}}$ are all adjacent ones in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_{k+1}$. It can also be inferred that the left-to-right order and the order defined by the magnitude relation coincide on MT_k.

Our first result states that this variation can be obtained from $\mathrm{SBT}^{\mathrm{L}}$ by iterating the following, one level at a time, from the zeroth level to deeper level: if the left (resp. right) child of a vertex at the specified level has even numerator, then bring up the subtree under that child by one level and place it on the left (resp. right) of the vertex. To make the statement more precise, let us make some preparations:

Definition 3. On the set $\left\{\frac{n}{m} \in \mathbb{Q} \cap (0,1) \,\middle|\, n \text{ is odd}\right\}$, define two functions φ_{SBT} and ψ_{SBT} by setting

$$\varphi_{\mathrm{SBT}}\left(\tfrac{n}{m}\right) := \begin{cases} \langle \frac{n'}{m'}, \frac{n}{m} \rangle & \text{if the left child } \frac{n'}{m'} \text{ of } \frac{n}{m} \text{ in} \\ & \quad \mathrm{SBT}^{\mathrm{L}} \text{ has even numerator} \\ \langle \frac{n}{m}, \frac{n''}{m''} \rangle & \text{if the right child } \frac{n''}{m''} \text{ of } \frac{n}{m} \text{ in} \\ & \quad \mathrm{SBT}^{\mathrm{L}} \text{ has even numerator} \end{cases}$$

and

$$\psi_{\mathrm{SBT}}\left(\tfrac{n}{m}\right) := \begin{cases} \langle \frac{n'}{m'}, \frac{n'''}{m'''}, \frac{n''''}{m''''} \rangle & \text{if the left child } \frac{n'}{m'} \text{ of } \frac{n}{m} \text{ in } \mathrm{SBT}^{\mathrm{L}} \text{ has} \\ & \text{odd numerator and } \frac{n'''}{m'''} \text{ and } \frac{n''''}{m''''} \text{ are} \\ & \text{the left and right children of the right} \\ & \text{child of } \frac{n}{m} \text{ in } \mathrm{SBT}^{\mathrm{L}}, \text{ respectively} \\ \langle \frac{n'''}{m'''}, \frac{n''''}{m''''}, \frac{n''}{m''} \rangle & \text{if the right child } \frac{n''}{m''} \text{ of } \frac{n}{m} \text{ in } \mathrm{SBT}^{\mathrm{L}} \text{ has} \\ & \text{odd numerator and } \frac{n'''}{m'''} \text{ and } \frac{n''''}{m''''} \text{ are} \\ & \text{the left a right children of the left} \\ & \text{child of } \frac{n}{m} \text{ in } \mathrm{SBT}^{\mathrm{L}}, \text{ respectively} \end{cases}.$$

Extend these two functions to the sequences of positive fractions with odd numerators by setting

$$\begin{aligned} \varphi_{\mathrm{SBT}}\left(\langle \tfrac{n_1}{m_1}, \tfrac{n_2}{m_2}, \ldots, \tfrac{n_\ell}{m_\ell} \rangle\right) &:= \varphi_{\mathrm{SBT}}\left(\tfrac{n_1}{m_1}\right)^\frown \varphi_{\mathrm{SBT}}\left(\tfrac{n_2}{m_2}\right)^\frown \cdots ^\frown \varphi_{\mathrm{SBT}}\left(\tfrac{n_\ell}{m_\ell}\right) \\ \psi_{\mathrm{SBT}}\left(\langle \tfrac{n_1}{m_1}, \tfrac{n_2}{m_2}, \ldots, \tfrac{n_\ell}{m_\ell} \rangle\right) &:= \psi_{\mathrm{SBT}}\left(\tfrac{n_1}{m_1}\right)^\frown \psi_{\mathrm{SBT}}\left(\tfrac{n_2}{m_2}\right)^\frown \cdots ^\frown \psi_{\mathrm{SBT}}\left(\tfrac{n_\ell}{m_\ell}\right) \end{aligned}$$

It is immediate from Proposition 2 that these two functions are well-defined. The same proposition also implies that fractions appearing in the sequence $\psi_{\mathrm{SBT}}\left(\frac{n}{m}\right)$ all have odd numerator. A simple induction on k then shows that any fraction from $\psi_{\mathrm{SBT}}^k\left(\frac{n}{m}\right)$ has odd numerator, which in turn implies (again by the same proposition) that precisely one child of it has odd numerator.

Here is the precise statement of how MT_k can be obtained from $\mathrm{SBT}^{\mathrm{L}}$ (see Figure 3):

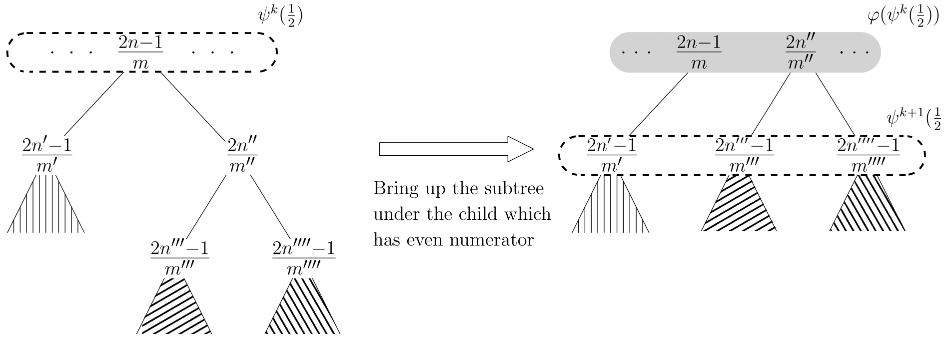

Figure 3: Illustration of how to define $\varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$ and $\psi^{k+1}\big(\frac{1}{2}\big)$.

Theorem 1. $\mathrm{MT}_k = \varphi_{\mathrm{SBT}}\big(\psi^k_{\mathrm{SBT}}\big(\frac{1}{2}\big)\big)$ *for any* $k \geq 0$.

Proof. Two remarks are in order before going to the main part of the proof: In this proof (and throughout this section), when we simply write φ and ψ, it means φ_{SBT} and ψ_{SBT}, respectively. Also, when we simply say that a fraction is the parent (or a child) of another fraction, we are referring to the parent and child relation in $\mathrm{SBT}^{\mathrm{L}}$.

Let us then go to the proof of the theorem. For technical reasons, we shall actually prove the following two statements about k simultaneously by induction on k:

$(k)'$ $\mathrm{MT}_k = \varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$.

$(k)''$ For any fraction $\frac{n}{m}$ from $\psi^k\big(\frac{1}{2}\big)$, if its left (resp. right) child has odd numerator, then three pairs $\frac{n_L}{m_L} < \frac{n}{m}$, $\frac{n}{m} < \frac{n+n_R}{m+m_R}$ and $\frac{n+n_R}{m+m_R} < \frac{n_R}{m_R}$ (resp. $\frac{n_L}{m_L} < \frac{n_L+n}{m_L+m}$, $\frac{n_L+n}{m_L+m} < \frac{n}{m}$ and $\frac{n}{m} < \frac{n_R}{m_R}$) are all adjacent ones in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \varphi\big(\psi^0\big(\frac{1}{2}\big)\big) \cup \varphi\big(\psi^1\big(\frac{1}{2}\big)\big) \cup \cdots \cup \varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$, where $\frac{n_L}{m_L}$ and $\frac{n_R}{m_R}$ are the left and right adjacent fractions to $\frac{n}{m}$ in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}^{\mathrm{L}}_0 \cup \mathrm{SBT}^{\mathrm{L}}_1 \cup \cdots \cup \mathrm{SBT}^{\mathrm{L}}_\ell$, respectively. ($\ell$ is the unique integer such that $\frac{n}{m}$ appears in $\mathrm{SBT}^{\mathrm{L}}_\ell$.)

Verification of the correctness for $k = 0, 1$ can be done by hand. For the induction step, assume that we have verified the correctness until k (≥ 1). We first claim that for any $\frac{n}{m} \in \psi^k\big(\frac{1}{2}\big)$, the (ordered) set MT_{k+1} contains every element of $\varphi\big(\psi\big(\frac{n}{m}\big)\big)$. To validate this claim, take a fraction $\frac{n}{m} \in \psi^k\big(\frac{1}{2}\big)$ arbitrarily and let $\frac{n_L}{m_L}$, $\frac{n_R}{m_R}$ be as in the statement $(k)''$. Then, by the mediant construction, the left and right children of $\frac{n}{m}$ are $\frac{n_L+n}{m_L+m}$ and $\frac{n+n_R}{m+m_R}$, respectively. As has been noted already, precisely one child of $\frac{n}{m}$ has odd numerator. Since proofs for the two cases run parallel to each other, we shall take up only the case where the left child $\frac{n_L+n}{m_L+m}$ of $\frac{n}{m}$ has odd numerator. Therefore, in the next paragraph, n_L and n_R are even and odd, respectively.

In view of the mediant construction and Proposition 1 (iv), it is evident that the left and right children of the right child of $\frac{n}{m}$ are $\frac{2n+n_R}{2m+m_R}$ and $\frac{n+2n_R}{m+2m_R}$, respectively. Consequently, $\psi\big(\frac{n}{m}\big) = \big\{\frac{n_L+n}{m_L+m}, \frac{2n+n_R}{2m+m_R}, \frac{n+2n_R}{m+2m_R}\big\}$. It is immediate from the mediant construction that the right child of $\frac{n_L+n}{m_L+m}$ is $\frac{n_L+2n}{m_L+2m}$, which has even numerator. Therefore, we have $\varphi\big(\frac{n_L+n}{m_L+m}\big) = \big\{\frac{n_L+n}{m_L+m}, \frac{n_L+2n}{m_L+2m}\big\}$. As the induction hypothesis $(0)', (1)', \ldots, (k)'$ and $(k)''$ combine to prove that $\frac{n_L}{m_L} < \frac{n}{m}$ are adjacent elements in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_k$, the definition of MT_{k+1} implies that MT_{k+1} contains both elements $\frac{n_L+n}{m_L+m}, \frac{n_L+2n}{m_L+2m}$ of $\varphi\big(\frac{n_L+n}{m_L+m}\big)$. The mediant construction also implies that the left child of $\frac{2n+n_R}{2m+m_R}$ is $\frac{3n+n_R}{3m+m_R}$, which has even numerator. Hence, $\varphi\big(\frac{2n+n_R}{2m+m_R}\big) = \big\{\frac{3n+n_R}{3m+m_R}, \frac{2n+n_R}{2m+m_R}\big\}$. From $(0)', (1)', \ldots, (k)'$ and $(k)''$, one can derive, as before, that MT_{k+1} contains both elements $\frac{3n+n_R}{3m+m_R}, \frac{2n+n_R}{2m+m_R}$ of $\varphi\big(\frac{2n+n_R}{2m+m_R}\big)$. By a similar line of reasoning, one can show that MT_{k+1} contains both elements of $\varphi\big(\frac{n+2n_R}{m+2m_R}\big) = \big\{\frac{n+2n_R}{m+2m_R}, \frac{n+3n_R}{m+2m_R}\big\}$. We thus conclude that $\mathrm{MT}_{k+1} \supset \varphi\big(\frac{n_L+n}{m_L+m}\big) \cup \varphi\big(\frac{2n+n_R}{2m+m_R}\big) \cup \varphi\big(\frac{n+2n_R}{m+2m_R}\big) = \varphi\big(\psi\big(\frac{n}{m}\big)\big)$, proving the claim.

Since MT_{k+1} and $\varphi\big(\psi^{k+1}\big(\frac{1}{2}\big)\big)$ comprise the same number of fractions, which can be proved by induction, the established claim implies that MT_{k+1} and $\varphi\big(\psi^{k+1}\big(\frac{1}{2}\big)\big)$ are identical as *sets*. The statement $(k+1)'$, which asserts that MT_{k+1} and $\varphi\big(\psi^{k+1}\big(\frac{1}{2}\big)\big)$ are identical as *ordered sets*, follows from this because the left-to-right order and the order defined by the magnitude relation coincide not only on MT_{k+1} but also on $\varphi\big(\psi^{k+1}\big(\frac{1}{2}\big)\big)$, which can be seen by induction.

Having completed the verification of the correctness of the statement $(k+1)'$, let us turn to the proof of the statement $(k+1)''$. Take an element $\frac{n}{m}$ of $\psi^{k+1}\big(\frac{1}{2}\big)$ arbitrarily and let ℓ be such that $\frac{n}{m}$ appears in $\mathrm{SBT}^{\mathrm{L}}_{\ell}$. Since $k \geq 1$, we have $\ell \geq 2$. As in the preceding argument for the statement $(k+1)'$, we shall take up only the case where the left child of $\frac{n}{m}$ has odd numerator. Then, as before, n_L is even and n_R is odd, where $\frac{n_L}{m_L}$ and $\frac{n_R}{m_R}$ are the left and right adjacent element to $\frac{n}{m}$ in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}^{\mathrm{L}}_0 \cup \mathrm{SBT}^{\mathrm{L}}_1 \cup \cdots \cup \mathrm{SBT}^{\mathrm{L}}_{\ell}$, respectively. Observe that one of $\frac{n_L}{m_L}$ and $\frac{n_R}{m_R}$ is the parent of $\frac{n}{m}$. Also, $\frac{n_L}{m_L} < \frac{n_R}{m_R}$ are adjacent elements in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}^{\mathrm{L}}_0 \cup \mathrm{SBT}^{\mathrm{L}}_1 \cup \cdots \cup \mathrm{SBT}^{\mathrm{L}}_{\ell-1}$, whose mediant $\frac{n_L+n_R}{m_L+m_R}$ is equal to $\frac{n}{m}$. There are two cases to consider:

Case 1: $\frac{n_R}{m_R}$ is the parent of $\frac{n}{m}$.

Being the parent of $\frac{n}{m} \in \psi^{k+1}\big(\frac{1}{2}\big)$, the fraction $\frac{n_R}{m_R}$ should be from $\psi^{k}\big(\frac{1}{2}\big)$. Since its left child $\frac{n}{m}$ has odd numerator and since $\frac{n_L}{m_L} < \frac{n_R}{m_R}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{SBT}^{\mathrm{L}}_0\cup \mathrm{SBT}^{\mathrm{L}}_1\cup\cdots\cup \mathrm{SBT}^{\mathrm{L}}_{\ell-1}$ as has been noted already, an application of the induction hypothesis $(0)', (1)', \ldots, (k)'$ and $(k)''$ proves that $\frac{n_L}{m_L} < \frac{n_R}{m_R}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_k$. It then follows that three pairs $\frac{n_L}{m_L} < \frac{n_L+n_R}{m_L+m_R}$, $\frac{n_L+n_R}{m_L+m_R} < \frac{n_L+2n_R}{m_L+2m_R}$ $\frac{n_L+2n_R}{m_L+2m_R} < \frac{n_R}{m_R}$ are all adjacent ones in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_{k+1}$. Combined with the equation $\frac{n_L+n_R}{m_L+m_R} = \frac{n}{m}$, the induction hypothesis $(0)', (1)', \ldots, (k)'$ and the verified statement $(k+1)'$, this proves the statement $(k+1)''$.

Case 2: $\frac{n_L}{m_L}$ is the parent of $\frac{n}{m}$.

Let $\frac{n_1}{m_1}$ be the parent of $\frac{n_L}{m_L}$, which exists because $\frac{n_L}{m_L}$ is from $\mathrm{SBT}^{\mathrm{L}}_{\ell-1}$ with $\ell\geq 2$. If $\frac{n_1}{m_1}$ is $\frac{n_R}{m_R}$, then it can be proved by using the induction hypothesis $(0)', (1)', \ldots, (k)'$ and $(k)''$ that $\frac{n_L}{m_L} < \frac{n_R}{m_R}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_k$. From the definition of MT_{k+1}, it follows that three pairs $\frac{n_L}{m_L} < \frac{n_L+n_R}{m_L+m_R}$, $\frac{n_L+n_R}{m_L+m_R} < \frac{n_L+2n_R}{m_L+2m_R}$ and $\frac{n_L+2n_R}{m_L+2m_R} < \frac{n_R}{m_R}$ are all adjacent ones in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_{k+1}$. The induction hypothesis $(0)', (1)', \ldots, (k)'$ and the verified statement $(k+1)'$ thus prove the statement $(k+1)''$. When $\frac{n_1}{m_1}$ is different from $\frac{n_R}{m_R}$, we argue to show the correctness as follows: Since its child $\frac{n_L}{m_L}$ has even numerator, the fraction $\frac{n_1}{m_1}$ should have odd numerator by Proposition 2. Also, as $\frac{n}{m}$ belongs to $\psi^{k+1}\left(\frac{1}{2}\right)$, it can be inferred from the definition of ψ that $\frac{n_1}{m_1}$ belongs to $\psi^{k}\left(\frac{1}{2}\right)$. Moreover, because $\frac{n_L}{m_L} < \frac{n_R}{m_R}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{SBT}^{\mathrm{L}}_0\cup \mathrm{SBT}^{\mathrm{L}}_1\cup\cdots\cup \mathrm{SBT}^{\mathrm{L}}_{\ell-1}$, the left adjacent element to $\frac{n_R}{m_R}$ in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{SBT}^{\mathrm{L}}_0\cup \mathrm{SBT}^{\mathrm{L}}_1\cup\cdots\cup \mathrm{SBT}^{\mathrm{L}}_{\ell-2}$ should be the parent $\frac{n_1}{m_1}$ of $\frac{n_L}{m_L}$ by Proposition 1 (iv). One can thus apply the induction hypothesis $(0)', (1)', \ldots, (k)', (k)''$ to see that $\frac{n_1+n_R}{m_1+m_R} < \frac{n_R}{m_R}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_k$. It follows that three pairs $\frac{n_1+n_R}{m_1+m_R} < \frac{n_1+2n_R}{m_1+2m_R}$, $\frac{n_1+2n_R}{m_1+2m_R} < \frac{n_1+3n_R}{m_1+3m_R}$, $\frac{n_1+3n_R}{m_1+3m_R} < \frac{n_R}{m_R}$ are all adjacent ones in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_{k+1}$. By applying the induction hypothesis $(0)', (1)', \ldots, (k)'$ and the verified statement $(k+1)'$, because we have $\frac{n_1+n_R}{m_1+m_R} = \frac{n_L}{m_L}$ and $\frac{n_L+n_R}{m_L+m_R} = \frac{n}{m}$ by the mediant construction, we see that the statement $(k+1)''$ is correct. □

Let us derive two properties of Mallows' variation here: The first property is that it contains each fraction from $\mathbb{Q}\cap(0,1)$ precisely once, which was proved as Theorem 1 in [6]. Indeed, a simple induction on k shows that for each fraction $\frac{n}{m}\in \mathrm{SBT}^{\mathrm{L}}_k$, there exists a $k'\leq k$ such that $\frac{n}{m}\in\varphi\left(\psi^{k'}\left(\frac{1}{2}\right)\right)$. This fact, Proposition 1 (i) and Theorem 1 then combine to prove that the Mallows' variation contains each fraction from $\mathbb{Q}\cap(0,1)$ at least once. That each fraction from $\mathbb{Q}\cap(0,1)$ appears *precisely* once can also be derived from Proposition 1 (i) and the definition of φ and ψ. The second property is that if $\frac{n_1}{m_1} < \frac{n_2}{m_2}$ are adjacent elements in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup \mathrm{MT}_0\cup \mathrm{MT}_1\cup\cdots\cup \mathrm{MT}_k$ for some k, then $n_1m_2 - m_1n_2 = -1$. This follows from Proposition 1 (iii) and the ensuing fact, which can be verified by induction on k: if two fractions are adjacent in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_k$, then they are adjacent in $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{SBT}_0^{\mathtt{L}} \cup \mathrm{SBT}_1^{\mathtt{L}} \cup \cdots \cup \mathrm{SBT}_{k'}^{\mathtt{L}}$ for some k'.

Let us temporarily write $\frac{n_{(k,L)}}{m_{(k,L)}}$ and $\frac{n_{(k,R)}}{m_{(k,R)}}$ ($k \geq 0$) for the rightmost element of the left half of MT_k and the leftmost element of the right half of MT_k, respectively. Note that, because the left-to-right order and the order defined by the magnitude relation coincide on MT_k and because two fractions $\frac{n_{(k+1,L)}}{m_{(k+1,L)}} < \frac{n_{(k+1,R)}}{m_{(k+1,R)}}$ are inserted between $\frac{n_{(k,L)}}{m_{(k,L)}}$ and $\frac{n_{(k,R)}}{m_{(k,R)}}$ when constructing MT_{k+1} from $\{\frac{0}{1}, \frac{1}{1}\} \cup \mathrm{MT}_0 \cup \mathrm{MT}_1 \cup \cdots \cup \mathrm{MT}_k$, they satisfy the relation $\frac{n_{(1,L)}}{m_{(1,L)}} < \frac{n_{(2,L)}}{m_{(2,L)}} < \frac{n_{(3,L)}}{m_{(3,L)}} < \cdots < \frac{n_{(3,R)}}{m_{(3,R)}} < \frac{n_{(2,R)}}{m_{(2,R)}} < \frac{n_{(1,R)}}{m_{(1,R)}}$. Two limits $\lim_k \frac{n_{(k,L)}}{m_{(k,L)}} \leq \lim_k \frac{n_{(k,R)}}{m_{(k,R)}}$ coincide and separate each of MT_k ($k \geq 0$) into its left and right half:

Proposition 3. *A fraction from* $\mathbb{Q} \cap (0,1)$ *appears in the left half of* MT_k *for some* $k \geq 0$ *if and only if it is smaller than* $2 - \sqrt{2}$*.*

Proof. It is plain that if a fraction from $\mathbb{Q} \cap (0,1)$ appears in the left (resp. right) half of MT_0, then it is smaller (resp. larger) than $2 - \sqrt{2}$. Also, if a fraction from $\mathbb{Q} \cap (0,1)$ appears in the left (resp. right) half of MT_k for some $k \geq 1$, then, since the left-to-right order and the order defined by the magnitude relation coincide on MT_k, it is smaller than or equal to $\frac{n_{(k,L)}}{m_{(k,L)}} < \lim_k \frac{n_{(k,L)}}{m_{(k,L)}}$ (resp. larger than or equal to $\frac{n_{(k,R)}}{m_{(k,R)}} > \lim_k \frac{n_{(k,R)}}{m_{(k,R)}}$). It is thus sufficient to prove that $\lim_k \frac{n_{(k,L)}}{m_{(k,L)}} = \lim_k \frac{n_{(k,R)}}{m_{(k,R)}} = 2 - \sqrt{2}$. For this purpose, let us calculate the continued fraction representations of $\frac{n_{(k,L)}}{m_{(k,L)}}$ and $\frac{n_{(k,R)}}{m_{(k,R)}}$ ($k \geq 1$).

By Theorem 1, $\frac{n_{(k,L)}}{m_{(k,L)}}$ is the rightmost element $\frac{n_{(k,R)}}{m_{(k,R)}}$ of the left half of $\varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$ and $\frac{n_{(k,R)}}{m_{(k,R)}}$ is the leftmost element of the right half of $\varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$. In view of the mediant construction of the Stern-Brocot tree and the definition of φ and ψ, the following equations can then be verified by induction:

$$\mathrm{Path}_{\mathrm{SBT}}\Big(\frac{2}{3}, \frac{n_{(k,L)}}{m_{(k,L)}}\Big) = \begin{cases} \langle \underbrace{\mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \ldots, \mathtt{R}, \mathtt{R}}_{2k-2}, \mathtt{L}, \mathtt{L} \rangle & \text{if } k \text{ is odd} \\ \langle \underbrace{\mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \ldots, \mathtt{R}, \mathtt{R}}_{2k-4}, \mathtt{L}, \mathtt{L}, \mathtt{R} \rangle & \text{if } k \text{ is even} \end{cases}$$

$$\mathrm{Path}_{\mathrm{SBT}}\Big(\frac{2}{3}, \frac{n_{(k,R)}}{m_{(k,R)}}\Big) = \begin{cases} \langle \underbrace{\mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \ldots, \mathtt{R}, \mathtt{R}}_{2k-2}, \mathtt{L} \rangle & \text{if } k \text{ is odd} \\ \langle \underbrace{\mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \mathtt{L}, \mathtt{L}, \mathtt{R}, \mathtt{R}, \ldots, \mathtt{R}, \mathtt{R}}_{2k} \rangle & \text{if } k \text{ is even} \end{cases}$$

It is known [2] that if the path from $\frac{1}{1}$ to a fraction in the Stern-Brocot tree is equal to

$$\langle \underbrace{U_1, U_1, \ldots, U_1}_{a_1}, \underbrace{U_2, U_2, \ldots, U_2}_{a_2}, \ldots, \underbrace{U_{\ell-1}, U_{\ell-1}, \ldots, U_{\ell-1}}_{a_{\ell-1}}, \underbrace{U_\ell, \ldots, U_\ell}_{a_\ell - 1} \rangle,$$

where $U_1\,(=\mathtt{L}), U_2, \ldots, U_{\ell-1}, U_\ell$ is an alternating sequence of L's and R's, then the continued fraction representation of that fraction is $[0; a_1, a_2, \ldots, a_{\ell-1}, a_\ell]$. Therefore,

$$\frac{n_{(k,L)}}{m_{(k,L)}} = \begin{cases} [0;1,1,\underbrace{2,2,\ldots,2}_{k-1},3] & \text{if } k \text{ is odd} \\ [0;1,1,\underbrace{2,2,\ldots,2}_{k}] & \text{if } k \text{ is even} \end{cases}$$

$$\frac{n_{(k,R)}}{m_{(k,R)}} = \begin{cases} [0;1,1,\underbrace{2,2,\ldots,2}_{k}] & \text{if } k \text{ is odd} \\ [0;1,1,\underbrace{2,2,\ldots,2}_{k-1},3] & \text{if } k \text{ is even} \end{cases}.$$

It follows that $\lim_k \frac{n_{(k,L)}}{m_{(k,L)}} = \lim_k \frac{n_{(k,R)}}{m_{(k,R)}} = [0;1,1,2,2,2,\ldots]$. Since the value of the infinite continued fraction $[0;1,1,2,2,2,\ldots]$ is equal to $2-\sqrt{2}$, this completes the proof. □

In the same spirit, one can show, for instance, that $\frac{4-\sqrt{2}}{7}$ separates each of MT_k $(k \geq 1)$ into its left one-sixth and right five-sixths.

Concerning the relationship between the left subtree $\mathrm{SBT}^{\mathrm{L}}$ of the Stern-Brocot tree and Mallows' variation, there are natural questions: Where is the j'th element of $\mathrm{MT}_{k'}$ placed in $\mathrm{SBT}^{\mathrm{L}}$? Conversely, where is the jth element of $\mathrm{SBT}^{\mathrm{L}}_k$ placed in Mallows' variation? Making use of Theorem 1 and the definition of φ and ψ, we shall present a way of answering these questions without actually constructing Mallows' variation. For the first question, we shall provide the following algorithm, which, when given $k' \geq 0$ and $j' \in \{1, 2, \ldots, 2\cdot 3^{k'}\}$, outputs integers k and j such that the jth element of $\mathrm{SBT}^{\mathrm{L}}_k$ is the j'th element of $\mathrm{MT}_{k'}$:

Algorithm 1

Step 1: Set $w := \langle\rangle$, $\ell := k'$, $i := j'$, $a := 0$, $b := 0$, $k := 0$, $j := 1$.
Step 2: Update w by iterating the following while $\ell \geq 0$:
· Set $a := \lfloor \frac{i-1}{6} \rfloor$ and $b := i - 6a$
· If a is odd and $b = 1$ then $w := \langle \mathtt{L}, \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if a is odd and $b = 2$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if a is odd and $b = 3$ then $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 1$
else if a is odd and $b = 4$ then $w := \langle \mathtt{R}, \mathtt{R} \rangle^\frown w$, $i := 2a + 1$
else if a is odd and $b = 5$ then $w := \langle \mathtt{R}, \mathtt{L} \rangle^\frown w$, $i := 2a + 2$
else if a is odd and $b = 6$ then $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
else if a is even and $b = 1$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if a is even and $b = 2$ then $w := \langle \mathtt{L}, \mathtt{R} \rangle^\frown w$, $i := 2a + 1$
else if a is even and $b = 3$ then $w := \langle \mathtt{L}, \mathtt{L} \rangle^\frown w$, $i := 2a + 2$
else if a is even and $b = 4$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 2$
else if a is even and $b = 5$ then $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
else $w := \langle \mathtt{R}, \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
· Set $\ell := \ell - 1$
Step 3: Set $k := \mathrm{lh}(w) - 1$, $i := 1$.
Step 4: Update j by iterating the following while $i \leq k$:
If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$ then $j := j + 2^{k-i}$, $i := i + 1$ else $i := i + 1$
Step 5: Output k and j.
(Here, $\mathrm{lh}(\cdot)$ and $\mathrm{proj}_i(\cdot)$ denote the functions that return the length and the ith letter of the input sequence, respectively.)

Proof of the correctness of the algorithm. We shall first verify the correctness of the following statement about $k' \geq 0$: For any $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'}\}$,

- the j'th element of $\varphi\big(\psi^{k'}\big(\frac{1}{2}\big)\big)$ has odd numerator if and only if $j' \equiv 1$ or 4 (mod 4);
- if we input k' and j' to the algorithm, then the final updated value of w is equal to the path in the Stern-Brocot tree from the root $\frac{1}{1}$ to the fraction which appears as the j'th element of $\varphi\big(\psi^{k'}\big(\frac{1}{2}\big)\big)$.

Verification of the correctness of the statement is by induction on k': For $k' = 0$ or 1, verification can be done by hand. For the induction step, assume that we have verified the correctness until $k' (\geq 1)$. Take a $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'+1}\}$ arbitrarily and write $\frac{n}{m}$ for the j'th element of $\varphi\big(\psi^{k'+1}\big(\frac{1}{2}\big)\big)$. Since the argument for the case where $\lfloor \frac{j'-1}{6} \rfloor$ is even runs in much the same way as the odd case, we shall take up the latter case only. Let $\frac{n'}{m'}$ and $\frac{n''}{m''}$ be the $\big(2\lfloor \frac{j'-1}{6} \rfloor + 1\big)$st and $\big(2\lfloor \frac{j'-1}{6} \rfloor + 2\big)$nd elements of $\varphi\big(\psi^{k'}\big(\frac{1}{2}\big)\big)$, respectively. An application of the induction hypothesis then implies that n' is even and n'' is odd, which in turn proves, in view of the definition of φ, that $\frac{n'}{m'}$ is the left child of $\frac{n''}{m''}$ in $\mathrm{SBT}^{\mathrm{L}}$. Note that, by Proposition 2, children of $\frac{n'}{m'}$ and the right child of $\frac{n''}{m''}$ all have odd numerator. It can be inferred from the mediant construction that the left child of the left child of $\frac{n'}{m'}$, the right child of the right child of $\frac{n'}{m'}$ and the left child of the right child of $\frac{n''}{m''}$ all have even numerator. From the definition of φ and ψ and the induction hypothesis, we thus conclude that

- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 1$, then $\frac{n}{m}$ is the left child of the left child of $\frac{n'}{m'}$, which has even numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{L}, \mathtt{L}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 2$, then $\frac{n}{m}$ is the left child of $\frac{n'}{m'}$, which has odd numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{L}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 3$, then $\frac{n}{m}$ is the right child of $\frac{n'}{m'}$, which has odd numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{R}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 4$, then $\frac{n}{m}$ is the right child of the right child of $\frac{n'}{m'}$, which has even numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{R}, \mathtt{R}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 5$, then $\frac{n}{m}$ is the left child of the right child of $\frac{n''}{m''}$, which has even numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{R}, \mathtt{L}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 6$, then $\frac{n}{m}$ is the right child of $\frac{n''}{m''}$, which has odd numerator, and the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{R}\rangle = \mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$.

These indicate the correctness of the statement for $k'+1$.

Using the verified statement, we can complete the proof of the correctness as follows: Let $\frac{n}{m}$ be the j'th element of $\mathrm{MT}_{k'}$. Then Theorem 1 and the verified statement combine to show that if we input k' and j' to the algorithm, then the final updated value of w is equal to $\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)$. Therefore, $k = \mathrm{lh}\big(\mathrm{Path}_{\mathrm{SBT}}\big(\frac{1}{1}, \frac{n}{m}\big)\big) - 1 = \mathrm{lh}\big(\mathrm{Path}_{\mathrm{SBT}^{\mathrm{L}}}\big(\frac{1}{2}, \frac{n}{m}\big)\big)$, which clearly indicates that $\frac{n}{m}$ indeed appears at the kth level of $\mathrm{SBT}^{\mathrm{L}}$. That the final updated value of j is the desired one follows from the ensuing general fact, which can be verified readily by induction on ℓ: For any vertex v from level ℓ of the binary tree T (with root v_0), the output of the following simple algorithm is such that v is the jth element of T_k:

$i \coloneqq 1;\ j \coloneqq 1$
while $i \leq \ell$ **do** :
  if $\mathrm{proj}_{i+1}\big(\mathrm{Path}_{\mathrm{T}}(v_0, v)\big) = \mathtt{R}$ **then** $j \coloneqq j + 2^{\ell-i}$, $i \coloneqq i+1$
    else $i \coloneqq i+1$
output j □

Having presented an algorithm for the first question, we then provide an algorithm for the converse question of where the jth element of $\mathrm{SBT}^{\mathtt{L}}_k$ is placed in Mallows' variation. The following algorithm, when given $k \geq 0$ and $j \in \{1, 2, \ldots, 2^k\}$, outputs integers k' and j' such that the j'th element of $\mathrm{MT}_{k'}$ is the jth element of $\mathrm{SBT}^{\mathtt{L}}_k$:

Algorithm 2

Step 1: Set $w \coloneqq \langle\rangle$, $k' \coloneqq 0$, $j' \coloneqq j$, $i \coloneqq 1$, $x \coloneqq 1$.
Step 2: Update w by iterating the following while $i \leq k$:
  If $j' > 2^{k-i}$ then $w \coloneqq w^\frown\langle\mathtt{R}\rangle$, $j' \coloneqq j' - 2^{k-i}$, $i \coloneqq i+1$
    else $w \coloneqq w^\frown\langle\mathtt{L}\rangle$, $i \coloneqq i+1$
Step 3: Set $j' \coloneqq 1$, $i \coloneqq 1$.
Step 4: Update k', j' by iterating the following while $i \leq k$:
  If $x = 1$ and $\mathrm{proj}_i(w) = \mathtt{L}$
    then $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 1$, $x \coloneqq 1$, $i \coloneqq i+1$
  else if $x = 1$ and $\mathrm{proj}_i(w) = \mathtt{R}$
    then $j' \coloneqq j'+1$, $x \coloneqq 4$, $i \coloneqq i+1$
  else if $x = 2$ and $\mathrm{proj}_i(w) = \mathtt{L}$
    then $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 2$, $x \coloneqq 3$, $i \coloneqq i+1$
  else if $x = 2$ and $\mathrm{proj}_i(w) = \mathtt{R}$
    then $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 3$, $x \coloneqq 1$, $i \coloneqq i+1$
  else if $x = 3$ and $\mathrm{proj}_i(w) = \mathtt{L}$
    then $j' \coloneqq j'-1$, $x \coloneqq 2$, $i \coloneqq i+1$
  else if $x = 3$ and $\mathrm{proj}_i(w) = \mathtt{R}$
    then $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 6$, $x \coloneqq 3$, $i \coloneqq i+1$
  else if $x = 4$ and $\mathrm{proj}_i(w) = \mathtt{L}$
    then $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 4$, $x \coloneqq 3$, $i \coloneqq i+1$
  else $k' \coloneqq k'+1$, $j' \coloneqq 6\lfloor\frac{j'-1}{2}\rfloor + 5$, $x \coloneqq 1$, $i \coloneqq i+1$
Step 5: Output k' and j'.

Proof of the correctness of the algorithm. For given k and j, let $\frac{n}{m}$ be the jth element of the kth level of $\mathrm{SBT}^{\mathtt{L}}$. In view of Theorem 1, it is sufficient to show that the outputs k' and j' of the algorithm is such that the fraction $\frac{n}{m}$ is the j'th element of $\varphi\big(\psi^{k'}\big(\frac{1}{2}\big)\big)$. Since this is evident when $k = 0$, we shall hereafter assume $k \geq 1$.

A simple induction on k shows that the variable w, when Step 2 of the algorithm has been completed, has length k and has value equal to $\mathrm{Path}_{\mathrm{SBT}^{\mathtt{L}}}\big(\frac{1}{2}, \frac{n}{m}\big)$. For $i \in \{0, 1, \ldots, k\}$, let $\frac{n_i}{m_i}$ be the fraction such that $\mathrm{Path}_{\mathrm{SBT}^{\mathtt{L}}}\big(\frac{1}{2}, \frac{n_i}{m_i}\big)$ is the length i initial subsequence of $\mathrm{Path}_{\mathrm{SBT}^{\mathtt{L}}}\big(\frac{1}{2}, \frac{n}{m}\big)$. Also, to make the ensuing argument precise, let us write $k'(i)$, $j'(i)$ and $x(i)$ for the values of k', j' and x when the ith round of the while-loop in Step 4 has been completed, respectively. (We set $k'(0) \coloneqq 0$, $j'(0) \coloneqq 1$ and $x(0) \coloneqq 1$.) We claim that for $i = 0, 1, \ldots, k$, the fraction $\frac{n_i}{m_i}$ is the $j'(i)$th element of $\varphi\big(\psi^{k'(i)}\big(\frac{1}{2}\big)\big)$ whose numerator n_i has the same parity as $x(i)$. This will complete the proof because we have $\frac{n_k}{m_k} = \frac{n}{m}$ and the outputs of the algorithm are $k'(k)$ and $j'(k)$.

Let us validate the claim by induction on i: For $i = 0$, the claim can be verified readily. For $i = 1$, since the initial value of x is 1, if $\mathrm{proj}_1(w) = \mathtt{L}$, then $\frac{n_1}{m_1} = \frac{1}{3}$, $k'(1) = 1$, $j'(1) = 1$ and $x(1) = 1$. From Figure 2, it is clear that $\frac{n_1}{m_1}$ is the $j'(1)$st element of $\varphi\big(\psi^{k'(1)}\big(\frac{1}{2}\big)\big)$. As $n_1 = 1 = x(1)$, the claim is indeed correct in this case. If $\mathrm{proj}_1(w) = \mathtt{R}$, then $\frac{n_1}{m_1} = \frac{2}{3}$, $k'(1) = 0$, $j'(1) = 2$ and $x(1) = 4$. Figure 2 shows that $\frac{n_1}{m_1}$ is the $j'(1)$st element of $\varphi\big(\psi^{k'(1)}\big(\frac{1}{2}\big)\big)$. Also, $n_1 = 2$ and $x(1) = 4$ have the same parity. Therefore, the claim is correct also in this case. For the induction step, assume that we have verified the correctness of the claim until $i\,(\geq 1)$. Observe that if $x(i)$ is even, then an application of the induction hypothesis proves that n_i is even too, which excludes by Proposition 2 the case where the parent $\frac{n_{i-1}}{m_{i-1}}$ of $\frac{n_i}{m_i}$ has even numerator. Therefore, there are the following six cases:

Case 1: $x(i) = 1$ and n_{i-1} is odd.

From the induction hypothesis, it is evident that n_i and $x(i-1)$ are both odd. In order to have $x(i) = 1$, it should be the case that $x(i-1) = 1$ and $\mathrm{proj}_i(w) = \mathtt{L}$. Being the left child of $\frac{n_{i-1}}{m_{i-1}}$, the fraction $\frac{n_i}{m_i}$ should be equal to $\frac{n_L + n_{i-1}}{m_L + m_{i-1}}$, where $\frac{n_L}{m_L}$ is the left adjacent element to $\frac{n_{i-1}}{m_{i-1}}$ in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}^{\mathtt{L}}_0 \cup \mathrm{SBT}^{\mathtt{L}}_1 \cup \cdots \cup \mathrm{SBT}^{\mathtt{L}}_{i-1}$. As n_i and n_{i-1} are both odd, it follows that n_L is even. There are two subcases: If $\mathrm{proj}_{i+1}(w) = \mathtt{L}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the left child $\frac{n_L + n_i}{m_L + m_i}$ of $\frac{n_i}{m_i}$, whose right child $\frac{n_L + 2n_i}{m_L + 2m_i}$ has even numerator. From the induction hypothesis and the definition of φ and ψ, it follows that $\frac{n_{i+1}}{m_{i+1}} = \frac{n_L + n_i}{m_L + m_i}$ is the $\big(6\lfloor\frac{j'(i)-1}{2}\rfloor + 1\big)$st element of $\varphi\big(\psi^{k'(i)+1}\big(\frac{1}{2}\big)\big)$. Since $x(i+1) = 1$ has the same parity as $n_{i+1} = n_L + n_i$, the claim is correct in this subcase. If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the right child $\frac{n_i+n_{i-1}}{m_i+m_{i-1}}$ of $\frac{n_i}{m_i}$, which has even numerator. The fraction $\frac{n_i}{m_i}$ being the $j'(i)$th element of $\varphi\big(\psi^{k'(i)}\big(\frac{1}{2}\big)\big)$ by the induction hypothesis, the definition of φ proves that $\frac{n_{i+1}}{m_{i+1}}$ is the $(j'(i)+1)$st element of $\varphi\big(\psi^{k'(i)}\big(\frac{1}{2}\big)\big)$. As the parity of $x(i+1) = 4$ and $n_{i+1} = n_i + n_{i-1}$ are the same, we conclude that the claim is true also in this subcase.

Case 2: $x(i) = 1$ and n_{i-1} is even.

The induction hypothesis and the algorithm show that n_i is odd, $x(i-1)$ is even and $\mathrm{proj}_i(w) = \mathtt{R}$. If $\mathrm{proj}_{i+1}(w) = \mathtt{L}$, then $\frac{n_{i+1}}{m_{i+1}}$ is equal to $\frac{n_{i-1}+n_i}{m_{i-1}+m_i}$, whose right child $\frac{n_{i-1}+2n_i}{m_{i-1}+2m_i}$ has even numerator. The induction hypothesis and the definition of φ and ψ imply that $\frac{n_{i+1}}{m_{i+1}}$ is the $\big(6\lfloor\frac{j'(i)-1}{2}\rfloor + 1\big)$st element of $\varphi\big(\psi^{k'(i)+1}\big(\frac{1}{2}\big)\big)$. Also, $\frac{n_{i+1}}{m_{i+1}}$ has odd numerator and $x(i+1) = 1$. If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$, then $\frac{n_{i+1}}{m_{i+1}}$ should be equal to the mediant $\frac{n_i+n_R}{m_i+m_R}$ of $\frac{n_i}{m_i}$ and the right adjacent element $\frac{n_R}{m_R}$ to $\frac{n_{i-1}}{m_{i-1}}$ in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}_0^{\mathtt{L}} \cup \mathrm{SBT}_1^{\mathtt{L}} \cup \cdots \cup \mathrm{SBT}_{i-1}^{\mathtt{L}}$. As we have $\frac{n_i}{m_i} = \frac{n_{i-1}+n_R}{m_{i-1}+m_R}$, the fraction $\frac{n_{i+1}}{m_{i+1}} = \frac{n_i+n_R}{m_i+m_R} = \frac{n_{i-1}+2n_R}{m_{i-1}+2m_R}$ has even numerator. From the induction hypothesis and the definition of φ, it follows that $\frac{n_{i+1}}{m_{i+1}}$ is the $(j'(i)+1)$st element of $\varphi\big(\psi^{k'(i)}\big(\frac{1}{2}\big)\big)$. Note that the parity of $\frac{n_{i+1}}{m_{i+1}}$ and $x(i+1) = 4$ are the same. Hence the claim is correct in this case.

Case 3: $x(i) = 2$ and n_{i-1} is odd.

From the induction hypothesis and the algorithm, it is immediate that n_i is even, $x(i-1) = 3$ and $\mathrm{proj}_i(w) = \mathtt{L}$. The left adjacent element $\frac{n_L}{m_L}$ to $\frac{n_{i-1}}{m_{i-1}}$ in $\big\{\frac{0}{1}, \frac{1}{1}\big\} \cup \mathrm{SBT}_0^{\mathtt{L}} \cup \mathrm{SBT}_1^{\mathtt{L}} \cup \cdots \cup \mathrm{SBT}_{i-1}^{\mathtt{L}}$ should have odd numerator because the mediant $\frac{n_L+n_{i-1}}{m_L+m_{i-1}}$ is equal to $\frac{n_i}{m_i}$. If $\mathrm{proj}_{i+1}(w) = \mathtt{L}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the left child $\frac{n_L+n_i}{m_L+m_i}$ of $\frac{n_i}{m_i}$, whose left child $\frac{2n_L+n_i}{2m_L+m_i}$ has even numerator. Since $\frac{n_i}{m_i}$ is the $j'(i)$th element of $\varphi\big(\psi^{k'(i)}\big(\frac{1}{2}\big)\big)$ by the induction hypothesis, the definition of φ and ψ implies that $\frac{n_{i+1}}{m_{i+1}}$ is the $\big(6\lfloor\frac{j'(i)-1}{2}\rfloor + 2\big)$nd element of $\varphi\big(\psi^{k'(i)+1}\big(\frac{1}{2}\big)\big)$. Note that $\frac{n_{i+1}}{m_{i+1}}$ has odd numerator and $x(i+1) = 3$. If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the right child $\frac{n_i+n_{i-1}}{m_i+m_{i-1}}$ of $\frac{n_i}{m_i}$, whose right child $\frac{n_i+2n_{i-1}}{m_i+2m_{i-1}}$ has even numerator. From the induction hypothesis and the definition of φ and ψ, it follows that $\frac{n_{i+1}}{m_{i+1}}$ is the $\big(6\lfloor\frac{j'(i)-1}{2}\rfloor + 3\big)$rd element of $\varphi\big(\psi^{k'(i)+1}\big(\frac{1}{2}\big)\big)$. Observe that $\frac{n_{i+1}}{m_{i+1}}$ has odd numerator and $x(i+1) = 1$. In either of the two subcases, $\frac{n_{i+1}}{m_{i+1}}$ is thus the $j'(i+1)$st element of $\varphi\big(\psi^{k'(i+1)}\big(\frac{1}{2}\big)\big)$ and its numerator has the same parity as $x(i+1)$.

Case 4: $x(i) = 3$ and n_{i-1} is odd.
The proof runs in much the same way as in Case 1.

Case 5: $x(i) = 3$ and n_{i-1} is even.
An analogous argument to Case 2 proves the claim also in this case.

Case 6: $x(i) = 4$ and n_{i-1} is odd.
The proof is similar to Case 3. □

There are closely related questions: Given $k' \geq 0$ and $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'}\}$, what is the j'th fraction of $\mathrm{MT}_{k'}$? Conversely, given a fraction $\frac{n}{m}$, where is it placed in Mallows' variation? The first question can be answered by combining Algorithm 1 with an algorithm [2] which calculates the jth fraction of $\mathrm{SBT}^{\llcorner}_k$. To answer the second question, one can first use a known algorithm [2] to calculate where $\frac{n}{m}$ is placed in the Stern-Brocot tree. Using the output as input, Algorithm 2 then calculates k' and j'.

4. The tree R-DT and its variation

Recall that the *Ducci map* D (over the triples) is the one defined by the equation $D(v_1, v_2, v_3) = (|v_1 - v_2|, |v_2 - v_3|, |v_3 - v_1|)$. Using this map, we introduced another labeled binary tree in [4]. To state its definition, observe that for each $\frac{n}{m} \in \mathbb{Q}_{>0} \setminus \{\frac{1}{1}\}$, there exist precisely two fractions $\frac{n'_1}{m'_1} < 1 < \frac{n'_2}{m'_2}$ such that $D\big(0, \frac{n'_i}{m'_i}, 1\big) \sim \big(0, \frac{n}{m}, 1\big)$ $(i = 1, 2)$, where $\sim$ is the smallest equivalence relation on $\mathbb{Q}^3$ satisfying the ensuing two conditions:

- $(v_1, v_2, v_3) \sim \lambda(v_1 - c, v_2 - c, v_3 - c)$ for any $\lambda \in \mathbb{Q}_{>0}$ and $c, v_1, v_2, v_3 \in \mathbb{Q}$;
- $(v_1, v_2, v_3) \sim (v_2, v_3, v_1)$ for any $v_1, v_2, v_3 \in \mathbb{Q}$.

Because of this property, we can make the following:

Definition 4 ([4])**.** The *Ducci tree* (DT) is the labeled binary tree constructed by the next rules:

- The root of the tree is labeled by $\frac{1}{1}$. Its left and right children are labeled by $\frac{1}{2}$ and $\frac{2}{1}$, respectively;
- If a vertex is labeled by a fraction $\frac{n}{m} \neq \frac{1}{1}$ (in simplest terms), then its left (resp. right) child is labeled by $\frac{n'_1}{m'_1}$ (resp. $\frac{n'_2}{m'_2}$), where fractions $\frac{n'_1}{m'_1} < 1 < \frac{n'_2}{m'_2}$ are such that $D\big(0, \frac{n'_i}{m'_i}, 1\big) \sim \big(0, \frac{n}{m}, 1\big)$ $(i = 1, 2)$.

Actually, it turned out that the labeled binary tree obtained by reversing the paths in the Ducci tree has closer relationship to the Stern-Brocot tree than the Ducci tree itself does. Here is its precise definition:

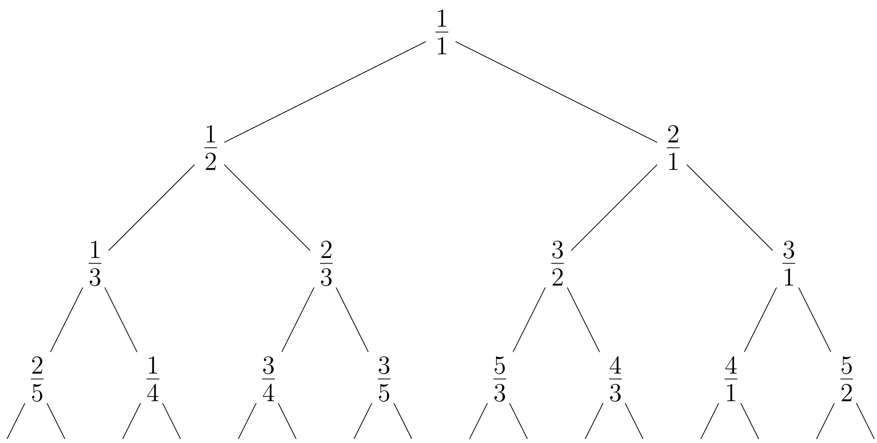

Figure 4: Top levels of the tree R-DT.

Definition 5 ([3])**.** R-DT is the labeled binary tree such that the equation $\mathrm{Path}_{\mathrm{DT}}\left(\frac{n}{m},\frac{1}{1}\right)=\mathrm{Path}_{\mathrm{R\text{-}DT}}\left(\frac{1}{1},\frac{n}{m}\right)$ is valid for any $\frac{n}{m}\in\mathbb{Q}_{>0}$. (See Figure 4.)

Note that in the statement, and also in what follows, a vertex is identified with its label. We can do so because different vertices get different labels in both trees [3].

In the Stern-Brocot tree, children of $[0;a_1,\dots,a_{\ell-1},a_\ell]$ are $[0;a_1,\dots,a_{\ell-1},a_\ell-1,2]$ and $[0;a_1,\dots,a_{\ell-1},a_\ell+1]$, which can be seen readily from the relationship between paths in the tree and continued fractions (referred already in the proof of Proposition 3). In the tree R-DT, we have:

Proposition 4 ([3])**.** *If* $[0;a_1,\dots,a_{\ell-1},a_\ell]$ *with* $\sum_{j=1}^{\ell}a_j>2$ *is*

- *the left child of a vertex, then its left and right children are* $[0;a_1,\dots,a_{\ell-1},a_\ell-1,2]$ *and* $[0;a_1,\dots,a_{\ell-1},a_\ell+1]$*, respectively;*
- *the right child of a vertex, then its left and right children are* $[0;a_1,\dots,a_{\ell-1},a_\ell+1]$ *and* $[0;a_1,\dots,a_{\ell-1},a_\ell-1,2]$*, respectively.* □

Consequently, $\frac{n}{m}$ and $\frac{n'}{m'}$ have the parent and child relation in the Stern-Brocot tree if and only if they have the parent and child relation in the tree R-DT. From Propositions 1 and 2, we can thus derive the following:

Corollary 1. (i) *The left subtree* R-DT^{L} *of the tree* R-DT *(i.e., the subtree of* R-DT *under* $\frac{1}{2}$*) contains each fraction from* $\mathbb{Q}\cap(0,1)$ *precisely once;*

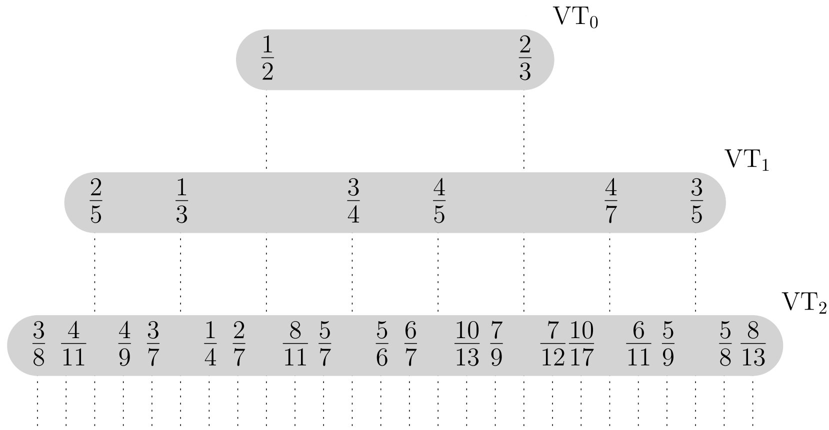

Figure 5: Top levels of VT_k $(k \geq 0)$.

(ii) *Suppose that in the tree* R-DT^{L}*, a vertex and its left and right children are labeled by* $\frac{n}{m}$*,* $\frac{n'}{m'}$ *and* $\frac{n''}{m''}$*, respectively. If* n *is odd, then precisely one of* n' *and* n'' *is even; If* n *is even, then both* n' *and* n'' *are odd.* □

Observe that the already-referred fact that different vertices get different labels in the tree R-DT (hence in R-DT^{L}) can also be inferred from the first item above.

As Mallows did for the Stern-Brocot tree, can we introduce a variation of the tree R-DT? Recall that the original formulation of Mallows' variation ("to insert two fractions between two adjacent fractions") was a natural variation of the mediant construction of the Stern-Brocot tree. Hence, because the tree R-DT does not have close relationship to the mediant construction, Mallows' original formulation is not so suggestive. However, there is a different yet equivalent formulation: Theorem 1 states that Mallows' variation can equivalently be defined as $\varphi_{\text{SBT}}\big(\psi^k_{\text{SBT}}\big(\frac{1}{2}\big)\big)$ $(k \geq 0)$. It is this formulation that inspired us to introduce a variation as follows:

Definition 6. For any $k \geq 0$, define $\mathrm{VT}_k \coloneqq \varphi_{\text{R-DT}}\big(\psi^k_{\text{R-DT}}\big(\frac{1}{2}\big)\big)$, where $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$ are resulting functions obtained by replacing all occurrences of "SBT^{L}" with "R-DT^{L}" in the definition of φ_{SBT} and ψ_{SBT} (see Definition 3).

(See Figure 5.) Note that two functions $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$ are well-defined by Corollary 1 (ii).

Proposition 5. *Let* $\frac{n}{m} \in \mathbb{Q} \cap (0,1)$ *and let* $\frac{n'}{m'}$ *and* $\frac{n''}{m''}$ *be the right child of the left child of* $\frac{n}{m}$ *in* R-DT^{L} *and the left child of the right child of* $\frac{n}{m}$ *in* R-DT^{L}*, respectively. Then* $n \equiv n' \equiv n'' \pmod 2$ *and* $m \equiv m' \equiv m'' \pmod 2$*.*

Proof. If $\frac{n}{m} = \frac{1}{2}$, then the assertion is evidently correct. If $\frac{n}{m} \neq \frac{1}{2}$ is the left (resp. right) child of a vertex, then we have

$$\frac{n'}{m'} = [0; a_1, \ldots, a_{\ell-1}, a_\ell - 1, 3] \quad \text{and} \quad \frac{n''}{m''} = [0; a_1, \ldots, a_{\ell-1}, a_\ell + 2]$$
$$(\text{resp. } \frac{n'}{m'} = [0; a_1, \ldots, a_{\ell-1}, a_\ell + 2] \quad \text{and} \quad \frac{n''}{m''} = [0; a_1, \ldots, a_{\ell-1}, a_\ell - 1, 3])$$

by Proposition 4, where $a_1, \ldots, a_{\ell-1}, a_\ell$ are positive integers such that $\frac{n}{m} = [0; a_1, \ldots, a_{\ell-1}, a_\ell]$. The validity of the asserted congruences can then be established by induction on ℓ. □

Proposition 6. *Let* $k \geq 0$ *and* $j \in \{1, 2, \ldots, 2 \cdot 3^k\}$.

(When k is odd) *The* jth *element of* VT_k *has odd numerator if and only if* $j \equiv 2$ *or* $3 \pmod 4$*;*

(When k is even) *The* jth *element of* VT_k *has odd numerator if and only if* $j \equiv 1$ *or* $4 \pmod 4$.

Proof. The proof is by induction on k: Figure 5 shows that the asserted equivalences are correct for $k = 0, 1$ and 2. Let us then show that if the stated equivalence for $2k\,(\geq 2)$ is true, then so is the equivalence for $2k+1$. Take an arbitrary $a \in \{0, 1, \ldots, \frac{3^{2k}-1}{2}\}$. By the induction hypothesis, the $(4a+1)$st element of VT_{2k} has odd numerator and the $(4a+2)$nd element of VT_{2k} has even numerator. From the definition of $\varphi_{\text{R-DT}}$, it follows that the latter is the right child of the former in the tree R-DT^{L}. By combining Corollary 1 (ii), Proposition 5 and the definition of $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$, we see that

- the left child of the $(4a+1)$st element of VT_{2k} has odd numerator and its left child has even numerator. Hence they are the $(12a+2)$nd and $(12a+1)$st elements of VT_{2k+1}, respectively;
- the left child of the $(4a+2)$nd element of VT_{2k} has odd numerator and its right child has even numerator. Hence they are the $(12a+3)$rd and $(12a+4)$th elements of VT_{2k+1}, respectively;
- the right child of the $(4a+2)$nd element of VT_{2k} has odd numerator and its left child has even numerator. Hence they are the $(12a+6)$th and $(12a+5)$th elements of VT_{2k+1}, respectively.

By a similar line of reasoning, we can show that for any $b \in \{0, 1, \ldots, \frac{3^{2k}-3}{2}\}$,

- the left child of the $(4b+3)$rd element of VT_{2k} has odd numerator and its right child has even numerator. Therefore they are the $(12b+7)$th and $(12b+8)$th elements of VT_{2k+1}, respectively;

- the right child of the $(4b+3)$rd element of VT_{2k} has odd numerator and its left child has even numerator. Therefore they are the $(12b+10)$th and $(12b+9)$th elements of VT_{2k+1}, respectively;
- the right child of the $(4b+4)$th element of VT_{2k} has odd numerator and its right child has even numerator. Hence they are the $(12b+11)$th and $(12b+12)$th elements of VT_{2k+1}, respectively.

Since every $j \in \{1, 2, \ldots, 2 \cdot 3^{2k+1}\}$ can be written as either $12a + i$ for some $a \in \{0, 1, \ldots, \frac{3^{2k}-1}{2}\}$ and $i \in \{1, 2, \ldots, 6\}$ or $12b + i$ for some $b \in \{0, 1, \ldots, \frac{3^{2k}-3}{2}\}$ and $i \in \{7, 8, \ldots, 12\}$, this indicates that the stated equivalence for $2k+1$ is correct.

To complete the proof of the induction step, we need to show that if the stated equivalence for $2k+1$ (≥ 3) is true, then so is the equivalence for $2k+2$. But that can be done by arguing similarly to the above. $\square$

In the same spirit to the preceding section, we ask the ensuing questions: Where is the j'th element of $\mathrm{VT}_{k'}$ placed in R-DT$^{\mathtt{L}}$? Conversely, where is the jth element of R-DT$^{\mathtt{L}}_k$ placed in $\mathrm{VT}_{k'}$ ($k' \geq 0$)? The following algorithm is for the first question, which, when given $k' \geq 0$ and $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'}\}$, outputs integers k and j such that the jth element of R-DT$^{\mathtt{L}}_k$ is the j'th element of $\mathrm{VT}_{k'}$:

Algorithm 3

Step 1: Set $w := \langle\rangle$, $\ell := k'$, $i := j'$, $a := 0$, $b := 0$, $k := 0$, $j := 1$.
Step 2: Update w by iterating the following while $\ell \geq 0$:
· Set $a := \lfloor \frac{i-1}{6} \rfloor$ and $b := i - 6a$
· If $a \equiv \ell \pmod 2$ and $b = 1$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if $a \equiv \ell \pmod 2$ and $b = 2$ then $w := \langle \mathtt{L}, \mathtt{R} \rangle^\frown w$, $i := 2a + 1$
else if $a \equiv \ell \pmod 2$ and $b = 3$ then $w := \langle \mathtt{R}, \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if $a \equiv \ell \pmod 2$ and $b = 4$ then $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 1$
else if $a \equiv \ell \pmod 2$ and $b = 5$ then $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
else if $a \equiv \ell \pmod 2$ and $b = 6$ then $w := \langle \mathtt{R}, \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
else if $a \not\equiv \ell \pmod 2$ and $b = 1$ then $w := \langle \mathtt{L}, \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if $a \not\equiv \ell \pmod 2$ and $b = 2$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 1$
else if $a \not\equiv \ell \pmod 2$ and $b = 3$ then $w := \langle \mathtt{L} \rangle^\frown w$, $i := 2a + 2$
else if $a \not\equiv \ell \pmod 2$ and $b = 4$ then $w := \langle \mathtt{L}, \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
else if $a \not\equiv \ell \pmod 2$ and $b = 5$ then $w := \langle \mathtt{R}, \mathtt{L} \rangle^\frown w$, $i := 2a + 2$
else $w := \langle \mathtt{R} \rangle^\frown w$, $i := 2a + 2$
· Set $\ell := \ell - 1$
Step 3: Set $k := \mathrm{lh}(w) - 1$, $i := 1$.
Step 4: Update j by iterating the following while $i \leq k$:
If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$ then $j := j + 2^{k-i}$, $i := i + 1$ else $i := i + 1$
Step 5: Output k and j.

Proof of the correctness of the algorithm. Since the rest of the proof runs along a similar line to that for Algorithm 1, we shall present a proof only for the induction part of the following claim: for any $k' \geq 0$ and $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'}\}$, if we input k' and j' to the algorithm, then the final updated value of w is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$, where $\frac{n}{m}$ is the j'th element of $\mathrm{VT}_{k'}$.

Suppose we have verified the correctness of the claim for $k'\,(\geq 1)$. Take an arbitrary $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'+1}\}$ and write $\frac{n}{m}$ for the j'th element of $\mathrm{VT}_{k'+1}$. Also, write $\frac{n'}{m'}$ and $\frac{n''}{m''}$ for the $\big(2\lfloor\frac{j'-1}{6}\rfloor + 1\big)$st and $\big(2\lfloor\frac{j'-1}{6}\rfloor + 2\big)$nd elements of $\mathrm{VT}_{k'}$, respectively. Assume that $\lfloor\frac{j'-1}{6}\rfloor$ and k' are both odd. (Proof for other three cases, i.e., either $\lfloor\frac{j'-1}{6}\rfloor$ or k' is even, run similarly and thus are not presented.) Then, by Proposition 6, n' is odd and n'' is even. In view of the definition of $\varphi_{\text{R-DT}}$, this indicates that $\frac{n''}{m''}$ is the right child of $\frac{n'}{m'}$ in R-DT^{L}. Corollary 1 (ii) shows that the left child of $\frac{n'}{m'}$ in R-DT^{L} has odd numerator, whose right child has odd numerator by Proposition 5. This, combined with Corollary 1 (ii), proves that the left child of the left child of $\frac{n'}{m'}$ has even numerator. Corollary 1 (ii) and Proposition 5 also imply that both children of $\frac{n''}{m''}$ have odd numerator and the right child of the left child of $\frac{n''}{m''}$ and the left child of the right child of $\frac{n''}{m''}$ both have even numerator. From the definition of $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$ and the induction hypothesis, it thus follows that

- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 1$, then $\frac{n}{m}$ is the left child of the left child of $\frac{n'}{m'}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{L}, \mathtt{L}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 2$, then $\frac{n}{m}$ is the left child of $\frac{n'}{m'}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n'}{m'}\big)^\frown\langle \mathtt{L}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 3$, then $\frac{n}{m}$ is the left child of $\frac{n''}{m''}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{L}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 4$, then $\frac{n}{m}$ is the right child of the left child of $\frac{n''}{m''}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{L}, \mathtt{R}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 5$, then $\frac{n}{m}$ is the left child of the right child of $\frac{n''}{m''}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{R}, \mathtt{L}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$;
- if $j' = 6\lfloor\frac{j'-1}{6}\rfloor + 6$, then $\frac{n}{m}$ is the right child of $\frac{n''}{m''}$ and thus the final updated value of w, when $k'+1$ and j' are given as input, is equal to $\mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n''}{m''}\big)^\frown\langle \mathtt{R}\rangle = \mathrm{Path}_{\text{R-DT}}\big(\frac{1}{1}, \frac{n}{m}\big)$.

The validity of the claim for $k'+1$ is evident from these. $\square$

We then take up the converse question of where the jth element of R-DT^{L}_k is placed in $\text{VT}_{k'}$ ($k' \geq 0$). The next algorithm, when given $k \geq 0$ and $j \in \{1, 2, \ldots, 2^k\}$, outputs integers k' and j' such that the j'th element of $\text{VT}_{k'}$ is the jth element of R-DT^{L}_k:

Algorithm 4

Step 1: Set $w := \langle\rangle$, $k' := 0$, $j' := j$, $i := 1$, $x := 1$.
Step 2: Update w by iterating the following while $i \leq k$:
 If $j' > 2^{k-i}$ then $w := w^\frown\langle\mathtt{R}\rangle$, $j' := j' - 2^{k-i}$, $i := i+1$
  else $w := w^\frown\langle\mathtt{L}\rangle$, $i := i+1$
Step 3: Set $j' := 1$, $i := 1$.
Step 4: Update k', j' by iterating the following while $i \leq k$:
 If $x = 1$ and $\text{proj}_i(w) = \mathtt{L}$
  then $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 2$, $x := 3$, $i := i+1$
 else if $x = 1$ and $\text{proj}_i(w) = \mathtt{R}$ then $j' := j'+1$, $x := 2$, $i := i+1$
 else if $x = 2$ and $\text{proj}_i(w) = \mathtt{L}$
  then $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 3$, $x := 1$, $i := i+1$
 else if $x = 2$ and $\text{proj}_i(w) = \mathtt{R}$
  then $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 6$, $x := 3$, $i := i+1$
 else if $x = 3$ and $\text{proj}_i(w) = \mathtt{L}$
  then $j' := j'-1$, $x := 4$, $i := i+1$
 else if $x = 3$ and $\text{proj}_i(w) = \mathtt{R}$
  then $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 5$, $x := 1$, $i := i+1$
 else if $x = 4$ and $\text{proj}_i(w) = \mathtt{L}$
  then $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 1$, $x := 1$, $i := i+1$
 else $k' := k'+1$, $j' := 6\lfloor\frac{j'-1}{2}\rfloor + 4$, $x := 3$, $i := i+1$
Step 5: Output k' and j'.

Proof of the correctness of the algorithm. The line of the proof being the same as that for Algorithm 2, we shall prove only the induction part of the following claim: for $i = 0, 1, \ldots, k$, the $j'(i)$th element of $\text{VT}_{k'(i)}$ is $\frac{n_i}{m_i}$ and its numerator n_i has the same parity as $x(i)$, where the fraction $\frac{n_i}{m_i}$ is such that $\text{Path}_{\text{R-DT}^{\text{L}}}\left(\frac{1}{2}, \frac{n_i}{m_i}\right)$ is the length i initial subsequence of the path in R-DT^{L} from $\frac{1}{2}$ to the jth element of R-DT^{L}_k and $k'(i)$, $j'(i)$ and $x(i)$ are the values of k', j' and x when the ith round of the while-loop in Step 4 has been completed, respectively. (We set $k'(0) := 0$, $j'(0) := 1$ and $x(0) := 1$.)

Suppose we have verified the correctness of the claim until i (≥ 1). The proof of the claim for $i+1$ is by case analysis. Observe that if $x(i)$ is even, then the induction hypothesis implies that the numerator n_i of the fraction

$\frac{n_i}{m_i}$ is also even, which proves by Corollary 1 (ii) that the parent $\frac{n_{i-1}}{m_{i-1}}$ of $\frac{n_I}{m_i}$ has odd numerator. The cases that we need to analyze is thus the same as the six cases from the proof for Algorithm 2. Let us present a proof for the first case (i.e., $x(i) = 1$ and n_{i-1} is odd) only, from which the reader will see how to prove the claim also in other five cases.

From the induction hypothesis and the algorithm, it follows that n_i is odd, $x(i-1) = 3$ and $\mathrm{proj}_i(w) = \mathtt{R}$. (Here, the value of the variable w is the final one, i.e., the value when Step 2 of the algorithm has been completed.) There are two subcases: If $\mathrm{proj}_{i+1}(w) = \mathtt{L}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the left child of the right child of $\frac{n_{i-1}}{m_{i-1}}$. As n_{i-1} is odd by assumption, Proposition 5 proves that n_{i+1} is odd too. The same proposition and Corollary 1 (ii) combine to show that the left child of $\frac{n_{i+1}}{m_{i+1}}$ and the right child of $\frac{n_i}{m_i}$ both have even numerator. From the definition of $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$, since $\frac{n_i}{m_i}$ is the $j'(i)$th element of $\mathrm{VT}_{k'(i)}$ by the induction hypothesis, we conclude that $\frac{n_{i+1}}{m_{i+1}}$ is the $\big(6\lfloor\frac{j'(i)-1}{2}\rfloor + 2\big)$nd element of $\mathrm{VT}_{k'(i)+1}$. As $x(i+1) = 3$ and n_{i+1} are both odd, the claim is thus correct in this subcase. If $\mathrm{proj}_{i+1}(w) = \mathtt{R}$, then $\frac{n_{i+1}}{m_{i+1}}$ is the right child of $\frac{n_i}{m_i}$ and has even numerator (as has been mentioned already in the preceding subcase). From the induction hypothesis and the definition of $\varphi_{\text{R-DT}}$, it can be inferred that $\frac{n_{i+1}}{m_{i+1}}$ is the $(j'(i)+1)$st element of $\mathrm{VT}_{k(i)}$. The parity of $x(i+1) = 2$ being the same as that of n_{i+1}, we see that the claim is correct also in this subcase. □

By combining Algorithm 3 with an algorithm [3] which calculates the jth fraction of R-DT_k^{L}, one can answer the ensuing question: given $k' \geq 0$ and $j' \in \{1, 2, \ldots, 2 \cdot 3^{k'}\}$, what is the j'th fraction of $\mathrm{VT}_{k'}$? The converse question can also be answered as follows: Given a fraction $\frac{n}{m}$, first use a known algorithm [3] to calculate where it is placed in the tree R-DT. The outputs can then be provided to Algorithm 4 to yield k' and j' such that $\frac{n}{m}$ is the j'th element of $\mathrm{VT}_{k'}$.

Unlike on $\varphi\big(\psi^k\big(\frac{1}{2}\big)\big)$, the left-to-right order and the order defined by the magnitude relation do not coincide on VT_k if $k \geq 1$. It will thus be useful if we can generate the ordered set VT_k directly without constructing $\mathrm{VT}_0, \mathrm{VT}_1, \ldots, \mathrm{VT}_{k-1}$. The next algorithm is for that purpose: given a $k \geq 2$, it outputs a sequence of continued fraction representations of elements of the ordered set VT_k.

Algorithm 5

Step 1: Set $w := f(1, k, \langle [0; 2, 1, 1, \ldots, 1, 2] \rangle)$ ($k + \lfloor \frac{k-3}{2} \rfloor$ consecutive 1's).
Step 2: Update w by iterating the following for i from 1 to $3^{k-1} - 1$:
$w := w^\frown f(i + 1, k, g(p(k, i), \mathrm{proj}_{6i}(w)))$
Step 3: Output w.

(Here, functions f, g and p are defined as follows:

$f(i, k, [0; a_1, \ldots, a_{\ell-1}, a_\ell]) =$
if $i \equiv k \pmod 2$ and $a_{\ell-1} = 1$ then
$\langle [0; a_1, \ldots, a_{\ell-1}, a_\ell], [0; a_1, \ldots, a_{\ell-2}, a_\ell],$
$[0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 3], [0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 4],$
$[0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 1, 3], [0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 1, 2] \rangle$
else if $i \equiv k \pmod 2$ then
$\langle [0; a_1, \ldots, a_{\ell-1}, a_\ell], [0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} + 1],$
$[0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 3], [0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 4],$
$[0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 1, 3], [0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 1, 2] \rangle$
else if $a_{\ell-1} = 1$ then
$\langle [0; a_1, \ldots, a_{\ell-1}, a_\ell], [0; a_1, \ldots, a_{\ell-1}, a_\ell + 1], [0; a_1, \ldots, a_{\ell-2}, 4],$
$[0; a_1, \ldots, a_{\ell-2}, 3], [0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 2], [0; a_1, \ldots, a_{\ell-3}, a_{\ell-2} + 1, 2] \rangle$
else
$\langle [0; a_1, \ldots, a_{\ell-1}, a_\ell], [0; a_1, \ldots, a_{\ell-1}, a_\ell + 1],$
$[0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} + 3], [0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} + 2],$
$[0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 2], [0; a_1, \ldots, a_{\ell-2}, a_{\ell-1} - 1, 1, 2] \rangle$

$g(p, [0; a_1, \ldots, a_{\ell-1}, a_\ell]) =$
if $p = 0$ and $a_{j_1} = 2$ then
$[0; a_1, \ldots, a_{j_1-2}, a_{j_1-1} + 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_1 - 1$ consecutive 1's)
else if $p = 0$ then
$[0; a_1, \ldots, a_{j_1-1}, a_{j_1} - 2, 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_1 - 1$ consecutive 1's)
else if $p = 1$ then
$[0; a_1, \ldots, a_{j_2-1}, a_{j_2} + 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_2 - 1$ consecutive 1's)
else if $p = 2$ then
$[0; a_1, \ldots, a_{j_2-1}, a_{j_2} + 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_2 - 3$ consecutive 1's)
else if $p = 3$ then
$[0; a_1, \ldots, a_{j_3-1}, a_{j_3} - 2, 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_3$ consecutive 1's)
else $[0; a_1, \ldots, a_{j_3-1}, a_{j_3} - 2, 2, 1, 1, \ldots, 1, 2]$ ($\ell - j_3 - 2$ consecutive 1's)
where $j_1 := \max\{j < \ell \mid a_j \neq 1\}$, $j_2 := \min\{j < \ell \mid$ only one of $a_{j+2}, a_{j+3}, \ldots, a_{\ell-1}$ is $2\}$ and $j_3 := \max\{j < \ell \mid a_j \neq 1, 2\}$.

$p(k, i) =$
if $i = 3^{k-2}$ then 3
else if $i = 2 \cdot 3^{k-2}$ then 0
else if $k \equiv e \pmod 2$ then $\mathrm{proj}_{a - \lfloor \frac{a}{3} \rfloor}(u)$
else $\mathrm{proj}_{a - \lfloor \frac{a}{3} \rfloor + 6}(u)$
where $i = a \cdot 3^e$ ($3 \nmid a$) and $u = \langle 1, 0, 0, 4, 3, 0, 0, 2, 1, 0, 0, 4 \rangle^{3^k}$.)

Proof of the correctness of the algorithm (Sketch). To prove the correctness, we need the following properties, which can be verified by using Propositions 4 and 6 and the definition of $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$:

(i) If $[0; a_1, \ldots, a_{\ell-1}, a_\ell]$ is the $(6i+1)$st ($i \in \{0, 1, \ldots, 3^{k-1}-1\}$) element of VT_k, then $f(i+1, k, [0; a_1, \ldots, a_{\ell-1}, a_\ell])$ is the sequence consisting of the $(6i+1)$st, $(6i+2)$nd, …, $(6i+6)$th elements of VT_k;

(ii) Let $i \in \{1, 2, \ldots, 3^{k-1}-1\}$ and write $\frac{n}{m}$ for the youngest common ancestor (in R-DT) of the $6i$th and $(6i+1)$st elements of VT_k. Then
- $p(k,i) = 0$ if and only if $\frac{n}{m}$ has even numerator;
- $p(k,i) = 1$ if and only if $\frac{n}{m}$ is different from $\frac{1}{2}$, is the left child of a vertex, has odd numerator and its right child has even numerator;
- $p(k,i) = 2$ if and only if $\frac{n}{m}$ is the left child of a vertex, has odd numerator and its left child has even numerator;
- $p(k,i) = 3$ if and only if either $\frac{n}{m} = \frac{1}{2}$ or $\frac{n}{m}$ is the right child of a vertex, has odd numerator and its right child has even numerator;
- $p(k,i) = 4$ if and only if $\frac{n}{m}$ is the right child of a vertex, has odd numerator and its left child has even numerator.

It will be sufficient to validate the claim that for any $i \in \{0, 1, \ldots, 3^{k-1}-1\}$, the sequence $w(i)$ is equal to the length $6i+6$ initial subsequence of VT_k, where $w(i)$ is the value of the variable w when the ith round of the loop in Step 2 has been completed. (We set $w(0)$ to be the initial value of w.) Let us do so by induction: Since the first element of VT_k is $[0; 2, 1, 1, \ldots, 1, 2]$ ($k + \lfloor \frac{k-3}{2} \rfloor$ consecutive 1's), which can be seen also by induction, the validity of the claim for $i = 0$ follows from (i). For the induction step, assume that the equality between $w(i)$ and the length $6i+6$ initial subsequence of VT_k has been verified. Proofs for other cases being analogous, we shall argue for the case $p(k, i+1) = 1$ only. As (ii) shows that the youngest common ancestor (in R-DT) of the $(6i+6)$th and $(6i+7)$th elements of VT_k is different from $\frac{1}{2}$, is the left child of a vertex, has odd numerator and its right child has even numerator, it can be inferred from Proposition 4 and the definition of $\varphi_{\text{R-DT}}$ and $\psi_{\text{R-DT}}$ that if the youngest common ancestor is $[0; a_1, \ldots, a_{\ell-1}, a_\ell]$, then the $(6i+6)$th and $(6i+7)$th elements of VT_k should be of the form

$$[0; a_1, \ldots, a_{\ell-1}, a_\ell - 1, 2, \underbrace{1, 1, \ldots, 1}_{j}, 2] \text{ and } [0; a_1, \ldots, a_{\ell-1}, a_\ell + 1, \underbrace{1, 1, \ldots, 1}_{j+1}, 2]$$

for some $j \geq 0$, respectively. Since they satisfy the equation

$$g(1,[0;a_1,\dots,a_{\ell-1},a_\ell-1,2,1,1,\dots,1,2]) \\ = [0;a_1,\dots,a_{\ell-1},a_\ell+1,1,1,\dots,1,2]$$

and since the $(6i+6)$th element $[0;a_1,\dots,a_{\ell-1},a_\ell-1,2,1,1,\dots,1,2]$ of VT_k is equal to $\mathrm{proj}_{6(i+1)}(w(i))$ by the induction hypothesis, the $(6i+7)$th element $[0;a_1,\dots,a_{\ell-1},a_\ell+1,1,1,\dots,1,2]$ of VT_k is equal to $g(1,\mathrm{proj}_{6(i+1)}(w(i))) = g(p(k,i+1),\mathrm{proj}_{6(i+1)}(w(i)))$. Therefore, $w(i+1) = w(i)^\frown f(i+2,k,g(p(k,i+1),\mathrm{proj}_{6(i+1)}(w(i))))$ is equal to the length $6i+12$ initial subsequence of VT_k by (i) and the induction hypothesis, completing the proof of the induction step in this case. □

5. Relationship between two variations

In this last section, we shall explain how Mallows' variation MT_k $(k \geq 0)$ of the Stern-Brocot tree and our variation VT_k $(k \geq 0)$ of the tree R-DT are related to each other.

Proposition 7. MT_k *and* VT_k *comprise the same fractions for any* $k \geq 0$.

Proof. It will be sufficient to prove that $k_1\bigl(\frac{n}{m}\bigr) = k_2\bigl(\frac{n}{m}\bigr)$ holds for any $\frac{n}{m} \in \mathbb{Q} \cap (0,1)$, where $k_1\bigl(\frac{n}{m}\bigr)$ and $k_2\bigl(\frac{n}{m}\bigr)$ denote the unique integers such that $\frac{n}{m}$ belongs to $\varphi_{\mathrm{SBT}}\bigl(\psi_{\mathrm{SBT}}^{k_1(\frac{n}{m})}\bigl(\frac{1}{2}\bigr)\bigr)$ and $\varphi_{\mathrm{R\text{-}DT}}\bigl(\psi_{\mathrm{R\text{-}DT}}^{k_1(\frac{n}{m})}\bigl(\frac{1}{2}\bigr)\bigr)$, respectively. To do so, take a fraction $\frac{n}{m} \in \mathbb{Q} \cap (0,1)$ arbitrarily. Then, since the parent and child (hence the ancestor and descendant) relations in the Stern-Brocot tree and the tree R-DT coincide, the sequence $\frac{n_0}{m_0}\bigl(=\frac{1}{2}\bigr), \frac{n_1}{m_1}, \dots, \frac{n_\ell}{m_\ell}\bigl(=\frac{n}{m}\bigr)$ is a descending sequence from $\frac{1}{2}$ to $\frac{n}{m}$ in the Stern-Brocot tree if and only if it is a descending sequence from $\frac{1}{2}$ to $\frac{n}{m}$ in the tree R-DT. We shall prove by induction that $k_1\bigl(\frac{n_i}{m_i}\bigr) = k_2\bigl(\frac{n_i}{m_i}\bigr)$ holds for any $i \in \{0,1,\dots,\ell\}$. The equation is evidently correct if $i = 0$. Then assume that we have verified the equation $k_1\bigl(\frac{n_i}{m_i}\bigr) = k_2\bigl(\frac{n_i}{m_i}\bigr)$. From the definition of φ_{SBT} and ψ_{SBT}, it follows that $k_1\bigl(\frac{n_{i+1}}{m_{i+1}}\bigr) = k_1\bigl(\frac{n_i}{m_i}\bigr)$ or $= k_1\bigl(\frac{n_i}{m_i}\bigr)+1$ according as n_{i+1} is even or odd. Similarly, it follows from the definition of $\varphi_{\mathrm{R\text{-}DT}}$ and $\psi_{\mathrm{R\text{-}DT}}$ that $k_2\bigl(\frac{n_{i+1}}{m_{i+1}}\bigr) = k_2\bigl(\frac{n_i}{m_i}\bigr)$ or $= k_2\bigl(\frac{n_i}{m_i}\bigr)+1$ according as n_{i+1} is even or odd. These and the induction hypothesis completes the proof. □

This brings us the question of how MT_k and VT_k can be converted to each other. For one direction (i.e., given a $j \in \{1,2,\dots,2\cdot 3^k\}$, what is j' such that the j'th element of VT_k is the jth element of MT_k?), we can furnish an algorithm as follows: Let $k \geq 0$ and $j \in \{1,2,\dots,2\cdot 3^k\}$ be given and let $\frac{n}{m}$ denote the jth element of MT_k. Following Steps 1 and 2 of

Algorithm 1, update the variable w. The final updated value of w is equal to $\mathrm{Path}_{\mathrm{SBT}}\left(\frac{1}{1}, \frac{n}{m}\right)$ as has been shown in the proof for Algorithm 1. Given $w = \mathrm{Path}_{\mathrm{SBT}}\left(\frac{1}{1}, \frac{n}{m}\right)$ as input, an algorithm from [3] then outputs the sequence $\mathrm{Path}_{\text{R-DT}}\left(\frac{1}{1}, \frac{n}{m}\right)$. Steps 3, 4 and 5 of Algorithm 4 with $\mathrm{Path}_{\text{R-DT}}\left(\frac{1}{1}, \frac{n}{m}\right)$ substituted into w then yield k' and j' such that $\frac{n}{m}$ is the j'th element of $\mathrm{VT}_{k'}$. (The above proposition guarantees $k' = k$; we need only j'.) The complexity of this algorithm is linear in k.

Likewise, we can provide an algorithm also for the opposite direction (i.e., given a $j \in \{1, 2, \ldots, 2 \cdot 3^k\}$, what is j' such that the j'th element of MT_k is the jth element of VT_k?): Let $k \geq 0$ and $j \in \{1, 2, \ldots, 2 \cdot 3^k\}$ be given and let $\frac{n}{m}$ denote the jth element of VT_k. Steps 1 and 2 of Algorithm 3 updates the variable w from $\langle\rangle$ to $\mathrm{Path}_{\text{R-DT}}\left(\frac{1}{1}, \frac{n}{m}\right)$. An algorithm from [3] then converts $\mathrm{Path}_{\text{R-DT}}\left(\frac{1}{1}, \frac{n}{m}\right)$ to $\mathrm{Path}_{\mathrm{SBT}}\left(\frac{1}{1}, \frac{n}{m}\right)$. Finally, by substituting $\mathrm{Path}_{\mathrm{SBT}}\left(\frac{1}{1}, \frac{n}{m}\right)$ into w in Steps 3, 4 and 5 of Algorithm 2, we obtain integers k' and j' such that $\frac{n}{m}$ is the j'th element of $\mathrm{MT}_{k'}$. (We again have $k' = k$ by the above proposition.) As before, the complexity of this algorithm is linear in k.

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