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Approximation of Frankl's conjecture in the complement family

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In this paper, we propose an approximation of Frankl's conjecture in the complement $\mathcal{C}$ of a union-closed family $\mathcal{F}$ in the power set of $U = \{1, \ldots, n\}$. Frankl's conjecture is the statement that at least half the members of $\mathcal{F}$ contain some common element k in U and it is equivalent to at most half the members of $\mathcal{C}$ containing some k. This paper proves that at most $1/2 + 1/2n$ of the members in $\mathcal{C}$ contain some common element k. In addition, we show that, for arbitrarily small $\epsilon > 0$ and any constant c such that $1 > c > 0$, there is an N such that whenever $\mathcal{F} \subseteq \mathcal{P}(U)$ is a union-closed family of size $|\mathcal{F}| > c \cdot 2^n$ for some $n \geq N$ then there exists an element that appears in at least $1/2 - \epsilon$ of the member sets.

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1. Introduction

Frankl's conjecture was proposed by Peter Frankl in 1979 as a problem concerning in extremal combinatorics. Despite the concise statement of the conjecture, it is extremely difficult and still remains open. Frankl's conjecture is the statement about a union-closed family $\mathcal{F}$ which is a subfamily of the power set of $U = \{1, \ldots, n\} = [n]$ such that $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$, and $\mathcal{F} \neq \{\emptyset\}$. The conjecture states that there is an element in U that belongs to at least half the members of $\mathcal{F}$. Namely,

Frankl's conjecture. *If $\mathcal{F}$ is union-closed, then there exists $k \in U$ which is contained in at least half the members of $\mathcal{F}$.*

There are several approaches to attempt to prove the conjecture and some partial results have been discovered. One of the best results to date without imposing any particular conditions on the union-closed family is due to Knill [4], and its estimate is improved by Wójcik [11], where both results show that the maximal frequency of the element is in the order of $1/\log_2 |\mathcal{F}|$. Up-compression is another approach introduced by Reimer [7] and this leads to estimating the minimal average set size in a union-closed family, that is, the set size is at least $\log_2 |\mathcal{F}|/2$. This approach is developed by Balla, Bollobás and Eccles [1] and they show Frankl's conjecture holds for $|\mathcal{F}| \geq 2/3 \cdot 2^n$. Furthermore, Poonen introduced FC-family [6] and this concept is followed and improved by Vaughan [8, 9] and Morris [5]. Živković and Vučković [10] prove that Frankl's conjecture holds for $n \leq 12$. Together with Faro [3], the conjecture is proved for $|\mathcal{F}| \leq 50$. A thorough survey of Frankl's conjecture is presented in Bruhn and Schaudt [2].

In this paper, we focus on the complement family $\mathcal{C}$ of $\mathcal{F}$ and discuss at most how many sets in the family can contain a common element. In [1], a similar argument is discussed and can be deduced from the result in [1] that it is not more than $|\mathcal{C}|(\log_2 |\mathcal{C}| + 2)/2n$. In this paper, we show that it is not more than $|\mathcal{C}|(n + 1)/2n$ as described in Theorem 2.3. If we compare it with the result that can be deduced from [1], our result is stronger when $|\mathcal{C}| > 2^{n-1}$.

In terms of complement family, Frankl's conjecture is equivalent to that at most half the members of $\mathcal{C}$ can contain a common element. Our result above gives a bound approximately close to this when n is large. Further, we prove that, for arbitrarily small $\epsilon > 0$ and any constant c such that $1 > c > 0$, almost all $\mathcal{F}$ have at least $1/2 - \epsilon$ of the members containing some common element when $|\mathcal{F}|$ is more than $c \cdot 2^n$. Accordingly, the next theorem is proved in Section 2.

Theorem 2.5. *For every $\epsilon > 0$ and every c with $0 < c < 1$ there is an N such that whenever $\mathcal{F} \subseteq \mathcal{P}([n])$ is a union-closed family of size $|\mathcal{F}| > c \cdot 2^n$ for some $n \geq N$ then there exists an element that appears in at least $\frac{1}{2} - \epsilon$ of the member sets.*

2. The complement family of union-closed sets

Let $U = [n]$ and $\mathcal{P}(U)$ be the power set of U. A union-closed family $\mathcal{F}$ is a subfamily of $\mathcal{P}(U)$ such that $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$, and $\mathcal{F} \neq \{\emptyset\}$. The complement family $\mathcal{C}$ of $\mathcal{F}$ in $\mathcal{P}(U)$ is defined as $\mathcal{C} = \mathcal{P}(U) \backslash \mathcal{F}$.

Let $\mathcal{B}_k$ be a family of $B \in \mathcal{C}$ such that $k \in B$ and $B \backslash \{k\} \in \mathcal{F}$.

Lemma 2.1. *$\mathcal{B}_k \cap \mathcal{B}_l = \emptyset$ for $1 \leq k < l \leq n$.*

Proof. If $X \in \mathcal{B}_k \cap \mathcal{B}_l$, then $X \backslash \{k\} \in \mathcal{F}$ and $X \backslash \{l\} \in \mathcal{F}$. Thus, $X = X \backslash \{k\} \cup X \backslash \{l\} \in \mathcal{F}$ and this leads to a contradiction. □

Corollary 2.2. $\min_{k \in U} |\mathcal{B}_k| \leq 1/n \cdot |\mathcal{C}|$

Proof. $\mathcal{B}_k \subseteq \mathcal{C}$ and $\mathcal{B}_k \cap \mathcal{B}_l = \emptyset$ lead to the statement. □

The property of the complement family $\mathcal{C}$ which is a weaker version of Frankl's conjecture is stated as follows.

Theorem 2.3. *$\mathcal{C}$ has at most $1/2 + 1/2n$ of the members which contain some $k \in U$.*

Proof. Since

$$|\mathcal{B}_k| \geq |\{X \in \mathcal{C} \mid k \in X\}| - |\{X \in \mathcal{C} \mid k \notin X\}|,$$

Corollary 2.2 indicates

$$min_{k \in U}(|\{X \in \mathcal{C} \mid k \in X\}| - |\{X \in \mathcal{C} \mid k \notin X\}|) \leq \frac{1}{n} \cdot |\mathcal{C}|.$$

On the other hand,

$$|\{X \in \mathcal{C} \mid k \in X\}| + |\{X \in \mathcal{C} \mid k \notin X\}| = |\mathcal{C}|.$$

Therefore, solving the above two formulas,

$$min_{k \in U}|\{X \in \mathcal{C} \mid k \in X\}| \leq \left(\frac{1}{2} + \frac{1}{2n}\right)|\mathcal{C}|$$

is deduced. □

For a union-closed family, we have the following lemma when $|\mathcal{F}|$ is sufficiently large.

Lemma 2.4. *Let $1 > c > 0$, $\mathcal{F} \subseteq \mathcal{P}(U)$, and $|\mathcal{F}| > c \cdot 2^n$. Then, $\mathcal{F}$ has at least $1/2 - 1/2n \cdot (1/c - 1)$ of the members which contain some $k \in U$.*

Proof. Theorem 2.3 deduces that at most

$$\left(\frac{1}{2} + \frac{1}{2n}\right) \cdot (2^n - |\mathcal{F}|)$$

members in $\mathcal{C}$ contain some k. Then $\mathcal{F}$ has at least

$$2^{n-1} - \left(\frac{1}{2} + \frac{1}{2n}\right) \cdot (2^n - |\mathcal{F}|) = \frac{1}{2} \cdot |\mathcal{F}| - \frac{1}{2n} \cdot (2^n - |\mathcal{F}|)$$

members which contain k. Thus, from $|\mathcal{F}| > c \cdot 2^n$, $\mathcal{F}$ has at least

$$\frac{1}{2} - \frac{1}{2n} \cdot \left(\frac{1}{c} - 1\right)$$

of the members which contain some $k \in U$. $\square$

The above lemma is an approximation of Frankl's conjecture for a union-closed family when $|\mathcal{F}| > c \cdot 2^n$ and considerably improves the observation of [2]. It states that if a union-closed family has the size of more than 2^{n-1} then at least 6/13 of the members contain some $k \in U$. Meanwhile, the conditions of Lemma 2.4 on $|\mathcal{F}|$ are more relaxed than their conditions when n is sufficiently large. To assure that at least $1/2 - \epsilon$ of the members of $\mathcal{F}$ contain some $k \in U$, c could be arbitrarily small provided that n is sufficiently large. We describe this statement in the following theorem.

Theorem 2.5. *For every $\epsilon > 0$ and every c with $0 < c < 1$ there is an N such that whenever $\mathcal{F} \subseteq \mathcal{P}([n])$ is a union-closed family of size $|\mathcal{F}| > c \cdot 2^n$ for some $n \geq N$ then there exists an element that appears in at least $\frac{1}{2} - \epsilon$ of the member sets.*

Proof. We can deduce that average frequency of an element is not smaller than $\frac{\log_2 |\mathcal{F}|}{|U|} \cdot \frac{|\mathcal{F}|}{2}$. (See (6) in [2].) Now, if $n = |U|$ and $|\mathcal{F}| > c \cdot 2^n$ then average frequency is not smaller than $\frac{\log_2 c+n}{n} \cdot \frac{|\mathcal{F}|}{2}$, and obviously $\frac{\log_2 c+n}{2n}$ will become larger than $1/2 - \epsilon$ if n is large enough. $\square$

3. Conclusion

We have shown that the complement of a union-closed family has a property which is easier to deal with in some context compared to that of $\mathcal{F}$ itself. The next goal of the study would be to prove

$$\lim_{n\to\infty} \frac{|\{\mathcal{F} \mid |\mathcal{F}| \leq c \cdot 2^n\}|}{|\{\mathcal{F} \mid |\mathcal{F}| > c \cdot 2^n\}|} = 0$$

for $\mathcal{F} \subseteq \mathcal{P}(U)$ when $c > 0$ is sufficiently small, which suggests almost all $\mathcal{F}$ satisfy the condition that at least $1/2 - \epsilon$ of the members contain some common element without any assumption.

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