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Pursuit-evasion games on Latin square graphs

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We investigate various pursuit-evasion parameters on Latin square graphs, including the cop number, metric dimension, and localization number. Bounds for the cop number are given for Latin square graphs and for similarly defined graphs corresponding to k mutually orthogonal Latin squares of order n. If $n > (k+1)^2$, then the cop number is shown to be $k+2$. Lower and upper bounds are provided for the metric dimension and localization number of Latin square graphs. An analysis of the metric dimension of back-circulant Latin squares shows that the lower bound is close to tight.

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1. Introduction

Pursuit-evasion games, including the well-known game of Cops and Robbers and the Localization game, are combinatorial models for detecting or neutralizing an adversary's activity on a graph. In such models, pursuers attempt to capture an evader loose on the vertices of a graph. There are numerous variants which dictate the rules for player movement. Such games are motivated by foundational topics in computer science, discrete mathematics, and artificial intelligence, such as robotics and network security. For a recent book on pursuit-evasion games, see [4]. For surveys of pursuit-evasion games, see [8, 9, 12], and see [7] for more background on Cops and Robbers.

In Cops and Robbers, the pursuers are *cops* and the evader is the *robber*. Both players move on vertices. The cops move first, followed by the robber; the players then alternate moves. The robber is visible, and players move to adjacent vertices or remain on their current vertex. The cops win if, after a finite number of rounds, they can land on the vertex of the robber; otherwise, the robber wins. The least number of cops needed to guarantee that the robber is captured on a graph G is the *cop number* of G, denoted by $c(G)$. Note that $c(G)$ is well-defined, as $c(G) \le \gamma(G)$, where $\gamma(G)$ is the domination number of G. For more background on the cop number of a graph, see [7].

In the *Localization game*, the robber moves first and is invisible to the cops during gameplay. As in Cops and Robbers, the robber occupies vertices and moves between vertices along edges. On their turn, the cops may move to any vertex of the graph. After each move, the cops occupy a set of vertices $u_1, u_2, \ldots, u_k$ and each cop sends out a *cop probe*, which gives their distance d_i, where $1 \le i \le k$, from u_i to the robber's vertex. The distances d_i are nonnegative integers or may be ∞. Hence, in each round, the cops determine a *distance vector* $D = (d_1, d_2, \ldots, d_k)$ of cop probes. The cops win if they have a strategy to determine, after a finite number of rounds, the vertex that the robber occupies, at which time we say that the cops *capture* the robber. We assume the robber is omniscient, in the sense that they know the entire strategy for the cops. The *localization number* of a graph G, written $\zeta(G)$, is the least positive integer k for which k cops have a winning strategy.

The minimum number of cops needed to win in the first round (that is, using only one set of cop probes in round 0) is equivalent to the *metric dimension*, written $\beta(G)$. Observe that $\zeta(G) \le \beta(G) \le |V(G)|$. A survey on metric dimension and related concepts may be found in [1], and a recent literature review on the localization number may be found in [6].

The present paper is the first to consider the cop number, localization number, and metric dimension of graphs arising from Latin squares. For a positive integer n, a *Latin square* of order n is an $n \times n$ array of cells with each cell containing a symbol from a set S with $|S| = n$, such that each symbol occurs exactly once in each row and in each column. Often rows are indexed by R, columns are indexed by C, and symbols are indexed by S. For a Latin square L, we write its set of *entries* as

$$\{(r, c, s) \in R \times C \times S : \text{ symbol } s \text{ occurs in row } r \text{ and column } c \text{ of } L\}.$$

We will take $R = C = S = [n] = \{1, 2, \ldots, n\}$. We call the elements of R the *row-indices*, of C the *column-indices*, and of S the *symbol-indices*. The elements of $R \cup C \cup S$ will be known as the *indices*. Define the *row-line* (or more simply, the row) of a row-index r as the subset of n entries of L that contain r, and analogously define *column-line* and *symbol-line*. Each of these is called simply a *line*. Given a Latin square L, we denote the symbol in row r and column c by $L[r, c]$.

The *Latin square graph* of a Latin square L of order n, written as $G(L)$, is the graph with n^2 vertices labeled with the cells of the Latin square, where distinct vertices are adjacent if they share a row, column, or symbol. See Figure 1 for the graph corresponding to the following Latin square of order 3:

$$L_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}.$$

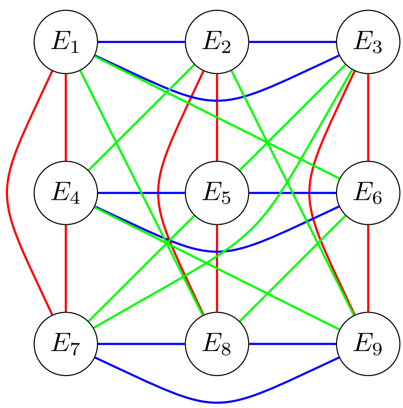

Figure 1: The graph arising from the Latin square L_3, where blue edges come from rows, red edges from columns, and green edges from symbols. Entries are labeled E_i, where $1 \leq i \leq 9$, for example where $E_1 = (1, 1, 1)$.

We may also consider graphs derived from mutually orthogonal Latin squares. A pair of Latin squares A and B of order n are *orthogonal* if the n^2 pairs $(A[i,j], B[i,j])$ are distinct. For positive integers n and k, a set of k Latin squares of order n are *mutually orthogonal*, written k-MOLS(n), if the Latin squares in the set are pairwise orthogonal. We may write an entry of a k-MOLS(n) as $(r, c, s_1, s_2, \ldots, s_k)$, where s_i is a symbol from the symbol set of the ith Latin square and $1 \leq i \leq k$. The maximum number of pairwise orthogonal Latin squares is $k = n - 1$. The existence of a set of $(n-1)$-MOLS(n) is equivalent to the existence of a (finite) projective plane of order n and an affine plane of order n; see [10].

If $\mathcal{L}$ is a set of k-MOLS(n), then define the *Latin square graph* of $\mathcal{L}$, written $G(\mathcal{L})$, to be the graph with n^2 vertices labeled with the cells of the Latin square, where distinct vertices are adjacent if the corresponding cells in the Latin square share a row, a column, or a symbol from any of the k

symbol sets. In the case $k = 1$, these are the Latin square graphs of Latin squares. The graph $G(\mathcal{L})$ is $(k+2)(n-1)$-regular.

The cop number of graphs arising from combinatorial designs was studied in [5], where bounds and exact values were determined for incidence graphs of designs, polarity graphs, block intersection graphs, and point graphs. That study was partially motivated by the search for new examples of so-called Meyniel extremal families of graphs, which have the conjectured largest asymptotic value of the cop number for connected graphs; see [2]. For a Latin square graph of order n, the domination number (which upper bounds the cop number) is bounded between $n/2$ and n, but an exact value is not known; see [13]. The localization number and metric dimension of designs were studied in [6], where these parameters were studied for incidence graphs of various balanced incomplete block designs such as projective planes, affine planes, and Steiner systems.

The present paper is organized as follows. After a subsection on notation, in Section 2 we consider the cop number of Latin square graphs arising from k-MOLS(n). For many instances of the parameters k and n, including the case $k = 1$, we determine the exact value of the cop number. In the remaining cases, we give bounds on the cop number. The metric dimension of Latin square graphs is discussed in the next section, and bounds are presented. In particular, for a Latin square L of order n, we derive in Theorem 3.5 that $n - \sqrt{n+\frac{5}{4}} - \frac{1}{2} \leq \beta(G(L))$. For the family of back-circulant Latin squares, we derive that for n sufficiently large with $2, 3, 5, 7 \nmid n$, $\beta(G(B_n)) \leq n-1$, which proves that the lower bound in Theorem 3.5 is close to tight. In Section 4, bounds are provided for the localization number of Latin square graphs. In particular, we show that $\frac{2}{3}(n-1) \leq \zeta(G(L)) \leq n+6$. Our final section presents several open problems on pursuit-evasion on Latin square graphs.

Throughout, all graphs considered are simple, undirected, connected, and finite. Note that the graphs studied are connected because Latin square (and k-MOLS(n)) graphs contain the $n \times n$ grid as a spanning subgraph. For a general reference on graph theory, see [15]. For background on Latin squares, see [10, 11, 14]. Unless otherwise stated, k and n are positive integers.

1.1. Notation

We think of $\mathcal{L}$ a set of k-MOLS(n) as being a $n \times n$ grid of *cells*, with cell (r, c) containing an *entry* $(r, c, s_1, \ldots, s_k)$. If $(r, c, s_1, \ldots, s_k) \in \mathcal{L}$, then we write $\mathcal{L}_i(r, c) = s_i$ for each $i \in [k]$. The *lines* of $\mathcal{L}$ are

$$R(\mathcal{L}, i) = \{(r, c) \in [n] \times [n] : r = i\},$$

$$C(\mathcal{L}, i) = \{(r, c) \in [n] \times [n] : c = i\},$$
$$S_j(\mathcal{L}, i) = \{(r, c) \in [n] \times [n] : \mathcal{L}_j(r, c) = i\}.$$

In the Latin square graph $G(\mathcal{L})$, each vertex is labeled by a *cell* (r, c) of $\mathcal{L}$. Two vertices in the Latin square graph, (r_1, c_1) and (r_2, c_2), are adjacent if either $r_1 = r_2$, $c_1 = c_2$, or $\mathcal{L}_i(r_1, c_1) = \mathcal{L}_i(r_2, c_2)$ for some $i \in [k]$. Each line ℓ of $\mathcal{L}$ (consisting of n cells of $\mathcal{L}$) can also be interpreted as a *line* of the Latin square graph consisting of n vertices of $G(\mathcal{L})$. Two lines ℓ_1 and ℓ_2 are *parallel* if $\ell_1 \cap \ell_2$. This only occurs if ℓ_1 and ℓ_2 are the same type of line; for example, both ℓ_1 and ℓ_2 are row-lines in $\mathcal{L}$. A cop is on or moves to a line ℓ if they are on or moves to a vertex v and v is contained in ℓ, and likewise for the robber.

To demonstrate this notation, we give three properties of Latin square graphs of $\mathcal{L}$ that will be useful throughout the paper.

(P1): Each vertex is contained in exactly $k + 2$ lines of $G(\mathcal{L})$.

(P2): Let $\{\ell_1, \ldots, \ell_{k+2}\}$ be the set of all lines of $G(\mathcal{L})$ that contain vertex v. Let ℓ be a line that is parallel to ℓ_1. The set $\ell \cap \ell_i$ is a singleton subset of ℓ when $i \neq 1$. The set $\bigcup_{1 \leq j \leq k+2}(\ell \cap \ell_j)$ is a $(k + 1)$-subset of ℓ.

(P3): Let $\{\ell_1, \ldots, \ell_{k+2}\}$ be the set of all lines of $G(\mathcal{L})$ that intersect v. A vertex $w \neq v$ is on either zero or one line in $\{\ell_1, \ldots, \ell_{k+2}\}$. This follows from the Latin property of the Latin square.

2. Cop number of Latin square graphs

For Latin squares of small orders, the cop number of their graphs may be directly computed. By directly checking, the cop number of a Latin square of order 1 or 2 is 1, order 3 is 2, and order 4 is 3. Interestingly, the cop number of Latin squares equals 3 for all $n \geq 5$, as we now demonstrate in the more general setting of MOLS.

Theorem 2.1. *If $\mathcal{L}$ is a set of k-MOLS(n), then we have that*

$$c(G(\mathcal{L})) \leq k + 2.$$

Proof. Suppose that $k+2$ cops are at play, which we label as $C_1, C_2, \ldots, C_{k+2}$. The idea of this proof is that these cops can use their first move to block off each of the robber's $k + 2$ possible escape routes, leading to the robber being captured on the next turn.

For their first move, the cops move to arbitrarily chosen vertices. The robber moves to the vertex $v = (r, c)$. If one of the $k + 2$ lines that intersect v also contains a cop, then the cop can win on their next move. Hence, we assume that the $k + 2$ lines that intersect v do not contain a cop.

A cop can move to any line incident with the robber. This follows from property (P2), since a line ℓ incident to the robber intersects with $k + 1$ of the $k + 2$ lines incident to each cop. Let $\{\ell_1, \ell_2, \ldots, \ell_{k+2}\}$ be the set of lines incident to the robber. For $1 \leq i \leq k + 2$, cop C_i moves to a vertex on the line ℓ_i. Thus, each line containing v now also contains a cop. If the robber moves, then it will still be on one of these lines which contains a cop, so will be captured. If the robber does not move, then it can be captured by any of the cops. □

If n is sufficiently large compared to k, then the upper bound in Theorem 2.1 has a matching lower bound.

Theorem 2.2. *Suppose that $n > (k+1)^2$. If $\mathcal{L}$ is a set of k-MOLS(n), then*

$$c(G(\mathcal{L})) = k + 2.$$

Proof. The upper bound follows by Theorem 2.1. For the lower bound, assume that $k+1$ cops are at play. Suppose during any point in play, the cops have just moved and did not capture the robber. We will show that there is a line containing the robber that does not contain a cop, and then show that this line must contain a vertex that the robber can move to without capture. Moving to this vertex means the robber will not be captured during the cops' next turn, completing a step that can be repeatedly applied. A proof that an initial placement is possible will be delayed until the end of the proof.

Suppose the $k+1$ cops have taken their turn and not been able to capture the robber, and that the robber is on vertex v. By property (P3), each cop is on at most one of the lines containing the robber's vertex v. As such, in the set of $k + 2$ lines that contain v there is at least one line ℓ that does not contain a cop. During the next round, the robber will move along line ℓ, and we proceed by showing that line ℓ contains a vertex such that the robber can move to this vertex without being captured on the next cop turn.

By property (P2), each cop is adjacent to exactly $k + 1$ vertices on ℓ. This means that the $k+1$ cops are adjacent to at most $(k+1)(k+1)$ vertices on ℓ. Since $n > (k+1)^2$, there is at least one vertex on ℓ that is not adjacent to a cop, and the robber moves to such a vertex. By repeating this strategy in subsequent rounds, the robber may avoid capture.

To see an initial placement is possible, the robber may apply the above analysis to any vertex v that does not contain a cop, and hence, find a vertex u that is adjacent to v but is not adjacent to any of the cops. The robber moves in the first round to u. □

We have the following immediate corollary in the case $k = 1$.

Corollary 2.3. *If L is Latin square of order $n \geq 5$, then $c(G(L)) = 3$.*

In the case that k is close to n, a lower bound is provided in our next theorem, although we do not know if it is tight.

Theorem 2.4. *Suppose that $n \leq (k+1)^2$. If $\mathcal{L}$ is a set of k-MOLS(n), then*

$$c(G(\mathcal{L})) \geq \left\lceil \frac{n}{k+1} \right\rceil .$$

Proof. Suppose that there are $\lceil \frac{n}{k+1} \rceil - 1$ cops. We show that the robber can be ensured of being on a line without a cop, and then show this line contains a "safe" vertex that the robber can move to.

Note that $\lceil \frac{n}{k+1} \rceil - 1 \leq k$ since $n \leq (k+1)^2$. By property (P3), at most k of the $k+2$ lines incident the robber also contain a cop. Therefore, the robber is always incident to at least one line, ℓ, that does not contain a cop. The robber will move along this line during its turn.

Each cop is incident to exactly $k+1$ vertices on ℓ by property (P2), and so there are $(\lceil \frac{n}{k+1} \rceil - 1)(k+1) < n$ vertices on ℓ adjacent to cops. Therefore, there is at least one vertex on ℓ that is not adjacent to a cop, and the robber moves to such a vertex on its turn. The robber can then employ the strategy found in the proof of Theorem 2.2 for its initial placement and to avoid capture indefinitely. □

The lower bound in Theorem 2.4 is tight, showing that the lower bound cannot be improved. We demonstrate this in the following lemma, in particular, when $k = n-1$ or $k = n-2$, which are the two largest possible values of k. It is known that $(n-1)$-MOLS(n) and $(n-2)$-MOLS(n) exist if and only if there exists a projective plane of order n, and so the following results only make sense for such integers n. This occurs at least when n is a prime power. Note that an $(n-2)$-MOLS(n) can always be extended to an $(n-1)$-MOLS(n).

Lemma 2.5. *If $\mathcal{L}$ is a set of $(n-1)$-MOLS(n), then $c(G(\mathcal{L})) = 1$.*

Proof. The graph $G(\mathcal{L})$ is the complete graph, which requires exactly one cop to capture the robber. □

Lemma 2.6. *If $\mathcal{L}$ is a set of $(n-2)$-MOLS(n), then $c(G(\mathcal{L})) = 2$.*

Proof. The lower bound is given by Theorem 2.4. We will play with two cops, in order to show that two cops are sufficient to capture the robber. By initializing the cops correctly, the robber can be captured when the cops first move from their initialized positions.

Note that every set $\mathcal{L}$ of $(n-2)$-MOLS(n) has a unique Latin square, say L', that can appended to $\mathcal{L}$ to form a set of $(n-1)$-MOLS(n). In the Cops and Robbers game on $\mathcal{L}$, if a cop is on vertex $(r, c) \in S_1(L', s)$, then it can move to any vertex except the other vertices in $S_1(L', s)$.

In the first round, move one cop to a vertex in $S_1(L', 1)$ and the other cop to a vertex in $S_1(L', 2)$. Therefore, if the first cop cannot capture the robber on the next move, the robber is on a vertex in $S_1(L', 1)$, and so is not in a vertex in $S_1(L', 2)$, and so can be captured by the second cop. $\square$

The upper bound in Theorem 2.1 is not tight when $k \in \{n-2, n-1\}$, and the lower bound in Theorem 2.4 is tight. It is possible that both could be tight for values $n < (k+1)^2$ with $k \notin \{n-2, n-1\}$, as it is possible that there is one Latin square that reaches the lower bound, and another Latin square of the same order that reaches the upper bound. We note that in the case of graphs from 2-MOLS(n), our results show that the cop number is 4 for $n \geq 11$. Analogous (but omitted) arguments improve this to show that the cop number of graphs from 2-MOLS(n) is 4 if $n \geq 7$ and 2-MOLS(n) exist.

3. Metric dimension of Latin square graphs

Let $\mathcal{L}$ be a set of k-MOLS(n). We note that $d(u, v) \in \{0, 1, 2\}$ for all pairs of vertices u, v in $G(\mathcal{L})$. We begin with general results on the metric dimension of graphs derived from MOLS.

Theorem 3.1. *If $\mathcal{L}$ is a set of k-MOLS(n), then*

$$\beta(G(\mathcal{L})) \leq (k+2)(2n-k-2).$$

Proof. After the robber makes their first move, the cops probe the vertices that correspond to the cells in first $k+2$ rows and the $k+2$ columns on the set on k-MOLS(n), which can also be written as:

$$S = \bigcup_{1 \leq j \leq k+2} (R(\mathcal{L}, j) \cup C(\mathcal{L}, j)).$$

There are $(k+2)(2n-k-2)$ vertices in this set. The robber could be on a vertex of S, or a vertex not on S. We will show that in both cases, the distances that the cops probe will uniquely determine the robber's position.

Case 1: The robber is on S.

In this case, a cop will probe a distance of 0, and so the cops will immediately capture the robber.

Case 2: The robber is on a vertex (r,c) not in S.

The $k+2$ cops on vertices in the same row-line as the robber, $R(\mathcal{L},r)\cap S = R(\mathcal{L},r)\cap\bigcup_{1\le j\le k+2}C(\mathcal{L},j)$, will all probe a distance of 1. Consider the $k+2$ cops $R(\mathcal{L},r')\cap S = R(\mathcal{L},r')\cap\bigcup_{1\le j\le k+2}C(\mathcal{L},j)$ on row-line $r'\neq r$, where $r'>k+2$. By property (P2), at most $k+1$ of these are incident with a line that contains (r,c), and so at most $k+1$ probe a distance of 1. Therefore, the row-line that contains the robber is uniquely identifiable from the cops that probe a distance of 1. A symmetric argument holds for the columns, and so the cops can find the exact location of the robber. □

Note in particular that $G(\mathcal{L})$ is the complete graph when $k=n-1$, and so Theorem 3.1 is tight in this case. Applying this result in the case for Latin squares of order n (with $k=1$) yields an upper bound of $6n-9$, which can be substantially improved.

Theorem 3.2. *If L is a Latin square of order n that contains a set of four entries of the form $\{(r_1,c_1,s_1),(r_1,c_2,s_2),(r_2,c_1,s_2),(r_2,c_2,s_3)\}$, where s_1,s_2,s_3 are each distinct and $n\ge 5$, then*

$$\beta(G(L))\le 2n-3.$$

Proof. We place $2n-3$ cops on the vertices of the two column-lines of c_1 and c_2 except for on the three vertices $(r_1,c_2),(r_2,c_1),(r_2,c_2)\}$. That is, there is a cop on each vertex in $(C(L,c_1)\cup C(L,c_2))\setminus\{(r_1,c_2),(r_2,c_1),(r_2,c_2)\}$. Note that this implies that $S(L,s_2)$ does not contain a cop, $S(L,s_3)$ contains exactly one cop, and all other symbol-lines contain exactly two cops. Similarly, $R(L,r_2)$ does not contain a cop, $R(L,r_1)$ contains exactly one cop, and all other row-lines contain exactly two cops.

We show how the cops capture the robber in two cases. The first is to show that if the robber in on column-line c_1 or c_2, then the cops can find the exact location of the robber and capture the robber. The second is to show that if the robber is not on one of these two column-lines, then the cops can also capture the robber.

Case 1: The robber is in column-line c_i for $i=1$ or $i=2$.

We show that the cops capture the robber. To see this, first note that if the robber is on the same vertex as a cop, then this cop probes 0, and so the robber is captured. Otherwise, the $n-2 \geq 3$ or $n-1 \geq 3$ cops in column-line c_i probe a distance of 1, for $i \in \{1,2\}$. By (P2), if the robber was on vertex (r,c) with $c \neq c_i$, then at most two vertices in column-line c_i would have distance 1 to the robber (one vertex on column-line c_i and row-line r, and the other vertex on column-line c_i and symbol-line $L(r,c)$). As at least three cops on column-line c_i probed a distance of 1, the cops know the robber is on column-line c_i.

Subcase 1a: $i = 1$.

In this case, the cops know the robber must be on (r_2, c_1), as this is the only vertex on column-line c_1 of distance 1 to all cops on column-line c_1.

Subcase 1a: $i = 2$.

In this case, either: 1) the cop on (r', c_1), where $L(r', c_1) = s_3$, probes a distance of 1, in which case the cops know the robber is on (r_2, c_2); or 2) no cop on column-line c_1 probes a distance of 1, in which case the cops know the robber is on a vertex of symbol-line s_2, which is only vertex (r_1, c_2). In either scenario, the robber is captured.

Case 2: The robber is in column-line c_i for $i \geq 3$.

By Case 1, the cops can identify that the robber is not on column-lines c_1 or c_2; otherwise, the robber would be captured. Suppose that the robber is on vertex (r,c). There are exactly four vertices in $C(L,c_1) \cup C(L,c_2)$ that have distance 1 to the robber; two in row-line r, and two in symbol-line $L(r,c)$. These four vertices are distinct by the Latin property of L, and as such the only pair of vertices in these four that share a row-line are those in row-line r, and the only pair of vertices in these four that share a symbol-line are those in symbol-line $L(r,c)$. If two cops in the same row-line probe a distance of 1, then the cops know the robber is on that row-line. If two cops in the same symbol-line probe a distance of 1, then the cops know the robber is on that symbol-line. Since exactly four vertices in $C(L,c_1) \cup C(L,c_2)$ that have distance 1 to the robber, at most four cops will probe a distance of 1, and we separate into cases for each possibility.

Subcase 2a: Suppose that exactly four cops probe 1.

If exactly four cops probe a distance of 1, then the cops know both the row-line and symbol-line of the robber, and so know that exact location of the robber by (P2), so the robber is captured.

Subcase 2b: Suppose that exactly three cops probe 1.

In this case, either: 1) two of these three cops share the same row-line, and so the cops know the robber is on that row-line, and the third cop is on the same symbol-line as the robber (which is symbol-line s_3); or 2) two of these three cops share the same symbol-line and so the cops know the robber is on that symbol-line. The third cop is on the same row-line as the robber (which is row-line r_1). The cops know exact location of the robber by (P2), and so the robber is captured.

Subcase 2c: Suppose that exactly two cops probe 1.

In this case, either: 1) both cops share the same row-line, so the cops know the robber is on that row-line, and also on symbol-line s_2 (which is the only symbol-line without a cop); 2) both cops share the same symbol-line so the cops know the robber is on that symbol-line, and also on row-line r_2 (which is the only row-line without a cop); or 3) neither cop shares a row-line or symbol-line, so the robber is on row-line r_1 and symbol-line s_3. The cops know exact location of the robber by (P2), so the robber is captured.

Subcase 2d: It is impossible to have less than two cops probe 1.

Otherwise, we would need to have either: 1) the robber on row-line r_1 and symbol-line s_2 (but only vertex (r_1, c_2) satisfies this); 2) the robber on row-line r_2 and symbol-line s_2 (but only vertex (r_2, c_1) satisfies this); or 3) the robber on row-line r_2 and symbol-line s_3 (but only vertex (r_2, c_2) satisfies this). Since the robber is not on column-line c_1 nor c_2, these cannot occur. □

There are some Latin squares that are not covered by Theorem 3.2, such as the Cayley table of addition for $\mathbb{Z}_2^k$ for $k \in \mathbb{N}$. A slight modification is applicable to all Latin squares.

Theorem 3.3. *If L is a Latin square of order $n \geq 4$, then*

$$\beta(G(L)) \leq 2n - 2.$$

Proof. By Theorem 3.2, all cases follow except the case where it is impossible for a subset of entries $\{(r_1, c_1, s_1), (r_1, c_2, s_2), (r_2, c_1, s_2), (r_2, c_2, s_3)\}$ to exist in L with s_1, s_2, s_3 each being unique.

Suppose that $s_1 = s_3$, and we now play on $G(L)$ as before, except that we also include an additional cop on vertex $(r_1, c_2) \in S(L, s_2)$. The proof of Theorem 3.2 is straightforwardly modified to show that the robber's row-line and symbol-line are determined by the cops, and so the robber's precise location is known. □

We present a lower bound for graphs arising from mutually orthogonal Latin squares.

Theorem 3.4. *If $\mathcal{L}$ is a set of k-MOLS(n), then*

$$\beta(G(\mathcal{L})) \geq \frac{2n^2-2}{(k+2)(n-1)+4}.$$

Proof. Let $G = G(\mathcal{L})$. Suppose there are c-many cops positioned on vertices $\mathcal{C} \subset V$, where c is a positive integer that is to be determined. Let $N_{\geq 2}(\mathcal{C}) = \{c \in V \setminus \mathcal{C} : |N_G(v) \cap \mathcal{C}| \geq 2\}$ be the collection of vertices with at least two cops in their neighborhood. Setting $e_G(\mathcal{C}, N_{\geq 2}(\mathcal{C}))$ as the number of edges in G between $\mathcal{C}$ and $N_{\geq 2}(\mathcal{C})$, and noting that G is $(k+2)(n-1)$-regular, it follows that

$$2|N_{\geq 2}(\mathcal{C})| \leq e_G(\mathcal{C}, N_{\geq 2}(\mathcal{C})) \leq |\mathcal{C}| \cdot (k+2)(n-1). \tag{1}$$

As a result of (1), it suffices to prove that to successfully capture the robber in one move, we necessarily need $n^2 - 2c - 1 \leq |N_{\geq 2}(\mathcal{C})|$.

Let $N_i(\mathcal{C}) = \{c \in V \setminus \mathcal{C} : |N_G(v) \cap \mathcal{C}| = i\}$ be the collection of vertices with exactly i cops in their neighborhood. We then have that $|V \setminus \mathcal{C}| = |N_0(\mathcal{C})| + |N_1(\mathcal{C})| + |N_{\geq 2}(\mathcal{C})|$ and also $|V \setminus \mathcal{C}| = n^2 - c$. As such, we are left to show that $n^2 - 2c - 1 \leq n^2 - c - |N_0(\mathcal{C})| - |N_1(\mathcal{C})|$, which is just that $|N_0(\mathcal{C})| + |N_1(\mathcal{C})| \leq c + 1$.

All vertices in $N_0(\mathcal{C})$ have distance 2 to each cop, so if $|N_0(\mathcal{C})| > 1$, then there are two vertices in $N_0(\mathcal{C})$ that cannot be distinguished by the cops. Therefore, $|N_0(\mathcal{C})| \leq 1$.

All vertices in $N_1(\mathcal{C})$ have distance 1 to one cop and distance 2 to all other cops. Using the pigeonhole principle, if $|N_1(\mathcal{C})| \geq c + 1$, then there are two vertices in $N_1(\mathcal{C})$ that both have distance 1 to the same cop and distance 2 to all other cops, and so these two vertices cannot be distinguished by the cops. Therefore, $|N_1(\mathcal{C})| \leq c$, and so $|N_0(\mathcal{C})| + |N_1(\mathcal{C})| \leq c + 1$, which completes the proof. $\square$

For Latin squares, Theorem 3.4 yields a lower bound of $\frac{2n^2-2}{3n+1} = \frac{2n}{3} - O(1)$ for their metric dimension. We improve this bound as follows.

Theorem 3.5. *If L is a Latin square of order n, then*

$$\beta(G(L)) \geq n - \sqrt{n + \frac{5}{4}} - \frac{1}{2}.$$

Proof. Let s be a positive integer that will be determined later in the proof. Suppose we play with $n-s$ cops. Independently of how the cops are employed, there is a set of s rows $\overline{R}$ and a set of s columns $\overline{C}$ such that their corresponding row-lines and column-lines each do not contain a vertex with a cop. Note also that there are at least s symbol-lines that do no contain a robber. The idea of this proof is that if $\overline{R}\times\overline{C}$ is large, then it must contain two vertices that the cops cannot distinguish.

Consider the set of vertices $\overline{R}\times\overline{C}$. Each of these vertices have distance 1 to a cop if and only if it is in the same symbol-line as that cop, by the definition of $\overline{R}$ and $\overline{C}$. As such, there can be at most one vertex in $\overline{R}\times\overline{C}$ that is not in the same symbol-line as a cop, or else there would be two vertices with distance 2 to all cops and so the cops could not distinguish these two vertices. Also, if a vertex in $\overline{R}\times\overline{C}$ is in symbol-line ℓ, then no other vertex of $\overline{R}\times\overline{C}$ can also be in symbol-line ℓ, or else there are two vertices that have distance 1 to all cops on the line ℓ and distance 2 to all other cops, and so cannot be distinguished by the cops.

As such, of the s^2 vertices in $\overline{R}\times\overline{C}$, there are at least s^2-1 that are in symbol-lines with some cop, and each of these s^2-1 symbol-lines are unique from each other. As such, there must be at least s^2-1 cops, since each cop is in at most one symbol-line and s^2-1 symbol-lines contain a cop. However, we are playing with only $n-s$ cops, so it must be that $n-s\geq s^2-1$. Solving for s, we find that $s\leq\sqrt{n+\frac{5}{4}}-\frac{1}{2}$, from which the result follows. $\square$

By Theorems 3.3 and 3.5, the metric dimension of a Latin square graph of order n will have metric dimension between somewhat below n and up to $2n$. Two different Latin squares graphs of the same order may have different metric dimension, so it is possible that both the upper and lower bounds we have given are tight. We proceed by showing that the lower bound is close to being tight.

The *back-circulant* Latin square B_n, is defined as $B_n[i,j]=i+j-1 \pmod{n}$, where we write n instead of 0 to remain consistent with our typical symbol set $[n]$. See Figure 2 for an example.

We need a few definitions. Suppose L is a Latin square of order n. For a non-negative integer d, a *partial transversal* of deficit d in L is a subset of $n-d$ entries $T\subseteq L$ such that each row, each column, and each symbol is represented at most once among the entries of T. A partial transversal of deficit $d=0$ is called a *transversal*. We note that in the following proof, we will commonly use $(r,c)\in S(L,s)$ to emphasize that the vertex in row-line r and column-line c is in symbol-line s, with corresponding entry $(r,c,s)\in L$.

$$B_{11} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{10} & 11 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & \mathbf{8} & 9 & 10 & 11 & 1 \\ \hline
3 & 4 & 5 & \mathbf{6} & 7 & 8 & 9 & 10 & 11 & 1 & 2 \\ \hline
\mathbf{4} & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & \mathbf{2} & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & \mathbf{11} & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & \mathbf{9} & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & \mathbf{7} \\ \hline
9 & 10 & 11 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & 7 & 8 \\ \hline
10 & 11 & 1 & 2 & \mathbf{3} & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & \mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
\end{array}.$$

Figure 2: The back-circulant Latin square B_{11}, where the 11 entries in bold are those chosen in Lemma 3.6.

Lemma 3.6. *For n sufficiently large with $2, 3, 5, 7 \nmid n$, we have that*

$$\beta(G(B_n)) \leq n - 1.$$

Proof. We begin by providing a set of n vertices of $G(B_n)$, and then will show that placing n cops on these vertices can capture the robber on the cops' first turn. In particular, the placement of cops will be made such that every row-line, column-line, and symbol-line contains exactly one cop. This means that exactly three cops will probe a distance of 1. Further, we will show that given the set of three cops that probe a distance of 1, the robber could only be on one particular vertex, and so will be captured on the first turn. After this, we will show that placing cops on only $n-1$ of these vertices provides identical information to if we had placed n, and so the cops can similarly capture the robber in one turn.

Place n cops on the vertices $(i, n+2-3i)$ for $i \in [n]$. Note that $(i, n+2-3i) \in S(B_n, n+1-2i)$. See Figure 2 for an example of this selection of vertices translated to the corresponding entries in B_{11}. As $2, 3 \nmid n$, note that each vertex containing a cop is on a unique row-line, unique column-line, and unique symbol-line. That is, the n corresponding entries of the Latin square L form a transversal of L. Therefore, if the robber is on a vertex that does not contain a cop, then exactly three cops will probe a distance of 1 to the robber (one for each line type). If we know that two particular cops probe a distance of 1 (and do not know the distances that the other cops probed), then there are at most six vertices of $G(L)$ that the robber may be on. We will show that for each of these six vertices, if the robber chose to be initialized on this vertex, there will be a distinct third cop of distance 1 from the robber that is associated with that choice, so the robber's location is known exactly.

Suppose that the first two cops are on vertices $C_i = (i, n+3-3i) \in S(L, n+2-2i)$ and $C_j = (j, n+3-3j) \in S(L, n+2-2j)$, where $i \neq j$. Table 1 provides the lines of the six vertices that the robber may be on, with each row of the table representing a distinct vertex. The first row of this table, for example, says that if the robber was on the vertex that is in the same row-line as C_i and the same column-line as C_j, then the robber is on the vertex $(i, n+3-3j) \in S(L, n+2-3j+i)$. Table 2 then provides the location of the third cop that also probes a distance of 1, given that the robber was on either of the six vertices that were possible. Note that the cop D_e corresponds to the case that the robber was on the vertex associated with the eth row of Table 1. For example, if the robber was on $(i, n+3-3j) \in S(L, n+2-3j+i)$, then the cop C_k with $k = (3j-i)/2$ would have distance 1 to the robber.

Table 1: The six possible locations of the robber, given that the cop on row-line i and cop on row-line j both probe a distance of 1 to the robber

Row-line	Column-line	Symbol-line
i	$n+3-3j$	$n+2-3j+i$
i	$n+3-2j-i$	$n+2-2j$
j	$n+3-3i$	$n+2-3i+j$
$3i-2j$	$n+3-3i$	$n+2-2j$
j	$n+3-2i-j$	$n+2-2i$
$3j-2i$	$n+3-3j$	$n+2-2i$

Finally, Table 3 shows the resulting equation if we assume that cop $D_e = D_f$, by equating the rows that D_e and D_f are in. As $2, 3, 5, 7 \nmid n$, each of these conditions would imply that $i = j$, giving a contradiction

Table 2: The lines of the vertices of the six additional cops that will probe a distance of 1 if the robber is on the corresponding locations given in Table 1

Cop	Row-line	Column-line	Symbol-line
D_1	$2^{-1}(3j-i)$	$n+3-3(2^{-1}(3j-i))$	$n+2-3j+i$
D_2	$3^{-1}(2j+i)$	$n+3-2j-i$	$n+2-(3^{-1}-1)(2j+i)$
D_3	$2^{-1}(3i-j)$	$n+3-3(2^{-1}(3i-j))$	$n+2-3i+j$
D_4	$3i-2j$	$n+3-3(3i-2j)$	$n+2-2(3i-2j)$
D_5	$3^{-1}(2i+j)$	$n+3-2i-j$	$n+2-(3^{-1}-1)(2i+j)$
D_6	$3j-2i$	$n+3-3(3j-2i)$	$n+2-2(3j-2i)$

Table 3: The equation (modulo n) that results when we assume that two cops in Table 2 share the same row-line

	D_1	D_2	D_3	D_4	D_5	D_6
D_1	–	$5i = 5j$	$4i = 4j$	$7i = 7j$	$7i = 7j$	$3i = 3j$
D_2	–	–	$7i = 7j$	$7i = 7j$	$i = j$	$9i = 9j$
D_3	–	–	–	$3i = 3j$	$5i = 5j$	$7i = 7j$
D_4	–	–	–	–	$7i = 7j$	$5i = 5j$
D_5	–	–	–	–	–	$8i = 8j$

of assumptions, and so each triple of cops that probe a distance of 1 will uniquely determine the location of the robber.

This completes the proof that placing cops on the n chosen vertices will capture the robber. To show that $n-1$ is sufficient, we may remove any one cop from this set of n vertices. Hence, either two or three cops will probe a distance of 1 to the robber. In the case that exactly two cops probe a distance of 1, we know that the removed cop would have probed a distance of 1 if we had not removed it. We therefore have the same information as if we had placed n cops on the set of n vertices, and so the robber's location is uniquely determined. □

4. Localization number of Latin square graphs

As the metric dimension is an upper bound on the localization number, by Theorem 3.3 we have the following.

Corollary 4.1. *If L is a Latin square of order n, then $\zeta(G(L)) \leq 2n-2$.*

The bound in Corollary 4.1 may be greatly improved, however. The following result demonstrates that using a little more than n cops, the cops may capture the robber.

Theorem 4.2. *For a Latin square L of order n, we have that*

$$\zeta(G(L)) \leq n+6.$$

Proof. We will play three rounds of the localization game on $G(L)$ with $n+6$ cops. In the first round, the cops will play such that they can identify two vertices of $G(L)$ that the robber must be residing on, although these two vertices may not share a common line. After the robber has taken its turn, the cops play their second turn and may identify two vertices of $G(L)$ that the robber must be residing on that are on a common line. After the robber moves and in the cops third and final turn, the cops are able to play to capture the robber.

For the first round, choose a symbol s and place n cops on the n vertices in symbol-line $S(L, s)$. We assume that the robber is not on one of these n vertices, or else it is immediately captured. By (P2), there must be exactly $k + 1 = 2$ cops that probe a distance of 1 to the robber, say the cops on vertices (r_1, c_1) and (r_2, c_2), where $r_1 \neq r_2$ and $c_1 \neq c_2$. The robber is either on the vertex (r_1, c_2) or (r_2, c_1).

For the second round, consider the symbol s_1 such that $(r_1, c_1) \in S(L, s_1)$ and place n cops on the n vertices in symbol-line $S(L, s_1)$. In addition, place a cop on (r_2, c_2) and a further four cops on the vertices of distance 1 from both (r_1, c_1) and (r_2, c_2) that do not yet contain cops. Note that if the robber just moved along line ℓ, then there are three cops on line ℓ. Since (P2) implies that at most two vertices on a line have distance 1 to the robber if the robber is not on that line, then when three cops on ℓ all probe a distance of 1, the cops know the robber is on this line. We may assume that ℓ is not the symbol-line $S(L, s_1)$, or else a cop would probe 0, and the robber would be captured. Now, as in the first round, since each vertex in a row-line contains a cop, there must be exactly $k + 1 = 2$ cops on this row-line that probe a distance of 1 to the robber, say the cops on vertices (r_3, c_3) and (r_4, c_4), where r_3 and r_4, and c_3 and c_4 may or may not be distinct. The robber is either on the vertex (r_3, c_4) or (r_4, c_3); however, these two vertices must be on the same line ℓ.

By symmetry, we may assume that ℓ is a symbol-line, and so that $r_3 \neq r_4$ and $c_3 \neq c_4$. For the third round, place n cops on the n vertices in symbol-line ℓ. In addition, place a cop on vertices (r_3, c_4) and (r_4, c_3) and a further four cops on vertices such that the lines of rows r_3, r_4 and columns c_3, c_4 each have three cops on their vertices. As in the second round, if the robber moved along some line ℓ', then the cops know this. If the robber moved along a symbol-line, then $\ell = \ell'$, and the robber is caught since a cop probed a distance of 0. Otherwise, we may assume by symmetry that ℓ' is a row-line. Further, exactly one cop on $\ell \setminus \{(r_3, c_3), (r_4, c_4)\}$ will probe a distance of 1 to the robber. This is because no vertex on $\ell \setminus \{(r_3, c_3), (r_4, c_4)\}$ can share a symbol-line or a row-line with the robber. As such, the row-line and column-line of the robber is known to the cops, and so by the Latin property, the cops capture the robber. $\square$

We also establish a lower bound on the localization number of MOLS.

Theorem 4.3. *If $\mathcal{L}$ is a set of k-MOLS(n), then*

$$\zeta(G(\mathcal{L})) \geq \frac{2(n-1)}{k+2}.$$

Proof. We play the game with c cops, and derive a lower bound on c such that these c cops can capture the robber (with c to be determined later). Suppose that the robber was not located during the cops' last turn, and after its turn, the robber informs the cops that the robber is on the vertices of some given row-line, say $R(\mathcal{L}, r)$.

This weakens the strategy for only the robber, and so cannot increase the number of cops required to capture the robber. Note that if the cops cannot capture the robber in this round, independent of the row-line on which the robber is located, then the cops will never be able to capture the robber in the standard game. Thus, a lower bound on c such that c cops are required to capture the robber during this single round will be a lower bound on $\zeta(G(\mathcal{L}))$. Similar to Theorem 3.4, we will analyze the number of vertices of $R(\mathcal{L}, r)$ that have distance 0 to a cop, have distance 2 to all cops not on $R(\mathcal{L}, r)$, have distance 1 to exactly one cop not on $R(\mathcal{L}, r)$, and have distance 1 to two or more cops not on $R(\mathcal{L}, r)$.

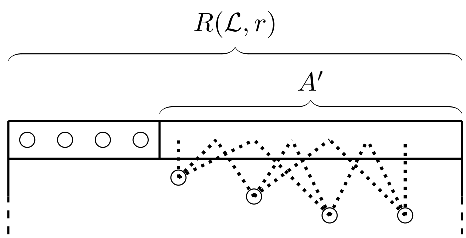

Figure 3: Eight cops attempting to locate a robber along a single row-line of vertices in the Latin square graph of a set of 2-MOLS(11).

Each cop is either on a vertex in $R(\mathcal{L}, r)$, or it is not on $R(\mathcal{L}, r)$ and is adjacent to $k+1$ vertices in $R(\mathcal{L}, r)$. Let A' denote the vertices of $R(\mathcal{L}, r)$ that do not contain cops. Each vertex on $R(\mathcal{L}, r) \setminus A'$ has distance 1 to each vertex in A', so cannot distinguish which vertex the robber is on if the robber is on a vertex of A'. Let C' denote the set of vertices containing the remaining cops on vertices not on $R(\mathcal{L}, r)$, which have some hope of distinguishing the remaining vertices of A', and let $c' = |C'|$. See Figure 3, which depicts a case with $k = 2$, $n = 11$, and where eight cops are at play.

Suppose the cops are able to determine the location of the robber on this turn. Each vertex in $R(\mathcal{L}, r) \setminus A'$ can be immediately localized, as these vertices contain a cop, which will probe a distance of 0. There can be at most one vertex in A' of distance 2 to all cops in C'. For each of the cops in C', there can be at most one vertex in A' of distance 1 to this cop and distance 2 to all other cops in C'. The most optimal situation for the cops is when each cop is adjacent to exactly one vertex in A' that has the property of being distance 2 to all other cops in C', so we assume that this is the case.

We therefore, have that $1 + c'$ vertices in A' have distance 1 to one or zero cops. The remaining $|A'| - c' - 1$ such vertices must each be adjacent to two cops each. Label the edges that directly connect these $|A'| - c' - 1$ vertices to the cops as E. This means that E contains at least $2(|A'| - c' - 1)$ edges. Each cop is adjacent to at most k such vertices, so $|E| \leq c'k$. Thus, we must have that $2(|A'| - c' - 1) \leq |E| \leq c'k$, and so

$$\frac{2(|A'| - 1)}{k + 2} \leq c'.$$

The total number of cops used is

$$c = n - |A'| + c' \geq n - |A'| + \frac{2(|A'| - 1)}{k + 2},$$

which is minimized when $|A'| = n$, yielding $c \geq \frac{2(n-1)}{k+2}$. The proof follows. $\square$

When k is close to n, the lower bound in Theorem 4.3 does not apply. In certain cases, when $k \geq n/2$, we may substantially improve the lower bound by observing certain properties of the set of MOLS. An *orthogonal array* $\mathrm{OA}(k+2, n)$ is a $(n^2) \times (k+2)$ array, with cells filled with symbols in $[n]$ such that the subarray formed by taking any two columns contain each pair in $[n] \times [n]$ precisely once. We say that two rows of an orthogonal array *intersect* in a column if both cells of that column in the two rows contain the same symbol. We note that there is a one-to-one correspondence between a set of k-MOLS(n) and an orthogonal array $\mathrm{OA}(k + 2, n)$; see [14].

Theorem 4.4. *If $\mathcal{M}$ is a set of k-*MOLS(n) *and $\mathcal{N}$ is a set of $(n - 1 - k)$-*MOLS(n) *such that the composition of the orthogonal arrays of $\mathcal{M}$ and $\mathcal{N}$ is the orthogonal array of a set of $(n - 1)$-*MOLS(n)*, then*

$$\zeta(G(\mathcal{M})) = \zeta(G(\mathcal{N})).$$

Proof. Let $\mathcal{O}(\mathcal{M})$ and $\mathcal{O}(\mathcal{N})$ denote the orthogonal arrays corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively. We write both of these arrays such that the side-by-side composition of the two arrays forms the orthogonal array of a set of $(n - 1)$-MOLS(n), say $\mathcal{O}(\mathcal{L})$. If a cop C probes a distance of 1 to the robber R on $G(\mathcal{M})$, then the rows of $\mathcal{O}(M)$ that correspond to the vertices of R and C will intersect, and since the corresponding two rows in $\mathcal{O}(\mathcal{L})$ can only intersect in one column, the two corresponding rows in $\mathcal{O}(\mathcal{N})$ do not intersect. Similarly, if a cop C probes a distance of 2 to the robber on R on $G(\mathcal{M})$, then the rows of $\mathcal{O}(\mathcal{M})$ that correspond to the vertices of R and C do not intersect, and since the corresponding two rows in $\mathcal{O}(\mathcal{L})$ do intersect, the two corresponding rows in $\mathcal{O}(\mathcal{N})$ must also intersect. Equivalent statements hold for $\mathcal{N}$.

We can define a Localization game on $\mathcal{O}(\mathcal{M})$ similar to the Localization game on graphs, except where the following rules apply.

1. The cops and robber are placed on rows of the orthogonal array.
2. The distance between a cop and robber is 0 if they are on the same row, 1 if their rows intersect, and 2 if their rows do not intersect.

By our observations in the first paragraph of this proof, the regular Localization game on $G(\mathcal{M})$ is equivalent to playing the new Localization game on $\mathcal{O}(\mathcal{M})$. An equivalent statement holds for $\mathcal{N}$.

By our observations in the first paragraph of this proof, the distance vectors obtained while playing the new Localization game on $\mathcal{O}(\mathcal{M})$ will differ from the distance vectors obtained while playing the new Localization game on $\mathcal{O}(\mathcal{N})$ only in that the 1's will be mapped to 2's, and vice versa. Thus, the information that the cops receive is equivalent, independent of whether the game is played on $\mathcal{O}(\mathcal{M})$ or $\mathcal{O}(\mathcal{N})$. As such, playing the Localization game on both $\mathcal{O}(\mathcal{M})$ and $\mathcal{O}(\mathcal{N})$ are equivalent. Since these games were equivalent to the Localization game played on $G(\mathcal{M})$ and $G(\mathcal{N})$, we have the desired result that $\zeta(G(\mathcal{M})) = \zeta(G(\mathcal{N}))$. □

By combining Theorems 4.3 and 4.4 we derive following result, which is an improvement when $k \geq n/2$. If $i < j$, then a set $\mathcal{M}$ of i-MOLS(n) is *completable* to a set of j-MOLS(n) if symbols may be added to $\mathcal{M}$ to form a j-MOLS(n).

Corollary 4.5. *If $\mathcal{M}$ is a set of k-MOLS(n) that is completable to a set of $(n-1)$-MOLS(n), then*

$$\zeta(G(\mathcal{M})) \geq \frac{2(n-1)}{n-k+1}.$$

It is well-known that $(n-1)$-MOLS(n) exist when n is a prime power; see for example, [14]. Thus, Corollary 4.5 shows that when n is a prime power and k is close to n, that a set of k-MOLS(n) exists such that the localization number is large. In particular, if $k = c$ or $k = n - c$, where c is a constant, then a set $\mathcal{M}$ of k-MOLS(n) exists such that $\zeta(G(\mathcal{M})) = \Theta(n)$.

5. Future directions

We determined the precise cop number of k-MOLS(n) when $n > (k+1)^2$. However, several other cases remain unresolved. For instance, it is unclear whether the bound on the cop number stated in Theorem 2.4 is tight. In Sections 3 and 4, for a Latin square L of order n, we established the bounds

$$n - \sqrt{\frac{n}{3} + \frac{37}{36}} + \frac{1}{6} \leq \beta(G(L)) \leq 2n - 2,$$

and

$$\frac{2}{3}(n-1) \leq \zeta(G(L)) \leq n + 6.$$

We do not know if these bounds are tight.

There are many other graph parameters in pursuit-evasion besides those studied in this paper, such as the 0-visibility cop number [16], the search number [12], and the burning number [3]. We will investigate these and other pursuit-evasion parameters on Latin square graphs in future work.

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