Journal of Combinatorics
Volume 15, Number 1, 41–57, 2024

Absolutely avoidable order-size pairs for induced subgraphs

Maria Axenovich* and Lea Weber

We call a pair (m, f) of integers, $m \geq 1$, $0 \leq f \leq \binom{m}{2}$, *absolutely avoidable* if there is n_0 such that for any pair of integers (n, e) with $n > n_0$ and $0 \leq e \leq \binom{n}{2}$ there is a graph on n vertices and e edges that contains no induced subgraph on m vertices and f edges. Some pairs are clearly not absolutely avoidable, for example $(m, 0)$ is not absolutely avoidable since any sufficiently sparse graph on at least m vertices contains independent sets on m vertices. Here we show that there are infinitely many absolutely avoidable pairs. We give a specific infinite set M such that for any $m \in M$, the pair $(m, \binom{m}{2}/2)$ is absolutely avoidable. In addition, among other results, we show that for any integer function $q(m)$ for which the limit $\lim_{m\to\infty} \frac{q(m)}{m}$ exists, there are infinitely many values of m such that the pair $(m, \binom{m}{2}/2 + q(m))$ is absolutely avoidable.

AMS 2000 subject classifications: Primary 05C35; secondary 05C75.
Keywords and phrases: Order, size, induced subgraphs, number of edges.

1. Introduction

One of the central topics of graph theory deals with properties of classes of graphs that contain no subgraph isomorphic to some given fixed graph, see for example Bollobás [5]. Similarly, graphs with forbidden induced subgraphs have been investigated from several different angles – enumerative, structural, algorithmic, and more.

Erdős, Füredi, Rothschild and Sós [8] initiated a study of a seemingly simpler class of graphs that do not forbid a specific induced subgraph, but rather forbid any induced subgraph on a given number m of vertices and number f of edges. Following their notation we say a graph G *arrows* a pair of non-negative integers (m, f) and write $G \to (m, f)$ if G has an induced

arXiv: 2106.14908
*The research was partially supported by the DFG grant FKZ AX 93/2-1.

subgraph on m vertices and f edges. We say that a pair (n, e) of non-negative integers *arrows* the pair (m, f), and write $(n, e) \rightarrow (m, f)$, if for any graph G on n vertices and e edges, $G \rightarrow (m, f)$.

As an example, if $t_{m-1}(n)$ denotes the number of edges in the balanced complete $(m-1)$-partite graph on n vertices, then by Turán's theorem [15] we know that any graph on n vertices with more than $t_{m-1}(n)$ edges contains K_m, a complete subgraph on m vertices. Equivalently stated, we have $(n, e) \rightarrow (m, \binom{m}{2})$ if $e > t_{m-1}(n)$.

For a fixed pair (m, f) let

$$S_n(m, f) = \{e : (n, e) \rightarrow (m, f)\} \quad \text{and} \quad \sigma(m, f) = \limsup_{n \rightarrow \infty} |S_n(m, f)| / \binom{n}{2}.$$

In [8] the authors considered $\sigma(m, f)$ for different choices of (m, f). One of their main results is

Theorem 1 ([8]). *If* $(m, f) \notin \{(2, 0), (2, 1), (4, 3), (5, 4), (5, 6)\}$, *then* $\sigma(m, f) \leq \frac{2}{3}$; *otherwise* $\sigma(m, f) = 1$.

He, Ma, and Zhao [9] improved the upper bound 2/3 to 1/2 and showed that there are infinitely many pairs for which the equality $\sigma(m, f) = \frac{1}{2}$ holds.

In [8] the authors also gave a construction demonstrating that "most of the"$\sigma(m, f)$ are 0, by showing that for large n almost all pairs (n, e) can be realized as the vertex disjoint union of a clique and a high-girth graph, and that for fixed m most pairs (m, f) cannot be realized as the vertex disjoint union of a clique and a forest. For some other results concerning sizes of induced subgraphs, see for example Alon and Kostochka [2], Alon, Balogh, Kostochka, and Samotij [1], Alon, Krivelevich, and Sudakov [3], Axenovich and Balogh [4], Bukh and Sudakov [6], Kwan and Sudakov [12, 11] and Narayanan, Sahasrabudhe, and Tomon [14].

In this paper we investigate the existence of pairs (m, f) for which we not only have $\sigma(m, f) = 0$, but the stronger property $S_n(m, f) = \emptyset$ for large n.

Definition 1. *A pair* (m, f) *is* ***absolutely avoidable*** *if there is* n_0 *such that for each* $n > n_0$ *and for any* $e \in \{0, \ldots, \binom{n}{2}\}$, $(n, e) \not\rightarrow (m, f)$.

Our results show that there are infinitely many absolutely avoidable pairs. Our first result gives an explicit construction of infinitely many absolutely avoidable pairs $(m, \binom{m}{2}/2)$. The second one provides an existence result of infinitely many absolutely avoidable pairs (m, f), where f is "close" to $\binom{m}{2}/2$. Finally, the last result shows that for every sufficiently large m congruent to 0 or 1 modulo 4, at least one of the pairs $(m, \binom{m}{2}/2)$ and $(m, \binom{m}{2}/2 - 6m)$ is absolutely avoidable.

For the first result we need to define the following set M of integers. Let

$$M = \left\{ \frac{1}{2} \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^s \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 5 \right) : s \in \mathbb{N}, s \geq 2 \right\}.$$

In particular, we have $M = \{40, 221, 1276 \dots\}$.

Theorem 2. *For any $m \in M$, $f = \binom{m}{2}/2$ is an integer and the pair (m, f) is absolutely avoidable.*

Theorem 3. *For any integer valued function $q(m)$ for which the limit $\lim\limits_{m\to\infty} \frac{q(m)}{m}$ exists, there are infinitely many values of m, such that the pair $(m, \binom{m}{2}/2 - q(m))$ is absolutely avoidable.*

Moreover, there are infinitely many values of m, such that for any integer $f' \in (\binom{m}{2}/2 - 0.175m, \binom{m}{2}/2 + 0.175m)$ the pair (m, f') is absolutely avoidable.

Theorem 4. *For any $m \geq 740$ with $m \equiv 0, 1 \pmod 4$ either $(m, \binom{m}{2}/2)$ or $(m, \binom{m}{2}/2 - 6m)$ is absolutely avoidable.*

The main idea of the proofs is that for certain pairs (m, f), there is no graph on m vertices and f edges which is a vertex disjoint union of a clique and a forest or a complement of a vertex disjoint union of a clique and a forest. In order to do so, we need several number theoretic statements that we prove in several lemmas. After that, we use the observation from [8], that for any $0 \leq c < 1$, for any sufficiently large n (depending on c), and any $e \leq c\binom{n}{2}$, there is a graph on n vertices with e edges that is the vertex disjoint union of a clique and a graph of girth greater than m. In particular, any m-vertex induced subgraph of such a graph is a disjoint union of a clique and a forest. Considering the complements, we deduce that (m, f) is absolutely avoidable.

The problem can also be considered in a bipartite setting. It would be interesting to show whether there are absolutely avoidable pairs. Unfortunately we cannot use our method to find such pairs, since any bipartite pair (m, f) with $f \leq m^2/2$ can be represented as the vertex disjoint union of a complete bipartite graph and a forest, see Section 4.

We state and prove the lemmas in Section 2 and prove the theorems in Section 3.

2. Lemmas and number theoretic results

For a positive real number x, let $[x] = \{0, 1, \dots, \lfloor x \rfloor\}$. We say that a pair (m, f) is *realizable* by a graph $H = (V, E)$ if $|V(H)| = m$ and $|E(H)| = f$.

For two reals x, y, $x \leq y$, we use the standard notation $(x, y), [x, y), (x, y]$, and $[x, y]$ for respective intervals of reals. For $x \in \mathbb{R}$ let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x, i.e. $\{x\} \in [0, 1)$ and $\{x\} = x \pmod 1$. A real valued sequence $(x_n)_{n\in\mathbb{N}}$ is called *uniformly distributed modulo 1* (we write u.d. mod 1) if for any pair of real numbers s, t with $0 \leq s < t \leq 1$ we have

$$\lim_{N\to\infty} \frac{|\{n : 1 \leq n \leq N, \{x_n\} \in [s, t)\}|}{N} = t - s.$$

The following lemma is used in the proof of Theorem 4:

Lemma 1. *(a) The sequence $(x_n) = \alpha n$ is u.d. mod 1 for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

(b) If a real valued sequence (x_n) is u.d. mod 1 and a real valued sequence (y_n) has the property $\lim\limits_{n\to\infty}(x_n - y_n) = \beta$, a real constant, then (y_m) is also u.d. mod 1.

For proofs of these facts see for example Theorem 1.2 and Example 2.1 in [10].

The following lemma is given in [8], we include it here for completeness.

Lemma 2. *Let $p \in \mathbb{N}$ and c be a constant $0 \leq c < 1$. Then for $n \in \mathbb{N}$ sufficiently large and any $e \in [c\binom{n}{2}]$, there exists a non-negative integer k and a graph on n vertices and e edges which is the vertex disjoint union of a clique of size k and a graph on $n - k$ vertices of girth at least p.*

Proof. Let $p > 0$ be given. We use the fact that for any v large enough there exists a graph of girth p on v vertices with $v^{1+\frac{1}{2p}}$ edges. For a probabilistic proof of this fact see for example Bollobás [5] and for an explicit construction see Lazebnik et al. [13]. Let n be a given sufficiently large integer. Let $e \in [c\binom{n}{2}]$. Let k be a non-negative integer such that $\binom{k}{2} \leq e \leq \binom{k+1}{2} - 1$. Note that since $e \leq c\binom{n}{2}$, and $\binom{k}{2} \leq c\binom{n}{2}$, thus $k \leq \sqrt{c}n + 1 \leq c'n$, where c' is a constant, $c' < 1$. We claim that (n, e) could be represented as a vertex disjoint union of a clique on k vertices and a graph of girth at least p. For that, consider a graph G' on $n - k$ vertices and girth at least p such that $|E(G')| \geq (n-k)^{1+\frac{1}{2p}}$. Consider G'', the vertex disjoint union of G' and K_k. Then $|E(G'')| \geq \binom{k}{2} + (n - k)^{1+\frac{1}{2p}} \geq \binom{k+1}{2} \geq e$. Here, the second inequality holds since $(n - k)^{1+\frac{1}{2p}} \geq k$ for $k \leq c'n$ and n large enough. Finally, let G be a subgraph of G'' on e edges, obtained from G'' by removing some edges of G'. Thus, G is the vertex disjoint union of a clique on k vertices and a graph of girth at least p. $\square$

We shall need two number theoretic lemmas for the proof of the main result. Below the set M is defined as in the introduction.

Lemma 3. *For any $m \in M$, m is a positive integer congruent to 0 or 1 modulo 4, and $\sqrt{2m^2 - 10m + 9}$ is an odd integer for each $m \in M$.*

Proof. Recall that $M = \left\{ \frac{1}{2} \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^s \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 5 \right) : s \in \mathbb{N}, s \geq 2 \right\}$. We see, that M corresponds to the following recursion: $(x_0, y_0) = (3, 1)$ and for $s \geq 0$

$$x_{s+1} = 3x_s + 4y_s$$
$$y_{s+1} = 2x_s + 3y_s.$$

I.e., for $s \geq 0$,

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^s \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Indeed, $M = \{(x_s + 5)/2 : s \geq 2\}$.

From the recursion we see that $x_{2s} \equiv 3 \pmod 8$, $x_{2s+1} \equiv 5 \pmod 8$, $y_{4s} = y_{4s+1} \equiv 1 \pmod 8$, and $y_{4s+2} = y_{4s+3} \equiv 5 \pmod 8$ for $s \in \mathbb{N}_0$. In particular y_s is an odd integer. Let $m_s = (x_s + 5)/2$, i.e., $M = \{m_s : s \geq 2\}$. When s is even, $m_s \equiv 0 \pmod 4$, and if s is odd, $m_s \equiv 1 \pmod 4$. This proves the first statement of the lemma.

Next, we observe that $(x, y) = (x_s, y_s)$ gives an integer solution to the generalized Pell's equation

$$(*) \qquad x^2 - 2y^2 = 7.$$

Indeed, $(x, y) = (x_0, y_0) = (3, 1)$ satisfies $(*)$. Assume that $(x, y) = (x_s, y_s)$ satisfies $(*)$. Let $(x, y) = (x_{s+1}, y_{s+1})$ and insert it into the left hand side of $(*)$. Then we have

$$x_{s+1}^2 - 2y_{s+1}^2 = 9x_s^2 + 24x_sy_s + 16y_s^2 - 8x_s^2 - 24x_sy_s - 18y_s^2 = x_s^2 - 2y_s^2 = 7.$$

Thus $(x, y) = (x_{s+1}, y_{s+1})$ also satisfies $(*)$.

Since (x_s, y_s) satisfies $(*)$, we have that $y_s = \sqrt{\frac{1}{2}(x_s^2 - 7)}$. Then $y_s = \sqrt{\frac{1}{2}((2m_s - 5)^2 - 7)} = \sqrt{\frac{1}{2}(4m_s^2 - 20m_s + 18)} = \sqrt{2m_s^2 - 10m_s + 9}$. Since y_s is an odd integer, the second statement of the lemma follows. $\square$

For the next lemmas and theorems we will need the following definitions. Let $m, q \in \mathbb{Z}$, $m \geq 5 + 2\sqrt{|q|}$. Let

$$y_q(m) = \frac{\sqrt{2m^2 - 10m - 8q + 9}}{2}, \qquad z_q(m) = \frac{\sqrt{2m^2 - 2m - 8q + 1}}{2},$$

$$t_q(m) = z_q(m) - y_q(m), \qquad d_q(m) = \frac{3}{2} - t_q(m),$$
$$L_q(m) = \left\lfloor \frac{5}{2} + y_q(m) \right\rfloor, \qquad R_q(m) = \left\lfloor \frac{1}{2} + z_q(m) \right\rfloor.$$

Note that since $m \geq 5 + 2\sqrt{|q|}$, we always have $y_q(m), z_q(m) \in \mathbb{R}$.

Lemma 4. *Let $q = q(m), m \in \mathbb{Z}$, $m \equiv 0, 1 \pmod 4$, $m \geq 5 + 2\sqrt{|q|}$, and $|q(m)| \in O(m)$.*

(a) We have $t_q(m) = \frac{2\sqrt{2}(1-\frac{1}{m})}{\sqrt{1-\frac{1}{m}+\frac{1-8q}{2m^2}}+\sqrt{1-\frac{5}{m}+\frac{9-8q}{2m^2}}}$. In particular, $\lim_{m\to\infty} d_q(m) = \frac{3}{2} - \sqrt{2}$.

(b) We have $L_q(m) > R_q(m)$ if and only if $\{y_q(m)\} \in [0, d_q(m)) \cup \left[\frac{1}{2}, 1\right)$. In particular, $L_0(m) > R_0(m)$ if $m \in M$.

Proof. We start by proving (a). By definition of $t_q(m)$ we have

$$\begin{aligned}
t_q(m) &= z_q(m) - y_q(m) \\
&= \frac{1}{2}\sqrt{2m^2 - 2m - 8q + 1} - \frac{1}{2}\sqrt{2m^2 - 10m - 8q + 9} \\
&= \frac{1}{2}\frac{2m^2 - 2m - 8q + 1 - 2m^2 + 10m + 8q - 9}{\sqrt{2m^2 - 2m - 8q + 1} + \sqrt{2m^2 - 10m - 8q + 9}} \\
&= \frac{2\sqrt{2}(1 - \frac{1}{m})}{\sqrt{1 - \frac{1}{m} + \frac{1-8q}{2m^2}} + \sqrt{1 - \frac{5}{m} + \frac{9-8q}{2m^2}}}.
\end{aligned}$$

This also shows that for $|q| = |q(m)| \in O(m)$, $\lim_{m\to\infty} d_q(m) = \frac{3}{2} - \lim\limits_{m\to\infty} t_q(m) = \frac{3}{2} - \sqrt{2}$, which concludes the proof of (a).

Now we can prove part (b). From part (a) we have in particular that $t_q(m) = \sqrt{2} + \epsilon_q(m)$, where for m sufficiently large $|\epsilon_q(m)| < 0.05$, and thus, $t_q(m) \in (1, \frac{3}{2})$. Thus, $d_q(m) = \frac{3}{2} - t_q(m) \in (0, \frac{1}{2})$ for sufficiently large m. We compare $L_q(m)$ and $R_q(m)$ using the expression $x = \lfloor x \rfloor + \{x\}$:

$$\begin{aligned}
L_q(m) &= \left\lfloor \frac{5}{2} + y_q(m) \right\rfloor \\
&= 2 + \lfloor y_q(m) \rfloor + \left\lfloor \frac{1}{2} + \{y_q(m)\} \right\rfloor \\
&= 2 + \lfloor y_q(m) \rfloor + \begin{cases} 0, & \{y_q(m)\} \in [0, \frac{1}{2}) \\ 1, & \{y_q(m)\} \in [\frac{1}{2}, 1) \end{cases},
\end{aligned}$$

$$\begin{aligned} R_q(m) &= \left\lfloor \frac{1}{2} + z_q(m) \right\rfloor \\ &= \left\lfloor \frac{1}{2} + y_q(m) + t_q(m) \right\rfloor \\ &= \lfloor y_q(m) \rfloor + \left\lfloor \frac{1}{2} + t_q(m) + \{y_q(m)\} \right\rfloor \\ &= \lfloor y_q(m) \rfloor + \begin{cases} 1, & t_q(m) + \{y_q(m)\} \in [1, \frac{3}{2}) \\ 2, & t_q(m) + \{y_q(m)\} \in [\frac{3}{2}, \frac{5}{2}) \end{cases}. \end{aligned}$$

Thus

$$\begin{aligned} &L_q(m) - R_q(m) \\ &= 2 + \begin{cases} 0-1, & \{y_q(m)\} \in [0, \frac{1}{2}) \text{ and } t_q(m) + \{y_q(m)\} \in [1, \frac{3}{2}) \\ 0-2, & \{y_q(m)\} \in [0, \frac{1}{2}) \text{ and } t_q(m) + \{y_q(m)\} \in [\frac{3}{2}, \frac{5}{2}) \\ 1-1, & \{y_q(m)\} \in [\frac{1}{2}, 1) \text{ and } t_q(m) + \{y_q(m)\} \in [1, \frac{3}{2}) \\ 1-2, & \{y_q(m)\} \in [\frac{1}{2}, 1) \text{ and } t_q(m) + \{y_q(m)\} \in [\frac{3}{2}, \frac{5}{2}) \end{cases}. \end{aligned}$$

So, $L_q(m) - R_q(m) > 0$ in all cases except for the second one, i.e., if and only if

$$\begin{aligned} \{y_q(m)\} &\in [0,1) \setminus \left(\left[0, \tfrac{1}{2}\right) \cap \left[\tfrac{3}{2} - t_q(m), \tfrac{5}{2} - t_q(m)\right)\right) \\ &= \left[\tfrac{1}{2}, 1\right) \cup ([0,1) \setminus [d_q(m), 1 + d_q(m))) \\ &= \left[\tfrac{1}{2}, 1\right) \cup [0, d_q(m)). \end{aligned}$$

Now let $m \in M$ and consider $y_0(m) = \frac{\sqrt{2m^2-10m+9}}{2}$. Then by Lemma 3, $2y_0(m)$ is an odd integer for all $m \in M$, i.e. $\{y_0(m)\} = \frac{1}{2}$. Thus, we have $L_0(m) > R_0(m)$ for all $m \in M$, which concludes the proof of (b). $\square$

Lemma 5. *If $q = q(m) \in \mathbb{Z}$, $m \in \mathbb{N}$, $m \equiv 0, 1 \pmod 4$, $m \geq 2\sqrt{|q|} + 5$, and $L_q(m) > R_q(m)$, then the pair $(m, \binom{m}{2}/2 - q)$ cannot be realized as the vertex disjoint union of a clique and a forest.*

Proof. Let $f = \binom{m}{2}/2 - q$. Suppose that (m, f) can be realized as the vertex disjoint union of a clique K on x vertices and a forest F on $m - x$ vertices. We shall show that $L_q(m) \leq R_q(m)$.

Claim 1: $x \geq L_q(m)$.

The forest F has $f - \binom{x}{2} = \binom{m}{2}/2 - q - \binom{x}{2}$ edges. Since F has $m - x$ vertices, it contains strictly less than $m - x$ edges. Thus $\binom{m}{2}/2 - q - \binom{x}{2} <$

$m-x$. Solving for x gives

$$x > \frac{3}{2} + \frac{1}{2}\sqrt{2m^2-10m-8q+9} \quad \text{or} \quad x < \frac{3}{2} - \frac{1}{2}\sqrt{2m^2-10m-8q+9}.$$

Since $m \geq 2\sqrt{|q|}+5$, we have $2m^2-10m-8q+9 \geq 9$. The second inequality gives $x < \frac{3}{2} - \frac{1}{2}\sqrt{2m^2-10m-8q+9}$, and thus $x < 0$, a contradiction. So only the first inequality for x holds and implies that

$$x \geq \left\lfloor \frac{3+\sqrt{2m^2-10m-8q+9}}{2} \right\rfloor + 1 = L_q(m),$$

which proves Claim 1.

Claim 2: $x \leq R_q(m)$.

The number of edges in the clique K is at most f and exactly $\binom{x}{2}$. Thus $\binom{x}{2} \leq f = \binom{m}{2}/2 - q$, which implies that $2x(x-1) \leq m(m-1) - 4q$. This in turn gives

$$x \leq \left\lfloor \frac{1+\sqrt{2m^2-2m-8q+1}}{2} \right\rfloor = R_q(m),$$

and proves Claim 2.

Claims 1 and 2 imply that $L_q(m) \leq R_q(m)$. □

Lemma 6. *Let $q = q(m) \in \mathbb{Z}$, $m \in \mathbb{N}$, $m \equiv 0, 1 \pmod 4$, $m \geq 2\sqrt{|q|}+5$. If both $L_q(m) > R_q(m)$ and $L_{-q}(m) > R_{-q}(m)$, then the pair $(m, f) = (m, \binom{m}{2}/2 - q)$ is absolutely avoidable.*

Proof. Let m satisfy the condition of the lemma and let $f_- = \binom{m}{2}/2 - q$ and $f_+ = \binom{m}{2}/2 + q$. Then by Lemma 5, neither (m, f_+) nor (m, f_-) can be represented as the vertex disjoint union of a clique and a forest.

By Lemma 2, for every sufficiently large n, and all $e \leq \lceil \binom{n}{2}/2 \rceil$ we can realize (n, e) as the vertex disjoint union of a clique and a graph of girth greater than m. Thus, for each $e \in \{0, 1, \ldots, \binom{n}{2}\}$ there is a graph G on n vertices and e edges such that either G or the complement $\overline{G}$ of G is a vertex disjoint union of a clique and a graph of girth greater than m.

If G is the vertex disjoint union of a clique and a graph of girth greater than m, then any m-vertex induced subgraph of G is a vertex disjoint union of a clique and a forest. Since (m, f_-) can not be represented as a clique and a forest, we have $G \not\to (m, f_-)$. If $\overline{G}$ is the vertex disjoint union of a clique and a graph of girth greater than m, then as above $\overline{G} \not\to (m, f_+)$. Since

$f_- = \binom{m}{2} - f_+$, we have that $G \not\to (m, f_-)$. Thus, (m, f_-) is absolutely avoidable. $\square$

3. Proofs of the Main Theorems

Proof of Theorem 2. Let $m \in M$. By Lemma 3 we have $m \equiv 0, 1 \pmod 4$, so $f = \binom{m}{2}/2$ is an integer. By Lemma 4(b) we have $L_0(m) > R_0(m)$. Now we can apply Lemma 6 with $q = 0$. Thus, the pair (m, f) is absolutely avoidable. $\square$

Proof of Theorem 3. Let $q = q(m) \in \mathbb{Z}$ and let $a = \lim_{m\to\infty} \frac{q(m)}{m}$.

Recall that $y_q(m) = \frac{1}{2}\sqrt{2m^2 - 10m + 9 - 8q}$.

Claim 1: $\lim_{m\to\infty} \left(\frac{m}{\sqrt{2}} - y_q(m)\right) = \frac{5}{2\sqrt{2}} + \sqrt{2}a$ and $\lim_{m\to\infty} \left(\frac{m}{\sqrt{2}} - y_{-q}(m)\right) = \frac{5}{2\sqrt{2}} - \sqrt{2}a$.

Observe that

$$\begin{aligned}
\lim_{m\to\infty} \left(\frac{m}{\sqrt{2}} - y_q(m)\right) &= \lim_{m\to\infty} \frac{m}{\sqrt{2}} \left(1 - \sqrt{1 - \frac{5}{m} + \frac{9-8q}{2m^2}}\right) \\
&= \lim_{m\to\infty} \frac{m}{\sqrt{2}} \frac{\frac{5}{m} - \frac{9-8q}{2m^2}}{1 + \sqrt{1 + \frac{5}{m} + \frac{9-8q}{2m^2}}} \\
&= \frac{5}{2\sqrt{2}} + \lim_{m\to\infty} \frac{\sqrt{2}q}{m} \\
&= \frac{5}{2\sqrt{2}} + \sqrt{2}a.
\end{aligned}$$

Doing a similar calculation for $y_{-q}(m)$ proves Claim 1.

Claim 2: $y_q(4m)$ and $y_{-q}(4m)$ are u.d. mod 1, and in particular, $y_0(4m)$ is u.d. mod 1.

Since $\frac{1}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$, by Lemma 1(a) the sequence $(x_{4m}) = (4m)/\sqrt{2}$ is u.d. mod 1. Since we have $\lim_{m\to\infty} (x_{4m} - y_q(4m)) = \frac{5+2\sqrt{2}a}{2\sqrt{2}} \in \mathbb{R}$ and $\lim_{m\to\infty} (x_{4m} - y_{-q}(4m)) = \frac{5-2\sqrt{2}a}{2\sqrt{2}} \in \mathbb{R}$, by Lemma 1(b) $(y_q(4m))$ and $(y_{-q}(4m))$ are also u.d. mod 1. This proves Claim 2.

Now, to prove the first part of the theorem, from Lemma 6 it suffices to find infinitely many integers m such that for $q = q(m)$, $L_q(m) > R_q(m)$ and $L_{-q}(m) > R_{-q}(m)$.

By Lemma 4(a), we have that $\lim_{m\to\infty} d_q(m) = \lim_{m\to\infty} d_{-q}(m) = 3/2-\sqrt{2}$. Let m_0 be large enough so that for any $m \geq m_0$, $d_q(m)$ and $d_{-q}(m)$ are close to these limits, i.e., $|d_q(m) - (3/2-\sqrt{2})| < (3/2-\sqrt{2})/3$ and $|d_{-q}(m) - (3/2-\sqrt{2})| < (3/2-\sqrt{2})/3$.

Let $\delta > 0$ be a small constant such that $\delta < (3/2-\sqrt{2})/2$, $2\delta < 1-\{\sqrt{2}a\}$ and if $\{\sqrt{2}a\} < 1/2$, then $\delta < 1/2 - \{\sqrt{2}a\}$. In addition assume that δ is sufficiently small that for any $m \geq m_0$, $\delta < d_q(m)/3$, and $\delta < d_{-q}(m)/3$. Using Claim 1, define m_δ to be sufficiently large, so that $m_\delta > m_0$ and for any $m \geq m_\delta$, $y_q(m) - \frac{m}{\sqrt{2}}$ and $y_{-q}(m) - \frac{m}{\sqrt{2}}$ are δ-close to the limiting values:

$$y_q(m) \in \left(\left(\frac{m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} - \sqrt{2}a\right) - \delta, \left(\frac{m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} - \sqrt{2}a\right) + \delta\right) \quad \text{and}$$
$$y_{-q}(m) \in \left(\left(\frac{m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} + \sqrt{2}a\right) - \delta, \left(\frac{m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} + \sqrt{2}a\right) + \delta\right).$$

We distinguish 2 cases based on the values of a:

Case 1: $\{\sqrt{2}a\} \in [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4})$, i.e. $\{2\sqrt{2}a\} \in [0, \frac{1}{2})$.

Since $\frac{4m}{\sqrt{2}} - \frac{5}{2\sqrt{2}}$ is a sequence u.d. mod 1, there is an infinite set M_1 of integers at least m_δ, such that for any $m \in M_1$

$$\frac{4m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} \in (k_m + 1/2 + \{\sqrt{2}a\} + \delta, k_m + 1/2 + \{\sqrt{2}a\} + 2\delta),$$

for some integer k_m. Then we have

$$\begin{aligned} y_q(4m) \in \Big(&(1/2 + k_m + \{\sqrt{2}a\} + \delta) - \sqrt{2}a - \delta,\\ &(1/2 + k_m + \{\sqrt{2}a\} + \delta) - \sqrt{2}a + \delta\Big),\\ y_{-q}(4m) \in \Big(&(1/2 + k_m + \{\sqrt{2}a\} + \delta) + \sqrt{2}a - \delta,\\ &(1/2 + k_m + \{\sqrt{2}a\} + \delta) + \sqrt{2}a + \delta\Big). \end{aligned}$$

This implies that

$$\{y_q(4m)\}, \{y_{-q}(4m)\} \in [1/2, 1)\,.$$

From Lemma 4(b), $L_q(4m) > R_q(4m)$ and $L_{-q}(4m) > R_{-q}(4m)$. Note that $f = \binom{4m}{2}/2 - q(4m)$ is an integer. By Lemma 6 the pair $\left(4m, \binom{4m}{2}/2 - q(4m)\right)$ is absolutely avoidable for any $m \in M_1$.

Case 2: $\{\sqrt{2}a\} \in [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1)$, i.e. $\{2\sqrt{2}a\} \in [\frac{1}{2}, 1)$.

Since $\frac{4m}{\sqrt{2}} - \frac{5}{2\sqrt{2}}$ is a sequence which is u.d. mod 1, there is an infinite set M_2 of integers at least m_δ, such that for any $m \in M_2$

$$\frac{4m}{\sqrt{2}} - \frac{5}{2\sqrt{2}} \in (k_m + \{\sqrt{2}a\} + \delta, k_m + \{\sqrt{2}a\} + 2\delta),$$

for some integer k_m. Then we have

$$y_q(4m) \in \Big((k_m + \{\sqrt{2}a\} + \delta) - \sqrt{2}a - \delta, (k_m + \{\sqrt{2}a\} + \delta) - \sqrt{2}a + \delta\Big)$$

and

$$y_{-q}(4m) \in \Big((k_m + \{\sqrt{2}a\} + \delta) + \sqrt{2}a - \delta, (k_m + \{\sqrt{2}a\} + \delta) + \sqrt{2}a + \delta\Big).$$

This implies that

$$\{y_q(4m)\} \in [0, 2\delta)\,, \{y_{-q}(4m)\} \in [1/2, 1)\,.$$

Recall that for any $m > m_\delta$, $\delta < d_q(m)/3$. Thus, $\{y_{-q}(4m)\} \in [1/2, 1)$ and $\{y_q(4m)\} \in [1/2, 1) \cup [0, d_q(4m))$. From Lemma 4(b), $L_q(4m) > R_q(4m)$ and $L_{-q}(4m) > R_{-q}(4m)$. Note that $f = \binom{4m}{2}/2 - q(4m)$ is an integer. Thus, by Lemma 6 the pair $\big(4m, \binom{4m}{2}/2 - q(4m)\big)$ is absolutely avoidable for any $m \in M_2$.

This proves the first part of the theorem.

For the second part, let $c = 0.175 < \frac{1}{4\sqrt{2}}$. We shall show that there is an infinite set M_0 of integers such that for any $m \in M_0$ and for all integers $q \in (-cm, cm)$, the pair $(m, \binom{m}{2}/2 - q)$ is absolutely avoidable. In order to do that, we shall show that $y_0(m)$ does not differ much from $y_q(m)$, for chosen values of m.

Recall that $\lim_{m\to\infty} d_q(m) = 3/2 - \sqrt{2} > 0$ for any $q \in (-cm, cm)$. Thus, the interval $\big[\frac{3}{4}, \frac{3}{4} + d_q(m)\big)$ has positive length for any such q and sufficiently large m. By Claim 2 the sequence $y_0(4m)$ is u.d. mod 1, thus there are infinitely many values of m that $m \equiv 0 \pmod 4$ and $\{y_0(m)\} \in \big[\frac{3}{4}, \frac{3}{4} + d_q(m)\big)$. Now our choice for m will allow us to use Lemmas 4, 5 and 6.

Let $q \in (-cm, cm)$. It will be easier for us to deal with $y_q(m) - y_0(m)$ instead of $y_q(m)$. Let $s_q(m) = y_q(m) - y_0(m)$. We have

$$\lim_{m\to\infty} s_q(m) \quad = \quad \lim_{m\to\infty} (y_q(m) - y_0(m))$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \frac{1}{2}\left(\sqrt{2m^2-10m+9-8q}-\sqrt{2m^2-10m+9}\right)\\
&= -\sqrt{2}\lim_{m\to\infty}\frac{q}{m}.
\end{aligned}
$$

Thus, since $q \in (-cm, cm)$, $c = 0.175 < \frac{1}{4\sqrt{2}}$, for m sufficiently large we have $s_q(m) \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. Since $y_q = s_q(m) + y_0(m)$, and $\{y_0(m)\} \in \left[\frac{3}{4}, \frac{3}{4} + d_q(m)\right)$, we have that $\{y_q\} = \{s_q(m) + y_0(m)\} \in [0, d_q(m)) \cup [\frac{1}{2}, 1)$. Lemma 4(b) implies that $L_q(m) > R_q(m)$ and $L_{-q}(m) > R_{-q}(m)$. Lemmas 5 and 6 then imply that $(m, \binom{m}{2}/2 - q)$ is absolutely avoidable. $\square$

Proof of Theorem 4. Let $m \geq 740$, $m \equiv 0, 1 \pmod 4$. If $L_0(m) > R_0(m)$, by Lemma 6 $(m, \binom{m}{2}/2)$ is absolutely avoidable, so we assume using Lemma 4(b) that $\{y_0(m)\} \in [d_0(m), \frac{1}{2})$.

We shall first make some observations about $y_{6m}(m)$ and $y_{-6m}(m)$ by comparing them to $y_0(m)$. From the definition we have

$$
y_0(m) = \frac{1}{2}\sqrt{2m^2-10m+9}, \quad y_{6m}(m) = \frac{1}{2}\sqrt{2m^2-58m+9},
$$
$$
y_{-6m}(m) = \frac{1}{2}\sqrt{2m^2+38m+9}.
$$

Thus

$$
\lim_{m\to\infty} y_0(m) - y_{6m}(m) = 6\sqrt{2} \quad \text{and} \quad \lim_{m\to\infty} y_0(m) - y_{-6m}(m) = -6\sqrt{2}.
$$

By Lemma 4(a),

$$
\lim_{m\to\infty} t_0(m) = \lim_{m\to\infty} t_{6m}(m) = \lim_{m\to\infty} t_{-6m}(m) = \sqrt{2}.
$$

This implies that

$$
\begin{aligned}
\lim_{m\to\infty} y_0(m) - y_{6m}(m) - t_{6m}(m) &= 5\sqrt{2} > 7\\
\lim_{m\to\infty} y_0(m) - y_{6m}(m) + t_0(m) &= 7\sqrt{2} < 10\\
\lim_{m\to\infty} -(y_0(m) - y_{-6m}(m)) + t_{-6m}(m) &= 7\sqrt{2} < 10\\
\lim_{m\to\infty} -(y_0(m) - y_{-6m}(m)) - t_0(m) &= 5\sqrt{2} > 7.
\end{aligned}
$$

Thus, for sufficiently large m we have

$$
y_{6m}(m) \quad < \quad y_0(m) - t_{6m}(m) - 7
$$

$$\begin{array}{rcl} y_{6m}(m) & > & y_0(m) + t_0(m) - 10 \\ y_{-6m}(m) & < & 10 + y_0(m) - t_{-6m}(m) \\ y_{-6m}(m) & > & 7 + y_0(m) + t_0(m). \end{array}$$

In particular, one can verify that for $m \geq 740$ the differences between the limits and the actual values are sufficiently small that the above inequalities hold.

Thus, combining these inequalities and recalling that $d_q(m) + t_q(m) = 3/2$, for any q, we have

$$y_0(m) - 8 - \frac{1}{2} - d_0(m) < y_{6m}(m) \leq y_0(m) - 8 - \frac{1}{2} + d_{6m}(m),$$
$$y_0(m) + 8 + \frac{1}{2} - d_0(m) < y_{-6m}(m) \leq y_0(m) + 8 + \frac{1}{2} + d_{-6m}(m).$$

Recall that by assumption $\{y_0(m)\} \in [d_0(m), \frac{1}{2})$. Recall also that by Lemma 4(a), $\lim_{m\to\infty} d_q(m) = \frac{3}{2} - \sqrt{2} \approx 0.086$, for $q \in \{0, 6m, -6m\}$. Then for sufficiently large m we have $\{y_{6m}(m)\} \in [0, d_{6m}(m)) \cup [\frac{1}{2}, 1)$ and $\{y_{-6m}(m)\} \in [0, d_{-6m}(m)) \cup [\frac{1}{2}, 1)$. In particular, one can again verify that this holds for $m \geq 740$.

This implies by Lemma 4(b) that $L_{6m}(m) > R_{6m}(m)$ and $L_{-6m}(m) > R_{-6m}(m)$. Therefore, by Lemma 6, the pair $(m, \binom{m}{2}/2 - 6m)$ is absolutely avoidable. $\square$

4. The bipartite setting

Our entire argument for the existence of absolutely avoidable pairs so far built on the fact that certain pairs (m, f) can not be realized as the disjoint union of a clique and a forest. A similar question can be asked in the bipartite setting:

We say a bipartite graph G *bipartite arrows* the pair (m, f), and write $G \overset{bip}{\to} (m, f)$ if G has an induced subgraph with parts of size m each, contained in the respective parts of G, with exactly f edges. We say that a pair (n, e) of non-negative integers *bipartite arrows* the pair (m, f), written $(n, e) \overset{bip}{\to} (m, f)$ if for any bipartite graph G with parts of size n each and with e edges, $G \overset{bip}{\to} (m, f)$.

We call a pair (m, f) *absolutely avoidable in a bipartite setting* if there exists n_0, such that for each $n \geq n_0$ and for any $e \in \{0, \dots, n^2\}$, $(n, e) \overset{bip}{\not\to} (m, f)$. We refer to a complete bipartite graph as a *biclique*. We say that a pair (m, f) is *bipartite representable* as a graph H if there is a bipartite graph H with m vertices in each part and f edges. The following lemma shows that our argument for the existence of such pairs in the non-bipartite case cannot be extended to the bipartite setting.

Here, a *biclique* is an induced subgraph of a complete bipartite graph, i.e., could be in particular an empty set or a single vertex.

Lemma 7. *For any positive integer m and any non-negative integer f, $f \leq \left\lfloor \frac{m^2}{2} \right\rfloor$, there is a bipartite graph H with m vertices in each part, f edges, which is the vertex disjoint union of a biclique and a forest.*

Proof. Fix a pair (m, f) with $f \leq \left\lfloor \frac{m^2}{2} \right\rfloor$. Let $x = \left\lfloor \frac{m}{2} \right\rfloor$ and let y be the largest integer such that $xy \leq f$. In particular

$$xy > f - x \quad \text{and} \quad y \leq \left\lfloor \frac{m^2}{2} \right\rfloor / \left\lfloor \frac{m}{2} \right\rfloor.$$

We shall use the fact that for any non-negative integers v' and e', with $e' < v'$ and for any partition $v' = v'' + v'''$, with v'', v''' positive integers, there is a forest with partite sets of sizes v'' and v''' and e' edges.

Case 1: $y < m$.

If $y = 0$ then $f < \left\lfloor \frac{m}{2} \right\rfloor$. In this case (m, f) is bipartite representable as a forest. So, assume that $y > 0$. We shall show that (m, f) is bipartite representable as a vertex disjoint union of $K_{x,y}$ and a forest. Let $e' = f - xy$, $v' = 2m - x - y$. We have that $e' \leq x - 1 = \left\lfloor \frac{m}{2} \right\rfloor - 1$. On the other hand, using the upper bound on y, we have that $v' \geq 2m - \left\lfloor \frac{m}{2} \right\rfloor - \left(\left\lfloor \frac{m^2}{2} \right\rfloor / \left\lfloor \frac{m}{2} \right\rfloor \right)$. Considering the cases when m is even or odd, one can immediately verify that $e' < v'$. Since $x + y + v' = 2m$ and $xy + e' = f$, we have that (m, f) is bipartite representable as a vertex-disjoint union of $K_{x,y}$ and a forest on v' vertices and e' edges. Note that in this case we needed $y < m$ so that $K_{x,y}$ doesn't span one of the parts completely.

Case 2: $y = m$.

In particular, we have that $f \geq \left\lfloor \frac{m}{2} \right\rfloor m$. If m is even, we have that $f \geq m^2/2$ and from our original upper bound $f \leq m^2/2$ it follows that $f = m^2/2$. Thus (m, f) is bipartite representable as $K_{m/2,m}$ and isolated vertices. If m is odd, let $m = 2k + 1$, $k \geq 1$. Then $f \leq \left\lfloor \frac{m^2}{2} \right\rfloor = 2k^2 + 2k$ and $f \geq y \left\lfloor \frac{m}{2} \right\rfloor = 2k^2 + k$. Consider $K_{k+1,2k-1}$ and let $e' = f - (k + 1)(2k - 1)$ and $v' = 2m - 3k$. Then $e' \leq 2k^2 + 2k - (2k^2 + k - 1) = k + 1$ and

$v' = 4k + 2 - 3k = k + 2$. Thus $v' > e'$. Therefore (m, f) is bipartite representable as a vertex disjoint union of $K_{k+1,2k-1}$ and a forest on v' vertices and e' edges.

Case 3:. $y = m + 1$.

This case could happen only if m is odd. Let $m = 2k + 1$. Then we have $x = k$ and $y = 2k + 2$ and $f = 2k^2 + 2k$. We see that (m, f) is bipartite representable by $K_{2k,k+1}$ and isolated vertices. □

5. Conclusion

We showed that there are infinite sets of absolutely avoidable pairs (m, f). One could further extend our results and provide more absolutely avoidable pairs.

A statement analogous to Theorem 4 statement holds for $m \equiv 2, 3 \pmod 4$, i.e. for any $m \geq m_0$ either $(m, \lfloor \binom{m}{2}/2 \rfloor)$ or $(m, \lfloor \binom{m}{2}/2 \rfloor - 6m)$ is absolutely avoidable. We omit the proof here but it can be obtained by a very similar method by slightly changing the constants in the calculations. The arguments in the proof of Theorem 4 should still hold if we deviate from $f_0 = \binom{m}{2}/2$ by a small term, as in Theorem 3. The reason here is that this change does not affect the limit computations for $d_q(m)$ and $y_q(m)$. Thus, for each large enough m, one should be able to obtain a small interval for f so that each (m, f) is absolutely avoidable. We cannot hope to do much better though: In infinitely many cases, if (m, f_0) is absolutely avoidable, then already for $(m, f_0 - m)$ or $(m, f_0 + m)$ our method does not give a contradiction. The constant 6 is the smallest integer for which the argument in the proof of Theorem 4 works (since $\{6\sqrt{2}\}$ is *close* to $\frac{1}{2}$ while $\{c\sqrt{2}\}$, $c \in [5]$ is not). We believe that one could show by an argument very similar to that used in the proof, that for sufficiently large m, for any constants a, b which satisfy that $\{a\sqrt{2} - b\sqrt{2}\}$ is *close enough* to $\frac{1}{2}$, we have that either $(m, f_0 - am)$ or $(m, f_0 - bm)$ is absolutely avoidable.

Recently, a similar question on avoidable order-size pairs was considered by Caro, Lauri, and Zarb [7] in the class of line graphs.

As mentioned in Section 4, the bipartite setting leaves the following:

Open Question: Are there any absolutely avoidable pairs (m, f) in the bipartite setting?

Acknowledgements

The authors thank Alex Riasanovsky for his careful reading of the manuscript and his suggestions. The authors also thank the referees for their careful reading of the manuscript.

References

[1] Noga Alon, József Balogh, Alexandr Kostochka, and Wojciech Samotij. Sizes of induced subgraphs of ramsey graphs. *Combinatorics, Probability and Computing*, 18(4):459–476, 2009. MR2507733

[2] Noga Alon and Alexandr V Kostochka. Induced subgraphs with distinct sizes. *Random Structures & Algorithms*, 34(1):45–53, 2009. MR2478538

[3] Noga Alon, Michael Krivelevich, and Benny Sudakov. Induced subgraphs of prescribed size. *Journal of Graph Theory*, 43(4):239–251, 2003. MR1994296

[4] Maria Axenovich and József Balogh. Graphs having small number of sizes on induced k-subgraphs. *SIAM Journal on Discrete Mathematics*, 21(1):264–272, 2007. MR2299710

[5] Béla Bollobás. *Extremal graph theory*. Courier Corporation, 2004. MR2078877

[6] Boris Bukh and Benny Sudakov. Induced subgraphs of ramsey graphs with many distinct degrees. *Journal of Combinatorial Theory, Series B*, 97(4):612–619, 2007. MR2325800

[7] Yair Caro, Josef Lauri, and Christina Zarb. The feasability problem for line graphs. *arXiv preprint arXiv:2107.13806*, 2021.

[8] Paul Erdős, Zoltán Füredi, Bruce Rothschild, and Vera Sós. Induced subgraphs of given sizes. *Discrete Mathematics*, 200(1-3):61–77, 1999. MR1692280

[9] Jialin He, Jie Ma, and Lilu Zhao. Improvements on induced subgraphs of given sizes. *arXiv preprint arXiv:2101.03898*, 2021.

[10] Lauwerens Kuipers and Harald Niederreiter. *Uniform distribution of sequences*. Courier Corporation, 2012. MR0419394

[11] Matthew Kwan and Benny Sudakov. Proof of a conjecture on induced subgraphs of ramsey graphs. *Transactions of the American Mathematical Society*, 372(8):5571–5594, 2019. MR4014288

[12] Matthew Kwan and Benny Sudakov. Ramsey graphs induce subgraphs of quadratically many sizes. *International Mathematics Research Notices*, 2020(6):1621–1638, 2020. MR4089430

[13] Felix Lazebnik, Vasiliy Ustimenko, and Andrew Woldar. A new series of dense graphs of high girth. *Bulletin of the American Mathematical Society*, 32(1):73–79, 1995. MR1284775

[14] Bhargav Narayanan, Julian Sahasrabudhe, and István Tomon. Ramsey graphs induce subgraphs of many different sizes. *Combinatorica*, 39(1):215–237, 2019. MR3936198

[15] Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok (in Hungarian)*, 48:436–452, 1941. MR0018405

Maria Axenovich
Karlsruhe Institute of Technology
Germany
E-mail address: maria.aksenovich@kit.edu

Lea Weber
Karlsruhe Institute of Technology
Germany
E-mail address: lea.weber@kit.edu

Received August 4, 2021