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The biker-hiker problem

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There are n travellers who have k bicycles and they wish to complete a journey in the shortest possible time. We investigate optimal solutions of this problem where each traveller cycles for $\frac{k}{n}$ of the journey. Each solution is represented by an $n \times n$ binary matrix M with k non-zero entries in each row and column. We determine when such a matrix gives an optimal solution. This yields an algorithm deciding the question of optimality of complexity $O(n^2 \log n)$. We introduce three symmetries of matrices that preserve optimality, allowing identification of minimal non-optimal members of this class. An adjustment to optimal solutions that eliminates unnecessary handovers of cycles is established, which maintains all other features of the solution. We identify two mutually transpose solution types, the first uniquely minimises the number of handovers, while the second keeps the number of separate cohorts to three while bounding their overall separation, in the case $2k \leq n$, to under $\frac{2}{n}$ of the journey.

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1. The problem: not enough bicycles

There are n friends who have k bicycles between them and the group needs to reach its destination as soon as possible. How should they go about doing this? An early allusion to this problem is in the novel *The Great House* by Cynthia Harnett [2]. Here a pair of 17th century travelling companions with only one horse between them adopt the 'ride and tie' method for their journey from Henley-on-Thames to London.

Assumptions. *Every person walks and cycles at the same speed as all the others, and cycling is faster than walking. We assume that the time required to swap from one form of locomotion to the other is negligible. For brevity, individual travellers will sometimes be referred to as 'he' while a set of travellers will be referred to as 'they'.*

Solution. Suppose that we devise a scheme, we shall call it an *optimal* scheme, in which each traveller cycles for $\frac{k}{n}$ of the length of the journey and never stops moving forward at any stage. Each will then have cycled and walked the same distance as each of their companions and so all n friends will arrive at their destination simultaneously. We claim that, if it exists, such a scheme is truly optimal in that it delivers the entire group to its destination in the least possible time, and that any non-optimal scheme is inferior in this respect.

It is convenient to consider the length of the journey to be n units, (although we will consider divisors other than n). To see that a optimal scheme is best, note that the maximum (net) forward progress by bicycle of any scheme is kn. It follows that if one member of the group of n travellers cycled more than k units, then some other member must cycle less than k units. This latter traveller would then take longer than others who have cycled k units (or more). Hence, any approach that involved any member cycling forward a total distance other than k units would take longer to deliver the entire group to their destination as opposed to an approach that adopted an optimal scheme.

That cycling is faster than walking makes the problem more interesting, a fact that is highlighted by considering the *Backpack-hiker* problem. Here there are k heavy backpacks to be transported to the finish and any traveller carrying a backpack walks more slowly than one that is unencumbered. The change in relative speeds makes this problem much simpler and less interesting as for *any* value of k $(1 \leq k \leq n)$ the minimum time for the group to complete the journey is the length of the journey divided by the speed of a backpack walker.

In Section 2, we list the properties of optimal schemes more formally through a discretised representation of the Biker-hiker problem based on square binary matrices. In Section 3, we characterise those matrices that correspond to optimal solutions and show that we may decide the question of optimality for a given matrix with an algorithm that involves $O(n^2 \log n)$ comparisons of partial sums of the rows of the matrix. We identify three symmetries of these optimal schemes, which leads to the discovery of minimal schemes that assign k cycled stages to each traveller and k cyclists to each stage but are nonetheless not optimal. In Sections 4 and 5, we identify and investigate a certain mutually transpose pair of optimal matrices for arbitrary values of the parameters n and k. Section 6 looks at certain facets of these special schemes.

2. k-uniform solutions

It will be convenient to allocate a measure of n units for the total length of the road the travellers will take, which we may take to be either linear or a circuit. Along the length of the journey we imagine there to be $n+1$ equally spaced *staging posts* $P_0, P_1, \cdots, P_n$, with P_0 and P_n marking the beginning and end of the trip respectively, so that the distance between successive signposts is 1 unit. We assign numbered symbols to each of the n *travellers* as we shall call them, $t_1, t_1, \cdots, t_n$.

Definition 2.1. (a) The problem of delivering the n travellers equipped with k bicycles $(0 \leq k \leq n)$ to their common destination in a way that minimizes the time of the last arrival will be called the (n,k)-*problem.*

(b) The leg of the journey from P_{j-1} to P_j is called *stage* j and is denoted by s_j $(1 \leq j \leq n)$.

(c) An *n-scheme* S is one in which each traveller t_i is directed to travel each stage s_j $(1 \leq j \leq n)$ either on foot, or by bicycle.

(d) The *incidence matrix* $M = M(S)$ of an n-scheme S is the $n \times n$ binary matrix $M = (m_{i,j})$ $(1 \leq i, j \leq n)$ where $m_{i,j} = 0$ or $m_{i,j} = 1$ according as traveller t_i is directed to walk or cycle respectively stage s_j from P_{j-1} to P_j. We shall write R_i and C_j for the ith row and jth column of M respectively.

(e) The *scheme* $S = S(M)$ of an $n \times n$ binary matrix $M = (m_{i,j})$ is that in which traveller t_i travels s_j on foot or by bicycle according as $m_{i,j} = 0$ or $m_{i,j} = 1$ $(1 \leq i, j \leq n)$. Note that $S(M(S)) = S$ and $M(S(M)) = M$.

Definition 2.2. An $n \times n$ binary matrix $M = (m_{i,j})$ is *k-uniform* if each row and each column contains exactly k entries equal to 1.

Proposition 2.3. *A scheme S is optimal if and only if*

(i) $M(S)$ is k-uniform and

(ii) whenever a set of travellers C arrives at a post P_j, the number of cycles at P_j is at least as great as the number of $t_i \in C$ such that $m_{i,j+1} = 1$.

Proof. If S is optimal then each traveller t_i rides exactly k stages so that R_i has exactly k entries which equal 1. There are then nk entries of M equal to 1. If it were not the case that each column had exactly k non-zero entries, then some column would contain more than k $1's$, which is impossible as no cycle may travel twice through the same stage. Therefore, M is k-uniform. As for Condition (ii), if it were violated then some traveller would have to stop at some stage to wait for a bicycle to arrive for their use. The time taken for their journey would then exceed the optimal time unless he cycled more than k stages, in which case some other traveller would cycle fewer than k stages, and the overall time for the group to complete the journey would exceed the optimal time. Hence, if S is optimal, both Conditions (i) and (ii) must be met.

Conversely, any scheme S represented by a k-uniform matrix M has exactly k entries of 1 in each row so that each traveller is scheduled to ride k stages. Condition (ii) ensures that the progress of each traveller is never stalled by a required cycle being unavailable upon arrival at a staging post. Therefore, S represents an optimal solution. □

Definition 2.4. We call a square k-uniform binary matrix M *optimal* if $S(M)$ is optimal.

3. Optimal matrices and their symmetries

Assignment mappings

We will now introduce *assignment mappings* ϕ_j for the each stage s_j $(1 \leq j \leq n-1)$ of a scheme S. Suppose $m_{i,j} = 1$, meaning that t_i cycles s_j. Then $\phi_j(i) = p$ conveys the information that t_p will cycle s_{j+1} on the cycle left behind at P_j by t_i.

Definition 3.1. Let S denote an $n \times n$ scheme with matrix $M = M(S) = (m_{i,j})$. A one-to-one partial mapping ϕ_j $(1 \leq j \leq n-1)$ is an *assignment mapping* for S if

$$\operatorname{dom} \phi_j = \{i : m_{i,j} = 1\}, \ \operatorname{ran} \phi = \{i : m_{i,j+1} = 1\}.$$

The main result of this section characterises optimal schemes in terms of the existence of a collection of assignment mappings that satisfy two constraints. The first is the optional constraint that allows a rider to stay on the same bike if he is required to ride two successive stages. The second constraint ensures that S is in accord with Proposition 2.3.

Theorem 3.2. *Let M be an $n \times n$ binary matrix. Then $S(M)$ is optimal if and only if M is k-uniform for some k $(0 \leq k \leq n)$ and for each j, $(1 \leq j \leq n-1)$ there exist assignment mappings ϕ_j such that*

$$\phi_j(i) = i \Leftrightarrow (m_{i,j} = m_{i,j+1} = 1) \text{ and} \tag{1}$$

(2) $$\sum_{l=1}^{j} m_{i',l} \leq \sum_{l=1}^{j} m_{i,l}, \text{ where } i' \text{ denotes } \phi_j(i).$$

Proof. Suppose that M is k-uniform and satisfies Conditions (1) and (2). Suppose inductively that the scheme $S(M)$ has not failed up to stage s_j, which holds when $j = 1$ as C_1 has k entries that equal 1, and so travellers assigned to cycle s_1 may do so.

Next consider stage s_{j+1} from P_j to P_{j+1}. For each i' such that $m_{i',j+1} = 1$ there exists a unique i such that $m_{i,j} = 1$ and $\phi_j(i) = i'$. By the inductive hypothesis, t_i has arrived at P_j by cycle without stalling. Condition (2) is then exactly the requirement that ensures that this has occurred no later than the arrival of $t_{i'}$ at P_j. Hence, S may continue with $t_{i'}$ riding s_{j+1} on the cycle that t_i has ridden on s_j. Therefore, s_{j+1} may be completed without stalling, and the induction continues. The process will therefore end with $S(M)$ being fully executed without stalling, and so $S(M)$ is indeed optimal.

Conversely, suppose that $S(M)$ is optimal. Then at stage s_{j+1} ($j \geq 0$), for each i' such that $m_{i',j+1} = 1$, it is possible for $t_{i'}$ to ride s_{j+1} on a cycle that has been left at P_j by some traveller t_i. It follows that Condition (2) is then met. This correspondence defines a partial one-to-one mapping:

$$\phi_j^{-1} : \{i : m_{i,j+1} = 1\} \to \{i : m_{i,j} = 1\}.$$

By uniformity, ϕ_j^{-1} is also surjective and so the partial one-to-one mapping ϕ_j is an assignment mapping which satisfies Condition (2). We now show that ϕ_j may be modified so that it also satisfies Condition (1). The forward direction of the implication in (1) follows from the definition of an assignment map, but the reverse implication does not follow from the optimality of $S(M)$.

Let us write ϕ for ϕ_j and, as before, abbreviate $\phi_j(i)$ to i'. Suppose then that $m_{i,j} = m_{i,j+1} = 1$ but $i \neq i'$. We consider the sequence $I = i, \phi(i), \phi^2(i), \cdots$. If I is a cycle, so that for some positive integer p, $\phi^p(i) = i$, then it follows that $m_{t,j} = 1 = m_{\phi(t),j}$ for all $t = \phi^k(i)\,(k \geq 0)$. In this case we may modify ϕ (while retaining the same symbol ϕ for the mapping) such that $\phi(t) = t$ for all $t = \phi^k(i)$, in accord with Condition (1). Moreover, applying Condition (2) repeatedly yields a cycle of inequalities that begins and ends with the same sum, and so are in fact equalities, indicating that all the travellers $t_i, t_{\phi(i)}, \cdots, t_{\phi^p(i)} = t_i$ arrive at P_j simultaneously. The original assignment mapping ϕ instructed this set of travellers to exchange bicycles in accord with the cycle I. The modified mapping simply allows each traveller to remain on the bike he is currently riding.

Alternatively, the sequence I does not generate a cycle. Then by definition of ϕ there exists a sequence of maximal length:

$$i_{-r}, i_{-r+1}, \cdots i_0 = i, i_1 = \phi(i), i_2, \cdots, i_{s-1}, i_s$$

$$\text{such that } \phi(i_p) = i_{p+1}, \ (-r \leq p \leq s-1), (r, s \geq 1). \tag{3}$$

In (3), $m_{i_{-r},j+1} = 0 = m_{i_s,j}$ and $m_{t,j} = m_{t,j+1} = 1$ for all $-r+1 \leq t \leq s-1$. We now modify ϕ by putting

$$\phi(t) = t \ \forall -r+1 \leq t \leq s-1 \tag{4}$$

$$\phi(i_{-r}) = i_s, \tag{5}$$

for then Condition (2) holds trivially for $i = t$ as in (4), and (2) also holds for (5) for $i = i_{-r}$, $i' = i_s$ as applying Condition (2) repeatedly for ϕ we have:

$$\sum_{l=1}^{j} m_{i_{-r},l} \geq \sum_{l=1}^{j} m_{i_{-r+1},l} \geq \cdots \geq \sum_{l=1}^{j} m_{i_{s-1},l} \geq \sum_{l=1}^{j} m_{i_s,l},$$

which, in the notation of Theorem 3.2, provides the required inequality concerning i_{-r} and $i_s = \phi_j(i_{-r}) = i'_{-r}$:

$$\sum_{l=1}^{j} m_{i'_{-r},j} \leq \sum_{l=1}^{j} m_{i_{-r},l}.$$

We modify ϕ for each such i, which is possible as the sequences as in (3) that arise are pairwise disjoint as ϕ is one-to-one, i_r is not in the range of ϕ, and i_s is not in the domain of ϕ. Modifying ϕ as necessary for each i such that $m_{i,j} = m_{i,j+1} = 1$ ensures that the partial one-to-one mapping ϕ satisfies both Conditions (1) and (2), thereby completing the proof. □

Definitions 3.3. Let M be a k-uniform matrix.

(i) For any j $(1 \leq j \leq n-1)$ we shall call an assignment mapping ϕ_j *optimal* if ϕ_j satisfies Conditions (1) and (2) of Theorem 3.2.
(ii) For any j $(1 \leq j \leq n-1)$ consider the partition of $X_n = \{1, 2, \cdots, n\}$ induced by M into the following four (possibly empty) disjoint subsets:

(6) $X_{1,1} = \{i : m_{i,j} = m_{i,j+1} = 1\}$, $X_{1,0} = \{i : m_{i,j} = 1, m_{i,j+1} = 0\}$,

$$X_{0,1} = \{i : m_{i,j} = 0, m_{i,j+1} = 1\},\ X_{0,0} = \{i : m_{i,j} = m_{i,j+1} = 0\}.$$

When necessary, we write $X^j_{1,0}$ etc. to indicate that the set refers to column C_j.
An assignment mapping ϕ_j then satisfies the conditions that:

(7) $$\operatorname{dom}\phi_j = X_{1,1} \cup X_{1,0},\ \operatorname{ran}\phi_j = X_{1,1} \cup X_{0,1}$$

with ϕ_j acting identically on $X_{1,1}$ if ϕ_j is optimal.

(iii) We shall denote the ith row sum up to column C_j by $S_{i,j}$:

(8) $$S_{i,j} = \sum_{l=1}^{j} m_{i,l}\ (1 \le i, j \le n).$$

Suppress the second subscript j by writing S_i for $S_{i,j}$, and form ordered sets, written in ascending order as:

(9)
$$\overline{X}_{1,0} = \{(i_1, S_{i_1}), \cdots, (i_p, S_{i_p}),\ S_{i_1} \le \cdots \le S_{i_p}, i_t \in X_{1,0},\ (1 \le t \le p)\}.$$

(10)
$$\overline{X}_{0,1} = \{(j_1, S_{j_1}), \cdots, (j_p, S_{j_p}),\ S_{j_1} \le \cdots \le S_{j_p}, j_t \in X_{0,1},\ (1 \le t \le p)\}.$$

To make each order unique, in the case of ties, we order by subscript value, so if $S_{i_1} = S_{i_2}$ then $(i_1, S_{i_1}) < (i_2, S_{i_2})$ for $\overline{X}_{1,0}$ if $i_1 < i_2$, and similarly for $\overline{X}_{0,1}$. We now meld these two lists to define a total order on $Y = \overline{X}_{1,0} \cup \overline{X}_{0,1}$. The order $(Y, \le)$ is equal to the order defined in (9) and (10) when restricted to $\overline{X}_{1,0}$ and to $\overline{X}_{0,1}$ respectively. For $(i, S_i) \in \overline{X}_{1,0}$ and $(j, S_j) \in \overline{X}_{0,1}$ we define $(i, S_i) < (j, S_j)$ if $|S_i| \le |S_j|$ and $(i, S_i) > (j, S_j)$ if $|S_i| > |S_j|$. In this way $\le$ is indeed a linear order on Y as transitivity is readily checked by cases.

Definition 3.4. (i) The reverse order, $(Y, \ge)$ of the linear order $(Y, \le)$ is the *canonical order* of Y.
Let $A = \{a, b\}$ be a two-letter alphabet.

(ii) The *canonical word* $w = a_1 a_2 \cdots a_{2p} \in A^{2p}$ $(a_r \in A, 1 \le p \le k)$ is defined by $a_r = a$ or $a_r = b$ according as the rth entry in the canonical order belongs to $\overline{X}_{1,0}$ or to $\overline{X}_{0,1}$.

(iii) For any word $w \in A^m (m \geq 0)$ we write $|w|_c$ for the number of instances of $c \in A$ in w. The *length* of w, denoted by $|w|$, is then $|w| = |w|_a + |w|_b$.
(iv) If $w \in A^m$ $(m \geq 0)$ has a factorization $w = uv$, we call u a *prefix* and v a *suffix* of w.
(v) A word $w \in A^{2m}$ $(m \geq 0)$ such that $|w|_a = |w|_b$ is called a *Dyck word* if for every prefix u of w, $|u|_a \geq |u|_b$.
(vi) For $w \in A^m (m \geq 0)$, the *dual reverse word* $\overline{w}$ is formed by taking the reverse word w^R of w and interchanging all instances of the letters a and b.

Remark 3.5. The set of all words of any length that satisfy the conditions of (v) is called the *Dyck language*. This is the language of *well-formed parentheses* in that replacing a and b by the left and right brackets '(' and ')' respectively, a Dyck word corresponds to a string of brackets that represents a meaningful bracketing of some binary operation. For further information, see [3].

Proposition 3.6. *(i) The dual reverse word $\overline{w}$ of a Dyck word w is also a Dyck word.*
(ii) There exists an optimal assignment mapping ϕ_j $(1 \leq j \leq n-1)$ if and only if the canonical word $w = w_j$ is a Dyck word.

Proof. (i) Let $\overline{w} = uv$, whence $w = \overline{\overline{w}} = \overline{v}\,\overline{u}$. Since w is a Dyck word, $|\overline{v}|_a \geq |\overline{v}|_b$, whence $|\overline{u}|_a \leq |\overline{u}|_b$, and so $|u|_a \geq |u|_b$. Hence, $\overline{w}$ is a Dyck word.
(ii) Suppose that $\phi = \phi_j$ is an optimal assignment mapping. The action of this mapping induces a bijection from letters $a_s = a$ in the canonical word w to letters $a_t = b$ in w, which acts, by Condition (2) of Theorem 3.2, so that a_s lies to the left of a_t in w. It follows that for any initial prefix u of $w = uv$, we must have $|u|_a \geq |u|_b$, for if $|u|_a < |u|_b$, there would be some instance of b in u that was not in the range of the induced mapping, contradicting that ϕ_j is one-to-one. Hence, w is a Dyck word.

Conversely, given that w is a Dyck word, we map $i \in X_{1,0}$ to $i' \in X_{0,1}$ whereby if i corresponds to the rth instance of a in w, then i' corresponds to the rth position of b in w. By the given condition, the rth a in w lies to the left of the rth b in w, whence $|S_i| \geq |S_{i'}|$. The map ϕ thereby defined satisfies Condition (2) of Theorem 3.2. Extending ϕ to act identically on $X_{1,1}$ then produces a required optimal assignment map. □

Theorem 3.7. *Algorithm to decide optimality of a k-uniform matrix M.*

For the columns C_j $(1 \le j \le n-1)$ of M:

1. Calculate the partial sums $S_{i,j}$ $(i \in X_{1,0} \cup X_{0,1})$;
2. Rank the $2p$ $(0 \le p \le k)$ partial sums from Step 1 in descending order, with members of $X_{1,0}$ taking precedence over members of $X_{0,1}$ in the case of a tie, as per Definition 3.3(iii).
3. Form the canonical word $w = w_j = a_1 \cdots a_{2p}$ where $a_r = a$ or b according as the rth member of this ranking lies in $X_{1,0}$ or $X_{0,1}$.
4. M is optimal if and only if w_j is a Dyck word for all $1 \le j \le n-1$.

However, it is not necessary to check the first two nor the last two assignment mappings for optimality by virtue of part (ii) of our next result.

Lemma 3.8. *(i) For a given j $(1 \le j \le n-1)$, all assignment mappings ϕ_j are optimal if and only if the canonical word $w_j = a^p b^p$, $(p = |X_{1,0}|)$.*

(ii) An assignment mapping ϕ_j is optimal if $j \in \{1, 2, n-2, n-1\}$ or if $k \in \{1, 2, n-2, n-1\}$.

Proof. (i) Every ϕ_j is optimal if and only if $S_{i_1,j} \ge S_{i_2,j}$ for all $i_1 \in X_{1,0}$ and $i_2 \in X_{0,1}$, which in turn is equivalent to $w_j = a^p b^p$, where $p = |X_{1,0}|$.

(ii) Let $i_1 \in X_{1,0}$ and $i_2 \in X_{0,1}$. For ϕ_1 and ϕ_2 we have $S_{i_1,j} \ge 1$ and $S_{i_2,j} \le 1$ $(j = 1, 2)$ whence it follows that $w_j = a^p b^p$. For ϕ_{n-2} or ϕ_{n-1} we have $S_{i_1,j} \ge k-1$ while $S_{i_2,j} \le k-1$, $(j = n-2, n-1)$ and again $w_j = a^p b^p$. The claim now follows from part (i).

Similarly, if $k \le 2$ then $S_{i_1,j} \ge 1$ and $S_{i_2,j} \le 1$, while if $k \ge n-2$ then $S_{i_1,j} \ge j-1$ and $S_{i_2,j} \le j-1$ and again the result follows. □

Corollary 3.9. *(i) An $n \times n$ uniform matrix M is optimal if $n \le 5$.*

(ii) For any non-optimal k-uniform matrix M, $3 \le k \le n-3$.

(iii) Optimality of a k-uniform matrix M is preserved under the exchange of columns C_1 and C_2, and under the exchange of columns C_{n-1} and C_n.

Proof. (i) For $n \le 5$, for any scheme there are at most $5-1 = 4$ assignment mappings which are among the four mappings listed in Lemma 3.8(ii), and so all are optimal.

(ii) This follows from Lemma 3.8(ii).

(iii) Indeed, we may replace C_1 and C_2 by any pair of binary columns that retains k-uniformity of M, for then the transformed matrix retains its status with respect to optimality by Lemma 3.8(ii). These correspond to exchanging adjacent instances of 0 and 1 in the two columns in opposite pairs. In particular, since complete exchange of C_1 and

C_2 retains k-uniformity, the result follows, as it does likewise for the exchange of the final column pair. □

Proposition 3.10. *Let $S = S(M)$ be an (n, k)-uniform scheme with a given set of assignment mappings ϕ_j $(1 \leq j \leq n-1)$. If all travellers complete c cycled stages of S without the scheme failing, (that is, without any traveller being stalled) then the scheme, with this set of assignment mappings, will not fail before some traveller is due to ride their $(c+3)$rd cycled stage.*

In particular, S will not fail prior to some traveller being due to ride their 3rd stage, and if all travellers complete $k-2$ stages without S failing, then S is an optimal scheme, which is realised by the given set of assignment mappings.

Proof. Suppose all travellers have completed c cycled stages without failure in S. Suppose a walking traveller $t_{i'}$ arrives at a staging post P_j $(1 \leq j \leq n-1)$, where s_j represents cycle stage number $c+1$ or $c+2$ for that traveller. Let $i = \phi_j^{-1}(i')$. Then $S_{i',j} = c$ in the first case, and $S_{i',j} = c+1$ in the second. If $t_{i'}$ stalls at P_j then it follows that $S_{i,j} \leq c$. However, since $m_{i,j} = 1$, it follows that t_i has not yet completed c cycled stages when the stall occurs, contrary to hypothesis. Therefore, if *all* travellers complete c cycle stages without the scheme failing, then the scheme will not fail prior to some traveller attempting to cycle a stage for the $(c+3)$rd occasion. The final statement simply draws attention to the special cases where $c = 0$, and where $c = k-2$. □

Examples 3.11. It follows from Corollary 3.9 that the smallest dimension n that might admit a non-optimal matrix M is $n = 6$. In this case, the inequality of Corollary 3.9(ii) becomes $3 \leq k \leq 6-3$, so that $k = 3$. Consider the simple scheme $S(M_1)$, where M_1 is given below. This scheme is clearly optimal: travellers t_1, t_2, t_3 ride the first three stages and then leave their bikes to be collected later by t_4, t_5, and t_6 who then ride together to the finish. The assignment mappings all act identically except for ϕ_3, which may be taken as any bijection such that $\phi_3(\{1,2,3\}) = \{4,5,6\}$. However, if we swap columns C_3 and C_4 in M_1, we have the array M_2. By Lemma 3.8, the only canonical word of M_2 that may fail to be a Dyck word is w_3. However, for $j = 3$ we have $X_{1,0} = \{4,5,6\}$ and $X_{0,1} = \{1,2,3\}$. For any $i_1 \in X_{1,0}$ and $i_2 \in X_{0,1}$ we have $S_{i_1,3} = 1 < 2 = S_{i_2,3}$ and so $w_3 = b^3a^3$, which is not a Dyck word. Therefore, M_2 is not optimal. Indeed, this example shows that the class of optimal matrices is not closed under permutation of columns, nor under the taking of transpositions.

$M_1 =$

P_0	P_1	P_2	P_3	P_4	P_5	P_6
t_1	1	1	1	0	0	0
t_2	1	1	1	0	0	0
t_3	1	1	1	0	0	0
t_4	0	0	0	1	1	1
t_5	0	0	0	1	1	1
t_6	0	0	0	1	1	1

$M_2 =$

P_0	P_1	P_2	P_3	P_4	P_5	P_6
t_1	1	1	0	1	0	0
t_2	1	1	0	1	0	0
t_3	1	1	0	1	0	0
t_4	0	0	1	0	1	1
t_5	0	0	1	0	1	1
t_6	0	0	1	0	1	1

Theorem 3.12. *The question of whether an $n \times n$ binary matrix M is optimal may be decided by an algorithm of complexity $O(n^2 \log n)$.*

Proof. 1. By inspecting rows and columns of M, decide whether M is uniform, an operation of order $O(n^2)$.
If M is uniform, we may decide optimality of M by carrying out the following procedure for each j with $1 \leq j \leq n-1$.

2. Compute $S_{i,j+1}$ from $S_{i,j}$ for all $1 \leq i \leq n-1$, which consists of n additions. This allows identification of the sets $X^j_{0,0}, X^j_{1,0}, X^j_{0,1}$ and $X^j_{1,1}$.
3. Form the two sets $\overline{X}^j_{1,0}$ and $\overline{X}^j_{0,1}$ and sort in descending order, a process which has time complexity $O(n \ln n)$, as this is the least possible for any comparison algorithm [1], from which may be read the canonical word, w_j.
4. At most $O(n)$ comparisons determine whether or not w_j is a Dyck word.

For each j, the total complexity of steps 2, 3, and 4 is $O(n)+O(n \ln n)+O(n) = O(n \ln n)$. These steps are carried out $n-1$ times, (strictly speaking, by Lemma 3.8(ii), at most $n-5$ applications are needed), which, including Step 1, yields an overall complexity of $O(n^2)+O(n^2 \log n) = O(n^2 \log n)$. □

Definition 3.13. Let $M = (m_{i,j})$ be an $n \times n$ k-uniform binary matrix. Let S_n denote the symmetric group on X_n. The $n \times n$ k-uniform matrices $M_\pi = (p_{i,j})$ $(\pi \in S_n), M_r = (r_{i,j})$, and $\overline{M} = (d_{i,j})$ are defined by:
(i) $p_{i,j} = m_{\pi(i),j}$, (ii) $r_{i,j} = m_{i,n-j+1}$, (iii) $d_{i,j} = (m_{i,j}+1) \pmod 2$.
We may denote $d_{i,j}$ by $\overline{m}_{i,j}$.

Theorem 3.14. *Suppose that $S(M)$ is an optimal scheme. Then so are the schemes (i) $S(M_\pi)$, (ii) $S(M_r)$, and (iii) $S(\overline{M})$.*

Lemma 3.15. *Let M be an $n \times n$ k-uniform matrix. Then*

(i) The jth canonical word of M_π $(\pi \in S_n)$ is w_j, the jth canonical word of M $(1 \leq j \leq n)$.

(ii) The jth canonical word of $\overline{M}$ is $\overline{w}_j$.
(iii) The jth canonical word of M_r is $\overline{w}_{n-j}$.

Proof. (i) The canonical words w_j $(0 \le j \le n-1)$ of M are defined by $(Y, \le)$ based on the partial orders as in (9) and (10). Replacing M by M_π, results in replacing each of the symbols i_t, j_s by $\pi^{-1}(i_t)$,$\pi^{-1}(j_t)$ in the sets (9) and (10). Since the value of w_j is independent of the naming of these symbols, each canonical word w_j is unaltered.

(ii) Write $\overline{S}_{i,j}$ for a typical partial sum of $\overline{M}$. Since for any matrix position (i,j), $\overline{S}_{i,j} = j - S_{i,j}$ the list of inequalities in (9) and (10), apart from tied sums, is reversed when passing from M to $\overline{M}$. Moreover, $i_1 \in X^j_{1,0}, i_2 \in X^j_{0,1}$ for M if and only if $i_1 \in X^j_{0,1}, i_2 \in X^j_{1,0}$ for $\overline{M}$. It follows from this pair of observations that the jth canonical word of $\overline{M}$ is $\overline{w}_j$, the dual reverse canonical word of w_j.

(iii) Denote the partial sums of M_r by $S^r_{i,j}$. Then $S^r_{i,j} + S_{i,n-j} = k$ $(1 \le j \le n$, taking $S_{i,0} = 0)$. Moreover, $i_1 \in X^j_{1,0}, i_2 \in X^j_{0,1}$ for M if and only if $i_1 \in X^{n-j}_{0,1}, i_2 \in X^{n-j}_{1,0}$ for M_r. Now

$$S^r_{i_1,j} \le S^r_{i_2,j} \Leftrightarrow k - S_{i_1,n-j} \le k - S_{i_2,n-j} \Leftrightarrow S_{i_2,n-j} \le S_{i_1,n-j}.$$

This pair of observations imply that the jth canonical word of M^r is $\overline{w}_{n-j}$. □

Proof of Theorem 3.14. Since M is optimal, by Theorem 3.7 all canonical words w_j of M are Dyck words. By Lemma 3.15, the corresponding canonical words of M_π, $\overline{M}$, and M^r are respectively w_j, $\overline{w}_j$, and $\overline{w}_{n-j}$. Since the reverse dual word of a Dyck word is a Dyck word (Proposition 3.6(i)) it follows, again by Theorem 3.7, that each of M_π, $\overline{M}$, and M^r is optimal. □

Definition 3.16. Define the *complementary assignment function* $\overline{\phi}_j$ of an assignment function ϕ_j by putting

(11) $$\operatorname{dom} \overline{\phi}_j = X_{0,0} \cup X_{0,1}, \ \operatorname{ran} \overline{\phi}_j = X_{00} \cup X_{1,0}$$

with $\overline{\phi}_j(i) = i$ if $i \in X_{0,0}$ and $\overline{\phi}_j(i) = \phi_j^{-1}(i)$ if $i \in X_{0,1}$.

Remark 3.17. We may prove Theorem 3.14 directly by identifying optimal assignment mappings ψ_j for the matrix of the transformed scheme in terms of given optimal assignment mappings ϕ_j of $M(S)$. In case (iii) for instance, put $\psi_j = \overline{\phi}_j$ $(1 \le j \le n-1)$, as per Definition 3.16. For $\overline{M}$ we have

$$\operatorname{dom} \psi_j = \{i : m_{i,j} = 0\} = \{i : d_{i,j} = 1\},$$

$$\operatorname{ran}\psi_j = \{i : m_{i,j+1} = 0\} = \{i : d_{i,j+1} = 1\},$$

whence it follows that the ψ_j qualify as assignment mappings for $S(\overline{M})$. Moreover, by definition, $\psi_j(i) = i$ if and only if $d_{i,j} = d_{i,j+1} = 1$, and so Condition (1) is satisfied. For $i \in X_{0,0}$ we have $\psi_j(i) = i$ and so in this case the inequality of Condition (2) becomes an equality, and is thus satisfied. Otherwise, $i \in X_{0,1}$. Then we have

$$\sum_{l=1}^{j} d_{\psi_j(i),l} = j - \sum_{l=1}^{j} m_{\psi_j(i),l} = j - \sum_{l=1}^{j} m_{\overline{\phi}_j(i),l} = j - \sum_{l=1}^{j} m_{\phi_j^{-1}(i),l}$$
$$\leq j - \sum_{l=1}^{j} m_{i,l} = \sum_{l=1}^{j} d_{i,l},$$

where the inequality comes from Condition (2) applied to the ϕ_j, thereby verifying Condition (2) for the ψ_j. For parts (i) and (ii) the corresponding assignment mappings are given respectively by $\psi_j = \pi^{-1}\phi_j\pi$, and $\psi_j = \phi_{n-j}^{-1}$, $(1 \leq j \leq n-1)$.

Removing unnecessary handovers from an optimal scheme

Optimal schemes may have unnecessary cycle handovers, which can be removed, resulting in a scheme that is still optimal and displays the same character as the original. Suppose that $S = S(M)$ is an optimal (n,k)-scheme and for some j we have $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ and $S_{i_1,j} = S_{i_2,j}$. Then t_{i_1} and t_{i_2} arrive at P_j simultaneously, the former by bike and the latter on foot, whereupon t_{i_2} takes one of the bikes parked at P_j and goes on to cycle s_{j+1}. However, one cycle handover could be avoided if the pair of travellers swapped labels at this point, with t_{i_1} taking on the mantle of t_{i_2} and vice-versa. In other words, t_{i_1} would complete the journey as instructed by the final part of R_{i_2} from $m_{i_2,j+1}$ onwards and similarly t_{i_2} would follow R_{i_1} from $m_{i_1,j+1}$ onwards, allowing t_{i_1} to remain on his bike for s_{j+1}.

This does not alter any column sums, and nor does it alter rows sums as the initial portions are equal: $S_{i_1,j} = S_{i_2,j}$, and hence so are the latter portions, as together they each sum to k. Applying this procedure repeatedly will lead to a more efficient scheme that will *appear* to be identical, meaning that if both schemes were to run simultaneously, at any given moment the set of positions of walking travellers and the set of positions of cycling travellers for the two schemes are identical. We shall call such a scheme *reduced*, with it being free of *excess handovers*. In summary, we have the following theorem.

Theorem 3.18. *Given any optimal scheme $S = S(M)$ for the (n,k)-problem we may construct an optimal scheme $S(M')$ that is free of unnecessary handovers by repetition of the rule that if for some j we have $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ and $S_{i_1,j} = S_{i_2,j}$ we replace R_{i_1} and R_{i_2} in M by*

$$R'_{i_1} = (m_{i_1,1}, \cdots, m_{i_1,j}, m_{i_2,j+1}, \cdots, m_{i_2,n}),$$

$$R'_{i_2} = (m_{i_2,1}, \cdots, m_{i_2,j}, m_{i_1,j+1}, \cdots, m_{i_1,n}). \tag{12}$$

Remark 3.19. Removal of unnecessary handovers yields a stronger form of Condition (2) of Theorem 3.2 in which all the associated inequalities for which $i' \neq i$ are strict, for *all* collections of optimal assignment mappings. However, this process does alter the scheme, whereas imposing Condition (1) merely chooses a special type of set of assignment maps for a given scheme.

Conversely, if $S(M)$ is optimal and *every* set of optimal assignment mappings yields strict inequalities in Condition (2), it follows that $S(M)$ has no unnecessary handover. However an optimal scheme may have *some* collection of assignment mappings for which the non-trivial inequalities in Condition (2) are all strict, yet the scheme still not be reduced. Such a collection of assignment mappings has the added feature that each traveller will find a parked cycle waiting for him whenever he is due to pick one up.

Simple comparison arguments like those in the proof of Theorem 3.14 give the following result.

Proposition 3.20. *For any optimal matrix M, the number $h = h(M)$ of excess handovers is the same for the optimal schemes M_π, M_r and $\overline{M}$.*

4. Solution to the biker-hiker problem

We now provide a particular solution type to the general Biker-hiker problem. Because of the cyclic nature of our solutions, it will be convenient in this section to label the travellers as $t_0, t_1, \cdots, t_{n-1}$ and the entries of an $n \times n$ matrix M as $m_{i,j}$ $(0 \leq i, j \leq n-1)$, and stages are labelled $s_0, s_1, \cdots, s_{n-1}$ also.

Definition 4.1 (The Cyclic Scheme)**.** We define the cyclic (n,k)-scheme $S = S_{n,k}$ with matrix $M(S) = M_{n,k}$ by assigning the cycling quota for t_i to consist of the k cyclically successive stages, which run from P_{ik} to $P_{(i+1)k}$, where arithmetic is conducted modulo n. The matrix $M_{n,k}$ of the $n \times n$ cyclic scheme will be called the *cyclic (n,k)-matrix.*

Since we are working modulo n, we identify P_0 and P_n, thereby making the journey a circuit. However, the following analysis holds whether the journey is linear or circular in nature.

Theorem 4.2. *The (n,k)-cyclic scheme $S_{n,k}$ is optimal.*

Proof. By construction, $M = M_{n,k}$ is row k-uniform. The entry $m_{i,j} = 1$ if and only if j belongs the cyclic sequence $ik, ik+1, \cdots, ik+k-1$ which is equivalent to the statement that $ki \pmod n$ lies in the cyclic interval $I_j = (j-k+1, j-k+2, \cdots, j)$. Therefore, the number of $1's$ in C_j is the number of solutions to the congruences $kx \equiv a \pmod n$, $a \in I_j$. Such a congruence has no solution if $d = \gcd(n,k)$ is not a divisor of a, otherwise there are d solutions. The number of a such that $d|a$ is the number of multiples of d in I_j, which is $\frac{k}{d}$, and so that there are exactly $d \cdot \frac{k}{d} = k$ non-zero entries in each column of M. (Indeed, every column of M represents the same cyclic sequence: see Prop. 4.9.)

To prove optimality of the matrix M of a cyclic scheme we note that at each dismount a rider has either been riding from P_0 or has just completed his kth cycled stage. In either instance, he is ahead of the traveller to whom he is assigned to transfer his bicycle. □

For $M = (m_{i,j})$, a square matrix, M_r, the matrix that results from reversing the rows of M is described by permuting the columns of M by $C_j \leftrightarrow C_{n-j-1}$. Similarly, we now define M_c by reversing the columns of M, which is effected by the row permutation whereby $R_i \leftrightarrow R_{n-i-1}$. Of course both these permutations are respectively involutions on the set of columns and the set of rows of M. Writing M_{rc} for $(M_r)_c$, and similarly defining M_{cr}, M_{r^2} and so on, we see that $M_{rc} = M_{cr} = (m_{n-1-i,n-1-j})$.

Proposition 4.3. *(i) For the cyclic (n,k)-matrix M,*

(i) $M_c = M_r$. (ii) $M_{cr} = M_{rc} = M$. (iii) $(M^T)_r = (M_r)^T$, $(M^T)_c = (M_c)^T$ (iv) $(M^T)_{rc} = M^T$.

Proof. We prove (i), from which (ii), (iii), and (iv) readily follow. For $M = (m_{i,j})$ we have $M_c = (c_{i,j})$ where $c_{i,j} = m_{n-1-i,j}$ and $M_r = (r_{i,j})$, where $r_{i,j} = m_{i,n-1-j}$. Then we have

$$c_{i,j}=1 \Leftrightarrow m_{n-1-i,j}=1 \Leftrightarrow j \equiv (n-1-i)k+a \pmod n \text{ for some } 0 \le a \le k-1$$

$$\Leftrightarrow j + ik + k \equiv a \pmod n \tag{13}$$

$$r_{i,j} = 1 \Leftrightarrow m_{i,n-j-1} = 1 \Leftrightarrow n-j-1 \equiv ik+b \pmod n \text{ for some } 0 \le b \le k-1$$

$$\Leftrightarrow j + ik + 1 \equiv -b \pmod{n}$$

$$\Leftrightarrow j + ik + k \equiv c \pmod{n}, \tag{14}$$

where $c = k - 1 - b$. Now

$$0 \leq b \leq k - 1 \Leftrightarrow -k + 1 \leq -b \leq 0 \Leftrightarrow 0 \leq c \leq k - 1.$$

We now note that the conditions of (13) and (14) are the same. It follows that $c_{i,j} = r_{i,j}$, allowing us to conclude that $M_c = M_r$. □

Theorem 4.4. *For the (n, k)-problem, $(1 \leq k \leq n - 1)$ on an n-circuit, the cyclic scheme matrix represents the unique solution, up to permutation of rows, in which each traveller mounts and dismounts a cycle only once.*

Proof. By construction $S(M_{n,k})$ instructs each traveller to mount and dismount a cycle exactly once on the circuit. On the other hand, a uniform scheme that has this property is the cyclic solution. To see this, take any traveller, label the traveller t_0 and label the post where t_0 mounts a cycle as P_0. Since t_0 has a single bike ride, he must pass posts that we may label, $P_1, P_2, \cdots$ until he alights at a post that we may label P_k, thereby completing his full quota. That bicycle is then picked up by another traveller, who we may label t_1, who rides between posts that we may label P_k to P_{2k} (subscripts modulo n). We continue this process with the traveller labelled t_i riding the k stages from P_{ik} to $P_{(i+1)k}$. But this is just the description of the cyclic solution of the (n, k)-problem. □

Remark 4.5. The feature of one ride per ciruit is preserved by any of the symmetries of Theorem 3.14. In the case of row reversal, the non-zero stages for t_i remain those between P_{ik} and $P_{(i+1)k}$ but are now ridden in reverse. Indeed, since by Proposition 4.3, $M_r = M_c$, we see that for the cyclic solution matrix M, M_r is a special case of permutation of the rows of M, and so M_r also represents an (n, k)-cyclic scheme. When we pass to the binary dual we find that $\overline{M}_{n,k} = (M_{n,n-k})_r$ and so by the previous observation it follows that $\overline{M}_{n,k}$ indeed represents a cylic solution to the $(n, n - k)$ problem. In detail, write $(M_{n,n-k})_r = (a_{i,j})$ whence $a_{i,j} = 1$ becomes

$$m_{i,n-1-j} = 1 \Leftrightarrow n-1-j \equiv i(n-k)+a \pmod{n} \text{ for some } 0 \leq a \leq n - k - 1$$

$$\Leftrightarrow j + 1 + a \equiv ik \pmod{n} \; 0 \leq a \leq n - k - 1. \tag{15}$$

For the left hand side we write $\overline{M}_{n,k} = (b_{i,j})$ and $M_{n,n-k} = (m_{i,j})$. Then $b_{i,j} = 1$ may be written as

$$m_{i,j} = 0 \Leftrightarrow j \equiv (i+1)k + b \pmod{n} \text{ for some } 0 \le b \le n-k-1$$

$$\Leftrightarrow j - (b+k) \equiv j + (n-b-k) \equiv ik \pmod{n}.$$

Now $0 \le n-1-b-k \le n-1-k$. Put $c = n-1-b-k$. Then

$$j + 1 + c \equiv ik \pmod{n} \quad 0 \le c \le n-k-1. \tag{16}$$

The agreement of (15) and (16) allow us to conclude that $\overline{M}_{n,k} = (M_{n,n-k})_r$ and so $\overline{M}_{n,k}$ represents a cyclic solution to the $(n, n-k)$-problem.

Proposition 4.6. *Consider the (n,k)-problem and let $d = gcd(n,k)$. Let R_i and C_j denote the ith row and jth column respectively of M, the matrix of the cyclic solution to the (n,k)-problem as defined in 4.1. Then*

(i) $R_i = R_j$ if and only if $i \equiv j \pmod{\frac{n}{d}}$;
(ii) $C_i = C_j$ if and only if $dq \le i, j \le d(q+1) - 1$ for some $q \in \{0, 1, \cdots, \frac{n}{d} - 1\}$.

Proof. (i) is trivially true if $k = 0$ or $k = n$. Otherwise, the cyclic intervals of entries that equal 1 in R_i and R_j respectively are defined by the corresponding cyclic lists of staging posts: $P_{ik}, P_{ik+1}, \cdots, P_{(i+1)k}$ and $P_{jk}, P_{jk+1}, \cdots, P_{(j+1)k}$. These lists are identical if and only if $ik \equiv jk \pmod{n} \Leftrightarrow i \equiv j \pmod{\frac{n}{d}}$.

(ii) We observe that the non-zero entries of each row R_i consist of two intervals: an initial interval I of R_i of length r say, and a terminal interval T of R_i of length $k - r$ $(0 \le r \le k)$. We may write $k = du$ and $n = dv$. Then for some $x \ge 0$ we have

$$ik \pmod{n} = dui - dvx = d(ui - xv).$$

If non-empty, the terminal interval T begins at P_{ik} and ends at P_n and so has length $|T|$ given by

$$|T| = n - ik (\text{mod } n) = d(v - ui + xv).$$

It follows that $d||T|$. The length $|I|$ of the initial interval is $|I| = k - |T| = du - |T|$, whence $d||I|$ also. In the case where both I and T are non-empty the (successive) zeros in R_i number $n-|I|-|T|$, which likewise is a multiple of d. Otherwise, there is an initial interval of zeros of length ik, which is a multiple of d, from which it follows that the terminal interval of zeros has length that is too a multiple of d. Therefore, within any row, counting left to right by columns, the entries from one multiple of d up to but not including the next, are equal, because each maximal list of identical entries begins at a multiple of d. Hence,

$$dq \leq i, j \leq d(q+1) - 1 \; (0 \leq q \leq \frac{n}{d} - 1) \Rightarrow C_i = C_j. \tag{17}$$

In order to prove the reverse implication, we introduce the following construction. By (17), the columns of M consist of $\frac{n}{d}$ blocks $A_1, A_2, \cdots, A_{\frac{n}{d}}$ of contiguous columns, with each A_i consisting of d identical columns. On the other hand M is partitioned into $\frac{n}{d}$ sets of d (non-contiguous) identical rows $B_1, B_2, \cdots, B_{\frac{n}{d}}$. We may permute the rows of M, giving a new optimal matrix M' in which the rows of M' are partitioned into $\frac{n}{d}$ blocks $B'_1, B'_2, \cdots, B'_{\frac{n}{d}}$ each consisting of d identical rows. The new column blocks, $A'_1, A'_2, \cdots, A'_{\frac{n}{d}}$ that result from this row permutation each consist of d columns, and the columns within each block remain identical. The pairwise intersections $A'_i \cap B'_j$ partition M' into $\frac{n^2}{d^2}$ square blocks, which are themselves $d \times d$ matrices. Each such block has identical columns and identical rows, whence it follows that all entries of any particular $A'_i \cap B'_j$ are identical. We can then form a *quotient matrix*, M'_d by identifying each of the $A'_i \cap B'_j$ with the common value (0 or 1) of all entries in that sub-matrix. Therefore, M'_d is the cyclic scheme for the $(\frac{n}{d} \times \frac{k}{d})$-problem in which the travellers and the bicycles are grouped into sets of order d, which move together as a block throughout the scheme.

If now the reverse implication in (17) were false, it would imply that there were two identical columns in the quotient matrix M'_d. It is possible to prove directly by analysing the cardinality of the intersection of sets of cylic intervals that in the case where n and d are coprime, no two columns are identical, which, since $(\frac{n}{d}, \frac{k}{d})$ is a pair of coprime integers, applies to $M_{d'}$. However, the desired result follows at once from the next proposition which shows that in the case of coprimality the deteminant of M corresponds to the number of bicycles. □

Proposition 4.7. *If n and k are coprime then $|det(M_{n,k})| = k$. Otherwise, $M_{n,k}$ is singular.*

Proof. Let $d = \gcd(k, n)$. If $d \geq 2$ then by Proposition 4.6(i), $M_{n,k}$ has a pair of identical rows and so $\det(M) = 0$. For $d = 1$, however, the rows are cyclically identical and no two are equal. It follows that the set of rows consists of all n different possibilities that arise from the cyclic sequence $(1, 1, \cdots, 1, 0, \cdots, 0)$, where the initial sequence of 1's has length k. By permuting the rows of $M_{n,k}$ we may obtain the circulant matrix $C_{n,k}$, where $R_i(C_{n,k})$ has for its non-zero entries $m_{i,i}, m_{i,i+1}, \cdots, m_{i,i+k-1}$, (addition modulo n). Hence, $\det(M_{n.k}) = \pm \det(C_{n,k})$. We may therefore complete the proof by showing that $\det(C_{n,k}) = k$.

By a standard result on circulant matrices (see, for example, [4]), with ω denoting any primitive nth root of unity:

$$\det(C_{n,k}) = \Pi_{i=0}^{n-1}(1 + \omega^i + \omega^{2i} + \cdots + \omega^{(k-1)i}). \tag{18}$$

For $i = 0$, the bracketed term is equal to k. It remains to show that the product of the other terms in (18) is equal to 1. By summing each of the geometric series we see that this claim is equivalent to the equation:

$$\Pi_{i=1}^{n-1}(\omega^{ki} - 1) = \Pi_{i=1}^{n-1}(\omega^i - 1). \tag{19}$$

However, since k and n are coprime, ω^k is also a primitive nth root of unity, and so it follows that the products in (19) are identical up to the order of their factors, thereby completing the proof. In particular, no two columns of $M_{n,k}$ are identical, thereby also completing the proof of Proposition 4.6. □

Remark 4.8. Note from the previous proof that for $\gcd(n, k) = 1$, $S(C_{n,k})$ is also the cyclic (n, k)-scheme. Moreover, the non-zero entries of $R_i(C_{n,k}^T)$ are $m_{i,i}, m_{i,i-1}, \cdots, m_{i,i-k+1}$. Hence, the non-zero entries of $R_{i+k-1}(C_{n,k}^T)$ are $m_{i+k-1,i+k-1}, m_{i+k-1,i+k-2}, \cdots, m_{i+k-1,i}$, which match those of $R_i(C_{n,k})$, and so $S(C_{n,k}^T)$ is also the cyclic (n, k)-scheme, with $C_{n,k}^T$ obtained by rotating the columns of $C_{n,k}$ forward by $k - 1$ places. This contrasts with $S(M_{n,k}^T)$, the subject of Section 5, which although optimal is of a different character to $S(M_{n,k})$.

The rows of $M_{n,k}$ represent the same cyclic sequence. The same is true of the columns.

Proposition 4.9. *Let $M = M_{n,k}$ be the cyclic (n, k)-matrix. Then every pair or columns of M represent the same cyclic sequence.*

Proof. Let $\gcd(n, k) = d$. For $M_{n,k} = (m_{i,j})$ we have, with addition modulo n, that $m_{i,j} = m_{i+1,j+k}$. Since $\gcd(n, k) = d$, there exists a value r such that $kr \equiv d \pmod n$; r-fold application of the previous equation then gives $m_{i,j} = m_{i+r,j+kr} = m_{i+r,j+d}$. It follows in particular that C_j and C_{j+d} define the same cyclic sequence, with one being transformed into the other through a rotation of r positions. By Proposition 4.7(ii), the columns $C_0, C_1, \cdots, C_{d-1}$ are identical, and so it now follows that every pair of columns of $M_{n,k}$ define the same cyclic sequence. □

5. The transpose solution

We have noted that optimality of a uniform matrix is generally not preserved under transposition. However, the cyclic scheme is an exception to this.

Theorem 5.1. *The transpose matrix* $M = M^T_{n,k}$ *of a cyclic* (n, k)*-matrix* $M_{n,k}$ *is also optimal.*

We shall call $S(M^T)$ a *transpose cyclic scheme* and similarly M^T is a *transpose cyclic matrix*. With subscripts calculated modulo n, the non-zero entries of column C_j of M^T are $m_{jk,j}, m_{jk+1,j}, \cdots, m_{jk+k-1,j}$ $(0 \leq j \leq n-1)$. The transpose cyclic matrix M^T is k-uniform, and so if $S(M^T)$ does not stall, we have optimality. By passing to the binary dual if necessary, we may suppose that $k \leq \frac{n}{2}$, for first note that for any binary matrix $M = (m_{i,j})$, we have $\overline{M}^T = \overline{M^T}$ as the (i, j)th entry in each of these matrices is $\overline{m}_{j,i}$. Now let us assume that for any cyclic (n, k)-matrix M with $n \geq 2k$, the transpose matrix M^T is optimal. Suppose that M is a cyclic (n, k)-matrix with $n < 2k$ and consider M^T. Then $\overline{M^T} = \overline{M}^T$, with $\overline{M}$ a cyclic $(n, n-k)$-matrix. Since $n < 2k$, it follows that $n > 2(n-k)$, and so by our assumption we have that $\overline{M}^T$ is optimal. But $\overline{M}^T = \overline{M^T}$, whence $\overline{\overline{M^T}} = M^T$ is also optimal.

We are therefore permitted to adopt the assumption that $2k \leq n$ in our proof that transpose cyclic matrices are optimal. For the remainder of the section we shall denote our transpose cyclic matrix by M (as opposed to M^T). For any t, at least one of the entries $m_{t,j}$ and $m_{t,j+1}$ of M is 0, as we now show.

For any $j \geq 0$, there is a unique $i\,(= jk \bmod n)$, such that the non-zero entries of columns C_j and C_{j+1} in M have the form:

$$(m_{i,j} = m_{i+1,j} = \cdots = m_{i+k-1,j} = 1)$$

$$\Leftrightarrow (m_{i+k,j+1} = m_{i+k+1,j+1} = \cdots = m_{i+2k-1,j+1} = 1). \tag{20}$$

Since the total number of entries listed in (20) is $2k \leq n$, it follows that there is no t such that $m_{t,j} = m_{t,j+1} = 1$, as claimed.

The non-zero entries of C_j form a cyclic block of length k. This will manifest itself either as a single linear block in C_j, or as a pair of *initial* and a *terminal* blocks. In the single block case, the initial and terminal blocks of non-zero entries are one and the same.

Lemma 5.2. *Let (i, j) be the final entry of the initial block of non-zero entries of C_j. We shall write $i = i(j)$. Then*

(21) $$S_{0,j} = S_{1,j} = \cdots = S_{i,j} = S_{i+1,j} + 1;\ S_{i+1,j} = S_{i+2,j} = \cdots = S_{n-1,j}.$$

Proof. We proceed by induction on j. For $j = 0$ we have $m_{0,0} = m_{1,0} = \cdots = m_{k-1,0} = 1$, $m_{k,0} = \cdots = m_{n-1,0} = 0$, in accord with (21), where $i(0) = k - 1$. Suppose now that (21) holds for some value of j and consider C_{j+1}. Suppose first that the non-zero entries of C_{j+1} form a single linear block: $m_{t,j+1} = m_{t+1,j+1} = \cdots = m_{t+k-1,j+1} = 1$. If $t = 0$ then $i(j) = n - 1$ in (21) and all the row sums for C_j in (21) are equal. It then follows that (21) holds for C_{j+1} as in the $j = 0$ case. Otherwise, $t \geq 1$ and so $i(j) = t-1$. By induction:

$$S_{0,j} = S_{1,j} = \cdots = S_{t-1,j} = S_{t,j} + 1,\ S_{t,j} = S_{t+1,j} = \cdots = S_{n-1,j}.$$

Since $S_{p,j} = S_{p,j+1}$ for all $0 \leq p \leq t - 1$ it follows that $S_{0,j+1} = S_{p,j+1}$ for all $0 \leq p \leq t - 1$. On the other hand for $t \leq p \leq t + k - 1$ we have $S_{p,j+1} = 1 + S_{p,j} = 1 + (S_{0,j} - 1) = S_{0,j} = S_{0,j+1}$. Therefore, $S_{0,j+1} = S_{p,j+1}$ for all $0 \leq p \leq t + k - 1$. Finally, for the case where $t + k \leq p$ we have $S_{p,j+1} = S_{p,j} = S_{0,j+1} - 1$ and so (21) is holds for the $S_{p,j+1}$ $(0 \leq p \leq n-1)$.

The alternative case is where the non-zero entries of C_{j+1} break into distinct initial and terminal blocks. The two blocks then have the respective forms:

(22) $$m_{0,j+1} = m_{1,j+1} = \cdots = m_{i,j+1} = 1$$
$$\&\ m_{n-k+i+1,j+1} = m_{n-k+i+2,j+1} = \cdots = m_{n-1,j+1} = 1\ (0 \leq i \leq k-2).$$

(Note that the total number of entries in (22) is indeed $(i + 1) + (n - 1 - (n - k + i)) = k$.) The single linear cyclic block of non-zero entries of C_j ends at $m_{n-k+i,j} = 1$ and begins at $m_{(n-k+i-(k-1)),j} = m_{n-2k+i+1,j} = 1$. By applying the inductive hypothesis to the row sums of C_j we infer that:

(23) $$S_{0,j+1} = S_{1,j+1} = \cdots = S_{i,j+1} = S_{0,j} + 1 = \cdots = S_{i,j} + 1.$$

Since $S_{0,j} = S_{1,j} = \cdots = S_{n-k+i,j}$, it follows that

(24) $$S_{i+1,j+1} = \cdots = S_{n-k+i,j+1} = S_{0,j}.$$

Finally, we have

$$S_{n-k+i,j} = 1 + S_{n-k+i+1,j},\ S_{n-k+i+1,j} = \cdots = S_{n-1,j}$$

$$(25) \qquad \Rightarrow S_{n-k+i+1,j+1} = \cdots = S_{n-1,j+1} = 1 + S_{n-k+i+1,j} = S_{0,j}.$$

Statements (23), (24), and (25) together give (22) as applied to C_{j+1}.

Now suppose that $n \leq 2k$. Recall from Remark 4.5 that the binary dual $\overline{M}$ of M is the cyclic transpose matrix of the $(n, n-k)$ problem with rows reversed.

Since $n \geq 2(n-k)$ it follows that (22) holds for the corresponding row sums of the columns of $\overline{M}$, (denoted $\overline{S}_{i,j}$). Note that $S_{i,j} + \overline{S}_{i,j} = j + 1$. Hence, for some i $(0 \leq i \leq n-1)$:

$$\overline{S}_{n-1,j} = \overline{S}_{n-2,j} = \cdots = \overline{S}_{n-i,j} = \overline{S}_{n-i-1,j} + 1;\ \overline{S}_{n-i-1,j} = \overline{S}_{n-i-2,j} = \cdots = \overline{S}_{0,j},$$

$$\Leftrightarrow S_{n-1,j} = S_{n-2,j} = \cdots = S_{n-i,j} = S_{n-i-1,j} - 1;$$

$$S_{n-i-1,j} = S_{n-i-2,j} \cdots = S_{0,j},$$

$$\Leftrightarrow S_{0,j} = \cdots = S_{n-i-1,j} = S_{n-i,j} + 1;\ S_{n-i,j} = \cdots = S_{n-1,j},$$

which is in accord with (22) with $i(j) = n - 1 - i$. This completes the proof. □

Proof of Theorem 5.1. As already observed, we may assume that $n \geq 2k$, in which case it is clear from (21) that for any column C_j,

$$(i_1 \in X^j_{1,0}, i_2 \in X^j_{0,1}) \Rightarrow S_{i_1,j} \geq S_{i_2,j},$$

from which it follows that the canonical word w_j is the Dyck word $w_j = a^k b^k$ $(2k \leq n \Rightarrow |X_{1,0}| = k)$. Hence, every assignment mapping ϕ_j is optimal, and therefore M is optimal. □

Proposition 5.3. *If $n \geq 2k$, then at any time point during the execution of $S(M)$, there are at most 3 distinct positions for the travellers. Moreover, the distance separating one cohort from the next is less than* 1 *unit.*

Proof. We begin with three useful observations.

- If $i \leq i'$ then t_i never trails $t_{i'}$. This follows easily from the fact that for any fixed j, the $S_{i,j}$ are monotonically decreasing in i (Lemma 5.2).

- Consider a typical column C_j of M. As explained prior to Lemma 5.2, C_j consists of three blocks, each of which consists of zeros or ones. Writing 0 for a block of zeros and 1 for a block of ones, the blocks of C_j have either of the two forms 010 or 101, although in the former case the second 0-block may be empty, as may be the second 1-block in the latter case.
- Two successive columns C_j and C_{j+1} cannot both have the 101-block structure. (This is a consequence of $n \geq 2k$, for since the combined length of the two 1-blocks is k, the length of the 0-block in the 101 case is at least k.)

When travelling in a common stage s_{j+1}, we shall refer to t_i and $t_{i'}$ as being *members of the block* if $m_{i,j}$ and $m_{i',j}$ are in the same block of C_j. A set of travellers who are currently moving together will be called a *cohort*.

We now prove inductively on j that, during the period when the leading cohort is between P_j and P_{j+1}, the following three conditions hold:

1. Any pair of members of the same block are in the same cohort.
2. There are at most 3 cohorts.
3. The distance between the members of two neighbouring cohorts is less than 1 unit.

Inductive verification of this trio of claims proves Proposition 5.3.

For $j = 0$ all three claims are clear and indeed there are only 2 cohorts. Consider C_j ($j \geq 1$) and suppose by way of induction that our claims hold for all lesser values of j.

Take the case where C_j has a 010 block structure. Suppose first that C_{j-1} also has a 010 structure. Then, by Lemma 5.2, the members of the joint 01-block of C_{j-1} arrive together at P_j and from the structure of the transpose cyclic scheme, form the first 0-block of C_j, so forming the lead cohort in s_{j+1}. By induction, the lead of this cohort over the second 0-block in C_{j-1} as it becomes the first 0-block of C_j is less than 1 unit. Since the leading cohort is walking, its lead over the next cohort remains less than 1 unit as the lead cohort traverses s_{j+1}. The 1-block of C_j consists of the first k entries of the members of the second 0-block of C_{j-1}, whose members arrived in unison at P_j. This 1-block cohort forms the second cohort, which then catches the leading cohort at P_{j+1}. During this time the lead of the second cohort is less than 1 unit over the third cohort which is the remainder of the second 0-block of C_{j-1}, which becomes the second 0-block of C_j upon arrival at P_j. These observation taken together demonstrate that Conditions 1-3 are respected throughout the time that the lead cohort walks s_{j+1}.

Next suppose that C_{j-1} has a 101 structure, in which case the members of the first 1-block of C_{j-1} arrive first at P_j and form the first 0-block of C_j. By induction, this cohort is less than 1 unit ahead of the next cohort. Since the leading block of C_j is walking, its lead over the following cohort cannot increase as it traverses s_{j+1}, and so remains less than 1 unit. By induction, the separation of the 0-block of C_{j-1} and the second 1-block of C_{j-1} is less than 1 unit up until the time the leading cohort of C_{j-1} reaches P_j. Their separation decreases after that and the two cohorts reach P_j in unison. After that the 01-block of C_{j-1} splits into two new cohorts, the first a cohort of size k is comprised of cyclists, which are the members of the 1-group of C_j, with the remainder of the joint 01-block of C_{j-1} becoming the second 0-block of C_j and the third cohort, (thus maintaining Conditions 1 and 2). The second 0-block of C_j will be less than 1 unit behind the second cohort until the leading cohort completes s_{j+1}. Hence, Conditions 1, 2, and 3 remain valid throughout the period where the leading cohort is travelling between P_j and P_{j+1}, thus continuing the induction.

Finally, we examine the case where C_j has the block form 101. By the third bullet point, C_{j-1} has the block form 010. By Lemma 5.2, the members of the 01-block of C_{j-1} arrive together at P_j, and by induction, the members of the second 0-block are the trailing cohort, which is less than 1 unit behind. The first 1-block of C_j is an initial segment of the joint 01-block of C_{j-1} and its members therefore proceed together as the lead cohort. By construction of the transpose cyclic scheme, the joint 01-block of C_{j-1} becomes the joint 10-block of C_j, with the walking members becoming a second cohort in s_{j+1}. Their distance behind the first cohort is always less than 1 unit. The second 0-block of C_{j-1} becomes the second 1-block of C_j, and so its members proceed together, as the third cohort. This cohort was also the third cohort of C_{j-1} and so was less than 1 unit behind the members of the 01-block of C_{j-1} (by Condition 3 and induction) when the joint block reached P_j. Hence, the separation between the two trailing cohorts is less than 1 unit (and decreases to 0 as these cohorts traverse s_{j+1}). Therefore, Conditions 1, 2, and 3 have been met, and so the induction continues, thereby completing the proof. □

Examples 5.4. Proposition 5.3 does not hold however when $2k > n$. For example, consider the transpose matrix M for the $(n, n-1)$ problem. Then the zeros consist of the non-leading diagonal running between entries $(n-1, 0)$ and $(0, n-1)$. If we let the ratio of the cycling speed to walking speed become arbitrarily large, then t_0 will reach P_{n-1} before t_{n-1} has reached P_1, so that the separation of t_0 and t_{n-1} approaches an upper limit of $n-1$ units.

6. Calculating features of cyclic schemes

Proposition 6.1. *In respect to the (n, k)-cyclic solution, let $n = r + qk$, $(0 \le r \le k-1)$, and let $d = gcd(n, k)$. Let $i_0, i_1, \cdots, i_{k-1}$ be the subscripts of the k travellers that ride stage s_0. Label the k bicycles as $b_0, b_1, \cdots, b_{k-1}$, where b_m is the bicycle ridden by t_{i_m} in s_0. Then during the execution of the scheme:*

(i) the total number of bicycle rides is $n + k - d$.

(ii) Each bicycle b_m is mounted on either $\lceil \frac{n}{k} \rceil$ or $\lceil \frac{n}{k} \rceil + 1$ occasions, with the first alternative applying if and only if $r \le c_m$, where $k(i_m + 1) \equiv c_m$ (mod n), $1 \le c_m \le k$.

Proof. (i) There are k travellers t_i that cycle s_0, and t_i completes their quota if and only if $ki \equiv 0 \pmod{n}$. There are d solutions i to this congruence. Therefore, $n-k+d$ travellers have a single ride while $k-d$ travellers have two separate rides, one beginning and the other ending their journey. The total number of cycle rides is therefore $n - k + d + 2(k - d) = n + k - d$.

(ii) Bicycle b_m $(0 \le m \le k-1)$ is mounted at P_0 by t_{i_m} who dismounts at P_{c_m}, where $k(i_m + 1) \equiv c_m \pmod{n}$ $(1 \le c_m \le k)$. If $c_m < r$ then

$$c_m + qk < r + qk = n. \tag{26}$$

Hence, b_m is ridden by $1+q+1 = q+2$ travellers. Since $1 \le r$ we have $\lceil \frac{n}{k} \rceil + 1 = (q+1) + 1 = q+2$, as required. Otherwise, $r \le c_m$ whence

$$c_m + (q-1)k \le qk \le r + qk = n. \tag{27}$$

If $1 \le r$ it follows from (27) that b_m is ridden by $1+(q-1)+1 = q+1$ travellers. If $r = 0$ then $c_m = k$ and this figure is $1 + (q-1) = q$, but in either event this number equals $\lceil \frac{n}{k} \rceil$, thus completing the proof. □

Proposition 6.2. *For the $M = M^T_{n,k}$ transpose cyclic matrix and scheme:*

(i) If $n \le 2k$ then traveller t_i has k cycle rides; if $2k \ge n$, then t_i has $n-k+1$ rides if $n-k \le i \le k-1$, and $n-k$ rides otherwise.

(ii) The total number of cycle rides is nk if $2k \le n$ and is $n(n-k-1)+2k$ if $2k \ge n$.

(iii) The number of excess handovers $h(M) = min(k(k-1), (n-k)(n-k-1))$.

(iv) The number of cycle rides after the elimination of excess handovers is, in all cases, $k(n-k+1)$.

Proof. (i) For the case where $2k \le n$, each traveller rides just one stage at a time, and so t_i has k rides. We analyse this case further. Applying Proposition 4.3(iv) to $M^T_{n,k}$, we have that $m_{i,j} = m_{n-1-i,n-1-j}$, and so

$$m_{0,0} = m_{1,0} = \cdots = m_{k-2,0} = m_{k-1,0} = 1,$$

$$m_{n-k,n-1} = m_{n-k+1,n-1} = \cdots = m_{n-2,n-1} = m_{n-1,n-1} = 1. \tag{28}$$

$$\begin{aligned} m_{k,0} = m_{k+1,0} = \cdots = m_{n-1,0} = 0 = m_{0,n-1} \\ = m_{1,n-1} = \cdots = m_{n-k-1,n-1}. \end{aligned} \tag{29}$$

It follows from (28) and (29) that for $0 \le i \le k-1$, row R_i has $m_{i,0} = 1$, and R_i has k (non-consecutive) entries equal to 1, each followed by a maximal sequence of positive length that consists of entries that equal 0. For $n-k \le i \le n-1$, the same is true for R_i but the statement applies for R_i considered in reverse order, beginning with $m_{i,n-1} = 1$. On the other hand, for $k \le i \le n-k-1$, R_i begins and ends with a sequence of zeros, and once again there are k entries equal to 1, but with no two consecutive entries equal to 1.

If we now pass from $M^T_{n,k}$ to $\overline{M^T_{n,k}}$, the rows indexed by $0 \le i \le k-1$ and $n-k \le i \le n-1$ each indicate k bicycle rides, while the remaining central rows each show $k+1$ cycle rides. By symmetry, the same conclusion applies to the matrix with rows reversed. Now by Remark 4.5 and Proposition 4.3(iii) we infer that

$$M^T_{n,k} = \overline{(M_{n,n-k})_r}^{\,T} = \overline{((M_{n,n-k})_r)^T} = \overline{((M^T_{n,n-k})_r} = (\overline{(M^T_{n,n-k})})_r. \tag{30}$$

Therefore, if $n < 2k$ we infer that the rows R_i of $M^T_{n,k}$ such that $0 \le i \le n-k-1$ or $k \le i \le n-1$ indicate $n-k$ bicycle rides, while those indexed by $n-k \le i \le k-1$ show $n-k+1$ cycle rides, as required.

(ii) If $2k \le n$ then each cycle is mounted on n separate occasions, and so the total number of cycle rides is nk. Otherwise, it follows by

(i) that the total number of cycle rides is given by:

$$n(n-k) + (k-1-(n-k-1)) = (n-k)(n-1) + k = n(n-k-1) + 2k.$$

(iii) The cyclic sequence of 1's in C_j has length k and may be written as $T_j = (i_j, i_j+1, \cdots, i_j+k-1)$ with addition mod n ($i_j = jk \pmod n$). Suppose that $n \geq 2k$, in which case $T_j = X^j_{1,0}$. By Lemma 5.2 it follows that s_j has no excess handovers unless for some t such that $0 \leq t \leq k-1$ we have $i_j + 2k - 1 = n + t$. In that case $j \leq n-2$ and T_{j+1} consists of two linear sequences, which are $I_1 = (0, 1, \cdots, t)$ and $I_2 = (n-k+t+1, n-k+t+1, \cdots, n-1)$, although the latter is empty if $t = k-1$. (Note that $|I_1| = t+1$, $|I_2| = n-1-(n-k+t) = k-t-1$, so that $|I_1| + |I_2| = k$.) Observe from Lemma 5.2 that the handovers to $t_0, \cdots, t_t$ at the completion of s_j are excess handovers, as are all those from $t_{n-k+t+1}, \cdots, t_{n-1}$ for s_{j+1}. There are no other unnecessary handovers in either s_j or in s_{j+1}. It follows that the number of excess handovers in the pair of stages s_j and s_{j+1} is then $|I_1| + |I_2| = k$.
It follows that the set of excess handovers of $S(M)$ is partitioned into sets of order k, with one such set for every $0 \leq j \leq n-1$ such that $0 \leq (j+1)k \pmod n \leq k-1$, with one exception. In the case where $j = n-1$ so that $j+1 \equiv 0 \pmod n$ there are no handovers from s_{n-1} to s_0. Let $d = \gcd(n,k)$. The number of multiples of d in the interval $[0, k-1]$ is $\frac{k}{d}$. Then there are d values of $(j+1)$ $(0 \leq j \leq n-1)$ such that $(j+1)k \equiv td \pmod n$ $(0 \leq t \leq \frac{n}{d})$. Hence, the number of sets in question is $d\frac{k}{d} = k$. Therefore, $h(M) = k^2 - k = k(k-1)$, as we subtract k in recognition of no handovers occurring from s_{n-1} to s_0.
On the other hand, if $n \leq 2k$ consider $\overline{M^T_{n,k}} = (M^T_{n,n-k})_r$ by (30). Since the latter matrix is also a reverse transpose matrix and $n \geq 2(n-k)$, it now follows from Proposition 3.20 that

(31)
$$h(M^T_{n,k}) = h(\overline{M^T_{n,k}}) = h(M^T_{n,n-k})_r = h(M^T_{n,n-k}) = (n-k)(n-k-1).$$

Therefore, $h(M) = k(k-1)$ if $n \geq 2k$ and $h(M) = (n-k)(n-k-1)$ otherwise. Combining these two cases we obtain the statement of (iii).

(iv) If $n \geq 2k$ we have the number of cycle rides after elimination of excess handovers is $kn - k(k-1) = k(n-k+1)$, as required. In the alternative case, by (i), the corresponding number has the same form:

$$n(n-k-1) + 2k - (n-k)(n-k-1)$$

$$= (n-k-1)(n-n+k) + 2k = k(n-k+1). \qquad \square$$

Examples 6.3. $M_{11,7}^T$

$M' =$

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
t_0	1	1	0	0	1	1	1	1	0	0	1
t_1	1	1	0	0	1	1	1	0	1	1	0
t_2	1	1	0	0	1	1	1	0	0	1	1
t_3	1	0	1	1	1	0	0	1	1	1	0
t_4	1	0	1	1	1	0	0	1	1	1	0
t_5	1	0	1	1	0	0	1	1	1	0	1
t_6	1	0	0	1	1	1	1	0	0	1	1
t_7	0	1	1	1	0	0	1	1	1	0	1
t_8	0	1	1	1	0	1	1	0	0	1	1
t_9	0	1	1	1	0	1	0	1	1	0	1
t_{10}	0	1	1	0	1	1	0	1	1	1	0

We expunge all excess handovers from $S(M_{11,7}^T)$ to yield M'. Since $2k \geq n$, from Proposition 6.2(iv) we find that the total ride number is $11(11-7-1)+2(7) = 47$. The number of rides by t_0 through to t_{10} is $(4+4+4+4)+(5+5+5)+(4+4+4+4) = 47$, in accord with part (i), as $n-k = 4$ and $n-k \leq i \leq k-1$ becomes $4 \leq i \leq 6$, so it is t_4, t_5, and t_6 who have the extra ride. We have $h(M) = (11-7)(11-7-1) = 12$. According to (iii), after elimination of excess handovers the total number of rides is $7(11-7+1) = 35$, which indeed equals $47 - h(M)$. All travellers have three rides in $S(M')$ except for t_5 and t_9 who each have four.

Throughout this paper we have placed the staging posts at intervals of one unit with the journey regarded as being of length n. We may however consider other partitions of the travellers' journey. Consider a putative scheme, $S = S_m$, based on partitions into m equal stages. Such a scheme S_m would then be represented by an $n \times m$ binary matrix $M = M(S_m)$.

Theorem 6.4. *For the (n,k)-problem, let $k = gcd(n,k)$ and put $n' = \frac{n}{d}$ and $k' = \frac{k}{d}$. Then an optimal scheme S_m defined by an $n \times m$ matrix exists for the (n,k)-problem if and only if $m = rn'$ for some $r \geq 1$, in which case each traveller cycles for $l = rk'$ of the m stages of S_m.*

Proof. As in the $m = n$ case, for S_m to be an optimal solution, we must have each column C_j of M containing exactly k instances of 1, and each row containing a common number, t say, of 1's. Counting the 1's by rows, and then by columns we equate to see that $tn = km$, whence $m = \frac{tn}{k} = \frac{tn'}{k'}$. Since $\gcd(n',k') = 1$, it follows that $k'|t$ so that $t = rk'$ say, and m necessarily has the form $m = rn'$, for some $r \geq 1$. Moreover, in any optimal scheme S_m, each traveller cycles the same number, l say, of stages of S_m. By optimality we then have $\frac{l}{m} = \frac{k}{n} = \frac{k'}{n'}$ so that $l = \frac{mk'}{n'} = \frac{rn'k'}{n'} = rk'$. In conclusion:

$$m = rn',\ l = rk'\ (r \geq 1). \tag{32}$$

Conversely, we now show that given that m satisfies (32), we may build a scheme S_m from copies of schemes for the (n', k')-problem to yield an optimal solution for the (n, k)-problem based on an $n \times m$ binary matrix M which is (l, k)-uniform, meaning that each row and each column contains exactly l and k non-zero entries respectively. To do this we take a $d \times r$ array and at each position in the array we place an optimal $n' \times n'$ matrix for the (n', k')-problem. (There is no need for these matrices to be identical solutions.) This yields a $(dn' \times rn') = (n \times m)$ binary matrix M with l entries of 1 in each row, and k entries of 1 in each column.

The first set of n' columns represents a scheme for the initial $\frac{1}{r}$ part of the journey. Executing this partial scheme will see d (disjoint) sets of travellers, with each set executing an optimal (n', k') scheme. Since these schemes are carried out in parallel, all n travellers will complete the first $\frac{1}{r}$ of the full journey simultaneously, as all these schemes are optimal. Each of these d sets of travellers will then repeat a similar process for the second and subsequent partial schemes, with all travellers completing each of the fractional journeys of lengths $\frac{1}{r}, \frac{2}{r}, \cdots, \frac{j}{r}, \cdots \frac{r}{r} = 1$ at the same time. All n travellers complete the journey simultaneously, having cycled l stages, as required to finish in the least time. □

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