Journal of Combinatorics
Volume 15, Number 2, 139–157, 2024

Low diameter monochromatic covers of complete multipartite graphs

Sean English, Grace McCourt*,
Connor Mattes, and Michael Phillips

Let the diameter cover number, $D_r^t(G)$, denote the least integer d such that under any r-coloring of the edges of the graph G, there exists a collection of t monochromatic subgraphs of diameter at most d such that every vertex of G is contained in at least one of the subgraphs. We explore the diameter cover number $D_2^2(G)$ when G is a complete multipartite graph. Specifically, we determine exactly the value of $D_2^2(G)$ for all complete tripartite graphs G, and almost all complete multipartite graphs with more than three parts.

AMS 2000 subject classifications: Primary 05C12; secondary 05C15.
Keywords and phrases: Diameter, covers, Ryser's conjecture.

1. Introduction

Given a hypergraph H, we say H is r-partite if there exists a partition $V(H) = V_1 \cup V_2 \cup \cdots \cup V_r$ such that no edge of H contains two or more vertices from V_i for any $1 \leq i \leq r$. The *vertex cover number*, $\tau(H)$, is the minimum cardinality of a set $S \subseteq V(H)$ such that S intersects every edge of H, and the *matching number*, $\nu(H)$, is the size of a largest set of pairwise disjoint edges in H. Ryser's Conjecture [8] attempts to relate these two parameters in a strong way for r-partite graphs.

Conjecture 1.1. *Let $r \geq 2$ and let H be an r-partite hypergraph. Then*

$$\tau(H) \leq (r-1)\nu(H).$$

Note that the classical König's Theorem [9], which states that the size of a minimum vertex cover in a bipartite graph G is no bigger than the size of a maximum matching in G, confirms the case $r = 2$ of Ryser's Conjecture. It was proved by Aharoni [1] that Ryser's Conjecture also holds for $r = 3$.

arXiv: 2105.07038
*Research supported in part by NSF RTG Grant DMS-1937241.

However, already for $r = 4, 5$ the only exact result is for the case $\nu(H) = 1$, shown by Tuza [11]. It has also been proven in the case of $r = 4, 5$, by Haxell and Scott [7], that there exists some $\epsilon > 0$ such that $\tau(H) \leq (r - \epsilon)\nu(H)$.

An equivalent formulation of Ryser's Conjecture involves coverings of edge-colored graphs with monochromatic connected components. More specifically, a *monochromatic connected subgraph cover* of an edge colored graph G is a collection of monochromatic connected subgraphs of G whose union contains every vertex. The *monochromatic tree cover number* $\mathrm{tc}_r(G)$ is the least integer such that for any r-coloring of the edges of G, there exists a monochromatic connected subgraph cover with $\mathrm{tc}_r(G)$ subgraphs. The *independence number* of G, $\alpha(G)$, is the size of the largest collection $S \subseteq V(G)$ such that no two vertices in S are adjacent. The following is an equivalent formulation of Conjecture 1.1, first observed by Gyárfás [5].

Conjecture 1.2. *For every graph G and all $r \geq 2$,*

$$\mathrm{tc}_r(G) \leq (r - 1)\alpha(G)$$

For a brief sketch of the proof that these two conjectures are equivalent, see [3], Section 1.

Phrasing Ryser's Conjecture in terms of edge colorings leads to many interesting questions and connections. In particular, this phrasing of Ryser's Conjecture sheds light on the fact that the conjecture fits into the large body of work on Ramsey theory. In particular, Ryser's Conjecture is closely related to problems involving finding large monochromatic components in edge-colored graphs. Given an r-coloring of the edges of the complete graph K_n, it is known that there always exists a monochromatic component of size at least $n/(r - 1)$ [5], and that this is sharp for infinitely many values of n and r. Conjecture 1.2, if true, implies this as well, and thus can be thought of as a strengthening of the question of the size of the largest monochromatic component. Large monochromatic components has been studied intensively in both graphs and hypergraphs [4], [6]

In this paper, we will explore structural properties of the monochromatic connected subgraphs in a cover, in particular we consider subgraphs with bounded diameter. Given a graph G, an integer $r \geq 2$, and an integer $t \geq \mathrm{tc}_r(G)$, define the *diameter cover number*, $D_r^t(G)$, to be the least integer d such that in every r-coloring of the edges of G, there exists a monochromatic connected subgraph cover of G with at most t subgraphs such that every subgraph has diameter at most d.

The diameter cover number was first studied by Milićević [10], who considered 3- and 4-colored complete graphs as well as 2-colored complete bipartite graphs, using the result for 3-colored complete graphs to prove a generalization of Banach's fixed point theorem. Note that it is known that $tc_3(K_n) \le 2$ [5], $tc_4(K_n) \le 3$, [5] $tc_2(K_{m,n}) \le 2$ [2], $tc_3(K_{m,n}) \le 4$ [2] and $tc_2(G) = 2$ for any graph G with $\alpha(G) = 2$.

Theorem 1.3 ([10]). *1. For all complete graphs K_n, $D_3^2(K_n) \le 8$.*

2. For all complete graphs K_n, $D_4^3(K_n) \le 80$.

3. For all complete bipartite graphs $K_{m,n}$, $D_2^2(K_{m,n}) \le 9$.

The above results were improved and extended by DeBiasio, Kamel, McCourt, and Sheats [3].

Theorem 1.4 ([3]). *1. For all complete graphs K_n with $n \ge 7$, $3 \le D_3^2(K_n) \le 4$.*

2. For all complete graphs K_n with $n \ge 5$, $2 \le D_4^3(K_n) \le 6$.

3. For all complete bipartite graphs $K_{m,n}$ with $m \ge 3$, $n \ge 4$, $3 \le D_2^2(K_{m,n}) \le 4$.

4. For all complete bipartite graphs $K_{m,n}$, $D_3^4(K_{m,n}) \le 6$.

5. For all graphs G on at least 7 vertices with $\alpha(G) = 2$, $3 \le D_2^2(G) \le 6$.

We focus on 2-colorings of complete multipartite graphs with at least three parts. Note that $tc_2(G) \le 2$ for any complete multipartite graph G. This was shown by Chen, Fujita, Gyárfás, Lehel, and Tóth [2], and we offer a sketch of a proof of it in Section 1.2.

We prove the following theorem in Section 2.

Theorem 1.5. *For any complete tripartite graph G,*

$$D_2^2(G) \le 3.$$

Note that if G is complete multipartite, then $D_2^4(G) \le 2$, since we can cover the entire graph with a red and blue star at a vertex in one part and another red and blue star at a vertex in another part. Therefore finding an upper bound on $D_2^t(G)$ for complete multipartite graphs with at least three parts is only interesting in the case of $t = 2, 3$.

It is worth noting that if F is a spanning subgraph of G and $\text{tc}_r(F) \ge t$, then $D_r^t(F) \ge D_r^t(G)$. Therefore, the bound in Theorem 1.5 also applies to all complete k-partite graphs with $k \ge 3$, as they contain a spanning complete tripartite graph.

In Section 3, we improve on Theorem 1.5, and provide a complete characterization of the value of $D_2^2(G)$ for all complete tripartite graphs G.

Theorem 1.6. *Let G be a complete tripartite graph.*

1. $D_2^2(G) = 3$ if and only if $K_{5,2,2} \subseteq G$ or $K_{4,3,2} \subseteq G$,

2. *$D_2^2(G) = 1$ if and only if $G = K_3$ or $G = K_{2,1,1}$, and*
3. *$D_2^2(G) = 2$ otherwise.*

In addition, for every $k \geq 4$, we determine the value of $D_2^2(G)$ for all but finitely many complete k-partite graphs G. We also explore the problem of categorizing more values of $D_2^2(G)$. In particular, it is easy to show that if G has a vertex adjacent to all other vertices then $D_2^2(G) \leq 2$. However, it is unclear if $D_2^2(G) \leq 2$ for any complete k-partite graphs without a part of size 1, when k is large. Let $K_k(2)$ denote the complete multipartite graph with k parts each of size 2. We propose the following conjecture:

Conjecture 1.7. *For all $k \geq 3$,*

$$D_2^2(K_k(2)) = 2.$$

In Section 4, we provide partial results towards proving this conjecture.

1.1. Definitions and notations

Call a graph a *double star* if it is isomorphic to graph with vertex set $a, b, x_1, \dots, x_n, y_1, \dots, y_m$ where ab, ax_i, by_j are edges for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Given a graph G and a vertex $v \in V(G)$, let $N(v)$ denote the open neighborhood of v, that is $N(v) = \{u \in V(G) : uv \in E(G)\}$. Given a graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$, a *blow-up* of G is a graph F with a partition $V(F) = V_1 \cup V_2 \cup \cdots \cup V_n$ of non-empty vertex sets such that for all $a \in V_i$ and $b \in V_j$, $ab \in E(F)$ if and only if $v_i v_j \in E(G)$. Given two disjoint subsets $A, B \subseteq V(G)$, we will let $G[A]$ denote the subgraph of G induced on the vertex set A, and let $G[A, B]$ denote the bipartite graph induced in G with parts A and B. Recall that the eccentricity of a vertex x in a connected graph F, denoted $\mathrm{ecc}_F(x)$ is defined as

$$\mathrm{ecc}_F(x) = \max\{d_F(x, y) \mid y \in V(F)\}.$$

Note that if a graph H is connected, then there exists a vertex $x \in V(H)$ such that $\mathrm{ecc}_H(x) = \mathrm{diam}(H)$.

1.2. For any complete multipartite graph G, $tc_2(G) \leq 2$

Note that adding edges can only decrease the tree cover number, so it suffices to show this for complete bipartite graphs. Let $G = (A, B, E)$ be a complete bipartite graph, and fix $v \in A$. Assume without loss of generality that v has at least one red edge incident to it, as otherwise we may swap the colors red and blue. Let A' and B' be the sets of vertices in A and B, respectively, which are covered by the red component containing v. If $A' = A$, then we can cover the graph with the maximum red and blue components containing v. If $B' = B$ then all vertices in $A \setminus A'$ are complete blue to B and therefore may be covered with a blue connected component. That blue component along with the red component containing v forms the desired cover. Otherwise, we may cover the graph with 2 connected blue components, $G[(A \setminus A') \cup B']$ and $G[A' \cup (B \setminus B')]$, both of which induce a complete blue bipartite graph.

2. Upper bound – proof of Theorem 1.5

In this section, we prove Theorem 1.5. To do this, we will first show in Section 2.1 that any hypothetical counterexample to the theorem must have some very specific structure, and then in Section 2.2 we explore this structure and show that we can find a suitable monochromatic subgraph cover.

Throughout this section, we will use the following result:

Lemma 2.1 (Lemma 4.18(P1) in [3])**.** *If all edges incident to a single vertex v in a two-edge coloring of a complete bipartite graph are all colored the same, then there is a monochromatic subgraph cover with a star and a double star.*

This can be easily seen by taking the double star at uv where u is some neighbor of v, and then the star in the other color centered at u.

Theorem 1.5. *For any complete tripartite graph G,*

$$D_2^2(G) \leq 3.$$

Proof. First note that if our complete multipartite graph has a part of size 1, then this vertex is the center of a blue star and a red star (one of these may be trivial), which cover the entire graph, so we have a cover with two subgraphs of diameter at most 2 in this case. Now, let $a, b, c \geq 2$ be fixed, and assume to the contrary that Theorem 1.5 does not hold for $G := K_{a,b,c}$. Let A, B and C be the partite sets of G with $|A| = a$, $|B| = b$ and $|C| = c$. Let $\chi : E(G) \to \{\text{red}, \text{blue}\}$ be a 2-coloring of the edges of G where χ has the property that there do not exist two monochromatic subgraphs of G of diameter at most 3 that cover $V(G)$. Let G_{red} and G_{blue} be the spanning subgraphs of G containing all the red edges and all the blue edges respectively, and define $N_{red}(x) := N_{G_{red}}(x), N_{blue}(x) := N_{G_{blue}}(x), d_{red}(x) := d_{G_{red}}(x)$, and $d_{blue}(x) := d_{G_{blue}}(x)$.

As noted in [3] following Lemma 4.18, the only case in which we do not know that a 2-colored complete bipartite graph has a monochromatic cover consisting of at most two subgraphs of diameter at most 3 occurs when both the red and blue spanning subgraphs are connected and of diameter exactly 4. This implies the following:

Claim 2.2. *Both graphs G_{red} and G_{blue} have diameter* 4.

Proof. Assume towards a contradiction this is not the case. If the diameter of G_{red} or G_{blue} is less than 4 then we can simply take that as our component, meaning that $D_2^2(G) \leq 3$. So, one of G_{red} or G_{blue} must have diameter at least 5 (in the case where one of G_{red} or G_{blue} is disconnected, we will consider it to have infinite diameter, and it would fall into this case). Note that the deletion of edges can only increase the diameter of a graph. Thus, if we delete all edges between two parts in G to create a complete bipartite graph G', we get that one of G'_{red} or G'_{blue} has diameter at least 5. Then, by Lemma 4.18 in [3], $3 \geq D_2^2(G') \geq D_2^2(G)$, a contradiction. □

Throughout the rest of this proof, we will fix a vertex $v \in V(G)$ such that $\mathrm{ecc}_{G_{red}}(v) = 4$. We may assume without loss of generality that $v \in A$.

We now will partition A, B and C based on red-distances from the vertex v. For $1 \leq i \leq 4$, let $A_i \subseteq A$, $B_i \subseteq B$ and $C_i \subseteq C$ be the sets of vertices that are red-distance exactly i from v in each of our three partite sets. This gives us the following partitions:

- $A = \{v\} \cup A_2 \cup A_3 \cup A_4$,
- $B = B_1 \cup B_2 \cup B_3 \cup B_4$, and
- $C = C_1 \cup C_2 \cup C_3 \cup C_4$,

where A_1 is omitted since $A_1 = \emptyset$ trivially.

2.1. Reducing the problem

Many of the sets in the above partitions of A, B and C may be empty. The goal of this section is to reduce the problem down to a case where we know exactly which sets are empty and which are non-empty. More specifically, we will show that we may focus on the case when A_3, B_2, B_4, C_2 and C_4 are all empty, while all the other sets in our partitions are non-empty.

Claim 2.3. *$B_1, C_1 \neq \emptyset$.*

Proof. First note by Lemma 2.1 that no vertex of G dominates an entire partite set in a single color, otherwise the bipartite subgraph between the dominated partite set and the rest of the graph would have a cover with a star and a double star. Therefore, there must exist a vertex in B that is adjacent to v in red, and also a vertex in C that is adjacent to v in red, giving us that $B_1 \neq \emptyset$ and $C_1 \neq \emptyset$. □

We now deal with all vertices at red-distance 4 from v.

Claim 2.4. *$A_4 \neq \emptyset$ and $B_4 = C_4 = \emptyset$.*

Proof. First suppose $B_4 \neq \emptyset$, and let $u_4 \in B_4$. The existence of u_4 immediately implies that there exists a vertex $u_3 \in A_3 \cup C_3$. Now by Claim 2.3 we know that $B_1 \neq \emptyset$, so let $u_1 \in B_1$. Note that the edges vu_4 and u_1u_3 are both blue since $d_{red}(v, u_4) = 4$ and $d_{red}(u_1, u_3) \geq 2$. Let S_1 be the largest blue double star with centers v and u_4 and let S_2 be the largest blue double star with centers u_1 and u_3.

We will show that S_1 and S_2 form a monochromatic subgraph cover of diameter at most 3, which will give us a contradiction. Indeed, every vertex in B_1 is a blue neighbor of u_3, while $B \setminus B_1 \subseteq N_{blue}(v)$, so $B \subseteq V(S_1) \cup V(S_2)$. Furthermore, every vertex in $A_3 \cup A_4 \cup C_3 \cup C_4$ is blue-adjacent to u_1, while every vertex in $\{v\} \cup A_2 \cup C_1 \cup C_2$ is blue-adjacent to u_4, giving us that $A, C \subseteq V(S_1) \cup V(S_2)$. This gives us the desired contradiction, so we must have that $B_4 = \emptyset$. By symmetry $C_4 = \emptyset$. Since v has eccentricity at least 4 in G_{red}, we must have $A_4 \neq \emptyset$, completing the proof. □

Next, we consider B_3 and C_3.

Claim 2.5. *$B_3, C_3 \neq \emptyset$.*

Proof. By Claim 2.4, we must have a vertex $x \in A_4$. Note that $N_{red}(x) \subseteq B_3 \cup C_3$. The contrapositive of Lemma 2.1 applied to $G[A \cup C, B]$ implies that $N_{red}(x) \cap B_3 \neq \emptyset$. In particular, $B_3 \neq \emptyset$, and by a symmetric argument, we can also conclude that $C_3 \neq \emptyset$. □

Before we show that $A_3 = \emptyset$, we deal with all vertices at red-distance 2 from v.

Claim 2.6. *$A_2 \neq \emptyset$ and $B_2 = C_2 = \emptyset$.*

Proof. First assume that $B_2 \neq \emptyset$. Note that by definition, for all distinct $X, Y \in \{A, B, C\}$ and $i, j \in \{1, 2, 3, 4\}$ with $|i - j| \geq 2$, $G[X_i, Y_j]$ is complete blue. Additionally, if $X \neq A$, and $i \in \{2, 3, 4\}$, $G[\{v\}, X_i]$ is complete blue. These blue complete bipartite graphs are enough to guarantee that

$H := G_{blue}[V(G) \setminus (A_2 \cup A_3)]$, as seen in Figure 1 has diameter 3, regardless of whether C_2 is empty or not, and further that every vertex in B_2 has eccentricity 2 in H.

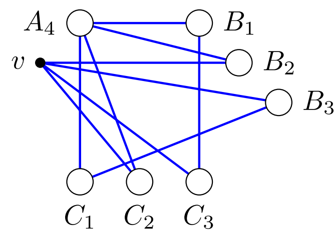

Figure 1: The blue subgraph H, which has diameter 3.

Let $x \in B_2$ and define $H' := G_{blue}[V(H) \cup N_{blue}(x)]$. Since x has eccentricity 2 in H, H' is a blue subgraph of G of diameter 3. Let S be the largest red star in G centered at x. Every vertex in $B \cup C$ is contained in H', and $A \setminus V(H') \subseteq N_{red}(x)$. Thus H' and S give us a monochromatic subgraph cover of diameter at most 3, a contradiction. Hence $B_2 = \emptyset$, and a symmetric argument shows that $C_2 = \emptyset$. Since B_3 is non-empty by Claim 2.5, there must exist a vertex at distance exactly 2 from v, so we must have that A_2 is non-empty, finishing the proof of the claim. □

The final set to consider is A_3.

Claim 2.7. *$A_3 = \emptyset$.*

Proof. Any vertex in A_3 must have a red neighbor in $B_2 \cup C_2$ in order to have red-distance 3 from v. By Claim 2.6, B_2 and C_2 are empty. Therefore A_3 must also be empty. □

Thus, we have shown that the sets A_2, A_4, B_1, B_3, C_1, C_3 are all non-empty, while all the other sets in our partitions of A, B and C are empty.

2.2. The final case of Theorem 1.5

Based on Section 2.1, we may assume that $A = \{v\} \cup A_2 \cup A_4$, $B = B_1 \cup B_3$ and $C = C_1 \cup C_3$, and furthermore each of these sets are non-empty. We will now explore this particular case and show that there must exist a monochromatic subgraph cover of diameter at most 3, which will conclude the proof of Theorem 1.5.

Note that, similar to the proof of Claim 2.6, $G[\{v\}, B_3]$, $G[B_3, C_1]$, $G[C_1, A_4]$, $G[A_4, B_1]$, $G[B_1, C_3]$ and $G[C_3, \{v\}]$ are all blue complete bipartite graphs due to the red-distance from v of each set. This implies that $G' := G[V(G) \setminus A_2]$ has a spanning blue C_6-blowup, call it C_6^+, and thus has diameter at most 3. Thus, our main goal of this section will be to show that the vertices in A_2 can either be added to C_6^+ without increasing the diameter, or can be included in a red subgraph of diameter 3.

Towards this, we first explore which edges are red in G'. By definition, we have that $G[\{v\}, B_1 \cup C_1]$ is a red star. Our next claim gives us a collection of red edges in G' that will help form a large red subgraph.

Claim 2.8. *The red subgraph $G_{red}[B_1, C_1]$, is a complete bipartite graph.*

Proof. Let $x \in B_1$ and $y \in C_1$. Assume to the contrary that the edge xy is blue. Then x has eccentricity 2 in $C_6^+ + xy$, so $G^* := G_{blue}[V(C_6^+) \cup N_{blue}(x)]$ has diameter at most 3. Let S be the largest red star with center x. Notice that G^* and S cover G since the only vertices that are not in G^* are in $A_2 \setminus N_{blue}(x)$, which is contained in $V(S)$. This gives us a monochromatic subgraph cover of diameter at most 3, a contradiction. Thus every edge in $G[B_1, C_1]$ is red. ☐

Note that using the same technique we can show that $G[B_3, C_3]$, $G[B_3, A_4]$, and $G[C_3, A_4]$ are complete red bipartite graphs as well, but only Claim 2.8 is necessary to complete the proof of Theorem 1.5. We now define a partition of A_2 into two sets, one of which can be added to the blue subgraph containing C_6^+, and the other which can be covered with a red subgraph of diameter at most 3. Let $A_{2,\text{red}} \subseteq A_2$ be the set of vertices, x, that satisfy at least one of the following properties:

(P1) x has only red neighbors in B_1,
(P2) x has only red neighbors in C_1, or
(P3) x has at least one red neighbor in each of B_1 and C_1.

Let $A_{2,\text{blue}} := A_2 \setminus A_{2,\text{red}}$, and note that every vertex in $A_{2,\text{blue}}$ has at least one blue neighbor in each of B_1 and C_1, and also has only blue neighbors in at least one of B_1 or C_1. First we show that the vertices in $A_{2,\text{blue}}$ can be included in a blue subgraph containing the blue C_6^+.

Claim 2.9. *The blue subgraph $G_{blue}[V(C_6^+) \cup A_{2,blue}]$ has diameter at most* 3.

Proof. We need only consider pairs of vertices x and y with at least one vertex in $A_{2,\text{blue}}$, say without loss of generality $x \in A_{2,\text{blue}}$. If y is also in $A_{2,\text{blue}}$, then since x has at least one neighbor in each of B_1 and C_1, and y

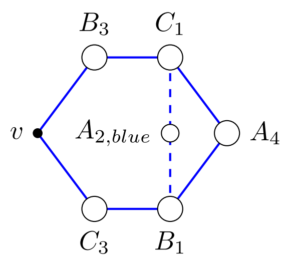

Figure 2: The blue subgraph $G_{blue}[V(C_6^+) \cup A_{2,\text{blue}}]$. Solid edges represent blue complete bipartite graphs. Dashed edges represent that every vertex in $A_{2,blue}$ is complete blue to one of B_1 and C_1, and has at least one blue neighbor in the other set.

is adjacent to every vertex in either B_1 or C_1, x and y are at distance 2 in $G_{blue}[V(C_6^+) \cup A_{2,\text{blue}}]$.

Now, consider a pair with $x \in A_{2,\text{blue}}$ and $y \in V(C_6^+)$. Assume without loss of generality that x has only blue neighbors in B_1, and let $z \in C_1$ be a blue neighbor of x. If $y \in B_3$, then (x, z, y) is a blue path of length 2. If $y \notin B_3$, then y is distance at most 2 from any vertex in B_1, so distance at most 3 from x, so in either case, x and y are at distance at most 3, so $G_{blue}[V(C_6^+) \cup A_{2,\text{blue}}]$ has diameter at most 3. □

The final step in the proof is to show that we can cover $A_{2,\text{red}}$ with a red subgraph.

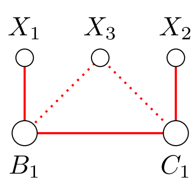

Figure 3: The red subgraph $G_{red}[B_1 \cup C_1 \cup A_{2,\text{red}}]$. Solid edges represent complete bipartite graphs while dotted edges represent that vertices in X_3 have at least one red neighbor in each set B_1 and C_1.

Claim 2.10. *The red subgraph $G_{red}[B_1 \cup C_1 \cup A_{2,red}]$ has diameter at most* 3.

Proof. By Claim 2.8, $G_{red}[B_1, C_1]$ is complete bipartite. Let $X_i \subseteq A_2$ be the set of vertices satisfying property (Pi) for $1 \leq i \leq 3$. Then note that $G_{red}[B_1 \cup C_1 \cup X_1 \cup X_2]$ has a spanning red P_4-blowup, call it P_4^+, which has diameter 3. Thus, to show that $G_{red}[B_1 \cup C_1 \cup A_{2,\text{red}}]$ has the desired diameter, we only need to consider distances between pairs of vertices with at least one vertex in X_3.

Let $x \in X_3$ and $y \in B_1 \cup C_1 \cup A_{2,\text{red}}$. Since $\text{ecc}_{P_4^+}(u) = 2$ for any vertex $u \in B_1 \cup C_1$, and x has a red neighbor in $B_1 \cup C_1$, $d_{red}(x, y) \leq 3$ if $y \in B_1 \cup C_1 \cup X_1 \cup X_2$. If instead $y \in X_3$, then let $x' \in B_1$ be a red neighbor of x, and $y' \in C_1$ be a red neighbor of y. The path (x, x', y', y) is a red path of length 3, so $d_{red}(x, y) \leq 3$, finishing the proof. □

This completes the proof of Theorem 1.5; in Section 2.1, we reduced the proof down to the case when $\{v\}, A_2, A_4, B_1, B_3, C_1$ and C_3 are the only non-empty sets in our original partition, and then via Claim 2.9 and Claim 2.10, we show that in this final case, we have a monochromatic subgraph cover using two subgraphs of diameter at most 3. □

3. Determining $D_2^2(G)$ exactly for complete tripartite graphs and others

In this section we provide a complete classification of $D_2^2(G)$ for all complete tripartite graphs, G, as well as prove some results towards a classification of $D_2^2(G)$ for any complete multipartite graph G. In light of Theorem 1.5, as the addition of edges can only decrease the value of $D_2^2(G)$, $D_2^2(G) \leq 3$ for all complete multipartite graphs with at least three parts. In Theorem 3.1 we prove that a particular complete multipartite graph G has $D_2^2(G) = 3$, and then in Proposition 3.2, we prove that if we add a vertex to an existing part in a complete multipartite graph G, the diameter cover number cannot decrease. For any fixed k, this gives us that there are finitely many complete k partite graphs with diameter cover number 2 or less. In the case of complete tripartite graphs, we then classify the remaining graphs via direct analysis.

Our first result in this section shows that for each $k \geq 2$, a complete $(k+1)$-partite graph with no part of size 1 and one large part will have diameter cover number 3. Let $K_{a,b*k}$ denote the complete $(k+1)$-partite graph with one part of size a and k parts of size b.

Theorem 3.1. *For all $k \geq 2$,*

$$D_2^2(K_{2k+1,2*k}) = 3.$$

Proof. The upper bound follows from Theorem 1.5 and the fact that $K_{2k+1,2k-2,2}$ is a spanning subgraph of $K_{2k+1,2*k}$. Now let us focus on the lower bound.

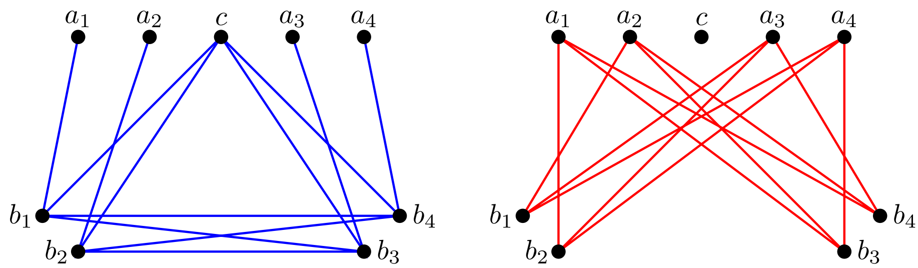

Figure 4: A coloring of $K_{5,2,2}$ that does not admit a monochromatic cover with two diameter 2 subgraphs.

Let $G = K_{2k+1,2*k}$, Let $A = \{a_1, a_2, \ldots, a_{2k}, c\}$ be the part of G of size $2k + 1$, and let $B = \{b_1, b_2, \ldots, b_{2k}\}$ denote the remaining vertices of G, where $\{b_{2i-1}, b_{2i}\}$ is a part of size 2 for all $1 \leq i \leq k$. Color the edges a_ib_i, cb_i, and b_ib_j blue for all $1 \leq i, j \leq 2k$ (if b_ib_j is not an edge, we do not assign it a color), and color all remaining edges of G red.

We claim that under this coloring, G does not have a monochromatic cover with two subgraphs of diameter at most 2. To see this, let us assume to the contrary that there exists such a cover. First note that since c is not incident to any red edges, there must be a blue subgraph containing c. Furthermore, the blue edges incident to the a_i's form a matching, so any blue subgraph of diameter 2 can contain at most one of the a_i's. and since there are $2k \geq 4$ such vertices a_i, there must be a red subgraph that contains all but at most one of the a_i's. This red subgraph must contain at least one of the b_i's, otherwise it would contain no edges, so without loss of generality, we can assume that b_1 is in the red subgraph. Since a_1 and b_1 are distance 3 from each other in red, a_1 cannot be in the red subgraph, and so the red subgraph contains the vertices in $\{a_2, a_3, \ldots, a_{2k}\}$. Now, note that b_2 is distance 3 from a_2 in red, and distance 3 from a_1 in blue, so b_2 cannot be in either the red or blue subgraph, a contradiction. Thus $D_2^2(G) \geq 3$. □

We now prove that adding vertices to a part in a complete multipartite graph does not decrease the diameter cover number.

Proposition 3.2. *Let $k \geq 2$, $b_1 \geq a_1, b_2 \geq a_2, \ldots, b_k \geq a_k$ be positive integers. If $D_2^2(K_{a_1,a_2,\ldots,a_k}) \geq 3$, then $D_2^2(K_{b_1,b_2,\ldots,b_k}) \geq 3$.*

Proof. Note that it will suffice to prove that $D_2^2(K_{a_1+1,a_2,\ldots,a_k}) \geq 3$. Let $G = K_{a_1+1,a_2,\ldots,a_k}$, and let G' be an induced subgraph of G isomorphic to $K_{a_1,a_2,\ldots,a_k}$. By assumption, there exists a 2-coloring $c : E(G') \to \{\text{red}, \text{blue}\}$

of the edges of G' such that there is no cover of G' using two monochromatic subgraphs, each with diameter at most 2. Let the edges of G' be colored according to this 2-coloring.

Let $x \in V(G')$ be a vertex in the part of size a_1. Let y be the vertex in $V(G) \setminus V(G')$. For each vertex $v \in V(G)$ such that $vy \in E(G)$, assign the color $c(vx)$ to the edge vy. We claim that there is no cover of G with two monochromatic subgraphs of diameter at most 2. Assume to the contrary that there was such a cover, say with monochromatic subgraphs G_1 and G_2, where the edges of G_i are colored $c_i \in \{\text{red}, \text{blue}\}$ for $i = 1, 2$. Let $V_i = V(G_i)$ for $i = 1, 2$. For $i = 1, 2$, if $y \in V_i$, let $V_i' = (V_i \setminus \{y\}) \cup \{x\}$, and if $y \notin V_i$, let $V_i' = V_i$. Let G_i' denote the subgraph of $G'[V_i']$ consisting of only edges of color c_i.

We claim that G_i' is connected and has diameter 2. Indeed, if $V_i' = V_i$, this follows immediately from our contrary assumption. If $V_i' = (V_i \setminus \{y\}) \cup \{x\}$, any path in G_i containing y can be replaced with a path of the same length in G_i' containing x, so $\text{diam}(G_i') \le \text{diam}(G_i) \le 2$. Finally, note that $V(G') = V_1' \cup V_2'$ since $V(G) = V_1 \cup V_2$, and $V_i \setminus \{y\} \subseteq V_i'$. This contradicts the original assumption that under the coloring c, there was no cover of G' with two monochromatic subgraphs with diameter at most 2. □

We note that our argument above can also prove the stronger fact that if G is a graph and H is a blow-up of G then $D_r^t(H) \ge D_r^t(G)$.

By combining Theorem 3.1 and Proposition 3.2, we see that given any $k \ge 3$, all but finitely many k-partite graphs G with no part of size 1 have $D_2^2(G) = 2$. Let us turn our attention to small complete tripartite graphs with no part of size 1. There are 10 complete tripartite graphs with no part of size 1 that our prior results do not imply a result for D_2^2, namely the graphs $K_{a,b,c}$ with $2 \le c \le b \le a \le 4$. Fortunately, to determine which complete tripartite graphs require diameter 3, we do not need to check all 10 of these graphs, but instead we need to find the "minimal" ones that require diameter 3 and the "maximal" ones that do not require diameter 3, and then we can apply Proposition 3.2 and its contrapositive to classify the rest.

Proposition 3.3. *We have the following:*

- $D_2^2(K_{4,3,2}) = 3$,
- $D_2^2(K_{4,2,2}) = 2$,
- $D_2^2(K_{3,3,3}) = 2$.

As the proofs for Proposition 3.3 involve simple logic and extensive case work, we only include a proof of the first equality, below. These graphs are small enough that we were able to verify these bounds using a brute force computer program (see supplemental files on arXiv).

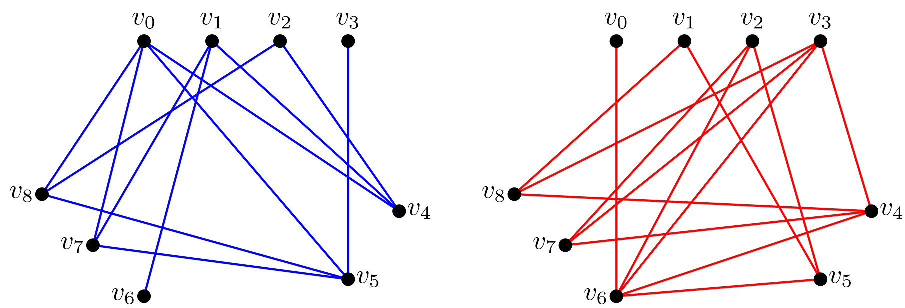

Figure 5: A coloring of $K_{4,3,2}$ that does not admit a monochromatic cover with two diameter 2 subgraphs.

proof that $D_2^2(K_{4,3,2}) = 3$. Note that the upper bound on $D_2^2(K_{4,3,2})$ follows from Theorem 1.5. For the lower bound, consider the coloring of $K_{4,3,2}$ shown in Figure 5. Assume towards a contradiction that there is some covering of the vertices with 2 monochromatic diameter 2 subgraphs. First we note that in the spanning blue subgraph vertices v_2, v_3, and v_6 are pairwise distance at least 3 from each other, and vertices v_0, v_1, and v_7 are pairwise distance 3 in the spanning red subgraph. Therefore, one of our subgraphs in our cover must be red, while the other must be blue. Let R be the set of vertices in the red subgraph, and B be the set of vertices in the blue subgraph. We consider two cases based on which subgraphs contains v_8.

Case 1: $v_8 \in R$. Since v_2 is distance 3 from v_8 in red, we must have that $v_2 \in B \setminus R$. Furthermore, as v_7 is distance 3 from v_2 in blue, $v_7 \in R \setminus B$. Since v_1 is distance 3 from v_7 in red, we have $v_1 \in B \setminus R$. Note that the only blue path of length 2 from v_1 to v_5 is through v_7, so since $v_1 \in B$, and $v_7 \notin B$, we have that $v_5 \notin B$. Similarly, the only red path of length 2 from v_7 to v_5 is through v_2, so since $v_7 \in R$ and $v_2 \notin R$, we have that $v_5 \notin R$, a contradiction.

Case 2: $v_8 \in B \setminus R$. If $v_3 \in B$, then v_1 and v_4 are in $R \setminus B$ since they are distance at least 3 in blue from v_3, but the only red path of length 2 from v_1 to v_4 uses v_8, which is not in R, a contradiction. Thus, $v_3 \in R \setminus B$. As v_8 and v_1 are distance 3 in blue, we know that $v_1 \in R \setminus B$, but the only red path of length 2 from v_1 to v_3 is through v_8, yielding another contradiction and completing the proof. □

Note that $D_2^2(K_3) = D_2^2(K_{2,1,1}) = 1$ since both graphs can be covered by two edges, but $D_2^2(K_{3,1,1}) \geq 2$ since $K_{3,1,1}$ cannot be covered by two diameter 1 subgraphs (even without regards to an edge coloring), and $D_2^2(K_{2,2,1}) \geq 2$

since any cover of $K_{2,2,1}$ with two diameter 1 subgraphs would necessarily need to contain at least one K_3, but if we color the edges incident with the vertex in the part of size 1 red and all other edges blue, there is no monochromatic triangle. We remind the reader that any complete multipartite graph with a part of size 1 has $D_2^2(G) \leq 2$ since a red and blue star centered at this vertex cover everything.

These results allow us to determine $D_2^2(G)$ for all complete tripartite graphs G. The results on complete tripartite graphs are summarized concisely below.

Theorem 3.4. *Let G be a complete tripartite graph.*

- $D_2^2(G) = 3$ *if and only if* $K_{5,2,2} \subseteq G$ *or* $K_{4,3,2} \subseteq G$*,*
- $D_2^2(G) = 1$ *if and only if* $G = K_3$ *or* $G = K_{2,1,1}$*, and*
- $D_2^2(G) = 2$ *otherwise.*

4. Progress towards Conjecture 1.7

In this section, we consider the family of complete multipartite graphs in which each partite set has size 2. Recall that we denote such a graph with k parts as $K_k(2)$.

Problem 4.1. *Determine the value of* $D_2^2(K_k(2))$*.*

Using the same program that we checked small examples of complete tripartite graphs with, we found that $D_2^2(K_3(2)) = D_2^2(K_4(2)) = D_2^2(K_5(2)) = 2$. In attempts to determine whether this value is 2 or 3 for $k \geq 6$, we proved the following results. These results serve as properties of a minimal example which requires diameter 3, if such a graph and coloring exists. Note that one difficulty in solving this problem is being unable to classify graphs of diameter 2. In particular, we often attempt to build a cover using familiar graphs of diameter 2 that are relatively easy to find by hand, such as stars and C_5 blow-ups. However, there are many diameter 2 graphs which are more difficult to find by hand. In particular, almost all graphs are diameter 2, and further we have pathological examples such as the Petersen graph and the graph pictured in Figure 6, which was in fact used in multiple covers produced by a program we ran to check small cases.

Throughout the section we will suppose that a 2-coloring of the edges of $K_k(2)$ exists that requires a subgraph of diameter 3 in every cover, and state necessary properties. We fix such a 2-coloring of $K_k(2)$ and let color 1 be blue and color 2 be red.

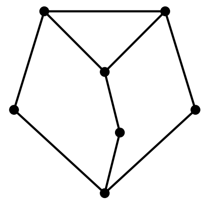

Figure 6: A diameter 2 graph used in some colorings of $K_{2,2,2,2,2}$.

For any $v \in V(K_k(2))$, we will denote by v' the unique vertex such that $vv' \notin E(K_k(2))$, and we call v' the *clone* of v. We begin with a result on the length 2 paths between clones.

Property 4.2. *For each $v \in V(K_k(2))$, every possible color combination of length 2 vv'-paths exist. That is, clones are distance 2 from each other in both colors 1 and 2, as well as have length 2 paths which alternate colors in both orders.*

Proof. For $i, j \in [2]$, let $X_{i,j}$ denote the vertices which send color i to v and color j to v'. Note that each of the four sets represent one of the four possible color combinations of length 2 vv'-paths. Suppose $X_{i,j} = \emptyset$ for some $i, j \in [2]$. Then the color j star at v and the color i star at j cover $V(K_k(2))$ and each have diameter 2. Hence $X_{i,j} \neq \emptyset$ for all $i, j \in [2]$. $\square$

Note that there must exist a vertex $x \in V(K_k(2))$ that has at least one vertex at distance at least 3 in blue, since otherwise the spanning blue subgraph is the desired cover of diameter at most 2. By the previous property, this other vertex is not x'. We will now partition the vertex set in terms of the distances in blue from x and x'.

For $i, j \in [3]$, let $A_{i,j}$ denote the vertices that are distance i in blue from x and distance j in blue from x', with the convention that when i or j is 3, we include vertices of distance at least 3 in blue instead of exactly 3. Let $A_{i,*} = A_{i,1} \cup A_{i,2} \cup A_{i,3}$ and $A_{*,j} = A_{1,j} \cup A_{2,j} \cup A_{3,j}$.

Next, we eliminate three of these nine sets.

Property 4.3. $A_{2,3} \cup A_{3,2} \cup A_{3,3} = \emptyset$.

Proof. First, suppose $A_{3,2} \cup A_{3,3} \neq \emptyset$ and let $y \in A_{3,2} \cup A_{3,3}$. Consider the red star at x and the red star at y. The red star at x covers $A_{2,*} \cup A_{3,*}$, and the red star at y covers $\{x'\} \cup (A_{1,*} \setminus \{y'\})$. If $y' \notin A_{1,*}$, then this gives the desired cover with diameter at most 2.

If $y' \in A_{1,1}$, then consider the red and blue stars at x'. Notice that the only red neighbors of x' which do not also send a red edge to x are $A_{1,2} \cup A_{1,3}$. However, all those vertices are red neighbors of y, so we can add x to the red star at x' while maintaining diameter 2. Thus we have the desired cover.

If $y' \in A_{1,2} \cup A_{1,3}$, then consider the red and blue stars at y'. Notice that the only vertex not covered by these two subgraphs is y. We will add y to the red subgraph by also adding x. As long as either $A_{3,*} \setminus \{y\} \neq \emptyset$ or y' has a red neighbor in $A_{2,*}$, this new red subgraph has diameter 2, and we have the desired cover. Otherwise, y is the only vertex in $A_{3,*}$ and every vertex in $A_{2,*}$ sends a blue edge to y'. Hence the red star at y and the blue star at y' give the desired cover.

Finally, we can switch the roles of x and x' in the above argument to show that if $A_{2,3} \neq \emptyset$, we get the desired cover. □

The previous two properties imply that the following sets must be non-empty.

Corollary 4.4. *$A_{1,1} \neq \emptyset$, $A_{2,2} \neq \emptyset$, $A_{1,2} \cup A_{1,3} \neq \emptyset$, and $A_{2,1} \cup A_{3,1} \neq \emptyset$.*

Note that by our choice of x and Property 4.3, we have more specifically that $A_{3,1} \neq \emptyset$

Now we give the location of the clones of vertices in $A_{1,3}$ and $A_{3,1}$.

Property 4.5. *For any $y \in A_{1,3}$, $y' \in A_{2,1}$. Similarly, for any $z \in A_{3,1}$, $z' \in A_{1,2}$.*

Proof. Fix $y \in A_{1,3}$. First, we will show that $y' \in A_{2,1} \cup A_{3,1}$. Consider the blue star at x along with the red C_5 blow up formed by x, $A_{2,2}$, x', y, and $(A_{2,1} \cup A_{3,1}) \setminus \{y'\}$. Note that $A_{2,2} \neq \emptyset$ and $A_{2,1} \cup A_{3,1} \neq \emptyset$ by Corollary 4.4. Thus if $y' \notin A_{2,1} \cup A_{3,1}$, those two monochromatic subgraphs form the desired cover.

Now suppose $y' \in A_{3,1}$. Consider the red star at y' along with the red C_5 blow up formed by x, $A_{2,2}$, x', y, and $(A_{2,1} \cup A_{3,1}) \setminus \{y'\}$. This gives the desired cover unless $(A_{2,1} \cup A_{3,1}) \setminus \{y'\} = \emptyset$, in which case we replace the C_5 blow up with the red star at x'. Thus we have the desired cover.

Therefore $y' \in A_{2,1}$. The second half of the property can be proven by switching the roles of x and x' in the above argument. □

5. Concluding results

In this paper we were able to determine the diameter cover number exactly for complete tripartite graphs, and for some classes of complete multipartite graphs, whenever two colors are used and two subgraphs are allowed. It would however be interesting to have a more thorough idea of the diameter cover number for other complete multipartite graphs. One of the most tangible questions in this regard would be work towards Conjecture 1.7, restated here:

Problem 4.1. *Determine the value of* $D_2^2(K_k(2))$*, where* $K_k(2)$ *is the complete* k*-partite graph in which each part is size* 2*.*

Section 4 covers some results we were able to show regarding the previous problem. Knowing this border case could also possibly help determine the diameter cover number for other complete multipartite graphs. In the other direction, one could also ask about complete bipartite graphs instead.

Problem 5.1. *Determine the value of* $D_2^2(K_{a,b})$*.*

It has been shown that $D_2^2(K_{a,b}) \leq 4$ in [3], however this is not known to be sharp, and one could work towards providing a complete classification as we did for complete tripartite graphs.

One could also explore this question while allowing for more subgraphs. As mentioned in the introduction, $D_2^t(G) \leq 2$ for $t \geq 4$. Thus we could achieve a complete classification for $t \geq 4$ by solving the following problem.

Problem 5.2. *Determine when* $D_2^t(G) = 1$ *for complete multipartite graphs* G *and* $t \geq 4$*.*

The behavior of $D_2^3(G)$ is less clear.

Problem 5.3. *Determine the value of* $D_2^3(G)$ *for complete multipartite graphs* G*.*

References

[1] R. Aharoni. Ryser's conjecture for tripartite 3-graphs. *Combinatorica*, 21(1):1–4, 2001. MR1805710

[2] G. Chen, S. Fujita, A. Gyárfás, J. Lehel, and Á. Tóth. Around a biclique cover conjecture, 2012.

[3] L. DeBiasio, Y. Kamel, G. McCourt, and H. Sheats. Generalizations and strengthenings of ryser's conjecture, 2020. MR4396459

[4] Z. Füredi and A. Gyárfás. Covering t-element sets by partitions. *European J. Combin.*, 12(6):483–489, 1991. MR1136390

[5] A. Gyárfás. Particiófedések és lefogóhalmazok hipergráfokban. *Tanulmányok—MTA Számitástech. Automat. Kutató Int. Budapest*, (71):62, 1977. MR0485565

[6] A. Gyárfás and P. Haxell. Large monochromatic components in colorings of complete 3-uniform hypergraphs. *Discrete Math.*, 309(10):3156–3160, 2009. MR2526733

[7] P. E. Haxell and A. D. Scott. On Ryser's conjecture. *Electron. J. Combin.*, 19(1):Paper 23, 10, 2012. MR2880654

[8] J. R. Henderson. *Permutation Decompositions of (0,1)-Matrices and Decomposition Transversals.* ProQuest LLC, Ann Arbor, MI, 1971. Thesis (Ph.D.)–California Institute of Technology. MR2620362

[9] D. Kőnig. Gráfok és mátrixok. *Matematikai és Fizikai Lapok*, 38:116–119, 1931.

[10] L. Milićević. Covering complete graphs by monochromatically bounded sets. *Appl. Anal. Discrete Math.*, 13(1):85–110, 2019. MR3948047

[11] Z. Tuza. Some special cases of Rysers conjecture. Unpublished manuscript, 1979.

Sean English
University of North Carolina Wilmington
United States
E-mail address: englishs@uncw.edu

Grace McCourt
university of Illinois Urbana Champaign
United States
E-mail address: mccourt4@illinois.edu

Connor Mattes
University of Colorado Denver
United States
E-mail address: connor.mattes@ucdenver.edu

Michael Phillips
University of Colorado, Denver
United States
E-mail address: michael.2.phillips@ucdenver.edu

Received May 20, 2021