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Absolutely avoidable order-size pairs in hypergraphs

Lea Weber*

For a fixed integer $r \geq 2$, we call a pair (m, f) of integers, $m \geq 1$, $0 \leq f \leq \binom{m}{r}$, *absolutely avoidable* if there is n_0, such that for any pair of integers (n, e) with $n > n_0$ and $0 \leq e \leq \binom{n}{r}$ there is an r-uniform hypergraph on n vertices and e edges that contains no induced sub-hypergraph on m vertices and f edges. Some pairs are clearly not absolutely avoidable, for example $(m, 0)$ is not absolutely avoidable since any sufficiently sparse hypergraph on at least m vertices contains independent sets on m vertices. Here we show that for any $r \geq 3$ and $m \geq m_0$, either the pair $(m, \lfloor \binom{m}{r}/2 \rfloor)$ or the pair $(m, \lfloor \binom{m}{r}/2 \rfloor - m - 1)$ is absolutely avoidable.

Next, following the definition of Erdős, Füredi, Rothschild and Sós, we define the *density* of a pair (m, f) as

$$\sigma_r(m, f) = \limsup_{n \to \infty} \frac{|\{e : (n, e) \to (m, f)\}|}{\binom{m}{r}},$$

where $(n, e) \to (m, f)$ if any n-vertex r-graph with e egdes contains an induced m-vertex subgraph with f edges. We show that for $r \geq 3$ most pairs (m, f) satisfy $\sigma_r(m, f) = 0$, and that for $m > r$, there exists no pair (m, f) of density 1.

1. Introduction

One of the central topics of graph theory deals with properties of classes of graphs that contain no subgraph isomorphic to some given fixed graph, see for example Bollobás [8]. Similarly, graphs with forbidden induced subgraphs have been investigated from several different angles – enumerative, structural, algorithmic, and more.

Erdős, Füredi, Rothschild and Sós [12] initiated a study of a class of graphs that do not forbid a specific induced subgraph, but rather forbid any induced subgraph on a given number m of vertices and number f of edges. Following their notation we say an r-uniform hypergraph (also referred to as an r-graph) G *arrows* a pair of non-negative integers (m, f) and write

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$G \to_r (m,f)$ if G has an induced sub-hypergraph on m vertices and f hyperedges. We say that a pair (n,e) of non-negative integers *arrows* (or simply induces) the pair (m,f), and write

$$(n,e) \to_r (m,f)$$

if for any r-graph G on n vertices and e hyperedges, $G \to_r (m,f)$. We say a pair (n,e) is *realised* by an r-graph G if G has n vertices and e edges. If r is clear from the context, we might omit the index and simply write $(n,e) \to (m,f)$.

As an example for $r = 2$, if $t_{m-1}(n)$ denotes the number of edges in the balanced complete $(m-1)$-partite graph on n vertices, then by Turán's theorem [20] we know that any graph on n vertices with more than $t_{m-1}(n)$ edges contains K_m, a complete subgraph on m vertices. Equivalently stated, we have $(n,e) \to (m, \binom{m}{2})$ if and only if $e > t_{m-1}(n)$.

Let r, m, f integers, $m \geq r \geq 3$ and $0 \leq f \leq \binom{m}{r}$. Following the notation in [12, 13], we define

$$\sigma_r(m,f) = \limsup_{n\to\infty} \frac{|\{e : (n,e) \to (m,f)\}|}{\binom{n}{r}}.$$

Erdős, Füredi, Rothschild and Sós [12] considered $\sigma_2(m,f)$ for different choices of (m,f). One of their main results is the following theorem.

Theorem 1 (Erdős, Füredi, Rothschild and Sós [12])**.** *If* $(m,f) \in \{(2,0),(2,1),(4,3),(5,4),(5,6)\}$, *then* $\sigma_2(m,f) = 1$*; otherwise,* $\sigma_2(m,f) \leq \frac{2}{3}$.

The upper bound $\frac{2}{3}$ was subsequently improved by He et al. [13] to $\frac{1}{2}$. On the other hand, they showed that there are infinitely many pairs for which the equality $\sigma(m,f) = \frac{1}{2}$ holds.

In [12], Erdős, Füredi, Rothschild and Sós also gave a construction that shows that "most of the" $\sigma_2(m,f)$ are 0, by showing that for large n almost all pairs (n,e) can be realised as the vertex disjoint union of a clique and a high-girth graph, and that for fixed m most pairs (m,f) cannot be realised as the vertex disjoint union of a clique and a forest. Axenovich and the author [6] investigated the existence of so-called absolutely avoidable pairs (m,f) for which we not only have $\sigma_2(m,f) = 0$, but the stronger property $\{e : (n,e) \to (m,f)\} = \emptyset$ for large n. Here, we extend this notion to hypergraphs:

Definition 1. *A pair* (m,f) *is* ***absolutely r-avoidable*** *if there is* n_0 *such that for each* $n > n_0$ *and for any* $e \in \{0,\dots,\binom{n}{r}\}$, $(n,e) \not\to_r (m,f)$.

In [6] we showed that for $r = 2$ there are infinitely many absolutely avoidable pairs and amongst others constructed an infinite family of absolutely avoidable pairs of the form $(m, \binom{m}{2}/2)$ and showed that for any sufficiently large m, there exists an f such that (m, f) is absolutely avoidable. Here, we extend this result to higher uniformities:

Theorem 2. *Let $r \geq 3$. Then there exists m_0 such that that for any $m \geq m_0$ either $(m, \lfloor\binom{m}{r}/2\rfloor)$ or $(m, \lfloor\binom{m}{r}/2\rfloor - m - 1)$ is absolutely avoidable.*

In [12] it was further claimed that "almost all pairs" have $\sigma_2(m, f) = 0$. Here we prove the following:

Proposition 1. *For $r, m \in \mathbb{N}$, $r, m \geq 3$, all but $O\left(m^{\frac{r}{r-1}}\right)$ of all possible $\binom{m}{r}$ pairs (m, f) satisfy $\sigma_r(m, f) = 0$.*

As seen in Theorem 1, for $r = 2$ there exist pairs with $\sigma_2(m, f) = 1$. This changes for $r \geq 3$, as seen in Proposition 2 below, for which we need some additional definitions and notation. An r-graph G is called l-partite if the vertex set can be partitioned into l parts $V(G) = V_1 \cup \cdots \cup V_l$, such that for any edge $e \in E(G)$ and $i \in [l]$ we have $|e \cap V_i| \leq 1$. By $T_r(n, l)$ we denote the complete l-partite r-graph with n vertices and part sizes $n_1, \ldots, n_l \in \{\lfloor\frac{n}{l}\rfloor, \lceil\frac{n}{l}\rceil\}$. Note that for $l < r$, $T_r(l, n)$ is empty, and for $r = 2$ this is just the Turán graph. The number of edges in $T_r(n, l)$ is denoted by $t_r(n, l)$ and for $l \geq r$ we have

$$t_r(n, l) = \sum_{S \in \binom{[l]}{r}} \prod_{i \in S} n_i = \frac{(l)_r}{l^r}\binom{n}{r} + o(n^r),$$

where $(l)_r = \prod_{i=0}^{r-1}(l - i) = l(l-1)\cdots(l-r+1)$. Note that for $r \geq 2$, $\frac{(l)_r}{l^r} = \frac{(l-1)_r}{(l-1)^r}(1-\frac{1}{l})^r\frac{l}{l-r}$, and by Bernoulli's inequality we have $(1-\frac{1}{l})^r > 1-\frac{r}{l}$, i.e. $\frac{(l)_r}{l^r} > \frac{(l-1)_r}{(l-1)^r}$, so $\frac{(l)_r}{l^r}$ is strictly increasing in l, and $\lim_{l\to\infty}\frac{(l)_r}{l^r} = 1$. Also note that we have $\frac{(r)_r}{r^r} = \frac{r!}{r^r} \leq \frac{1}{r}$.

Let $l_{m,r}$ be the largest $l \in \mathbb{N}$ for which $t_r(m, l) < \frac{1}{2}\binom{m}{r}$. Note that this is well-defined by the previous observation, in particular, $l_{m,r} \geq r$. For example, one can verify that for $r = 3$, we have $l_{m,r} = \begin{cases} 3, & 4 \leq m \leq 11, \\ 4, & 12 \leq m \leq 72, \\ 5, & m \geq 73. \end{cases}$

Proposition 2. *Let $m > r \geq 3$, $0 \leq f \leq \binom{m}{r}$. Then $\sigma_r(m, f) < 1$.*

In particular, we can give the following upper bounds on σ_r for any $l \in \mathbb{N}$ satisfying $l \leq l_{m,r}$:

(a) *We have* $\sigma_r(m, f) \le 1 - \frac{(l)_r}{l^r}$.
(b) *If* $t_r(m, l) < f < \binom{m}{r} - t_r(m, l)$ *for some* l, *then* $\sigma_r(m, f) < 1 - 2\frac{(l)_r}{l^r}$.

Note that He, Ma and Zhao [13] mentioned in their conclusion without proof, that for pairs (m, f) with $m > r \ge 3$, $0 \le f \le \binom{m}{r}$, the bound $\sigma_r(m, f) \le 1 - \frac{r!}{r^r}$ holds.

For some other results concerning sizes of induced subgraphs (of 2-graphs), see for example Alon and Kostochka [2], Alon, Balogh, Kostochka, and Samotij [1], Alon, Krivelevich, and Sudakov [3], Axenovich and Balogh [4], Bukh and Sudakov [9], Kwan and Sudakov [14, 15] and Narayanan, Sahasrabudhe, and Tomon [18]. A similar question on avoidable order-size pairs was considered by Caro, Lauri, and Zarb [10] for the class of line graphs.

In Section 2 of this paper we will build on one of the proof ideas used in [6] for 2-graphs, and extend these methods to higher uniformities in order to prove Theorem 2. In Section 3 we will make some observations on the r-density σ_r and prove Proposition 1 and Proposition 2.

2. Existence of absolutely avoidable pairs

For a positive real number x, let $[x] = \{0, 1, \dots, \lfloor x \rfloor\}$. We will call an r-graph G *m-sparse* if every subset of m vertices in G induces at most m edges. We denote the complete r-graph or clique on n vertices by $K_n^{(r)}$. We call an r-graph with at most m edges an *$\le m$-edge ("at most m-edge") r-graph.*

In order to prove results in the 2-uniform case, the following fact is used in [6, 12]:

Let $m > 0$ be given. Then for any v large enough there exists a graph of girth at least m on v vertices with $v^{1+\frac{1}{2m}}$ edges.

For a probabilistic proof of this fact see for example Bollobás [8] and for an explicit construction see Lazebnik et al. [16].

The proof of the first lemma follows a standard probabilistic argument:

Lemma 1. *Let $m > 0$, $r \ge 2$ be given. Then for any n large enough there exists an n-vertex r-graph with $\Omega(n^{r-1+\frac{1}{m+1}})$ edges which is m-sparse.*

Proof. Let $c_{m,f,r} = \left(\frac{m}{e}\right)^{\frac{m}{f}} \frac{r!f}{m^r e 2^{1/f}}$. Consider a random r-graph $G \in G_r(n, p)$, for $p < c_{m,r,f} n^{-m/f}$. Then the probability that some m-subset contains at least f edges is less than

$$\binom{n}{m}\binom{\binom{m}{r}}{f} p^f \le \left(\frac{ne}{m}\right)^m \left(\frac{m^r ep}{r!f}\right)^f < \frac{1}{2}.$$

Now let X be the number of edges in G. Using the computations above and the standard bound $\binom{n}{r} \geq \left(\frac{n}{r}\right)^r$, we have

$$\mathbb{E}[X] = \binom{n}{r} p \geq c_{m,f,r} \frac{n^{r-m/f}}{r^r}.$$

Using Chernoff's bound for $\mathrm{Bin}(n,p)$ distributed random variables, we obtain that for $\delta \in (0,1)$ the probability that G has fewer than a $(1-\delta)$-fraction of the expected number of edges is

$$\mathbb{P}(X \leq (1-\delta)\mathbb{E}(X)) \leq \exp\left(-\tfrac{\delta^2}{2}\mathbb{E}[X]\right) \leq \exp\left(-c_{m,f,r}\tfrac{\delta^2 n^{r-\frac{m}{f}}}{2r^r}\right) < \frac{1}{2},$$

where the last inequality holds for $\frac{m}{f} \leq r$ and n sufficiently large. Thus, there exists an r-graph on n vertices with at least $(1-\delta)\binom{n}{r}p$ edges in which each m-subset spans at most $f-1$ edges.

Note that, by choosing $f = m+1$, we obtain the existence of an r-graph with $c'_{r,f,m} n^{r-1+\frac{1}{m+1}}$ hyperedges and no m-subset which spans more than m hyperedges, i.e. an m-sparse graph. $\square$

The next two lemmata show that for many possible order-size pairs (n,e) we can find an r-graph which realises this pair and has a "nice", i.e. easy to analyse structure. We will use them repeatedly throughout the paper.

Lemma 2. *Let $p, r \in \mathbb{N}$, $p, r \geq 2$, and c be a constant, $0 \leq c < 1$. Then for $n \in \mathbb{N}$ sufficiently large and any $e \in [c\binom{n}{r}]$, there exists a non-negative integer k and an r-graph on n vertices and e edges which is the vertex disjoint union of a $K_k^{(r)}$ and a p-sparse r-graph on $n-k$ vertices.*

Proof. Let $p, r > 0$ be given and let n be a given sufficiently large integer. Let $e \in [c\binom{n}{r}]$. Let k be the non-negative integer such that $\binom{k}{r} \leq e \leq \binom{k+1}{r} - 1$. Note that since $e \leq c\binom{n}{r}$, $\binom{k}{r} \leq c\binom{n}{r}$, and thus, $k \leq \sqrt[r]{c}n + 1 \leq c'n$, where c' is a constant with $c' < 1$. We claim that the pair (n,e) could be represented as the vertex disjoint union of a $K_k^{(r)}$ and a p-sparse r-graph.

Let G' be a p-sparse graph on $n-k$ vertices with exactly $e - \binom{k}{r}$ edges. Lemma 1 guarantees the existence of such a graph, since

$$e - \binom{k}{r} < \binom{k+1}{r} - \binom{k}{r} = \binom{k}{r-1} \overset{k \leq c'n}{\leq} (n-k)^{r-1+\frac{1}{p+1}}.$$

Now let G be the vertex disjoint union of $K_k^{(r)}$ and G. $\square$

Lemma 3. *Let $p, r \in \mathbb{N}$, $p, r \geq 2$, and c be a constant, $0 < c \leq 1$. Then for $n \in \mathbb{N}$ sufficiently large and any integer e with $c\binom{n}{r} \leq e \leq \binom{n}{r}$, there exists a non-negative integer $k \leq n$ and an r-graph on k vertices and e edges which is the complement of a p-sparse r-graph.*

Proof. Note that adding isolated vertices to the complement of an m-sparse graph results in the complement of the vertex disjoint union of a clique and an m-sparse graph. Thus, the statement immediately follows from Lemma 2 by taking complements. $\square$

Lemma 4. *If for some integers m, r, f with $m \geq r \geq 2$ and $0 \leq f \leq \binom{m}{r}$ neither (m, f) nor $(m, \binom{m}{r} - f)$ can be realised as an r-graph which is the vertex disjoint union of a complete r-graph and an $\leq$$m$-edge r-graph, then the pair (m, f) is absolutely avoidable.*

Proof. Assume we can realise neither (m, f) nor $(m, \binom{m}{r} - f)$ as the vertex disjoint union of a complete r-graph and an $\leq$$m$-edge r-graph.

By the previous lemma, for n sufficiently large and any $e \leq \lceil \binom{n}{r}/2 \rceil$, there exists an r-graph G with e hyperedges which is the vertex disjoint union of a clique and an r-graph which is m-sparse. In particular, for every $e \in \{0, 1, \ldots, \binom{n}{r}\}$, there is an r-graph G on n vertices with e edges, such that either G or its complement is the vertex disjoint union of a clique and an m-sparse r-graph.

If G is the union of a clique and an m-sparse r-graph, then clearly $G \not\to_r (m, f)$, since (m, f) cannot be realised as the union of a clique and an $\leq$$m$-edge r-graph.

If $\bar{G}$ is the union of a clique and an m-sparse r-graph, then any induced r-graph on m vertices is the complement of the vertex disjoint union of a clique and an $\leq$$m$-edge r-graph. Since $(m, \binom{m}{r} - f)$ cannot be realised as the union of a clique and an $\leq$$m$-edge r-graph, the pair $(m, f) = (m, \binom{m}{r} - (\binom{m}{r} - f))$ cannot be realised by a graph whose complement is the union of a clique and an $\leq$$m$-edge r-graph. Thus, $G \not\to_r (m, f)$. $\square$

In the 2-uniform case we used a slightly stronger statement (i.e. no m-subset spans more than $m - 1$ edges), to find absolutely avoidable pairs. For $r > 2$, it suffices to find pairs (m, f), which cannot be realised as the vertex disjoint union of a clique $K_x^{(r)}$ and an $\leq$$m$-edge r-graph.

Good candidates for such pairs (m, f) are again, as in the 2-uniform case, pairs which look roughly like $(m, \binom{m}{r}/2 + o(1))$.

We will use the following Lemmata several times:

Lemma 5. *Let $r \geq 2, m, f$ be integers with $m \geq r$, $0 \leq f \leq \binom{m}{r}$. If for some $k \in \mathbb{N}$, $\binom{k}{r} + m < f < \binom{k+1}{r}$, then the pair (m, f) cannot be realised* *as an r-graph which is the vertex disjoint union of a clique and an $\leq m$-edge hypergraph.*

Proof. Assume (m, f) is realised as $K_l^{(r)} + H$, where $l \geq 0$ and H is an $\leq m$-edge r-graph. Then from the lower bound on f, we have that $l > k$, and from the upper bound on f, we see that $l < k+1$. Thus, no such l exists. $\square$

Lemma 6. *Let $r \geq 2, m, f$ be integers with $m \geq r$, $0 \leq f \leq \binom{m}{r}$. If for some $k \in \mathbb{N}$, $\binom{k-1}{r} < f < \binom{k}{r} - m$, then the pair (m, f) cannot be realised as an r-graph which is the union of the complement of an $\leq m$-edge hypergraph and some isolated vertices.*

Proof. Assume (m, f) is realised as $K_l^{(r)} - H$, where $l \geq 0$ and H is an $\leq m$-edge r-graph, and some isolated vertices. Then from the upper bound on f, we have that $l < k$, and from the lower bound on f, we see that $l > k - 1$. Thus, no such l exists. $\square$

Proof of Theorem 2. Let $r \geq 3$, $m \geq m_0$ and let $f_0 = \lfloor \binom{m}{r}/2 \rfloor$.

Using Lemma 4, we need to show that either (m, f_0) or both $(m, f_0 - (m+1))$ and $(m, \binom{m}{r} - f_0 + (m+1))$ are not realisable as the vertex disjoint union of a clique and an $\leq m$-edge r-graph. To this end we will show that the condition of Lemma 5 is satisfied.

Let x be an integer such that $\binom{x}{r} \leq \lfloor \binom{m}{r}/2 \rfloor < \binom{x+1}{r}$. By standard bounds we observe the following:

$$\frac{1}{2}\left(\frac{m}{r}\right)^r \leq \left\lfloor \frac{1}{2}\binom{m}{r} \right\rfloor < \binom{x+1}{r} < \left(\frac{(x+1)e}{r}\right)^r,$$

and thus,

$$x + 1 > \frac{1}{2^{1/r}e} m.$$

Thus, by choosing m_0 sufficiently large, we have for $m \geq m_0$ that $x - 1 > \frac{m}{4}$, since for $r \geq 3$ $\sqrt[r]{2}e \geq \sqrt[3]{2}e > 4$, and

$$(*) \qquad \binom{x-1}{r-1} \geq \left(\frac{x-1}{r-1}\right)^{r-1} \geq \left(\frac{m}{4(r-1)}\right)^{r-1} > 2m + 2.$$

Let $f_- = \lfloor \binom{m}{r}/2 \rfloor - (m+1)$ and $f_+ = \lceil \binom{m}{r}/2 \rceil + (m+1)$.

Case 1: $\binom{x}{r} + m < f_0$. Then by Lemma 5, (m, f_0) cannot be realised by $K_k^{(r)} + H$, where $k \in \mathbb{N}$ and H has at most m edges. If $\lceil \binom{m}{r}/2 \rceil < \binom{x+1}{r}$, then again by Lemma 5, $(m, \binom{m}{r} - f_0)$ cannot be realised as the disjoint union of a clique and an $\leq m$-edge r-graph, i.e. by Lemma 4, (m, f_0) is absolutely avoidable.

Otherwise, we have $\lceil \binom{m}{r}/2 \rceil = \binom{x+1}{r} = \lfloor \binom{m}{r}/2 \rfloor + 1$. We clearly have $f_- < \binom{x+1}{r}$ and $f_+ > \binom{x+1}{r} + m$, so it remains to show that $f_- > \binom{x}{r} + m$ and $f_+ < \binom{x+2}{r}$. Indeed, we have

$$f_- - \binom{x}{r} = \binom{x+1}{r} - m - 1 - \binom{x}{r} = \binom{x}{r-1} - (m+1) \overset{(*)}{>} 2m+1-m-1 = m,$$

i.e. $f_- > \binom{x}{r} + m$, and also

$$f_+ = \binom{x+1}{r} + m + 1 < \binom{x+1}{r} + 2m + 1 \overset{(*)}{<} \binom{x+1}{r} + \binom{x-1}{r-1} < \binom{x+2}{r},$$

and thus, $f_+ < \binom{x+2}{r}$.

Case 2: $\binom{x}{r} \leq \lfloor \binom{m}{r}/2 \rfloor \leq \binom{x}{r} + m$.

It remains to check that neither (m, f_-) nor (m, f_+) can be realised as the vertex disjoint union of a clique and an $\leq m$-edge r-graph. On the one hand we have $f_- \leq \binom{x}{r} + m - (m+1) < \binom{x}{r}$. Thus, in order to use Lemma 5, it remains to verify that we also have $f_- > \binom{x-1}{r} + m$. Indeed, we have

$$\begin{aligned} f_- - \binom{x-1}{r} &\geq \binom{x}{r} - (m+1) - \left(\binom{x}{r} - \binom{x-1}{r-1} \right) = \binom{x-1}{r-1} - (m+1) \\ &\overset{(*)}{>} 2m+1-m-1 = m \end{aligned}$$

for $m \geq m_0$, i.e. we have $\binom{x-1}{r} + m < f_- < \binom{x}{r}$, so by Lemma 5, (m, f_-) cannot be realised as the vertex disjoint union of a clique and an $\leq m$-edge r-graph.

On the other hand, we clearly have $f_+ > \binom{x}{r} + m$, and also,

$$f_+ \leq \binom{x}{r} + m + 1 + (m+1) \overset{(*)}{<} \binom{x}{r} + \binom{x-1}{r-1} < \binom{x}{r} + \binom{x}{r-1} = \binom{x+1}{r},$$

so by Lemma 5, (m, f_+) cannot be realised as $K_k^{(r)} + H$, where $k \in \mathbb{N}$ and H is an $\leq m$-edge r-graph.

Thus, by Lemma 4 the pair (m, f_-) is absolutely avoidable. $\square$

3. Density observations

Let $r \geq 3$, m, $f \leq \binom{m}{r}$. Recall from the introduction that the density is defined as

$$\sigma_r(m,f) = \limsup_{n\to\infty} \frac{|\{e : (n,e) \to (m,f)\}|}{\binom{n}{r}}.$$

By considering complementary pairs, it immediately follows that $\sigma_r(m,f) = \sigma_r(m, \binom{m}{r} - f)$. Recall that for a family of r-graphs $\mathcal{G}$, $\mathrm{ex}_r(n,\mathcal{G})$ denotes the extremal number, i.e. the maximum number of edges an r-graph on n vertices can have without containing any member of $\mathcal{G}$. For an r-graph H, the *Turán-density* is defined as $\pi_r(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\{H\})}{\binom{n}{r}}$.

Note that for $f = 0$, σ_r corresponds to the Turán density, i.e. $\sigma_r(m,0) = \sigma_r(m,\binom{m}{r}) = \pi_r(K_m^{(r)})$, where the currently best known general bounds on the Turán density are

$$1 - \left(\frac{r-1}{m-1}\right)^{r-1} \leq \pi(K_m^r) \leq 1 - \binom{m-1}{r-1}^{-1},$$

due to Sidorenko [19] and de Caen [11]. Also note that $\sigma_r(r,1) = \sigma_r(r,0) = 1$. Thus, the only non-trivial cases are $m > r$, which are dealt with in Proposition 1 and Proposition 2.

Before we prove Proposition 1, we show the following auxiliary lemma:

Lemma 7. *Let $m,r,f \in \mathbb{N}$ with $m \geq r \geq 3$ and $0 \leq f \leq \binom{m}{r}$.*

(a) If (m,f) cannot be realised as the disjoint union of a clique and an $\leq m$-edge r-graph, then $\sigma_r(m,f) = 0$. In particular, if there is no $x \in [m]$, such that $0 \leq f - \binom{x}{r} < m$, then $\sigma_r(m,f) = 0$.

(b) If (m,f) cannot be realised as the complement of an $\leq m$-edge r-graph and some isolated vertices, then $\sigma_r(m,f) = 0$. In particular, if there is no $x \in [m]$, such that $0 \leq \binom{x}{r} - f < m$, then $\sigma_r(m,f) = 0$.

(c) If $\sigma_r(m,f) > 0$, then there exist $x, \bar{x} \in [m]$ such that $0 \leq f - \binom{x}{r} < m$ and $0 \leq (\binom{m}{r} - f) - \binom{\bar{x}}{r} < m$.

(d) If for some $l \in \mathbb{N}$ we have $\binom{l}{r-1} > 2m$, then for $f > \binom{l}{r}$ and $f \neq \binom{x}{r}$ for $x \in [m]$ we have $\sigma_r(m,f) = 0$.

Proof. (a) By Lemma 2, for any $0 < c' < 1$, n sufficiently large, and $e \in \mathcal{E}_n := \left[c'\binom{n}{r}\right]$ there exists an r-graph G on n vertices with e edges which is the vertex disjoint union of a clique and an m-sparse r-graph. Note that any induced subgraph on m vertices of G is the union of a clique and an r-graph with at most m edges. Thus, by definition of σ_r, if a pair (m,f) cannot be realised by a clique and an $\leq m$-edge r-graph, we have

$$\sigma_r(m,f) = \limsup_{n\to\infty} \frac{|\{e : (n,e) \to (m,f)\}|}{\binom{n}{r}} \leq \limsup_{n\to\infty} \frac{|\binom{[n]}{r} - \mathcal{E}_n|}{\binom{n}{r}} = 1 - c'.$$

Letting c' go to one proves the statement.
If there is no $x \in [m]$, such that $0 \leq f - \binom{x}{r} \leq \binom{m}{r}$, then the pair (m, f) cannot be realised as the union of a clique and an $\leq$$m$-edge r-graph. Thus, in this case we have $\sigma_r(m, f) = 0$.

(b) By Lemma 3, for any $0 < c' < 1$, n sufficiently large, and $e \in \mathcal{E}_n := \left[\binom{n}{r}\right] - \left[c'\binom{n}{r}\right]$ there exists an r-graph G on n vertices with e edges which is the complement of an m-sparse r-graph and some isolated vertices.
Note that any induced subgraph on m vertices of G is the union of a clique with at most m edges removed and an empty graph. Thus, by definition of σ_r, if a pair (m, f) cannot be realised as the complement of an $\leq$$m$-edge r-graph and some isolated vertices, we have

$$\sigma_r(m, f) = \limsup_{n\to\infty} \frac{|\{e : (n, e) \to (m, f)\}|}{\binom{n}{r}} \leq \limsup_{n\to\infty} \frac{|\binom{[n]}{r} - \mathcal{E}_n|}{\binom{n}{r}} = c'.$$

The statement follows by letting c' go to zero.
The "in particular" part follows similarly as in part (a).

(c) The first part is the contrapositive of the "in particular" statement in part (a). The second statement follows trivially using $\sigma_r(m, f) = \sigma_r(m, \binom{m}{r} - f)$.

(d) Let $f > \binom{l}{r}$, $f \neq \binom{x}{r}$ for $x \in [m]$, and let t be the unique integer satisfying $\binom{t}{r} < f < \binom{t+1}{r}$. Since $\binom{l}{r-1} > 2m$, it implies that $\binom{t+1}{r} - \binom{t}{r} = \binom{t}{r-1} \geq \binom{l}{r-1} > 2m$. Thus, we either have $f > \binom{t}{r} + m$ or $\binom{t+1}{r} - m > f$. In particular, by part (a) or (b), we have $\sigma_r(m, f) = 0$.
Note: The results by Axenovich, Balogh, Clemen and the author in [5] imply that the condition $f > \binom{l}{r}$ might not be needed. It is shown there for $r = 3$. $\square$

Proof of Proposition 1. Let m be fixed and $f \leq \binom{m}{r}$; write f uniquely as $f = \binom{l}{r} + l'$, where $l \in [m]$ and $0 \leq l' < \binom{l}{r-1}$. By Lemma 7(d) it follows that we have $\sigma_r(m, f) = 0$ if $\binom{l}{r} > 2m$ and $l' > 0$. In particular, any pair (m, f) with $\sigma_r(m, f) > 0$ must satisfy either $l' = 0$ or $\binom{l}{r} \leq 2m$. In the first case, there are exactly $m + 1$ possible choices for f (i.e. $f = \binom{x}{r}$ for some $x \in \{0, \ldots, m\}$). In the second case, we obtain that $\left(\frac{l}{r}\right)^{r-1} \leq \binom{l}{r-1} \leq 2m$, i.e. $l \leq (2m)^{1/r-1}r$, i.e. $f \leq \binom{l}{r} \leq \left(\frac{el}{r}\right)^r \leq e^r(2m)^{r/r-1}$. Thus, at most $(m + 1) + e^r(2m)^{r/r-1} \in O\left(m^{\frac{r}{r-1}}\right)$ of all possible pairs (m, f) satisfy $\sigma_r(m, f) > 0$. Note that for $r \geq 3$, we have $m^{\frac{r}{r-1}} \in o(\binom{m}{r})$. $\square$

For the proof of Proposition 2 we will use the following extension of Turán's theorem to hypergraphs by Mubayi [17]. For fixed $l, r \geq 2$ let $\mathcal{F}_l^{(r)}$ be the family of r-graphs with at most $\binom{l}{2}$ edges, that contain a *core* S of l vertices, such that every pair of vertices in S is contained in an edge.

Theorem 3 (Mubayi [17]). *Let $r, l, n \geq 2$. Then*

$$\mathrm{ex}(n, \mathcal{F}_{l+1}^{(r)}) = t_r(n, l)$$

and the unique r-graph on n vertices containing no copy of any member of $\mathcal{F}_{l+1}^{(r)}$ for which equality holds is $T_r(n, l)$, the complete balanced l-partite r-graph on n vertices.

Proof of Proposition 2. Now let $m > r \geq 2$, $0 \leq f \leq \binom{m}{r}$.

Let $l \in \mathbb{N}$, such that $t_r(m, l) < \frac{1}{2}\binom{m}{r}$. Note that for $r \geq 3$, such an l always exists, since we have $t_r(m, r) < \frac{1}{2}\binom{m}{r}$, so we can always choose $l = r$.

Thus, in particular, we are in one of two cases: Either we have $f \geq \frac{1}{2}\binom{m}{r} > t_r(m, l)$, or we have $f \leq \frac{1}{2}\binom{m}{r}$, i.e. $f - \binom{m}{r} \geq \frac{1}{2}\binom{m}{r} > t_r(m, l)$.

Case 1: $f > t_r(m, l)$. Then by Theorem 3, any r-graph that realises the pair (m, f) contains a member of $\mathcal{F}_{l+1}^{(r)}$. If $e \leq t_r(n, l)$, we have $(n, e) \not\to (m, f)$, since taking any subgraph of $T_r(n, l)$ with e edges yields an (n, e) graph not containing any member of $\mathcal{F}_{l+1}^{(r)}$, and thus, a graph not containing induced (m, f). In particular, this implies that

$$\sigma_r(m, f) \leq \lim_{n\to\infty} \frac{\binom{n}{r} - t_r(n, l)}{\binom{n}{r}} < 1.$$

Case 2: $\binom{m}{r} - f > t_r(m, l)$. Then by Theorem 3 any graph that realises $(m, \binom{m}{r} - f)$ contains a member of $K_{l+1}^{(r)}$. Then any graph G with $G \to_r (m, \binom{m}{r} - f)$ must contain a member of $\mathcal{F}_{l+1}^{(r)}$, i.e. $|E(G)| > t_r(n, l)$. Thus, for each $e \leq t_r(n, l)$, $(n, e) \not\to_r (m, \binom{m}{r} - f)$, and thus, by considering the complement, for each $e \geq \binom{n}{r} - t_r(n, l)$, $(n, e) \not\to_r (m, f)$. In particular, we have

$$\sigma_r(m, f) \leq \lim_{n\to\infty} \frac{\binom{n}{r} - t_r(n, l)}{\binom{n}{r}} < 1.$$

Thus, in either case, we have

$$\sigma_r(m, f) \leq 1 - \limsup_{n\to\infty} \frac{t_r(n, l)}{\binom{n}{r}} = 1 - \frac{(l)_r}{l^r} < 1.$$

This proves part (a).

To obtain part (b), assume that for some $l \in \mathbb{N}$ we have $t_r(m,l) < f < \binom{m}{r} - t_r(m,l)$. Then by Cases 1 and 2, we see that $(n,e) \to_r (m,f)$ requires $t_r(n,l) < e < \binom{n}{r} - t_r(n,l)$. Thus, we obtain that

$$\sigma_r(m,f) \leq 1 - 2\limsup_{n\to\infty} \frac{t_r(n,l)}{\binom{n}{r}} = 1 - 2\frac{(l)_r}{l^r},$$

which completes the proof. □

Corollary 1. *For $r = 3$, $m > 3$, $0 < f < \binom{m}{3}$, we have the following upper bounds on $\sigma_3(m,f)$:*

1. $\sigma_3(m,f) \leq \frac{7}{9}$.
2. *If $t_3(m,3) < f < \binom{m}{r} - t_3(m,3)$, then $\sigma_3(m,f) \leq \frac{5}{9}$.*
3. *If $m \geq 12$, then $\sigma_3(m,f) \leq \frac{5}{8}$.*
4. *If $m \geq 73$, then $\sigma_3(m,f) \leq \frac{13}{25}$.*

For $r = 4$, $m > 4$, $0 < f < \binom{m}{4}$, we have the following upper bounds on σ_4:

1. $\sigma_4(m,f) \leq \frac{29}{32}$,
2. *There is m_0, such that for all $m \geq m_0$ we have $\sigma_4(m,f) \leq \frac{131}{243} \approx 0.54$.*

Proof. We start with $r = 3$. In order to obtain our bounds, we can compute the fraction $\frac{(l)_3}{l^3}$ for different $l \geq 3$. We have that

$$\frac{(3)_3}{3^3} = \frac{2}{9}, \qquad \frac{(4)_3}{4^3} = \frac{3}{8}, \qquad \frac{(5)_3}{5^3} = \frac{12}{25}, \qquad \frac{(6)_3}{6^3} = \frac{5}{9}.$$

Note that $\frac{(6)_3}{6^3} > \frac{1}{2}$, so for $r = 3$, the best possible upper bound one can achieve for any pair using Proposition 2 will use $l = 5$.

Now (1) and (2) immediately follow from Proposition 2, by setting $l = r = 3$ and observing that for $r \geq 3$, we always have $t_r(m,r) < \frac{1}{2}\binom{m}{r}$. Then

$$\sigma_3(m,f) \leq 1 - \lim_{n\to\infty} \frac{t_3(n,3)}{\binom{n}{3}} = 1 - \lim_{n\to\infty} \left(\frac{(l)_r}{l^r} \frac{\binom{n}{r}}{\binom{n}{r}} + \frac{o(n^r)}{\binom{n}{r}} \right) = 1 - \frac{(3)_3}{3^3} = \frac{7}{9},$$

and for (2), if $t_3(m,3) < f < \binom{m}{r} - t_3(m,3)$,

$$\sigma_3(m,f) \leq 1 - 2\lim_{n\to\infty} \frac{t_3(n,3)}{\binom{n}{3}} = \frac{5}{9}.$$

Now for (3) note that for $m \geq 12$, we have $l_{m,3} = 4$, and thus, by Proposition 2(a), for all pairs (m, f) with $m \geq 12$ we have

$$\sigma_3(m,f) \leq 1 - \left(\frac{(4)_3}{4^3}\right) = 1 - \frac{6}{16} = \frac{5}{8}.$$

For (4) note that for $m \geq 73$, we have $l_{m,3} = 5$, so using Proposition 2(a), for all pairs (m, f) with $m \geq 73$ we have

$$\sigma_3(m,f) \leq 1 - \left(\frac{(5)_3}{5^3}\right) = 1 - \frac{12}{25} = \frac{13}{25}.$$

For the case $r = 4$, we obtain the first part by computing $\frac{(4)_4}{4^4} = \frac{3}{32}$. The second part is obtained by computing $\frac{(9)_4}{9^4} = \frac{112}{243}$ and noting that $l_{m,4} = 9$ for $m \geq m_0$. $\square$

4. Concluding remarks

We have shown that for $r \geq 3$ and m sufficiently large there always exists an f such that (m, f) is absolutely r-avoidable; however, in the cases considered f is always roughly $\binom{m}{r}/2$. This inspires the following interesting question:

Question. Are there absolutely avoidable pairs where $f/\binom{m}{r}$ is bounded away from $\frac{1}{2}$ in the limit?

We have proven that for $m > r \geq 3$ and f with $0 \leq f \leq \binom{m}{r}$, we always have $\sigma_r(m, f) < 1$. We have also shown that for fixed r, most pairs (m, f) satisfy $\sigma_r(m, f) = 0$. On the other hand, for $r \geq 3$ there is no pair (m, f) with $0 < f < \binom{m}{r}$ for which we can show that $\sigma_r(m, f) > 0$. This inspires the following question:

Open problem. For $m > r \geq 3$, are there any f with $0 < f < \binom{m}{r}$ such that $\sigma_r(m, f) > 0$?

Next, we will use Lemma 7 to identify candidate pairs (m, f) for $r = 3, m \leq 15$ which might satisfy $\sigma_3(m, f) > 0$:

Lemma 8. *Let $r = 3$, $4 \leq m \leq 15$ and $0 < f < \binom{m}{r}$. If $(m, f) \neq (6, 10)$, then $\sigma_3(m, f) = 0$.*

Proof. Let (m, f) be a pair with $4 \leq m \leq 15$, $0 < f < \binom{m}{r}$ with $\sigma_3(m, f) > 0$. One can verify that for $4 \leq m \leq 15$, we have $2\binom{m-3}{3} + 4 \leq \binom{m}{3}$, i.e. $\binom{m-3}{3} + 2 \leq \binom{m}{3} - \binom{m-3}{3} - 2$. Thus, for any $f \leq \binom{m}{3}$, we either have $f \geq$

$\binom{m-3}{3}+2$ or $\binom{m}{3}-f \geq \binom{m-3}{3}+2$, assume w.l.o.g. that we have $f \geq \binom{m-3}{3}+2$. Let x be the unique value with $\binom{x}{3} \leq f < \binom{x+1}{3}$ and write $f = \binom{x}{3} + x'$, $0 \leq x' < \binom{x}{2}$. Then by assumption we have $x \in \{m-3, m-2, m-1\}$.

By Lemma 7(a) we know that if we cannot realise the pair (m, f) as the vertex disjoint union of a clique and an $\leq m$-edge 3-graph, then we have $\sigma_3(m, f) = 0$. Thus, we can assume that the pair (m, f) can be realised as the vertex disjoint union of a clique and an $\leq m$-edge 3-graph. By Lemma 5, it follows that (m, f) can be realised as the disjoint union of $K_x^{(3)}$ and an $\leq m$-edge 3-graph.

Assume we have $x = m - 3$. Since by assumption $f \geq \binom{x}{3} + 2$, clearly the pair (m, f) cannot be realised as the vertex disjoint union of $K_x^{(3)}$ and a graph on at most 1 edge. Since $m - x = 3$, it also cannot be realised as the vertex disjoint union of $K_x^{(3)}$ and a graph with 2 edges, a contradiction. Thus, for $x = m - 3$ we have $\sigma_3(m, f) = 0$.

So assume $x \in \{m-2, m-1\}$. Note that since $m - x \leq 2$, the pair (m, f) cannot be realised as the vertex disjoint union of a clique on x vertices and a graph with at least 1 edge. Thus, the only pairs (m, f) which might satisfy $\sigma_3(m, f) > 0$ have $x' = 0$, i.e. $f \in \{\binom{m-2}{3}, \binom{m-1}{3}\}$. Then $(m, f) \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{(6, 10), (10, 84), (13, 165), (15, 169), (15, 91)\}$ with

$$\mathcal{A}_1 = \{(5,4), (7,20), (8,35), (9,56), (11,120), (12,165), (13,220), (14,286)\},$$
$$\mathcal{A}_2 = \{(5,1), (6,4), (7,10), (8,20), (9,35), (10,56), (11,84), (12,120), (14,220)\}.$$

Now let $(m, f) \in \mathcal{A}_1 \cup \mathcal{A}_2$. Let $\bar{f} = \binom{m}{3} - f$ and let $y \in [m]$ such that $\binom{y}{3} \leq \bar{f} < \binom{y+1}{3}$, i.e. $\bar{f} = \binom{y}{3} + y'$ for some $y' \leq \binom{y}{2}$. Then it is easy to verify that we are in one of three cases:

- $y \in \{m-1, m-2\}$ and $y' > 0$,
- $y = m - 3$ and $y' > 1$,
- $y \leq m - 4$ and $y' > m$.

In each case, by Lemma 5 $(m, \bar{f})$ canot be realised as the disjoint union of a clique and an $\leq m$-edge 3-graph, and thus, by Lemma 7(a), $\sigma_3(m, f) = \sigma_3(m, \bar{f}) = 0$.

The pair $(6, 10)$ is self-complementary with $10 = \binom{5}{3} = \binom{6}{3}/2$.

For the pair $(10, 84)$ we have $\binom{10}{3} - 84 = 36 = \binom{7}{3} + 1 = \binom{10}{3} - \binom{9}{3}$. Note that for $m = 10$, we have $2m < \binom{7}{2}$, i.e. by Proposition 1, $\sigma_3(10, 36) = 0$.

For the pair $(13, 165)$ we have $\binom{13}{3} - 165 = 121 = \binom{10}{3} + 1 = \binom{13}{3} - \binom{11}{3}$. Note that for $m = 13$, we have $2m < \binom{10}{2}$, i.e. by Proposition 1, $\sigma_3(13, 121) = 0$.

For the pair $(15, 91)$ we have $91 = \binom{15}{3} - \binom{14}{3} = \binom{9}{3} + 7$, and for the pair $(15, 169)$ we have $169 = \binom{15}{3} - \binom{13}{3} = \binom{11}{3} + 4$. Note that for $m = 15$, we have $2m < \binom{9}{2} < \binom{11}{2}$, i.e. by Proposition 1, $\sigma_3(15, 91) = \sigma_3(15, 169) = 0$. $\square$

Lemma 8 implies that for $r = 3$, the smallest value of m, for which the first open problem has no answer is $m = 6$. In this case, $f = 10$ is the only possible value for which we might have $\sigma_3(6, f) > 0$. This leads to the following sub-problem of the first open problem: What is $\sigma_3(6, 10)$?

From Corollary 1 we obtain that $\sigma_3(6, 10) \leq \frac{5}{9}$, but we can do slightly better, as can be seen by considering the following construction:

Let $G_1 = K_3^{(3)}$, i.e. the 3-graph on 3 vertices with one edge. Assume G_{k-1} has been constructed. We obtain $G_k = (V_k, E_k)$ by taking 3 copies of G_{k-1} and adding all edges using exactly one vertex from each copy of G_{k-1}, i.e. inserting the edges of a complete 3-partite graph. Thus, for G_k we have $|V_k| = 3|V_{k-1}| = 3^k$ and $|E_k| = |V_{k-1}|^3 + 3|E_{k-1}|$. One can show that $\frac{|E_k|}{\binom{|V_k|}{3}} = \frac{1}{4} - o(1)$, see for example Bárány and Füredi [7].

Thus, for sufficiently large $n = 3^k$, this gives a construction with $(\frac{1}{4} - o(1))\binom{n}{3}$ edges. It is easy to verify that every 6-set in G_n spans at most 8 edges, so every 6-set in its complement spans at least 12 edges, i.e. G_n and its complement are induced $(6, 10)$-free, so for n sufficiently large and any $e \leq (\frac{1}{4} - o(1))\binom{n}{3}$ or $e \geq (\frac{3}{4} - o(1))\binom{n}{3}$ there exists a 3-graph on n vertices with e edges, which does not arrow $(6, 10)$. Thus, $\sigma_3(6, 10) \leq \frac{1}{2}$.

Note that this construction is not the only one that shows $\sigma_3(6, 10) \leq \frac{1}{2}$: Blowing up a $C_5^{(3)}$ in the same manner as $K_3^{(3)}$ above is also induced $(6, 10)$-free and achieves the same bound.

Very recently it was shown by Axenovich, Balogh, Clemen and the author [5] that indeed we have $\sigma_3(6, 10) > 0$. They also found a better construction which shows that $\sigma_3(6, 10) \leq$.

It would be interesting to further investigate this problem, as currently there is no other known pair (m, f) for $r \geq 3$ which satisfies $\sigma_r(m, f) > 0$.

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Lea Weber
Karlsruhe Institute of Technology
Karlsruhe
Germany
E-mail address: lea.weber@kit.edu

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