Journal of Combinatorics
Volume 15, Number 2, 217–282, 2024

Sharp minimum degree conditions for disjoint doubly chorded cycles

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In 1963, Corrádi and Hajnal proved that if G is an n-vertex graph where $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles, and furthermore, the minimum degree condition is best possible for all n and k where $n \geq 3k$. This serves as the motivation behind many results regarding best possible conditions that guarantee the existence of a fixed number of disjoint structures in graphs. For doubly chorded cycles, Qiao and Zhang proved that if $n \geq 4k$ and $\delta(G) \geq \lfloor \frac{7k}{2} \rfloor$, then G contains k vertex-disjoint doubly chorded cycles. However, the minimum degree in this result is sharp for only a finite number of values of k. Later, Gould Hirohata, and Horn improved upon this by showing that if $n \geq 6k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint doubly chorded cycles. Furthermore, this minimum degree condition is best possible for all n and k where $n \geq 6k$. In this paper, we prove two results. First, we extend the result of Gould et al. by showing their minimum degree condition guarantees k disjoint doubly chorded cycles even when $n \geq 5k$, and in addition, this is best possible for all n and k where $n \geq 5k$. Second, we improve upon the result of Qiao and Zhang by showing that every n-vertex graph G with $n \geq 4k$ and $\delta(G) \geq \lceil \frac{10k-1}{3} \rceil$, contains k vertex-disjoint doubly chorded cycles. Moreover, this minimum degree is best possible for all $k \in \mathbb{Z}^+$.

AMS 2000 subject classifications: Primary 05C35, 05C38.
Keywords and phrases: Cycles, chorded cycles, doubly chorded cycles, minimum degree.

1. Introduction

All graphs in this paper are simple with no loops and no multiple edges. Given a graph G, we use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively, and for a vertex v, we often use $v \in G$ to denote

*The research of both authors was supported by the Grand Valley State University Student Summer Scholars Program.

$v \in V(G)$. For a subgraph H of G, and for a vertex $v \in G$ (where v is not necessarily in H), the neighborhood of v in H is denoted by $N_H(v)$, and the number of neighbors of v in H will be written by $d_H(v)$. We use $|G|$ for $|V(G)|$, $\overline{G}$ for the complement of G, and $\delta(G)$ for the minimum degree of G. Furthermore, $\sigma_2(G)$ denotes the minimum Ore degree of G (sometimes called the minimum degree sum), which is given by the minimum of $d_G(x) + d_G(y)$ over all non-adjacent pairs of distinct vertices x and y in G (when G is complete, we say $\sigma_2(G) = \infty$).

K_n is used to denote the complete graph on n vertices, and $K_{k_1,\ldots,k_t}$ is the complete t-partite graph with parts of size $k_1, \ldots, k_t$. Also, the Paw is the 4-vertex graph formed by adding an edge to $K_{1,3}$.

If a graph H contains a spanning cycle C and $|E(H)| > |E(C)|$, then H is called a chorded cycle, and every edge in $E(H) \setminus E(C)$ is called a chord. If a chorded cycle H has at least two chords, then we say H is a doubly chorded cycle. Lastly, two graphs are said to be 'disjoint' if they have no vertices in common.

In 1963, Corrádi and Hajnal proved the following theorem, which verified a conjecture of Erdős.

Theorem 1 (Corrádi and Hajnal [1]). *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

The condition on the number of vertices in this result is clearly best possible as every cycle requires at least three vertices. The minimum degree condition is also best possible as there exist n-vertex graphs with $n \geq 3k$ and minimum degree $2k - 1$ that do not have k disjoint cycles (see [5] for a complete characterization). In fact, for every $k \in \mathbb{Z}^+$ and every $n \geq 3k$, there exists an n-vertex graph with minimum degree $2k - 1$ that does not have k disjoint cycles. Thus, the minimum degree condition in Theorem 1 is not just best possible in general, but is actually best possible for all $n, k \in \mathbb{Z}^+$ where $n \geq 3k$.

Theorem 1 has been extended in a number of ways, and serves as the motivation behind finding best possible conditions that guarantee the existence of a fixed number of similar objects in a graph. One such extension is an analogue for chorded cycles proved by Finkel in 2008.

Theorem 2 (Finkel [2]). *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.*

The condition on the number of vertices in this result is clearly best possible as every chorded cycle requires at least four vertices. Furthermore, the minimum degree condition is also sharp, as there exist n-vertex graphs with $n \geq 4k$ and minimum degree $3k-1$ that do not contain k disjoint chorded cycles. The complete characterization of such graphs is given by the following result.

Theorem 3 (Molla, Santana, and Yeager [6]). *For all $k \in \mathbb{Z}^+$ with $k \geq 2$, if G is an n-vertex graph where $n \geq 4k$ and $\sigma_2(G) \geq 6k-2$, then G contains k disjoint chorded cycles unless either:*

- *$G \cong K_{3k-1,n-3k+1}$ with $n \geq 6k-2$, or*
- *$G \cong K_{1,3k-2,3k-2}$ where $n = 6k-3$.*

One consequence of Theorem 3 is that every n-vertex graph with minimum degree $3k-1$ that does not contain k disjoint chorded cycles, must have $n \geq 6k-3$. Therefore, the minimum degree condition in Theorem 2 is not best possible when $4k \leq n \leq 6k-4$, and it is currently unknown as to what the best possible minimum degree condition might be for n in this range. A possible answer to this is the following conjecture from [7]. As a note, the authors from [7] actually pose a more general conjecture in regards to finding both cycles and chorded cycles in a graph, and they prove an approximate version of the following conjecture as well as the more general version.

Conjecture 4 (Molla, Santana, and Yeager [7]). *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $4k \leq n \leq 6k-4$ and $\delta(G) \geq \frac{3k}{2} + \frac{n}{4}$, then G contains k disjoint chorded cycles.*

We now turn our attention to doubly chorded cycles, which is main concern of this paper. To begin, a well-known theorem by Hajnal and Szemerédi on packings of cliques yields the following.

Theorem 5 (Hajnal and Szemererédi [4]). *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n = 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint copies of K_4.*

As K_4 is the smallest doubly chorded cycle, Theorem 5 guarantees the existence of k disjoint doubly chorded cycles. Furthermore, the minimum degree condition is known to be best possible for all $k \in \mathbb{Z}^+$.

In 2010, Qiao and Zhang sought to extend this result for graphs on at least $4k$ vertices.

Theorem 6 (Qiao and Zhang [8]). *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 4k$ and $\delta(G) \geq \lfloor \frac{7k}{2} \rfloor$, then G contains k disjoint doubly chorded cycles.*

The condition on the number of vertices in this result is clearly best possible as every doubly chorded cycle requires at least four vertices. However, the only sharpness examples for the minimum degree condition are $K_{2,2}$, $K_{3,3,3}$, $K_{4,5,5}$, and $K_{8,8,8}$, which show it best possible for $k = 1, 2, 3$, and 5, respectively.

This result was later improved upon by Gould, Hirohata, and Horn in 2013, who proved the following Ore degree version and subsequent minimum degree corollary.

Theorem 7 (Gould, Hirohata, and Horn [3])**.** *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 6k$ and $\sigma_2(G) \geq 6k-1$, then G contains k disjoint doubly chorded cycles.*

Corollary 8 (Gould, Hirohata, and Horn [3])**.** *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 6k$ and $\delta(G) \geq 3k$, then G contains k disjoint doubly chorded cycles.*

The minimum degree condition in Corollary 8 is best possible for all $n, k \in \mathbb{Z}^+$ where $n \geq 6k$, and we will show this in Section 2.

The main purpose of this paper is attempt to determine the best possible minimum degree condition for n-vertex graphs with $4k < n < 6k$ that guarantees the existence of k disjoint doubly chorded cycles. In particular, we prove the following two results.

Theorem 9. *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 5k$ and $\delta(G) \geq 3k$, then G contains k disjoint doubly chorded cycles.*

Theorem 10. *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph where $n \geq 4k$ and $\delta(G) \geq \lceil \frac{10k-1}{3} \rceil$, then G contains k disjoint doubly chorded cycles.*

Theorem 9 extends the result of Gould, Hirohata, and Horn in Corollary 8 by showing that the minimum degree condition in Corollary 8 also suffices for n-vertex graphs with $n \geq 5k$. Furthermore, in Section 2 we show that this minimum degree condition is best possible for all $n, k \in \mathbb{Z}^+$ where $n \geq 5k$.

Theorem 10 improves upon the result of Qiao and Zhang in Theorem 6. In particular, in Section 2 we show that our minimum degree is best possible for all $k \in \mathbb{Z}^+$, while the minimum degree condition in Theorem 6 is sharp only when $k \in \{1, 2, 3, 5\}$.

That said, the sharpness examples that we will construct in Section 2 for Theorem 10 will all be n-vertex graphs with $n = 5k - 1$. These graphs will also demonstrate that the condition on the number of vertices in Theorem 9 is best possible for the minimum degree condition of $3k$. That is, it is impossible to replace the condition '$n \geq 5k$' in Theorem 9 with '$n \geq x$' where $x < 5k$ and still guarantee k disjoint doubly chorded cycles.

This still leaves the question as to what is the best possible minimum degree condition for n-vertex graphs with $4k < n < 5k$ that guarantees the existence of k disjoint doubly chorded cycles? We pose the following conjecture, which if true, would completely answer this question, and we prove an approximate version of this conjecture in Section 9.

Conjecture 11. *For all $k \in \mathbb{Z}^+$, if G is an n vertex graph where $4k \leq n \leq 5k$ and $\delta(G) \geq \lceil \frac{5k+n}{3} \rceil$, then G contains k disjoint doubly chorded cycles.*

The remainder of the paper is structured as follows. As mentioned, in Section 2, we construct the sharpness examples to Theorems 9 and 10. In addition, we construct graphs which show that if Conjecture 11 is true, it is best possible. The proofs of Theorems 9 and 10 are spread across Sections 3–8, and in some sense, are proved simultaneously. In Section 3, we define some notation and begin the setup of our proofs, and in Section 4, we prove several structural lemmas that are foundational to our proofs. Sections 5 and 6 deal with separate cases and culminate in a proof of Theorem 9, subject to a lemma, whose detailed proof is contained in Section 7. Theorem 10 is proved in Section 8 based on all our prior work. Lastly, we address Conjecture 11 in Section 9, and there prove an approximate version of this conjecture.

2. Sharpness examples

In this section, we construct sharpness examples which show that Theorems 9 and 10 are sharp. Furthermore, we construct examples that show that if Conjecture 11 is true, then it is also sharp.

The following observations will be used in our arguments. In complete bipartite graphs, every doubly chorded cycle requires at least three vertices from each partite set. In complete tripartite graphs, every 5-vertex doubly chorded cycle requires exactly one vertex from one partite set and exactly two vertices from each of the other two partite sets.

Observe that for all $k \in \mathbb{Z}^+$ and $n \geq 6k - 2$, $K_{3k-1,n-3k+1}$ is an n-vertex graph with minimum degree $3k - 1$. Furthermore, $K_{3k-1,n-3k+1}$ does not have k disjoint doubly chorded cycles, as each doubly chorded cycle requires at least three vertices from each partite set. This construction shows that the minimum degree condition in Theorem 9 is best possible for all $n, k \in \mathbb{Z}^+$ where $n \geq 6k - 2$. This same construction shows the minimum degree condition in Corollary 8 is also best possible for such k and n.

For $k, n \in \mathbb{Z}^+$ such that $5k \le n \le 6k-1$, let $H_{k,n} = K_{6k-n-1,n-3k,n-3k+1}$. Since $5k \le n \le 6k-1$, the smallest partite set has size $6k-n-1$, where $0 \le 6k-n-1 \le k-1$, so that $H_{k,n}$ could be bipartite. Observe that $H_{k,n}$ is an n-vertex graph, and $\delta(H_{k,n}) = 6k-n-1+n-3k = 3k-1$.

We claim that $H_{k,n}$ does not have k disjoint doubly chorded cycles. If on the contrary, $H_{k,n}$ contains k disjoint doubly chorded cycles, then each one has either five vertices, or at least six. By our observations above, the maximum number of doubly chorded cycles that have exactly five vertices is $6k-n-1$. This means we need to still find $k-(6k-n-1) = n-5k+1$ more disjoint doubly chorded cycles, each with at least six vertices. So in total, the number of vertices we need is at least $5(6k-n-1)+6(n-5k+1) = n+1$, which is impossible as $H_{k,n}$ is an n-vertex graph. So $H_{k,n}$ is a sharpness example to Theorem 9. Thus, for every $n, k \in \mathbb{Z}^+$ where $n \ge 5k$, we can construct an n-vertex graph with minimum degree $3k-1$ that does not have k disjoint doubly chorded cycles.

The last family of graphs we will construct will be sharpness examples to both Theorem 10 and Conjecture 11 (if true) Let $t, n \in \mathbb{Z}$ such that $0 \le t \le \frac{n}{4}$. Define the graph

$$G(t,n) = K_{t,t+\lfloor\frac{n-4t}{3}\rfloor,t+\lfloor\frac{n-4t}{3}\rfloor+\alpha,t+\lfloor\frac{n-4t}{3}\rfloor+\beta}$$

where if $n-4t \equiv 0 \mod 3$, then $\alpha = \beta = 0$; if $n-4t \equiv 1 \mod 3$, then $\alpha = 0$ and $\beta = 1$; if $n-4t \equiv 2 \mod 3$, then $\alpha = \beta = 1$. Observe that in each case, the number of vertices in $G(t,n)$ is exactly n, and furthermore, if $t = 0$, then $G(t,n)$ is tripartite.

Lemma 12. *If $4k \le n \le 5k$, then $G' = G(5k-n, n)$ contains k disjoint doubly chorded cycles, and furthermore the only way to find k disjoint doubly chorded cycles is to use every vertex.*

Proof. Since $4k \le n \le 5k$, we have $0 \le 5k-n \le \frac{n}{4}$. If $n = 5k$, then G' is tripartite, and if $n < 5k$, then the smallest partite set of G' has size $5k-n$. Regardless, the maximum number of disjoint copies of K_4 we can find in G' is $5k-n$. If the number of disjoint copies of K_4 we create is say $\ell < 5k-n$, then in order to find k disjoint doubly chorded cycles in G', each of the remaining $k-\ell$ disjoint doubly chorded cycles must have at least five vertices. This requires

$$n \ge 4\ell + 5(k-\ell) = 5k-\ell > 5k-(5k-n) = n,$$

which is a contradiction. Thus, the only way to find k disjoint doubly chorded cycles in G' is create $5k-n$ disjoint copies of K_4.

Now if we remove these $5k-n$ disjoint copies of K_4 from G', this leaves a new graph G'' that is a complete tripartite graph with partite sets of size

$$\left\lfloor \frac{5(n-4k)}{3} \right\rfloor, \left\lfloor \frac{5(n-4k)}{3} \right\rfloor + \alpha, \text{ and } \left\lfloor \frac{5(n-4k)}{3} \right\rfloor + \beta.$$

Let $x = n - 4k$. Since $4k \le n$, we have $x \ge 0$. We induct on x to show that G'' contains x disjoint copies of $K_{1,2,2}$ and this covers all of the vertices of G''. This is clear if $x = 0, 1, 2$ as G'' is empty, $K_{1,2,2}$, or $K_{3,3,4}$, respectively (recall that α and β are defined based on $5(n-4k)$, as $t = 5k - n$). For $x \ge 3$, we remove a copy of $K_{5,5,5}$ from G'', which contains 3 disjoint copies of $K_{1,2,2}$, and induct on the remaining graph.

Thus, G' will contain $5k-n$ disjoint copies of K_4 and $x = n-4k$ disjoint copies of $K_{1,2,2}$, and so G' will have $5k-n+n-4k = k$ disjoint doubly chorded cycles, and furthermore the only way to find k disjoint doubly chorded cycles is to use every vertex. □

Let $k, n' \in \mathbb{Z}^+$ such that $4k < n' \le 5k$, and let $H' = G(5k-n', n')$. Note that $t = 5k - n' > 0$ so that H' is a 4-partite graph, and furthermore, the sizes of the partite sets of H' depend on $n' - 4t = 5n' - 20k$. If $5n' - 20k \equiv 0$ or $2 \mod 3$, then form the graph H from H' by deleting a vertex from the smallest partite set with size $5k - n'$; in these cases $\delta(H) = \delta(H') - 1$. If $5n' - 20k \equiv 1 \mod 3$, then form the graph H from H' by deleting a vertex from the largest partite set with size $5k - n' + \lfloor \frac{5n'-20k}{3} \rfloor + \beta$; recall that in this case, $\alpha = 0$ and $\beta = 1$ so that $\delta(H) = \delta(H')$.

By Lemma 12, H does not contain k disjoint doubly chorded cycles as it does not have enough vertices. Let $n = |V(H)|$ so that $n = n' - 1$ and $4k \le n < 5k$. We claim $\delta(H) = \lceil \frac{5k+n}{3} \rceil - 1$, which will show that H is a sharpness example to Theorem 10 (when $n = 5k - 1$) and Conjecture 11.

If $5n' - 20k \equiv 0 \mod 3$, then $\lfloor \frac{5n'-20k}{3} \rfloor = \frac{5n'-20k}{3}$, and $\delta(H') = 3(5k - n') + 2\lfloor \frac{5n'-20k}{3} \rfloor = \frac{5k+n'}{3}$, so that $3|(5k+n')$ and $3|(5k+n+1)$. Therefore,

$$\delta(H) = \delta(H') - 1 = \frac{5k+n'}{3} - 1 = \frac{5k+n+1}{3} - 1 = \left\lceil \frac{5k+n}{3} \right\rceil - 1.$$

Similarly, if $5n' - 20k \equiv 2 \mod 3$, then $\delta(H') = \frac{5k+n'-1}{3}$ so that $3|(5k+n)$. Therefore,

$$\delta(H) = \delta(H') - 1 = \frac{5k+n'-1}{3} - 1 = \frac{5k+n}{3} - 1 = \left\lceil \frac{5k+n}{3} \right\rceil - 1.$$

Laslty, if $5n' - 20k \equiv 1 \mod 3$, then $\delta(H') = \frac{5k+n'-2}{3}$ so that $3|(5k+n-1)$ and $3|(5k+n+2)$. Note that in this case,

$$\delta(H) = \delta(H') = \frac{5k+n'-2}{3} = \frac{5k+n-1}{3} = \frac{5k+n+2}{3} - 1 = \left\lceil \frac{5k+n}{3} \right\rceil - 1.$$

So for all $n, k \in \mathbb{Z}^+$ where $4k \leq n \leq 5k-1$, we can construct an n-vertex graph with minimum degree $\lceil \frac{5k+n}{3} \rceil - 1$ that does not contain k disjoint doubly chorded cycles. These graphs are sharpness examples to Conjecture 11, if it is true. Furthermore, for all $k \in \mathbb{Z}^+$ and $n = 5k-1$, these graphs will have minimum degree $\lceil \frac{10k-1}{3} \rceil - 1$ and so are sharpness examples to Theorem 10.

3. Setup and notation

In this section, we provide the setup behind our proofs of Theorems 9 and 10. To start, we present notation that will be used throughout our proofs.

3.1. Notation

Let G be a graph, $v \in V(G)$, and A and B be two subsets of $V(G)$, not necessarily disjoint. We let $N_B(v)$ denote $N_G(v) \cap B$, and let both $\|v, B\|$ and $d_B(v)$ denote $|N_B(v)|$. We also let $\|A, B\| = \sum_{v \in A} \|v, B\|$. For every collection of subgraphs $\mathcal{H}$ of G, we let $V(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} V(H)$. If H is a subgraph of G, we often replace $V(H)$ with H in the above notation (e.g., $N_H(v) = N_{V(H)}(v)$, $\|v, H\| = \|v, V(H)\|$, and $\|A, H\| = \|A, V(H)\|$). Similarly, we often replace $V(\mathcal{H})$ with $\mathcal{H}$ when $\mathcal{H}$ is a collection of subsets of G (e.g., $\|A, \mathcal{H}\| = \|A, V(\mathcal{H})\|$). Furthermore, this notation is commutative so that $\|A, B\| = \|B, A\|$.

If G is a graph and $A \subseteq V(G)$, we let $G[A]$ denote the subgraph of G induced by the vertices of A. If H is a subgraph of G, we let $H + A = G[V(H) \cup A]$ and $H - A = G[V(H) \setminus A]$. If $|A|$ is small, we often replace A with the vertices of A in the above notation (e.g., if $A = \{v\}$, we use $H + v = H + A$ and $H - v = H - A$). If F is a subgraph of G, we let $H + F = H + V(F)$ and $H - F = H - V(F)$.

For each doubly chorded cycle $C \in \mathcal{C}$, we fix a spanning cycle and assume an inherent orientation of this cycle, say clockwise. So for any $v_i, v_j \in C$, there are exactly two paths from v_i to v_j along the spanning cycle of C. We let $v_i C v_j$ denote the path that follows the orientation of the spanning cycle and let $v_i \overleftarrow{C} v_j$ denote the path that follows the reverse orientation. Similarly, given a path P, we assume an inherent orientation of this path, say from left-to-right. So in following the orientation of P, if v_i appears before v_j, then define v_iPv_j (resp. $v_j\overleftarrow{P}v_i$) as the unique subpath of P that starts at v_i (resp. v_j) and ends at v_j (resp. v_i).

We also let $[v_i, v_j]_C$ and $[v_i, v_j]_P$ denote $V(v_iCv_j)$ and $V(v_iPv_j)$, respectively. We also let $(v_i, v_j)_C$ and $(v_i, v_j)_P$ denote $V(v_iCv_j) \setminus \{v_i, v_j\}$ and $V(v_iPv_j) \setminus \{v_i, v_j\}$, respectively. We similarly define $(v_i, v_j]_C, [v_i, v_j)_C$, $(v_i, v_j]_P$, and $[v_i, v_j)_P$. When it is clear from context what the host object is, we will often supress the subscripts (e.g., $[v_i, v_j]$). Note that $[v_i, v_j]_C \cap [v_j, v_i]_C = \{v_i, v_j\}$.

At times we will identify a doubly chorded cycle by first describing its spanning cycle and then providing at least two chords. For example, if $C = v_1 \dots v_t v_1$ is a cycle with $t \geq 6$ and $v_1v_3, v_1v_4, v_1v_5 \in E(G)$, then we say $v_1Cv_5v_1$ is a doubly chorded cycle with chords v_1v_3 and v_1v_4.

Given a fixed path P, a hop is an edge in $E(G[P]) \setminus E(P)$; that is, a hop is an edge whose endpoints are both on P, but are not consecutive along P. Given a vertex $v \in P$, a hop neighbor of v is a vertex adjacent to v via a hop.

Lastly, to keep from writing 'doubly chorded cycle' throughout the rest of this paper, we will often use 'DCC' in its place.

3.2. Setup

We now begin the proofs of Theorems 9 and 10. Suppose that for some $k \in \mathbb{Z}^+$, there exist n-vertex graphs with $n \geq 4k$ and minimum degree at least $3k$ that do not contain k disjoint DCCs. Among these graphs choose G to be one that is edge-maximal with respect to not having k disjoint DCCs. That is, G does not contain k disjoint DCCs, however for each edge $e \notin E(G)$, $G + e$ does contain k disjoint DCCs. Since G cannot be complete (otherwise it would contain k disjoint DCCs as $n \geq 4k$), there exists an edge $e \notin E(G)$.

Since $G + e$ contains k disjoint DCCs, G must contain $k - 1$ disjoint DCCs, and furthermore these DCCs cover all but at least four vertices of G. That is, we can partition G into a collection of $k - 1$ disjoint DCCs and some nonempty remainder with at least four vertices.

Over all possible collections of $k - 1$ disjoint chorded cycles in G, we say an optimal collection $\mathcal{C}$ is a collection of $k - 1$ disjoint DCCs which satisfies the following conditions, where R is the graph $G - V(\mathcal{C})$:

(O1) the number of vertices in $\mathcal{C}$ is minimum,

(O2) subject to (O1), the total number of chords in the DCCs of $\mathcal{C}$ is maximum,

(O3) subject to (O1) and (O2), the length of the longest path in R is maximum, and

(O4) subject to (O1), (O2), and (O3), the number of edges in R is maximum.

In the rest of this paper, we fix an optimal collection $\mathcal{C}$ and remainder $R = G - V(\mathcal{C})$. We will also refer to this as an optimal partition of G. As we already know that G has a partition into a collection of $k-1$ disjoint DCCs and some nonempty remainder with at least four vertices, (O1) implies that given our optimial collection $\mathcal{C}$, we have $|R| \geq 4$. Furthermore, by (O1) and (O2), $G[C] \cong C$ for all $C \in \mathcal{C}$.

Our goal is to first show that $n < 5k$, which will prove Theorem 9 due to the following. Any counterexample to Theorem 9 is an n-vertex graph H with $n \geq 5k$ and $\delta(H) \geq 3k$ that does not contain k disjoint DCCs. From H, we can construct a sequence of graphs $H = H_0, H_1, H_2, \dots,$ such that for each $i \geq 1$, H_i is obtained from H_{i-1} by adding an edge to H_{i-1} that does not result in H_i containing k disjoint DCCs. At some point this process must stop and the resulting graph, say H_t, will be an n-vertex graph with $n \geq 5k$ and $\delta(H_t) \geq 3k$ that is edge-maximal with respect to not having k disjoint DCCs. This will contradict our showing that every such graph will have less than $5k$ vertices.

Once we have shown that $n < 5k$, we will then assume that in fact, $\delta(G) \geq \frac{10k-1}{3}$. As $\frac{10k-1}{3} \geq 3k$ for all $k \in \mathbb{Z}^+$, all of the previous properties proven for G will still hold. We then show arrive at contradictions in all possible situations, showing that G does not exist, and by the above argument, no counterexample to Theorem 10 exists.

4. Structural lemmas

In this section, we prove several structural lemmas that will be used throughout the remaining sections.

An immediate corollary of (O1) is that, for any $C \in \mathcal{C}$, no vertex of C is incident to three chords; otherwise, we could replace C with a DCC on fewer vertices. So every vertex in C is incident to at most two chords.

Lemma 13. *If $C \in \mathcal{C}$, then C contains at most one vertex incident to two chords, and furthermore, if such a vertex exists and $|C| \geq 6$, then there is another vertex in C that is not incident to any chord in C.*

Proof. Let $C \in \mathcal{C}$ and let $x \in C$ such that x is incident to two chords xx_1 and xx_2, where $x_1 \in (x, x_2)$. Suppose e is a chord in C other than xx_1 and xx_2. Both endpoints of e cannot be in $[x, x_2]$, otherwise xCx_2x is a DCC with chords xx_1 and e, on fewer vertices than C, contradicting (O1). By symmetry, both endpoints of e cannot be in $[x_1, x]$. Therefore, every chord in C other than xx_1 and xx_2 must have one endpoint in (x, x_1) and the other in (x_2, x).

Suppose there exists $y \in C - x$ such that y is incident to two chords, yy_1 and yy_2, where $y_1 \in (y, y_2)$. By symmetry, we may assume $y \in (x, x_1)$ and $y_1, y_2 \in (x_2, x)$. If there exists $z \in (y, x_1)$, then is not an edge, then $xx_1Cy_2y\overleftarrow{C}x$ is a DCC with chords xx_2 and yy_1, on fewer vertices than C, contradicting (O1). Hence, $(y, x_1) = \emptyset$. However, $xx_1Cy_1y\overleftarrow{C}x$ is a DCC with chords xx_2 and yx_1, on fewer vertices than C, contradicting (O1). Thus, C contains at most one vertex incident to two chords.

Now suppose $|C| \geq 6$. We now show there exists a vertex in $C - x$ that is not incident to a chord. If there exists $z \in (x_1, x_2)$, then as shown above, it cannot be incident to a chord, as the other endpoint would either be in $[x, x_2]$ or $[x_1, x]$. So we may assume $(x_1, x_2) = \emptyset$, and without loss of generality, (x, x_1) has at least two vertices. So let w_1 and w_2 be two such vertices such that x, w_1, and w_2 are consecutive along C.

So each w_i is incident to a chord w_iw_i', where $w_i' \in (x_2, x)$. Note also that $w_1' \neq w_2'$ as otherwise we have two vertices in C that are incident to two chords. If $w_2' \in (w_1', x)$, then $xx_2Cw_1'w_1Cx_1x$ is a DCC with chords xw_1 and x_1x_2 without w_2'. If $w_1' \in (w_2', x)$, then $x\overleftarrow{C}w_1'w_1Cx_2x$ is a DCC with chords xw_1 and xx_1 without w_2'. In either case, we contradict (O1). □

Note that the following three lemmas apply to collections of $k-1$ disjoint DCCs that satisfy (O1) and possibly (O2). So while they apply to our optimal collection $\mathcal{C}$, they may also apply to other collections of $k-1$ disjoint DCCs.

Lemma 14. *Let $\mathcal{C}'$ be a collection of $k-1$ DCCs that satisfies (O1), and let $R' = G \setminus V(\mathcal{C}')$. For all $v \in R'$ and $C \in \mathcal{C}'$, $\|v, C\| \leq 4$ and if equality holds, $|C| \leq 5$.*

Proof. We will start by showing that $\|v, C\| \leq 4$, so suppose $\|v, C\| \geq 5$. If there exists a $c_1, c_2 \in C$ that are adjacent along the cycle of C such that $\|v, C - c_1 - c_2\| \geq 4$, then $G[C - c_1 - c_2 + v]$ contains a DCC with strictly fewer vertices than C, contradicting (O1). Since $\|v, C\| \geq 5$ this implies $\|v, C\| = 5$ and $|C| = 5$. Since $|C| = 5$, then every chord in C will form a triangle, and so v together with this triangle in C will form a K_4, contradicting (O1). Hence, $\|v, C\| \leq 4$.

Suppose $\|v, C\| = 4$. We will prove that $|C| \leq 5$ by considering cases.

Case 1. $|C| \geq 9$

In this case, we can always find $c_1, c_2 \in C$ that are adjacent along the cycle of C, such that $\|v, C - c_1 - c_2\| \geq 4$, which as we stated, leads to a contradiction.

Case 2. $|C| = 8$

Label the vertices so that $C = v_1v_2v_3v_4v_5v_6v_7v_8v_1$. To avoid having $c_1, c_2 \in C$ that are adjacent along the cycle of C, such that $\|v, C - c_1 - c_2\| \geq 4$, we may assume $N_C(v) = \{v_1, v_3, v_5, v_7\}$. If C has a chord with both endpoints in $[v_1, v_5]$, then vv_1Cv_5v is a DCC with this chord and vv_3, contradicting (O1). So by symmetry, we may assume the chords in C are v_2v_6 and v_4v_8. However, $vv_3Cv_6v_2v_1v$ forms a DCC with chords v_2v_3 and vv_5, that contradicts (O1).

Case 3. $|C| = 7$

Label the vertices so that $C = v_1v_2v_3v_4v_5v_6v_7v_1$. To avoid having $c_1, c_2 \in C$ that are adjacent along the cycle of C, such that $\|v, C - c_1 - c_2\| \geq 4$, we can conclude without loss of generality that $N_C(v) = \{v_1, v_2, v_4, v_6\}$. If C has a chord with both endpoints in $[v_1, v_4]$, then vv_1Cv_4v is a DCC with this chord and vv_2, contradicting (O1). If C has a chord with both endpoints in $[v_4, v_1]$, then vv_4Cv_1v is a DCC with this chord and vv_6, contradicting (O1). By symmetry, C has no chords with both endpoints in $[v_2, v_6]$ or in $[v_6, v_2]$. However, this leaves C with only one possible chord, v_3v_7, a contradiction.

Case 4. $|C| = 6$

Label so that $C = v_1v_2v_3v_4v_5v_6v_1$. To avoid having $c_1, c_2 \in C$ that are adjacent along the cycle of C, such that $\|v, C - c_1 - c_2\| \geq 4$, we can conclude without loss of generality that either $N_C(v) = \{v_1, v_2, v_4, v_5\}$ or $N_C(v) = \{v_1, v_2, v_3, v_5\}$.

Suppose first that $N_C(v) = \{v_1, v_2, v_4, v_5\}$. If C has a chord with both endpoints in $[v_1, v_4]$, then vv_1Cv_4v is a DCC with this chord and vv_2, contradicting (O1). If C has a chord with both endpoints in $[v_4, v_1]$, then vv_4Cv_1v is a DCC with this chord and vv_2 that contradicts (O1). So v_1, v_4, and by symmetry, v_2, v_5 are not incident to chords in C. Yet this implies the only possible chord is v_3v_6, a contradiction.

Now suppose $N_C(v) = \{v_1, v_2, v_3, v_5\}$. If C has a chord with both endpoints in $[v_1, v_4]$, then vv_1Cv_4v is a DCC with this chord and vv_2, contradicting (O1). By symmetry, C has no chord with both endpoints in $[v_4, v_1]$. If $v_2v_i \in E(G)$ for $i \in \{5, 6\}$, then $vv_2v_iCv_1v$ is a DCC with chords v_1v_2 and vv_6, contradicting (O1). So v_1, v_2, v_4, and by symmetry, v_6, are not incident to any chords in C. However, this implies the only possible chord is v_3v_5, a contradiction.

This completes all cases and proves the lemma. □

Lemma 15. *Let $\mathcal{C}'$ be a collection of $k-1$ disjoint DCCs that satisfies (O1) and (O2), and let $R' = G \setminus V(\mathcal{C}')$. For all $v \in R'$ and $C \in \mathcal{C}'$, $\|v, C\| \leq 4$, and if equality holds, either $C \cong K_4$ or $C \cong K_{1,2,2}$ and $G[C + v] \cong K_{2,2,2}$. As a result, for all $x \in C$, $G[C - x + v] \cong C$.*

Proof. By Lemma 14, we can conclude that $\|v, C\| \leq 4$ and that $|C| \leq 5$.

Suppose $\|v, C\| = 4$. If $|C| = 4$, then $C \cong K_4$, and we are done. So assume $|C| = 5$, and label $C = v_1v_2v_3v_4v_5v_1$ where v_5 is the non-neighbor of v. Observe that vv_1Cv_4v forms a DCC including all chords with both endpoints in $[v_1, v_4]_C$ and two additional chords, vv_2 and vv_3. The number of chords in C is exactly the number of chords with both endpoints in $[v_1, v_4]_C$ together with those incident to v_5. Thus, if v_5 is not incident to two chords, we contradict (O2). Hence $v_5v_2, v_5v_3 \in E(G)$.

These cannot be the only chords in C, otherwise $vv_3v_5Cv_2v$ forms a DCC with chords v_2v_5, v_2v_3, and vv_1, contradicting (O2). If there exists a triangle in $G[N_C(v)]$, then we can replace C with a copy of K_4, contradicting (O1). Therefore, the only other chord in C is v_1v_4, so that $C \cong K_{1,2,2}$. □

We will often encounter the situation in which for some $v \in R$ and $C \in \mathcal{C}$, $\|v, C\| = 4$. Therefore, as a consequence of Lemma 15, we will use the following labels for the vertices of C in the situation where $C \in \{K_4, K_{1,2,2}\}$. If $C \cong K_4$ label the vertices a_1, a_2, a_3, and a_4. If $C \cong K_{1,2,2}$, then label the vertex in the part of size one as b, label the two vertices in one of the parts of size two as c_1 and c_2, and label the remaining two vertices in the final part as d_1 and d_2.

Lemma 16. *Let $\mathcal{C}'$ be a collection of $k-1$ disjoint doubly chorded cycles that satisfies (O1) and (O2), and let $R' = G \setminus V(\mathcal{C}')$. Suppose there exists $C \in \mathcal{C}'$ such that $C \cong K_{1,2,2}$. Then for any edge $xy \in E(R')$, we have $\|\{x, y\}, C\| \leq 7$, and if equality holds, then without loss of generality, $N_C(x) = \{c_1, c_2, d_1, d_2\}$ and $N_C(y) \in \{\{b, c_1, c_2\}, \{b, d_1, d_2\}\}$. Furthermore, if $\|x, C\| = \|y, C\| = 3$, then $|N_C(x) \cap N_C(y)| \leq 2$, and if equality holds, then $N_C(x) \cap N_C(y) \in \{\{c_1, c_2\}, \{d_1, d_2\}\}$.*

Proof. Let $C \in \mathcal{C}'$ be such that $C \cong K_{1,2,2}$, and let $xy \in E(R')$. Observe the following:

(1) if x and y have two common neighbors in C, say u and v, such that $uv \in E(G)$, then $G[\{x, y, u, v\}] \cong K_4$, which contradicts (O1).

Therefore, if we say $\|x, C\| = 4$ and $\|y, C\| \geq 3$, then by Lemma 15, $N_C(x) = \{c_1, c_2, d_1, d_2\}$, and the only way to avoid (1) is for $N_C(y) \in \{\{b, c_1, c_2\}, \{b, d_1, d_2\}\}$. Similarly, if $\|x, C\| = \|y, C\| = 3$ and $|N_C(x) \cap N_C(y)| \geq 2$, then the only way to avoid (1) is for $N_C(x) \cap N_C(y) \in \{\{c_1, c_2\}, \{d_1, d_2\}\}$. ☐

We now return to our optimal collection $\mathcal{C}$ with $R = G \setminus V(\mathcal{C})$.

Lemma 17. *Suppose P_1 and P_2 are two disjoint, non-trivial paths in G. If there exist $u, v \in P_1$ such that $\|\{u, v\}, P_2\| \geq 5$, then $G[P_1 + P_2]$ contains a DCC. Furthermore, if $\|\{u, v\}, P_2\| \geq 4$, then $G[P_1 + P_2]$ contains a DCC, unless one of the following configurations exists up to symmetry and relabelling of vertices:*

1. *$N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1\}$, and $v_1 \in (u_1, u_3)_{P_2}$;*
2. *$N_{P_2}(u) = \{u_1, u_2\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(u) \cap N_{P_2}(v) = \emptyset$, and u_1, v_1, v_2, u_2 appear in this order along P_2 (not necessarily consecutive).*

Proof. Suppose $\|\{u, v\}, P_2\| \geq 4$, and without loss of generality, suppose $\|v, P_2\| \leq \|u, P_2\|$. Let w_L and w_R be the endpoints of P_2 where $P_2 = w_L P_2 w_R$.

If $\|u, P_2\| \geq 4$, then $G[P_2 + u]$ contains a DCC. So $1 \leq \|v, P_2\| \leq \|u, P_2\| \leq 3$.

If $\|u, P_2\| = 3$, let $N_{P_2}(u) = \{u_1, u_2, u_3\}$ where u_1, u_2, and u_3 appear in this order along P_2 (not necessarily consecutive). Let $v_1 \in N_{P_2}(v)$. If $v_1 \in [w_L, u_1]_{P_2}$, then $uP_1vv_1P_2u_3u$ is a DCC with chords uu_1 and uu_2. So $v_1 \notin [w_L, u_1]_{P_2}$, and by symmetry, $v_1 \notin [u_3, w_R]_{P_2}$. So $v_1 \in (u_1, u_3)_{P_2}$. If $\|v, P_2\| \geq 2$ so that $v_2 \in N_{P_2}(v)$ exists, then by the same argument, $v_2 \in (u_1, u_3)_{P_2}$. Without loss of generality, we may assume $v_1 \in (u_1, v_2)_{P_2}$.

If $v_2 \in [u_2, u_3)_{P_2}$, then $uu_1P_2v_2vP_1u$ forms a DCC with chords uu_2 and vv_1. If $v_2 \in (u_1, u_2)_{P_2}$, then $vv_1P_2u_3uP_1v$ forms a DCC with chords uu_2 and vv_2. Thus, v_2 does not exist, and configuration 1 holds.

So $\|v, P_2\| = \|u, P_2\| = 2$. Let $N_{P_2}(u) = \{u_1, u_2\}$ where u_1 and u_2 appear in this order along P_2 (not necessarily consecutive), and similarly define $N_{P_2}(v) = \{v_1, v_2\}$. If $v_1 \in [u_2, w_R]_{P_2}$, then $uu_1P_2v_2v\overleftarrow{P}_1u$ is a DCC with chords uu_2 and vv_1. So $v_1 \notin [u_2, w_R]_{P_2}$ and by symmetry, $v_2 \notin [w_L, u_1]_{P_2}$.

Suppose $v_1 = u_1$. If $v_2 \in (u_1, u_2)_{P_2}$, then $v_1 P_2 u_2 u P_1 v v_1$ is a DCC with chords uu_1 and vv_2. If $v_2 \in [u_2, w_R]_{P_2}$, then $u_1 P_2 v_2 v \overleftarrow{P}_1 u u_1$ is a DCC with chords vv_1 and uu_2. So $v_1 \neq u_1$ and by symmetry $v_2 \neq u_2$. Thus, either $v_1 \in [w_L, u_1)_{P_2}$ or $v_1 \in (u_1, u_2)_{P_2}$, and either $v_1 \in (u_1, u_2)_{P_2}$ or $v_2 \in (u_2, w_R]_{P_2}$.

If $v_1 \in [w_L, u_1)_{P_2}$ and $v_2 \in (u_1, u_2)_{P_2}$, then $u P_1 v v_1 P_2 u_2 u$ is a DCC with chords uu_1 and vv_2. A symmetric argument holds if $v_1 \in (u_1, u_2)_{P_2}$ and $v_2 \in (u_2, w_R]_{P_2}$. So either $v_1 \in (u_1, u_2)_{P_2}$ and $v_2 \in (u_1, u_2)_{P_2}$, or $v \in [w_L, u_1)_{P_2}$ and $v_2 \in (u_2, w_R]_{P_2}$. The former immediately gives configuration 2, while the latter gives configuration 2 after switching u_i with v_i. □

Lemma 18. *Suppose P_1 and P_2 are two disjoint, non-trivial paths in G. If there exist $u, v \in P_1$ such that $\|\{u, v\}, P_2\| \geq 6$, then $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1| + |P_2|$ vertices. Furthermore, if $\|\{u, v\}, P_2\| \geq 5$, then $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1| + |P_2|$ vertices, unless one of the following configurations exists up to symmetry and relabelling of vertices:*

1. *$N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(u) \cap N_{P_2}(v) = \emptyset$, u and v are the endpoints of P_1, v_1 and u_3 are the endpoints of P_2, and v_1, u_1, u_2, v_2, u_3 appear in this order on P_2 (not necessarily consecutive) so that $|P_2| \geq 5$. Furthermore, if $|P_1| = 2$, then $(u_1, u_2)_{P_2} \neq \emptyset$, $(u_2, v_2)_{P_2} = \emptyset$, and in particular, $|P_2| \geq 6$;*
2. *$N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $u_1 = v_1$, $u_3 = v_2$, u and v are the endpoints of P_1, $u_1 = v_1$ and $u_3 = v_2$ are the endpoints of P_2, and $u_1 = v_1, u_2, u_3 = v_2$ appear in this order on P_2 (not necessarily consecutive). Furthermore, if say $|P_1| = 2$, then $P_2 = u_1 u_2 u_3$; that is,$|P_2| = 3$.*

Proof. Suppose $\|\{u, v\}, P_2\| \geq 5$ for some $u, v \in P_1$. If say $\|u, P_2\| \geq 4$, then we can easily form a DCC in $G[P_2 + u]$, which avoids v. So $2 \leq \|u, P_2\| \leq 3$ and by symmetry $2 \leq \|v, P_2\| \leq 3$.

Let $X = N_{P_2}(u) \cup N_{P_2}(v)$, and label the vertices in $X = \{x_1, x_2, \ldots, x_{|X|}\}$ such that $x_1, x_2, \ldots, x_{|X|}$ appear in this order along P_2 (not necessarily consecutive). If $|X| = 3$, then as $\|\{u, v\}, P_2\| \geq 5$, without loss of generality, $X = N_{P_2}(u)$. If $x_1, x_2 \in N_{P_2}(v)$, then $x_1 u P_1 v x_2 \overleftarrow{P}_2 x_1$ is a DCC with chords ux_2 and vx_1 that avoids x_3. A symmetric argument holds if $x_2, x_3 \in N_{P_2}(v)$. So $N_{P_2}(v) = \{x_1, x_3\}$ and $u P_1 v x_1 P_2 x_3 u$ is a DCC with chords ux_1 and ux_2. Thus, we must have $\|\{u, v\}, P_2\| = 5$. Further, u, v, x_1, and x_3 must be the endpoints of their respective paths, otherwise we have a DCC with fewer than $|P_1| + |P_2|$ vertices. This yields the first part of configuration 2.

To complete configuration 2, assume $|P_1| = 2$. If $(x_1, x_2)_{P_2} \neq \emptyset$, then $x_1 u x_2 P_2 x_3 v x_1$ is a DCC with chords uv and ux_3 that avoids all the vertices in $(x_1, x_2)_{P_2}$. So $(x_1, x_2)_{P_2} = \emptyset$ and by symmetry $(x_2, x_3)_{P_2} = \emptyset$. Thus, $|P_2| = 3$. So when $|X| = 3$, configuration 2 holds.

Now suppose $|X| = 4$. Since $\|\{u, v\}, P_2\| \geq 5$, either $\|\{u, v\}, [x_1, x_3]_{P_2}\| \geq 4$, or $\|\{u, v\}, [x_2, x_4]_{P_2}\| \geq 4$. Without loss of generality, suppose $\|\{u, v\}, [x_1, x_3]_{P_2}\| \geq 4$. By Lemma 17, either configuration 1 or 2 from Lemma 17 holds, otherwise $G[P_1 + [x_1, x_3]_{P_2}]$ contains a DCC that avoids x_4. Since u and v only have three neighbors all together on $[x_1, x_3]_{P_2}$, only configuration 1 from Lemma 17 holds, and so without loss of generality, u is adjacent to x_1, x_2, and x_3, and x_2 is the only neighbor of v in $[x_1, x_3]_{P_2}$. As $2 \leq \|v, P_2\|$, we must have $x_4 \in N_{P_2}(v)$. However, $x_4 v \overleftarrow{P}_1 u x_2 P_2 x_4$ is a DCC with chords ux_3 and vx_2 that avoids x_1.

Lastly suppose $|X| \geq 5$. So by the definition of X, $\|\{u, v\}, [x_1, x_4]_{P_2}\| \geq 4$ and $\|\{u, v\}, [x_2, x_5]_{P_2}\| \geq 4$. In each, either configuration 1 or 2 from Lemma 17 holds, otherwise $G[P_1 + [x_1, x_4]_{P_2}]$ contains a DCC that avoids x_5 or $G[P_1 + [x_2, x_5]_{P_2}]$ contains a DCC that avoids x_1.

Suppose configuration 1 from Lemma 17 holds for $[x_1, x_4]_{P_2}$ so that without loss of generality, u have exactly three neighbors in $[x_1, x_4]$, namely x_1, x_4, and exactly one vertex from $\{x_2, x_3\}$, and v is only adjacent to the vertex from $\{x_2, x_3\}$ that u is not adjacent to. Since $\|u, P_2\| \leq 3$, we know $x_5 \in N_{P_2}(v)$, and furthermore, $N_{P_2}(u) = \{x_1, x_4, x_i\}$ where $i \in \{2, 3\}$ and $N_{P_2}(v) = \{x_5, x_{5-i}\}$. However, we know either configuration 1 or 2 from Lemma 17 holds for $[x_2, x_5]_{P_2}$. As u and v only have two neighbors each in $[x_2, x_5]_{P_2}$, we must have configuration 2 from Lemma 17. This implies $N_{P_2}(u) = \{x_1, x_3, x_4\}$ and $N_{P_2}(v) = \{x_2, x_5\}$. Now $x_1 P_2 x_5 v \overleftarrow{P}_1 u x_1$ is a DCC with chords ux_3 and ux_4. Thus, we must have u, v, x_1, and x_5 be the endpoints of their respective paths, otherwise we have a DCC with fewer than $|P_1| + |P_2|$ vertices. This yields the first part of configuration 1 in our lemma. We will deal with the case where $|P_1| = 2$ in a moment.

If configuration 2 from Lemma 17 holds for $[x_1, x_4]_{P_2}$, then without loss of generality, u is only adjacent to x_2 and x_3 from $[x_1, x_4]_{P_2}$, and v is only adjacent to x_1 and x_4. However, we know either configuration 1 or 2 from Lemma 17 holds for $[x_2, x_5]_{P_2}$. Configuration 2 from Lemma 17 cannot hold as neither u or v is adjacent to both x_3 and x_4. So configuration 1 from Lemma 17 holds, and $N_{P_2}(u) = \{x_2, x_3, x_5\}$ and $N_{P_2}(v) = \{x_1, x_4\}$. Just as above, u, v, x_1, and x_5 must be the endpoints of their respective paths, otherwise we have a DCC with fewer than $|P_1| + |P_2|$ vertices. This yields the first part of configuration 1 in our lemma.

Now to complete configuration 1, suppose $|P_1| = 2$, and relabel the vertices so that $N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(u) \cap N_{P_2}(v) = \emptyset$, u and v are the endpoints of P_1, v_1 and u_3 are the endpoints of P_2, and

v_1, u_1, u_2, v_2, u_3 appear in this order on P_2 (not necessarily consecutive). If $(u_1, u_2)_{P_2} = \emptyset$ so that $u_1u_2 \in E(G)$, then $v_1P_2u_1uu_2P_2v_2vv_1$ is a DCC with chords uv and u_1u_2 with fewer vertices than $|P_1| + |P_2|$ as it skips u_3. Thus, $(u_1, u_2)_{P_2} \neq \emptyset$ so that $|P_2| \geq 6$. Furthermore, if $(u_2, v_2)_{P_2} \neq \emptyset$, then $v_2P_2u_3uu_2\overleftarrow{P}_2v_1vv_2$ is a DCC with chords uv and uu_1 with fewer vertices than $|P_1| + |P_2|$ as it skips all the vertices in $(u_2, v_2)_{P_2}$. This completes configuration 2, and prove the lemma. □

Lemma 19. *Suppose P_1 and P_2 are two disjoint, non-trivial paths in G. If there exist $u, v, w \in P_1$ such that $\|\{u, v, w\}, P_2\| \geq 6$, then $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1|+|P_2|$ vertices, unless one of the following configurations exists up to symmetry and relabelling of vertices.*

1. *$N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1\}$, $N_{P_2}(u) \cap N_{P_2}(v) \cap N_{P_2}(w) = \emptyset$, u and v are the endpoints of P_1, v_1 and u_3 are the endpoints of P_2, and $v_1, u_1, u_2, v_2, w_1, u_3$ appear in this order along P_2 (not necessarily consecutive).*
2. *$N_{P_2}(u) = \{u_1, u_2\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1, w_2\}$, $N_{P_2}(u) \cap N_{P_2}(v) \cap N_{P_2}(w) = \emptyset$, $w \in (u, v)_{P_1}$, and $w_1, u_1, v_1, v_2, u_2, w_2$ appear in this order along P_2 (not necessarily consecutive).*
3. *$N_{P_2}(u) = \{u_1, u_2\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1, w_2\}$, u and v are the endpoints of P_1, $u_1 = v_1$, $u_2 = v_2$, u_1 and u_2 are the endpoints of P_2, and u_1, w_1, w_2, u_2 appear in this order along P_2 (not necessarily consecutive).*

In particular, if $\|\{u, v, w\}, P_2\| \geq 6$, then $G[P_1 + P_2]$ contains a DCC (not necessarily on fewer than $|P_1|+|P_2|$ vertices) unless $\|\{u, v, w\}, P_2\| = 6$ and configuration 2 occurs.

Proof. Suppose $\|\{u, v, w\}, P_2\| \geq 6$, and let x_L and x_R be the endpoints of P_2 such that $P_2 = x_LP_2x_R$. If say $\|u, P_2\| \geq 4$, then $G[P_2 + u]$ contains a DCC that avoids v and w. Suppose in the following, $\|u, P_2\| = 3$. Since $\|\{u, v, w\}, P_2\| \geq 6$, either $\|w, P_2\| \geq 2$ or $\|v, P_2\| \geq 2$. Suppose $\|v, P_2\| \geq 2$. If $\|v, P_2\| \geq 3$, then $\|\{u, v\}, P_2\| \geq 6$, and we are done by Lemma 18. So $\|v, P_2\| = 2$. Since $\|\{u, v\}, P_2\| = 5$, by Lemma 18 either configuration 1 or 2 from Lemma 18 holds. Furthermore, we must have $w \in (u, v)_{P_1}$, otherwise we are done by Lemma 17, and since $\|\{u, v, w\}, P_2\| \geq 6$, there exists $w_1 \in N_{P_2}(w)$.

We claim configuration 1 from Lemma 18 holds. If on the contrary, configuration 2 from Lemma 18 holds, then $N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $u_1 = v_1$, $u_3 = v_2$, u and v are the endpoints of P_1, $u_1 = v_1$ and $u_3 = v_2$ are the endpoints of P_2, and $u_1 = v_1, u_2, u_3 = v_2$ appear in this order on P_2 (not necessarily consecutive). If $w_1 \in [u_1, u_2]_{P_2}$, then $uP_1vu_1P_2u_2u$ is a DCC with chords uu_1 and ww_1 that avoids u_3. A symmetric argument holds if $w_1 \in [u_2, u_3]_{P_2}$, so we must have configuration 1 from Lemma 18.

So $N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(u) \cap N_{P_2}(v) = \emptyset$, u and v are the endpoints of P_1, v_1 and u_3 are the endpoints of P_2, and v_1, u_1, u_2, v_2, u_3 appear in this order on P_2 (not necessarily consecutive) so that $|P_2| \geq 5$. Note that $\|\{u, w\}, P_2\| \geq 4$. If $G[P_2 + [u, w]_{P_1}]$ contains a DCC, then it avoids v; so by Lemma 17, $w_1 \in (u_1, u_3)_{P_2}$. If $w_1 \in (u_1, v_2]_{P_2}$, then $uP_1vv_2\overleftarrow{P}_2u_1u$ is a DCC with chords uu_2 and ww_1 that avoids v_1. If $w_1 = u_3$, then $w_1w\overleftarrow{P}_1uu_1P_2w_1$ is a DCC with chords uu_2 and uu_3 that avoids v_1. So $w_1 \in (v_2, u_3)_{P_2}$. Suppose there exists a $w_2 \in N_w(P_2)$. By a similar argument $w_2 \in (v_2, u_3)$. Then, $vv_2P_2u_3uP_1v$ forms a DCC with chords ww_1 and ww_2 that avoids v. Therefore $N_w(P_2) = \{w_1\}$. Furthermore, $uP_1vv_1P_2u_3u$ is a DCC with chords uu_1 and uu_2. Therefore, u, v, v_1, and u_3 are the endpoints of their respective paths, otherwise we have a DCC with fewer than $|P_1|+|P_2|$ vertices. This yields configuration 1 in this lemma.

This completes the case when $\|u, P_2\| = 3$. So without loss of generality, as $\|\{u, v, w\}, P_2\| \geq 6$, we have $\|u, P_2\| = \|v, P_2\| = \|w, P_2\| = 2$. Suppose without loss of generality that $w \in (u, v)_{P_1}$. If either $G[P_2 + [u, w]_{P_1}]$ or $G[P_2 + [w, v]_{P_1}]$ contain a DCC then we are done, as either would avoid v or u, respectively. So by Lemma 17, we must have configuration 2 from Lemma 17 hold for each.

As a result, $N_{P_2}(u) = \{u_1, u_2\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1, w_2\}$, $N_{P_2}(u) \cap N_{P_2}(w) = \emptyset$, and $N_{P_2}(v) \cap N_{P_2}(w) = \emptyset$. We now have two cases depending on the order of u_1, u_2, w_1, w_2 along P_2.

Case 1. u_1, w_1, w_2, u_2 appear in this order along P_2 (not necessarily consecutive).

If $v_1 \in (w_1, w_2)_{P_2}$, then we must have w_1, v_1, v_2, w_2, in this order, so that $v_2 \in (v_1, w_2)_{P_2}$. However, $uP_1vv_1P_2u_2u$ is a DCC with chords ww_2 and vv_2 that avoids u_1. So $v_1 \notin (w_1, w_2)_{P_2}$, and by symmetry, $v_2 \notin (w_1, w_2)_{P_2}$.

Now suppose $v_1 = u_1$. Note that by Lemma 17, $v_2 \in (w_2, x_R]_{P_2}$. If $v_2 \in (u_2, x_R]_{P_2}$, then $uP_1vv_1P_2u_2u$ is a DCC with chords ww_1 and ww_2 that avoids v_2. If $v_2 \in (w_2, u_2)_{P_2}$, then $v_2v\overleftarrow{P}_1uu_1P_2v_2$ is a DCC with chords ww_1 and ww_2 that avoids u_2. So $v_2 = u_2$. Note that $uP_1vu_1P_2u_2u$ is a DCC with chords ww_1 and ww_2. So u, v, u_1, and u_2 must be the endpoints of their respective paths, otherwise we have a DCC with fewer than $|P_1| + |P_2|$ vertices. This yields configuration 3 in this lemma.

A symmetric argument holds if $v_2 = u_2$. So either $v_1 \in [x_L, u_1)_{P_2}$ or $v_1 \in (u_1, w_1)_{P_2}$, and by symmetry, $v_2 \in (w_2, u_2)_{P_2}$ or $v_2 \in (u_2, x_R]_{P_2}$. If

$v_1 \in (u_1, w_1)_{P_1}$, then $uP_1vv_1P_2u_2u$ is a DCC with chords ww_1 and ww_2 that avoids u_1. So we must have $v_1 \in [x_L, u_1)_{P_2}$, and by symmetry, $v_2 \in (u_2, x_R]_{P_2}$. However, $uP_1vv_1P_2u_2u$ is a DCC with chords ww_1 and ww_2 that avoids v_2. This completes the case.

Case 2. w_1, u_1, u_2, w_2 appear in this order along P_2 (not necessarily consecutive).

Recall that by Lemma 17, configuration 2 from Lemma 17 holds for $G[P_2 + [w, v]_{P_2}]$. In particular, either v_1, w_1, w_2, v_2 or w_1, v_1, v_2, w_2 appear in this order along P_2 (not necessarily consecutive).

If $v_1 \in [x_L, w_1]_{P_2}$, then $uP_1vv_1P_2u_2u$ is a DCC with chords uu_1 and ww_1 that avoids w_2. So $v_1 \notin [x_L, w_1]_{P_2}$ and by a symmetric argument $v_1 \notin [w_2, x_R]_{P_2}$. So we must have $v_1 \in (w_1, w_2)_{P_2}$, and by symmetry, $v_2 \in (w_1, w_2)_{P_2}$. Now, $G[P_1 + (w_1, w_2)_{P_2}]$ cannot have a DCC, as it would avoid w_1. So as $\|\{u, v\}, (w_1, w_1)_{P_2}\| \geq 4$, by Lemma 17, either configuration 1 or 2 from Lemma 17 holds, and in particular, it must be configuration 2. So either u_1, v_1, v_2, u_2 or v_1, u_1, u_2, v_2 appear in this order along P_2 (not necessarily consecutive). In either case, we get configuration 2 in this lemma. This completes the case, and proves the lemma. □

Lemma 20. *Suppose P_1 and P_2 are two disjoint, non-trivial paths in G such that $|P_1| = 3$. If $\|P_1, P_2\| \geq 6$, then $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1| + |P_2|$ vertices.*

Proof. Suppose $P_1 = uwv$. By Lemma 19, we are done unless one of the three configurations in Lemma 19 holds. If configuration 1 holds, then $v_1P_2u_2uww_1\overleftarrow{P}_2v_2vv_1$ is a DCC with chords uu_1 and wv on fewer than $|P_1|+|P_2|$ vertices as it skips u_3. If configuration 2 holds, then $w_1P_2u_1uu_2\overleftarrow{P}_2v_1vw$ with chords uw and vv_2 on fewer than $|P_1| + |P_2|$ vertices as it skips u_3. If configuration 3 holds, then $u_1P_2w_1wvv_2uu_1$ is a DCC with chords uw and vu_1 on fewer than $|P_1| + |P_2|$ vertices as it skips w_2.

Thus, in every case $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1| + |P_2|$ vertices. □

Lemma 21. *Suppose P_1 and P_2 are two disjoint paths such that $\min\{|P_1|, |P_2|\} \geq 4$. If $\|P_1, P_2\| \geq \min\{|P_1|, |P_2|\} + 4$, then $G[P_1 + P_2]$ contains a DCC on fewer than $|P_1| + |P_2|$ vertices.*

Proof. Suppose without loss of generality that $|P_1| \leq |P_2|$. Therefore, if we show that satisfying $\|P_1, P_2\| \geq |P_1| + 4$ implies the existence of a DCC on fewer than $|P_1| + |P_2|$ vertices in $G[P_1 + P_2]$, then we are done.

If there exists $u \in P_1$ such that $\|u, P_2\| \geq 4$, then $G[P_2 + u]$ contains a DCC, and we are done. Thus we assume $\|u, P_2\| \leq 3$ for all $u \in P_1$. Suppose there exists $u \in P_1$ such that $\|u, P_2\| = 3$. If $\|v, P_2\| \leq 1$ for all $v \in P_1 - u$, then $\|P_1, P_2\| \leq |P_1| + 2$, which is a contradiction. So there exists $v \in P_1 - u$ such that $\|v, P_2\| \geq 2$. If $\|v, P_2\| = 3$, then $\|\{u, v\}, P_2\| \geq 6$ and by Lemma 18 we are done; so $\|v, P_2\| = 2$. Again, there must exist $w \in P_1 - u - v$ such that $\|w, P_2\| \geq 1$. If $\|w, P_2\| = 2$, then $\|\{u, v, w\}, P_2\| \geq 7$ and by Lemma 19 we are done; so $\|w, P_2\| = 1$. Similarly, there exists $x \in P_1 - u - v - w$ such that $\|x, P_2\| \geq 1$. So $\|\{u, v, w\}, P_2\| \geq 6$, and by configuration 1 in Lemma 19, $N_{P_2}(u) = \{u_1, u_2, u_3\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1\}$, $N_{P_2}(u) \cap N_{P_2}(v) \cap N_{P_2}(w) = \emptyset$, u and v are the endpoints of P_1, v_1 and u_3 are the endpoints of P_2, and $v_1, u_1, u_2, v_2, w_1, u_3$ appear in this order along P_2 (not necessarily consecutive). Similarly, $\|\{u, v, x\}, P_2\| \geq 6$, so that $x \in (u, v)_{P_1}$, $N_{P_2}(x) = \{x_1\}$, and $x_1 \in (v_2, u_3)_{P_2}$. However, $uP_1vv_2P_2u_3u$ is a DCC with chords ww_1 and xx_1 on strictly fewer vertices than $|P_1| + |P_2|$.

Thus in the remainder of this proof we assume that for all $u \in P_1$, $\|u, P_2\| \leq 2$. Since $\|P_1, P_2\| \geq |P_1| + 4$, there exist distinct vertices $u, v, w, x \in P_1$ such that $\|u, P_2\| = \|v, P_2\| = \|w, P_2\| = \|x, P_2\| = 2$. Suppose there exist three distinct vertices from $\{u, v, w, x\}$ such that they form configuration 3 in Lemma 19, and without loss of generality suppose it is u, v, w. Then $N_{P_2}(u) = \{u_1, u_2\}$, $N_{P_2}(v) = \{v_1, v_2\}$, $N_{P_2}(w) = \{w_1, w_2\}$, u and v are the endpoints of P_1, $u_1 = v_1$, $u_2 = v_2$, and $u_1 = v_1, w_1, w_2, u_2 = v_2$ appear in this order along P_2 (not necessarily consecutive). Since u and v are the endpoints of P_1, without loss of generality, we may assume $x \in (u, w)_{P_1}$. When we consider x, w, v, either configuration 2 or 3 in Lemma 19 holds. If configuration 2 holds, we must have $v_1, v_2 \in (w_1, w_2)_{P_2}$, which is a contradiction as $w_1, w_2 \in (v_1, v_2)_{P_2}$. So configuration 3 holds, and $x_1 = v_1$ and $x_2 = v_2$. However, $xP_1vv_2\overleftarrow{P}_2x_1x$ is a DCC with chords xx_2 and vv_1 with fewer vertices than $|P_1| + |P_2|$ as it does not include u.

So for every three vertices from $\{u, v, w, x\}$, configuration 2 in Lemma 19 holds. Without loss of generality, suppose u, w, x, v appear in this order along P_1. When we consider u, w, x, we see that $x_1, x_2 \in (w_1, w_2)_{P_2}$, however when we consider w, x, v, we must have $w_1, w_2 \in (x_1, x_2)_{P_2}$, a contradiction.

This completes the proof of the lemma. □

Lemma 22. *Suppose Q_1 and Q_2 are disjoint subgraphs in G such that $Q_1 \cong K_3$ and Q_2 contains a nontrivial, spanning path. Then*

1. *if $\|Q_1, Q_2\| \geq 5$, then $G[Q_1 + Q_2]$ contains a DCC on fewer than $|Q_1| + |Q_2|$ vertices;*
2. *if $\|Q_1, Q_2\| \geq 4$, then $G[Q_1 + Q_2]$ contains a DCC;*

3. *for any* $x, y \in Q_2$*, if* $\|Q_1, \{x, y\}\| \geq 3$*, then* $G[Q_1 + Q_2]$ *contains a DCC;*
4. *if* $|Q_2| = 3$ *and* $\|Q_1, Q_2\| \geq 4$*, then* $G[Q_1 + Q_2]$ *contains a DCC on fewer than* $|Q_1| + |Q_2|$ *vertices.*

Proof. Suppose $V(Q_1) = \{u, v, w\}$, and let P_2 be a spanning path of Q_2 with endpoints q and q'. The following claim will be useful in proving the above statements.

Claim 22.1. For any $e \in E(Q_1)$, if $\|e, Q_2\| \geq 4$, then $G[Q_1 + Q_2]$ contains a DCC with fewer vertices than $|Q_1| + |Q_2|$.

Proof. If $G[uv + Q_2]$ contains a DCC, then we are done as we skip w. So by Lemma 18, we assume $\|uv, Q_2\| \leq 4$. If $\|uv, Q_2\| = 4$, then by Lemma 17, $G[uv + Q_2]$ is one of the two configurations in Lemma 17. If the first configuration holds, then without loss of generality we may assume $v_1 \in [u_2, u_3)_{Q_2}$. However, $u_1 P_2 v_1 v w u u_1$ is a DCC with chords uv and uu_2 with fewer than $|Q_1| + |Q_2|$ vertices as it skips u_3. If the second configuration holds, then $u_1 P_2 v_2 v w u u_1$ is a DCC with chords uv and vv_1 with fewer than $|Q_1| + |Q_2|$ vertices as it skips u_3. □

Proof of 1. Suppose $\|Q_1, Q_2\| \geq 5$. Then there exists some edge in Q_1, say uv, such that $\|uv, Q_2\| \geq 4$. So by the claim, we are done. □

Proof of 2. Suppose $\|Q_1, Q_2\| \geq 4$. By the claim, if we consider the edge uv, then $\|uv, Q_2\| \leq 3$ otherwise we are done. As this holds for every edge in Q_1, we get $\|Q_1, Q_2\| \leq 4$, so that in fact equality holds. In particular, we may assume $\|u, Q_2\| = 2$ and $\|v, Q_2\| = \|w, Q_2\| = 1$. Let $N_{Q_2}(u) = \{u_1, u_2\}, N_{Q_2}(v) = \{v_1\}$, and $N_{Q_2}(w) = \{w_1\}$. If $v_1 \in [u_2, q']_{Q_2}$, then $u_1 P_2 v_1 v w u u_1$ is a DCC with chords uv and uu_2. By symmetry, we may assume $v_1 \in (u_1, u_2)_{Q_2}$, and furthermore, $w_1 \in (u_1, u_2)_{Q_2}$. Without loss of generality, suppose $v_1 \in (u_1, w_1]_{Q_2}$. Then $u_1 P_2 w_1 w v u u_1$ is a DCC with chords vv_1 and uv.

So in any case we get a DCC in $G[Q_1 + Q_2]$. □

Proof of 3. Fix $x, y \in Q_2$. If either $\|x, Q_1\| = 3$ or $\|y, Q_1\| = 3$, then we are done as $G[x + Q_1]$ or $G[y + Q_1]$ contain a DCC, respectively. So $\|x, Q_1\| \leq 2$ and $\|y, Q_2\| \leq 2$. Thus, if $\|\{x, y\}, Q_2\| \geq 3$, then without loss of generality, $\|x, Q_1\| = 2$ with $N_{Q_1}(x) = \{u, v\}$. If $yw \in E(G)$, then $xuvwyx$ is a DCC with chords uw and xv. So y has a neighbor in $\{u, v\}$, say u. Then $yuwvxy$ is a DCC with chords uv and xv. A similar DCC exists when $yv \in E(G)$. □

Proof of 4. Let $Q_2 = qxq'$. By the Proof of 3 above, $\|Q_1, qx\| \leq 2$ and $\|Q_1, xq'\| \leq 2$, otherwise we find a DCC that skips q' and q, respectively. So if $\|Q_1, Q_2\| \geq 4$, we must have $\|q, Q_1\| = \|q', Q_1\| = 2$ and $\|x, Q_1\| = 0$. Without loss of generality, suppose $N_{Q_1}(q) = \{u, v\}$. If $N_{Q_1}(q') = \{u, v\}$ as well, then $uvq'xqu$ is a DCC with chords uq' and vq that skips w. So we may assume $N_{Q_2}(q') = \{w, v\}$. However, $q'wuqvq'$ is a DCC with chords uv and wv that skips x. □

This proves all the statements, and so proves the lemma. □

Lemma 23. *Suppose Q is a subgraph of G such that $G[Q] \cong K_4^-$ where $V(Q) = \{q_1, q_2, q_3, q_4\}$ and q_1q_3 is its chord. Let $v \in G$ be disjoint of Q. If $\|v, Q\| \geq 3$, or $\|v, Q\| = 2$ and at most one of the edges vq_1 and vq_3 exist, then $G[Q + v]$ contains a DCC.*

Proof. Suppose that $\|v, Q\| \geq 3$. Without loss of generality either $\{q_1, q_2, q_3\} \subseteq N_Q(v)$ or $\{q_1, q_2, q_4\} \subseteq N_Q(v)$. In both cases, $G[Q + v]$ contains a DCC.

Suppose $\|v, Q\| = 2$ and at most one of the edges vq_1 and vq_3 exist. Suppose $vq_1 \in E(G)$, so that $vq_3 \notin E(G)$. Without loss of generality, assume $N_Q(v) = \{q_1, q_2\}$. Then $vq_1q_4q_3q_2v$ forms a DCC with chords q_1q_3 and q_1q_2. A similar DCC exists if $vq_3 \in E(G)$. If neither edge vq_1 and vq_3 exists, $N_q(v) = \{q_2, q_4\}$. Then $vq_2q_3q_1q_4v$ forms a DCC with chords q_1q_2 and q_3q_4. □

Lemma 24. *Suppose Q is a subgraph of G such that $|Q| = 4$ and $G[Q]$ contains a cycle on four vertices. Let xy be an edge disjoint from Q. If $\|xy, Q\| \geq 3$ such that $N_Q(x) \cap N_Q(y) = \emptyset$, then $G[Q + xy]$ contains a DCC.*

Proof. Since Q has a spanning cycle, we can label it as $c_1c_2c_3c_4c_1$. If $\|x, Q\| \geq 3$, then without loss of generality $\{c_1, c_2, c_3\} \subseteq N_Q(x)$. Then $xc_2c_3c_4c_1x$ forms a DCC with chords xc_2 and c_1c_2. So, $\|x, Q\| \leq 2$ and by symmetry $\|y, Q\| \leq 2$. Since $\|\{x, y\}, Q\| \geq 3$, we may assume $\|x, Q\| = 2$ and $\|y, Q\| \geq 1$. Since $N_Q(x) \cap N_Q(y) = \emptyset$, without loss of generality, we can assume that $yc_1, xc_2 \in E(G)$. Then $xc_2c_3c_4c_1yx$ forms a DCC with chord c_1c_2 and another chord incident to x. □

Lemma 25. *Suppose Q is a subgraph of G such that $G[Q] \cong Paw$, and let $v \in G$ disjoint from Q. If $\|v, Q\| \geq 3$, then $G[Q + v]$ contains a DCC.*

Proof. Label $V(Q) = \{x_1, x_2, x_3, x_4\}$ where $d_Q(x_1) = 1$ and $d_Q(x_2) = 3$. If v has three neighbors in $Q - x_1$, then $G[Q + v]$ contains a K_4. So we may assume $vx_1 \in E(G)$, and without loss of generality, $vx_4 \in E(G)$. Then $vx_1x_2x_3x_4v$ is a DCC with chords x_2x_4 and another chord incident to v. □

5. $V(R) \neq V(P)$

In this section, we assume that $V(R) \neq V(P)$ with the goal of arriving at a contradiction. Note that since $V(R) \neq V(P)$, there exists $v \in V(R) \setminus V(P)$. In addition, we define $\mathcal{P}$ to be the set of all vertices $\tilde{p}$ in R such that $\tilde{p}$ is an endpoint of a path $\tilde{P}$ where $V(\tilde{P}) = V(P)$. In other words, $\mathcal{P}$ contains all the endpoints of every spanning path of $G[V(P)]$. Furthermore, p is always assumed to be an endpoint of P.

Lemma 26. *Let $v \in V(R)\setminus V(P)$, then $\|\{v,p\},C\| \leq 6$ for all $C \in \mathcal{C}$.*

Proof. Suppose there exists a $C \in \mathcal{C}$ such that $\|\{v,p\},C\| \geq 7$. By Lemma 15 either $\|v,C\| = 4$ or $\|p,C\| = 4$. Suppose that $\|v,C\| = 4$ and $\|p,C\| \geq 3$. Let $x \in N_C(p)$. By Lemma 15, we can replace C with $G[C - x + v] \cong C$ so that we obtain a new partition $\mathcal{C}'$ and $R' = R - v + x$ that satisfies (O1) and (O2). However, since $xp \in E(G)$, the longest path in R' is longer than the longest path in R, which contradicts (O3).

So suppose that $\|v,C\| = 3$ and $\|p,C\| = 4$. If $C \cong K_4$, then the same argument above holds. So suppose $C \cong K_{1,2,2}$. Since $\|v,C\| = 3$, then up to symmetry, either $N_C(v) = \{b, c_1, c_2\}$ or $N_C(v) = \{c_1, c_2, d_1\}$, otherwise $G[C + v]$ will contain a copy of K_4, contradicting (O1). By Lemma 15, $pd_2 \in E(G)$, so that in either case, we can replace C with $G[C - d_2 + v] \cong C$ so that we obtain a new partition $\mathcal{C}'$ and $R' = R - v + d_2$ that satisfies (O1) and (O2). However, since $d_2 p \in E(G)$, the longest path R' is longer than the longest path in R, which contradicts (O3). This concludes all cases and proves the lemma. □

Lemma 27. *There exists a $\tilde{p} \in \mathcal{P}$ such that $\|\tilde{p}, R\| \leq 2$.*

Proof. Note that for each $\tilde{p} \in \mathcal{P}$, there exists a path $\tilde{P}$ in R such that $V(\tilde{P}) = V(P)$ and $\tilde{p}$ is an endpoint of $\tilde{P}$. Observe that $\|\tilde{p}, R\| = \|\tilde{p}, P\|$, as otherwise we can construct a longer path than P in R, contradicting (O3). For all $\tilde{p} \in \mathcal{P}$, we assume $\|\tilde{p}, P\| \geq 3$ so that in particular, $|P| \geq 2$.

Let p and p' be the endpoints of P. Since $\|p,P\| \geq 3$, let p_1, p_2, and p_3 are the neighbors of p on P such that p, p_1, p_2, and p_3 appear in this order (not necessarily consecutive) along P.

Let $\tilde{p}$ be the vertex immediately preceding p_2 in $[p, p_2]$ (note that perhaps $\tilde{p} = p_1$). Observe that $\tilde{P} = \tilde{p}\overleftarrow{P}pp_2Pp'$ is a path such that $V(\tilde{P}) = V(P)$. So $\tilde{p} \in \mathcal{P}$, and $\|\tilde{p}, R\| \geq 3$. We know that $\tilde{p}$ is already adjacent to p_2, as well as the vertex immediately preceding it on P. So $\tilde{p}$ must be adjacent to a third vertex $\tilde{p}'$.

If $\tilde{p}' \in [p, p_3]$, then pPp_3 is a DCC with chords pp_2 and $\tilde{p}\tilde{p}'$. If $\tilde{p}' \in (p_3, p]$, then $pP\tilde{p}\tilde{p}'\overleftarrow{P}p_2p]$ is a DCC with chords pp_3 and $\tilde{p}p_2$. Either case yields a contradiction, which proves the lemma. □

By Lemma 27, we may assume that P and $p \in P$ are chosen so that $\|p, R\| \leq 2$.

Lemma 28. *For every $v \in V(R) \backslash V(P)$, $\|v, R\| \geq 4$.*

Proof. It follows from our minimum degree constraint and Lemma 26 that

$$2(3k) \leq d_G(v) + d_G(p) = \|\{v, p\}, \mathcal{C}\| + \|\{v, p\}, R\| \leq 6(k-1) + \|\{v, p\}, R\|,$$

so $\|\{v, p\}, R\| \geq 6$. Recall that p was chosen so that $\|p, R\| \leq 2$, and hence $\|v, R\| \geq 4$. □

We now look to complete the case where $V(R) \neq V(P)$. Observe that for all $x \in V(R) \setminus V(P)$, $\|x, P\| \leq 3$, otherwise $G[P + x]$ contains a DCC. Consequently, Lemma 28 implies every such x must have a neighbor in $R \backslash P$, which implies the existence of nontrivial paths in $R \setminus P$, and furthermore implies $|P| \geq 2$.

Now let Q be a longest path in $R \setminus P$, and let v and v' be its endpoints. Since Q is nontrivial, v and v' are distinct vertices. Furthermore, every neighbor of v and v' that is in R, is specifically contained in P or Q, as otherwise we contradict the construction of Q.

By Lemma 28, $\|\{v, v'\}, R\| \geq 8$. By Lemma 17, $\|\{v, v'\}, P\| \leq 4$, otherwise R will contain a DCC. So $\|\{v, v'\}, Q\| \geq 4$. If either $\|v, Q\| \geq 4$ or $\|v', Q\| \geq 4$, then $G[Q]$ contains a DCC. So $\|v, P\| \geq 1$ and $\|v', P\| \geq 1$. Yet, because $\|\{v, v'\}, Q\| \geq 4$, $G[P + Q]$ will contain a DCC with chords incident to either v or v'.

This leads us to our contradication and completes the case when $V(R) \neq V(P)$.

6. $V(R) = V(P)$

In this section, we assume $V(P) = V(R)$. So for every $v \in V(G)$, $\|v, P\| = \|v, R\|$. Since $|R| \geq 4$, we can specify p, q, q', p' as the vertices in R such that $P = pq \ldots q'p'$. If $|R| \geq 5$, then we let r denote the vertex immediately following q along P, and if $|R| \geq 6$, then we also let r' denote the vertex immediately preceeding q' on P. Since R has no DCC, we see that $\|p, R\| \leq 3$, $\|q, R\| \leq 4$, and $\|r, R\| \leq 5$. The same bounds hold for p', q', and r', respectively. Furthermore, for every $v \in P \setminus \{p, q, r, r', q', p'\}$, $\|v, R\| \leq 6$.

Lemma 29. *If $|P| \geq 6$, then there exists $Q = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ along P such that $\|Q, R\| \leq 17$.*

Proof. Label the vertices of P as $P = pqr \cdots r'q'p'$. For each $\alpha \in \{p, q, r\}$, let $\alpha_1, \alpha_2, \ldots$ denote the neighbors of α in $(r, p']$ such that for each $i \geq 2$, if α_{i+1} exists, then $\alpha_{i+1} \in (\alpha_i, p']$. In particular, r_1 always exists and is possibly r'.

In the following, we will often consider pqr and r_1Pp' as two separate non-trivial paths and apply lemmas from Section 4 regarding the number of edges between two non-trivial paths. We will also use the fact that $\|p, (r, p']\| \leq 2$ and $\|q, (r, p']\| \leq 2$, as otherwise we get a DCC in R.

Claim 29.1. If $pr \in E(G)$, then $\|pqr, R\| \leq 9$ and if equality holds then either $\|p, R\| = \|q, R\| = 2$ and $\|r, R\| = 5$, or $\|r, R\| = 4$ and without loss of generality, $\|p, R\| = 3$, $\|q, R\| = 2$, and further, $p_1 \in (r_1, r_2)$.

Proof. Suppose $pr \in E(G)$. Then $G[pqr] \cong K_3$, and p and q are similar vertices as they are both endpoints of a path spanning R. So by Lemma 22.2, we have $\|pqr, (r, p']\| \leq 3$, otherwise R contains a DCC. As $\|pqr, pqr\| = 6$, we get $\|pqr, R\| \leq 9$.

Suppose $\|pqr, R\| = 9$. If $\|r, R\| = 5$, then $\|p, R\| = \|q, R\| = 2$, which proves one part of the claim. Now as $pr \in E(G)$, we have $\|r, R\| \geq 3$. If $\|r, R\| = 3$, then $\|\{p, q\}, (r, p']\| = 2$, and as we noted at the beginning of this section, $\|p, R\| \leq 3$. So $\|q, (r, p']\| \geq 1$ and q_1 exists. If q_2 also exists, then $qq_2\overleftarrow{P}rpq$ is a DCC with chords qq_1 and qr. If p_1 exists, then as p and q are similar vertices, without loss of generality, $p_1 \in [q_1, p']$. However, $pp_1\overleftarrow{P}p$ is a DCC with chords pr and qq_1.

So we must have $\|r, R\| = 4$, and r_2 exists. Furthermore, $\|\{p, q\}, (r, p']\| = 1$ and either p_1 or q_1 exists. As p and q are similar, suppose without loss of generality, that p_1 exists. If $p_1 \in [r_2, p']$, then $pp_1\overleftarrow{P}p$ is a DCC with chords pr and rr_2. If $p_1 = r_1$, then $rr_2\overleftarrow{P}r_1pqr$ is a DCC with chords pr and rr_1. So we must have $p_1 \in (r_1, r_2)$, which completes the proof of the claim. □

Claim 29.2. If $pr \notin E(G)$, then $\|pqr, R\| \leq 8$.

Proof. Suppose $pr \notin E(G)$, and suppose on the contrary $\|pqr, R\| \geq 9$. So $\|pqr, pqr\| = 4$ and $\|pqr, (r, p']\| \geq 5$. By Lemma 20, $\|pqr, (r, p']\| \leq 5$, otherwise R contains a DCC. So $\|pqr, (r, p']\| = 5$. As a result, we must have a pair of distinct vertices $x, y \in \{p, q, r\}$, such that $\|\{x, y\}, (r, p']\| \geq 4$. In fact, equality must hold as otherwise if $\|\{x, y\}, (r, p']\| \geq 5$, R will contain a DCC by Lemma 17. Thus for all $\alpha \in \{p, q, r\}$, $\|\alpha, (r, p']\| \geq 1$.

Note that we cannot have $\{x, y\} = \{p, q\}$ as otherwise, $\|\{p, q\}, [r, p']\| \geq 5$ and R will contain a DCC by Lemma 17. So $r \in \{x, y\}$ and furthermore, as

$rr_1 \in E(G)$, r must play the role of u in Lemma 17 in both configurations, where $u_1 = r_1$.

If configuration 1 holds, then as $\|q,(r,p']\| \|p,(r,p']\| = 1$, both q and p play the role of v so that $q_1, p_1 \in (r_1, r_3)$. Suppose $p_1 \in (r_1, r_2]$. If $q_1 \in [r_1, p_1]$, then $rr_1Pq_1qpp_1Pr_3r$ is a DCC with chords qr and rr_2. If $q_1 \in (p_1, r_3)$, then $rqpp_1Pr_3r$ is a DCC with chords rr_2 and qq_1. So we must have $p_1 \in (r_2, r_3)$. If $q_1 \in (r_1, p_1]$ then pPp_1p is a DCC with chords qq_1 and rr_2, and if $q_1 \in (p_1, r_3)$, then $rr_1Pp_1pqq_1Pr_3r$ is a DCC with chords qr and rr_2. So in all cases we get contradictions so that configuration 2 holds from Lemma 17 where r plays the role of u and either p or q plays the role of v.

Suppose p plays the role of v so that r_1, p_1, p_2, r_2 appear in this order along P (not necessarily consecutive) and $|\{r_1, p_1, p_2, r_2\}| = 4$. If $q_1 \in [r_1, p_1)$, then pPp_2p is a DCC with chords qq_1 and pp_1. If $q_1 \in [p_1, r_2]$, then $rr_2\overleftarrow{P}p_1pqr$ is a DCC with chords qq_1 and pp_2. Lastly, if $q_1 \in (r_2, p']$, then $qq_1\overleftarrow{P}p_2pp_1\overleftarrow{P}r_1rq$ is a DCC with chords pq and rr_2.

So we must have q playing the role of v in Lemma 17. If $p_1 \in [q_2, p']$, then pPp_1p is a DCC with chords qq_1 and qq_2. If $p_1 \in [r_1, q_1]$, then pp_1Pr_2rqp is a DCC with chords qq_1 and qq_2. So we must have $p_1 \in (q_1, q_2)$, however $rPq_1qpp_1Pr_2r$ is a DCC with chords qq_2 and qr. This completes all the cases and proves the claim. □

Now by Claims 29.1 and 29.2, we must have $\|pqr, R\| = \|r'q'p', R\| = 9$, otherwise we are done, and furthermore $pr, r'p' \in E(G)$. Suppose first that $\|p, R\| = \|q, R\| = 2$ and $\|r, R\| = 5$, and so r_2 and r_3 exist. Note that by Claim 29.1, $r_1 \neq r'$ as otherwise $r_2 = q'$, $r_3 = p'$, and as a result both $\|q', R\|, \|p', R\| \geq 3$.

We must have $\|r_1, R\| \geq 5$, otherwise $\|\{p, q, r_1, r', q', p'\}, R\| \leq 17$, and we are done. Now the only neighbors of r_1 are r and those in $(r_1, p']$. So r_1 has two hop neighbors, say x_1 and x_2 in $(r_1, p']$ where $x_2 \in (x_1, p']$. If $x_1 \in (r_1, r_3]$, then $rr_3\overleftarrow{P}r$ is a DCC with chords rr_2 and r_1x_1. So $x_1, x_2 \in (r_3, p']$. However, $r_1x_2\overleftarrow{P}r_2rr_1$ is a DCC with chords rr_3 and r_1x_1.

So by Claim 29.1, we must have $\|r, R\| = 4$ and without loss of generality, $\|p, R\| = 3$ and $\|q, R\| = 2$, with $p_1 \in (r_1, r_2)$. As before, $r_1 \neq r'$, otherwise $p_1 = q'$, $r_2 = p'$, and both $\|q', R\|, \|p', R\| \geq 3$. We also must have $\|r_1, R\| \geq 4$, otherwise $\|\{p, q, r_1, r', q', p'\}, R\| \leq 17$, and we are done. So r_1 has two hop neighbors, say x_1 and $x_2 \in (r_1, p']$ where $x_2 \in (x_1, p']$. If $x_2 \in (r_1, r_2]$, then rPr_2r is a DCC with chords r_1x_1 and r_1x_2. So $x_2 \in (r_2, p']$. However, $r_1x_2\overleftarrow{P}p_1pPr_1$ is a DCC with chords pr and rr_2.

So in all cases we get a contradiction, which proves the lemma. □

Lemma 30. *If $|P| \geq 5$, then there exists $Q = \{v_1, v_2, v_3, v_4, v_5\}$ along P such that $\|Q, R\| \leq 14$.*

Proof. If $|P| = |R| = 5$, then we claim that $|E(R)| \leq 7$. If for all $v \in R$, $\|v, R\| \leq 3$, then $|E(R)| \leq \frac{15}{2}$ and we are done. So there exists $x \in R$ such that $\|x, R\| = 4$, that is, x is a dominating vertex in R. Since R has no DCC, $R - x$ must be acyclic. Thus, $|E(R - x)| \leq 3$ and $|E(R)| \leq 7$. Therefore, if $|P| = 5$, then we can let $Q = V(P)$ to obtain $\|Q, R\| \leq 14$.

So we may assume $|P| \geq 6$. By Lemma 29, there exists $Q \subseteq V(P)$ such that $|Q| = 6$ and $\|Q, R\| \leq 17$. If $\|Q, R\| \leq 14$, then for any $x \in Q$, $\|Q - x, R\| \leq 14$ and we are done. So $\|Q, R\| \geq 15$ and there exists $y \in Q$ such that $\|y, R\| \geq 3$. However, $\|Q - y, R\| \leq 14$, and we are done. □

Lemma 31. *Let $|P| \geq 5$, let $Q = \{v_1, v_2, v_3, v_4, v_5\} \subseteq V(P)$, and let $C \in \mathcal{C}$. If $\|Q, C\| \geq 16$, then $\|Q, C\| \leq 17$, and furthermore, one of the following configurations occurs.*

1. *$C \cong K_4$ with $N_C(\{v_1, v_3, v_5\}) \subseteq \{a_1, a_2, a_3\}$ and $N_C(\{v_2, v_4\}) \subseteq V(C)$,*
2. *$C \cong K_{1,2,2}$ with $N_C(\{v_1, v_3, v_5\}) \subseteq \{b, c_1, c_2\}$ and $N_C(\{v_2, v_4\}) \subseteq \{c_1, c_2, d_1, d_2\}$, or*
3. *$C \cong K_{1,2,2}$ with $N_C(\{v_1, v_3, v_5\}) = \{b, c_1, c_2\}$, $N_C(v_4) = \{b, d_1, d_2\}$, and $N_C(v_2) = \{c_1, c_2, d_1, d_2\}$, or*
4. *$C \cong K_{1,2,2}$ with $N_C(\{v_1, v_3, v_4, v_5\}) = \{b, c_1, c_2\}$, and $N_C(v_2) = \{c_1, c_2, d_1, d_2\}$.*

Note that in configurations 1 and 2, $\|Q, C\| \in \{16, 17\}$, and in configurations 3 and 4 $\|Q, C\| = 16$.

Proof. Suppose that $|P| \geq 5$, and let $Q = \{v_1, v_2, v_3, v_4, v_5\} \subseteq V(P)$, labeled so that v_1, v_2, v_3, v_4, and v_5 appear in this order (not necessarily consecutive) along P. Suppose also that $\|Q, C\| \geq 16$ for some $C \in \mathcal{C}$. Thus $\|v, C\| = 4$ for some $v \in Q$, and by Lemma 15, $C \cong K_4$ or $C \cong K_{1,2,2}$. Recall that if $C \cong K_4$, then $V(C) = \{a_1, a_2, a_3, a_4\}$, and if $C \cong K_{1,2,2}$, then $V(C) = \{b, c_1, c_2, d_1, d_2\}$, where c_1 and c_2 are the vertices in one partite set of size two, d_1 and d_2 are the vertices in the other partite set of size two, and b is the dominating vertex.

Case 1. $C \cong K_4$.

Our goal in this case is to prove that configuration 1 of this lemma holds.

Claim 31.1. There exists $a_i \in C$ such that either $\|\{v_1, v_2\}, C - a_i\| = 6$ or $\|\{v_4, v_5\}, C - a_i\| = 6$.

Proof. Suppose on the contrary that for all $a_i \in C$, we satisfy $\|\{v_1, v_2\}, C - a_i\| \leq 5$ and $\|\{v_4, v_5\}, C - a_i\| \leq 5$. We claim $\|\{v_1, v_2\}, C\| \leq 6$. Indeed, if $\|\{v_1, v_2\}, C\| \geq 7$, then we may assume $\|v_1, C\| = 4$ and $a_1, a_2, a_3 \in N_C(v_2)$. However, $\|\{v_1, v_2\}, C - a_4\| = 6$, a contradiction.

So $\|\{v_1, v_2\}, C\| \leq 6$ and by symmetry, $\|\{v_4, v_5\}, C\| \leq 6$. Since $\|Q, C\| \geq 16$, we have $\|v_3, C\| = 4$. So, in fact, we must have $\|\{v_1, v_2\}, C\| = \|\{v_4, v_5\}, C\| = 6$, else $\|Q, C\| < 16$. Since $\|\{v_1, v_2\}, C\| = 6$, v_1 and v_2 have two common neighbors, say a_1 and a_2; thus, $G[v_1Pv_2+a_1+a_2]$ contains a DCC. If we can show $G[v_3Pv_5+a_3+a_4]$ also contains a DCC, then we are done by contradiction. Since $\|\{v_4, v_5\}, C-a_i\| \leq 5$ for each $i \in \{3, 4\}$, and $\|\{v_4, v_5\}, C\| = 6$, we deduce that $\|\{v_4, v_5\}, \{a_3, a_4\}\| \geq 2$. So $\|\{a_3, a_4\}, v_3Pv_5\| \geq 4$. By Lemma 17, we must have equality and either configuration 1 or 2 occurs where a_3a_4 plays the role of P_1 and v_3Pv_5 plays the role of P_2. However, in neither configuration is $\|v_3, a_3a_4\| = 2$. So $G[v_3Pv_5 + a_3 + a_4]$ contains a DCC by Lemma 17, and we are done. □

Claim 31.2. If $\|\{v_1, v_2\}, C - a_i\| = 6$ for some $a_i \in C$, then each of the following hold:

1. for all $a_\ell \in C - a_i$, $\|a_\ell a_i, \{v_3, v_4, v_5\}\| \leq 4$,
2. $\|\{v_1, v_2\}, C\| \leq 7$, and
3. $N_C(\{v_3, v_5\}) \subseteq V(C - a_i)$.

Symmetric statements hold if $\|\{v_4, v_5\}, C - a_i\| = 6$.

Proof. In all the following we assume without loss of generality that $\|\{v_1, v_2\}, C - a_4\| = 6$. Observe $G[v_1Pv_2+a_i+a_j]$ contains a DCC for all $1 \leq i < j \leq 3$.

We cannot have $G[v_3Pv_5+a_4+a_\ell]$ contain a DCC for all $\ell \in \{1, 2, 3\}$, else we get two disjoint DCCs in $G[R+C]$. Thus, by Lemma 18, $\|a_\ell a_4, v_3Pv_5\| \leq 4$, and in particular, $\|a_\ell a_4, \{v_3, v_4, v_5\}\| \leq 4$, for all $\ell \in \{1, 2, 3\}$. This proves the first item in the claim.

We now prove item 2. Suppose on the contrary that $\|\{v_1, v_2\}, C\| = 8$, which is the most it can possibly be by Lemma 15. This implies that for all $a_j \in C$, $\|\{v_1, v_2\}, C - a_j\| = 6$, and so by item 1 of this claim, $\|a_ia_j, \{v_3, v_4, v_5\}\| \leq 4$ for all $1 \leq i < j \leq 4$. As a result, $\|C, \{v_3, v_4, v_5\}\| \leq 8$. However, as $\|\{v_1, v_2\}, C\| = 8$ and $\|Q, C\| \geq 16$, we must have equality so that $\|C, \{v_3, v_4, v_5\}\| = 8$ and furthermore, $\|a_ia_j, \{v_3, v_4, v_5\}\| = 4$ for all $1 \leq i < j \leq 4$.

In particular, $\|a_1a_2, \{v_3, v_4, v_5\}\| = 4$, and since we can form a DCC in with $G[v_1Pv_2 + a_3 + a_4]$, Lemma 17 implies either configuration 1 or 2 holds with a_1a_2 as P_1 and v_3Pv_5 as P_2. Configuration 1 must hold otherwise we would need four vertices in $\{v_3, v_4, v_5\}$. So without loss of generality, $\|a_1, \{v_3, v_4, v_5\}\| = 3$ and $\|a_2, \{v_3, v_4, v_5\}\| = 1$. However, since $\|a_ia_j,$

$\{v_3, v_4, v_5\}\| = 4$ for all $1 \le i < j \le 4$ and $\|a_1, \{v_3, v_4, v_5\}\| = 3$, we would have $\|C, \{v_3, v_4, v_5\}\| \le 6$, a contradiction. This proves item 2 of the claim.

To prove item 3, suppose that $\|a_4, \{v_3, v_5\}\| \ge 1$. Since $\|\{v_1, v_2\}, C\| \le 7$ by item 2 of this claim, we must have $\|\{v_3, v_4, v_5\}, C\| \ge 9$. Additionally, as $\|a_\ell a_4, \{v_3, v_4, v_5\}\| \le 4$, for all $\ell \in \{1, 2, 3\}$, if $\|a_4, \{v_3, v_4, v_5\}\| \ge 2$, then $\|C, \{v_3, v_4, v_5\}\| \le 8$, a contradiction. So $\|a_4, \{v_3, v_4, v_5\}\| \le 1$, and in fact equality holds. Further, $\|\{a_1, a_2, a_3\}, \{v_3, v_4, v_5\}\| \ge 8$. Since $\|a_i, \{v_3, v_4, v_5\}\| \le 3$ for each i, we may assume $\|a_1, \{v_3, v_4, v_5\}\| = \|a_2, \{v_3, v_4, v_5\}\| = 3$. So $\|a_1 a_4, \{v_3, v_4, v_5\}\| = 4$.

Recall that $G[v_1 P v_2 + a_i + a_j]$ contains a DCC for all $1 \le i < j \le 3$. In particular, $G[v_1 P v_2 + a_2 + a_2]$ contains a DCC so that $G[v_3 P v_5 + a_1 + a_4]$ cannot contain a DCC. So as $\|a_1 a_4, \{v_3, v_4, v_5\}\| = 4$, by Lemma 17, either configuration 1 or 2 holds. Similar to the above, we must have configuration 1. So as $\|a_4, \{v_3, v_4, v_5\}\| = 1$, a_1 plays the role of u and a_4 plays the role of v. In particular, v_3 and v_5 are not adjacent to a_4, a contradition to $\|a_4, \{v_3, v_5\}\| \ge 1$. Thus, $N_C(\{v_3, v_5\}) \subseteq \{a_1, a_2, a_3\}$, which proves item 3, and finishes the proof of the claim. ☐

By Claim 31.1, we may assume without loss of generality that $\|\{v_1, v_2\}, C - a_4\| = 6$. By item 3 in Claim 31.2, we know $N_C(\{v_3, v_5\}) \subseteq \{a_1, a_2, a_3\}$. So it remains to show $N_C(v_1) \subseteq \{a_1, a_2, a_3\}$ to complete this case. So suppose $v_1 a_4 \in E(G)$. By item 2 in Claim 31.2, $\|\{v_1, v_2\}, C\| \le 7$, and since $N_C(v_3) \subseteq \{a_1, a_2, a_3\}$, we have $\|\{v_1, v_2, v_3\}, C\| \le 10$, which implies $\|\{v_4, v_5\}, C\| \ge 6$. If $\|\{v_4, v_5\}, C - a_4\| = 6$, then by item 3 in Claim 31.2, $N_C(\{v_1, v_3\}) \subseteq \{a_1, a_2, a_3\}$, a contradiction as we assumed $v_1 a_4 \in E(G)$. So we must have $\|\{v_4, v_5\}, C - a_4\| \le 5$, and as $N_C(\{v_3, v_5\}) \subseteq \{a_1, a_2, a_3\}$, we deduce that $\|\{v_3, v_4, v_5\}, C\| \le 9$ and furthermore, the only way equality holds is if $v_4 a_4 \in E(G)$ and $N_C(v_3) = \{a_1, a_2, a_3\}$. Since $\|\{v_1, v_2\}, C\| \le 7$ and $\|Q, C\| \ge 16$, we must have $\|\{v_3, v_4, v_5\}, C\| = 9$, and consequently, $\|\{v_1, v_2\}, C\| = 7$, $v_4 a_4 \in E(G)$, $N_C(v_3) = \{a_1, a_2, a_3\}$, and $\|\{v_4, v_5\}, C - a_4\| = 5$.

Recall that we are assuming $v_1 a_4 \in E(G)$. If $v_2 a_4 \in E(G)$, then as $\|\{v_1, v_2\}, C\| = 7$, we may assume without loss of generality that $G[v_1 P v_2 + a_1 + a_4]$ contains a DCC. As $\|\{v_3, v_4, v_5\}, C\| = 9$ and $N_C(\{v_3, v_5\}) \subseteq V(C) - a_4$ by item 3 in Claim 31.2, we get $\|\{v_3, v_4, v_5\}, a_2 a_3\| \ge 5$. However, Lemma 18 implies $G[v_3 P v_5 + a_2 + a_3]$ contains a DCC.

So $v_2 a_4 \notin E(G)$, which implies $\|v_1, C\| = 4$ and $\|v_2, C - a_4\| = 2$. Thus $\|Q - v_1, C - a_4\| \ge 10$, and in particular, there exists $a_i \in C - a_4$ such that $\|a_i, Q - v_1\| = 4$. However, this results in two disjoints DCCs in $C - a_i + v_1$ and $v_2 P v_5 + a_i$ in $G[R + C]$, a contradiction.

Thus, we must have $N_C(\{v_1, v_3, v_5\}) \subseteq \{a_1, a_2, a_3\}$, which completes this case.

Case 2. $C \cong K_{1,2,2}$.

A (T, e)-partition is a partition of C into two subgraphs, T and e, in which T is a triangle and e is an edge.

Claim 31.3. For every (T, e)-partition, $G[v_1Pv_2 + e]$ (and by symmetry $G[v_4Pv_5 + e]$) does not contain a DCC.

Proof. Fix a (T, e)-partition of C. Suppose $G[v_1Pv_2 + e]$ contains a DCC, and suppose also that $G[v_1Pv_2+T]$ also contains a DCC. Since $G[R+C]$ does not contain two disjoint DCCs, we cannot have $G[v_3Pv_5+T]$ or $G[v_3Pv_5+e]$ contain a DCC. So by Lemmas 22 and 18, $\|v_3Pv_5, T\| \leq 3$ and $\|v_3Pv_5, e\| \leq 4$. As a result, $\|\{v_3, v_4, v_5\}, C\| \leq 7$. However, since $\|Q, C\| \geq 16$, this implies $\|\{v_1, v_2\}, C\| \geq 9$, which contradicts Lemma 15.

So suppose $G[v_1Pv_2 + e]$ contains a DCC, but $G[v_1Pv_2 + T]$ does not. Again, by Lemma 22, $\|\{v_3, v_4, v_5\}, T\| \leq 3$. The same lemma implies that since $G[v_1Pv_2 + T]$ does not contain a DCC, $\|\{v_1, v_2\}, T\| \leq 2$. So $\|\{v_1, v_2\}, T\| \leq 2$, and $\|\{v_1, v_2\}, C\| \leq 6$. However, since $\|\{v_3, v_4, v_5\}, T\| \leq 3$, we have $\|\{v_3, v_4, v_5\}, e\| \geq 7$, a contradiction as this can be at most six. □

Claim 31.4. For every (T, e)-partition, $G[v_1Pv_2 + T]$ (and by symmetry $G[v_4Pv_5 + T]$) must contain a DCC. Furthermore, $\|e, Q\| \leq 7$.

Proof. Fix a (T, e)-partition of C, say $T = bc_1d_1b$ and $e = c_2d_2$. By Claim 31.3, $G[v_1Pv_2 + e]$ does not contain a DCC so that $\|\{v_1, v_2\}, e\| \leq 3$. We wish to show that $G[v_1Pv_2 + T]$ contains a DCC; so if not, then by Lemma 22, $\|\{v_1, v_2\}, T\| \leq 2$. Thus, $\|\{v_1, v_2\}, C\| \leq 5$ and as $\|Q, C\| \geq 16$, $\|\{v_3, v_4, v_5\}, C\| \geq 11$. Note that if $\|\{v_4, v_5\}, C\| \geq 8$, then $G[v_4Pv_5 + e]$ would contain a DCC, contradicting Claim 31.3. So we must have $\|\{v_4, v_5\}, C\| = 7$, $\|v_3, C\| = 4$, and by Lemma 15, $N_C(v_3) = V(C - b)$. Furthermore, $\|\{v_1, v_2\}, C\| = 5$.

Since $\|\{v_1, v_2\}, C\| = 5$, the inequalities of $\|\{v_1, v_2\}, T\| \leq 2$ and $\|\{v_1, v_2\}, e\| \leq 3$ must be equality. As $N_C(v_3) = V(C - b)$, $G[v_1Pv_3 + e]$ contains a DCC. However, as $\|\{v_4, v_5\}, C\| = 7$, we have $\|v_4Pv_5, T\| \geq 3$, which by Lemma 22 implies $G[v_4Pv_5 + T]$ contains a DCC. So $G[R + C]$ contains two disjoint DCCs, a contradiction. This proves that $G[v_1Pv_2 + T]$ contains a DCC.

To show $\|e, Q\| \leq 7$, recall that by Claim 31.3 neither $G[v_1Pv_2 + e]$ or $G[v_4Pv_5 + e]$ contain a DCC. Therefore, $\|e, \{v_1, v_2\}\| \leq 3$ and $\|e, \{v_4, v_5\}\| \leq 3$. If $\|e, v_3\| = 2$, then $\|v_3Pv_5, e\| \geq 5$ and by Lemma 17, $G[v_3Pv_5 + e]$ contains a DCC. However, we just showed that $G[v_1Pv_2 + T]$ contains a DCC. This completes the proof of the claim. □

By Claim 31.4, $\|\{c_i, d_i\}, Q\| \leq 7$ for each $i \in \{1, 2\}$. Hence $\|C - b, Q\| \leq 14$, which implies $\|b, Q\| \geq 2$. Note that by Lemma 15, if $\|b, Q\| = 5$, then for all $v \in Q$, $\|v, C\| \leq 3$ contradicting $\|Q, C\| \geq 16$. So $2 \leq \|b, Q\| \leq 4$.

Subcase 2.1. $\|b, Q\| = 4$.

We will show that configuration 3 or 4 of this lemma holds. By Lemma 15 and to satisfy $\|Q, C\| \geq 16$, there exists only one vertex, call it $v \in Q$, such that $\|v, C\| = 4$, and all others are adjacent to b. Without loss of generality, $v \in \{v_1, v_2, v_3\}$ so that $\|\{v_1, v_2, v_3\}, b\| = 2$. For $i \in \{1, 2\}$, let $T_i = bc_id_ib$, and $e_i = c_id_i$. By Claim 31.4, $G[v_4Pv_5 + T_i]$ contains a DCC for each $i \in \{1, 2\}$. So $G[v_1Pv_3 + e_i]$ cannot contain a DCC. By Lemma 17, $\|\{v_1, v_2, v_3\}, e_i\| \leq 4$ for each i. So $\|\{v_1, v_2, v_3\}, C - b\| \leq 8$ and $\|\{v_1, v_2, v_3\}, C\| \leq 10$. Since $\|Q, C\| \geq 16$, we have $\|\{v_4, v_5\}, C\| \geq 6$; however since $v \in \{v_1, v_2, v_3\}$, we know $\|\{v_4, v_5\}, C\| = 6$ so that the previous inequalities must be equality. For example, $\|\{v_1, v_2, v_3\}, e_i\| = 4$ for each i, and in particular, when $i = 1$. By Lemma 17, we must have configuration 1 or 2 in which e_i plays the role of P_1 and v_1Pv_3 plays the role of P_2. Since configuration 2 requires $\{v_1, v_2, v_3\}$ to have at least four vertices, we must have configuration 1. Thus, v_2 is adjacent to both c_1 and d_1, and v_1 and v_3 have the same neighbor, say c_1.

Note that bc_2d_1b and c_1d_2 is another (T, e)-partition for which all the previous arguments hold. In particular, v_2 is adjacent to both c_1 and d_2, and v_1 and v_3 are not adjacent to d_2 as they are already both adjacent to c_1. Again, when considering $T = bc_1d_1b$ and $T = c_2d_2$, we get $N_C(v_2) = V(C - b)$, and $N_C(v_1) = N_C(v_3) = \{b, c_1, c_2\}$.

Recall that $bv_4, bv_5 \in E(G)$. We cannot have $v_5d_i \in E(G)$ for some $i \in \{1, 2\}$, as otherwise $v_5d_ibv_3Pv_5$ is a DCC with chords v_5b and v_4b, and $v_2c_1d_{3-i}c_2v_1Pv_2$ is a DCC with chords v_1c_1 and v_2c_2. So we must have $N_C(v_5) = \{b, c_1, c_2\}$.

To show that either configuration 3 or 4 of this lemma holds, we only need to show $N_C(v_4)$ is either $\{b, c_1, c_2\}$ or $\{b, d_1, d_2\}$. Suppose on the contrary that without loss of generality, $N_C(v_4) = \{b, c_1, d_1\}$; however, this results in replacing C with the K_4 in $G[\{v_4, b, c_1, d_1\}]$ which contradicts (O1). This completes the case when $\|b, Q\| = 4$.

Subcase 2.2. $2 \leq \|b, Q\| \leq 3$.

Here we will show configuration 2 of this lemma holds. Since $\|Q, C\| \geq 16$, we have $\|C - b, Q\| \geq 13$. So for $e_i = c_id_i$ where $i \in \{1, 2\}$, we may assume without loss of generality, $\|e_1, Q\| \geq 7$. However, recall by Claim 31.4 that $\|e, Q\| \leq 7$ for all e in a (T, e)-partition, which each e_i is. Thus,

$\|e_1, Q\| = 7$. By Claim 31.3, $\|e_1, \{v_4, v_5\}\| \leq 3$ so that $\|e_1, \{v_1, v_2, v_3\}\| \geq 4$. However, if $G[v_1Pv_3 + e_1]$ contains a DCC, then we get a contradiction as Claim 31.4 implies $G[v_4Pv_5 + b + c_2 + d_2]$ contains a DCC. So by Lemma 17, $\|e_1, \{v_1, v_2, v_3\}\| = 4$, and one of two configurations holds where e_1 plays the role of P_1 and v_1Pv_3 plays the role of P_2. Since configuration 2 of Lemma 17 requires at least four vertices in $\{v_1, v_2, v_3\}$, we must have configuration 1; furthermore, v_2 is adjacent to both c_1 and d_1, and without loss of generality, v_1 and v_3 are both adjacent to c_1 and not adjacent to d_1

Recall $\|e_1, Q\| = 7$, so we may also argue that $\|e_1, \{v_3, v_4, v_5\}\| \geq 4$, and by symmetry, v_4 is adjacent to both c_1 and d_1, and v_3 and v_5 are both adjacent to c_1 and not adjacent to d_1. We now let $e_1^* = c_1d_2$ and $e_2^* = c_2d_1$. As $\|C - b, Q\| \geq 13$ and $\|e, Q\| = 7$ for all e in a (T, e)-partition, either $\|e_1^*, Q\| = 7$ or $\|e_2^*, Q\| = 7$. In either case, all the above arguments apply.

If $\|e_1^*, Q\| = 7$, then because we already know every vertex in Q is adjacent to c_1, the above argument implies that v_1, v_3, v_5 are not adjacent to d_2, but v_2 and v_4 are. Since v_2 and v_4 are both adjacent to c_1 and d_1, we cannot have bv_2 or $bv_4 \in E(G)$, otherwise we can replace C with a copy of K_4, contradicting (O1). This yields configuration 2, as we allow any vertex in Q to be ajdacent to c_2.

If $\|e_2^*, Q\| = 7$, then because we already know that the only vertices in Q that are adjacent to d_1 are v_2 and v_4, the same argument implies that c_2 is adjacent to all the vertices in Q. As a result, b and d_2 cannot have common neighbors in Q, otherwise we can replace C with a copy of K_4 that includes c_2. Since v_2 and v_4 are both adjacent to c_1 and d_1, then $bv_2, bv_4 \notin E(G)$, otherwise we can replace C with a copy of K_4 contradicting (O1). So the only vertices possibly adjacent to b are v_1, v_3, and v_5. So if $\|b, Q\| = 3$, then because every vertex in Q is adjacent to c_2, d_2 can only be adjacent to v_2 and v_4, which yields configuration 2. If $\|b, Q\| = 2$, then we actually have $\|C - b, Q\| \geq 14$, from which we can conclude $\|e_1^*, Q\| = 7$, and we again get configuration 2.

This completes all cases and proves the lemma. □

Lemma 32. $|P| \leq 5$.

Proof. Let $|P| \geq 6$. Then by Lemma 29 there exists $Q = \{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq V(P)$ such that $\|Q, R\| \leq 17$. As $\delta(G) \geq 3k$, we get $6(3k) \leq \|Q, R\| + \|Q, \mathcal{C}\| \leq 17 + \|Q, \mathcal{C}\|$. Therefore, $\|Q, \mathcal{C}\| > 18(k-1)$, which implies there exists a $C \in \mathcal{C}$ such that $\|Q, \mathcal{C}\| \geq 19$. Thus $\|x, C\| = 4$ for some $x \in Q$, and by Lemma 15, $C \cong K_4$ or $C \cong K_{1,2,2}$. If there exists $u \in Q$ such that $\|u, C\| \leq 1$, then $\|Q - u, C\| \geq 18$ which contradicts Lemma 31. So, $\|u, C\| \geq 2$ for all $u \in Q$.

Suppose there exists $v \in Q$ such that $\|v, C\| = 2$. Relabel the vertices in $Q-v$ as u_1, u_2, u_3, u_4 and u_5 so that they appear in this order, not necessarily consecutive, along P. Without loss of generality, either $v \in [p, u_1)$, $v \in (u_1, u_2)$, or $v \in (u_2, u_3)$. Since $\|v, C\| = 2$ and $\|Q, C\| \geq 19$, we have $\|Q - v, C\| \geq 17$. So by Lemma 31, equality holds and $G[(Q - v) + C]$ is either configuration 1 or 2 in Lemma 31.

Suppose first that we have configuration 1. Since $\|Q - v, C\| = 17$, we must have $N_C(\{u_1, u_3, u_5\}) = \{a_1, a_2, a_3\}$ and $N_C(\{u_2, u_4\}) = V(C)$ by Lemma 31. Since $\|v, C\| = 2$, without loss of generality, $va_1 \in E(G)$. Note that $G[u_4Pu_5 + C - a_1]$ contains a DCC. If $v \in (u_1, u_2)$ or $v \in (u_2, u_3)$, then $u_1Pu_3a_1u_1$ is a DCC with chords va_1 and u_2a_1. If $v \in [p, u_1)$, then vPu_3a_1v is a DCC with chords u_1a_1 and u_2a_1. In either case, we get two disjoint DCCs, which is a contradiction. This completes the case when configuration 1 holds.

Now suppose we have configuration 2. Since $\|Q-v, C\| \geq 17$, by Lemma 31, we know equality holds and furthermore, $N_C(\{u_1, u_3, u_5\}) = \{b, c_1, c_2\}$ and $N_C(\{u_2, u_4\}) = \{c_1, c_2, d_1, d_2\}$. Since $\|v, C\| = 2$, we may assume that v is adjacent to either c_1 or d_1. In either case, note that $G[u_4Pu_5 + c_2 + d_2 + b]$ contains a DCC. Suppose first that v is adjacent to c_1. If $v \in (u_1, u_2)$ or $v \in (u_2, u_3)$, then $u_1Pu_3c_1u_1$ is a DCC with chords u_2c_1 and vu_2. If $v \in [p, u_1)$, then vPu_3c_1v is a DCC with chords u_1c_1 and u_2c_1. Now suppose v is adjacent to d_1. If $v \in [p, u_1)$ or $v \in (u_1, u_2)$, then $vd_1c_1u_3\overleftarrow{P}v$ is a DCC with chords u_2c_1 and u_2d_1. If $v \in (u_2, u_3)$, then $u_1Pvd_1c_1u_1$ is a DCC with chords u_2c_1 and u_2d_1. In any case, we get two disjoint DCCs, which is a contradiction. This completes the case when configuration 2 holds.

This implies that for all $v \in Q$, $\|v, C\| \geq 3$. We now return to our original labeling of the vertices of Q as $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let's now assume that v_1, v_2, v_3, v_4, v_5, and v_6 appear in this order (not necessarily consecutive) along P. Suppose $\|v_6, C\| = 3$ so that $\|Q - v_6, C\| \geq 16$. In this case, Lemma 31 holds, and one of the configurations listsed occurs. Note that in each configuration, at least one of v_2 or v_4 has four neighbors on C, and furthermore, v_1, v_3, and v_5 each have at most three neighbors on C. Since we showed above that for all $v \in Q$, $\|v, C\| \geq 3$, we know in particular that $\|v_1, C\| = 3$. Yet this implies that $\|Q - v_1, C\| \geq 16$, so that one of the configurations in Lemma 31 holds here as well. However, this would imply that at least one of v_3 or v_5 would need to have four neighbors on C, which cannot happen as we just saw that each has at most three neighbors.

So $\|v_6, C\| = 4$ and by symmetry, $\|v_1, C\| = 4$. Since $\|v, C\| \geq 3$ for all $v \in Q$, we see that $\|Q - v_6\| \geq 16$, which implies that one of the configurations in Lemma 31 holds. However, in none of the configurations is $\|v_1, C\| = 4$, a contradiction. This proves the lemma. $\square$

Lemma 33. *If* $|P| = 5$, *then* $R \cong K_{1,1,3}$.

Proof. Let $|P| = 5$. Since $|P| = |R| = 5$, by Lemma 30, $\|R, R\| \leq 14$. We claim there exists $C \in \mathcal{C}$ such that $\|R, C\| \geq 16$. If not, then as $\delta(G) \geq 3k$, we get: $5(3k) \leq \|R, R\| + \|R, \mathcal{C}\| \leq 14 + 15(k-1)$. However, this implies $15k \leq 15k - 1$, a contradiction. So $\|R, C\| \geq 16$ for some $C \in \mathcal{C}$, and by Lemma 31, one of the four configurations hold. First note that if configuration 4 holds, then we can replace $C \cong K_{1,2,2}$ with $G[\{b, c_1, v_3, v_4\}] \cong K_4$, contradicting (O1). So we only need to consider configurations 1–3.

Now since $\|R, R\| \leq 14$, we know $|E(R)| \leq 7$. Furthermore, by inspection, the only 5-vertex graph with seven edges and no DCC is $K_{1,1,3}$. So if $R \ncong K_{1,1,3}$, then $|E(R)| < 7$. Our goal in the following is to consider each of the remaining three configurations from Lemma 31, and show that in each one, we can find disjoint graphs H_1 and H_2 in $R + C$ such that $H_1 \cong C$ and $H_2 \cong K_{1,1,3}$. This results in a new collection that will satisfy (O1), (O2), and (O3), but contradict (O4).

First, if configuration 3 occurs, then $G[\{b, c_1, v_3, v_4, v_5]\} \cong K_{1,2,2}$, and $G[\{v_1, v_2, c_2, d_1, d_2\}] \cong K_{1,1,3}$. Now consider configuration 1. Since $\|Q, C\| \geq 16$, at most one edge is missing between Q and C. If a_1 is not adjacent to v_1 or v_2, then $G[\{v_1, v_2, a_2, a_3\}] \cong K_4$ and $G[\{a_1, a_4, v_3, v_4, v_5\}] \cong K_{1,1,3}$. If a_1 is not adjacent to v_3 or if a_4 is not adjacent to v_2, then $G[\{a_1, a_2, v_1, v_2\}] \cong K_4$ and $G[\{a_3, a_4, v_3, v_4, v_5\}] \cong K_{1,1,3}$. This covers all the cases by symmetry.

Lastly, consider configuration 2. Once again since $\|Q, C\| \geq 16$, at most one edge is missing between Q and C. There are a few cases to consider. Suppose c_1 is not adjacent to either v_1, v_2, or v_3, then $G[\{v_1, v_2, v_3, c_2, d_2\}] \cong K_{1,1,3}$ and $G[\{v_4, v_5, c_1, d_1, b\}] \cong K_{1,2,2}$. Notice these same structures exist if $bv_1 \notin E(G)$ or $bv_3 \notin E(G)$. If d_1 is not adjacent to v_2, then then $G[\{v_1, v_2, v_3, c_1, b\}] \cong K_{1,2,2}$ and $G[\{v_4, v_5, c_2, d_1, d_2\}] \cong K_{1,1,3}$. By symmetry, this covers all cases and proves the lemma. ◻

Lemma 34. $|P| = 4$.

Proof. By Lemma 32, $|P| \leq 5$. So suppose $|P| = 5$. By Lemma 33, $G[P] \cong K_{1,1,3}$. Let v_1, v_2, v_3 be the vertices in P such that $d_R(v_i) = 2$ for each i, and let $F = \{v_1, v_2, v_3\}$. We claim that for all $C \in \mathcal{C}$, $\|F, C\| \leq 9$.

Suppose on the contrary that $\|F, C\| \geq 10$ for some $C \in \mathcal{C}$. Without loss of generality, suppose $\|v_1, C\| = 4$. By Lemma 15, $|C| \leq 5$. Since $\|F, C\| \geq 10$, v_2 and v_3 have a common neighbor in C, say x. Then $G[C - x + v_1]$ and $G[P - v_1 + x]$ each contain a DCC, a contradiction. So $\|F, C\| \leq 9$.

However, this yields the following contradiction: $3(3k) \leq \|F, \mathcal{C}\| + \|F, R\| \leq 9(k-1) + 6 = 9k - 3$. ◻

6.1. $|R| = |P| = 4$

In the following, we assume the vertices of R are labeled so that $P = v_1v_2v_3v_4$.

Lemma 35. *There exists $C \in \mathcal{C}$ such that $\|R, C\| \geq 13$, and consequently, $R \cong K_{1,1,2}$.*

Proof. Suppose that for all $C \in \mathcal{C}$, $\|R, C\| \leq 12$. Note that $\|R, R\| \leq 10$ as $R \not\cong K_4$. However, this yields the following contradiction: $4(3k) \leq \sum_{i=1}^{4} d_G(v_i) = \|R, R\| + \|R, \mathcal{C}\| \leq 10 + 12(k-1) = 12k - 2$.

This proves the first part of the statement of Lemma 35. So suppose $\|R, C\| \geq 13$ for some $C \in \mathcal{C}$, and suppose $R \not\cong K_{1,1,2}$. Since $\|R, C\| \geq 13$, there exists $v_i \in R$ such that $\|v_i, C\| = 4$. So, by Lemma 15, $C \in \{K_4, K_{1,2,2}\}$.

Note that $K_{1,1,2}$ is the only 4-vertex graph with five edges. So if $R \not\cong K_{1,1,2}$, then $|E(R)| < 5$. In each of the following cases we will find disjoint graphs H_1 and H_2 in $R + C$ such that $H_1 \cong C$ and $H_2 \cong K_{1,1,2}$. This results in a new collection that will satisfy (O1), (O2), and (O3), but contradict (O4).

Case 1. $C \cong K_4$.

Suppose $\|v_1, C\| = 4$. Then $\|R - v_1, C\| \geq 9$. So without loss of generality, $\|a_1, R - v_1\| = 3$. Thus, we can replace C and R with $G[C - a_1 + v_1] \cong K_4$ and $G[R - v_1 + a_1] \cong K_{1,1,2}$, respectively.

So $\|v_1, C\| \leq 3$, and by symmetry, $\|v_4, C\| \leq 3$. Without loss of generality, suppose $\|v_2, C\| = 4$. Then, as in the previous case, we may assume $\|a_1, R - v_2\| = 3$. Observe that if we replace C and R with $G[C - a_1 + v_2] \cong K_4$ and $G[R - v_2 + a_1]$, respectively, then $G[R - v_2 + a_1]$ has at least four edges. So R must have at least four edges, otherwise we contradict (O4). Thus $R \in \{C_4, Paw\}$. However, if $R \cong C_4$, as $\|v_2, C\| = 4$, then by symmetry we are done by the previous case. Therefore $R \cong Paw$.

Note that in the Paw, three of the four vertices are endpoints of paths spanning the Paw. As we have assumed $\|v_2, C\| = 4$ and have shown that $\|v_1, C\| \leq 3$ above, we may assume that $d_R(v_2) = 3$, so that $\|v_i, C\| \leq 3$ for $i \in \{1, 3, 4\}$, otherwise we are again done by the previous case. In fact, equality must hold as $\|R, C\| \geq 13$.

As a result, v_3 and v_4 have a common neighbor, say a_1, in C. However, we can replace C and R with $G[R - v_1 + a_1] \cong K_4$ and $G[C - a_1 + v_1] \in \{K_4, K_{1,1,2}\}$, respectively. This completes the case when $C \cong K_4$.

Case 2. $C \cong K_{1,2,2}$.

In this case, we first prove the following claim.

Claim 35.1. $|E(R)| \geq 4$.

Proof. Suppose that $|E(R)| < 4$. As $|R| = |P| = 4$, we have $R \cong P_4$. If $\|v_1, C\| = 4$, then $\|R - v_1, C\| \geq 9$, which implies that there exists distinct $v_i, v_j \in R - v_1$ such that v_i and v_j have a common neighbor, say $x \in C$. However, we can then replace C and R with $G[C - x + v_1] \cong C$ and $G[R - v_1 + x]$, respectively, where $G[R - v_1 + x]$ has at least four edges, contradicting (O4).

So $\|v_1, C\| \leq 3$, and by symmetry, $\|v_4, C\| \leq 3$. Since $\|R, C\| \geq 13$, we may assume $\|v_2, C\| = 4$ and $\|R - v_2, C\| \geq 9$. Note that if $\|v_3, C\| = 4$, then $\|v_2 v_3, C\| = 8$, which contradicts Lemma 16. So $\|v_i, C\| = 3$ for all $i \in \{1, 3, 4\}$. Also by Lemma 16, we may assume $N_C(v_1) = \{b, c_1, c_2\}$ and $N_C(v_3) \in \{\{b, c_1, c_2\}, \{b, d_1, d_2\}\}$. If $v_4 b \in E(G)$, then we can replace C and R with $G[C - b + v_2] \cong C$ and $G[R - v_2 + b]$, respectively, where $G[R - v_2 + b]$ has at least four edges, contradicting (O4). So $N_C(v_4) \subseteq \{c_1, c_2, d_1, d_2\}$ and in fact, v_4 is adjacent to some c_i. If $N_C(v_3) = \{b, c_1, c_2\}$, then we can replace C and R with $G[C - c_i + v_2] \cong C$ and $G[R - v_2 + c_i]$, respectively, where $G[R - v_2 + c_i]$ has at least four edges. So $N_C(v_3) = \{b, d_1, d_2\}$. Now v_4 must also be adjacent to some d_j. However, we can then replace C and R with $G[C - d_j + v_1] \cong C$ and $G[R - v_1 + d_j]$, respectively, where $G[R - v_1 + d_j]$ has at least four edges. This completes all cases where $R \cong P_4$, and proves the claim. □

So $|E(R)| \geq 4$. As we are assuming $R \ncong K_{1,1,2}$, we have $R \in \{C_4, Paw\}$. We now show that if $R \cong C_4$, then in fact, we may assume $R \cong Paw$. Indeed, if $R \cong C_4$, then as $\|R, C\| \geq 13$, we may assume without loss of generality that $\|v_1, C\| = 4$. Note that either v_2 and v_3 have a common neighbor in C, or v_3 and v_4 have a common neighbor, call it x. Then we can replace C and R with $G[C - x + v_1]$ and $G[R - v_1 + x]$, respectively, where $G[R - v_1 + x] \cong Paw$.

So $R \cong Paw$, and we may assume $d_R(v_2) = 3$. If $\|v_2, C\| = 4$, then by Lemma 16 and the assumption $\|R, C\| \geq 13$, we have $\|v_i, C\| = 3$ for $i \in \{1, 3, 4\}$. Again, by Lemma 16, we may assume without loss of generality $N_C(v_1) = N_C(v_3) = \{b, c_1, c_2\}$ and $N_C(v_4) = \{b, d_1, d_2\}$. However, we can replace C and R with $G[C - c_1 + v_4] \cong C$ and $G[R - v_4 + c_1] \cong K_{1,1,2}$, respectively, contradicting (O4).

So $\|v_2, C\| \leq 3$. If $\|v_1, C\| = 4$, then we have $\|\{v_3, v_4\}, C\| \geq 6$, which means v_3 and v_4 have a common neighbor, say $x \in C$. We can then replace

C and R with $G[C-x+v_1] \cong C$ and $G[R-v_1+x] \cong K_{1,1,2}$, respectively, contradicting (O4). So $\|v_1, C\| \leq 3$, and by Lemma 16, $\|\{v_3, v_4\}, C\| \leq 7$. Since $\|R, C\| \geq 13$, equality must hold in each case. So, by Lemma 16, we may assume $\|v_4, C\| = 4$, $N_C(v_2) = \{b, c_1, c_2\}$ and $N_C(v_3) = \{b, d_1, d_2\}$. If $v_1 b \in E(G)$, then we can replace C and R with $G[C-b+v_4] \cong C$ and $G[R-v_4+b] \cong K_{1,1,2}$, respectively, contradicting (O4). If $v_1 b \notin E(G)$, then v_1 must be adjacent to some c_i. However, we can replace C and R with $G[C-c_j+v_3] \cong C$ and $G[R-v_3+c_j] \cong K_{1,1,2}$, respectively, again contradicting (O4). This completes the proof of the lemma. ☐

Since $R \cong K_{1,1,2}$, we may assume the vertices of R are labeled so that $d_R(v_1) = d_R(v_4) = 2$.

Lemma 36. *Let $C \in \mathcal{C}$. For $i \in \{1, 4\}$, if $\|v_i, C\| = 4$ and $C \cong K_4$, then $\|R - v_i, C\| \leq 8$.*

Proof. Without loss of generality, suppose $\|v_1, C\| = 4$ for some $C \in \mathcal{C}$, and suppose $C \cong K_4$. If we have $\|a_i, R\| = 4$ for any $a_i \in V(C)$, then $G[C-a_i+v_1]$ and $G[R-v_1+a_i]$ each form K_4, a contradiction. Thus $\|a_i, R\| \leq 3$ for all i. So $\|R, C\| \leq 12$, and since $\|v_1, C\| = 4$, the lemma is proved. ☐

For a graph H, we let H^- to denote any graph that is obtained from H be removing a single edge. That is, H^- represents any arbitrary graph from a particular family of graphs.

Lemma 37. *For all $C \in \mathcal{C}$, $\|R, C\| \leq 14$, and if $\|R, C\| \geq 13$, then one of the following configurations holds:*

1. $\|R, C\| = 14$, $\|v_1, C\| = \|v_4, C\| = 3$, *and* $G[R+C] \cong K_5 \vee \overline{K_3}$,
2. $\|R, C\| = 13$, $5 \leq \|\{v_1, v_4\}, C\| \leq 6$, *and* $G[R+C] \cong (K_5 \vee \overline{K_3})^-$,
3. $\|R, C\| = 13$, $\|v_1, C\| = \|v_4, C\| = 3$, *and* $G[R+C] \cong K_{2,3,4}$,
4. $\|R, C\| = 14$, $\|v_1, C\| = \|v_4, C\| = 4$, *and* $G[R+C] \cong K_{3,3,3}$, *or*
5. $\|R, C\| = 13$, $7 \leq \|\{v_1, v_4\}, C\| \leq 8$, *and* $G[R+C] \cong K^-_{3,3,3}$.

Proof. Fix $C \in \mathcal{C}$ such that $\|R, C\| \geq 13$. There exists some $u \in R$ such that $\|u, C\| \geq 4$, so by Lemma 15 equality holds and $C \in \{K_4, K_{1,2,2}\}$.

Case 1. $C \cong K_4$.

By Lemma 36, $\|v_1, C\| \leq 3$ and $\|v_4, C\| \leq 3$. So we may assume $\|v_2, C\| = 4$.

Suppose that $\|v_3, C\| = 4$. Since $\|R, C\| \geq 13$, we may assume $N_C(v_1) = \{a_1, a_2, a_3\}$ and further $\|v_4, C\| \geq 2$ with $a_3 \in N_C(v_4)$. If $v_4 a_4 \in E(G)$, then we can replace C with two disjoint DCCs in $C - a_3 - a_4 + v_3 + v_4$ and $R - v_3 - v_4 + a_3 + a_4$, a contradiction. So $N_C(a_4) \subseteq \{a_1, a_2, a_3\}$, which yields either configuration 1 or 2.

Now suppose $\|v_3, C\| = 3$. Then as $\|R, C\| \geq 13$, we must have $\|v_i, C\| = 3$ for $i \in \{1, 3, 4\}$. Without loss of generality, we may assume $N_C(v_1) = \{a_1, a_2, a_3\}$. Suppose $N_C(v_4) \neq N_C(v_1)$, so that we may assume $N_C(v_4) = \{a_2, a_3, a_4\}$. Without loss of generality, $a_2 \in N_C(v_3)$. If $v_3a_1 \in E(G)$, then we can replace C with two disjoint DCCs in $C - a_1 + v_4$ and $R - v_4 + a_1$. So $N_C(v_3) = \{a_2, a_2, a_4\}$. However, we can again replace C with two disjoint DCCs in $C - a_2 - a_4 + v_1 + v_2$ and $R - v_1 - v_2 + a_2 + a_4$. So we must have $N_C(v_4) = N_C(v_1)$. However, this yields configuration 2.

Note that in each situation $\|R, C\| \leq 14$, which completes this case.

Case 2. $C \cong K_{1,2,2}$.

Note that by Lemma 16, $\|\{v_i, v_{i+1}\}, C\| \leq 7$ for $i \in \{1, 3\}$. Thus, $\|R, C\| \leq 14$.

Suppose $\|v_2, C\| = 4$. By Lemma 16, $\|v_i, C\| \leq 3$ for $i \in \{1, 3, 4\}$, and since $\|R, C\| \geq 13$, we must have equality. So, by Lemma 16, we may assume $N_C(v_1) = N_C(v_4) = \{b, c_1, c_2\}$ and $N_C(v_3) = \{b, d_1, d_2\}$. This yields configuration 3.

So $\|v_2, C\| \leq 3$ and by symmetry $\|v_3, C\| \leq 3$. Without loss of generality, $\|v_1, C\| = 4$, and so $\|v_i, C\| = 3$ for some $i \in \{2, 3\}$. As $R \cong K_{1,1,2}$, we may assume without loss of generality, that $\|v_2, C\| = 3$, and by Lemma 16, we may assume $N_C(v_2) = \{b, c_1, c_2\}$. Note that if v_3 is adjacent to some c_i, then we can replace C with $G[\{c_i, v_1, v_2, v_3\}] \cong K_4$, contradicting (O1). So $N_C(v_3) \subseteq \{b, d_1, d_2\}$.

Now since $\|R, C\| \geq 13$, $\|\{v_3, v_4\}, C\| \geq 6$. Since $\|v_3, C\| \leq 3$, we have $\|v_4, C\| \geq 3$. If $\|v_4, C\| = 4$, then because $N_C(v_3) \subseteq \{b, d_1, d_2\}$, we get either configuration 4 or 5. If $\|v_4, C\| = 3$, then by Lemma 16, $N_C(v_4) = \{b, d_1, d_2\}$. Howevever, this would mean $\|v_3, C\| = 3$ as well as $N_C(v_3) \in \{\{b, c_1, c_2\}, \{b, d_1, d_2\}\}$, which in either case contradicts Lemma 16.

This completes both cases and proves the lemma. □

Let $\tilde{C} \in \mathcal{C}$ such that $|\tilde{C}|$ is largest amongst all DCCs in $\mathcal{C}$. The proof of the following lemma requires many structural lemmas and cases, and is proven in Section 7.

Lemma 38. *For all $C \in \mathcal{C} \setminus \{\tilde{C}\}$, if $|\tilde{C}| \geq 6$, then $\|R + \tilde{C}, C\| \leq 3(|\tilde{C}| + 4)$.*

Using this lemma we can prove the following.

Lemma 39. *For all $C \in \mathcal{C}$, $|C| \leq 5$.*

Proof. First observe that $\|R, R\| = 10$. By the definition of $\tilde{C}$, if $|\tilde{C}| \leq 5$, then we are done. So we may assume $|\tilde{C}| \geq 6$, and by Lemma 15 $\|v, \tilde{C}\| \leq 3$ for all $v \in R$; so $\|R, \tilde{C}\| \leq 12$.

We now claim $\|\tilde{C}, \tilde{C}\| \leq 3|\tilde{C}|$. Indeed, if for all $v \in \tilde{C}$, $\|v, \tilde{C}\| \leq 3$, then we are done. So suppose we have $\|v, \tilde{C}\| = 4$ for some $v \in \tilde{C}$ so that v is incident to two chords in $\tilde{C}$. By Lemma 13, v is the only vertex incident to two chords, and since $|\tilde{C}| \geq 6$, there is at least one other vertex in $\tilde{C}$ that is not incident to a chord. Thus, $\|\tilde{C}, \tilde{C}\| \leq 4 + 2 + 3(|\tilde{C}| - 2) = 3|\tilde{C}|$.

This together with Lemma 38 yields the following:

$$\begin{aligned}
3k(|\tilde{C}| + 4) &\leq \sum_{v \in R + \tilde{C}} d_G(v) \\
&= \|R + \tilde{C}, R\| + \|R + \tilde{C}, \tilde{C}\| + \|R + \tilde{C}, \mathcal{C} \setminus \{\tilde{C}\}\| \\
&\leq (10 + 12) + (12 + 3|\tilde{C}|) + 3(|\tilde{C}| + 4)(k - 2) \\
&= 22 + 3(|\tilde{C}| + 4)(k - 1)
\end{aligned}$$

This simplifies to $3(|\tilde{C}| + 4) \leq 22$. However, since $|\tilde{C}| \geq 6$, we get $30 \leq 22$, which is a contradiction. □

We are now able to prove Theorem 9.

Proof of Theorem 9. By Lemma 35, $|R| = 4$, and by Lemma 39, $\sum_{C \in \mathcal{C}} |V(C)| \leq 5(k - 1)$. Thus, $n \leq 4 + 5(k - 1) < 5k$. So every n-vertex graph H with $n \geq 4k$ and $\delta(H) \geq 3k$ without k disjoint DCCs satisfies $n < 5k$. □

7. Proof of Lemma 38

The goal of this section is to prove Lemma 38. So let $\tilde{C} \in \mathcal{C}$ be such that $|\tilde{C}|$ is largest amongst all doubly chorded cycles in $\mathcal{C}$, and assume $|\tilde{C}| \geq 6$. We show that for all $C \in \mathcal{C} - \{\tilde{C}\}$, $\|R + \tilde{C}, C\| \leq 3(|\tilde{C}| + 4)$. We first show this holds if any vertex in the remainder has four neighbors on C.

Lemma 40. *Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$ and $v \in R$. If $\|v, C\| = 4$, then $\|R + \tilde{C}, C\| \leq 3(|\tilde{C}| + 4)$.*

Proof. Fix $C \in \mathcal{C} \setminus \{\tilde{C}\}$ and $v \in R$ such that $\|v, C\| = 4$. By Lemma 15, $C \in \{K_4, K_{1,2,2}\}$. Observe that for all $x \in C$, $C - x + v$ creates a DCC C' such that replacing C with C' yields a new collection of $k - 1$ disjoint DCCs, call it $\mathcal{C}'$, that satisfies (O1). Let R' denote $G \setminus \mathcal{C}'$, and note that $x \in R'$. So, by Lemma 14, since $|\tilde{C}| \geq 6$, we must have $\|x, \tilde{C}\| \leq 3$. Thus, $\|C, \tilde{C}\| \leq 3|C| \leq 15$.

Now by Lemma 37, $\|R, C\| \leq 14$. Thus $\|R + \tilde{C}, C\| \leq 29 \leq 3(6+4) \leq 3(|\tilde{C}| + 4)$. ☐

As a result of Lemma 40, we may assume that for all $v \in R$ and $C \in \mathcal{C}\backslash\{\tilde{C}\}$, $\|v, C\| \leq 3$, and furthermore, $\|R, C\| \leq 12$. In the rest of this section, we consider $\|C, \tilde{C}\|$ for each the following cases: when $C \cong K_4$, when $|C| = 5$, when $|C| \geq 6$ and there exists $xy \in E(R)$ such that $\|xy, C\| \geq 5$, and when $|C| \geq 6$ and $\|R, C\| \leq 8$.

Lemma 41. *Let $C_1, C_2 \in \mathcal{C}$. If $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$, then for each $i \in [2]$, there exist consecutive vertices x and y along the cycle of C_i such that $\|\{x, y\}, C_{3-i}\| \geq 7$.*

Proof. Let $C_1, C_2 \in \mathcal{C}$ so that $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$. Label the vertices of C_i as $v_1v_2 \cdots v_{|C_i|}$. Suppose first that $|C_i|$ is even. Consider the set of consecutive pairs of vertices along the cycle of C_i, $\{v_1v_2, v_3v_4, \ldots v_{|C_i|-1}v_{|C_i|}\}$. If $\|\{v_j, v_{j+1}\}, C_{3-i}\| \leq 6$ for all $v_jv_{j+1} \in \{v_1v_2, v_3v_4, \ldots, v_{|C_i|-1}v_{|C_i|}\}$, then $\|C_1, C_2\| \leq 3(\max\{|C_1|, |C_2|\})$ which is a contradiction. Therefore, there exists at least one pair of consecutive vertices $xy \in \{v_1v_2, v_3v_4, \ldots, v_{|C_i|-1}v_{|C_i|}\}$ such that $\|\{x, y\}, C_{3-i}\| \geq 7$.

Now suppose that $|C_i|$ is odd. Consider the consecutive pairs of vertices $\{v_2v_3, v_4v_5, \ldots, v_{|C_i|-1}v_{|C_i|}\}$. If there exists a pair $v_jv_{j+1} \in \{v_2v_3, v_4v_5, \ldots, v_{|C_i|-1}v_{|C_i|}\}$ such that $\|\{v_j, v_{j+1}\}, C_{3-i}\| \geq 7$, we are done. So let $\|\{v_j, v_{j+1}\}, C_{3-i}\| \leq 6$, for all $v_jv_{j+1} \in \{v_2v_3, v_4v_5, \ldots, v_{|C_i|-1}v_{|C_i|}\}$. Then $\|C_i - v_1, C_{3-i}\| \leq 3(|C_i| - 1)$. Since we assumed that $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$, we can conclude that $\|v_1, C_{3-i}\| \geq 4$. If $\|v_2, C_{3-i}\| \geq 3$, then $\|\{v_1, v_2\}, C_{3-i}\| \geq 7$ as desired. So we have $\|v_2, C_{3-i}\| \leq 2$. Now consider the consecutive pairs of vertices $\{v_{|C_i|}v_1, v_3v_4, \ldots v_{|C_i|-2}v_{|C_i|-1}\}$. The same argument which shows that $\|v_1, C_{3-i}\| \geq 4$, shows that $\|v_2, C_{3-i}\| \geq 4$. However, this contradicts $\|v_2, C_{3-i}\| \leq 2$. Hence there must exist a consecutive pair of vertices x and y along the cycle of C_i such that $\|\{x, y\}, C_{3-i}\| \geq 7$. ☐

Lemma 42. *Let $C_1, C_2 \in \mathcal{C}$. If $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$, then for all $z \in C_i$, $\|z, C_{3-i}\| \leq 6$.*

Proof. Let $C_1, C_2 \in \mathcal{C}$ so that $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$ and suppose there exists a vertex $z \in C_i$ such that $\|z, C_{3-i}\| \geq 7$. Label the neighbors of z as $z_1, z_2, z_3, z_4, z_5, z_6, z_7, \ldots$ in this order along the cycle of C_{3-i} not necessarily consecutive. From Lemma 41, we can conclude that there exist consecutive vertices $x, y \in C_{3-i}$ along the cycle portion such that $\|\{x, y\}, C_i\| \geq 7$. Both x and y could be neighbors of z, and so $\|\{x, y\}, C_i - z\| \geq 5$. By Lemma 17 we can conclude that $G[(C_{3-i} - z) + x + y]$ contains a doubly chorded cycle. Since $\|z, C_{3-i}\| \geq 7$ there exists at least five neighbors of z that are not x or y. Without loss of generality, suppose these five are z_1, z_2, z_3, z_4 and z_5. Hence $G[z + z_1 Q_{3-i} z_4]$ and $G[(C_{3-i} - z) + x + y]$ form DCCs on fewer vertices than $|C_1|$ and $|C_2|$, contradicting (O1). Therefore, $\|z, C_{3-i}\| \leq 6$ for all $z \in C_i$ as desired. □

Lemma 43. *Let $C \in \mathcal{C}$. If $C \cong K_4$, then, $\|C, \tilde{C}\| \leq 3|\tilde{C}|$.*

Proof. Let $C \in \mathcal{C}$ and $C \cong K_4$. Suppose $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. By Lemma 41, there exist consecutive vertices x and y along the cycle of $\tilde{C}$, such that $\|\{x, y\}, C\| \geq 7$. Suppose $\|\{x, y\}, C\| = 8$ so that each edge in C forms a K_4 with xy. Let $e \in E(C)$. The remaining vertices in C, form a K_4 with xy, and so if e forms a DCC with $\tilde{C} - x - y$ on fewer vertices than $|V(e + \tilde{C} - x - y)|$, this contradicts (O1). Therefore, by Lemma 18 $\|e, \tilde{C} - x - y\| \leq 5$ for all $e \in E(C)$. If there is a $v \in C$, such that $\|v, \tilde{C} - x - y\| \geq 4$, then for any $v' \in (C - v)$, $\|v', \tilde{C} - x - y\| \leq 1$ and consequently $\|C, \tilde{C} - x - y\| \leq 7$. If there is a $v \in C$, such that $\|v, \tilde{C}\| = 3$, then for any $v' \in (C - v)$, $\|v', \tilde{C} - x - y\| \leq 2$ and consequently $\|C, \tilde{C} - x - y\| \leq 9$. If for all $v \in C$, $\|v, \tilde{C} - x - y\| \leq 2$, then $\|C, \tilde{C} - x - y\| \leq 8$. Therefore, in all cases $\|C, \tilde{C} - x - y\| \leq 9$, meaning $\|C, \tilde{C}\| \leq 17 \leq 3|\tilde{C}|$, which contradicts our assumption that $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$.

Suppose that $\|\{x, y\}, C\| = 7$, and without loss of generality, $\|x, C\| = 4$ and $\|y, C\| = 3$. Recall that the vertices of $C \cong K_4$ are labelled a_1, a_2, a_3 and a_4. Without loss of generality, suppose that $ya_1, ya_2, ya_3 \in E(G)$. Note that for all $e \in \{a_1a_2, a_1a_3, a_2a_3\}$, $G[e + x + y] \cong K_4$. Therefore, if a_1a_4, a_2a_4, or a_3a_4 form a DCC with $\tilde{C} - x - y$ on strictly fewer vertices than $|\tilde{C}|$, this would contradict (O1). Thus, by Lemma 18, $\|\{a_i, a_4\}, \tilde{C} - x - y\| \leq 5$, for each $i \in \{1, 2, 3\}$, and furthermore if equality holds for each i, then $\|a_4, \tilde{C}\| \geq 2$ by the configurations in Lemma 18.

If $\|a_4, \tilde{C} - x - y\| = 5$, then $\|\{a_1, a_2, a_3\}, \tilde{C} - x - y\| = 0$ and $\|C, \tilde{C}\| \leq 12$. If $\|a_4, \tilde{C} - x - y\| = 4$, then $\|\{a_1, a_2, a_3\}, \tilde{C} - x - y\| \leq 3$, meaning $\|C, \tilde{C}\| \leq 14$. If $\|a_4, \tilde{C} - x - y\| = 3$, then $\|\{a_1, a_2, a_3\}, \tilde{C} - x - y\| \leq 6$, meaning $\|C, \tilde{C}\| \leq 16$. If $\|a_4, \tilde{C} - x - y\| = 2$, then $\|\{a_1, a_2, a_3\}, \tilde{C} - x - y\| \leq 9$, meaning $\|C, \tilde{C}\| \leq 18$. In all of these cases $\|C, \tilde{C}\| \leq 18 \leq 3|\tilde{C}|$, which contradicts our assumption that $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. So let $\|a_4, \tilde{C} - x - y\| \leq 1$. As noted above, Lemma 18 implies that $\|\{a_i, a_4\}, \tilde{C} - x - y\| \leq 4$, for each $i \in \{1, 2, 3\}$. Then if $\|a_4, \tilde{C} - x - y\| = 1$, $\|\{a_1, a_2, a_3\}, \tilde{C} - x - y\| \leq 9$, meaning $\|C, \tilde{C}\| \leq 17$, which contradicts our assumption that $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. So $\|a_4, \tilde{C} - x - y\| = 0$. Recall that $C - a_j + x \cong K_4$ for each $j \in \{1, 2, 3\}$. So if $G[\tilde{C} - x + a_j]$ contains a DCC on less than $|\tilde{C}|$ vertices, this contradicts (O1). Thus, $\|a_j, \tilde{C} - x\| \leq 4$ so that $\|a_j, \tilde{C}\| \leq 5$. However, as $\|a_4, \tilde{C}\| \leq 3$, we get $\|C, \tilde{C}\| \leq 18$, which contradicts our assumption that $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. □

Lemma 44. *Let $C_1, C_2 \in \mathcal{C}$. If $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$, then for all $v \in C_i$, $\|v, C_{3-i}\| \leq 5$.*

Proof. Let $C_1, C_2 \in \mathcal{C}$ so that $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$. Suppose there exists a vertex $v \in C_i$ such that $\|v, C_{3-i}\| \geq 6$. By Lemma 42, we can conclude that $\|v, C_{3-i}\| = 6$. By Lemma 41, we know that there exists consecutive vertices x and y along the cycle of C_{3-i} such that $\|\{x, y\}, C_i\| \geq 7$. Notice that $\|\{x, y\}, (C_i - v)\| \geq 5$. If $G[(C_i - v) + x + y]$ contains a DCC on fewer vertices than $|(C_i - v) + x + y|$, then $G[(C_i - v) + x + y]$ and $G[(C_{3-i} - x - y) + v]$ contain DCCs on fewer vertices, contradicting (O1). Therefore, by Lemma 18, we can conclude that $\|\{x, y\}, (C_i - v)\| = 5$ and so $vx, vy \in E(G)$.

Let v_1, v_2, v_3, and v_4 be the four remaining neighbors of v so that y, x, v_1, v_2, v_3, and v_4 appear in this order along the cycle of C_{3-i}, not necessarily consecutive, and so that $x, y \in (v_4, v_1)_{C_{3-i}}$. Furthermore, from Lemma 17, we can conclude that $G[(C_i - v) + x + y]$ contains a DCC. Therefore, if there exists $z \in (x, v_1)_{C_{3-i}}$ or $z \in (v_4, y)_{C_{3-i}}$, then $G[(C_i - v) + x + y]$ and $G[(C_{3-i} - x - y - z) + v]$ contain DCCs on fewer vertices than $|C_1| + |C_2|$, contradicting (O1). Hence v_4, y, x, v_1 are consecutive along C_{3-i}.

If $\|v_4 y, C_i\| \geq 8$, then $\|v_4 y, C_i - v\| \geq 6$ and by Lemma 18, $G[C_i - v + v_4 + y]$ contains a DCC on fewer than $|C_i - v + v_4 + y|$ vertices. However, $G[C_{3-i} - v_4 - y + v]$ also contains a DCC, contradicting (O1). So $\|v_4 y, C_i\| \leq 7$ and by symmetry, $\|v_1 x, C_i\| \leq 7$. Therefore, $\|\{x, y, v_1, v_4\}, C_i\| \leq 14$.

Note that $G[\{v, v_4, y, x, v_1\}]$ forms a DCC. So $H = G[(v_1 v_4)_{C_{3-i}} + (C_i - v)]$ cannot contain a DCC on fewer vertices than $|H|$. Suppose first that $|C_{3-i}| \geq 8$ so that $|(v_1, v_4)_{C_{3-i}}| \geq 4$. By Lemma 21:

$$\begin{aligned}
\|(v_1, v_4)_{C_{3-i}}, C_i - v\| &\leq \min\{|(v_1, v_4)_{C_{3-i}}|, |C_i - v|\} + 3 \\
&\leq |(v_1, v_4)_{C_{3-i}}| + 3 \\
&\leq |C_{3-i}| - 4 + 3 \\
&\leq |C_{3-i}| - 1.
\end{aligned}$$

Since $\|v, C_{3-i}\| = 6$, we know that v is only adjacent to v_2 and v_3 in $(v_1, v_4)_{C_{3-i}}$, and so

$$\|(v_1, v_4)_{C_{3-i}}, C_i\| \leq |C_{3-i}| + 1.$$

Since $\|\{x, y, v_1, v_4\}, C_i\| \leq 14$,

$$\|C_{3-i}, C_i\| \leq |C_{3-i}| + 15.$$

However $|C_{3-i}| \geq 8$, which implies $\|C_{3-i}, C_i\| \leq 3|C_{3-i}|$, a contradiction to $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$. Therefore $|C_{3-i}| = 6$ or 7.

If $|C_{3-i}|$, then a similar argument to the above holds so that by Lemma 20:

$$\|(v_1, v_4)_{C_{3-i}}, C_i - v\| \leq 5.$$

Since $\|v, C_{3-i}\| = 6$, we know that v is only adjacent to v_2 and v_3 in $(v_1, v_4)_{C_{3-i}}$, and so

$$\|(v_1, v_4)_{C_{3-i}}, C_i\| \leq 7.$$

Since $\|\{x, y, v_1, v_4\}, C_i\| \leq 14$,

$$\|C_{3-i}, C_i\| \leq 21.$$

However, $|C_{3-i}| = 7$, which implies $\|C_{3-i}, C_i\| \leq 3|C_{3-i}|$, a contradiction to $\|C_1, C_2\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$. Therefore, $|C_{3-i}| = 6$.

Note that for any pair of consecutive vertices along C_{3-i}, say w and z, $G[C_{3-i} - w - z + v]$ contains a DCC. So, by Lemma 18, we must have $\|wz, C_i - v\| \leq 5$, else $G[C_i - v + w + z]$ will contain a DCC on less than $|C_i - v + w + z|$ vertices, contradicting (O1). Therefore, $\|C_{3-i}, C_i - v\| \leq 15$ and so $\|C_{3-i}, C_i\| \leq 21$. As $\|C_{3-i}, C_i\| \geq 3(\max\{|C_1|, |C_2|\}) + 1$, we get $|C_i| \leq 6$.

Recall that $\|xy, C_i - v\| = 5$ and $G[C_{3-i} - x - y + v]$ contains a DCC. So we must either have configuration 1 or 2 in Lemma 18. Configuration 1 cannot occur as it requires $|C_i - v| \geq 6$. So configuration 2 holds, which implies $|C_i - v| = 3$ so that $C_i \cong K_4$, and without loss of generality, $\|x, C_i - v\| = 3$. However, $G[C_i - v + x] \cong K_4$ and $G[C_{3-i} - x - y + v]$ form DCCs on fewer than $|C_1| + |C_2|$ vertices, contradicting (O1).

This completes the proof of the lemma. □

Lemma 45. *Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$. If $|C| = 5$, then $\|C, \tilde{C}\| \leq 3|\tilde{C}|$.*

Proof. Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$ such that $|C| = 5$, and suppose $\|C, \tilde{C}\| > 3|\tilde{C}|$.

Claim 45.1. For all $c \in \tilde{C}$, $\|c, C\| \leq 4$.

Proof. Let $c \in \tilde{C}$ such that $\|c, C\| = 5$, and let $v \in C$. Label $C = vv_1v_2v_3v_4v$. We will consider the number of chords incident to v. Note that by Lemma 13 v is incident to at most two chords.

Suppose v is incident to two chords so that $vv_2, vv_3 \in E(G)$. Label $e_1 = v_1v_2$, $e_2 = v_3v_4$, T_1 as triangle vv_3v_4v and T_2 as triangle vv_1v_2v.

Note that $G[T_i + c] \cong K_4$. So if $G[\tilde{C} - c + e_i]$ contains a DCC on fewer than $|\tilde{C} - c + e_i|$ vertices, this contradicts (O1). Hence by Lemma 18 $\|e_i, \tilde{C} - c\| \leq 5$, implying $\|e_i, \tilde{C}\| \leq 7$. Therefore, $\|\{e_1, e_2\}, \tilde{C}\| \leq 14$. Note that $\|C, \tilde{C}\| \geq 19$, by our assumption that $\|C, \tilde{C}\| > 3|\tilde{C}|$, and so $\|v, \tilde{C}\| \geq 5$. By Lemma 44, $\|v, \tilde{C}\| = 5$, implying $\|C, \tilde{C}\| = 19$, $\|\{e_1, e_2\}, \tilde{C}\| = 14$, and more specifically $\|e_i, \tilde{C} - c\| = 5$.

As argued above e_i cannot form a DCC with $\tilde{C} - c$ on fewer vertices that $|\tilde{C} - c + e_i|$. Therefore, by Lemma 18, either configuration 1 or 2 occurs, implying $|\tilde{C} - c| \geq 6$ or $|\tilde{C} - c| = 3$. The latter cannot hold as $|\tilde{C}| \geq 6$. If the former holds, then $|\tilde{C}| \geq 7$, so that $\|C, \tilde{C}\| \geq 22$. However, we showed $\|C, \tilde{C}\| = 19$, a contradiction. So v cannot be incident to two chords in C, and by symmetry, the same holds for all vertices in C.

As C is a DCC, we can assume without loss of generality that vv_2 and v_1v_4 are the only chords in C. We now label $e_1 = v_2v_3$, $e_2 = v_3v_4$, T_1 as triangle vv_1v_4v, and T_2 as triangle vvv_1v_2v. As above $G[T_i + c] \cong K_4$. So we must have $\|e_{3-i}, \tilde{C} - c\| \leq 5$, otherwise by Lemma 18, $G[\tilde{C} - c + e_{3-i}]$ will contain a DCC on fewer than $|\tilde{C} - c + e_{3-i}|$ vertices, contradicting (O1). Similarly, $\|v_j, \tilde{C} - c\| \leq 3$ for each $j \in \{2, 3, 4\}$. From here we can deduce $\|\{v_2, v_3, v_4\}, \tilde{C} - c\| \leq 8$.

Note that $G[C - v + c] \cong K_{1,2,2}$, which has three chords. If $\|v, \tilde{C} - c\| = 4$, then $G[\tilde{C} - c + v]$ contains a DCC that is either on fewer vertices than $\tilde{C}$, or has the same number of chords as $\tilde{C}$, as c can only be incident to at most two chords by Lemma 13. However, this either contradicts (O1) or (O2). So $\|v, \tilde{C} - c\| \leq 3$, and the same argument shows $\|v_1, \tilde{C} - c\| \leq 3$.

Thus, $\|C, \tilde{C} - c\| \leq 14$ so that $\|C, \tilde{C}\| \leq 19$. However, as $\|C, \tilde{C}\| > 3|\tilde{C}|$ and $|\tilde{C}| \geq 6$, we must have $|\tilde{C}| = 6$, and furthermore, we must have equality in our prior inequalities. In particular, $\|\{v_2, v_3, v_4\}, \tilde{C} - c\| = 8$ so that we may assume without loss of generality that $\|v_2v_3, \tilde{C} - c\| = 5$. Therefore, either configuration 1 or 2 in Lemma 18 holds, and either $|\tilde{C} - c| \geq 6$ or $|\tilde{C} - c| = 3$, respectively. However, both yields contradictions as $|\tilde{C}| = 6$.

This proves the claim. □

Claim 45.2. For every edge e along the cycle of C, and for every edge xy along the cycle of $\tilde{C}$, $G[e + xy] \ncong K_4$. In particular, $\|e, xy\| \leq 3$.

Proof. Let e be an edge along the cycle of C, let H be the remaining 3-vertex path along the cycle of C (perhaps $G[H] \cong K_3$), and let xy be an edge along the cycle of $\tilde{C}$.

Suppose first that both $G[H + xy]$ and $G[e + xy]$ contain DCCs. If $G[\tilde{C} - x - y + H]$ contains a DCC on fewer vertices than $|\tilde{C} - x - y + H|$, this contradicts (O1). Therefore, $\|H, \tilde{C} - x - y\| \leq 5$ by Lemma 20. Similarly,

$G[e+\tilde{C}-x-y]$ cannot contain a DCC on fewer vertices than $|\tilde{C}-x-y+e|$, so by Lemma 18 $\|e,\tilde{C}-x-y\| \leq 5$. Together we get $\|C,\tilde{C}-x-y\| \leq 10$. Since we assumed that $\|C,\tilde{C}\| > 3|\tilde{C}|$, we have $\|C,\{x,y\}\| \geq 9$. However, this implies either $\|x,C\| \geq 5$ or $\|y,C\| \geq 5$, contradicting Claim 45.1.

Next suppose $G[e+xy]$ contains a DCC, but $G[H+xy]$ does not. As above, $\|H,\tilde{C}-x-y\| \leq 5$ by Lemma 20. As $G[H+xy]$ does not contain a DCC, Lemma 17 implies $\|H,xy\| \leq 4$ and if equality holds, then either configuration 1 or 2 holds. Configuration 2 cannot hold as $|H| = 3$, and if configuration 1 holds, then either x or y is adjacent to all the vertices of H, say x. However, $G[e+xy] \cong K_4$, so that $\|x,C\| = 5$, contradicting Claim 45.1. So we must have $\|H,xy\| \leq 3$.

This implies $\|H,\tilde{C}\| \leq 8$, and since we assumed $\|C,\tilde{C}\| > 3|\tilde{C}|$, $\|e,\tilde{C}\| \geq 11$. However, this implies there exists $x \in T$ such that $\|x,\tilde{C}\| \geq 6$ contradicting Lemma 44. □

Claim 45.3. Given a partitioning of C into a triangle, T, and a disjoint edge, e, there exists an edge xy along the cycle of $\tilde{C}$ such that $G[T+xy]$ contains a DCC.

Proof. Let T and e be such a partition of C. Suppose T does not form a DCC with any edge xy along the cycle of $\tilde{C}$. By Lemma 22.3, $\|xy,T\| \leq 2$ and by Claim 45.2 $\|xy,e\| \leq 3$. Therefore $\|xy,C\| \leq 5$. However, since this is for all edges xy along the cycle of $\tilde{C}$ this contradicts Lemma 41. □

Claim 45.4. Given a partitioning of C into a triangle, T, and a disjoint edge, e, $\|e,\tilde{C}\| \leq 7$.

Proof. Let T and e be such a partition of C and suppose that $\|e,\tilde{C}\| \geq 8$. By Claim 45.3, there exists an edge xy along the cycle of $\tilde{C}$ such that $G[T+xy]$ contains a DCC. So if $G[\tilde{C}-x-y+e]$ contains a DCC on strictly fewer vertices than $|\tilde{C}-x-y+e|$ this contradicts (O1). Therefore, by Lemma 18, $\|e,\tilde{C}-x-y\| \leq 5$ and so $\|e,\tilde{C}\| \leq 8$ by Claim 45.2. As $\|e,\tilde{C}\| \geq 8$, we must have equality, and furthermore, $\|e,\tilde{C}-x-y\| = 5$. Therefore, either configuration 1 or 2 from 18 holds, implying that $|\tilde{C}-x-y| \geq 6$ or $|\tilde{C}-x-y| = 3$, respectively. The latter cannot hold as $|\tilde{C}| \geq 6$, so that the former holds and $|\tilde{C}| \geq 8$.

As $\|C,\tilde{C}\| > 3|\tilde{C}| \geq 24$ and $\|e,\tilde{C}\| = 8$, we have $\|T,\tilde{C}\| \geq 17$. However, this implies there exists a vertex $x \in T$ such that $\|x,\tilde{C}\| \geq 6$, contradicting Lemma 44. □

Claim 45.5. $|\tilde{C}| = 6$

Proof. Partition C into triangle, T, and disjoint edge, e. By Claim 45.4, $\|e, \tilde{C}\| \leq 7$, and so if $|\tilde{C}| \geq 8$, $\|T, \tilde{C}\| \geq 18$ contradicting Lemma 44.

So suppose $|\tilde{C}| = 7$. By our assumption that $\|C, \tilde{C}\| > 3|\tilde{C}|$ and by Claim 45.4, we can conclude that $\|T, \tilde{C}\| \geq 15$. This implies that there is a vertex $v \in \tilde{C}$ such that $\|T, v\| = 3$, and so $G[T + v] \cong K_4$. If $G[\tilde{C} - v + e]$ contains a DCC on strictly fewer vertices than $|\tilde{C} - v + e|$ this contradicts (O1). Therefore, by Lemma 18 $\|e, \tilde{C} - v\| \leq 5$. As $\|v, T\| = 3$ and $\|v, C\| \leq 4$ Claim 45.1, we get $\|e, \tilde{C}\| \leq 6$. However, this implies $\|T, \tilde{C}\| \geq 16$, and furthermore, there is a vertex $x \in T$ such that $\|x, \tilde{C}\| \geq 6$ which contradicts Lemma 44. Therefore, $|\tilde{C}| = 6$. □

Claim 45.6. Let H be a 3-vertex path along the cycle of $\tilde{C}$ (perhaps $G[H] \cong K_3$). Given a partition of C into triangle T and disjoint edge e, $G[e + H]$ does not contain a DCC.

Proof. Partition C into a triangle T and disjoint edge e, and let H be a 3-vertex path along the cycle of $\tilde{C}$, where possibly $G[H] \cong K_3$. Suppose on the contrary that $G[e + H]$ contains a DCC. If $G[\tilde{C} - H + T]$ contains a DCC on strictly fewer vertices than $|\tilde{C} - H + T|$ this contradicts (O1). By Claim 45.5, the vertices along the cycle of $\tilde{C}$ disjoint from H form a $K_{1,2}$ so that by Lemma 22.4, $\|T, \tilde{C} - H\| \leq 3$.

By Claim 45.4, $\|e, \tilde{C}\| \leq 7$. So as $\|C, \tilde{C}\| > 3|\tilde{C}|$, we have $\|T, \tilde{C}\| \geq 12$, and further $\|T, H\| \geq 9$. This implies that each vertex in T is adjacent to all vertices in the H. However, this contradicts Claim 45.2. □

Claim 45.7. Given a partitioning of C into a triangle, T, and a disjoint edge, e, $\|e, \tilde{C}\| \leq 6$ and and $\|T, \tilde{C}\| \geq 14$.

Proof. Let C be partitioned into a triangle T and disjoint edge e. By Claim 45.4 $\|e, \tilde{C}\| \leq 7$. Suppose that $\|e, \tilde{C}\| = 7$. By Claim 45.2 $\|e, xy\| \leq 3$ for all edges xy along the cycle of $\tilde{C}$. Since $\|e, \tilde{C}\| = 7$ and $|\tilde{C}| = 6$ by Claim 45.5, there exists an edge xy along the cycle of $\tilde{C}$ such that $\|e, xy\| \geq 3$, and by Claim 45.2, equality holds.

Label $\tilde{C} = xyv_1v_2v_3v_4$. Without loss of generality, we can assume that $\|e, x\| = 2$ and $\|e, y\| = 1$. We must have $\|e, v_1\| = 0$ otherwise $G[e + xyv_1]$ will contain a DCC, contradicting Claim 45.6. Similarly, we must have $\|e, v_4\| \leq 1$. If $\|e, v_4\| = 0$, then as $\|e, v_2v_3\| \leq 3$ by Lemma 45.2, we get $\|e, \tilde{C}\| \leq 6$, a contradiction as we assumed $\|e, \tilde{C}\| = 7$. So $\|e, v_4\| = 1$. Yet to avoid contradicting Claim 45.6, we must have $\|e, v_3\| = 0$, which again gives $\|e, \tilde{C}\| \leq 6$ as $\|e, v_2\| \leq 2$.

So we may assume $\|e, \tilde{C}\| \leq 6$. Since $\|C, \tilde{C}\| > 3|\tilde{C}|$, this implies that $\|T, \tilde{C}\| \geq 13$. As $|\tilde{C}| = 6$, there exists a vertex $x \in \tilde{C}$ such that $\|T, x\| = 3$

and hence $G[T+x] \cong K_4$. If e forms a DCC with $\tilde{C}-x$ on strictly fewer vertices than $|\tilde{C}-x+e|$ this contradicts (O1). Therefore, by Lemma 18 we can conclude that $\|e, \tilde{C}-x\| \leq 5$. However, if equality holds, then either configuration 1 or 2 of Lemma 18 occurs, implying $|\tilde{C}| = 7$ or $|\tilde{C}| = 4$, contradicting $|\tilde{C}| = 6$. Hence $\|e, \tilde{C}-x\| \leq 4$, and as $\|T, x\| = 3$, we get $\|e, \tilde{C}\| \leq 5$ by Claim 45.1. So, in fact, $\|T, \tilde{C}\| \geq 14$. □

Label the vertices of C so that $C = r_1r_2t_1t_2t_3r_1$, where $e = r_1r_2$ and $t_1t_2t_3t_1$ is T; in particular, t_1t_3 is a chord of C. C must have at least one more chord, and up to symmetry it is r_1t_1 or r_2t_2. In either case, Claim 45.7 implies $\|e, \tilde{C}\| \leq 6$ and $\|T, \tilde{C}\| \geq 14$.

Suppose $r_1t_1 \in E(G)$. Then we can apply Claim 45.7 to the edge t_2t_3 and triangle $r_1r_2t_2r_1$ to get $\|t_2t_3, \tilde{C}\| \leq 6$. This together with $\|C, \tilde{C}\| > 3|\tilde{C}|$, implies that $\|t_1, \tilde{C}\| \geq 7$, which contradicts Lemma 44.

So we may assume $r_2t_2 \in E(G)$. Here we apply Claim 45.7 to the edge t_3r_1 and triangle $r_2t_1t_2r_2$ to get $\|r_2t_1t_2, \tilde{C}\| \geq 14$. By Lemma 44, $\|z, \tilde{C}\| \leq 5$ for all $z \in \{r_2, t_1, t_2, t_3\}$. However, the only way for $\|T, \tilde{C}\| \geq 14$ and $\|r_2t_1t_2, \tilde{C}\| \geq 14$, is for some edge in $e' \in \{r_2t_1, t_1t_2, t_2t_3\}$ to have $\|e', \tilde{C}\| \geq 10$. Thus, for some edge e'' along the spanning cycle of $\tilde{C}$, we must have $\|e', e''\| \geq 4$, however, this contradicts Claim 45.2.

As all cases result in contradictions, this proves the lemma. □

Lemma 46. *Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$. If $xy \in E(R)$ such that $\|\{x, y\}, C\| \geq 5$, then $\|\tilde{C}, C\| \leq 3|\tilde{C}|$.*

Proof. Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$. Note that if $|C| \leq 5$, then by Lemmas 43 and 45, $\|C, \tilde{C}\| \leq 3|\tilde{C}|$, and we are done. So suppose in all the following that $|\tilde{C}| \geq |C| \geq 6$.

Let $xy \in E(R)$ such that $\|\{x, y\}, C\| \geq 5$, and suppose on the contrary that $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. Without loss of generality suppose $\|x, C\| \geq 3$. By Lemma 15, if $\|x, C\| = 4$, then $|C| \leq 5$, a contradiction. So $\|x, C\| = 3$ and $\|y, C\| \geq 2$. Note that it suffices to consider $\|y, C\| = 2$, as when $\|y, C\| = 3$, we can delete and edge incident to y and still obtain our results below.

Let $N_C(x) = \{x_1, x_2, x_3\}$, and $N_C(y) = \{y_1, y_2\}$, such that x_1, x_2, x_3 appear along C in this order, but not necessarily consecutive, and similarly order y_1 and y_2.

In many of the following arguments we will use the following observation:

Observation. If we replace C with a DCC C' contained in $G[C + xy]$ such that $|C| = |C'|$, then by Lemma 14, for all $z \in V(C) - V(C')$, $\|z, \tilde{C}\| \leq 3$ as otherwise $|\tilde{C}| \leq 5$.

Using this observation, we will in many cases show that for all $z \in C$, $\|z, \tilde{C}\| \leq 3$, which will contradict $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$. We now proceed based on the size of $|N_C(x) \cap N_C(y)|$.

Case 1. $|N_C(x) \cap N_C(y)| \geq 2$.

Without loss of generality, suppose $x_1 = y_1$ and $x_2 = y_2$. Since xyx_1Cx_2x is a DCC with chords xx_1 and yx_2, there is at most one vertex in (x_2, x_1) other than x_3, else we would get a DCC with fewer vertices than C, contradicting (O1). Similarly, since $yx_2Cx_3xx_1y$ is a DCC with chords xy and xx_2, there are at most two vertices (x_3, x_2) other than x_1. By symmetry, there are at most two vertices (x_1, x_3) other than x_2. Lastly, since xx_2Cx_1yx is a DCC with chords xx_3 and yx_2, there are at most two vertices in (x_1, x_2).

However, due to these restrictions on the number of vertices in C, we deduce that $|C| \leq 5$, which contradicts the assumption that $|C| \geq 6$.

Case 2. $|N_C(x) \cap N_C(y)| = 1$.

Without loss of generality, suppose $x_1 = y_1$. Up to symmetry, we have two cases to consider here: either $y_2 \in (x_1, x_2)$ or $y_2 \in (x_2, x_3)$.

Subcase 2.1. $y_2 \in (x_1, x_2)$

Since xyx_1Cx_2x is a DCC with chords xx_1 and yy_2, there is at most one vertex in (x_2, x_1) other than x_3; since xx_3Cy_2yx is a DCC with chords xx_1 and yx_1, there is at most one vertex in (y_2, x_3) other than x_2; since xx_2Cx_1yx is a DCC with chords xx_3 and xx_1, there is at most one vertex in (x_1, x_2) other than y_2. By these inequalities, $|C| \leq 6$ so that equality holds. Let $\{v_1, v_2\} = V(C) - N_C(\{x, y\})$ so that v_1 and v_2 appear in this order along C (not necessarily consecutive). We have three cases:

1. $C = x_1v_1y_2x_2v_2x_3x_1$,
2. $C = x_1y_2v_1x_2x_3v_2x_1$, and
3. $C = x_1v_1y_2x_2x_3v_2x_1$.

In each of these situations we can replace C with the DCC xx_2Cx_1yx with chords xx_3 and xx_1; call it C'. Now $|C'| = |C|$, so by the observation $\|\{y_2, v_1\}, \tilde{C}\| \leq 6$. Furthermore, the number of chords in C is equal to the number of chords incident to vertices in $\{y_2, v_1\}$ together with the number of hops in $[x_2, x_1]$. As the number of chords in C' is equal to the number of hops in $[x_2, x_1]$ plus two, there must be at least two chords with an endpoint in $\{y_2, v_1\}$ otherwise replacing C with C' yields a collection that satisfies (O1) but contradicts (O2).

Similarly, xyx_1Cx_2x is a DCC with chords xx_1 and yy_2 that implies $\|\{x_3, v_2\}, \tilde{C}\| \leq 6$ and there are at least two chords in C with an endpoint in $\{x_3, v_2\}$. Lastly, xx_2Cx_1x is a chorded cycle with xx_3 as a chord; so there are no hops in $[x_2, x_1]$ otherwise we replace C with a DCC with fewer vertices.

We now consider each option for C separately and in each, we either contradict (O1) or show $\|\{x_1, x_2\}, \tilde{C}\| \leq 6$, which implies $\|C, \tilde{C}\| \leq 3|\tilde{C}|$, a contradiction to $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$.

Subcase 2.1.1. $C = x_1v_1y_2x_2v_2x_3x_1$.

If $x_3v_1 \in E(G)$, then $xx_3v_1x_1yx$ is a DCC with chords x_1x_3 and xx_1 that contradicts (O1). If $x_3y_2 \in E(G)$, then $xx_1x_3y_2yx$ is a DCC with chords yx_1 and xx_3 that contradicts (O1). Therefore, since C has at least two chords with an endpoint in $\{x_3, v_2\}$ and there are no hops in $[x_2, x_1]$, we must have v_2y_2 and v_2v_1. Now replacing C with the DCC $xx_1v_1y_2v_2x_3x$ with chords v_1v_2 and x_1x_3, and the DCC $xx_3v_2v_1y_2x_2x$ with chords x_2v_2 and y_2v_2, yields $\|\{x_1, x_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

Subcase 2.1.2. $C = x_1y_2v_1x_2x_3v_2x_1$.

If $y_2v_2 \in E(G)$, then $xx_1v_2y_2yx$ is a DCC with chords x_1y_2 and yx_1 that contradicts (O1). If $y_2x_3 \in E(G)$, then $xyx_1y_2x_3x$ is a DCC with chords xx_1 and yy_2 that contradicts (O1). Therefore, since C has at least two chords with an endpoint in $\{x_3, v_2\}$ and there are no hops in $[x_2, x_1]$, we must have v_1x_3 and v_1v_2. However, $xx_3v_2v_1x_2x$ is a DCC with chords v_1x_3 and x_2x_3 that contradicts (O1).

Subcase 2.1.3. $C = x_1v_1y_2x_2x_3v_2x_1$.

If $y_2x_3 \in E(G)$, then $xx_3x_2y_2yx$ is a DCC with chords xx_2 and x_3y_2 that contradicts (O1). Since C has at least two chords with an endpoint in $\{x_3, v_2\}$ and there are no hops in $[x_2, x_1]$, we must have at least two of the edges in $\{x_3v_1, v_2y_2, v_2v_1\}$. Suppose we have $v_2y_2, v_2v_1 \in E(G)$. Now replacing C with the DCC $xx_3v_2v_1y_2x_2x$ with chords x_2x_3 and v_2y_2, and the DCC $xx_3v_2y_2v_1x_1x$ with chords v_1v_2 and x_1v_2, yields $\|\{x_1, x_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

So we must have $x_3v_1 \in E(G)$. If $v_2v_1 \in E(G)$, then $xx_1v_2v_1x_3x$ is a DCC with chords x_1v_1 and x_3v_2 that contradicts (O1). So $v_2y_2 \in E(G)$. Now replacing C with the DCC $xx_1v_2y_2v_1x_3x$ with chords x_1v_1 and x_3v_2, and the DCC $yy_2v_1x_3x_2xy$ with chords y_2x_2 and xx_3, yields $\|\{x_1, x_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

This completes all cases when $y_2 \in (x_1, x_2)$.

Subcase 2.2. $y_2 \in (x_2, x_3)$.

Since xx_1Cy_2yx is a DCC with chords xx_2 and yx_1, there is at most one vertex in (y_2, x_1) other than x_3. By symmetry, there is at most one vertex in (x_1, y_2) other than x_2. Since $|C| \geq 6$, we must have exactly one vertex in (x_1, y_2) other than x_2, and exactly one vertex in (y_2, x_1) other than x_3. By these inequalities, $|C| \leq 6$ so that equality holds. Let $\{v_1, v_2\} = V(C) - N_C(\{x, y\})$ so that v_1 and v_2 appear in this order along C (not necessarily consecutive). Up to symmetry, we have three cases:

1. $C = x_1v_1x_2y_2x_3v_2x_1$,
2. $C = x_1x_2v_1y_2x_3v_2x_1$, and
3. $C = x_1x_2v_1y_2v_2x_3x_1$.

In each of these situations we can replace C with the DCC yy_2Cx_1xy with chords yx_1 and xx_3; call it C'. Now $|C'| = |C|$, so by the observation, $\|\{v_1, x_2\}, \tilde{C}\| \leq 6$. Furthermore, the number of chords in C is equal to the number of chords with an endpoint in $\{v_1, x_2\}$ together with the number of hops in $[y_2, x_1]$. As the number of chords in C' is equal to the number of hops in $[y_2, x_1]$ plus two, there must be at least two chords with an endpoint in $\{v_1, x_2\}$, otherwise $\mathcal{C}'$ yields a collection that satisfies (O1) but contradicts (O2). Similarly, yxx_1Cy_2y is a DCC with chords yx_1 and xx_2 that implies $\|\{x_3, v_2\}, \tilde{C}\| \leq 6$ and there are at least two chords in C with an endpoint in $\{x_3, v_2\}$.

We now consider each option for C separately and in each, we either contradict (O1) or show $\|\{x_1, y_2\}, \tilde{C}\| \leq 6$, which implies $\|C, \tilde{C}\| \leq 3|\tilde{C}|$, a contradiction to $\|C, \tilde{C}\| \geq 3|\tilde{C}| + 1$.

Subcase 2.2.1.
$C = x_1v_1x_2y_2x_3x_2x_1$

If $x_2x_3 \in E(G)$, then $xx_2x_3y_2yx$ is a DCC with chords xx_3 and x_2y_2 contradicting (O1). If $x_1x_2 \in E(G)$, then $xyy_2x_2x_1x$ is a DCC with chords xx_2 and yx_1 contradicting (O1), and a symmetric argument holds if $x_1x_3 \in E(G)$. So $x_2x_3, x_1x_2, x_1x_3 \notin E(G)$. Suppose $v_1x_3 \in E(G)$. Now replacing C with the DCC $xx_2v_1x_3y_2yx$ with chords xx_3 and x_2y_2, and the DCC xx_3Cx_2x with chords v_1x_3 and xx_1, yields $\|\{x_1, y_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

So $v_1x_3 \notin E(G)$, and by symmetry $v_2x_2 \notin E(G)$. Since C has at least two chords with an endpoint in $\{x_2, v_1\}$ and at least two chords with an endpoint in $\{v_2, x_3\}$, we must have $v_1y_2, v_2y_2, v_1v_2 \in E(G)$. Now replacing C with the DCC $xx_3v_2y_2v_1x_2x$ with chords x_2y_2 and y_2x_3, and the DCC xx_3Cx_2x with chords xx_1 and v_1v_2, yields $\|\{x_1, y_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

Subcase 2.2.2.
$C = x_1x_2v_1y_2x_3v_2x_1$

Since xx_3Cx_2x is a chorded cycle with chord xx_1, there can be no hops in $[x_3, x_2]$, otherwise it is a DCC contradicting (O1). Suppose $x_3v_1 \in E(G)$. Now replacing C with the DCC $xyx_1x_2v_1x_3x$ with chords xx_1 and xx_2, and the DCC $xx_2v_1x_3y_2yx$ with chords v_1y_2 and xx_3, yields $\|\{x_1, y_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction. So $x_3v_1 \notin E(G)$.

Since C has at least two chords with an endpoint in $\{x_3, v_2\}$ and there are no hops in $[x_3, x_2]$, we must have v_1v_2 and v_2y_2. Now replacing C with the DCC $xx_3v_2y_2v_1x_2x$ with chords v_1v_2 and y_2x_3, and the DCC $xx_2v_1v_2x_1yx$ with chords x_1x_2 and xx_1, yields $\|\{x_1, y_2\}, \tilde{C}\| \leq 6$ by the observation, a contradiction.

Subcase 2.2.3.
$C = x_1x_2v_1y_2v_2x_3x_1$.

If $x_2y_2 \in E(G)$, then $xx_1x_2y_2yx$ is a DCC with chords xx_2 and yx_1 contradicting (O1). If $x_1v_1 \in E(G)$, then $xyx_1v_1x_2x$ is a DCC with chords xx_1 and x_1x_2 contradicting (O1). If $x_2v_2 \in E(G)$, then $xx_1x_2v_2x_3x$ is a DCC with chords xx_2 and x_1x_3 contradicting (O1). So $x_2y_2, x_1v_1, x_2v_2 \notin E(G)$, and by symmetry, $x_3y_2, x_1v_2, v_1x_3 \notin E(G)$ respectively. However, C has at least two chords with an endpoint in $\{v_2, x_3\}$, so that we must have x_2x_3 and v_1v_2. However, $G[x_1Cx_3 + x] \cong K_4$, contradicting (O1).

This completes the case when $y_2 \in (x_2, x_3)$, and completes the case when $|N_C(x) \cap N_C(y)| = 1$.

Case 3. Suppose that $N_C(x) \cap N_C(y) = \emptyset$

To complete this final case, we proceed based on $|C|$.

Subcase 3.1. $|C| = 6$

Here we relabel the vertices of C so that $C = v_1v_2v_3v_4v_5v_6v_1$ with $xv_1, yv_1 \notin E(G)$. Note that if $v_2v_4 \in E(G)$, then $G[\{v_2, v_3, v_4\}] \cong K_4$ and as $\|xy, \{v_2, v_3, v_4\}\| = 3$, Lemma 22.3 implies $G[xy + v_2Cv_4]$ contains a DCC that contradicts (O1). Therefore, $v_2v_4 \notin E(G)$, and similarly, $v_3v_5, v_4v_6 \notin E(G)$. Since C is a DCC, we know that there must exists two additional edges from $\{v_1v_3, v_1v_4, v_1v_5, v_2v_5, v_2v_6, v_3v_6\}$.

Claim 46.1. $v_1v_3, v_1v_5 \notin E(G)$.

Proof. Suppose $v_1v_3 \in E(G)$. Note that either x or y is adjacent to at least two vertices from v_2, v_3, v_6. So if $v_2v_6 \in E(G)$, then $G[\{v_1, v_2, v_3, v_6\}] \cong K_4^-$ with chord v_1v_2. So, by Lemma 23, either $G[x + v_6Cv_3]$ or $G[y + v_6Cv_3]$ contain a DCC that contradicts (O1). Therefore, $v_2v_6 \notin E(G)$. A similar argument shows how $v_3v_6, v_1v_4 \notin E(G)$.

Hence either $v_2v_5 \in E(G)$ or $v_1v_5 \in E(G)$. Suppose first that $v_2v_5 \in E(G)$ so that $v_2Cv_5v_2$ is a 4-cycle. Then, by Lemma 24, $G[xy + v_2Cv_5]$ contains a DCC on $|C|$ vertices, otherwise we contradict (O1). In particular, this is a triply chorded cycle, so that C must have three chords, as it would contradict (O2). Therefore, $v_1v_5 \in E(G)$. However, either x or y is adjacent to at least two vertices from v_2, v_5, v_6, and $G[\{v_1, v_2, v_5, v_6\}] \cong K_4^-$ with chord v_1v_5. So, by Lemma 23, either $G[x + v_5Cv_2]$ or $G[y + v_5Cv_2]$ contains a DCC on fewer vertices than C. Hence $v_2v_5 \notin E(G)$, so that $v_1v_5 \in E(G)$, and C has exactly two chords.

Now we consider which vertices are the neighbors of x. If $N_C(x) = \{v_3, v_4, v_5\}$, then $xv_3v_1v_5v_4x$ forms a DCC with chords xv_5 and v_3v_4 contradicting (O1). So x is adjacent to at least one of v_2 or v_6. Without loss of generality, suppose it is v_2. If $xv_3 \in E(G)$, then note that $G[v_1Cv_3 + v_j]$ contains a *Paw* for each $j \in \{4, 5, 6\}$. Therefore, for some $j \in \{4, 5, 6\}$, $\|x, \{v_2, v_3, v_j\}\| = 3$ so that by Lemma 25, $G[v_1Cv_3 + v_j]$ will contain a DCC contradicting (O1). So $xv_3 \notin E(G)$, and by a similar argument we cannot have both xv_5 and xv_6. So we must have $xv_4 \in E(G)$.

If $xv_5 \in E(G)$, then $xv_5v_1Cv_4x$ is a triply chorded cycle on $|C|$ vertices with chords v_4v_5, v_1v_3, and xv_2. However, this contradicts (O2) as C has exactly two chords. So $N_C(x) = \{v_2, v_4, v_6\}$ and $N_C(y) = \{v_3, v_5\}$.

We now use the observation to show $\|C, \tilde{C}\| \le 3|C| \le 3|\tilde{C}|$. Observe that xv_2Cv_5yx is a DCC on $|C|$ vertices with chords yv_3 and xv_4, avoiding v_1 and v_6. By symmetry, we obtain a similar DCC avoiding v_2. Also yv_5Cv_2xy is a DCC on $|C|$ vertices with chords xv_6 and v_1v_5, avoiding v_3 and v_4. By symmetry, we obtain a similar DCC avoiding v_5.

Therefore, $v_1v_3 \notin E(G)$ and by symmetry $v_1v_5 \notin E(G)$. □

Claim 46.2. $v_2v_5, v_3v_6 \notin E(G)$.

Proof. Suppose $v_2v_5 \in E(G)$. If $v_2v_6 \in E(G)$, then $v_2v_5v_6v_2$ is a K_3, and by Lemma 3, $G[xy + v_2v_5v_6]$ contains a DCC on strictly fewer vertices than C, contradicting (O1). Hence $v_2v_6 \notin E(G)$.

We now use the observation. Note that $G[v_2Cv_5]$ and $G[v_5Cv_2]$ are each 4-cycles. By Lemma 24, $G[v_2Cv_5 + xy]$ and $G[v_5Cv_2 + xy]$ contain DCCs on $|C|$ vertices, else we contradict (O1). These avoid v_1, v_6, and v_3, v_4, respectively. So we only need to find DCCs on $|C|$ vertices that avoid v_2 and v_5.

As $v_2v_6 \notin E(G)$, the only other possible chords are v_1v_4 or v_3v_6. However, either of these will allow us to create 4-cycles in C such that we can repeat the above argument to form our desired DCCs that avoid v_2 and v_5. Therefore, $v_2v_5 \notin E(G)$ and by symmetry $v_3v_6 \notin E(G)$. □

By the previous two claims, the only available chords are v_1v_4 and v_2v_6. We now use the observation. Note that $G[v_1Cv_4]$ and $G[v_4Cv_1]$ are both 4-cycles. By Lemma 24, $G[v_1Cv_4+xy]$ and $G[v_4Cv_1+xy]$ contain DCCs on $|C|$ vertices (else we contradict (O1)). These avoid, v_5, v_6 and v_2, v_3, respectively. So we only need to find DCCs on $|C|$ vertices that avoid v_1 and v_4.

Note that if v_3 and v_5 different neighbors in $\{x, y\}$, say xv_3 and yv_5, then $xv_3v_2v_6v_5yx$ is a DCC with chords incident to v_2 and v_6 on $|C|$ vertices avoiding v_1 and v_4. So v_3 and v_5 have the same neighbor in $\{x, y\}$, say u. Both v_2 and v_6 cannot both be adjacent to u as well, so without loss of generality, suppose v_2 is adjacent to v, where $\{x, y\} = \{u, v\}$. However, uv_5Cv_2vu is a DCC on $|C|$ vertices with chords incident to v_6, and vv_2Cv_5uv is a DCC on $|C|$ vertices with chords incident to v_3 and v_4, avoiding v_4 and v_1, respectively.

This completes all cases when $|C| = 6$, so $7 \leq |C| \leq |\tilde{C}|$.

Subcase 3.2. $|C| = 7$

As in the case where $|C| = 6$, we relabel the vertices of C as $C = v_1v_2 \dots v_7v_1$ where v_1 is not adjacent to either x or y. We know another vertex in C, say v^*, is not adjacent to either x or y, so we proceed based on those cases.

Claim 46.3. $v^* \notin \{v_2, v_7\}$.

Proof. Suppose v_2 is not adjacent to either x or y. Note that xy and $[v_3, v_7]$ are nontrivial paths such that $\|xy, [v_3, v_7]\| \geq 5$. So, by Lemma 18, either $G[xy+[v_3, v_7]]$ contains a DCC on at most six vertices, contradicting (O1), or configuration 1 or 2 holds. However, neither configuration holds as $|xy| = 2$, and $[v_3, v_7]$ has only five vertices. So v_2, and by symmetry v_7, is adjacent to either x or y. □

Claim 46.4. $v^* \notin \{v_3, v_6\}$.

Proof. Suppose v_3 is not adjacent to either x or y. Note that xy and $[v_4, v_7]$ are nontrivial paths such that $\|xy, [v_4, v_7]\| \geq 4$. So, by Lemma 17, either $G[xy + [v_4, v_7]]$ contains a DCC on at most six vertices, or configuration 1 or 2 holds. Suppose configuration 2 holds, and without loss of generality, v_4, v_7 are neighbors of x and v_5, v_6 are neighbors of y. Then $xv_4v_5yv_6v_7x$ is a DCC with chords v_5v_6 and xy, contradicting (O1). So configuration

1 holds. As $\|x, C\| = 3$ and $\|y, C\| = 2$, we must have $\|x, [v_4, v_7]\| = 3$ and $\|y, [v_4, v_7]\| = 1$. So along with symmetry, we may assume $N_C(x) = \{v_4, v_6, v_7\}$ and $N_C(y) = \{v_2, v_5\}$.

Note that yv_2Cv_6xy is a DCC on $|C|$ vertices in which the number of chords is exactly 2 plus the number of chords with both endpoints in $[v_2, v_6]$. Since the number of chords in C is exactly the number of chords with both endpoints in $[v_2, v_6]$ plus the number of chords with at least one endpoint in $\{v_1, v_7\}$, C must have at least two chords with an endpoint in $\{v_1, v_7\}$, else we contradict (O2). Similarly, $yv_5v_6xv_7Cv_2y$ shows that C has at least two chords with an endpoint in $\{v_3, v_4\}$. We now show that v_7 and v_4 cannot be incident to a chord.

No chord in C has both endpoints in $[v_4, v_7]$ otherwise xv_4Cv_7x is a DCC with xv_6 and this additional chord, contradicting (O1). Similarly, all of the following edges result in DCCs that contradict (O1). If $v_2v_7 \in E(G)$, then xv_6Cv_2yx is a DCC with chords xv_7 and v_2v_7. If $v_3v_7 \in E(G)$, then $xv_7v_3Cv_6x$ is a DCC with chords xv_4 and v_6v_7. If $v_1v_4 \in E(G)$, then $xv_7v_1v_4Cv_6x$ is a DCC with chords xv_4 and v_6v_7. If $v_2v_4 \in E(G)$, then $yv_2v_4Cv_6xy$ is a DCC with chords yv_5 and xv_4.

So v_4 and v_7 cannot be incident to a chord, however this implies that both v_1 and v_3 are incident to two chords, which contradicts Lemma 13. ☐

By the previous two claims, $v^* \in \{v_4, v_5\}$. By symmetry, we may assume $v^* = v_4$ so that v_4 is not adjacent to either x or y.

Claim 46.5. v_2 and v_5 have the same neighbor in $\{x, y\}$, and by symmetry v_3 and v_7 have the same neighbor in $\{x, y\}$.

Proof. Let $\{u, v\} = \{x, y\}$, and suppose $uv_2, vv_5 \in E(G)$. Note vv_5Cv_2uv is a DCC on $|C|$ vertices, in which the number of chords is exactly the number of chords in C with both endpoints in $[v_5, v_2]$ plus two. The number of chords in C is exactly the number of chords with both endpoints in $[v_5, v_2]$ plus the number of chords with at least one endpoint in $\{v_3, v_4\}$. So C must have at least two chords with an endpoint in $\{v_3, v_4\}$, otherwise we contradict (O2). We will show this cannot happen.

Note that every chord of the form v_iv_{i+3} modulo 7 creates a 4-cycle in C. So unless the chord is v_1v_4, we get a DCC by Lemma 24 that contradicts (O1). So every chord in C is either v_1v_4 or it creates a K_3 in C. If there exists a chord in C with both endpoints in $[v_2, v_5]$, then uv_2Cv_5vu is a DCC with this chord and either uv_3 or vv_3, contradicting (O1). So the only chords with an endpoint in $\{v_3, v_4\}$ are v_1v_3, v_1v_4, and v_4v_6.

We cannot have both v_1v_3 and v_1v_4. If so, and v_3 is adjacent to u, then $uv_2v_1v_4v_3u$ is a DCC with chords v_2v_3 and v_1v_3, contradicting (O1). Also if

v_3 is adjacent to v, then $uv_2v_1v_4v_3vu$ is a DCC with chords v_2v_3 and v_1v_3, contradicting (O1).

So $v_4v_6 \in E(G)$, and either v_1v_4 or v_1v_3 exists. Now v_3 must be a neighbor of v, otherwise $vv_5v_6v_4v_3uv$ is a DCC with chords v_4v_5 and either uv_6 or vv_6, contradicting (O1). Also v_6 must a be a neighbor of u, otherwise $G[v_3Cv_6]$ contains a Paw, and $\|v, [v_3Cv_6]\| = 3$. So, by Lemma 25, $G[v_3Cv_6 + v]$ contains a DCC contradicting (O1). However, vv_3Cv_6uv is a DCC with chords vv_5 and v_4v_6, contradicting (O1).

This completes the proof of the claim. □

By the claim, we may assume without loss of generality, $N_C(x) = \{v_2, v_5, v_6\}$ and $N_C(y) = \{v_3, v_7\}$. If any chord exists in $[v_2, v_6]$, then xv_2Cv_6x forms a DCC containing this chord and xv_5 that contradicts (O1). Similarly, if any chord exists in $[v_5, v_2]$, then xv_5Cv_2x forms a DCC containing this chord and xv_6, contradicting (O1). Hence the only possible chords in C have one endpoint in $\{v_3, v_4\}$ and the other endpoint in $\{v_7, v_1\}$. If $v_3v_1 \in E(G)$, then $yv_7v_1v_3v_2x$ forms a DCC with chords v_1v_2 and yv_3, and if $v_3v_7 \in E(G)$, then by Lemma 24 $G[xy + [v_7, v_3]]$ contains a DCC, each contradicting (O1).

Therefore, $v_4v_7, v_4v_1 \in E(G)$. But then, $xv_5v_4v_1v_7v_6x$ forms a DCC contradicting (O1).

This completes the case when $|C| = 7$.

Subcase 3.3. $|C| \geq 8$.

In our final case we return to the labelling of $N_C(x) = \{x_1, x_2, x_3\}$ and $N_C(y) = \{y_1, y_2\}$. Up to symmetry, we have two cases for how x_1, x_2, x_3, y_1, y_2 appear along C (not necessarily consecutive): x_1, x_2, x_3, y_1, y_2, or x_1, y_1, x_2, x_3, y_2.

Claim 46.6. x_1, y_1, x_2, x_3, y_2 appear in this order along C (not necessarily consecutive).

Proof. Suppose on the contrary that x_1, x_2, x_3, y_1, y_2 appear in this order along C (not necessarily consecutive). Since xx_2Cy_2yx is a DCC with chords xx_3 and yy_1, there is at most one vertex in (y_2, x_2) other than x_1. By symmetry, there is at most one vertex in (x_2, y_1) other than x_3. Since xx_1Cy_1yx is a DCC with chords xx_2 and xx_3, there is at most one vertex in (y_1, x_1) other than y_2. By symmetry, there is at most one vertex in (x_3, y_2) other than y_1.

If there is a vertex in (y_2, x_1), then $|C| \leq 7$. Similarly, if there is a vertex in (x_3, y_1). As $|C| \geq 8$, we must have exactly one vertex in each of (x_1, x_2), (x_2, x_3), and (y_1, y_2); label these vertices v_1, v_2, and v_3, respectively. In particular, $|C| = 8$ in this case.

Now xx_2Cy_2yx is a DCC on $|C|$ vertices whose number of chords is exactly two plus the number of chords in C with both endpoints in $[x_2, y_2]$. As the number of chords in C is exactly the number of chords with both endpoints in $[x_2, y_2]$ plus the number of chords with at least one endpoint in $\{x_1, v_1\}$, C has at least two chords with an endpoint in $\{x_1, v_1\}$, else we contradict (O2). By symmetry, C has at least two chords with an endpoint in $\{v_2, x_3\}$. We now show x_1 and x_3 are not incident to any chords in C.

If there exists a chord in C with both endpoints in $[x_1, x_3]$, then xx_1Cx_3x is a DCC with this chord and xx_2, contradicting (O1). If there exists a chord in C with both endpoints in $[y_1, x_1]$, then yy_1Cx_1xy is a DCC with this chord and yy_2, contradicting (O1). So x_1 is not incident to a chord in C, and by symmetry, the same holds for x_3. However, this implies v_1 and v_2 are both incident to two chords, contradicting Lemma 13. □

By the above claim, we assume x_1, y_1, x_2, x_3, y_2 appear in this order along C (not necessarily consecutive). Since yy_2Cx_2xy is a DCC with chords xx_1 and yy_1, there is at most one vertex in (x_2, y_2) other than x_3. By symmetry, there is at most one vertex in (y_1, x_3) other than x_2. Since yy_1Cx_1xy is a DCC with chords xx_2, xx_3, yy_2, there are at most two vertices in (x_1, y_1). By symmetry, there are at most two vertices in (y_2, x_1).

Claim 46.7. $(x_2, x_3) \neq \emptyset$.

Proof. If $(x_2, x_3) = \emptyset$, then $yy_1Cx_2xx_3Cy_2y$ is a DCC with chords x_2x_3 and xy, so that there is at most one vertex in (y_2, y_1) other than x_1. As $|C| \geq 8$, we may label C by symmetry as $C = x_1v_1y_1v_2x_2x_3v_3y_2x_1$. Now xx_3Cy_1yx is a DCC on $|C|$ vertices whose number of chords is exactly two plus the number of chords with two endpoints in $[x_3, y_1]$. As the number of chords in C is exactly the number of chords with two endpoints in $[x_3, y_1]$ plus the number of chords with at least one endpoint in $\{v_2, x_2\}$, C must have at least two chords with an endpoint in $\{v_2, x_2\}$, else we contradict (O2). Similarly, yy_2Cx_2xy implies there are at least two chords in C with an endpoint in $\{x_3, v_3\}$. We claim no chord is incident to either x_2 or x_3.

Indeed, if C has a chord with both endpoints in $[x_1, x_3]$, then xx_1Cx_3x is a DCC with this chord and xx_2, contradicting (O1). Similarly, if C has a chord with both endpoints in $[x_2, x_1]$, then xx_2Cx_1x is a DCC with this chord and xx_3. Thus, neither x_2 nor x_3 can be incident to a chord. So v_2 and v_3 must both be incident to two chords each, contradicting Lemma 13. □

Let $v_1 \in (x_2, x_3)$. This implies that y_1, x_2, v_1, x_3, y_2 are all consecutive along C. Now yy_2Cx_2xy is a DCC on $|C|$ vertices whose number of chords is exactly two plus the number of chords with both endpoints in $[y_2, x_2]$.

As the number of chords in C is exactly the number of chords with both endpoints in $[y_2, x_2]$ plus the number of chords with at least one endpoint in $\{v_1, x_3\}$, C must have at least two chords with an endpoint in $\{v_1, x_3\}$, else we contradict (O2). By symmetry, C must have at least two chords with an endpoint in $\{v_1, x_2\}$. As a result, v_1 must be incident to a chord, otherwise x_2 and x_3 are both incident to two chords, contradicting Lemma 13.

Since $|C| \geq 8$, without loss of generality, there exists $v_2 \in (x_1, y_1)$. C has no chords with both endpoints in $[x_2, x_1]$, otherwise xx_2Cx_1x is a DCC with this chord and xx_3, contradicting (O1). Similarly, if C has a chord with both endpoints in $[y_1, x_3]$, then yy_1Cx_3xy is a DCC with this chord and xx_2, contradicting (O1). Therefore, every chord with an endpoint in $\{v_1, x_3\}$ (recall that there are at least two such chords) has its other endpoint in (x_1, y_1), and the same holds for every chord with an endpoint in $\{v_1, x_2\}$ (recall that there are at least two such chords).

As a result, $(y_2, x_1) = \emptyset$, otherwise xx_1Cx_3x is a DCC with chords xx_2 and at least one chord with its endpoints in $\{v_1, x_3\}$ and (x_1, y_1), contradicting (O1). Therefore, we can label the vertices of C as $C = x_1v_2v_3y_1x_2v_1x_3y_2x_1$. So, in particular, every chord with an endpoint in either $\{v_1, x_2\}$ or $\{v_1, x_3\}$ has its other endpoint in $\{v_2, v_3\}$, and we know there are at least two such chords.

Recall that v_1 must be incident to a chord. If $v_1v_2 \in E(G)$, then $xx_3v_1v_2Cx_2x$ is a DCC with chords x_2v_1 and at least one other chord with its endpoints in $\{v_1, x_2\}$ and $\{v_2, v_3\}$, contradicting (O1). So $v_1v_3 \in E(G)$. As C has at least two chords with an endpoint in $\{v_1, x_2\}$, x_2 is incident to a chord with an endpoint in $\{v_2, v_3\}$, and the same holds for x_3. If either x_2 or x_3 is adjacent to v_3, then $xx_3v_1v_3Cx_2x$ is a DCC with this chord and x_2v_1, contradicting (O1). However, this implies both $x_2v_2, x_3v_2 \in E(G)$, yet $v_2Cx_3v_2$ is a DCC with chords v_1v_3 and x_2v_2, contradicting (O1).

This completes all cases and proves the lemma. □

Lemma 47. *Let $C_1, C_2 \in \mathcal{C}$ such that $|C_1| \geq 6$ and $|C_2| \geq 6$, then $\|C_1, C_2\| \leq 3\max\{|C_1|, |C_2|\} + 4$.*

Proof. Let $C_1, C_2 \in \mathcal{C}$ with $|C_1| \geq 6$ and $|C_2| \geq 6$, and assume that $\|C_1, C_2\| \geq 3\max\{|C_1|, |C_2|\} + 5$.

Claim 47.1. Let $v \in C_i$ and $u \in C_{3-i}$. If $\|v, C_{3-i} - u\| \geq 4$ and $\|u, C_i - v\| \geq 4$, then equality holds, and furthermore, u and v are each incident to two chords in their respective DCCs. Consequently, if $\|v, C_{3-i}\| \geq 5$ and $\|u, C_i\| \geq 5$, then equality holds, $uv \in E(G)$, and u and v are each incident to two chords in their respective DCCs.

Proof. Let $v \in C_i$ and $u \in C_{3-i}$ such that $\|v, C_{3-i} - u\| \geq 4$ and $\|u, C_i - v\| \geq 4$. Label four neighbors of v in C_{3-i} as v_1, v_2, v_3 and v_4 such that they appear in this order along C_{3-i} (not necessarily consecutive), where $N_{C_{3-i}}(v) \cap [v_1, v_4]_{C_{3-i}} = \{v_1, v_2, v_3, v_4\}$ and $u \in (v_4, v_1)_{C_{3-i}}$. Similarly, label four neighbors of u in C_i as u_1, u_2, u_3, and u_4.

If $(u, v_1)_{C_{3-i}} \cup (v_4, u)_{C_{3-i}} \neq \emptyset$, then $G[u + [v_1, v_4]_{C_{3-i}}]$ and $G[v + [u_1, u_4]_{C_i}]$ contain DCCs on strictly fewer vertices than $|C_1| + |C_2|$, contradicting (O1). Therefore v_4, u, and v_1 are consecutive along C_{3-i} and by symmetry, u_4, v, and u_1 are consecutive along C_i. This implies $\|v, C_{3-i} - u\| = \|u, C_i - v\| = 4$.

Note that $vv_1C_{3-i}v_4v$ forms a DCC with chords vv_2 and vv_3 and $uu_1C_iu_4u$ forms a DCC with chords uu_2 and uu_3, call these C_v and C_u respectively. Furthermore, the number of chords in C_v is exactly two more than the number of chords in C_{3-i} not incident to u, and the number of chords in C_u is exactly two more than the number of chords in C_i not incident to v. Therefore, u and v must both be incident to at least two chords otherwise C_v and C_u forms DCC on the same number of vertices as $|C_1|$ and $|C_2|$, but with more chords, contradicting (O2). ☐

Claim 47.2. Let $v \in C_i$ and xy be an edge along the cycle of C_{3-i} such that $\|v, C_{3-i}\| \geq 5$ and $\|\{x, y\}, C_i\| \geq 7$. Then, $xv, yv \in E(G)$.

Proof. Let $v \in C_i$ and xy be an edge along the cycle of C_{3-i} such that $\|v, C_{3-i}\| \geq 5$ and $\|\{x, y\}, C_i\| \geq 7$. Suppose that either $xv \notin (E)G$ or $yv \notin E(G)$. This means that $\|v, C_{3-i} - x - y\| \geq 4$ and so $G[C_{3-i} - x - y + v]$ contains a DCC. Furthermore, $\|\{x, y\}, C_i - v\| \geq 6$ and by Lemma 18, $G[C_i - v + xy]$ contains a DCC on strictly fewer vertices than $|C_i - v + xy|$, contradicting (O1). Therefore, $xv, yv \in E(G)$. ☐

We will consider the following cases:

Case 1. Suppose there exists a vertex $v \in C_i$ such that $\|v, C_{3-i}\| \geq 5$.

By Lemma 44, $\|v, C_{3-i}\| = 5$. Label the neighbors of v in C_{3-i} as v_1, v_2, v_3, v_4 and v_5 in this order along the cycle, not necessarily consecutive. Note that by Claim 47.1, for all $x \in V(C_{3-i}) - N_{C_{3-i}}(v)$, we have $\|x, C_i\| \leq 3$. As a result, we must have $\|\{v_1, v_2, v_3, v_4, v_5\}, C_i\| \geq 20$, otherwise $\|C_{3-i}, C_i\| \leq 3(|C_{3-i}| - 5) + 19 \leq 3|C_{3-i}| + 4$, a contradiction.

Therefore, if $\|v_i, C_{3-i}\| = 4$, for all $1 \leq i \leq 5$, then $\|\{v_1, v_2, v_3, v_4, v_5\}, C_i\| = 20$ and $\|x, C_i\| = 3$ for all vertices $x \in V(C_{3-i}) - N_{C_{3-i}}(v)$. Since $|C_{3-i}| \geq 6$, there does exist $x \in V(C_{3-i}) - N_{C_{3-i}}(v)$, and we can assume without loss of generality that $x \in (v_5, v_1)_{C_{3-i}}$ and that xv_1 is an edge along the cycle of C. However $\|\{x, v_1\}, C_i\| \geq 7$, which contradicts Claim 47.2.

Therefore, suppose that $\|v_1, C_{3-i}\| = 5$. By Claim 47.1, v and v_1 are each incident to two chords in their respective DCCs. Therefore, for $j \in \{2,3,4,5\}$, $\|v_j, C_i\| \leq 4$, otherwise v_j would be incident to two chords by Claim 47.1, which will contradict Lemma 13. Since $\|C_1, C_2\| \geq 3\max\{|C_1|, |C_2|\} + 5$, we get that for all $j \in \{2,3,4,5\}$, $3 \leq \|v_j, C_i\| \leq 4$, and further, there can only be one $j \in \{2,3,4,5\}$ where $\|v_j, C_i\| = 3$. Lastly, if $\|v_j, C_i\| = 3$ for some $j \in \{2,3,4,5\}$, then for all $z \in V(C_{3-i}) - N_{C_{3-i}}(v)$, $\|z, C_i\| = 3$.

Since $|C_{3-i}| \geq 6$, there exists a vertex $u \in V(C_{3-i}) - N_{C_{3-i}}(v)$. So $u \in (v_j, v_{j+1})_{C_{3-i}}$ for some $j \in \{1,2,3,4,5\}$ where j is taken modulo 5. As noted, either $\|v_j, C_i\| \geq 4$ or $\|v_{j+1}, C_i\| \geq 4$. Without loss of generality, we may assume it is v_j, and furthermore, we may assume v_j and u are consecutive along C_{3-i}. However, this implies $\|u, C_i\| \leq 2$, otherwise we contradict Claim 47.2. However, in order to satisfy $\|C_1, C_2\| \geq 3\max\{|C_1|, |C_2|\} + 5$, we must have $|C_{3-i}| = 6$, $\|u, C_i\| = 2$, and $\|v_j, C_i\| = 4$ for all $j \in \{2,3,4,5\}$. Also u cannot be adjacent to v_1 along the cycle of C_{3-i}, otherwise we contradict Claim 47.2.

By symmetry, we may assume $u \in (v_2, v_4)_{C_{3-i}}$. Recall that v_1 is incident to two chords in C_{3-i}. There can be at most one chord in $[v_4, v_1]_{C_{3-i}}$, meaning v_1 must be incident to a chord in $[v_1, v_4)_{C_{3-i}}$. Note that regardless of the location of the chord in $[v_1, v_4)_{C_{3-i}}$, and regardless of the location of $u \in (v_2, v_4)_{C_{3-i}}$, $G[v + [v_1, v_4)_{C_{3-i}}]$ contains a DCC. However, since $\|\{v_4, v_5\}, C_i - v\| = 6$, by Lemma 18, $G[C_i - v + v_4 + v_5]$ contains a DCC on strictly fewer vertices than $|C_i - v + v_4 + v_5|$, contradicting (O1). This completes the case.

Case 2. Suppose for all vertices $v \in C_i$, $\|v, C_{3-i}\| \leq 4$.

By symmetry, we may assume that for all $z \in C_{3-i}$, $\|z, C_i\| \leq 4$. As $\|C_1, C_2\| \geq 3\max\{|C_1|, |C_2|\} + 5$, there exists $v \in C_i$ such that $\|v, C_{3-i}\| = 4$. Label the neighbors of v in C_{3-i} as v_1, v_2, v_3, and v_4 in this order along the cycle, not necessarily consecutive. Since $\|C_1, C_2\| \geq 3\max\{|C_1|, |C_2|\} + 5$ and for each j, $\|v_j, C_i\| \leq 4$, there exists $u \in V(C_{3-i}) - N_{C_{3-i}}(v)$, such that $\|u, C_i\| = 4$, and we can label the neighbors of u in C_i as u_1, u_2, u_3 and u_4 in this order along the cycle, not necessarily consecutive, with $u \in (v_4, v_1)_{C_{3-i}}$ and $v \in (u_4, u_1)_{C_i}$.

By Claim 47.1, u and v are both incident to two chords in their respective DCCs. If there exists another vertex $u' \in V(C_{3-i}) - N_{C_{3-i}}(v)$, then $\|u', C_i\| \leq 3$ otherwise by Claim 47.1, u' is also incident to two chords, contradicting Lemma 13. Thus as $\|C_1, C_2\| \geq 3|C_i| + 5$, we have $\|v_j, C_{3-i}\| = 4$ and by symmetry, $\|u_j, C_i\| = 4$ for each $j \in \{1,2,3,4\}$.

Now consider v_1. If $v_1u_1 \notin E(G)$, then by Claim 47.1, v_1 is incident to two chords, contradicting Lemma 13. Hence, $v_1u_1, v_1u_2, v_1u_3, v_1u_4 \in E(G)$. However, this implies $\|v_1, C_i\| = 5$, which is a contradiction.

This completes all cases and proves the lemma. □

We can now prove Lemma 38.

Proof of Lemma 38. Let $C \in \mathcal{C} \setminus \{\tilde{C}\}$. We show that in every possibility, $\|R+\tilde{C}, C\| \leq 3(|\tilde{C}|+4)$.

If there exists $v \in R$ such that $\|v, C\| \geq 4$, then by Lemma 15 equality holds, and by Lemma 40 $\|R+\tilde{C}, C\| \leq 3(|\tilde{C}|+4)$. So we may assume that $\|R, C\| \leq 12$.

If there exists an edge $xy \in E(R)$ such that $\|xy, C\| \geq 5$, then by Lemma 46 $\|\tilde{C}, C\| \leq 3|\tilde{C}|$. So $\|R+\tilde{C}, C\| \leq 12+3|\tilde{C}| = 3(|\tilde{C}|+4)$. So we may assume that $\|R, C\| \leq 8$.

If $|C| = 4$ so that $C \cong K_4$, then by Lemma 43, $\|\tilde{C}, C\| \leq 3|\tilde{C}|$. So $\|R+\tilde{C}, C\| \leq 8+3|\tilde{C}| \leq 3(|\tilde{C}|+4)$.

If $|C| = 5$, then by Lemma 45, $\|\tilde{C}, C\| \leq 3|\tilde{C}|$. So $\|R+\tilde{C}, C\| \leq 8+3|\tilde{C}| \leq 3(|\tilde{C}|+4)$.

If $|C| \geq 6$, then by Lemma 47, $\|\tilde{C}, C\| \leq 3|\tilde{C}|+4$. So $\|R+\tilde{C}, C\| \leq 8+3|\tilde{C}|+4 = 3(|\tilde{C}|+4)$.

This completes the proof of the lemma. □

8. Proof of Theorem 10

In this section we prove Theorem 10. So we are assuming G is an n-vertex graph with $n \geq 4k$ such that $\delta(G) \geq \frac{10k-1}{3}$, and furthermore, G is edge-maximal with respect to not having k disjoint doubly chorded cycles. It is important to note that in all of the previous lemmas we were assuming that G was an n-vertex with $n \geq 4k$ and $\delta(G) \geq 3k$. Since $\frac{10k-1}{3} \geq 3k$ for $k \geq 1$, all of the previous lemmas apply in this section as well. So, in particular, by Lemma 35, $R \cong K_{1,1,2}$. We will also heavily rely on Lemma 37 below.

Lemma 48. *If there exists $C' \in \mathcal{C}$ such that $\|R, C'\| \geq 13$, then $G[R+C'] \notin \{K_{3,3,3}^-, K_{3,3,3}\}$.*

Proof. Suppose on the contrary that there exists $C' \in \mathcal{C}$ such that $\|R, C'\| \geq 13$ and $G[R+C'] \in \{K_{3,3,3}^-, K_{3,3,3}\}$. Note that by Lemma 37, $G[C'] \cong K_{1,2,2}$. In the following we will assume that $G[R+C'] \cong K_{3,3,3}^-$, as all the arguments will hold if $G[R+C'] \cong K_{3,3,3}$. Let u and v be the nonadjacent vertices in $R+C'$ such that if we added the edge uv, we would have $G[R+C'] \cong K_{3,3,3}$.

Claim 48.1. For all $C \in \mathcal{C} - \{C'\}$, $\|R+C', C\| \leq 30$.

Proof. Suppose there exists $C \in \mathcal{C} - \{C'\}$ such that $\|R+C', C\| \geq 31$. Note that for every vertex $z \in R+C'$, we can arrange $G[R+C']$ into two disjoint subgraphs R^* and C^* such that $z \in R^*$, $R^* \cong R$, and $G[C^*] \cong G[C'] \cong K_{1,2,2}$. Therefore, R^* and $\mathcal{C}^* = (\mathcal{C} \cup \{C^*\}) - \{C'\}$ is an optimal partition with $z \in R^*$. So, by Lemma 15, $\|z, C\| \leq 4$ for all $z \in R+C'$.

Since $\|R+C', C\| \geq 31$, there exist at least four vertices in $R+C'$, say v_1, v_2, v_3, v_4, such that $\|v_i, C\| = 4$ for each i. So, by Lemma 15, $C \in \{K_4, K_{1,2,2}\}$, and furthermore, for each i and $y \in C$, $G[C - y + v_i] \cong C$.

Note that regardless of whether $C \cong K_4$ or $C \cong K_{1,2,2}$, since $\|R+C', C\| \geq 31$, there exists a vertex $x \in C$ such that $\|x, R+C'\| \geq 7$. In particular, as $G[R+C'] \cong K^-_{3,3,3}$, x must be adjacent to all three vertices of a triangle in $G[R+C']$. Furthermore, some v_i is not one of these three vertices. Therefore, x with these three vertices forms a K_4 and $G[C - x + v_i] \cong C$. So replacing C' and C with these DCCs respectively, contradicts (O1). This proves the claim. □

By the above claim:

$$9\left(\frac{10k-1}{3}\right) \leq \|R+C', R+C'\| + \|R+C', \mathcal{C} - \{C'\}\| \leq 54 + 30(k-2).$$

However, this yields $30k - 3 \leq 30k - 6$, a contradiction. □

We are now ready to prove Theorem 10.

Proof of Theorem 10. For each i where $0 \leq i \leq 8$, let $\mathcal{C}_i = \{C \in \mathcal{C} : \|\{v_1, v_4\}, C\| = i\}$. Note that $\sum_{i=0}^{8} |\mathcal{C}_i| = k-1$. By the definition of $\mathcal{C}_i$:

$$2\left(\frac{10k-1}{3}\right) \leq d_G(v_1) + d_G(v_4) = \|\{v_1, v_4\}, R\| + \|\{v_1, v_4\}, \mathcal{C}\| = 4 + \sum_{i=0}^{8} i \cdot |\mathcal{C}_i|.$$

This yields:

$$2\left(\frac{10k-1}{3}\right) \leq 4 + \sum_{i=0}^{8} i \cdot |\mathcal{C}_i|. \tag{2}$$

By Lemma 37, $\|R, C\| \leq 14$ for all $C \in \mathcal{C}$. If $C \in \mathcal{C}_8$ and $\|R, C\| \geq 13$, then by Lemma 37, $G[R+C] \in \{K^-_{3,3,3}, K_{3,3,3}\}$. However, this contradicts Lemma 48. So if $C \in \mathcal{C}_8$, then $\|R, C\| \leq 12$. Similarly, if $C \in \mathcal{C}_7$ and

$\|R, C\| \geq 13$, then by Lemma 37, $G[R+C] \cong K^-_{3,3,3}$. Again this contradicts Lemma 48 so that for all $C \in \mathcal{C}_7$, $\|R, C\| \leq 12$. These yield the following:

$$4\left(\frac{10k-1}{3}\right) \leq \|R, R\| + \|R, \mathcal{C}\| \leq 10 + \left(14\sum_{i=0}^{6} |\mathcal{C}_i|\right) + 12|\mathcal{C}_7| + 12|\mathcal{C}_8|.$$

This gives us:

$$4\left(\frac{10k-1}{3}\right) \leq 10 + \left(14\sum_{i=0}^{6} |\mathcal{C}_i|\right) + 12|\mathcal{C}_7| + 12|\mathcal{C}_8|. \tag{3}$$

However, adding (2) to (3) yields the following contradiction.

$$\begin{aligned} 20k - 2 &\leq 14 + 14|\mathcal{C}_0| + 15|\mathcal{C}_1| + 16|\mathcal{C}_2| + 17|\mathcal{C}_3| + 18|\mathcal{C}_4| + 19|\mathcal{C}_5| \\ &\quad + 20|\mathcal{C}_6| + 19|\mathcal{C}_7| + 20|\mathcal{C}_8| \\ &\leq 14 + 20\sum_{i=0}^{8} |\mathcal{C}_i| \\ &= 14 + 20(k-1) \\ &= 20k - 6. \end{aligned}$$

This completes the proof of Theorem 10. □

9. Exploring Conjecture 11

In this final section we provide some evidence to support Conjecture 11. In particular, we prove an approximate version of Conjecture 11 using a result on near-tilings of graphs by Shokoufandeh and Zhao in [9]. To state their result, we first need a few definitions.

For any graph H, let $\sigma(H)$ denote the size of the smallest color class over all proper $\chi(H)$-colorings of H. Define the critical chromatic number of H, denoted by $\chi_{cr}(H)$, to be

$$\chi_{cr}(H) = \frac{(\chi(H) - 1)|H|}{|H| - \sigma(H)}.$$

As examples, $\chi_{cr}(K_{1,2,2}) = \frac{5}{2}$, $\chi_{cr}(K_{1,1,2}) = \frac{8}{3}$, and $\chi_{cr}(K_4) = 4$.

Theorem 49 (Shokoufandeh and Zhao [9])**.** *For every H with $\chi(H) > 2$, there exists $n_0 = n_0(H)$ such that for every $n \geq n_0$ the following holds. If G is an n-vertex graph and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n,$$

then G contains a collection of disjoint copies of H that covers all but at most

$$\frac{5(\chi(H) - 2)(|H| - \sigma(H))^2}{\sigma(H)(\chi(H) - 1)}$$

vertices.

Using Theorem 49, we now prove a proposition that shows an approximate version of Conjecture 11 holds.

Proposition 50. *For every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that for all $n \geq n_0$, if G is an n-vertex graph with $4k + \epsilon n \leq n < 5k$ and $\delta(G) \geq \left(\frac{5k}{3n} + \frac{1}{3} + \epsilon\right) n$, then G contains k disjoint doubly chorded cycles.*

Proof. Fix $\epsilon > 0$, and let $C = \frac{7}{\epsilon} + 3$. For all graphs H on at most $\lceil C \rceil$ vertices, Theorem 49 returns an $n_0(H)$. Denote the maximum of these $n_0(H)$ as n_0^*, and let our n_0 be the maximum of n_0^* and $\left\lceil \frac{10}{3} C^3 \right\rceil$.

Let G be an n-vertex graph where $n \geq n_0$ with $4k + \epsilon n \leq n < 5k$ and $\delta(G) \geq \left(\frac{5k}{3n} + \frac{1}{3} + \epsilon\right) n$. Define h and k' as follows:

$$h = 2 \left\lfloor \frac{C}{2} \cdot \frac{n - \frac{10}{3}C^2}{n} \right\rfloor, \text{ and } k' = \left\lceil C \cdot \frac{k}{n} \right\rceil. \tag{4}$$

We now derive two useful inequalities involving h. First, as $\dfrac{n - \frac{10}{3}C^2}{n} \leq 1$ and $h = 2 \left\lfloor \dfrac{C}{2} \cdot \dfrac{n - \frac{10}{3}C^2}{n} \right\rfloor$, we get:

$$h \leq C. \tag{5}$$

Second, observe that

$$h = 2 \left\lfloor \frac{C}{2} \cdot \frac{n - \frac{10}{3}C^2}{n} \right\rfloor \geq 2 \left(\frac{C}{2} \cdot \frac{n - \frac{10}{3}C^2}{n} - 1 \right) = C - 2 - \frac{\frac{10}{3}C^3}{n}.$$

As n was chosen so that $n \geq n_0 \geq \frac{10}{3}C^3$, we get:

$$h \geq C - 3. \tag{6}$$

Recall the definition of the graph $G(t, n)$ from Section 2. Let $H = G(5k' - h, h)$ so that by the construction of $G(5k' - h, h)$, H will have exactly h vertices and by (5), $h \leq C$ so that by our choice of $n \geq n_0$, we can apply Theorem 49 to H.

Claim 50.1. $4k' \leq h < 5k'$.

Proof. We first show $h < 5k'$ by showing $\frac{5k'}{h} \geq \frac{5k}{n}$. Since $k' = \lceil \frac{Ck}{n} \rceil \geq \frac{Ck}{n}$ and $h \leq C$ by (5), we have:

$$\frac{5k'}{h} \geq \frac{5\frac{Ck}{n}}{C} = \frac{5k}{n}.$$

Since n was chosen so that $n < 5k$, we get $\frac{5k}{n} > 1$ so that $\frac{5k'}{h} > 1$. Thus, $h < 5k'$.

We now show $h \geq 4k'$. Since $k' = \lceil \frac{Ck}{n} \rceil \leq \frac{Ck}{n} + 1$ and $h \geq C - 3$ by (6), we have:

$$\frac{k'}{h} \leq \frac{\frac{Ck}{n} + 1}{C - 3} = \frac{k}{n} + \frac{\frac{3k}{n} + 1}{C - 3}.$$

Since n was chosen so that $n \geq 4k + \epsilon n$, we know that $\frac{k}{n} < \frac{1}{4}$. By this and the fact that $C = \frac{7}{\epsilon} + 3$, we get:

$$\frac{k}{n} + \frac{\frac{3k}{n} + 1}{C - 3} < \frac{k}{n} + \frac{\frac{7}{4}}{C - 3} = \frac{k}{n} + \frac{\epsilon}{4}.$$

So

$$\frac{k'}{h} \leq \frac{k}{n} + \frac{\epsilon}{4}, . \tag{7}$$

As mentioned above, $\frac{k}{n} < \frac{1}{4}$ so that $\frac{k'}{h} < \frac{1+\epsilon}{4}$ and consequently $\frac{4k'}{h} \leq 1$. Thus, $h \geq 4k'$. ☐

Thus by this claim and Lemma 12, H contains k' disjoint doubly chorded cycles. Furthermore, as the claim states $h < 5k'$, we get $5k' - h > 0$ so that H is 4-partite. Thus, $\chi(H) = 4$ and $\sigma(H) = 5k' - h$. So $\chi_{cr}(H) = \frac{(4-1)h}{h-(5k'-h)} = \frac{3h}{2h-5k'}$. This gives:

$$\left(1 - \frac{1}{\chi_{cr}(H)}\right) n = \left(1 - \frac{2h - 5k'}{3h}\right) n = \left(\frac{5k'}{3h} + \frac{1}{3}\right) n.$$

Recall that $\delta(G) \geq \left(\frac{5k}{3n} + \frac{1}{3} + \epsilon\right) n$. By (7),

$$\frac{5k}{3n} + \frac{1}{3} + \epsilon \geq \frac{5}{3}\left(\frac{k'}{h} - \frac{\epsilon}{4}\right) + \frac{1}{3} + \epsilon \geq \frac{5k'}{3h} + \frac{1}{3}.$$

So $\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n$, and by Theorem 49, G contains a collection of disjoint copies of H that covers at least $n - \frac{10(2h-5k')^2}{3(5k'-h)}$ vertices.

As $4k' \leq h < 5k'$, we have $5k' - h > 0$ and $0 \leq 2h - 5k' < h$. So $\frac{(2h-5k')^2}{5k'-h} \leq h^2 \leq C^2$, where the last inequality is due to (5). Therefore, G contains a collection of disjoint copies of H that covers at least $n - \frac{10}{3}C^2$ vertices. As each copy of H contains has h vertices and contains k' disjoint doubly chorded cycles, G contains at least $(n - \frac{10}{3}C^2)(\frac{k'}{h})$ disjoint doubly chorded cycles.

By (4),

$$k' = \left\lceil \frac{Ck}{n} \right\rceil \geq \frac{Ck}{n} \text{ and } h = 2\left\lfloor \frac{C}{2} \cdot \frac{n - \frac{10}{3}C^2}{n} \right\rfloor \leq C\left(\frac{n - \frac{10}{3}C^2}{n}\right).$$

Therefore, the number of disjoint doubly chorded cycles in G is at least:

$$\left(n - \frac{10}{3}C^2\right)\frac{k'}{h} \geq \left(n - \frac{10}{3}C^2\right)\frac{Ck}{n}\frac{n}{C(n - \frac{10}{3}C^2)} = k.$$

This proves the proposition. □

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Received May 15, 2020