REGULAR DEFORMATIONS OF COMPLETELY INTEGRABLE SYSTEMS

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We study several aspects of the regular deformations of completely integrable systems. Namely, we prove the existence of a Hamiltonian normal form for these deformations and we show the necessary and sufficient conditions a perturbation has to satisfy in order for the perturbed Hamiltonian to be a first order deformation.

1. Introduction

This article presents some results concerning the deformations of regular completely integrable (CI) systems. These are the dynamical systems defined by a Hamiltonian $H_0 \in C^\infty(\mathcal{M})$ on a symplectic manifold $\mathcal{M}$ admitting a momentum map, i.e., a collection $\boldsymbol{A} = (A_1, \ldots, A_d) : \mathcal{M} \to \mathbb{R}^d$ of d smooth functions, d being half of the dimension of $\mathcal{M}$, satisfying $\{A_j, H\} = 0$ and $\{A_j, A_k\} = 0$ for all $j, k : 1, \ldots d$, and whose differentials dA_j are linearly independent almost everywhere. Then, the Arnol'd–Mineur–Liouville Theorem [**2**, **5**, **7**] insures that in a neighborhood of any connected component of any compact regular fiber $\boldsymbol{A}^{-1}(a)$, $a \in \mathbb{R}^d$, of the momentum map, there exists a fibration in Lagrangian tori along which H_0 is constant. These tori are thus invariant by the dynamics generated by the associated Hamiltonian vector field X_H.

Despite the "local" character of the Arnol'd–Mineur–Liouville Theorem, it is tempting to try to glue together these "local" fibrations in the case of *regular* CI Hamiltonians, i.e., those for which there exists, near each point of $\mathcal{M}$, a local fibration in invariant Lagrangian tori. Unfortunately, this is not always possible. Some CI Hamiltonians do not admit any (global) fibration in Lagrangian tori and some others admit several different ones[1]. Nevertheless, these examples belong to the non-generic (within the class

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[1]For example, the free particle moving on the sphere S^2 is such a system.

of regular CI Hamiltonians) class of *degenerate* Hamiltonians and one can show that imposing a non-degeneracy condition insures that there exists a fibration of $\mathcal{M}$ in Lagrangian tori along which H_0 is constant, and moreover that it is unique. The genericity of non-degeneracy conditions thus motivates the study of fibrations in Lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$. Such a fibration actually gives rise to several natural geometric structures which we review in the first section.

Starting from a regular CI Hamiltonian $H_0 \in C^\infty(\mathcal{M})$, it is well-known since Poincaré's work [9] that adding a small perturbation εH_1 will destroy its integrable character and yield chaotic behaviors. Nevertheless, it is relevant to investigate the space of all CI Hamiltonians, since they are the starting point of any perturbation theory, like the celebrated K.A.M. Theory [1, 4, 8] which tells us that one can actually say a lot about the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ when ε is small.

A first step toward the understanding of the space of all CI systems is to restrict ourselves to regular deformations of regular CI hamiltonians, i.e., smooth families of Hamiltonians H_ε which are CI and regular for each ε. After introducing a few necessary tools in Section 3.2, we prove in Section 3.3 a normal form for regular deformations of CI Hamiltonians. Finally, Section 3.4 is devoted to the study of the first order deformations. We give there the condition on the perturbation H_1 for the perturbed system $H_0 + \varepsilon H_1$ to be CI up to ε^2.

2. Geometric structures of regular CI systems

In this section, we review several geometric structures which are naturally associated with any fibration in Lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$. In particular, we show that there exists a natural process of averaging any tensor field in the direction of the fibers. This process then allows us to prove (Proposition 2.6) that each symplectic vector field splits into two parts : the first is Hamiltonian and the second is symplectic and preserves the fibration. This will be used in Section 3.3 to prove the Hamiltonian normal form (Theorem 3.9) for regular deformations.

First, let us fix some basic notations. We denote by $\mathcal{V}(\mathcal{M})$ the space of smooth vector fields on the manifold $\mathcal{M}$. A symplectic form ω on $\mathcal{M}$ provides a isomorphism $\omega : \mathcal{V}(\mathcal{M}) \to \Omega^1(\mathcal{M})$, also denoted by ω, i.e., $\omega(X) = \omega(X,.)$ for each $X \in C^\infty(\mathcal{M})$. The inverse is denoted by $\omega^{-1} : \Omega^1(\mathcal{M}) \to \mathcal{V}(\mathcal{M})$. For each vector field, we denote by ϕ_X^t its flow at time t. Let $O \subset \mathcal{M}$ be any subset. We say that a vector field X is symplectic (resp. Hamiltonian) in $\mathcal{O}$ if its associated 1-form $\omega(X)$ is closed (resp. exact) in $\mathcal{O}$. To each Hamiltonian $H \in C^\infty(\mathcal{M})$, we can associate a vector field $X_H = -\omega^{-1}(dH)$. Now, given a fibration $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$, we say that a vector field $\tilde{X} \in \mathcal{V}(M)$ is a lift of a vector field $X \in \mathcal{V}(\mathcal{B})$ if for each $b \in \mathcal{B}$ and each $m \in \pi^{-1}(b)$ we have $\pi_*(\tilde{X}_m) = X_b$.

2.1. The period bundle. Let $(\mathcal{M}, \omega)$ be a symplectic manifold of dimension $2d$ and $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ a locally trivial fibration in Lagrangian tori, whose fibers are denoted by $\mathcal{M}_b = \pi^{-1}(b)$, $b \in \mathcal{B}$. The tangent spaces $L_m = T_m \mathcal{M}_{\pi(m)}$ of the fibers form an integrable vector subbundle $L = \bigcup_{m \in \mathcal{M}} L_m$ of $T\mathcal{M}$. A theorem due to Weinstein [13] insures that each leaf of a Lagrangian foliation (not necessarily a fibration) is naturally endowed with an affine structure. This affine structure on a leaf $\mathcal{M}_b$ can actually be expressed in a very convenient way (see [14]) in terms of the torsion-free and flat connection $\nabla : \mathcal{V}(\mathcal{M}_b) \times \mathcal{V}(\mathcal{M}_b) \to \mathcal{V}(\mathcal{M}_b)$ defined by

$$\nabla_X Y = \omega^{-1}(\widetilde{X} \lrcorner d(\widetilde{Y} \lrcorner \omega)),$$

where $\widetilde{X} \in \Gamma(L)$ and $\widetilde{Y} \in \Gamma(L)$ extend X and Y in $\mathcal{V}(\mathcal{M})$ and are everywhere tangent to L. We denote by $\mathcal{V}_\nabla(\mathcal{M}_b)$ the space of parallel vector fields on $\mathcal{M}_b$. One can see easily from the definition of ∇ that a vector field $X \in \mathcal{V}(\mathcal{M})$ is vertical and parallel on each fiber if and only if its associated 1-form $\omega(X)$ is a pull-back of a 1-form on $\mathcal{B}$.

Now, since the foliation under consideration actually defines a fibration, the holonomy of ∇ must vanish. Indeed, for each $b \in \mathcal{B}$, any collection of smooth functions $f_1, \ldots, f_d \in C^\infty(\mathcal{B})$ whose differentials df_j are linearly independent near b provides d Hamiltonian vector fields $X_{f_1 \circ \pi}, \ldots, X_{f_d \circ \pi} \in \mathcal{V}(\mathcal{M})$ everywhere tangent to the fibers, parallel on each fiber and linearly independent in a neighborhood of $\mathcal{M}_b$. Therefore, they form a global parallel frame on $\mathcal{M}_b$, implying that the holonomy of ∇ vanishes and that each fiber $\mathcal{M}_b$ is endowed with the structure of a standard[2] affine torus. This implies that the space $\mathcal{V}_\nabla(\mathcal{M}_b)$ is a d-dimensional vector space and that the union $\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ is naturally endowed with a structure of a smooth vector bundle over $\mathcal{B}$.

Since each fiber $\mathcal{M}_b$ is isomorphic to the standard torus $\mathbb{T}^d$, we can consider among the parallel vector fields on $\mathcal{M}_b$, those whose dynamics is 1-periodic. We denote this set by Λ_b. It is then easy to prove that it is a lattice in $\mathcal{V}_\nabla(\mathcal{M}_b)$. We call it the *period lattice*. The genuine geometric content of the Arnol'd–Mineur–Liouville Theorem [2, 5, 7], which is often hidden by the formulation in coordinates, amounts to saying that the union $\Lambda = \bigcup_{b \in \mathcal{B}} \Lambda_b$, called the *period bundle*, is a smooth lattice subbundle of $\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$. This can be proved by constructing explicit smooth sections of this bundle which are 1-periodic, namely Hamiltonian vector fields $X_{\xi \circ \pi}$ where the function $\xi \in C^\infty(\mathcal{B})$ is called *action* and given by $b \to \xi(b) = \int_{\gamma(b)} \theta$, with θ any symplectic potential and $b \to \gamma(b)$ a smooth family of vertical cycles. Furthermore, this shows that smooth (local) sections of Λ are Hamiltonian.

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[2]Here, "standard" means holonomy-free.

The smoothness of the period bundle Λ provides a way to relate the spaces $\mathcal{V}_\nabla(\mathcal{M}_b)$ for neighboring points b and implies the existence of a natural integer flat connection on the vector bundle $\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$. This connection may have a non-vanishing holonomy, called the *monodromy*. Now, the symplectic form ω provides an isomorphism between the sections of $\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$ and those of $T^*\mathcal{B}$. This gives the base space $\mathcal{B}$ a natural structure of an affine space, as discovered by Duistermaat in [**3**].

2.2. The torus action bundle. Our discussion so far shows that given a fibration in Lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$, there exists a natural associated torus bundle acting on it. Indeed, for each $b \in \mathcal{B}$, the quotient

$$\mathcal{G}_b = \mathcal{V}_\nabla(\mathcal{M}_b)/\Lambda_b$$

is a Lie group isomorphic to the torus $\mathbb{T}^d$. This isomorphism is not canonical, but it can be realized by choosing a basis of Λ_b. We will denote the elements of $\mathcal{G}_b$ by $[X_b]$, with $X_b \in \mathcal{V}_\nabla(\mathcal{M}_b)$, since they are equivalence classes. Taking the union over all b, we get a torus bundle $\mathcal{G} = \bigcup_{b \in \mathcal{B}} \mathcal{G}_b$. It is a smooth bundle since the period bundle Λ is so. We stress the fact that $\mathcal{G}$ is in general *not* a principal bundle since there might not exist any global action of $\mathbb{T}^d$ on $\mathcal{G}$, because of the presence of monodromy which precisely prevents us from choosing a global basis of Λ. On the other hand, there exists a distinguished global section, since each fiber is a group with a well-defined identity element.

Although we cannot apply the general theory of connections on principal bundles, there is a natural way to speak about local parallel sections of $\mathcal{G}$ over a subset $\mathcal{O} \subset \mathcal{B}$. These sections are simply local sections $b \to [X_b]$ of $\mathcal{G}$, with $b \to X_b$ being a local parallel section of $\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b)$. We denote the set of local parallel sections by $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$.

Lemma 2.1. *For each simply contractible subset $\mathcal{O} \subset \mathcal{B}$, the space $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ is a Lie group isomorphic to the torus $\mathbb{T}^d$.*

Proof. If $\mathcal{O}$ is simply connected, then the monodromy vanishes in $\mathcal{O}$ and there exist local sections $X_1, \ldots, X_d \in \Gamma(\mathcal{O}, \Lambda)$ with $\{X_j(b)\}$ generating the lattice Λ_b at each $b \in \mathcal{O}$. To each element $(t_1, \ldots, t_d) \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, we associate $[X] = [t_1 X_1 + \cdots + t_d X_d] \in \Gamma_\nabla(\mathcal{O}, \mathcal{G})$. One easily verifies that this provides an isomorphism. $\square$

Let us now describe how the bundle $\mathcal{G}$ acts on $\mathcal{M}$. First, for each b, the group $\mathcal{G}_b$ acts naturally on $\mathcal{M}_b$ in the following way.

$$\mathcal{G}_b \times \mathcal{M}_b \to \mathcal{M}_b$$
$$([X_b], m) \to [X_b](m) = \phi^1_{X_b}(m),$$

where $X_b \in \mathcal{V}_\nabla(\mathcal{M}_b)$ is a representative of the class $[X_b]$. One can see easily that this action is commutative, free, transitive and affine with respect

to Weinstein's connection on $\mathcal{M}_b$. Now, given any section $g \in \Gamma(\mathcal{G})$, its restriction $g|_{\mathcal{O}}$ to any simply connected subset $\mathcal{O} \subset \mathcal{B}$ is of the form $g|_{\mathcal{O}} = [X]$, where $X \in \Gamma(\mathcal{O}, \bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$. We can then extend the previous fiberwise action of the groups $\mathcal{G}_b$ to a vertical action of the sections of the toric bundle $\mathcal{G}$ on $\mathcal{M}$ by

$$\Gamma(\mathcal{G}) \times \mathcal{M} \to \mathcal{M}$$

$$(g, m) \to [X](m) = \phi^1_X(m),$$

where $X \in \Gamma(\mathcal{O}, \bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$ for any simply connected neighborhood $\mathcal{O}$ of $b = \pi(m)$. This is well-defined since another representative X' of the class $[X]$ would differ from X only by an element of $\Gamma(\mathcal{O}, \Lambda)$ which would provide $\phi^1_{X'-X} = \mathbb{I}$. This action naturally inherits the properties of the fiberwise action and we can show that the following additional property holds when we restrict ourselves to the parallel sections of $\mathcal{G}$.

Lemma 2.2. *For any simply connected subset $\mathcal{O} \subset \mathcal{B}$, the group $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ acts vertically on $\mathcal{M}$ in a symplectic way.*

We call this action the *toric action* of $\mathcal{G}$ on $\mathcal{M}$. Even if this action is local, it provides a way to average any tensor field on $\mathcal{M}$. Indeed, according to Lemma 2.1, $\Gamma_\nabla(\mathcal{O}, \mathcal{G})$ is a compact Lie group provided $\mathcal{O} \subset \mathcal{B}$ is simply connected. It is thus endowed with its Haar measure $\mu_\mathcal{G}$ and for any tensor field T of any type on $\mathcal{M}$, we can define its *vertical average* $\langle T \rangle$ in the following way. For each $m \in \mathcal{M}$, we set

$$\langle T \rangle_m = \int_{\Gamma_\nabla(\mathcal{O}, \mathcal{G})} \left((\phi^1_X)_* T \right)_m d\mu_\mathcal{G},$$

where $\mathcal{O} \subset \mathcal{B}$ is any simply connected neighborhood of $b = \pi(m)$. We can check that this definition does not depend on the choice of $\mathcal{O}$. Choosing a basis $X_1, \ldots, X_d$ of $\Gamma(\mathcal{O}, \Lambda)$ provides an explicit expression for the averaged tensor, namely

$$\langle T \rangle_m = \int_0^1 dt_1 \cdots \int_0^1 dt_d \left(\left(\phi^{t_1}_{X_1} \right)_* \circ \cdots \circ \left(\phi^{t_d}_{X_d} \right)_* T \right)_m.$$

A tensor field T is called *invariant under the toric action of $\mathcal{G}$*, or simply *$\mathcal{G}$-invariant*, if for each local parallel section $X \in \Gamma_\nabla(\mathcal{O}, \mathcal{G})$ we have $(\phi^1_X)_* (T) = T$, or equivalently $\mathcal{L}_X T = 0$. The following properties can be proved in a straightforward way.

Lemma 2.3. *We have the following basic properties:*
   1) *T is $\mathcal{G}$-invariant if and only if $\langle T \rangle = T$.*
   2) *$\langle \langle T \rangle \rangle = \langle T \rangle$.*
   3) *Each p-form $\alpha \in \Omega^p(\mathcal{M})$ verifies $\langle d\alpha \rangle = d\langle \alpha \rangle$.*
   4) *Let T and S be two tensor fields. If T is $\mathcal{G}$-invariant, then the contraction $T \lrcorner S$ with respect to any two indices verifies $\langle T \lrcorner S \rangle = T \lrcorner \langle S \rangle$.*

5) *In particular, if $X \in \mathcal{V}(\mathcal{M})$ is a vector field and $\alpha = \omega(X)$ its associated 1-form, then we have $\omega(\langle \alpha \rangle) = \langle X \rangle$.*

2.3. Decomposition of symplectic vector fields. The averaging process presented in the previous section provides a way to decompose any symplectic vector field into the sum of a Hamiltonian vector field and a symplectic vector field preserving the fibration. The key step is the following lemma.

Lemma 2.4. *If α is a closed 1-form on $\mathcal{M}$ whose vertical average vanishes, then it is exact. Moreover, one can choose the primitive $f \in C^\infty(\mathcal{M})$, $\alpha = df$, with the property $\langle f \rangle = 0$.*

Proof. Let us work locally in a simply connected subset $\mathcal{O} \subset \mathcal{B}$. There exists a basis $(X_1, \ldots, X_d)$ of $\Gamma(\mathcal{O}, \Lambda)$. Choosing an "initial point" $m(b)$ depending smoothly on $b \in \mathcal{O}$, i.e., a smooth section of the restricted bundle $\pi^{-1}(\mathcal{O}) \xrightarrow{\pi} \mathcal{O}$, let us consider the smooth family of cycles $\gamma_j(b)$ consisting of the orbits $t \to \phi_{X_j}^t(m(b))$. The homology classes $[\gamma_j(b)]$ form for each $b \in \mathcal{O}$ a basis of $H_1(\mathcal{M}_b)$. On the other hand, since the fibration $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ is locally trivial and $\mathcal{O}$ is contractible, the classes $[\gamma_j(b)]$ form a basis of the homology of $\widetilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$.

Then, we show that for each $j = 1, \ldots, d$ and each $b \in \mathcal{O}$, one has $\int_{\gamma_j(b)} \langle \alpha \rangle = \int_{\gamma_j(b)} \alpha$. Indeed, one has

$$\int_{\gamma_j(b)} \langle \alpha \rangle = \int_0^1 dt \, \langle \alpha \rangle(X_j) \circ \phi_{X_j}^t(m(b)) = \int_0^1 dt \, X_j \lrcorner \left(\phi_{X_j}^{-t} \right)_* \langle \alpha \rangle.$$

Moreover, expressing the average $\langle \alpha \rangle$ in terms of the generators X_j, one obtains

$$\int_{\gamma_j(b)} \langle \alpha \rangle = \int_0^1 dt_1 \cdots \int_0^1 dt_d \int_0^1 dt \left(\phi_{X_1}^{t_1} \right)_* \circ \cdots \circ \widehat{\left(\phi_{X_j}^{t_j} \right)_*}$$
$$\times \circ \cdots \circ \left(\phi_{X_d}^{t_d} \right)_* \left(X_j \lrcorner \left(\phi_{X_j}^{t_j - t} \right)_* \alpha \right),$$

where the entry below $\widehat{}$ has been omitted. Then, we check with a trivial change of variable that

$$\int_0^1 dt_j \int_0^1 dt \left(X_j \lrcorner \left(\phi_{X_j}^{t_j - t} \right)_* \alpha \right) = \int_{\gamma_j(b)} \alpha.$$

This implies that $\int_{\gamma_j(b)} \langle \alpha \rangle = \int_{\gamma_j(b)} \alpha$.

Finally, the hypothesis $\langle \alpha \rangle = 0$ yields $\int_{\gamma_j(b)} \alpha = 0$, where the classes $[\gamma_j(b)]$ form a basis of the homology of $\widetilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$, as shown before. Since α is closed, this implies that it is actually exact. Thus, there exists a function $f \in C^\infty(\widetilde{\mathcal{O}})$ such that $\alpha = df$ in $\widetilde{\mathcal{O}}$. This function is unique up to a constant. On the other hand, we deduce from the property $\langle df \rangle = d\langle f \rangle$ and the hypothesis $\langle \alpha \rangle = 0$ that $\langle f \rangle$ is a constant function. This allows us to

choose the primitive f in a unique way by requiring that $\langle f \rangle = 0$. This criterion is independent of the choice of the basis $(X_1, \ldots, X_d)$ and thus allows us to find a primitive f of α globally defined on $\mathcal{M}$. $\qquad \square$

We need also the following property, which will be proved later in the slightly more general case of time-dependent vector fields (Lemma 3.7).

Lemma 2.5. *If $\widetilde{Y} \in \mathcal{V}(\mathcal{M})$ is a symplectic lift of a vector field $Y \in \mathcal{V}(\mathcal{B})$, then it is $\mathcal{G}$-invariant.*

We now state the announced decomposition of symplectic vector fields. We stress the fact that this result still holds in the presence of monodromy.

Proposition 2.6. *Any symplectic vector field $X \in \mathcal{V}(\mathcal{M})$ decomposes in a unique way as*
$$X = X_1 + X_2,$$
where

- *X_1 is a Hamiltonian vector field, $X_1 = X_A$, with $\langle A \rangle = 0$, where $\langle A \rangle$ is the vertical average of the Hamiltonian A.*
- *X_2 is a symplectic lift of a vector field on $\mathcal{B}$. Namely, it is the vertical average of X, i.e., $X_2 = \langle X \rangle$.*

Proof. Let $\alpha = \omega(X, .)$ be the 1-form associated with X, which is closed since X is symplectic. Let $\alpha_2 = \langle \alpha \rangle$ be the vertical average of α and let $\alpha_1 = \alpha - \alpha_2$. The 1-forms α_1 and α_2 are closed since $d\langle \alpha \rangle = \langle d\alpha \rangle$. Thus, the vector fields X_1 and X_2, associated with α_1 and α_2, are symplectic. On the other hand, one has $\langle \alpha_1 \rangle = 0$ and Lemma 2.4 then implies that X_1 is Hamiltonian, $X_1 = X_A$, with $\langle A \rangle = 0$. Finally, $\langle \alpha_2 \rangle = \alpha_2$ implies that $\langle X_2 \rangle = X_2$. Now, any $\mathcal{G}$-invariant vector field must be a lift of a vector field on $\mathcal{B}$, since the toric action of $\mathcal{G}$ is vertical and transitive on each fiber. This proves the second point of the proposition.

Moreover, the decomposition $X = X_1 + X_2$ is the unique one of this type. Indeed, suppose that there is a second decomposition $X = X_1' + X_2'$ with the same properties. Taking the vertical average of both expressions, we obtain $\langle X_1 + X_2 \rangle = \langle X_1' + X_2' \rangle$ and thus $\langle X_2 \rangle = \langle X_2' \rangle$. Now, by Lemma 2.5, both X_2 and X_2' are $\mathcal{G}$-invariant. It follows that $X_2 = X_2'$ and thus $X_1 = X_1'$. $\quad \square$

3. Deformations of completely integrable systems

3.1. Regular deformations of completely integrable systems. Let $(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B})$ be a *regular CI system* composed of a fibration in Lagrangian tori $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ and a Hamiltonian $H_0 \in C^\infty(\mathcal{M})$ constant along the fibers. As we mentioned in the introduction, we will restrict ourselves to regular deformations of H_0, i.e., smooth families of Hamiltonians H_ε which are CI and regular for each ε. At this point, we would like to stress the fact that this

does *not* imply that H_ε is constant along the fibers of a family of fibrations $\mathcal{M} \xrightarrow{\pi_\xi} \mathcal{B}$ *depending smoothly on* ε. Nevertheless, we conjecture that it is true for the generic class of *non-degenerate* Hamiltonians. We will discuss the non-degeneracy conditions in Section 3.2 and we now restrict our study to the following class of deformations.

Definition 3.1. Let $(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B})$ be a regular CI system and $H_\varepsilon \in C^\infty(\mathcal{M})$ a smooth family of Hamiltonians. We say that H_ε is a *regular deformation* of H_0 if it has the form

$$H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon,$$

where $I_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ is a smooth family of functions with $I_0 = H_0$ and $\phi^\varepsilon : \mathcal{M} \to \mathcal{M}$ is a smooth family of symplectomorphisms with $\phi^0 = \mathbb{I}$.

For our purposes, we will need to work from now on with time-dependent vector fields since each smooth family of diffeomorphisms ϕ^ε with $\phi^0 = \mathbb{I}$ is the flow at time ε of the time-dependent vector field X_ε defined by

$$\frac{d(f \circ \phi^\varepsilon(m))}{d\varepsilon} = X_\varepsilon(f) \circ \phi^\varepsilon(m)$$

for each smooth function $f \in C^\infty(\mathcal{M})$ and each point $m \in \mathcal{M}$. We denote this flow by $\phi_{X_\varepsilon}^\varepsilon$. In all the following, all the considered families ϕ^ε of diffeomorphisms will implicitly depend smoothly on ε and satisfy $\phi^0 = \mathbb{I}$. We refer e.g., to [**6**] for a review of the properties of time-dependent vector fields.

3.2. Non-degenerate CI systems. Non-degeneracy conditions are those used in K.A.M. theories, like for example those introduced by Arnol'd, Kolmogorov, Bryuno or Rüssmann. We refer to [**10**] for a review of different non-degeneracy conditions together with their properties and we will focus on two of them. But first of all, we need to define a few notions.

Since the CI Hamiltonian H_0 is constant along the fibers which are connected, it must be of the form $H_0 = F_0 \circ \pi$, with $F_0 \in C^\infty(\mathcal{B})$. Denote by ∇ the Duistermaat's affine connection which exists naturally on the base space $\mathcal{B}$. For any subset $\mathcal{O} \subset \mathcal{B}$, we denote by $\mathcal{V}_\nabla(\mathcal{O})$ the space of parallel vector fields. Since the holonomy of ∇ may not vanish, the space $\mathcal{V}_\nabla(\mathcal{O})$ might be empty. Nevertheless, when $\mathcal{O}$ is simply connected, this space is a d-dimensional vector space. All the non-degeneracy conditions, including those presented here, are local : F_0 (or H_0) is said to be *non-degenerate* if is non-degenerate at each $b \in \mathcal{B}$. Moreover, these conditions involve the space of parallel vector fields, but the mentioned local character means that one needs actually only the spaces $\mathcal{V}_\nabla(\mathcal{O})$ for a neighborhood $\mathcal{O} \subset \mathcal{B}$ of each point $b \in \mathcal{B}$. We will use a slight misuse of language and say "for each $X \in \mathcal{V}_\nabla(\mathcal{B})$" instead of "for each $b \in \mathcal{B}$, each neighborhood $\mathcal{O} \subset \mathcal{B}$ of b and each $X \in \mathcal{V}_\nabla(\mathcal{O})$".

For each $X \in \mathcal{V}_\nabla(\mathcal{B})$, let us define the function $\Omega_X \in C^\infty(\mathcal{B})$ by $\Omega_X = dF_0(X)$ and the associated *resonance set*

$$\Sigma_X = \{b \in \mathcal{B} \mid \Omega_X(b) = 0\}.$$

Definition 3.2. The function F_0 is *Rüssmann non-degenerate* if for each non-vanishing $X \in \mathcal{V}_\nabla(\mathcal{B})$, the resonant set Σ_X has an empty interior.

Among the non-degeneracy conditions used in the literature, Rüssmann's condition [12] is the weakest one and has nevertheless the following important consequence (see e.g., [10] for a proof).

Lemma 3.3. *If $(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B})$ is a "Rüssmann" non-degenerate C.I system, then $\mathcal{M} \xrightarrow{\pi} \mathcal{B}$ is the unique fibration in Lagrangian tori such that H_0 is constant along the fibers.*

This non-degeneracy condition is enough to insure the unicity of the normal form of Theorem 3.9 which will be proved in Section 3.3 but for the study of first order deformations developed in Section 3.4, we will need a stronger one, which is nevertheless weaker than Kolmogorov's or Arnold's ones.

Definition 3.4. The function F_0 is *weakly non-degenerate* if for each non-vanishing $X \in \mathcal{V}_\nabla(\mathcal{B})$ and each point $b \in \Sigma_X$, one has

$$d(\Omega_X)_b \neq 0.$$

This condition implies among other that the resonant sets Σ_X are 1-codimensional submanifolds of $\mathcal{B}$.

3.3. Normal form for regular deformations. The aim of this section is to prove Theorem 3.9 which insures that, by changing the function I_ε, one may assume that ϕ^ε is a Hamiltonian flow. This result is based on Proposition 3.8 which states that any family of symplectomorphisms ϕ^ε can be written as the composition of a Hamiltonian flow with a family of fiber-preserving symplectomorphisms. Let us first define precisely these two notions.

Definition 3.5. A family of symplectomorphisms ϕ^ε is called *Hamiltonian* if its vector field X_ε is Hamiltonian, $X_\varepsilon = X_{A_\varepsilon}$, with $A_\varepsilon \in C^\infty(\mathcal{M})$ depending smoothly on ε.

Definition 3.6. A family of diffeomorphisms $\phi^\varepsilon : \mathcal{M} \to \mathcal{M}$ is called *fiber-preserving* if there exists a family of diffeomorphisms on the base space $\varphi^\varepsilon : \mathcal{B} \to \mathcal{B}$ such that

$$\pi \circ \phi^\varepsilon = \varphi^\varepsilon \circ \pi.$$

We say that ϕ^ε is *vertical* whenever $\varphi^\varepsilon = \mathbb{I}$ for all ε.

Whenever a vector field on $\mathcal{M}$ is both symplectic and a lift of a vector field on $\mathcal{B}$, then we have the following property.

Lemma 3.7. *If $\widetilde{Y}_\varepsilon \in \mathcal{V}(\mathcal{M})$ is symplectic for each ε and is a lift of a time-dependent vector field $Y_\varepsilon \in \mathcal{V}(\mathcal{B})$, then it is $\mathcal{G}$-invariant and for each tensor field T one has*

$$\left\langle \left(\phi_{\widetilde{Y}_\varepsilon}^\varepsilon \right)_* T \right\rangle = \left(\phi_{\widetilde{Y}_\varepsilon}^\varepsilon \right)_* \langle T \rangle.$$

Proof. Let denote by $\phi^\varepsilon = \phi_{\widetilde{Y}_\varepsilon}^\varepsilon$ the flow of $\widetilde{Y}_\varepsilon$. This flow is fiber-preserving and thus verifies $\pi \circ \phi^\varepsilon = \varphi^\varepsilon \circ \pi$ with $\varphi^\varepsilon : \mathcal{B} \to \mathcal{B}$ a family of diffeomorphisms. One can easily show that φ^ε is actually the flow of Y_ε.

First of all, for each vertical and parallel vector field $X \in \Gamma(\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$, one has $\phi_*^\varepsilon X \in \Gamma(\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$. Indeed, as mentioned in Section 2.1, $\phi_*^\varepsilon X$ is vertical and parallel if and only if the 1-form $\omega(\phi_*^\varepsilon X)$ is a pull-back. Now, one has $\omega(\phi_*^\varepsilon X) = ((\phi^\varepsilon)^{-1})^*(\omega(X))$ since ϕ^ε is symplectic for each ε. On the other hand, $\omega(X) = \pi^* \beta$ with $\beta \in \Omega^1(\mathcal{B})$, since by hypothesis X is vertical and parallel. Consequently, one has

$$\omega(\phi_*^\varepsilon X) = \left((\phi^\varepsilon)^{-1}\right)^* \pi^* \beta = \pi^* \left((\varphi^\varepsilon)^{-1}\right)^* \beta.$$

This proves that $\omega(\phi_*^\varepsilon X)$ is a pull-back and, therefore, $\phi_*^\varepsilon X$ is vertical and parallel.

If in addition $X \in \Gamma(\mathcal{O}, \Lambda)$, with $\mathcal{O} \subset \mathcal{B}$ a subset, i.e., X is 1-periodic in $\pi^{-1}(\mathcal{O})$, then so is $\phi_*^\varepsilon X$ in $\phi^\varepsilon(\pi^{-1}(\mathcal{O}))$. Now, the smooth bundle Λ has discrete fibers and $\phi_*^\varepsilon X$ depends smoothly on ε. This implies that for all ε, one has $\phi_*^\varepsilon X = \phi_*^{\varepsilon=0} X$ and thus $\phi_*^\varepsilon X = X$. Then, the derivative with respect to ε shows that $[\widetilde{Y}, X] = 0$, i.e., $\widetilde{Y}$ is $\mathcal{G}$-invariant. By linearity, this is true as well for all $X \in \Gamma(\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$.

Therefore, for each $X \in \Gamma(\bigcup_{b \in \mathcal{B}} \mathcal{V}_\nabla(\mathcal{M}_b))$ and each ε, ϕ^ε commutes with the flow ϕ_X^t. This implies that ϕ^ε commutes with the toric action of $\mathcal{G}$ and thus with the averaging process, i.e.,

$$\left\langle \left(\phi_{\widetilde{Y}_\varepsilon}^\varepsilon \right)_* T \right\rangle = \left(\phi_{\widetilde{Y}_\varepsilon}^\varepsilon \right)_* \langle T \rangle$$

for any tensor field T.   □

We can now give the following decomposition result for families of symplectomorphisms.

Proposition 3.8. *Each family of symplectomorphisms ϕ^ε decomposes in a unique way as follows:*

$$\phi^\varepsilon = \Phi^\varepsilon \circ \phi_{Z_\varepsilon}^\varepsilon,$$

where

- *Φ^ε is a fiber-preserving family of symplectomorphisms.*
- *$Z_\varepsilon = X_{G_\varepsilon}$ is a time-dependent Hamiltonian vector field with $\langle G_\varepsilon \rangle = 0$.*

Moreover, the vector field of Φ^ε is equal to the average $\langle X_\varepsilon \rangle$, where X_ε is the vector field of ϕ^ε.

Proof. Let X_ε be the vector field of ϕ^ε. Proposition 2.6 insures that for each ε, X_ε decomposes into $X_\varepsilon = \widetilde{Y}_\varepsilon + W_\varepsilon$, where $\widetilde{Y}_\varepsilon$ is a lift of a vector field $Y_\varepsilon \in \mathcal{V}(\mathcal{B})$ and W_ε is Hamiltonian. Moreover, by looking more carefully at the proof of Proposition 2.6, one can easily check that $\widetilde{Y}_\varepsilon$ and W_ε depend smoothly on ε, since $\widetilde{Y}_\varepsilon$ is nothing but the vertical average of X_ε.

Let Ψ^ε be the family of symplectomorphisms defined by $\phi^\varepsilon_{\widetilde{Y}_\varepsilon + W_\varepsilon} = \phi^\varepsilon_{\widetilde{Y}_\varepsilon} \circ \Psi^\varepsilon$ and let Z_ε be its vector field. On the one hand, $\Phi^\varepsilon = \phi^\varepsilon_{\widetilde{Y}_\varepsilon}$ is fiber-preserving since $\widetilde{Y}_\varepsilon$ is a lift of a vector field on $\mathcal{B}$. On the other hand, one can check in a straightforward way that the vector field X^3_ε of a composition of flows $\phi^\varepsilon_{X^1_\varepsilon} \circ \phi^\varepsilon_{X^2_\varepsilon}$ is given by the formula $X^3_\varepsilon = X^1_\varepsilon + (\phi^\varepsilon_{X^1_\varepsilon})_* X^2_\varepsilon$. Therefore, in our case we have $\widetilde{Y}_\varepsilon + W_\varepsilon = \widetilde{Y}_\varepsilon + \phi^\varepsilon_{\widetilde{Y}_\varepsilon}(Z_\varepsilon)$ and thus

$$Z_\varepsilon = \left(\phi^\varepsilon_{\widetilde{Y}_\varepsilon}\right)^{-1}_* (W_\varepsilon).$$

According to Proposition 2.6, W_ε is Hamiltonian and verifies $\langle W_\varepsilon \rangle = 0$. First, this insures that Z_ε is Hamiltonian. Second, Lemma 3.7 implies that

$$\langle Z_\varepsilon \rangle = \left(\phi^\varepsilon_{\widetilde{Y}_\varepsilon}\right)^{-1}_* \langle W_\varepsilon \rangle = 0$$

since $\widetilde{Y}_\varepsilon$ is symplectic and a lift of a vector field on $\mathcal{B}$.

Finally, we show that this decomposition is unique. Indeed, suppose that we have a second decomposition $\phi^\varepsilon_{X_\varepsilon} = \phi^\varepsilon_{\widetilde{Y}'_\varepsilon} \circ \phi^\varepsilon_{Z'_\varepsilon}$ with the same properties. The vector field $\widetilde{Y}'_\varepsilon$ must be a lift of a vector field on $\mathcal{B}$ since $\phi^\varepsilon_{\widetilde{Y}'_\varepsilon}$ is fiber-preserving. On the other hand, as we mentioned before, we have the relation $\tilde{X}_\varepsilon = \widetilde{Y}'_\varepsilon + \phi^\varepsilon_{\widetilde{Y}'_\varepsilon}(Z'_\varepsilon)$. Arguing as before, we can show that $\phi^\varepsilon_{\widetilde{Y}'_\varepsilon}(Z'_\varepsilon)$ is a Hamiltonian vector field with vanishing vertical average. Now, Theorem 2.6 tells us that the decomposition $X_\varepsilon = \widetilde{Y}_\varepsilon + W_\varepsilon$ is unique and thus $\widetilde{Y}'_\varepsilon = \widetilde{Y}_\varepsilon$ and $Z'_\varepsilon = Z_\varepsilon$. $\square$

We have now all the necessary material to state the following theorem which gives a normal form for regular deformations of a given regular CI system.

Theorem 3.9. *Let $(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B})$ a regular CI system. If H_ε is a regular deformation of H_0, then there exist a family of functions $I_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ and a family of Hamiltonian symplectomorphisms $\phi^\varepsilon_{X_{G_\varepsilon}}$, with $\langle G_\varepsilon \rangle = 0$ such that*

$$H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$$

for each ε.

Moreover, if H_0 is Rüssmann non-degenerate, then the families I_ε and $\phi^\varepsilon_{X_{G_\varepsilon}}$ are unique.

Proof. By definition, H_ε is a regular deformation of H_0 if there exist a family of functions $J_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ and a family of symplectomorphisms ϕ^ε such that $H_\varepsilon = J_\varepsilon \circ \phi^\varepsilon$. On the other hand, Proposition 3.8 insures that ϕ^ε decomposes into $\phi^\varepsilon = \Phi^\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$, where Φ^ε is fiber-preserving and $\langle G_\varepsilon \rangle = 0$. Therefore, we have $H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}}$, where the function $I_\varepsilon = J_\varepsilon \circ \Phi^\varepsilon$ is indeed an element of $\pi^*(C^\infty(\mathcal{B}))$ since Φ^ε is fiber-preserving.

Let us now show the unicity in case H_0 is Rüssmann non-degenerate. Suppose there is another family of functions $I'_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ and another family of symplectomorphims $\phi^\varepsilon_{X_{G'_\varepsilon}}$, with $\langle G'_\varepsilon \rangle = 0$ and such that $H_\varepsilon = I'_\varepsilon \circ \phi^\varepsilon_{X_{G'_\varepsilon}}$. We thus have $I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}} = I'_\varepsilon \circ \phi^\varepsilon_{X_{G'_\varepsilon}}$ and if we define the flow $\Phi^\varepsilon = \phi^\varepsilon_{X_{G_\varepsilon}} \circ \left(\phi^\varepsilon_{X_{G'_\varepsilon}} \right)^{-1}$, then we obtain $I_\varepsilon \circ \Phi^\varepsilon = I'_\varepsilon$.

First of all, since Φ^ε is a family of symplectomorphisms, the fibration $\mathcal{M} \xrightarrow{\pi_\varepsilon} \mathcal{B}$ given by $\pi_\varepsilon = \pi \circ (\Phi^\varepsilon)^{-1}$ is also Lagrangian. Then, we can see that the function I_ε is also constant along the fibers of the deformed fibration. Indeed, by hypothesis it has the form $I_\varepsilon = I'_\varepsilon \circ (\Phi^\varepsilon)^{-1}$. Using then the fact that I'_ε has the form $I'_\varepsilon = f_\varepsilon \circ \pi$ with $f_\varepsilon \in C^\infty(\mathcal{B})$, it follows that $I_\varepsilon = f_\varepsilon \circ \pi_\varepsilon$. This proves that I_ε is constant along the fibers of both fibrations π and π_ε.

Moreover, for $\varepsilon = 0$ the function I_ε is equal to H_0 which is non-degenerate. This implies that I_ε is also non-degenerate for small enough ε since non-degeneracy is an open condition. Therefore, Lemma 3.3 insures that there is a unique fibration such that I_ε is constant along the fibers. The two fibrations π and π_ε thus coincide, this proves that Φ^ε preserves the initial fibration π. Consequently, we have the decomposition $\phi^\varepsilon_{X_{G_\varepsilon}} = \Phi^\varepsilon \circ \phi^\varepsilon_{X_{G'_\varepsilon}}$ with Φ^ε preserving the fibration π. Now, Proposition 3.8 insures that this decomposition is unique. Accordingly we have $G_\varepsilon = G'_\varepsilon$ and thus $I_\varepsilon = I'_\varepsilon$. $\qquad\square$

3.4. First order deformations. In this last section, we address the problem of finding what are the necessary and sufficient conditions on a perturbation $H_1 \in C^\infty(\mathcal{M})$ which insure that the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ is CI up to ε^2, i.e., has the form $H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}} + O(\varepsilon^2)$ with $I_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ and $\langle G_\varepsilon \rangle = 0$.

Most of the work here will be achieved with the help of Fourier series. Let us begin by expliciting the geometric status of the object we will consider[3] . First, we will work locally in some $\mathcal{O} \subset \mathcal{B}$, with an action-angle coordinates system (ξ, x) and consider the Fourier series with respect to the periodic variable x. For any smooth function $f(\xi, x)$ we will denote by $\tilde{f}(\xi, k)$ its

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[3]We refer to [**11**] for a detailled description of this issue.

Fourier series defined by the usual expression

$$f(\xi, x) = \sum_{k \in E} e^{ik(x-x_0)} \tilde{f}(\xi, k).$$

The discrete set E in which the Fourier variable k lives is naturally a lattice of the vector space $\mathcal{V}_\nabla(\mathcal{O})$ of parallel vector fields on $\mathcal{O}$. This can be seen as follows. First, if ξ denotes the coordinates of a point $b \in \mathcal{O}$, then $x - x_0$ can be understood as an element of $\mathcal{V}_\nabla(\mathcal{M}_b)$ well-defined up to elements of Λ_b, the lattice of 1-periodic parallel vector fields on the fiber $\mathcal{M}_b$. On the other hand, its dual Λ_b^* is a lattice of the space $\Omega_\nabla^1(\mathcal{M}_b)$ of parallel 1-forms on $\mathcal{M}_b$. The Fourier variable k lives naturally in Λ_b^*. Moreover, this family Λ_b^* depends smoothly on b as Λ_b does. Now, the symplectic form provides an isomorphism $\iota_b : \Omega_\nabla^1(\mathcal{M}_b) \to T_b\mathcal{B}$, depending smoothly on b. Under this identification, k can be seen as an element of the lattice $\iota_b(\Lambda_b^*)$ of the vector space $T_b\mathcal{B}$. Finally, if we identify $\iota_b(\Lambda_b^*)$ with the space $E \subset \mathcal{V}_\nabla(\mathcal{O})$ of sections of the associated lattice bundle $\bigcup_{b \in \mathcal{O}} \iota_b(\Lambda_b^*)$, we obtain a suitable space for the Fourier variable k to live in. Accordingly, for each $k \in E$ the Fourier series $\tilde{f}(\xi, k)$ is a smooth (with respect to ξ) function, well-defined up to a phase, due to an arbitrary choice of the family of origin points $b \to x_0(b)$.

Definition 3.10. A function $f \in C^\infty(\mathcal{M})$ is called *non-resonant* if for each non-vanishing $k \in E$ and each $\xi \in \Sigma_k$ one has $\tilde{f}(\xi, k) = 0$.

The resonant manifolds Σ_k were defined in Section 3.2. We have the following equivalent criterion which has to be checked on each torus $\mathcal{M}_b$ on which the dynamics of X_{H_0} is periodic.

Lemma 3.11. *A function $f \in C^\infty(\mathcal{M})$ is non-resonant if and only if for each T-periodic torus $\mathcal{M}_b$ the average of f along the trajectories of X_{H_0}*

$$\bar{f} := \frac{1}{T} \int_0^T f|_{\mathcal{M}_b} \circ \phi_{X_{H_0}}^t \, dt$$

is a constant function on $\mathcal{M}_b$.

Proof. First, one can show (see e.g., [11, Prop. A.62]) that for each $k \neq 0$ the set of periodic tori in Σ_k is dense in Σ_k. This implies that the non-resonance condition is equivalent to

$$\forall \xi \text{ periodic}, \forall k \in E \backslash 0, dF_0(k)_\xi = 0 \implies \tilde{f}(\xi, k) = 0,$$

where $F_0 \in C^\infty(\mathcal{B})$ is the function defined by $H_0 = F_0 \circ \pi$. On the other hand, a short calculation shows that for each ξ the Fourier series $\tilde{\bar{f}}(\xi, k)$ of the average $\bar{f}$ is given by

$$\tilde{\bar{f}}(\xi, k) = \begin{cases} \tilde{f}(x, k) & \text{if } dF_0(k) = 0 \\ 0 & \text{if } dF_0(k) \neq 0 \end{cases}.$$

Therefore, the non-resonance condition indeed amounts to requiring that the averaged function $\bar{f}$ is constant on the torus $\mathcal{M}_b$.

This non-resonance condition is the right one to control the complete integrability up to ε^2, as it is shown in the following theorem. $\qquad\square$

Theorem 3.12. *Let $(H_0, \mathcal{M} \xrightarrow{\pi} \mathcal{B})$ be a weakly non-degenerate regular CI system and $H_1 \in C^\infty(\mathcal{M})$ a perturbation. The perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ is CI up to ε^2 if and only if H_1 is non-resonant.*

Proof. First, the complete integrability up to ε^2 means that H_ε has the form $H_\varepsilon = I_\varepsilon \circ \phi^\varepsilon_{X_{G_\varepsilon}} + O(\varepsilon^2)$ with $I_\varepsilon \in \pi^*(C^\infty(\mathcal{B}))$ and $\langle G_\varepsilon \rangle = 0$. In this expression, the terms of order ε^0 give simply $H_0 = I_0$ and the ε^1 terms yield the equation $H_1 = I_1 + X_{G_0}(H_0)$. By definition of the Poisson bracket, this is equivalent to

(3.1) $$\{H_0, G_0\} = I_1 - H_1.$$

In Fourier coordinates, this equation reads

$$idF_0(k)\widetilde{G}_0(\xi, k) = \widetilde{I}_1(\xi, k) - \widetilde{H}_1(\xi, k)$$

for each ξ and each $k \in E$. For $k = 0$, we have $\widetilde{I}_1(\xi, 0) = I_1(\xi)$ since I_1 is a function constant along the fibers. We thus have to set $\widetilde{H}_1(\xi, 0) = I_1(\xi)$. Remark that $\widetilde{H}_1(\xi, 0)$ is nothing but the vertical average of the function H_1. The Fourier coefficient $\widetilde{G}_0(\xi, 0)$ is free and can be set to 0, which means that $\langle G_0 \rangle = 0$. Now, for all non-vanishing k, we have $\tilde{I}_1(\xi, k) = 0$ and we need to solve the equation

$$dF_0(k)\widetilde{G}_0(\xi, k) = i\widetilde{H}_1(\xi, k).$$

The non-resonance condition is certainly necessary, since in order to divide by the function $\Omega_k = dF_0(k)$, $\tilde{H}_1(\xi, k)$ needs to vanish at least where Ω_k does, i.e., on the resonance manifold Σ_k. The solution $\tilde{G}_0$ is thus defined by the quotient $\frac{\tilde{H}_1(\xi, k)}{\Omega_k(\xi)}$ and it still remains to prove that the non-resonance condition is sufficient to insure that $\tilde{G}_0(\xi, k)$ is smooth with respect to ξ uniformly with respect to k, and with a fast decay in k. This will mean that $G_0(\xi, x)$ is smooth with respect to (ξ, x). For this purpose, we will show that for any compact set $\mathcal{K} \subset \mathcal{B}$, there are two positive constants T and C such that

(3.2) $$|\Omega_k| < T \implies |d\Omega_k| > C$$

uniformly with respect to $k \in E \backslash 0$. If this holds, then for each k we decompose $\mathcal{K}$ into two parts defined by $|\Omega_k| < T$ and $|\Omega_k| \geq T$. Away from the resonance manifold Σ_k, i.e., for ξ such that $|\Omega_k(\xi)| \geq T$, we can simply divide by Ω_k and $\widetilde{G}_0$ will satisfy the estimate $|\widetilde{G}_0(\xi, k)| \leq \frac{\tilde{H}_1(\xi, k)}{T}$. On the other hand, close to the resonance manifold Σ_k, i.e., for ξ such that

$|\Omega_k(\xi)| < T$, we consider X_k the gradient of Ω_k (for some fixed Riemannain metric). Its norm verifies $|X_k| > C$ and is transversal to the submanifold Σ_k. It is thus suitable to parametrize the "distance" to Σ_k, thru its flow $\phi_{X_k}^t$. Indeed, since X_k is the dual vector of Ω_k, i.e., $d\Omega_k(X_k) = 1$, we have $\Omega_k \circ \phi_{X_k}^t(\xi) = t + \Omega_k(\xi)$. Therefore, we can compute the first order Taylor expansion with integral rest of $\widetilde{H}_1(\xi, k)$. This yields

$$\widetilde{H}_1(\xi, k) = \widetilde{H}_1\left(\phi_{X_k}^{-\Omega_k(\xi)}, k\right) + \int_0^{\Omega_k(\xi)} dt\, X_k(\widetilde{H}_1) \circ \phi_{X_k}^{t-\Omega_k(\xi)}(\xi).$$

By construction, the point $\phi_{X_k}^{-\Omega_k(\xi)}$ is on Σ_k and $\widetilde{H}_1$ vanishes at this point according to the non-resonance condition. Then, the change of variable $u = t/\Omega_k(\xi)$ gives

$$\widetilde{H}_1(\xi, k) = \Omega_k(\xi) \int_0^1 du\, X_k(\widetilde{H}_1) \circ \phi_{X_k}^{(u-1)\Omega_k(\xi)}(\xi)$$

and we can solve the equation $\Omega_k(\xi)\widetilde{G}_0(\xi, k) = i\widetilde{H}_1(\xi, k)$ by dividing by Ω_k. The fast decay of $\widetilde{H}_1(\xi, k)$ implies the fast decay of the solution $\widetilde{G}_0(\xi, k)$ for both cases $|\Omega_k| < T$ and $|\Omega_k| \geq T$, and thus proves the smoothness of $G_0(\xi, x)$.

The last point is to prove the existence of the constants T and C in equation (3.2). In fact we will prove that this equation holds for k living in the space $P = \{X \in \mathcal{V}_\nabla(\mathcal{B}), |X| \geq 1\}$ and this will imply the result for $k \in E\backslash 0$. Because of non-degeneracy, one has $d\Omega_k \neq 0$ and thus $\frac{d\Omega_k}{|k|} \neq 0$ for all point in $\Sigma_k \cap \mathcal{K}$. Therefore, there is a constant $C(k)$ such that $\left|d\Omega_{\frac{k}{|k|}}\right| = \frac{|d\Omega_k|}{|k|} > 2C(k)$ in $\Sigma_k \cap \mathcal{K}$. Now, the smoothness of $\Omega_{\frac{k}{|k|}}$ implies that there is a constant $T(k)$ such that $\frac{|d\Omega_k|}{|k|} > C(k)$ whenever $\left|\Omega_{\frac{k}{|k|}}\right| < T(k)$. Let us now decompose the elements k into their angular and radial parts, i.e., $k := \left(\frac{k}{|k|}, k\right) \in S^{d-1} \times [1, \infty]$. Taking the minimum of $T(k)$ and $C(k)$ over the compact set S^{d-1}, we obtain positive constants $T'(|k|)$ and $C'(|k|)$ such that $\frac{|d\Omega_k|}{|k|} > C'(|k|)$ whenever $\left|\Omega_{\frac{k}{|k|}}\right| < T'(|k|)$. Using again $\Omega_{\frac{k}{|k|}} = \frac{\Omega_k}{|k|}$ and setting $T = T'(1)$ and $C = C'(1)$, we see that the following implication holds.

$$\left|\frac{\Omega_k}{|k|}\right| < T \implies \frac{|d\Omega_k|}{|k|} > C.$$

Finally, using the fact that $|k| \leq 1$, we obtain equation (3.2).   □

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