

Shortening the Hofer length of Hamiltonian circle actions

Yael Karshon and Jennifer Slimowitz Pearl

A Hamiltonian circle action on a compact symplectic manifold is known to be a closed geodesic with respect to the Hofer metric on the group of Hamiltonian diffeomorphisms. If the momentum map attains its minimum or maximum at an isolated fixed point with isotropy weights not all equal to plus or minus one, then this closed geodesic can be deformed into a loop of shorter Hofer length. In this paper we give a lower bound for the possible amount of shortening, and we give a lower bound for the index (“number of independent shortening directions”). If the minimum or maximum is attained along a submanifold B , then we deform the circle action into a loop of shorter Hofer length whenever the isotropy weights have sufficiently large absolute values and the normal bundle of B is sufficiently un-twisted.

1	Introduction	210
2	Preliminaries	215
3	New results and further questions	218
4	Shortening a non-effective circle action	225
5	Shortening a combined action	226

Yael Karshon is partially supported by an NSERC Discovery Grant. Part of this research was conducted at the Hebrew University of Jerusalem and was partially supported by the Israel Science Foundation, founded by the Academy of Sciences and Humanities. Jennifer Slimowitz acknowledges support by an NSF*/NATO Postdoctoral Fellowship during the time that this work was done.

*Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

6	Disjoining discs in \mathbb{C}	232
7	Disjoining a family of discs	235
8	Shortening on a manifold with an isolated maximum	237
9	Symplectic vector bundles	240
10	Shortening on a manifold with an arbitrary maximum	248
11	The index of the positive Hofer length functional	252
	References	257

1. Introduction

A Hamiltonian circle action on a compact symplectic manifold (M, ω) can be viewed as a loop in the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms, parametrized by \mathbb{R}/\mathbb{Z} . We are concerned with deforming this loop, through loops in $\text{Ham}(M, \omega)$, into a loop whose Hofer length is smaller. In short, we call such a deformation “shortening”. The reader may go directly to Section 3 for the precise statements of the new results that we prove in this paper.

One motivation for such a study comes from Riemannian geometry, where one studies the action functional $\gamma \mapsto \int_0^1 \|\dot{\gamma}(t)\|^2 dt$ on spaces of loops or paths with fixed endpoints in a compact Riemannian manifold. Critical points are geodesics; the index of a critical point is finite and is equal to the number (counting multiplicity) of conjugate points along the geodesic; every homotopy class contains a minimal geodesic. Morse theory in this context has played a major role in the study of the topology of compact Lie groups. See Milnor’s book [17].

The infinite dimensional group $\text{Ham}(M, \omega)$ is a central object in symplectic geometry. It has a natural Finsler metric, introduced by Hofer, which induces a non-trivial distance function. The length of a path $\gamma: [0, 1] \rightarrow \text{Ham}(M, \omega)$ with respect to this Finsler metric is called the *Hofer length* of γ . It is equal to the sum of the *positive Hofer length* and the *negative Hofer length*; see Section 2.

There have been various attempts to extend to this setting notions that had been applied successfully to the action functional of Riemannian geometry. The situation here is more difficult and it is not yet clear to what

extent the analogy can be carried out. But the partial results that exist are beautiful and deep. A main theme is to express Morse properties of Hofer's functional on a space of paths in $\text{Ham}(M, \omega)$ in terms of the dynamics of the flows on M given by these paths.

There are several notions of "geodesic path" in $\text{Ham}(M, \omega)$. With all of these notions, every time-independent Hamiltonian flow is geodesic. (But not every geodesic is time-independent.) Thus, every Hamiltonian circle action is a geodesic loop in $\text{Ham}(M, \omega)$.

Starting with a Hamiltonian circle action $\{\phi_t: M \rightarrow M\}_{0 \leq t \leq 1}$ on a compact connected symplectic manifold, here is a sample of known results. Considering the opposite circle action gives the analogous results with "maximum" replaced by "minimum" and "positive Hofer length" replaced by "negative Hofer length".

- 1) Suppose that the circle action is semi-free near the set where the momentum map is maximal. (*Semi-free* means that only the trivial group and the entire circle occur as stabilizers.) Then its positive Hofer length is minimal among homotopic loops. See [19], the paragraph after Thm. 1.21, whose proof relies on [15, Thm. 1.9].

In particular, if the circle action is semi-free on M then it is length-minimizing among homotopic loops in $\text{Ham}(M, \omega)$. This was already shown by McDuff-Slimowitz [14, Theorem 1.4].

Another consequence is that, if the circle action is semi-free near the set where the momentum map is maximal, then the circle action is non-contractible in $\text{Ham}(M, \omega)$. This was already shown by McDuff-Tolman [15, Theorem 1.1].

- 2) Suppose that the momentum map attains its maximum at an isolated fixed point and that the circle action is not semi-free near the maximum. Then the circle action can be deformed to a loop in $\text{Ham}(M, \omega)$ whose Hofer length is smaller. See Ustilovsky [24, Theorem 1.2.E].
- 3) If n is a sufficiently large integer, then the non-effective circle action $\{\phi_{nt}\}_{0 \leq t \leq 1}$ can be deformed into a loop of smaller Hofer length. See Polterovich [18, p. 86].

Let B be the set where the momentum map attains its maximum. If there exists a Hamiltonian diffeomorphism g such that $g(B) \cap B = \emptyset$, then the non-effective circle action $\{\phi_{2t}\}_{0 \leq t \leq 1}$ can be deformed to a loop in $\text{Ham}(M, \omega)$ of smaller Hofer length. See Polterovich [18, Theorem 8.2.H].

Many results in the literature apply to spaces of paths in $\text{Ham}(M, \omega)$ with fixed endpoints and to geodesic loops or paths that are possibly time-dependent. We refer the reader to Hofer [6, 7], Hofer and Zehnder [8], Bialy-Polterovich [3], Yi Ming Long [12], Siburg [22], Ustilovsky [24], Lalonde-McDuff [10, 11], McDuff-Slimowitz [14], McDuff [13], and Savelyev [19, 20]. Polterovich's book [18] contains systematic explanations of many results in the subject.

Example 1.1. Let $e^{2\pi it} \in S^1$ act on $\mathbb{C}\mathbb{P}^2$ by $[u, z, w] \mapsto [u, e^{-2\pi it}z, e^{2\pi it}w]$, with the Fubini Study form normalized such that the momentum map is $(|w|^2 - |z|^2)/(|u|^2 + |z|^2 + |w|^2) + \text{constant}$. Take an equivariant blow-up at the point $[0, \star, 0]$ where the momentum map is minimal. This yields a Hamiltonian circle action on a symplectic manifold (M, ω) whose isotropy weights where the momentum map is maximal are $-1, -2$ and whose isotropy weights where the momentum map is minimal are $1, 1$. It is an effective circle action that represents a non-trivial element of $\pi_1(M, \omega)$ (by the above item (1), applied to the minimum) but that can be deformed into a shorter loop in $\text{Ham}(M, \omega)$ (by the above item (2), applied to the maximum).

Typically, to show that a geodesic cannot be deformed to a loop of smaller Hofer length requires “hard” holomorphic curve techniques, whereas to show that a geodesic can be shortened is possible with “soft” techniques.

In this paper we use “soft” techniques to explore the behaviour of the Hofer length functional as we deform a Hamiltonian circle action within the space of loops in $\text{Ham}(M, \omega)$. Starting with a circle action is a very restrictive assumption, but it allows us to obtain new results that do not follow from general existing results. Our new results are sketched in the abstract and stated in Section 3. In brief,

- ★ Our Theorem 3.1 is the first quantitative shortening estimate for effective Hamiltonian circle actions.
- ★ Our Theorem 3.7, Theorem 3.9, and Corollary 3.12 are the first shortening results that apply when the extrema of the momentum map are not necessarily displaceable and the circle action is effective.
- ★ Our Theorem 3.17 gives a lower bound on the index of the Hofer length functional. It confirms a prediction of Yasha Savelyev. (But see Remark 1.2.)

We now give an overview of the paper.

In Section 2 we review standard background material. The purpose of this review is to remind equivariant symplectic geometers of facts about time-dependent Hamiltonian flows and to remind symplectic topologists of facts about momentum maps. In Section 3 we state our results, give examples, and pose further questions.

In Sections 4–7 we develop the main tools for the later proofs of our main results. We warm up by describing, in Section 4, Polterovich’s shortening trick for non-effective circle actions (item (3) in the above sample of known results). In our later proofs we use variations of this Polterovich trick; we give such variations in Section 5: take a Hamiltonian circle action with momentum map H , and assume that, near the set where H is maximal, we can write $H = K + F$ where K and F also generate circle actions. Also assume that the sets where K and F are maximal have non-empty intersection, and that there exists a symplectic isotopy that disjoins the first of these sets from the second. Finally, assume that the infimum of the sum of K and F is equal to the sum of their infima. We deduce that the circle action can be deformed to a shorter loop in $\text{Ham}(M, \omega)$, and we estimate the amount by which this loop is shorter. Our actual statement is slightly more technical; see Lemma 5.1. The qualitative part of this result can be strengthened if H decomposes, near its maximum, into a sum of more than two momentum maps; see Lemma 5.10. Sections 6 and 7 contain variations on the simple fact that a disc of area A_1 can be disjointed from a disc of area A_2 inside a disc of area greater than $A_1 + A_2$. See Lemmas 6.1 and 7.1.

In Section 8 we prove our first new result, Theorem 3.1: when the momentum map attains its maximum at an isolated fixed point and one of the isotropy weights has absolute value ≥ 2 , we give a lower bound for the amount of possible shortening, in terms of the gap between the two largest critical values of the momentum map. The proof consists of the following steps. There exists an equivariant Darboux chart whose image is the entire set of points where the momentum map is larger than the second-largest critical value; this set can be identified with an ellipsoid in \mathbb{C}^n . We view this ellipsoid as a family of discs in \mathbb{C} parametrized by points in a subset of \mathbb{C}^{n-1} , where the circle action on the \mathbb{C} component is non-effective. From the results of Section 5 we get shortening by an amount that depends on the size of a family of discs that can be disjointed from another such family through a Hamiltonian isotopy supported in the ellipsoid. Such an estimate, in turn, is obtained from the results of Section 7. For effective circle actions, these are the first shortening results that are quantitative.

In Sections 9 and 10 we prove our second new result (more precisely, set of results). We now allow the momentum map to attain its maximum

along a submanifold B of positive dimension. To describe a neighbourhood of B in M , we use Sternberg's minimal coupling procedure and Weinstein's symplectic tubular neighbourhood theorem. Let E denote the normal bundle of B in M . In the simplest case, we may assume that E has k sections that are nowhere all vanishing, where $k + 1$ is the smallest absolute value of an isotropy weight in E . In the general case, we make a similar assumption for a *sub-bundle* E' of E . The idea of the proof is to express the circle action as a composition of $k + 1$ circle actions on a neighbourhood of B and to then apply the results of Section 5. For the precise results, and for some special cases, see Theorems 3.7 and 3.9 and Corollary 3.12. These are the first shortening results when the maximum is attained along a manifold that is not necessarily displaceable and the circle action is effective.

Finally, in Section 11, we prove our third new result, Theorem 3.17: we give a lower bound for the index of the Hofer length functional, assuming that the momentum map attains its maximum at an isolated fixed point. Composing with an equivariant Darboux chart, we may work with a linear circle action on \mathbb{C}^n that rotates the coordinates of \mathbb{C}^n with speeds $-k_1, \dots, -k_n$, where the k_j s are positive integers. We write the action on the j th coordinate as a composition of k_j terms, $\phi_t \circ \dots \circ \phi_t$, where ϕ_t generates the scalar multiplication circle action. We obtain a family of shortenings by applying Polterovich's trick of Section 4 to the different factors of these compositions: in each coordinate, we conjugate all but one of the ϕ_t s by a translation of \mathbb{C} . Varying over the two dimensional family of translations of \mathbb{C} , independently in each one of the $\sum(k_j - 1)$ conjugations, we obtain a family of loops in $\text{Ham}(M, \omega)$ that depends on $2 \sum(k_j - 1)$ real parameters and such that the original circle action corresponds to the origin in the parameter space $\mathbb{R}^{2 \sum(k_j - 1)}$. An explicit computation then shows that the Hofer length of these loops, as a function on $\mathbb{R}^{2 \sum(k_j - 1)}$, achieves its maximum at the origin, and its Hessian is negative definite at the origin. This confirms one direction of a prediction for the index that was made by Yasha Savelyev.

Remark 1.2. While completing this paper, we learned that Savelyev [21] can now confirm his conjecture with the methods of Ustilovsky [24].

This paper is the outgrowth of a joint project that took place in the years 1999-2001. Yael Karshon takes the responsibility for any errors that might be present.

Acknowledgement. We thank Leonid Polterovich for inspiring conversations and helpful suggestions. We thank Yasha Savelyev for suggesting to

compute the index of the Hofer length functional and telling us his conjecture for its value. We also thank Fabian Ziltener and Brian Lee for helpful discussions. Finally, we are grateful to the referee for his or her extremely thorough report and numerous helpful comments.

2. Preliminaries

The purpose of this section is to remind equivariant symplectic geometers of facts about time-dependent Hamiltonian flows and to remind symplectic topologists of facts about momentum maps.

Hamiltonian isotopies

Let (M, ω) be a symplectic manifold. A *smooth path* of symplectomorphisms is a map $t \mapsto \psi_t$, from an interval I to the group of symplectomorphisms of (M, ω) , such that the map $(t, m) \mapsto \psi_t(m)$ from $I \times M$ to M is smooth. Such a path is a *Hamiltonian isotopy* if its velocity vector field, defined by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, is generated by a smooth function $H: I \times M \rightarrow \mathbb{R}$, by Hamilton's equations $dH_t = -\iota(X_t)\omega$, where $H_t(\cdot) = H(t, \cdot)$. The function $H = \{H_t\}_{t \in I}$ is called *the Hamiltonian*. If the Hamiltonian isotopy $\{\psi_t\}$ is compactly supported, its *Hofer length* is

$$\text{length}\{\psi_t\}_{t \in I} = \int_I \left(\max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt.$$

A symplectomorphism is *Hamiltonian* if it can be connected to the identity map by a Hamiltonian isotopy.

The positive and negative Hofer length functionals

Suppose that M is compact. The Hamiltonian $\{H_t\}$ is *normalized* if

$$\int_M H_t \omega^n = 0 \quad \text{for all } t.$$

The *positive Hofer length functional* associated to a Hamiltonian isotopy $\{\varphi_t\}_{a \leq t \leq b}$ in $\text{Ham}(M, \omega)$ is the number

$$\ell_+(\varphi) = \int_a^b \max_M H_t dt$$

where $\{H_t\}_{a \leq t \leq b}$ is the normalized Hamiltonian that generates the isotopy. The *negative Hofer length functional* is, similarly,

$$\ell_-(\varphi) = \int_a^b \max_M -H_t dt.$$

The Hofer length of φ is then $\ell_+(\varphi) + \ell_-(\varphi)$.

Conjugation

If $\{\psi_t\}$ is a Hamiltonian isotopy, generated by $\{H_t\}$, and b is a symplectomorphism, then $\{b\psi_t b^{-1}\}$ is a Hamiltonian isotopy, generated by $\{H_t \circ b^{-1}\}$. If H_t is normalized then $H_t \circ b^{-1}$ is also normalized.

Reparametrization

Let $\{\psi_t\}$ be a Hamiltonian isotopy, generated by a function $\{H_t\}$. Let $t = t(\tau)$ be a smooth function. Then $\bar{\psi}_\tau := \psi_{t(\tau)}$ is a Hamiltonian isotopy, generated by the product $\bar{H}_\tau := H_{t(\tau)} \frac{dt}{d\tau}$.

Indeed, let ξ_t be the vector field that generates the isotopy, so that $\frac{d}{dt}\psi_t = \xi_t \circ \psi_t$ and $dH_t = -\iota(\xi_t)\omega$. Let $\bar{\xi}_\tau = \xi_{t(\tau)} \frac{dt}{d\tau}$. Then $d\bar{H}_\tau = dH_{t(\tau)} \frac{dt}{d\tau} = -\iota(\bar{\xi}_\tau)\omega$ and $\frac{d}{d\tau}\bar{\psi}_\tau = \frac{d}{dt}\psi_t|_{t=t(\tau)} \frac{dt}{d\tau} = \bar{\xi}_\tau \circ \bar{\psi}_\tau$.

Composition and inverse

If $\{\psi_t^K\}_{t \in I}$ is a Hamiltonian isotopy generated by $\{K_t\}_{t \in I}$, and $\{\psi_t^F\}_{t \in I}$ is a Hamiltonian isotopy generated by $\{F_t\}_{t \in I}$, then $\{\psi_t^K \circ \psi_t^F\}_{t \in I}$ is a Hamiltonian isotopy generated by $\{K_t + F_t(\psi_t^K)^{-1}\}_{t \in I}$, and $(\Psi_t^K)^{-1}$ is a Hamiltonian isotopy generated by $-K_t \circ \Psi_t^K$. If K_t and F_t are normalized, so are $K_t + F_t(\psi_t^K)^{-1}$ and $-K_t \circ \Psi_t^K$.

The group of Hamiltonian symplectomorphisms is denoted $\text{Ham}(M, \omega)$.

Paths in the Hamiltonian group

If $\{\psi_t\}$ is a smooth path in the group of symplectomorphisms of (M, ω) , and for each t the symplectomorphism ψ_t is in the subgroup $\text{Ham}(M, \omega)$, then $\{\psi_t\}$ is a Hamiltonian isotopy. This is a result due to Banyaga [2, p.190, prop. II.3.3]. In textbooks, the proof is often omitted or is intertwined with proofs of more difficult facts. We recall the proof.

Let ξ_t be the velocity vector field, defined by $\frac{d}{dt}\psi_t = \xi_t \circ \psi_t$. The path $\{\psi_t\}$ is a Hamiltonian isotopy if and only if $\iota(\xi_t)\omega$ is exact for all t , if and only if the $H^1(M; \mathbb{R})$ -valued integral $\text{Flux}(\{\psi_\tau\}_{0 \leq \tau \leq t}) := \int_0^t [\iota(\xi_\tau)\omega] d\tau$ is zero for all t . (We then set $H_t(x) = -\int_{x_0}^x \iota(\xi_t)\omega$.)

The evaluation of the flux on a smooth loop $\gamma: S^1 \rightarrow M$ is equal to the symplectic area of the surface $\tilde{\gamma}: S^1 \times [0, t] \rightarrow M$ given by $\tilde{\gamma}(s, \tau) := \psi_\tau(\gamma(s))$; this follows from Stokes's formula. It follows that the flux of a closed loop of symplectomorphisms is a class in $H^1(M; \mathbb{R})$ whose evaluation on elements of $H_1(M; \mathbb{Z})$ takes values in the countable set $\langle \omega, H_2(M; \mathbb{Z}) \rangle$. Such classes make up a countable subgroup of $H^1(M; \mathbb{R})$.

Suppose that $\{\psi_t\}$ is a path of symplectomorphisms and that each ψ_t is Hamiltonian. Then for each t the path $\{\psi_\tau\}_{0 \leq \tau \leq t}$ can be completed to a closed loop of symplectomorphisms by a Hamiltonian isotopy. This completion does not effect the flux. So $t \mapsto \text{Flux}(\{\psi_\tau\}_{0 \leq \tau \leq t})$ is a continuous map that takes values in a countable subgroup of $H^1(M; \mathbb{R})$. Such a map must be constant. Since it is zero for $t = 0$, it is zero for all t .

Hamiltonian circle actions

Let (M, ω) be a symplectic manifold with a circle action generated by a Hamiltonian $H: M \rightarrow \mathbb{R}$. The *momentum map* for the circle action is the Hamiltonian H .

The critical points of H are the fixed points for the circle action. Let $p \in M$ be such a point. The linearized isotropy action on $T_p M$ is linearly equivariantly symplectomorphic to \mathbb{C}^n with the standard symplectic form and with the circle acting as a subgroup of $(S^1)^n$, namely, $e^{2\pi i t} \cdot (z_1, \dots, z_n) = (e^{2\pi i k_1 t} z_1, \dots, e^{2\pi i k_n t} z_n)$. The integers k_1, \dots, k_n , which measure the speeds at which the linearized S^1 -action rotates its eigenspaces, are called the *isotropy weights* at p . They are unique up to permutation. This isomorphism $T_p M \cong \mathbb{C}^n$ carries the Hessian of H to the quadratic form $\pi k_1 |z_1|^2 + \dots + \pi k_n |z_n|^2$. So if H attains its maximum at p then all the isotropy weights at p are non-positive.

The *equivariant Darboux* theorem asserts that a neighbourhood of p in M is equivariantly symplectomorphic to a neighbourhood of the origin in $T_p M$. In particular, if p is an isolated fixed point then the isotropy weights at p are all non-zero. Thus, at least one isotropy weight at p has absolute value greater than one if and only if every neighbourhood of p contains a point whose stabilizer is finite and non-trivial. Also, the set B_{\max} where H

is maximal is a symplectic submanifold, and the circle acts on the fibres of its normal bundle with negative weights.

The momentum map H is a Bott-Morse function with even indices and coindices. Assuming that M is compact and connected, this implies that the level sets of H are connected and that preimages of intervals are connected. Thus, the maximum of H is attained along a connected component of the fixed point set. If a fixed point p is a local maximum for H then it is also a global maximum. If a neighbourhood of p in M is equivariantly symplectomorphic to the ellipsoid $\{z \mid \sum k_j \pi |z_j|^2 < \epsilon\}$ with the momentum map $-\sum k_j \pi |z_j|^2$, then this neighbourhood contains *all* the points in M where H is ϵ -close to its maximum.

See [1, 5].

3. New results and further questions

In this section we state our main results. We prove them in Sections 8, 10, and 11. We also give some examples, corollaries, and further questions.

In each of these results, we start with a Hamiltonian circle action, viewed as a loop in the group of Hamiltonian diffeomorphisms, and we deform it to a loop whose Hofer length is smaller. Moreover, throughout the deformation, the negative Hofer length remains constant, and the positive Hofer length eventually becomes smaller without ever becoming larger than the initial one.

Shortening on a manifold with an isolated maximum

Theorem 3.1. *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point and that at least one of the isotropy weights at that point has absolute value greater than one. Let d be a positive number that is smaller than the gap between the two largest critical values of the momentum map. Then the circle action can be deformed to a loop in $\text{Ham}(M, \omega)$ whose Hofer length is smaller than that of the original circle action by $2d/9$.*

Remark 3.2. In Theorem 3.1, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map, let p_{\max} denote the point where H attains its maximum, let α denote the gap between the two largest critical values of H , and consider the subset of M

given by

$$(3.3) \quad \{m \mid H(m) > H(p_{\max}) - \alpha\}.$$

We can choose the deformation such that the normalized deformed Hamiltonians remain between $H(p_{\max}) - \alpha$ and $H(p_{\max})$ on the set (3.3) and coincide with H outside this set.

Remark 3.4. In Theorem 3.1, we actually obtain a better estimate than $\frac{2}{9}d$. If one of the isotropy weights is even then we can shorten by $\frac{1}{4}d$. Otherwise, let $-k_1$ be the weight whose absolute value is largest; then we can shorten by $\frac{1}{4}(1 - \frac{1}{k_1^2})d$. When $k_1 = 3$, this becomes $\frac{2}{9}d$.

This is the first quantitative result for shortening of loops in $\text{Ham}(M, \omega)$ that are effective circle actions. We prove it in Section 8.

The estimate of Theorem 3.1 is far from sharp, as we see in the following example.

Example 3.5. Let a, b be positive integers that are relatively prime. Let $e^{2\pi it} \in S^1$ act on $\mathbb{C}\mathbb{P}^2$ by $[u, z, w] \mapsto [u, e^{-2\pi ita}z, e^{2\pi itb}w]$, with the Fubini-Study form normalized such that the momentum map is $(-a|z|^2 + b|w|^2)/(|u|^2 + |z|^2 + |w|^2) + \text{constant}$. The Hofer length of this action is $a + b$. The isotropy weights at the maximum are $-b, -(a + b)$; the isotropy weights at the minimum are $a, a + b$. Applying the shortening result of Theorem 3.1 and the analogous result with maximum replaced by minimum, for any $\epsilon > 0$ we can deform the circle action to a loop of Hofer length $< 7/9(a + b) + \epsilon$. This upper bound is greater than one. But the action can be deformed to a loop of Hofer length one or zero, so the bound is not optimal. (To see that the action can be deformed to a loop of Hofer length one or zero, notice that the action extends to an action of $\text{PU}(3)$, that $\pi_1(\text{PU}(3)) \cong \mathbb{Z}/3\mathbb{Z}$, and that the non-trivial elements of $\pi_1(\text{PU}(3))$ are represented by the actions $[u, z, w] \mapsto [u, e^{-2\pi it}z, w]$ and $[u, z, w] \mapsto [u, z, e^{2\pi it}w]$, which have Hofer length equal to one.)

Remark 3.6. The above example is inspired by Proposition 1.3 of the paper [16] of McDuff and Tolman: if G is a compact Lie group with trivial centre and M is a coadjoint orbit of G , then every non-trivial element of $\pi_1(G)$ is represented by a sub-circle of G that acts on M semi-freely.

Shortening on a manifold with an arbitrary maximum

Recall that, on a compact connected symplectic manifold with a Hamiltonian circle action, the set where the momentum map is maximal is a connected symplectic submanifold.

In the following theorem, a subset B of a symplectic manifold M is *symplectically k -displaceable* in a neighbourhood V if there exist k symplectomorphisms b_1, \dots, b_k of M , each connected to the identity through a path of symplectomorphisms supported in V , such that $B \cap b_1(B) \cap \dots \cap b_k(B) = \emptyset$.

In the symplectic literature, “ B is displaceable” often means that there exists a Hamiltonian symplectomorphism g such that $B \cap g(B) = \emptyset$. Thus, a set can be symplectically k -displaceable in every neighbourhood without being displaceable (even when $k = 1$).

Theorem 3.7. *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Let B_{\max} be the set where the momentum map is maximal. Let $-k_1, \dots, -k_s$ denote the distinct weights for the circle action on the normal bundle of B_{\max} , and let $k := \min\{k_1, \dots, k_s\} - 1$. Suppose that B_{\max} is symplectically k -displaceable in every neighbourhood. Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ into a loop of smaller Hofer length.*

Remark 3.8. In Theorem 3.7, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonians remain between $H(B_{\max}) - \alpha$ and $H(B_{\max})$ on the set $\{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and coincide with H outside this set.

In practice, the easiest way to show that a symplectic submanifold is symplectically k -displaceable in every neighbourhood is to show that its normal bundle has k sections that are nowhere all vanishing. (k -displaceability then follows from the symplectic tubular neighbourhood theorem (9.5) and Lemma 9.10.) More generally, we have the following theorem.

Theorem 3.9. *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Let B_{\max} be the set where the momentum map is maximal. Let $E' \rightarrow B_{\max}$ be an S^1 -invariant subbundle of the normal bundle to B_{\max} in M , and let $-k'_1, \dots, -k'_s$ be the distinct weights for the circle*

action on E' . Let

$$k' = \min\{k'_1, \dots, k'_s\} - 1.$$

Suppose that

(3.10) E' has k' sections that are nowhere all vanishing.

Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ into a loop of smaller Hofer length.

Remark 3.11. In Theorem 3.9, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonians remain between $H(B_{\max}) - \alpha$ and $H(B_{\max})$ on the set $\{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and coincide with H outside this set.

Theorem 3.9 has the following corollaries.

Corollary 3.12. *Let (M, ω) be a compact symplectic manifold with a Hamiltonian circle action. Let B_{\max} be the subset of M where the momentum map attains its maximum. Let E denote the normal bundle of B_{\max} in M and $-k_1, \dots, -k_s$ the distinct weights for the circle action on E . Suppose that one of the following conditions holds.*

1) *The normal bundle of B_{\max} in M has a trivial sub-bundle on which the circle acts with a weight of absolute value ≥ 2 .*

(In particular, this holds under the assumptions of Theorem 3.1.)

2) *We have*

$$k_j > 1 + \frac{\dim B_{\max}}{\text{codim } B_{\max}}$$

for all $j \in \{1, \dots, s\}$.

3) *After a possible relabeling, assume that $k_1 < \dots < k_s$. Let E_j denote the sub-bundle where the circle acts with weight $-k_j$. There exists $j \in \{1, \dots, s\}$ such that*

$$(k_j - 1) \text{rank}(E_j \oplus \dots \oplus E_s) > \dim B_{\max}.$$

(For $j = 1$, this condition amounts to case (2).)

Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ to a loop of smaller Hofer length.

Moreover, we can choose a deformation with the properties described in Remark 3.11.

Proof of Corollary 3.12. In each of the three cases, (3.10) holds for an appropriate choice of sub-bundle E' :

- 1) Let E' be a trivial sub-bundle of E on which S^1 acts with a weight $-k'_1$ of absolute value $k'_1 \geq 2$. Then $k' = k'_1 - 1 \geq 1$, and (3.10) holds because E' is trivial.
- 2) Let $E' = E$. Then $\text{rank } E' = \text{codim } B_{\max}$. Our assumption on the weights give $k' > \frac{\dim B_{\max}}{\text{codim } B_{\max}}$. So $k' \text{rank } E' > \dim B_{\max}$, and (3.10) holds for generic sections.
- 3) Let $E' = E_j \oplus \cdots \oplus E_s$. Then $k' = k_j - 1$. The assumption on the weights gives $k' \text{rank } E' > \dim B_{\max}$, and again (3.10) holds for generic sections.

In each of these cases, the shortening result follows from Theorem 3.9. \square

Remark 3.13. Let k and k' be as in Theorems 3.7 and 3.9. Existence of k' sections of E' that are nowhere all vanishing does not necessarily imply the existence of k sections of E that are nowhere all vanishing, because k' can be larger than k . So Theorem 3.9 does not follow from Theorem 3.7.

Remark 3.14. Informally, the assumptions of Theorems 3.7 and 3.9 and Corollary 3.12 mean that the normal bundle is “sufficiently untwisted”.

Remark 3.15. There seems to be an interesting relationship with the work of McDuff-Tolman. In the terminology of Section 1.1 of McDuff-Tolman’s paper [15], the component B_{\max} is *homologically visible* if and only if the Euler class of the *obstruction bundle* $(E_1 \otimes \mathbb{C}^{k_1-1}) \oplus \cdots \oplus (E_s \otimes \mathbb{C}^{k_s-1})$ is non-zero. Here, k_1, \dots, k_s are the weights of the circle action on the normal bundle, and E_j is the subbundle on which the circle acts with weight k_j . By Theorem 1.2 of [15], if B_{\max} is homologically visible then the circle action is non-contractible in $\text{Ham}(M, \omega)$. (We are not sure whether B_{\max} being homologically visible further implies that the circle action is minimal among homotopic loops. Proposition 3.3 of [15] might be relevant.) The condition (3.10) of our Theorem 3.9 implies that B_{\max} is not homologically visible. (k' sections of E' that are nowhere all vanishing can be considered

as a non-vanishing section of $E' \otimes \mathbb{C}^{k'}$. Writing $E' = E'_1 \oplus \cdots \oplus E'_{s'}$ where the circle acts on E'_j with weight k'_j , we get a nonvanishing section of $(E'_1 \otimes \mathbb{C}^{k'}) \oplus \cdots \oplus (E'_{s'} \otimes \mathbb{C}^{k'})$. Because $k' \leq k'_j - 1$ for all $j = 1, \dots, s'$, we get a non-vanishing section of $(E'_1 \otimes \mathbb{C}^{k'_1-1}) \oplus \cdots \oplus (E'_{s'} \otimes \mathbb{C}^{k'_{s'}-1})$. Because this is a subbundle of the obstruction bundle, the obstruction bundle has a non-vanishing section, and hence its Euler class is zero.)

We prove Theorems 3.7 and 3.9 in Section 10. These are the first shortening results that apply to situations when the maximum is not necessarily displaceable.

Example 3.16. Let $e^{2\pi it} \in S^1$ act on $\mathbb{C}\mathbb{P}^4$ by

$$[z_0, z_1, z_2, z_3, z_4] \mapsto [z_0, z_1, e^{-2\pi it} z_2, e^{-2\pi it \cdot 2} z_3, e^{-2\pi it \cdot 3} z_4],$$

with momentum map $(-|z_2|^2 - 2|z_3|^2 - 3|z_4|^2)/(|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + \text{constant}$. The momentum map attains its maximum along the submanifold $B_{\max} = \{[z_0, z_1, 0, 0, 0]\}$, which is isomorphic to $\mathbb{C}\mathbb{P}^1$. The negative isotropy weights along B_{\max} are $-1, -2, -3$. The momentum map attains its minimum at the isolated fixed point $[0, 0, 0, 0, z_4]$. The isotropy weights at the minimum are $1, 2, 3, 3$. In the notation of item (3) of Corollary 3.12, applied at B_{\max} , we have the following values.

j	k_j	rank E_j
1	1	2
2	2	2
3	3	2

We can apply item (3) of Corollary 3.12 to either $E_2 \oplus E_3$ or E_3 . Also applying Theorem 3.1 at the minimum, we conclude that the circle action can be deformed to a loop whose positive Hofer length and negative Hofer length are both smaller. Nevertheless, the circle action is not contractible in $\text{Ham}(M, \omega)$: it is homotopic (in $\text{PU}(5)$) to the circle action $[z_0, z_1, z_2, z_3, z_4] \mapsto [e^{-2\pi it} z_0, z_1, z_2, z_3, z_4, z_5]$, which is semi-free and thus non-contractible (by item (1) on p. 211).

The index of the Hofer positive length functional

Theorem 3.17. *Let (M, ω) be a $2n$ dimensional compact connected symplectic manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point. Let $-k_1, \dots, -k_n$*

be the isotropy weights at the maximum, with possible repetitions. Then there exists a neighbourhood D of the origin in $\mathbb{R}^{\sum 2(k_j-1)}$, and, for each $\lambda \in D$, a loop $\{\psi_t^{(\lambda)}\}_{0 \leq t \leq 1}$ in $\text{Ham}(M, \omega)$, such that the following properties hold. For $\lambda = 0$, the loop $\{\psi_t^{(0)}\}_{0 \leq t \leq 1}$ is the given circle action. The function $\lambda \mapsto \text{length}(\{\psi_t^{(\lambda)}\})$ is smooth, $\lambda = 0$ is a critical point of this function, and the Hessian of this function at $\lambda = 0$ is negative definite.

Remark 3.18. In fact, we can choose the deformation such that the negative Hofer length remains constant. Specifically, let H denote the momentum map, and let p_{\max} denote the point where H attains its maximum. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonian remains between $H(p_{\max}) - \alpha$ and $H(p_{\max})$ on the set $\{m \in M \mid H(m) > H(p_{\max}) - \alpha\}$ and coincides with H outside this set.

Savelyev [19, 20], in the setup of Theorem 3.17, predicted that the index of the positive Hofer length functional on the space of loops in $\text{Ham}(M, \omega)$ at the given Hamiltonian circle action is equal to $\sum 2(k_i - 1)$. Theorem 3.17 is a sense in which the index of the positive Hofer length functional is *at least* $\sum_i 2(k_i - 1)$. So Theorem 3.17 confirms the \geq direction of Savelyev's prediction. (But see Remark 1.2.)

Further questions

- 1) Are there examples where the estimate of Theorem 3.1 is sharp?
- 2) Can the estimate of Theorem 3.1 be improved in the presence of high isotropy weights?
As noted by Polterovich, this question is also interesting for non-effective circle actions obtained by iterating an effective circle action, as it may provide information on the asymptotic norm of the homotopy class of the effective circle action (see [18]).
- 3) Under the assumptions of Theorem 3.7 or of Theorem 3.9, can one obtain a quantitative estimate?
- 4) In Theorem 3.9, can the condition (3.10) be replaced by the condition that B_{\max} not be homologically visible? (Cf. Remark 3.15.)
- 5) Can Theorem 3.17 be extended to cases in which the maximum is attained along a manifold of positive dimension?
- 6) Savelyev's prediction for the index of the positive Hofer length functional is an even number. Polterovich expects the index to be even in

additional cases. On a compact connected manifold, a Morse function with even indices and coindices has nice topological properties: every local minimum is a global minimum, and the level sets are connected. Palais-Smale-Morse theory does not apply to the space of loops in $\text{Ham}(M, \omega)$, but we may seek analogies.

If a loop in the Hamiltonian group is a local minimum for the positive Hofer length functional, is it also a global minimum in its homotopy class?

Suppose that a homotopy class in $\pi_1(\text{Ham}(M, \omega))$ contains two circle actions that are semi-free near the set where the momentum map is maximal. (Cf. item (1) on p. 211.) Can we deform one to the other through loops of constant positive Hofer length?

4. Shortening a non-effective circle action

To warm up, we describe a procedure, which we learned from Leonid Polterovich, for shortening a non-effective Hamiltonian circle action by displacing the set where its momentum map is maximal.

Let $H: M \rightarrow \mathbb{R}$ be a time independent Hamiltonian. Suppose that $\frac{1}{2}H$ generates a circle action, $\{\phi_t\}_{0 \leq t \leq 1}$, so that H generates the non-effective circle action

$$(4.1) \quad \psi_t^H = \phi_{2t} = \phi_t \circ \phi_t.$$

Let $b: M \rightarrow M$ be a symplectomorphism. Then

$$(4.2) \quad \psi_t^{\bar{H}} := \{\phi_t b \phi_t b^{-1}\}_{0 \leq t \leq 1}$$

is a loop in $\text{Ham}(M)$, based at the identity, and generated by the Hamiltonian

$$(4.3) \quad \bar{H}_t = \frac{1}{2}H + \frac{1}{2}Hb^{-1}\phi_t^{-1}.$$

If H is normalized, so is \bar{H}_t .

If b can be connected to the identity through a path of symplectomorphisms, then the loop (4.2) is a deformation of the loop (4.1). Since $\bar{H}_t(x) \leq \max H$ and $\bar{H}_t(x) \geq \min H$ for all $x \in M$ and all t ,

$$(4.4) \quad \text{length}(\{\psi_t^{\bar{H}}\}) \leq \text{length}(\{\psi_t^H\}),$$

and this is a strict inequality if $\max \bar{H}_t < \max H$ for some t .

Denote by B_{\max}^H the subset of M where H takes its maximal value. If $B_{\max}^H \cap b(B_{\max}^H) = \emptyset$ then, for $t = 0$, the maximum of $\overline{H}_0 = \frac{1}{2}H + \frac{1}{2}Hb^{-1}$ is strictly smaller than that of H .

We have shown that, if there exists a symplectic isotopy that displaces the set where the momentum map is maximal, then the non-effective circle action can be deformed to a loop of shorter Hofer length. A similar argument gives a quantitative result: if there exists a symplectic isotopy that displaces the set where the momentum map is ϵ -close to its maximum, then the non-effective circle action can be deformed to a loop whose Hofer length is shorter by ϵ from that of the original circle action.

5. Shortening a combined action

In this section we give variations of Polterovich's trick that apply to effective circle actions. We begin with a lemma that we will use to prove Theorem 3.1. The lemma has two parts – a qualitative part and a quantitative part.

We say that a symplectomorphism b *disjoins* a set A from a set B if $b(A) \cap B = \emptyset$.

Lemma 5.1. *Let (M, ω) be a compact symplectic manifold. Let $H: M \rightarrow \mathbb{R}$ be the momentum map for a circle action $\{\psi_t^H\}_{0 \leq t \leq 1}$. Let*

$$i: (U, \omega_0) \rightarrow (M, \omega)$$

be an open symplectic embedding. Suppose that

$$(5.2) \quad i^*H = K + F \quad \text{on } U,$$

where K and F generate commuting circle actions $\{\psi_t^K\}_{0 \leq t \leq 1}$ and $\{\psi_t^F\}_{0 \leq t \leq 1}$ on U .

Let $\hat{N}: U \rightarrow \mathbb{R}$ be a function that Poisson commutes with K and F and satisfies

$$(5.3) \quad \inf_{\hat{N}^{-1}(\alpha)} (K + F) = \inf_{\hat{N}^{-1}(\alpha)} K + \inf_{\hat{N}^{-1}(\alpha)} F \quad \text{for every } \alpha \in \mathbb{R}.$$

Also suppose that

$$(5.4) \quad i(U) \text{ does not meet the set where } H \text{ is minimal.}$$

1) Let

$$\begin{aligned} B_{\max}^H &= \{m \in M \mid H \text{ is maximal at } m\}, \\ B_{\max}^K &= \{u \in U \mid K \text{ is maximal at } u\}, \\ \text{and } B_{\max}^F &= \{u \in U \mid F \text{ is maximal at } u\}. \end{aligned}$$

Suppose that

$$(5.5) \quad B_{\max}^H \subset i(U) \quad \text{and} \quad B_{\max}^K \cap B_{\max}^F \neq \emptyset.$$

Suppose that there exists a symplectomorphism $b: U \rightarrow U$ that disjoins B_{\max}^F from B_{\max}^K and such that b can be connected to the identity through a path of symplectomorphisms that are compactly supported in U and that preserve the function \hat{N} .

Then the circle action $\{\Psi_t^H\}$ can be deformed, through loops in $\text{Ham}(M, \omega)$, to a loop $\{\Psi_t^{\bar{H}}\}$ whose Hofer length is smaller.

2) Let $\epsilon > 0$. Let

$$\begin{aligned} \mathcal{N}_\epsilon^H &= \{m \in M \mid H(m) > \max H - \epsilon\}, \\ \mathcal{N}_\epsilon^K &= \{u \in U \mid K(u) > \max K - \epsilon\}, \\ \text{and } \mathcal{N}_\epsilon^F &= \{u \in U \mid F(u) > \max F - \epsilon\}. \end{aligned}$$

Suppose that

$$(5.6) \quad \mathcal{N}_\epsilon^H \subset i(U) \quad \text{and} \quad B_{\max}^K \cap B_{\max}^F \neq \emptyset.$$

Suppose that there exists a symplectomorphism $b: U \rightarrow U$ that disjoins \mathcal{N}_ϵ^F from \mathcal{N}_ϵ^K and such that b can be connected to the identity through a path of symplectomorphisms that are compactly supported in U and that preserve the function \hat{N} .

Then the circle action $\{\Psi_t^H\}$ can be deformed, through loops in $\text{Ham}(M, \omega)$, to a loop $\{\Psi_t^{\bar{H}}\}$ whose Hofer length is smaller than that of $\{\Psi_t^H\}$ by at least ϵ .

In each of these two cases, the deformation can be chosen such that the normalized deformed Hamiltonians remain between $\inf H(i(U))$ and $\max H(M)$ on the set $i(U)$ and coincide with H outside this set, so the positive Hofer length never exceeds the initial one and the negative Hofer length remains the same throughout the deformation.

Proof. Because K and F generate commuting circle actions, $K \circ \psi_t^F = K$ and $F \circ \psi_t^K = F$ for all t .

Because $B_{\max}^H \subset i(U)$, $i^*H = K + F$, and $B_{\max}^K \cap B_{\max}^F \neq \emptyset$,

$$(5.7) \quad B_{\max}^H = i(B_{\max}^K \cap B_{\max}^F).$$

Let $b: U \rightarrow U$ be a compactly supported symplectomorphism of U .

The loop

$$\psi_t^{\overline{H}} = \begin{cases} i \circ (\psi_t^K \circ b\psi_t^F b^{-1}) \circ i^{-1} & \text{on } i(U) \\ \psi_t^H & \text{outside } i(U) \end{cases}$$

is generated by the Hamiltonian $\overline{H}_t: M \rightarrow \mathbb{R}$ that is given by

$$(5.8) \quad \overline{H}_t = \begin{cases} (K + Fb^{-1}(\psi_t^K)^{-1}) \circ i^{-1} & \text{on } i(U) \\ H & \text{outside } i(U). \end{cases}$$

If H is normalized, so is \overline{H}_t .

If b can be connected to the identity through a compactly supported symplectic isotopy, then the loop $\{\psi_t^{\overline{H}}\}$ is a deformation of the loop $\{\psi_t^H\}$ in $\text{Ham}(M, \omega)$.

By (5.7) and (5.8), we have $\max \overline{H}_t \leq \max H$ for all t . So the positive Hofer length of $\{\psi_t^{\overline{H}}\}$ does not exceed that of the circle action $\{\psi_t^H\}$.

If b preserves \hat{N} , then, by (5.3), (5.4), (5.8), and because ψ_t^K preserves \hat{N} , we have $\min \overline{H}_t = \min H$ for all t . So the negative Hofer length of $\{\psi_t^{\overline{H}}\}$ is equal to that of the circle action $\{\psi_t^H\}$.

Suppose that b disjoins B_{\max}^F from B_{\max}^K . Then, for $t = 0$, we have $\max \overline{H}_0 = \max(K + Fb^{-1}) < \max K + \max F = \max H$. This strict inequality is because $B_{\max}^K \cap B_{\max}^{Fb^{-1}}$, being equal to $B_{\max}^K \cap b(B_{\max}^F)$, is empty. (In fact, $\max \overline{H}_t < \max H$ for all t ; this follows from $K + Fb^{-1}(\psi_t^K)^{-1} = (K + Fb^{-1})(\psi_t^K)^{-1}$ and $B_{\max}^K \cap b(B_{\max}^F) = \emptyset$.) Because the inequality $\max \overline{H}_t \leq \max H$ holds for all t and is strict for $t = 0$, the positive Hofer length of $\{\psi_t^{\overline{H}}\}$ is strictly smaller than that of the circle action $\{\psi_t^H\}$.

This completes the proof of the first, qualitative, part of the lemma.

To prove the second, quantitative, part of the lemma, suppose now that b disjoins \mathcal{N}_ϵ^F from \mathcal{N}_ϵ^K , let m be a point in M , and we will show that $\overline{H}_t(m) \leq \max H - \epsilon$ for all t .

Suppose that m is not in $i(U)$, and, thus, also not in \mathcal{N}_ϵ^H . Then $\overline{H}_t(m) = H(m)$ and $H(m) \leq \max H - \epsilon$, so

$$\overline{H}_t(m) \leq \max H - \epsilon.$$

Now suppose that m is in $i(U)$, say, $m = i(u)$ for $u \in U$. By (5.8),

$$\overline{H}_t(m) = K(u) + F(u')$$

where $u' = b^{-1}(\psi_t^K)^{-1}(u)$. If $\overline{H}_t(m)$ is ϵ -close to $\max H$, then $u \in \mathcal{N}_\epsilon^K$ and $u' \in \mathcal{N}_\epsilon^F$, that is,

$$(5.9) \quad u \in \mathcal{N}_\epsilon^K \cap \psi_t^K(b(\mathcal{N}_\epsilon^F)).$$

Because $K = K \circ \psi_t^K$, the set \mathcal{N}_ϵ^K is preserved under ψ_t^K , so we can rewrite (5.9) as $u \in \psi_t^K(\mathcal{N}_\epsilon^K \cap b(\mathcal{N}_\epsilon^F))$. But, by assumption, $\mathcal{N}_\epsilon^K \cap b(\mathcal{N}_\epsilon^F)$ is empty. So $\overline{H}_t(m)$ cannot be ϵ -close to $\max H$, and again we conclude that $\overline{H}_t(m) \leq \max H - \epsilon$. So $\overline{H}_t(m) \leq \max H - \epsilon$ for all $m \in M$ and for all t . Since \overline{H}_t is normalized, we conclude that the positive Hofer lengths satisfy $\ell_+(\{\psi_t^H\}) \leq \ell_+(\{\psi_t^K\}) - \epsilon$, as required. \square

The second, quantitative, part of Lemma 5.1 does not automatically extend to more than two summands. The first, qualitative, part does extend to more than two summands. We now give such an extension. We will use it to prove Theorems 3.7 and 3.9.

Lemma 5.10. *Let (M, ω) be a compact symplectic manifold. Let $H: M \rightarrow \mathbb{R}$ be the momentum map for a circle action $\{\psi_t^H\}_{0 \leq t \leq 1}$. Let*

$$i: (U, \omega_0) \rightarrow (M, \omega)$$

be an open symplectic embedding. Suppose that

$$(5.11) \quad i^*H = H_{(0)} + H_{(1)} + \cdots + H_{(k)} \quad \text{on } U$$

where $H_{(0)}, H_{(1)}, \dots, H_{(k)}$ generate commuting circle actions on U .

Let

$$B_{\max}^H = \{m \in M \mid H \text{ is maximal at } m\}$$

and, for each $0 \leq j \leq k$,

$$B_{\max}^{(j)} = \{u \in U \mid H_{(j)} \text{ is maximal at } u\}.$$

Suppose that

$$(5.12) \quad B_{\max}^H \subset i(U) \quad \text{and} \quad B_{\max}^{(0)} \cap B_{\max}^{(1)} \cap \cdots \cap B_{\max}^{(k)} \neq \emptyset.$$

Suppose that

$$(5.13) \quad \inf_U (H_{(0)} + H_{(1)} + \cdots + H_{(k)}) = \sum_{j=0}^k \inf_U H_{(j)}$$

and that

$$(5.14) \quad i(U) \text{ does not meet the set where } H \text{ is minimal.}$$

Suppose that there exist symplectomorphisms b_1, \dots, b_k of U such that

$$(5.15) \quad B_{\max}^{(0)} \cap b_1(B_{\max}^{(1)}) \cap \cdots \cap b_k(B_{\max}^{(k)}) = \emptyset$$

and such that each b_j can be connected to the identity through a path of symplectomorphisms that is compactly supported in U .

Then the circle action $\{\psi_t^H\}$ can be deformed, through loops in $\text{Ham}(M, \omega)$, to a loop $\{\bar{\psi}_t\}$ whose Hofer length is smaller.

The deformation can be chosen such that the normalized deformed Hamiltonians remain between $\inf H(i(U))$ and $\max H(M)$ on the set $i(U)$ and coincide with H outside this set, so the positive Hofer length never exceeds the initial one and the negative Hofer length remains the same.

Proof. Denote by

$$\{\psi_t^{(0)}\}_{0 \leq t \leq 1}, \dots, \{\psi_t^{(k)}\}_{0 \leq t \leq 1}$$

the circle actions on U that are generated by $H_{(0)}, \dots, H_{(k)}$. Let b_1, \dots, b_k be arbitrary compactly supported symplectomorphisms of U . For each $j \in \{0, \dots, k\}$, let

$$(5.16) \quad \bar{\psi}_t^{(j)} := \psi_t^{(0)} \circ \left(b_1 \psi_t^{(1)} b_1^{-1} \right) \circ \cdots \circ \left(b_j \psi_t^{(j)} b_j^{-1} \right).$$

This is a loop in $\text{Ham}(U, \omega_0)$. We have the following three facts:

1) $\{\bar{\psi}_t^{(j)}\}$ is generated by the Hamiltonian

$$(5.17) \quad \bar{H}_t^{(j)} := H_{(0)} + H_{(1)} b_1^{-1} (\bar{\psi}_t^{(0)})^{-1} + \cdots + H_{(j)} b_j^{-1} (\bar{\psi}_t^{(j-1)})^{-1}.$$

2) Outside a compact subset of U ,

$$\bar{\psi}_t^{(j)} = \psi_t^{(0)} \circ \psi_t^{(1)} \circ \cdots \circ \psi_t^{(j)} \quad \text{and} \quad \bar{H}_t^{(j)} = H_{(0)} + H_{(1)} + \cdots + H_{(j)}.$$

3) For all t ,

$$(5.18) \quad \int_U \left(\overline{H}_t^{(j)} - (H_{(0)} + H_{(1)} + \cdots + H_{(j)}) \right) \omega_0^n = 0.$$

Facts (1) and (2) are shown by induction on j . As for Fact (3), by (5.17), the left hand side of (5.18) is the sum over $m = 1, \dots, j$ of

$$(5.19) \quad \int_U H_{(m)} \circ \left(b_m^{-1} \circ (\overline{\psi}_t^{(m-1)})^{-1} \right) \omega_0^n - \int_U H_{(m)} \omega_0^n.$$

The map $b_m^{-1} \circ (\overline{\psi}_t^{(m-1)})^{-1}$ is a symplectomorphism, so it preserves the volume form ω_0^n , and so (5.19) vanishes.

By the above items (1) and (2), we can define a loop in $\text{Ham}(M, \omega)$ by

$$\overline{\psi}_t := \begin{cases} i \circ \overline{\psi}_t^{(k)} \circ i^{-1} & \text{on } i(U) \\ \psi_t^H & \text{on } M \setminus i(U), \end{cases}$$

and it is generated by the Hamiltonian

$$(5.20) \quad \overline{H}_t := \begin{cases} \overline{H}_t^{(k)} \circ i^{-1} & \text{on } i(U), \\ H & \text{on } M \setminus i(U). \end{cases}$$

By the above item (3), if H is normalized, so is \overline{H}_t .

From (5.11), (5.13), (5.14), (5.17), and (5.20), we deduce that $\min_M \overline{H}_t = \min_M H$. So the negative Hofer length of $\{\overline{\psi}_t\}$ is equal to that of the circle action $\{\psi_t^H\}$.

From (5.11), (5.12), (5.17), and (5.20), we see that $\max_M \overline{H}_t \leq \max_M H$ for all t , with a strict inequality when $t = 0$ if b_1, \dots, b_k satisfy (5.15). Integrating, we get the inequality $\int_0^1 \max_M \overline{H}_t dt \leq \int_0^1 \max_M H dt$, and if b_1, \dots, b_k satisfy (5.15) then this inequality is strict (because in this case the pointwise inequality between the integrands is strict in a neighbourhood of $t = 0$). That is, the positive Hofer length of $\{\overline{\psi}_t\}$ does not exceed that of $\{\psi_t^H\}$, and, if b_1, \dots, b_k satisfy (5.15), then the positive Hofer length of $\{\overline{\psi}_t\}$ is strictly smaller than that of $\{\psi_t^H\}$.

The lemma is obtained by applying this argument as b_1, \dots, b_k vary over smooth paths of symplectomorphisms that start with the identity, are compactly supported in U , and end with b_1, \dots, b_k that satisfy (5.15). \square

6. Disjoining discs in \mathbb{C}

For every neighbourhood of the origin in \mathbb{C} , there exists a Hamiltonian isotopy, supported in this neighbourhood, that disjoins the origin from itself. Such an isotopy can be generated by a function $F: \mathbb{C} \rightarrow \mathbb{R}$ whose Hamiltonian vector field is zero outside the given neighbourhood, is equal to $-\frac{\partial}{\partial x}$ on a neighbourhood of the origin, and is equal to a non-negative multiple of $-\frac{\partial}{\partial x}$ everywhere on the x -axis. For example, if the given neighbourhood contains the set $\{z \mid \pi|z|^2 \leq 3\epsilon\}$, we may take the function $F(z) = y\rho(\pi|z|^2/\epsilon - 1)$ where y is the imaginary part of z and where $\rho(s)$ is equal to one for $s \leq 0$, to zero for $s \geq 1$, and to $(\int_s^1 e^{-\frac{1}{\tau(1-\tau)}} d\tau)/(\int_0^1 e^{-\frac{1}{\tau(1-\tau)}} d\tau)$ for $0 \leq s \leq 1$.

For our quantitative results we will need to keep track of the sizes of neighbourhood of the origin that get disjoined. We do so in the following lemma.

Lemma 6.1. *There exists a smooth family of functions*

$$F_t^{A,\epsilon}: \mathbb{C} \rightarrow \mathbb{R}, \quad 0 \leq t \leq 1,$$

parametrized by $A \geq 0$ and $0 < \epsilon < 1$, such that $F_t^{A,\epsilon}(z) = 0$ for z outside the disc $\{z \mid \pi|z|^2 < 1 + A + \epsilon\}$ and such that the Hamiltonian flow $b_t^{A,\epsilon}: \mathbb{C} \rightarrow \mathbb{C}$ of $F_t^{A,\epsilon}$, at time $t = 1$, carries the disc $\{z \mid \pi|z|^2 \leq 1\}$ into the annulus $\{z \mid A < \pi|z|^2 < 1 + A + \epsilon\}$.

Remark 6.2. *Smooth family* means that the function $(A, \epsilon, t, z) \mapsto F_t^{A,\epsilon}(z)$ from $[0, \infty) \times (0, 1) \times [0, 1] \times \mathbb{C}$ to \mathbb{R} is smooth in the usual sense. It follows that the function $(A, \epsilon, t, z) \mapsto b_t^{A,\epsilon}(z)$ is smooth.

Lemma 6.1 is used in the next section, in the proof of Lemma 7.1. The rest of this section is devoted to the proof of Lemma 6.1. The reader may skip the proof and proceed to the next section.

Lemma 6.3. *There exists a smooth family of (time independent) functions $H^\epsilon: \mathbb{C} \rightarrow \mathbb{R}$, parametrized by $0 < \epsilon < 1$, whose Hamiltonian flows b_t^ϵ have the following properties.*

- *For every t , the diffeomorphism b_t^ϵ is the identity map outside the disc*

$$(6.4) \quad \{z \mid \pi|z|^2 < 1 + \epsilon\}.$$

- For $t > 1$, the diffeomorphism b_t^ϵ carries the disc

$$(6.5) \quad \{z \mid \pi|z|^2 \leq 1\}$$

into the slit disc

$$(6.6) \quad \{z \mid \pi|z|^2 < 1 + \epsilon\} \setminus \text{the non-negative } x\text{-axis.}$$

Proof. We choose H^ϵ such that, for each ϵ , the Hamiltonian vector field of H^ϵ is zero outside the disc (6.4), is equal to $-\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial x}$ on the segment $[0, \frac{1}{\sqrt{\pi}}]$ of the x-axis, and is equal to a non-negative multiple of $-\frac{\partial}{\partial x}$ on the rest of the x-axis.

For example, we may take $H^\epsilon(z) = \frac{1}{\sqrt{\pi}}y\rho((\pi|z|^2 - 1)/\epsilon)$ where y is the imaginary part of z and where $\rho(s)$ is equal to 1 for $s \leq 0$, to 0 for $s \geq 1$, and is non-negative.

For each t , the Hamiltonian flow generated by the function H^ϵ , at times $t \geq 0$, carries the segment $[-\sqrt{\frac{1+\epsilon}{\pi}}, \frac{1}{\sqrt{\pi}}]$ to a segment of the form $[-\sqrt{\frac{1+\epsilon}{\pi}}, x_t]$; if $t > 1$, then $x_t < 0$. Also, this flow carries the segment $[\frac{1}{\sqrt{\pi}}, \sqrt{\frac{1+\epsilon}{\pi}}]$ to the segment $[x_t, \sqrt{\frac{1+\epsilon}{\pi}}]$, and it is the identity map outside the disc (6.4). Thus, it satisfies the requirements of the lemma. \square

Lemma 6.7. *There exists a smooth family of functions $H_t^{\epsilon, \delta}: \mathbb{C} \rightarrow \mathbb{R}$, parametrized by $0 < \epsilon < 1$ and $0 < \delta < \epsilon$, whose Hamiltonian flows $b_t^{\epsilon, \delta}: \mathbb{C} \rightarrow \mathbb{C}$ have the following properties.*

- The diffeomorphism $b_t^{\epsilon, \delta}$ is the identity map outside the disc $\{z \mid \pi|z|^2 < t + 1 + \epsilon\}$.
- The diffeomorphism $b_t^{\epsilon, \delta}$ carries the set

$$(6.8) \quad \{re^{i\theta} \mid \delta \leq \pi r^2 \leq 1 + \epsilon - \delta, \delta \leq \theta \leq 2\pi - \delta\}$$

to the set

$$(6.9) \quad \{re^{i\theta} \mid t + \delta \leq \pi r^2 \leq t + 1 + \epsilon - \delta, \delta \leq \theta \leq 2\pi - \delta\}.$$

Proof. We choose $H_t^{\epsilon, \delta}$ whose Hamiltonian vector field vanishes outside the annulus $\{z \mid t < \pi|z|^2 < 1 + t + \epsilon\}$ and, on the set (6.9), it is equal to $\frac{1}{2\pi r}\frac{\partial}{\partial r}$. (Then, on the set (6.9), the derivative of the function $\pi|z|^2$ along this vector field is equal to one.) The Hamiltonian flow generated by the function $H_t^{\epsilon, \delta}$, at time $t \geq 0$, will carry the set (6.8) to the set (6.9), as required.

To be explicit, we may take $H_t^{\epsilon, \delta}(re^{i\theta}) = -\frac{\theta}{2\pi}\rho_1^\delta(\theta)\rho_2^{\epsilon, \delta}(\pi r^2 - t)$ where $0 \leq \theta < 2\pi$ and $t \leq \pi r^2 \leq 1 + t + \epsilon$, where $\rho_1^\delta: [0, 2\pi] \rightarrow \mathbb{R}$ is a smooth family of functions that vanish to all orders at $\theta = 0$ and $\theta = 2\pi$ and such that ρ_1^δ is equal to one on $[\delta, 2\pi - \delta]$, and where $\rho_2^{\epsilon, \delta}: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth family of functions that vanish outside $[0, 1 + \epsilon]$ and such that $\rho_2^{\epsilon, \delta}$ is equal to one on $[\delta, 1 + \epsilon - \delta]$. □

Proof of Lemma 6.1. Every closed subset of \mathbb{C} which is contained in the slit disc

$$(6.10) \quad \{z \mid \pi|z|^2 < 1 + \epsilon\} \setminus \text{the non-negative x-axis}$$

is also contained in a set of the form

$$\{re^{i\theta} \mid \delta \leq \pi r^2 \leq 1 + \epsilon - \delta, \delta \leq \theta \leq 2\pi - \delta\}$$

for some $\delta > 0$. Moreover, for every closed subset of $(0, 1) \times \mathbb{C}$ which is contained in the product of the open segment $(0, 1)$ with the slit disc (6.10), there exists a positive smooth function $\epsilon \mapsto \delta_\epsilon$ such that the closed subset is also contained in the subset

$$(6.11) \quad \{(\epsilon, re^{i\theta}) \mid 0 < \epsilon < 1, \delta_\epsilon \leq \pi r^2 \leq 1 + \epsilon - \delta_\epsilon, \delta_\epsilon \leq \theta \leq 2\pi - \delta_\epsilon\}$$

of $(0, 1) \times \mathbb{C}$.

Let b_t^ϵ be the Hamiltonian flows of Lemma 6.3. Because $(\epsilon, z) \mapsto (\epsilon, b_t^\epsilon(z))$ is a diffeomorphism of $(0, 1) \times \mathbb{C}$, it carries the set $\{(\epsilon, z) \mid \pi|z|^2 \leq 1\}$ to a closed subset of $(0, 1) \times \mathbb{C}$. For $t > 1$, by the second item of Lemma 6.3, this closed subset is contained in the product of $(0, 1)$ with the slit disc (6.10). Fix $T > 1$, and fix a smooth function $\epsilon \mapsto \delta_\epsilon$ such that the set $\{(\epsilon, b_T^\epsilon(z)) \mid \pi|z|^2 \leq 1\}$ is contained in the set (6.11).

Let $b_t^{\epsilon, \delta}$ be the Hamiltonian flows of Lemma 6.7.

Let $\rho: [0, 1] \rightarrow [0, 1]$ be a smooth function that takes 0 to 0 and 1 to 1 and whose derivatives of all orders vanish at the endpoints 0 and 1. For example, we may take $\rho(s) = \int_0^s e^{-\frac{1}{s(1-s)}} ds / \int_0^1 e^{-\frac{1}{s(1-s)}} ds$. The Hamiltonian flow

$$b_\tau^{A, \epsilon} = \begin{cases} b_{T\rho(2\tau)}^\epsilon & 0 \leq \tau \leq 1/2 \\ b_{A\rho(2\tau-1)}^{\epsilon, \delta_\epsilon} & 1/2 \leq \tau \leq 1 \end{cases}$$

satisfies the requirements of Lemma 6.1. □

7. Disjoining a family of discs

The purpose of this section is to prove the following quantitative result, which is later used in the proof of Theorem 3.1. Recall that a function from a topological space X to a topological space Y is *proper* if the preimage of every compact subset of Y is a compact subset of X .

Lemma 7.1. *Let W be a symplectic manifold and $N: W \rightarrow [0, r)$ a proper function. Let $A_1, A_2: [0, r) \rightarrow \mathbb{R}$ be smooth functions such that the set*

$$C := \{x \in [0, r) \mid A_1(x) \geq 0 \text{ and } A_2(x) \geq 0\}$$

is compact. Let U be an open subset of $\mathbb{C} \times W$ that contains the set

$$(7.2) \quad \{(z, w) \mid N(w) \in C \text{ and } \pi|z|^2 \leq A_1(N(w)) + A_2(N(w))\}.$$

Then there exists a symplectic isotopy, compactly supported in U , that disjoins the set

$$\{(z, w) \mid \pi|z|^2 \leq A_1(N(w))\}$$

from the set

$$\{(z, w) \mid \pi|z|^2 \leq A_2(N(w))\}$$

and that preserves the function $(z, w) \mapsto N(w)$.

Lemma 7.1 is, more or less, a parametrized version of the fact that a disc of area A_1 can be disjoined from a disc of area A_2 inside any disc of area greater than $A_1 + A_2$. The symplectic isotopy moves the w coordinate but it does not change the value of $N(w)$. For a fixed value of $N(w)$, the effect of the isotopy on the z coordinate is independent of the w coordinate.

Proof. The map $(z, w) \mapsto (\pi|z|^2, N(w))$ from $\mathbb{C} \times W$ to $[0, \infty) \times [0, r)$ is proper, hence closed. So the image of the complement of U under this map is a closed subset of $[0, \infty) \times [0, r)$. This image is disjoint from the compact set

$$\{(s, x) \in [0, \infty) \times [0, r) \mid x \in C \text{ and } s \leq A_1(x) + A_2(x)\}$$

so it has a positive distance from this set. Let $\bar{\epsilon}$ be a positive number that is smaller than half of this distance and is smaller from the difference $r - \max C$. Then $V := (C + (-\bar{\epsilon}, \bar{\epsilon})) \cap [0, r)$ is an open neighbourhood of C in

$[0, r)$ whose closure in $[0, r)$ is compact, and U contains the set

$$(7.3) \quad \{(z, w) \mid N(w) \in \text{closure}(V) \text{ and } \pi|z|^2 \leq A_1(N(w)) + A_2(N(w)) + \bar{\epsilon}\}.$$

Let $\bar{F}_t^{A_1, A_2} : \mathbb{C} \rightarrow \mathbb{R}$ be a smooth family of functions, defined for $t \in [0, 1]$ and $A_1, A_2 \in \mathbb{R}$, such that, when $A_1 \geq 0$ and $A_2 \geq 0$, the time-dependent function $z \mapsto \bar{F}_t^{A_1, A_2}(z)$ vanishes outside the set given by $\pi|z|^2 \leq A_1 + A_2 + \bar{\epsilon}$ and its Hamiltonian flow $\bar{b}_t^{A_1, A_2}$, at time $t = 1$, carries the set $\{z \mid \pi|z|^2 \leq A_1\}$ into the set $\{z \mid A_2 < \pi|z|^2 \leq A_1 + A_2 + \bar{\epsilon}\}$. For example, we may set

$$\bar{F}_t^{A_1, A_2}(z) := cF_t^{A, \epsilon}(z/c)$$

where $c = \sqrt{A_1 + \bar{\epsilon}/2}$, $A = \frac{A_2}{A_1 + \bar{\epsilon}/2}$, and $\epsilon = \frac{\bar{\epsilon}/2}{A_1 + \bar{\epsilon}/2}$, and where $F_t^{A, \epsilon}$ is as in Lemma 6.1. (The Hamiltonian flow $\bar{b}_t^{A_1, A_2}$ of $\bar{F}_t^{A_1, A_2}$ relates to the Hamiltonian flow $b_t^{A, \epsilon}$ of $F_t^{A, \epsilon}(z)$ by $\bar{b}_t^{A_1, A_2}(z) = cb_t^{A, \epsilon}(z/c)$.)

Let $\rho : [0, r) \rightarrow \mathbb{R}$ be a smooth function that vanishes outside V and such that $\rho|_C \equiv 1$. Let

$$H_t(z, w) = \rho(N(w))\bar{F}_t^{A_1(N(w)), A_2(N(w))}(z).$$

This function vanishes outside the compact set (7.3). Its Hamiltonian flow preserves the function $N(w)$. We will show, for every N_0 , that this flow, at time $t = 1$, disjoins

$$(7.4) \quad \{(z, w) \mid \pi|z|^2 \leq A_1(N_0) \text{ and } N(w) = N_0\}$$

from

$$(7.5) \quad \{(z, w) \mid \pi|z|^2 \leq A_2(N_0) \text{ and } N(w) = N_0\}.$$

When $N_0 \in C$, the restriction of the Hamiltonian flow of H_t to the level set $\{N(w) = N_0\}$ is

$$b_t(z, w) = \left(\bar{b}_t^{A_1, A_2}, b_{T(z, w, t)}^N(w) \right)$$

where $A_1 = A_1(N_0)$, where $A_2 = A_2(N_0)$, where $\bar{b}_t^{A_1, A_2}$ is the Hamiltonian flow of $\bar{F}_t^{A_1, A_2}$ on \mathbb{C} , where b_T^N is the Hamiltonian flow of N on W (with time parameter T), and where $T(z, w, t)$ is some real valued function of z, w, t .

At $t = 1$, this flow carries the set

$$\{(z, w) \mid \pi|z|^2 \leq A_1(N_0) \text{ and } N(w) = N_0\}$$

into the set

$$\{(z, w) \mid A_2(N_0) < \pi|z|^2 \leq A_1(N_0) + A_2(N_0) + \bar{\epsilon} \text{ and } N(w) = N_0\},$$

so it disjoins (7.4) from (7.5).

When $N_0 \notin C$, one of the sets (7.4) and (7.5) is empty. So the flow trivially disjoins the set (7.4) from the set (7.5). \square

8. Shortening on a manifold with an isolated maximum

In this section we prove Theorem 3.1 and Remarks 3.2 and 3.4. We recall the statement:

Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point and that at least one of the isotropy weights at that point has absolute value greater than one. Let d be a positive number that is smaller than the gap between the two largest critical values of the momentum map. Then the circle action can be deformed to a loop in $\text{Ham}(M, \omega)$ whose Hofer length is smaller than that of the original circle action by $2d/9$.

We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map, let p_{\max} denote the point where H attains its maximum, let α denote the gap between the two largest critical values of H , and consider the subset of M given by

$$(8.1) \quad \{m \mid H(m) > H(p_{\max}) - \alpha\}.$$

We can choose the deformation such that the normalized deformed Hamiltonians remain greater than $H(p_{\max}) - \alpha$ on the set (8.1) and coincide with H outside this set.

We actually obtain a better estimate than $\frac{2}{9}d$. If one of the isotropy weights is even then we can shorten by $\frac{1}{4}d$. Otherwise, let $-k_1$ be the weight whose absolute value is

largest; then we can shorten by $\frac{1}{4}(1 - \frac{1}{k_1^2})d$. When $k_1 = 3$, this becomes $\frac{2}{9}d$.

Proof of Theorem 3.1 and Remarks 3.2 and 3.4. Let $-k_1, \dots, -k_n$ be the isotropy weights at p_{\max} . The equivariant Darboux theorem implies that the integers k_j are positive. We assume that at least one of them is greater than one. Without loss of generality,

$$(8.2) \quad k_1 = a + b$$

where a and b are positive integers.

The point p_{\max} is the only critical point of H in the set (8.1), and the momentum map H is proper as a map from this set to the ray $(H(p_{\max}) - \alpha, \infty)$. There exists an equivariant symplectomorphism from the subset of \mathbb{C}^n given by

$$\mathcal{N}_\alpha^H = \{z \mid \pi(k_1|z_1|^2 + \dots + k_n|z_n|^2) < \alpha\},$$

with the circle action generated by the momentum map $H(z_1, \dots, z_n) = -k_1\pi|z_1|^2 - \dots - k_n\pi|z_n|^2$ to the subset (8.1) of M ; denote it

$$i: \mathcal{N}_\alpha^H \rightarrow M.$$

This follows from [9, Proposition 2.8] (which should have been stated with $|z_j|^2$ instead of the typo $|z_j|$).

Let

$$N(z_2, \dots, z_n) := \pi(k_2|z_2|^2 + \dots + k_n|z_n|^2).$$

We will apply Lemma 5.1 with $U = \mathcal{N}_\alpha^H$,

$$K(z_1, \dots, z_n) := H(p_{\max}) - \pi b|z_1|^2,$$

and

$$F(z_1, \dots, z_n) := -\pi a|z_1|^2 - N(z_2, \dots, z_n).$$

Let $0 < d < \alpha$ and let $s = \frac{2}{9}d$. By Lemma 5.1, to shorten by amount s , it is enough to disjoin the set $\mathcal{N}_s^K = \{z \mid \pi b|z_1|^2 < s\}$ from the set $\mathcal{N}_s^F = \{z \mid \pi a|z_1|^2 + N(z_2, \dots, z_n) < s\}$ by a compactly supported symplectic isotopy

of \mathcal{N}_α^H that preserves the function

$$\hat{N}(z_1, z_2, \dots, z_n) := N(z_2, \dots, z_n).$$

Write the sets \mathcal{N}_s^K and \mathcal{N}_s^F as

$$\mathcal{N}_s^K = \left\{ z \mid \left| \pi |z_1|^2 < \frac{s}{b} \right. \right\} \quad \text{and} \quad \mathcal{N}_s^F = \left\{ z \mid \left| \pi |z_1|^2 < \frac{s - N}{a} \right. \right\}.$$

By Lemma 7.1, with $A_1(N) = s/b$ and $A_2(N) = (s - N)/a$, for any open neighbourhood U of the set

$$(8.3) \quad \left\{ z \mid \left| \pi |z_1|^2 \leq \frac{s}{b} + \frac{s - N}{a} \right. \right\}$$

there exists a symplectic isotopy, compactly supported in U , that disjoins \mathcal{N}_s^F from \mathcal{N}_s^K and that preserves the function \hat{N} . So it is enough to show that the set (8.3) is contained in the set \mathcal{N}_α^H .

Because $a + b = k_1$, we can rewrite the set (8.3) as

$$(8.4) \quad \left\{ z \mid \left| \frac{ab}{sk_1} \pi |z_1|^2 + \frac{b}{sk_1} N \leq 1 \right. \right\}.$$

Also, we rewrite the set \mathcal{N}_α^H as

$$(8.5) \quad \left\{ z \mid \left| \frac{k_1}{\alpha} \pi |z_1|^2 + \frac{1}{\alpha} N < 1 \right. \right\}.$$

It remains to show that the set (8.4) is contained in the set (8.5). This is equivalent to requiring that

$$\frac{ab}{sk_1} \geq \frac{k_1}{\alpha} \quad \text{and} \quad \frac{b}{sk_1} \geq \frac{1}{\alpha}.$$

Because $a \leq k_1$, the first of these inequalities implies the second.

If k_1 is even, write $k_1 = a + b$ where $b = a$. Then $ab/k_1^2 = 1/4$, which is greater than $2/9$. If k_1 is odd, write $k_1 = a + b$ where $b = a + 1$. Then $ab/k_1^2 = 1/4(1 - 1/k_1^2)$ which, because $k_1 \geq 3$, is greater than or equal to $1/4(1 - 1/9) = 2/9$. In either case, if $s = 2/9d$ and $d < \alpha$, then $s < (ab/k_1^2)\alpha$, which is equivalent to the required inequality. \square

9. Symplectic vector bundles

In this section we apply the symplectic tubular neighbourhood theorem to describe a neighbourhood of a fixed component in a manifold with Hamiltonian circle action in terms of a symplectic vector bundle, and we give some results about symplectic vector bundles that we will use in the next section.

Lemma 9.1. *Let*

$$\pi: E \rightarrow B$$

be a symplectic vector bundle, i.e., a vector bundle with a fibrewise symplectic form. There exists on the total space E a closed two-form ω_Θ with the following properties.

- 1) *The pullback of ω_Θ to the fibres of E coincides with the fibrewise symplectic forms.*
- 2) *The pullback of ω_Θ by the zero section of E is zero.*
- 3) *At the points of the zero section, the fibres of E are ω_Θ -orthogonal to the zero section.*

Moreover, if a compact Lie group K acts on E by bundle automorphisms, then ω_Θ can be chosen to be K invariant.

Proof. Let $2n = \text{rank} E$, and let

$$G = \text{Sp}(\mathbb{R}^{2n})$$

be the group of symplectic linear transformations of \mathbb{R}^{2n} . Let $P \rightarrow B$ be the principal G bundle (with a right G action) whose fibre E_b over a point b of B consists of the set of linear symplectic isomorphisms from \mathbb{R}^{2n} to E_b .

We identify E with the associated bundle

$$P \times_G \mathbb{R}^{2n},$$

in which $[pa, z] = [p, az]$ for all $p \in P$, $z \in \mathbb{R}^{2n}$, and $a \in G$.

Let $\Theta \in \Omega^1(P, \mathfrak{g})$ be a connection one-form; use the same symbol, Θ , to denote its pullback to $P \times \mathbb{R}^{2n}$. Let ω_0 be the standard symplectic form on \mathbb{R}^{2n} . Let $\Phi_{\mathbb{R}^{2n}}: \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$ be the quadratic momentum map for the G -action on \mathbb{R}^{2n} . Let $\langle \cdot, \cdot \rangle$ denote the pairing between \mathfrak{g}^* and \mathfrak{g} . Then $\langle \Phi_{\mathbb{R}^{2n}}, \Theta \rangle$ is a G -invariant real valued one-form on $P \times \mathbb{R}^{2n}$. The difference $\omega_0 - d\langle \Phi_{\mathbb{R}^{2n}}, \Theta \rangle$ descends to a closed two-form ω_Θ on $P \times_G \mathbb{R}^{2n}$ with the desired properties.

If a compact Lie group K acts on E by bundle automorphisms, then, by averaging, we can choose the connection one-form Θ to be K -invariant. The resulting two-form ω_Θ is then K -invariant. \square

Let (B, ω_B) be a symplectic manifold, let $\pi: E \rightarrow B$ be a symplectic vector bundle, and let a compact Lie group K act on E by bundle automorphisms that descend to symplectomorphisms of (B, ω_B) . Let ω_Θ be a closed two-form as in Lemma 9.1. The corresponding *coupling form* is

$$(9.2) \quad \omega_E := \pi^* \omega_B + \omega_\Theta.$$

It has the following properties.

- 1) Its pullback to the fibres coincides with the fibrewise symplectic forms.
- 2) Its pullback by the zero section coincides with ω_B .
- 3) At the points of the zero section, the fibres of E are ω_E -orthogonal to the zero section.

Consequently, ω_E is non-degenerate near the zero section, and the normal bundle of the zero section in (E, ω_E) is isomorphic to $E \rightarrow B$ as symplectic vector bundles.

Remark 9.3. The construction that we just described is Shlomo Sternberg's *minimal coupling* [23]. For more details, see chapter 1 of the book [4] by Guillemin, Lerman, and Sternberg.

Lemma 9.4. *Let $\pi: E \rightarrow B$ be a symplectic vector bundle, and let the circle group act on E by fibrewise linear symplectic transformations. Let ω_E be an invariant closed two-form on the total space of E whose pullback to the fibres coincides with the fibrewise symplectic form. Let $H: E \rightarrow \mathbb{R}$ be the fibrewise quadratic momentum map for the fibrewise circle action. Then H is also a momentum map for the circle action on the total space (E, ω_E) .*

Proof. Let ξ denote the vector field that generates the circle action. The one-form $\iota(\xi)\omega_E$ is closed and vanishes on the zero section, so it is exact; so there exists a momentum map

$$H_E: E \rightarrow \mathbb{R}$$

for the circle action on (E, ω_E) . (ω_E need not be nondegenerate.) Because $\xi = 0$ along the zero section, H_E is locally constant along the zero section.

Adding a locally constant function, we may assume that H_E vanishes along the zero section. Because the pullback of ω_E to the fibres coincides with the fibrewise symplectic form, the restriction of H_E to each fibre is also a momentum map for the fibrewise circle action; because it vanishes on the zero section, it must coincide with the fibrewise quadratic momentum map, H . \square

Now let B be a connected component of the fixed point set of a circle action on a symplectic manifold (M, ω) , and let E be the normal bundle of B in M . Then B is a symplectic submanifold, and E can be identified with the symplectic orthocomplement of TB in $TM|_B$. Moreover, the circle action on M induces a circle action on E by bundle automorphisms. Let ω_E be an invariant coupling form on E , as described above. In this setting, we have the following symplectic tubular neighbourhood theorem.

(9.5) There exists an invariant neighbourhood U of the zero section in E and an equivariant symplectic open embedding $(U, \omega_E) \rightarrow (M, \omega)$ whose restriction to the zero section is the identity map on B and, under the natural identification of $TE|_B$ with $TM|_B$, whose differential is the identity map at every point of B .

(This follows from the classical tubular neighbourhood theorem in differential topology, combined with Theorem 4.1 of Alan Weinstein's paper [25], keeping track of a group action as explained in the last paragraph of Section 3 of [25]. Also see [26].)

Lemma 9.6. *Let $E \rightarrow B$ be a symplectic vector bundle with a fibrewise circle action. Suppose that all the weights are negative; denote the weights by $-k_1, \dots, -k_s$ and let*

$$k = \min\{k_1, \dots, k_s\} - 1.$$

Then there exist $k + 1$ commuting fibrewise circle actions on E , with fibrewise quadratic momentum maps

$$H_{(j)}: E \rightarrow \mathbb{R}, \quad 0 \leq j \leq k,$$

with the following properties.

(i) The H_j are negative outside the zero section, and their sum

$$H_{(0)} + H_{(1)} + \cdots + H_{(k)}$$

is the fibrewise quadratic momentum map for the given circle action on E .

(ii) There exist arbitrarily small neighbourhoods U of the zero section that are invariant under the $k + 1$ commuting fibrewise circle actions, whose intersections with the fibres of E are connected, and that satisfy

$$(9.7) \quad \inf_U \sum_{j=0}^k H_{(j)} = \sum_{j=0}^k \inf_U H_{(j)}.$$

Remark 9.8. This lemma is about fibrewise structures. It does not require a two-form on the total space of E .

Proof. Fix an invariant fibrewise compatible complex structure and Hermitian metric on E . (Because the weights are all negative, such a metric is unique.) After eliminating repetitions, we may assume that the positive numbers k_1, \dots, k_s are distinct. Let E_j denote the $-k_j$ th weight space for the circle action on E , so that E decomposes as $E_1 \oplus \cdots \oplus E_s$. The function

$$z \mapsto \pi \|z_j\|^2,$$

for $z = (z_1, \dots, z_s) \in E_1 \oplus \cdots \oplus E_s$, is a momentum map for the scalar multiplication circle action on the E_j factor. The quadratic fibrewise momentum map for the given circle action on E is

$$(9.9) \quad -k_1 \pi \|z_1\|^2 - \cdots - k_s \pi \|z_s\|^2.$$

For each j , because $k_j \geq k + 1$, we can decompose k_j as

$$k_j = a_{0j} + a_{1j} + \cdots + a_{kj}$$

where a_{ij} are positive integers. We get a decomposition of (9.9) into the sum

$$H_{(0)} + H_{(1)} + \cdots + H_{(k)},$$

where

$$H_{(i)}(z) = -a_{i1} \pi \|z_1\|^2 - \cdots - a_{is} \pi \|z_s\|^2$$

generates the fibrewise circle action on E that rotates the j th coordinate with speed $-a_{ij}$. Because the numbers a_{ij} are positive, the functions $H_{(j)}$ are negative outside the zero section.

Finally, every neighbourhood of the zero section contains a neighbourhood of the form $U = \bigcap_{j=1}^s \{z \mid \pi \|z_j\|^2 < \epsilon\}$, and such a U satisfies the requirements of part (ii) of the lemma. \square

Lemma 9.10. *Let (B, ω_B) be a symplectic manifold, let $E \rightarrow B$ be a symplectic vector bundle, let ω_E be a coupling form (see (9.2)), and let U be an open subset of E where ω_E is non-degenerate. Let k be a positive integer. Suppose that there exist k sections of E that are nowhere all vanishing. Then there exist smooth functions $g_1, \dots, g_k: U \rightarrow \mathbb{R}$ whose Hamiltonian vector fields are nowhere all tangent to the zero section of E .*

Proof. Let ξ_1, \dots, ξ_k be sections of $E \rightarrow B$ that are nowhere all vanishing. Fix $j \in \{1, \dots, k\}$. Let $g_j: E \rightarrow \mathbb{R}$ be the function whose restriction to the fibre over b is the linear functional $\iota(\xi_j(b))\omega_{E_b}$, where ω_{E_b} is the symplectic form on the fibre over b . Identifying E with the vertical subbundle of $TE|_B$, the Hamiltonian vector field of $g_j|_U$ is equal to ξ_j at the points of the zero section. Because ξ_1, \dots, ξ_k are vertical and nowhere all vanishing, they are nowhere all tangent to the zero section. \square

Remark 9.11. Each of the following assumptions on an oriented vector bundle $E \rightarrow B$ implies that there exist k sections that are nowhere all vanishing. Let $e(E)$ denote the Euler class of the bundle.

- (a) $\dim B < k \operatorname{rank} E$.
- (b) $\dim B = k \operatorname{rank} E$ and $e(E)^k = 0$.
- (c) $\operatorname{rank} E = 2$, $e(E) = 0$, and $k \geq 1$.

Proof. If $\dim B < k \operatorname{rank} E$ then, for any generic choice of k sections of E , these sections are nowhere all vanishing. This gives (a). If the rank of a bundle is equal to the dimension of its base, or if the rank of a bundle is equal to 2, then the Euler class being zero implies the existence of a non-vanishing section. Applying this to the bundle $\underbrace{E \oplus \dots \oplus E}_{k \text{ times}}$ gives (b); applying it to E gives (c). \square

In preparation for the proof of Theorem 3.9, which involves a *subbundle* of the normal bundle, we now give a refinement of the minimal coupling

construction. Let (B, ω_B) be a symplectic manifold, let $\pi: E \rightarrow B$ be a symplectic vector bundle, and let a compact Lie group K act on E by symplectic bundle automorphisms. Let E' be a K -invariant symplectic sub-bundle of E .

Let E'' be the symplectic orthocomplement of E' in E . Then $E = E' \oplus E''$, and we have a pullback diagram:

$$(9.12) \quad \begin{array}{ccc} E & \xrightarrow{p''} & E'' \\ p' \downarrow & & \downarrow \pi'' \\ E' & \xrightarrow{\pi'} & B. \end{array}$$

Let $\omega_{\Theta'}$ and $\omega_{\Theta''}$ denote K -invariant two-forms on E' and E'' with the properties listed in Lemma 9.1. The corresponding coupling forms on E' and on E are

$$\omega_{E'} = \pi'^* \omega_B + \omega_{\Theta'}$$

and

$$(9.13) \quad \begin{aligned} \omega_E &= \pi^* \omega_B + p'^* \omega_{\Theta'} + p''^* \omega_{\Theta''} \\ &= p'^* \omega_{E'} + p''^* \omega_{\Theta''}. \end{aligned}$$

The proof of Theorem 3.9 will use the following lemma. In it, we view E' as the subset of E consisting of the set of points whose fibrewise E'' coordinate is zero, i.e., the preimage under p'' of the zero section of E'' .

Lemma 9.14. *Let (B, ω_B) be a symplectic manifold, let $E \rightarrow B$ be a symplectic vector bundle, and let E' and E'' be symplectic sub-bundles such that $E = E' \oplus E''$. Let $\omega_{E'}$ be a coupling form on E' and let ω_E be a coupling form on E that is compatible with $\omega_{E'}$ in the sense of (9.13). Let U' be a neighbourhood of the zero section in E' where $\omega_{E'}$ is non-degenerate, and let U be a neighbourhood of the zero section in E where ω_E is non-degenerate and that is contained in $p'^{-1}(U')$.*

Let $g'_1, \dots, g'_k: U' \rightarrow \mathbb{R}$ be smooth functions whose Hamiltonian vector fields in U' are nowhere all tangent to the zero section of E' . Let $g_1, \dots, g_k: U \rightarrow \mathbb{R}$ be their pullbacks under $p'|_U: U \rightarrow U'$. Then, at each point of the zero section of E , the Hamiltonian vector fields of g_1, \dots, g_k are tangent to E' and are not all tangent to the zero section of E .

Proof. Viewing E' as a subset of E (namely, the p'' -preimage of the zero section of E''), for every point $q \in E'$ we have a natural decomposition

$$(9.15) \quad TE|_q = TE'|_q \oplus E''|_{\pi(q)}.$$

Let ξ'_1, \dots, ξ'_k be the Hamiltonian vector fields of g'_1, \dots, g'_k in U' . Let $\tilde{\xi}'_1, \dots, \tilde{\xi}'_k$ be the corresponding sections of $TE|_{E' \cap U}$. That is, under the decomposition (9.15), we have

$$\tilde{\xi}'_j = \xi'_j \oplus 0.$$

In particular, $\tilde{\xi}'_j$ are tangent to the subset E' of E , and they satisfy $p'_* \tilde{\xi}'_j = \xi'_j$. The vectors $p''_* \tilde{\xi}'_j$ are tangent to the zero section of E'' , so they are in the null space of $\omega_{\Theta''}$. It is enough to show that, along $E' \cap U$, the Hamiltonian vector fields of g_1, \dots, g_k are equal to ξ'_1, \dots, ξ'_k . Fix $j \in \{1, \dots, k\}$. We now compute: along $E' \cap U$,

$$\begin{aligned} dg_j &= p'^* dg'_j && \text{by the definition of } g_j \\ &= -p'^* \iota(\xi'_j) \omega_{E'} && \text{by the definition of } \xi'_j \\ &= -\iota(\tilde{\xi}'_j) p'^* \omega_{E'} && \text{because } p'_* \tilde{\xi}'_j = \xi'_j \\ &= -\iota(\tilde{\xi}'_j) (\omega_E - p''^* \omega_{\Theta''}) && \text{by (9.13)} \\ &= -\iota(\tilde{\xi}'_j) \omega_E && \text{because } p''_* \tilde{\xi}'_j \text{ is in the null space of } \omega_{\Theta''}. \end{aligned}$$

So $dg_j = -\iota(\tilde{\xi}'_j) \omega_E$, as required. □

We will also use the following variant of Lemma 9.6.

Lemma 9.16. *Let $E \rightarrow B$ be a symplectic vector bundle with a fibrewise circle action whose weights are all negative. Let*

$$E = E' \oplus E''$$

be a decomposition of E into two invariant symplectic subbundles that are fibrewise symplectically orthogonal. Let $-k_1, \dots, -k_s$ be the distinct weights for the circle action on E' . Let

$$k = \min\{k_1, \dots, k_s\} - 1.$$

Then there exist $k + 1$ commuting fibrewise circle actions on E , with fibrewise quadratic momentum maps

$$H_{(j)}: E \rightarrow \mathbb{R}, \quad 0 \leq j \leq k,$$

that preserve the decomposition $E = E' \oplus E''$, that commute with the given circle action on each of E' and E'' , and that have the following properties.

- (i) The function $H_{(0)}$ vanishes on the zero section on E and is negative outside it, the functions $H_{(1)}, \dots, H_{(k)}$ all vanish on E'' and are negative outside it, and the sum

$$H_{(0)} + H_{(1)} + \dots + H_{(k)}$$

is the fibrewise quadratic momentum map for the given circle action on E .

- (ii) There exist arbitrarily small neighbourhoods U of the zero section in E that are invariant under these $k + 1$ circle actions and under the given circle actions on E' and on E'' , whose intersections with the fibres of E are connected, and that satisfy

$$(9.17) \quad \inf_U \sum_{j=0}^k H_{(j)} = \sum_{j=0}^k \inf_U H_{(j)}.$$

Proof. Fix an invariant fibrewise compatible complex structure and Hermitian metric on each of E' and E'' . Let E_j denote the $-k_j$ th weight space for the circle action on E' , so that E decomposes as $E_1 \oplus \dots \oplus E_s \oplus E''$. The function

$$z \mapsto \pi \|z_j\|^2,$$

for $z = (z_1, \dots, z_s, z'') \in E_1 \oplus \dots \oplus E_s \oplus E''$, is a momentum map for the scalar multiplication circle action on the E_j factor. The quadratic fibrewise momentum map for the given circle action on E is

$$-k_1 \pi \|z_1\|^2 - \dots - k_s \pi \|z_s\|^2 - \hat{N}(z'')$$

where $\hat{N}: E'' \rightarrow \mathbb{R}$ is the negative of the fibrewise quadratic momentum map for the fibrewise circle action on E'' . Decompose each k_j as

$$k_j = a_{0j} + a_{1j} + \dots + a_{kj}$$

where a_{ij} are positive integers. The lemma then holds with

$$H_{(i)}(z) = -a_{i1} \pi \|z_1\|^2 - \dots - a_{is} \pi \|z_s\|^2$$

for $1 \leq i \leq k$, with

$$H_{(0)}(z) = -a_{01} \pi \|z_1\|^2 - \dots - a_{0s} \pi \|z_s\|^2 - \hat{N}(z''),$$

and with neighbourhoods of the zero section of the form

$$U = \{z \mid \pi \|z_j\|^2 < \epsilon \text{ for } j = 1, \dots, s, \text{ and } \hat{N}(z'') < \epsilon\}$$

for small $\epsilon > 0$. □

10. Shortening on a manifold with an arbitrary maximum

In this section we prove Theorems 3.7 and 3.9, with Remarks 3.8 and 3.11. We begin by recalling the statement of Theorem 3.7 and Remark 3.8:

Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Let B_{\max} be the set where the momentum map is maximal. Let $-k_1, \dots, -k_s$ denote the distinct weights for the circle action on the normal bundle of B_{\max} , and let $k := \min\{k_1, \dots, k_s\} - 1$. Suppose that B_{\max} is symplectically k -displaceable in every neighbourhood. Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ into a loop of smaller Hofer length.

We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonians remain between $H(B_{\max}) - \alpha$ and $H(B_{\max})$ on the set $\{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and coincide with H outside this set.

Proof of Theorem 3.7 and Remark 3.8. Let E denote the normal bundle of B_{\max} in (M, ω) ; recall that its fibrewise symplectic structure and circle action are induced from M . Apply Lemma 9.6 to obtain $k + 1$ commuting fibrewise circle actions on E , with fibrewise quadratic momentum maps

$$H'_{(j)}: E \rightarrow \mathbb{R}, \quad 0 \leq j \leq k$$

that are negative outside the zero section, whose sum

$$H'_{(0)} + H'_{(1)} + \dots + H'_{(k)}$$

is the fibrewise quadratic momentum map for the given circle action on E (the one induced from M), and such that part (ii) of Lemma 9.6 holds (with “ $H_{(j)}$ ” replaced by “ $H'_{(j)}$ ”).

These circle actions fit together into a (not necessarily faithful) action of the torus $T := (S^1)^{k+1}$, whose diagonal acts by the given circle action on E . Apply Lemma 9.1 and Equation (9.2), with $K = T$, to obtain a T -invariant coupling form ω_E on E . Consider the given circle action on M and the induced circle action on E , and apply the symplectic tubular neighbourhood theorem (9.5), to obtain an invariant neighbourhood U of the zero section in E on which ω_E is nondegenerate and an equivariant open symplectic embedding

$$i: (U, \omega_E) \rightarrow (M, \omega)$$

whose restriction to the zero section is the identity map on B_{\max} and, under the natural identification of $TE|_{B_{\max}}$ with $TM|_{B_{\max}}$, whose differential is the identity map at every point of B_{\max} .

Fix a positive number, $\alpha > 0$. After possibly shrinking U , we may assume that $i(U) \subset \{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$, and that U satisfies the requirements in part (ii) of Lemma 9.6: U is invariant under the entire T action on E , the intersections of U with the fibres of E are connected, and

$$\inf_U \sum_{j=0}^k H'_{(j)} = \sum_{j=0}^k \inf_U H'_{(j)}.$$

Let $H: M \rightarrow \mathbb{R}$ be the momentum map on M . We have

$$(10.1) \quad \begin{aligned} i^*H &= \left(H(B_{\max}) + H'_{(0)} + H'_{(1)} + \cdots + H'_{(k)} \right) \Big|_U \\ &= H_{(0)} + H_{(1)} + \cdots + H_{(k)}, \end{aligned}$$

where $H_{(0)} = H(B_{\max}) + H'_{(0)}|_U$ and where $H_{(j)} = H'_{(j)}|_U$ for each $1 \leq j \leq k$. (The intersection with U of each fibre of E is an invariant connected open neighbourhood of the origin in the fibre. In this neighbourhood, the left and right hand sides of (10.1) are momentum maps for the same circle action with the same value at the origin, so they are equal.)

Because the T action on E preserves ω_E and by Lemma 9.4, the maps $H'_{(0)}, H'_{(1)}, \dots, H'_{(k)}$ are momentum maps for the $k+1$ commuting circle actions on the total space (E, ω_E) and not only fibrewise. So $H_{(0)}, H_{(1)}, \dots, H_{(k)}$ generate commuting circle actions on (U, ω_E) .

We are now in the setup of Lemma 5.10. The sets $B_{\max}^{(j)}$ where the functions $H_{(j)}$ attain their maximum are all equal to the zero section of E . This implies the condition (5.12) of Lemma 5.10. It also reduces the last condition of Lemma 5.10 to the condition that the zero section is symplectically

k -displaceable in U . This condition then follows from that B_{\max} is symplectically k -displaceable in $i(U)$. Theorem 3.7 and Remark 3.8 then follows from Lemma 5.10. \square

Next, we prove Theorem 3.9 and Remark 3.11. We recall the statement:

Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian circle action. Let B_{\max} be the set where the momentum map is maximal. Let $E' \rightarrow B_{\max}$ be an S^1 -invariant subbundle of the normal bundle to B_{\max} in M , and let $-k'_1, \dots, -k'_{s'}$ be the distinct weights for the circle action on E' . Let

$$k' = \min\{k'_1, \dots, k'_{s'}\} - 1.$$

Suppose that

(10.2) E' has k' sections that are nowhere all vanishing.

Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ into a loop of smaller Hofer length.

We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let H denote the momentum map. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonians remain between $H(B_{\max}) - \alpha$ and $H(B_{\max})$ on the set $\{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and coincide with H outside this set.

Proof of Theorem 3.9 and Remark 3.11. Let E denote the normal bundle of B_{\max} in (M, ω) ; recall that its fibrewise symplectic structure and circle action are induced from M . Let E'' be the fibrewise symplectic orthocomplement of E' in E , so $E = E' \oplus E''$. Apply Lemma 9.16 to obtain $k' + 1$ commuting fibrewise circle actions on E , with fibrewise quadratic momentum maps

$$H'_{(j)}: E \rightarrow \mathbb{R} \quad , \quad 0 \leq j \leq k'$$

such that $H'_{(0)}$ vanishes on the zero section of E and is negative outside it, $H'_{(1)}, \dots, H'_{(k')}$ vanish on E'' and are negative outside it, the sum $H'_{(0)} + H'_{(1)} + \dots + H'_{(k')}$ is the fibrewise quadratic momentum map for the given circle action on E , and part (ii) of Lemma 9.16 holds (with “ k ” and “ $H_{(j)}$ ” replaced by “ k' ” and “ $H'_{(j)}$ ”).

These $k' + 1$ circle actions, together with the given circle actions on E' and on E'' , fit together into a fibrewise action of a torus T . Apply Lemma 9.1 and Equations (9.2) and (9.13) to obtain T invariant coupling forms $\omega_{E'}$ on E' and ω_E on E . Consider the given circle action on M and the induced circle action on E , and apply the symplectic tubular neighbourhood theorem (9.5) to obtain an invariant neighbourhood U of the zero section in E on which ω_E is nondegenerate and an equivariant symplectic open embedding

$$i: (U, \omega_E) \rightarrow (M, \omega)$$

whose restriction to the zero section is the identity map on B_{\max} and, under the natural identification of $TE|_{B_{\max}}$ with $TM|_{B_{\max}}$, whose differential is the identity map at every point of B_{\max} .

Fix a positive number, $\alpha > 0$. After possibly shrinking U , we may assume that $i(U) \subset \{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and that U satisfies the requirements in part (ii) of Lemma 9.16: U is T invariant, the intersections of U with the fibres of E are connected, and

$$\inf_U \sum_{j=0}^{k'} H'_{(j)} = \sum_{j=0}^{k'} \inf_U H'_{(j)}.$$

Let $H: M \rightarrow \mathbb{R}$ denote the momentum map on M . We have

$$(10.3) \quad \begin{aligned} i^* H &= \left(H(B_{\max}) + H'_{(0)} + \cdots + H'_{(k')} \right) \Big|_U \\ &= H_{(0)} + H_{(1)} + \cdots + H_{(k')}, \end{aligned}$$

where

$$H_{(0)} = \left(H(B_{\max}) + H'_{(0)} \right) \Big|_U$$

and where

$$H_{(j)} = \left(H'_{(j)} \right) \Big|_U \quad \text{for each } 1 \leq j \leq k'.$$

(The intersection of U with each fibre of E is an invariant connected open neighbourhood of the origin in the fibre. In this neighbourhood, the left and right hand sides of (10.3) are momentum maps for the same circle action with the same value at the origin, so they are equal.)

Because the T action on E preserves ω_E and by Lemma 9.4, the maps $H'_{(0)}, H'_{(1)}, \dots, H'_{(k')}$ are momentum maps for the $k' + 1$ commuting circle actions on the total space (E, ω_E) and not only fibrewise. So $H_{(0)}, H_{(1)}, \dots, H_{(k')}$ generate commuting circle actions on (U, ω_E) .

We are now in the setup of Lemma 5.10. The set $B_{\max}^{(0)}$ where $H_{(0)}$ attains its maximum is equal to the zero section, and, for $1 \leq j \leq k'$, the set $B_{\max}^{(j)}$ where $H_{(j)}$ attains its maximum is the intersection of U with the image of E'' in E . Slightly abusing notation, we write these sets as B_{\max} and as $E'' \cap U$. To apply Lemma 5.10, we need to find symplectomorphisms $b_1, \dots, b_{k'}$ of U , each connected to the identity through a path of compactly supported symplectomorphisms of U , such that

$$(10.4) \quad B_{\max} \cap b_1(E'' \cap U) \cap \dots \cap b_{k'}(E'' \cap U) = \emptyset.$$

Let U' be a neighbourhood of the zero section in E' where $\omega_{E'}$ is non-degenerate. By Lemma 9.10, there exist functions $g'_1, \dots, g'_{k'}$ on U' whose Hamiltonian vector fields are nowhere all tangent to the zero section of E' . For $j = 1, \dots, k'$, let $g_j = \rho(g'_j \circ p')$ where $p': E \rightarrow E'$ is the projection map and where $\rho: U \rightarrow \mathbb{R}$ is a cutoff function that is equal to 1 near the zero section of E and has compact support in U . Lemma 9.14 implies that, at each point of B_{\max} , the Hamiltonian vector fields of $g_1, \dots, g_{k'}$ are tangent to E' and are not all tangent to B_{\max} . Let $\psi_t^{g_1}, \dots, \psi_t^{g_{k'}}$ be the corresponding flows. If t is a sufficiently small positive number, the intersection $B_{\max} \cap \psi_t^{g_1}(E'' \cap U) \cap \dots \cap \psi_t^{g_{k'}}(E'' \cap U)$ is empty. Lemma 5.10, with $b_1, \dots, b_{k'}$ taken to be $\psi_t^{g_1}, \dots, \psi_t^{g_{k'}}$, gives the result of Theorem 3.9 and Remark 3.11. □

11. The index of the positive Hofer length functional

In this section we prove Theorem 3.17 and Remark 3.18. We recall the statement, with slight change of notation (H_M instead of H):

Let (M, ω) be a $2n$ dimensional compact connected manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point. Let $-k_1, \dots, -k_n$ be the isotropy weights at the maximum, with possible repetitions. Then there exists a neighbourhood D of the origin in $\mathbb{R}^{\sum 2(k_j-1)}$, and, for each $\lambda \in D$, a loop $\{\psi_t^{(\lambda)}\}_{0 \leq t \leq 1}$ in $\text{Ham}(M, \omega)$, such that the following properties hold. For $\lambda = 0$, the loop $\{\psi_t^{(0)}\}_{0 \leq t \leq 1}$ is the given circle action. The function $\lambda \mapsto \text{length}(\{\psi_t^{(\lambda)}\})$ is smooth, $\lambda = 0$ is a critical point of this function, and the Hessian of this function at $\lambda = 0$ is negative definite.

In fact, we can choose the deformation such that the negative Hofer length remains constant. Specifically, let H_M denote the momentum map, and let p_{\max} denote the point where H_M attains its maximum. For every positive number α , we can choose the deformation such that the normalized deformed Hamiltonian remains between $H_M(p_{\max}) - \alpha$ and $H_M(p_{\max})$ on the set $\{m \in M \mid H_M(m) > H_M(p_{\max}) - \alpha\}$ and coincides with H_M outside this set.

Proof of Theorem 3.17 and Remark 3.18. We set the following notation.

$$\begin{aligned} \phi_t: \mathbb{C} &\rightarrow \mathbb{C}, & \phi_t(z) &= e^{-2\pi it} z & \text{for } t \in \mathbb{R}. \\ \beta_\lambda: \mathbb{C} &\rightarrow \mathbb{C}, & \beta_\lambda(z) &= z + \lambda & \text{for } \lambda \in \mathbb{C}. \end{aligned}$$

Let p_{\max} denote the point where the momentum map H_M attains its maximum. By the equivariant Darboux theorem, near p_{\max} , we can identify the manifold with \mathbb{C}^n , the action with the product action

$$(11.1) \quad (\phi_{k_1 t}) \times \cdots \times (\phi_{k_n t}),$$

and the momentum map with the function

$$z \mapsto H_M(p_{\max}) - N(z),$$

where

$$(11.2) \quad N(z) = \pi (k_1 |z_1|^2 + \cdots + k_n |z_n|^2).$$

(We will use variants of the symbol N to denote variations and combinations of norm-square functions, and we will use variants of the symbol H to denote momentum maps.)

If k is an integer greater than 1, the non-effective circle action ϕ_{kt} on \mathbb{C} has the deformation

$$(11.3) \quad \overline{\phi}_t^{(\lambda)} := \phi_t \circ \beta_{\lambda_{k-1}} \circ \phi_t \circ \cdots \circ \beta_{\lambda_2} \circ \phi_t \circ \beta_{\lambda_1} \circ \phi_t \circ \beta_{-\lambda_1 - \cdots - \lambda_{k-1}},$$

parametrized by $(\lambda) = (\lambda_1, \dots, \lambda_{k-1}) \in \mathbb{C}^{k-1}$ and generated by

$$(11.4) \quad \overline{H}_t^{(\lambda)} := H + H\beta_{\lambda_{k-1}}^{-1}\phi_t^{-1} + H\beta_{\lambda_{k-2}}^{-1}\phi_t^{-1}\beta_{\lambda_{k-1}}^{-1}\phi_t^{-1} + \cdots \\ \cdots + H\beta_{\lambda_1}^{-1}\phi_t^{-1}\beta_{\lambda_2}^{-1}\phi_t^{-1} \cdots \beta_{\lambda_{k-1}}^{-1}\phi_t^{-1},$$

where $H(z) = -\pi|z|^2$. (We will use the symbol (λ) , with brackets, for a parameter in \mathbb{C}^{k-1} or in $\mathbb{C}^{\Sigma(k_i-1)}$ and the symbol λ , without brackets, for a parameter in \mathbb{C} .)

Applying such a deformation to each factor of (11.1), we get a family of deformations of the product action. This family is parametrized by elements of $\mathbb{C}^{\Sigma(k_i-1)}$. When we view the parameter (λ) as an element $((\lambda)_1, \dots, (\lambda)_n)$ of the n -fold product $\mathbb{C}^{k_1-1} \times \dots \times \mathbb{C}^{k_n-1}$, the deformation is

$$(11.5) \quad \overline{\phi}^{(\lambda)_1} \times \dots \times \overline{\phi}^{(\lambda)_n},$$

and it is generated by

$$(11.6) \quad H_t^{(\lambda)}(z) := \overline{H}_t^{(\lambda)_1}(z_1) + \dots + \overline{H}_t^{(\lambda)_n}(z_n).$$

With the help of cut-off functions, this deformation can be plugged into the original manifold to obtain a family of deformations of the original circle action, parametrized by λ s in a neighbourhood D of the origin in $\mathbb{C}^{\Sigma(k_i-1)}$, and generated by a function that in a neighbourhood of p can be identified with $z \mapsto H_M(p_{\max}) + H_t^{(\lambda)}(z)$ and that attains its maximum in that neighbourhood. We give the details later.

We now turn to the relevant computation on \mathbb{C}^n . We consider the function

$$(\lambda) \mapsto \int_0^1 \left(\max_{z \in \mathbb{C}^n} H_t^{(\lambda)}(z) \right) dt$$

from $\mathbb{C}^{\Sigma(k_i-1)}$ to \mathbb{R} . By (11.4) and (11.6), this function is everywhere non-positive and vanishes at the origin. We will show that this function is quadratic in the real and imaginary parts of λ , and that it is strictly negative when $\lambda \neq 0$. This will imply that the function is smooth and that its Hessian at the origin is negative definite, as required.

By (11.6), it is enough to consider separately each of the functions $(\lambda)_i \mapsto \int_0^1 \max_{z \in \mathbb{C}} \overline{H}_t^{(\lambda)_i} dt$. Omitting the index i , for each $0 \leq t \leq 1$ we consider the function

$$(11.7) \quad (\lambda) \mapsto \int_0^1 \left(\max_{z \in \mathbb{C}} \overline{H}_t^{(\lambda)} \right) dt$$

from \mathbb{C}^{k-1} to \mathbb{R} . We claim that it is a quadratic function of the real and imaginary parts of λ whose coefficients are trigonometric polynomials in t , and that it is everywhere non-positive, vanishes at $\lambda = 0$, and strictly negative for $\lambda \neq 0$ when $t = 0$.

Expanding the expression (11.4), we get that $\overline{H}_t^{(\lambda)}(z)$ is equal to $-\pi$ times

$$(11.8) \quad |z|^2 + |e^{2\pi it}z - \lambda_{k-1}|^2 + |e^{2\pi i \cdot 2t}z - e^{2\pi i \cdot t}\lambda_{k-1} - \lambda_{k-2}|^2 + \dots \\ + |e^{2\pi i(k-1)t}z - e^{2\pi i(k-2)t}\lambda_{k-1} - \dots - e^{2\pi i \cdot 2t}\lambda_3 - e^{2\pi it}\lambda_2 - \lambda_1|^2.$$

Expanding further, this is equal to

$$|z|^2 + |z|^2 + |\lambda_{k-1}|^2 - 2\text{Re}(e^{2\pi it}z \cdot \lambda_{k-1}) \\ + |z|^2 + |e^{2\pi it}\lambda_{k-1} + \lambda_{k-2}|^2 - 2\text{Re}(e^{2\pi i \cdot 2t}z \cdot (e^{2\pi it}\lambda_{k-1} + \lambda_{k-2})) \\ + \dots + |z|^2 + |e^{2\pi i(k-2)t}\lambda_{k-1} + \dots + e^{2\pi i \cdot 2t}\lambda_3 + e^{2\pi it}\lambda_2 + \lambda_1|^2 \\ - 2\text{Re}(e^{2\pi i(k-1)t}z \cdot (e^{2\pi i(k-2)t}\lambda_{k-1} + \dots + e^{2\pi i \cdot 2t}\lambda_3 + e^{2\pi it}\lambda_2 + \lambda_1)).$$

Thus, $\overline{H}_t^{(\lambda)}(x + iy)$ is equal to $-\pi$ times $kx^2 + ky^2 + ax + by + c$, where each of a, b is a linear function of the real and imaginary parts of $\lambda_1, \dots, \lambda_{k-1}$ whose coefficients are trigonometric polynomials in t and where c is a quadratic function of the real and imaginary parts of $\lambda_1, \dots, \lambda_{k-1}$ whose coefficients are trigonometric polynomials in t . Completing the square, this function attains its minimum when $x = -\frac{a}{2k}$ and $y = -\frac{b}{2k}$, and its minimal value is $c - \frac{a^2}{4k} - \frac{b^2}{4k}$ which, as required, is a quadratic function of the real and imaginary parts of $\lambda_1, \dots, \lambda_{k-1}$ whose coefficients are trigonometric polynomials in t .

Finally, if $\lambda \neq 0$, then at least one of the summands in (11.8) is strictly positive when $z = 0$, whereas the first summand is strictly positive when $z \neq 0$, so the minimal value of the function (11.8) is strictly positive.

This completes our computation in \mathbb{C}^n . We now return to our manifold M . Fix a symplectomorphism $f: U \rightarrow M$ from a neighbourhood U of the origin in \mathbb{C}^n onto a neighbourhood of p_{\max} in M such that $f(0) = p_{\max}$ and such that $(f^*H_M)(z) = H_M(p_{\max}) - N(z)$, where $N(\cdot)$ is given in (11.2). Given a positive number α , we may choose U such that $i(U) \subset \{m \in M \mid H_M(m) > H_M(p_{\max}) - \alpha\}$.

Let $\bar{\epsilon}$ be a positive number that is sufficiently small so that the set $\{z \in \mathbb{C}^n \mid N(z) \leq \bar{\epsilon}\}$ is contained in U . Then the subset of M where H_M is $\bar{\epsilon}$ -close to its maximum $H_M(p)$ is contained in $f(U)$. (See the facts about Hamiltonian circle actions in §2.)

Let $\rho: [0, \infty) \rightarrow [0, 1]$ be a smooth function that is equal to one on $[0, \bar{\epsilon}/2]$ and is equal to zero outside $[0, \bar{\epsilon}]$. Use the same symbol, ρ , to denote the function $z \mapsto \rho(N(z))$ on \mathbb{C}^n . Then the function $H_t^{(\rho \cdot \lambda)}(z)$ is equal to $H_t^{(\lambda)}(z)$ when $N(z) \leq \bar{\epsilon}/2$ and is equal to $-N(z)$ when $N(z) > \bar{\epsilon}$.

The function on M

$$(11.9) \quad H_{M_t}^{(\lambda)} := \begin{cases} H_M(p_{\max}) + H_t^{(\rho \cdot \lambda)}(\cdot) \circ f^{-1} & \text{when } H_M \text{ is } \bar{\epsilon}\text{-close to its maximum} \\ H_M & \text{elsewhere} \end{cases}$$

is smooth and it generates a deformation of our loop.

If the original momentum map H_M is normalized, so is the deformed Hamiltonian (11.9). The argument is similar to (5.18).

Let D be a neighbourhood of the origin in $\mathbb{C}^{\Sigma(k_i-1)}$ that is small enough so that, for every $\lambda \in D$, the following facts are true.

- For every t , the function $H_t^{(\lambda)}(z)$ attains its maximum at a unique point of \mathbb{C}^n , and this point lies in $\{z \mid N(z) \leq \bar{\epsilon}/4\}$.
- This maximum is greater than $-\bar{\epsilon}/4$.
- For every ν and z such that $0 \leq \nu \leq 1$ and $\bar{\epsilon}/2 \leq N(z) \leq \bar{\epsilon}$, we have $H_t^{(\nu \cdot \lambda)}(z) \leq -\bar{\epsilon}/4$ for all t .

This is possible because $H_t^{(\lambda)}(z)$ is smooth in the variables λ , z , and t , is quadratic in z , and is equal to $-N(z)$ when $\lambda = 0$.

Then, for every λ and t , the functions $H_t^{(\rho \cdot \lambda)}(z)$ and $H_t^{(\lambda)}(z)$ have the same maximal value, and they attain it at the same (unique) point z .

Denote by $\psi_t^{(\lambda)}$ the family of loops in $\text{Ham}(M, \omega)$ that are generated by the functions (11.9). By our choice of D , for every $\lambda \in D$, the positive Hofer length of the corresponding loop is

$$\ell_+(\{\psi_t^{(\lambda)}\}) = H_M(p_{\max}) + \int_0^1 \max_{z \in \mathbb{C}^n} H_t^{(\lambda)}(z) dt.$$

By this and our computation in \mathbb{C}^n , the function

$$\lambda \mapsto \ell_+(\{\psi_t^{(\lambda)}\})$$

from D to \mathbb{R} takes its maximal value at the origin and its Hessian at the origin is negative definite. This completes the proof of Theorem 3.17 and Remark 3.18. □

References

- [1] M. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.
- [2] A. Banyaga, *Sur la structure du groupe des diffeomorphismes qui prservent une forme symplectique*, Comment. Math. Helv. **53** (1978), 174–227.
- [3] M. Bialy and L. Polterovich, *Geodesics of Hofer’s metric on the group of Hamiltonian diffeomorphisms*, Duke Math. J. **76** (1994), 273–292.
- [4] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, New York, 1996.
- [5] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
- [6] H. Hofer, *On the topological properties of symplectic maps*, Proceedings of the Royal Society of Edinburgh 115 A (1990), 25–28.
- [7] H. Hofer, *Estimates for the energy of a symplectic map*, Comment. Math. Helv. **68** (1993), 48–72.
- [8] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhauser, Boston, MA (1994).
- [9] Y. Karshon and S. Tolman, *The Gromov width of complex Grassmannians*, Algebraic and Geometric Topology **5** (2005), 911–922.
- [10] F. Lalonde and D. McDuff, *Hofer’s L^∞ -geometry: energy and stability of Hamiltonian flows parts I and II*, Invent. Math. **122** (1995), 1–33 and 35–69.
- [11] F. Lalonde and D. McDuff, *Local non-squeezing theorems and stability*, Geom. and Func. Analysis vol. **5**, No. 2 (1995), 364–386.
- [12] Yi Ming Long, *Geodesics in the compactly supported Hamiltonian diffeomorphisms group*, Math. Z. **220** (1995), no. 2, 279–294.
- [13] D. McDuff, *Geometric variants of the Hofer norm*, J. Sympl. Geom. **1** (2002), p.197–252.
- [14] D. McDuff and J. Slimowitz, *Hofer-Zehnder capacity and length minimizing Hamiltonian paths*, Geom. Topol. **5** (2001), 799–830.

- [15] D. McDuff and S. Tolman, *Topological properties of Hamiltonian circle actions*, IMRP Int. Math. Res. Pap., **2006** (2006), 1–77.
- [16] D. McDuff and S. Tolman, *On nearly semifree circle actions*, [arXiv: math/0503467](https://arxiv.org/abs/math/0503467) [math.SG].
- [17] J. Milnor, *Morse theory*, Princeton University Press, Princeton, N.J., 1969.
- [18] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics ETH Zrich Birkhäuser Verlag, Basel, 2001.
- [19] Y. Savelyev, *Quantum characteristic classes and the Hofer metric*, Geom. Topol. **12** (2008), no. 4, 2277–2326.
- [20] Y. Savelyev, *Virtual Morse theory on $\Omega\text{Ham}(M, \omega)$* , J. Diff. Geom. **84** (2010), no. 2, 409–425.
- [21] Y. Savelyev, *Proof of the index conjecture in Hofer geometry*, Math. Res. Lett. **20**(5) (2013), 981–984.
- [22] K. Siburg, *New minimal geodesics in the group of symplectic diffeomorphisms*, Calc. Var. Part. Diff. Eq. **3** (1995), 299–309.
- [23] S. Sternberg, *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang–Mills field*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 23, 5253–5254.
- [24] I. Ustilovsky, *Conjugate points on geodesics of Hofer’s metric*, Differential Geometry and its Applications **6** (1996), 327–342.
- [25] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Mathematics **6**, 329–346 (1971).
- [26] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Lecture Notes **29**, A.M.S., Providence RI, 1977.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO
40 ST. GEORGE STREET, TORONTO ON M5S 2E4, CANADA
E-mail address: `karshon@math.toronto.edu`

DIVISION OF MATHEMATICAL SCIENCES, NATIONAL SCIENCE FOUNDATION
4201 WILSON BOULEVARD, ARLINGTON, VA 22230, USA
E-mail address: `jslimowi@nsf.gov`

RECEIVED DECEMBER 24, 2009

ACCEPTED OCTOBER 16, 2013

