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A note on the symplectic topology of b-manifolds

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A Poisson manifold (M^{2n}, π) is b-symplectic if $\bigwedge^n \pi$ is transverse to the zero section. We prove an h-principle for open, b-symplectic manifolds, which shows that an open, orientable manifold M is b-symplectic if and only if $M \times \mathbb{C}$ has an almost-complex structure. For closed, oriented manifolds, we observe that a cosymplectic manifold is the singular locus of a b-symplectic manifold if and only if it is symplectically fillable. We use this observation to prove that every 3-dimensional, closed, orientable cosymplectic manifold is the singular locus of a closed, orientable 4-manifold. We also discuss extensions of this result to higher dimensions.

1. Introduction and statement of main results

A Poisson structure on a manifold M can be described as a bivector $\pi \in \mathfrak{X}^2(M)$ which obeys the partial differential equation $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields. The image of the induced bundle map $\pi^\sharp : T^*M \to TM$ is an involutive distribution, of possibly varying rank, each of whose integral submanifolds carries an induced symplectic form.

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Symplectic structures are those Poisson structures whose underlying foliation has M^{2n} as its only leaf; equivalently, they are those Poisson structures for which $\pi^\sharp$ is invertible – i.e., for which $\bigwedge^n \pi$ does not meet the zero section.

In [9], this nondegeneracy condition has been relaxed in a very natural way, by demanding that $\bigwedge^n \pi$ be *transverse* to the zero section instead of avoiding it:

Definition 1.1. A Poisson manifold (M^{2n}, π) is of b**-symplectic type** if $\bigwedge^n \pi$ is transverse to the zero section $M \subset \bigwedge^{2n} TM$.

Such structures were first defined, in the case of dimension two, by Radko [17], who called them *topologically stable* Poisson structures. Poisson structures of b-symplectic type have also appeared under the name *log symplectic* [3, 7, 12, 13].

Symplectic structures are those Poisson structures of b-symplectic type whose **singular locus** $Z(\pi) := (\bigwedge^n \pi)^{-1} M \subset M$ is empty. Quite crucially for what follows is that general Poisson structures of b-symplectic type do not stay too far from being symplectic.

The transversality condition $\bigwedge^n \pi \pitchfork M$ ensures that the singular locus $Z = Z(\pi)$ is a codimension-one submanifold of M, which by the Poisson condition is itself foliated in codimension one by symplectic leaves of π. Hence, Z is a corank-one Poisson submanifold. Those vector fields $v \in \mathfrak{X}(M)$ which are tangent to Z form the space of all sections of a vector bundle ${}^bT(M, Z) \to M$, called the b**-tangent bundle** [15]. The bundle ${}^bT(M, Z)$ has a canonical structure of Lie algebroid, and a Poisson structure of b-symplectic type π on M with singular locus Z can be described alternatively by a closed, nondegenerate section of $\bigwedge^2 \left({}^bT^*(M, Z)\right)$, in complete analogy with the symplectic case. This viewpoint motivates the nomenclature adopted in [9, 16] [1], and to which we adhere. Henceforth, we will refer to Poisson structures of b-symplectic type as b**-symplectic** structures. With this perspective, it is not surprising that many tools from Symplectic Topology can be adapted to this b-setting. In fact, the purpose of this paper is to use such tools to discuss the following existence problems in b-symplectic geometry:

(P1) Which manifolds M carry a structure of b-symplectic manifold?

(P2) Which corank-one Poisson manifolds Z appear as singular loci of closed b-symplectic manifolds ?

[1] Closed, nondegenerate sections of $\bigwedge^2 \left({}^bT^*(M, Z)\right)$ were introduced in [16] for $Z = \partial M$.

For closed manifolds the answer to (P1) is unknown, even in the symplectic case. For *open* manifolds (i.e., whose connected components either have non-empty boundary or are non-compact), we show:

Theorem A. *An orientable, open manifold M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

In fact, the story here is completely analogous to the symplectic case: supporting a b-symplectic structure imposes restrictions on the de Rham cohomology of a closed manifold [3, 12], but these do not apply to open manifolds. There, we show the existence of b-symplectic structures is a purely homotopical question, which abides by a version of the h-principle of Gromov [6]. In some very special cases, the finer control granted by having an h-principle description allows one to prescribe the singular locus Z of the ensuing b-symplectic manifold. (However, in the case where Z bounds a compact region in M, these techniques break down completely.)

The singular locus Z of a b-symplectic manifold is not just a general corank-one Poisson manifold, in that it can be defined by a **cosymplectic structure** $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$ (i.e., a pair of closed forms for which $\theta \wedge \eta^{n-1}$ is a volume form; cf. [9, Proposition 10] or Section 4). Thus, the Existence Problem (P2) is about describing which cosymplectic structures appear as the singular locus of a closed b-symplectic manifold.

Our first result says that (P2) can be rephrased as a problem of Symplectic Topology; namely, that of determining those (closed) cosymplectic manifolds which admit symplectic fillings (see Section 2).

Lemma 1.1. *A cosymplectic manifold (Z, η, θ) is the singular locus of a closed, orientable b-symplectic manifold if and only if (Z, η, θ) is symplectically fillable.*

Symplectic fillings of contact manifolds – and more generally symplectic cobordisms with concave/convex boundaries – are central to Symplectic Topology, whereas the case of cosymplectic (or flat) boundaries has received comparatively little attention. A notable exception is Eliashberg's result that 3-dimensional **symplectic mapping tori** (i.e., suspensions of a symplectomorphisms of surfaces) are symplectically fillable [4].

Our second result follows from observing that symplectic fillability of all cosymplectic 3-manifolds is a consequence of symplectic fillability of all symplectic mapping tori. This solves the cosymplectic existence problem in dimension 3:

Theorem B. *Any cosymplectic manifold of dimension* 3 *is the singular locus of orientable, closed, b-symplectic manifolds.*

Our third result describes a class of symplectomorphisms φ which yield symplectically fillable symplectic mapping tori in arbitrary dimensions: namely, those built out of **Dehn twists** around parametrized Lagrangian spheres (see Definition 4.3) and their inverses:

Theorem C. *If Z is a symplectic mapping torus defined by a symplectomorphism which is Hamiltonian isotopic to a word on Dehn twists and their inverses, then Z is the singular locus of orientable, closed, b-symplectic manifolds.*

Remark 1.1. Symplectic fillability of a sympletic mapping torus is a property that only depends on the Hamiltonian isotopy class of the symplectomorphism. The key result in Eliashberg's argument in [4] is that, for surfaces, the fillability of a symplectic mapping torus depends just on the symplectic isotopy class of the symplectomorphism. We do not know whether this is true in higher dimensions.

While this project was being completed the authors learned of research by G. Cavalcanti which has some overlap with theirs. More precisely, the idea of constructing b-symplectic manifolds without boundary by gluing cosymplectic cobordisms appeared independently in [3].

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2. Cosymplectic cobordisms and b-symplectic structures

We summarize below basic facts and conventions about b-symplectic manifolds and cosymplectic cobordisms, and describe the relation between both structures. For a more detailed account we refer the reader to [7–9, 13, 15, 16].

2.1. b-manifolds

The Lie subalgebra $\mathfrak{X}(M,Z) \subset \mathfrak{X}(M)$ consisting of those vector fields v which are tangent to Z can be identified with the space of smooth sections of the **b-tangent bundle** ${}^bT(M,Z) \to M$. By its very construction, ${}^bT(M,Z)$ comes equipped with a bundle map ${}^bT(M,Z) \to TM$ covering id_M, which is the identity outside Z. Its restriction to Z defines an epimorphism ${}^bT(M,Z)|_Z \to TZ$, whose kernel ${}^bN(M,Z)$ has a canonical trivialization $\nu \in \Gamma(Z, {}^bN(M,Z))$: if one expresses Z locally as $x_1 = 0$ in a coordinate chart $(x_1, ..., x_n)$, then $x_1 \frac{\partial}{\partial x_1}$ is a no-where vanishing local section of ${}^bT(M,Z)$ independent of choices along Z.

The bundle dual to ${}^bT(M,Z)$ will be denoted by ${}^bT^*(M,Z)$; sections of its p-th exterior power will be called b-forms (of degree p) on (M,Z), and we write ${}^b\Omega^p(M,Z)$ for the space of all such forms.

Since $\mathfrak{X}(M,Z) \subset \mathfrak{X}(M)$ is a Lie subalgebra, ${}^bT(M,Z)$ has a natural structure of Lie algebroid, and as such, it carries a differential

$$ {}^b\mathrm{d} : {}^b\Omega^p(M,Z) \to {}^b\Omega^{p+1}(M,Z) $$

given by the usual Koszul-type formula. Note that ${}^b\mathrm{d}$ agrees with d outside Z, and that we have a short exact sequence of chain complexes:

$$ (1) \qquad 0 \longrightarrow (\Omega^\bullet(M), \mathrm{d}) \longrightarrow ({}^b\Omega^\bullet(M,Z), {}^b\mathrm{d}) \xrightarrow{\flat} (\Omega^{\bullet-1}(Z), \mathrm{d}) \longrightarrow 0, $$

where $\flat$ maps a b-form ω to its contraction with the canonical ν.

2.2. Cosymplectic and b-symplectic structures

Mimicking the usual terminology, $\omega \in {}^b\Omega^2(M,Z)$ will be called **b-symplectic** if ω^{top} is nowhere vanishing and ω is closed, ${}^b\mathrm{d}\omega = 0$.

We recall from [9, Proposition 20] that there is a bijective correspondence between b-symplectic forms on (M,Z), and Poisson structures of b-symplectic type with singular locus Z. We can thus speak unambiguously of *b-symplectic manifolds*; that is, if (M,π) is a Poisson manifold of b-symplectic type, then $(\pi|_{M \setminus Z(\pi)})^{-1}$ extends to a b-symplectic form ω on the b-manifold $(M, Z(\pi))$.

Let then (M,ω) be a b-symplectic manifold, with singular locus $Z \subset M$. By the very definition, $\omega|_{M\setminus Z}$ is a symplectic manifold in the usual sense.

Before we describe the Poisson structure around points in the singular locus Z, we recall from [8] how *cosymplectic structures* appear in our context.

Recall that if $(\theta,\eta) \in \Omega^1(Z)\times\Omega^2(Z)$ has the property that $\theta\wedge\eta^{\mathrm{top}-1}$ is a volume form, then to such a pair there corresponds a pair $(R,\nu)\in\mathfrak{X}(Z)\times\mathfrak{X}^2(Z)$, where R is the *R*eeb vector field

$$\iota_R\theta = 1,\ \ \iota_R\eta = 0,$$

and ν is characterized by:

$$\iota_\theta\nu = 0,\ \ \nu^\sharp\eta^\sharp + \theta\otimes R = \mathrm{id}_{TZ}\,.$$

Then:

$$\mathrm{d}\theta = 0,\ \mathrm{d}\eta = 0 \qquad\Longleftrightarrow\qquad [R,\nu]=0,\ [\nu,\nu]=0.$$

Hence from a Poisson-theoretic perspective, a cosymplectic structure (θ,η) is a pair (R,ν) consisting of a corank-one Poisson structure $\nu\in\mathrm{Poiss}(Z)$, together with a Poisson vector field $R\in\mathfrak{X}(Z)$ transverse to the leaves of ν.

A natural way cosymplectic structures occur in Symplectic Geometry is as hypersurfaces transverse to symplectic vector fields. The setting is the following: one is given a hypersurface X in a symplectic manifold (W,Ω), and we assume that on an open neighborhood $U\subset W$ of X there exists a symplectic vector field $v\in\mathfrak{X}(U)$ transverse to X. Denote by

$$\varphi : U\times\mathbb{R}\supset \mathrm{dom}(\varphi)\longrightarrow W,\quad \frac{d}{dt}\varphi_t(x) = v\circ\varphi_t(x)$$

the local flow of v, and consider the induced map:

$$c : X\times\mathbb{R}\cap\mathrm{dom}(c)\longrightarrow W,\quad c(x,t) := \varphi_t(x).$$

Then the *adapted collar* c embeds in W the open neighborhood

$$V_\varepsilon\subset X\times\mathbb{R},\quad V_\varepsilon := \{(x,t)| -\varepsilon(x) < t < \varepsilon(x)\}$$

for some small enough positive function $\varepsilon : U\to\mathbb{R}_+$, and c pulls Ω back to:

$$c^*\Omega = \mathrm{d}t\wedge\theta + \eta, \tag{2}$$

where $(\theta,\eta) := (c^*\iota_v\Omega|_X, c^*\Omega|_X)$ and we have abused notation and omitted the first projection map on $V_\varepsilon\subset X\times\mathbb{R}$.

More to the point, a description similar to (2) exists for a b-symplectic structure ω in a neighborhood of its singular locus $Z(\omega)$. When $Z(\omega)$ is coorientable, it goes as follows: start with any collar $c : V_\varepsilon\hookrightarrow M$ extending

$\mathrm{id}_{Z(\omega)}$ and regard c as a b-embedding $c : (V_\varepsilon, Z(\omega)) \hookrightarrow (M, Z(\omega))$. If we let θ stand for $\flat(c^*\omega)$, then $\widetilde{\eta} := c^*\omega - \mathrm{d}\log|t| \wedge \theta$ is an honest closed two-form $\widetilde{\eta} \in \Omega^2(V_\varepsilon)$; in particular, $\eta := \widetilde{\eta}|_{Z(\omega)}$ makes sense as a closed two-form on $Z(\omega)$, and note that (θ, η) defines a cosymplectic structure on $Z(\omega)$.

Define on V_ε the b-symplectic form

$$\omega_0 := \mathrm{d}\log|t| \wedge \theta + \eta \tag{3}$$

and observe that the path of b-forms $\omega_t := \omega_0 + t(c^*\omega - \omega_0)$ is b-symplectic on $V_{\varepsilon'}$, for some $0 < \varepsilon' < \varepsilon$. Now, $c^*\omega - \omega_0 = \mathrm{d}\alpha$, for some $\alpha \in \Omega^1(V_{\varepsilon'})$ which vanishes along $Z(\omega)$. Hence the time-dependent b-vector field v_t defined by $\iota_{v_t}\omega_t + \alpha = 0$ vanishes along $Z(\omega)$, and hence the local flow of v_t is defined up to time one on some $V_{\varepsilon''}$ for some $0 < \varepsilon'' < \varepsilon'$, and satisfies $\phi_t^*\omega_t = \omega_0$. Therefore, we obtain the *adapted collar*:

$$\overline{c} := c\phi_1 : V_{\varepsilon''} \hookrightarrow M$$

pulling ω back to ω_0. Note that in formula (3) the one-form θ is intrinsically defined, while η depends on the collar, this choice not affecting its restriction to the symplectic distribution $\ker\theta$. Observe also that the inverse $\pi_0 := \omega_0^{-1} \in \mathrm{Poiss}(M)$ is given by:

$$\pi_0 := t\frac{\partial}{\partial t} \wedge R + \nu,$$

where (R, ν) is the pair determined by (θ, η). In particular, on its singular locus $Z(\omega)$, a b-symplectic form ω determines a corank-one Poisson structure ν which comes from a cosymplectic structure (θ, η) on $Z(\omega)$. This is still the case when $Z(\omega)$ is not coorientable in M (as in the example below), where a $\mathbb{Z}_2$-equivariant version of the normal form (3) can be proved for the pullback of ω to the orientation covering of M.

Example 2.1 (Radko's sphere). The b-form $\omega = \frac{1}{h}\mathrm{d}h \wedge \mathrm{d}\theta$ on $\mathbb{S}^2$, where h, θ stand for cylindrical coordinates, is b-symplectic. Its symplectic leaves are either points in the equator $\mathbb{S}^1 \subset \mathbb{S}^2$, or components of $\mathbb{S}^2 \setminus \mathbb{S}^1$. This b-form is invariant under the antipodal map on the sphere and, thus, induces a b-symplectic form on the projective plane $\mathbb{RP}^2$ which is a non-orientable b-symplectic manifold.

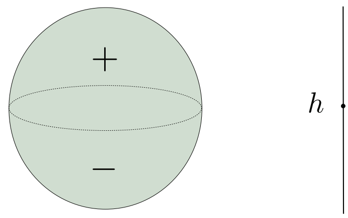

Figure 1: Radko sphere $\mathbb{S}^2$ with the equator as critical hypersurface and the upper and lower hemisphere as positive and negative symplectic leaf, respectively.

2.3. Cobordisms

An alternative perspective on b-symplectic manifolds is that they arise by gluing certain cobordisms in the symplectic category.

Definition 2.1. A **cosymplectic cobordism** (M, ω, θ) is a compact symplectic manifold (M, ω), together with a cosymplectic structure of the form $(\theta, \omega|_{\partial M})$ on its boundary ∂M.

As pointed out in the previous subsection, the cosymplectic structure $(\theta, \omega|_{\partial M})$ is induced by a symplectic vector field v defined in a neighborhood of ∂M and transverse to it. A connected component X of ∂M is called *incoming* or *outgoing* according to whether such a defining transverse symplectic vector field v points into M or out of it, respectively.

Observe that if $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$ are cosymplectic cobordisms, and there exists a diffeomorphism:

$$\varphi : \partial M_0 \xrightarrow{\sim} \partial M_1, \quad \varphi^*\theta_1 = \theta_0, \quad \varphi^*(\omega|_{\partial M_1}) = \omega|_{\partial M_0},$$

then a b-symplectic manifold (M_{01}, ω_{01}) and symplectic embeddings $\varphi_i : (M_i, \omega_i) \hookrightarrow (M_{01}, \omega_{01})$ exist, with:

- $Z(\omega_{01}) = \varnothing$ and

$$\varphi_0 M_0 \cup \varphi_1 M_1 = M_{01}$$

  if ∂M_0 is incoming and ∂M_1 is outgoing, or ∂M_0 is outgoing and ∂M_1 is incoming;

- $Z(\omega_{01}) \simeq \partial M_0 \simeq \partial M_1$ and outside a neighborhood $U \simeq Z(\omega_{01}) \times (-1, 1)$ of $Z(\omega_{01})$, we have:

$$\varphi_0 M_0 \coprod \varphi_1 M_1 = M_{01} \setminus U$$

if $\partial M_0, \partial M_1$ are both incoming or both outgoing.

In the first case one constructs collars as in (3) and glues the result into a symplectic manifold; in the second case the orientations do not match, and so one mediates the previous gluing by a collar of the form:

$$(Z(\omega) \times [-1, 1], \mathrm{d}f \wedge \theta + \eta),$$

where $f = f(t)$ is a monotone function with $\mathrm{d}f = \mathrm{d}\log|t|$ around $t = 0$ and $\mathrm{d}f = \pm\mathrm{d}t$ around $t = \pm 1$.

Remark 2.1. Just as in the symplectic case, a b-symplectic version of Moser's argument [9] shows that the isomorphism type of a b-symplectic manifold obtained by gluing symplectic cobordisms with equal boundary orientations does not depend on the choices of collars.

Recall that the **double** $M\#\overline{M}$ of an orientable manifold with boundary can be given, up to isomorphism, a unique smooth structure in which $M \hookrightarrow M\#\overline{M}$ and $\overline{M} \hookrightarrow M\#\overline{M}$ are smooth embeddings. An illustration of the natural way b-symplectic structures occur when trying to pass from compact symplectic manifolds to closed ones is given by the following result:

Corollary 2.1. *If (M, ω, θ) is a cosymplectic cobordism, then its double $M\#\overline{M}$ is a b-symplectic manifold.*

For example, Radko's sphere is the *b-symplectic double* of any closed symplectic disk $(\mathbb{D}^2, \omega)$ endowed with a nowhere zero 1-form on $\mathbb{S}^1$ of period 2π.

3. h-principle

In this section, we use standard h-principle arguments to provide a complete answer to the Existence Problem (P1) under the additional assumption that M is open.

A necessary condition for a manifold M^{2n} to be symplectic is that it carry a nondegenerate two-form, or, equivalently, an almost-complex structure. If

M is compact, we have a further necessary condition, namely, that there be a degree-two cohomology class $\tau \in H^2(M)$ with $\tau^n \neq 0$.

For *open* manifolds M a classical theorem of Gromov [6] states that the sole obstruction to the existence of a symplectic structure is that M be almost-complex. More precisely, given any non-degenerate two-form ω_0 and a cohomology class $\tau \in H^2(M)$, there is a path $\omega : [0,1] \to \Omega^2(M)$ of nondegenerate two-forms connecting ω_0 to a closed two-form ω_1 representing $[\omega_1] = \tau$.

We consider now the case of b-symplectic structures. Recall that b-symplectic manifolds need *not* be oriented as usual manifolds, so in particular they may fail to be almost-complex. However:

Lemma 3.1. *If an orientable M admits a b-symplectic structure ω, then $M \times \mathbb{C}$ is almost-complex.*

Proof. The b-tangent bundle ${}^bT(M,Z)$ admits an almost-complex structure, and hence so does ${}^bT(M,Z) \times \mathbb{C} \to M \times \mathbb{C}$, and a straightforward adaptation of the argument in [1, §4] shows that for M orientable, ${}^bT(M,Z) \times \mathbb{C} \cong T(M \times \mathbb{C})$. □

Just as in the symplectic case, if we demand that M be compact, the existence of a b-symplectic structure is obstructed:

1) there exists a cohomology class $\tau \in H^2(M)$ with $\tau^{n-1} \neq 0$ [12];
2) furthermore, if M is orientable, a non-trivial $\vartheta \in H^2(M)$ must exist squaring to zero [3].

None of these obstructions appear when M is open, so one wonders if, in that case, $M \times \mathbb{C}$ being almost-complex is sufficient to ensure that M carries a b-symplectic structure. We answer the question in the affirmative:

Theorem A. *Let M be an orientable, open manifold. Then M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

We need to introduce the analogs of nondegenerate two-forms. To do that, observe that a bivector $\pi \in \mathfrak{X}^2(M^{2n})$ whose top exterior power $\bigwedge^n \pi$ is transverse to the zero section defines a b-manifold $(M, Z(\pi))$, $Z(\pi) := (\bigwedge^n \pi)^{-1}M \subset M$. Since b-bivectors $\mathfrak{X}^2(M, Z(\pi))$ sit inside the space of all bivectors $\mathfrak{X}^2(M)$, it makes sense to require π to be a b-bivector in the b-manifold it defines, in which case π is nondegenerate *as a b-bivector. For a bivector $\pi \in \mathfrak{X}^2(M)$ whose top power is transverse to M, the condition* *that $\pi \in \mathfrak{X}^2(M, Z(\pi))$ is not automatically satisfied (as a simple coordinate check shows); those which do fulfill the condition will be called* **transversally nondegenerate b-bivectors**.

In the sequel we show that transversally nondegenerate b-bivectors can be homotoped into b-symplectic ones, provided that the manifold be open:

Theorem 3.1. *On an* open *manifold M, a transversally nondegenerate b-bivector π_0 is homotopic through transversally nondegenerate b-bivectors to a Poisson bivector π_1. Moreover, one can arrange that $Z(\pi_1)$ be non-empty if $Z(\pi_0)$ is non-empty.*

This statement is a result of checking that 1-jets of Poisson bivectors of exact b-symplectic type form a microflexible differential relation, invariant under the pseudogroup of local diffeomorphisms of M, cf. [6]. We opted instead to follow the somewhat more visual scheme of proof of [5].

Proof. Take $\pi_0 \in \mathfrak{X}^2_{\pitchfork}(M)$ a transversally nondegenerate b-bivector, $\pi_0 \in \mathfrak{X}(M, Z_0)$, and let $\omega_0 \in {}^b\Omega^2(M, Z_0)$ be the corresponding b-symplectic form.

Observe that the b-differential ${}^b\mathrm{d} : {}^b\Omega^p(M, Z_0) \longrightarrow {}^b\Omega^{p+1}(M, Z_0)$ can be factored as a composition ${}^b\mathrm{d} = \widetilde{\mathrm{symb}}({}^b\mathrm{d}) \circ j_1$, where j_1 denotes the 1-jet map

$$j_1 : \Gamma\left(M, \bigwedge^p {}^bT^*(M, Z_0)\right) \longrightarrow \Gamma\left(M, J_1 \bigwedge^p {}^bT^*(M, Z_0)\right)$$

and

$$\widetilde{\mathrm{symb}}({}^b\mathrm{d}) : \Gamma\left(M, J_1 \bigwedge^p {}^bT^*(M, Z_0)\right) \longrightarrow \Gamma\left(M, \bigwedge^{p+1} {}^bT^*(M, Z_0)\right)$$

is induced by a bundle map

$$\mathrm{symb}({}^b\mathrm{d}) : J_1 \bigwedge^p {}^bT^*(M, Z_0) \longrightarrow \bigwedge^{p+1} {}^bT^*(M, Z_0).$$

As one easily checks, $\mathrm{symb}({}^b\mathrm{d})$ is an epimorphism with contractible fibres; in particular, we can lift ω_0 to $\widetilde{\omega}_0 \in \Gamma(M, J_1{}^bT^*(M, Z_0))$.

Now, since M is an open manifold, there exists a subcomplex K of a smooth triangulation of M, of positive codimension, with the property that, for an arbitrarily small open set $U \subset M$ around K, there exists an isotopy of open embeddings $g_t : M \hookrightarrow M$, $g_0 = \mathrm{id}_M$, with $g_1(M) \subset U$ and $g_t|_K = \mathrm{id}_K$. We will call K a core for M, and say that g_t compresses M

into the neighborhood U of the core K. Note in passing that one can always find a core K of M meeting Z_0.

Fix then a core K of M. The Holonomic Approximation theorem of [5] then says that we can find

- an isotopy h_t of M;
- an open set V around $h_1(K)$;
- a section $\alpha \in \Gamma(V, {}^bT^*(M, Z_0))$

such that $j_1\alpha$ is so C^0-close to $\widetilde{\omega}_0$ that we can find a homotopy

$$\widetilde{\omega}_t \in \Gamma(V, J_1{}^bT^*(M, Z_0)),$$

connecting $\widetilde{\omega}_0|_V$ to $j_1\alpha$, and with $\widetilde{\mathrm{symb}}({}^b\mathrm{d})\widetilde{\omega}_t$ nondegenerate b-forms on V. Moreover, the scheme of proof in [5] shows that one can require in addition that the new core $h_1(K)$ also meet Z_0.

Now let us choose a compression g_t of M into the neighborhood V of the core $h_1(K)$. Regard g_t as a smooth family of b-maps

$$g_t : (M, Z_t) \longrightarrow (M, Z_0), \quad Z_t := g_t^{-1} Z_0,$$

and set $\omega_1 := {}^b\mathrm{d}(g_1^*\alpha) \in {}^b\Omega^2(M, Z_1)$. Observe now that $\widehat{\omega}_t^1 := g_t^*\widetilde{\omega}_0$ connects $\widetilde{\omega}_0$ to $g_1^*(\widetilde{\omega}_0|_V)$, and $\widehat{\omega}_t^2 := g_1^*\widetilde{\omega}_t$ connects $g_1^*(\widetilde{\omega}_0|_V)$ to a lift of ω_1. Let $\widehat{\omega}_t$ denote the concatenation of $\widehat{\omega}_t^1$ and $\widehat{\omega}_t^2$:

$$\widehat{\omega}_t := \begin{cases} \widehat{\omega}_{2t}^1 & 0 \leqslant t \leqslant 1/2, \\ \widehat{\omega}_{2t-1}^2 & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $t \mapsto \pi_t := \widehat{\omega}_t^{-1} \in \mathfrak{X}(M, Z_t)$ defines a homotopy of transversally nondegenerate b-bivectors between π_0 and a Poisson bivector π_1. □

A few remarks are in order:

- If ω_0 could be C^0-approximated by a closed $\omega_1 \in {}^b\Omega^2(M, Z_0)$, we would be done. However, such an approximation is severely obstructed, in that it would imply that Z_0 admits a structure of cosymplectic manifold.(For compact Z, the existence of a cosymplectic structure implies e.g. that it has non-trivial cohomology in all possible degrees).
- We get around this problem by changing the topology of Z_0 rather drastically; observe in particular that Z_1 may be disconnected even if Z_0 is connected. One should perhaps think of Z_1 as Z_0 with those

places 'blown to infinity' where ω_0 cannot be approximated by closed b-forms.

- Of course, when M is itself almost-complex, Gromov's theorem allows us to produce an honest symplectic structure.
- If in the statement of Theorem 3.1 we further assume that:
  - $Z = Z(\pi_0)$ is a regular fibre $f^{-1}(0)$ of a proper Morse function $f : M \to \mathbb{R}$, unbounded from above and from below, and
  - ω_0 is already ^{b}d-closed around Z,

  then one can impose that the homotopy π_t above be stationary around Z [5, Theorem 7.2.4].

Proof of Theorem A. It remains to show that the existence of an almost-complex structure on $M \times \mathbb{C}$ ensures the existence of a transversally nondegenerate b-bivector. But according to [2], the former guarantees the existence of a *folded* symplectic form $\phi \in \Omega^2(M)$, namely, a closed two-form, whose top power is transverse to the zero section, thus defining a smooth *folding locus* $Z = (\phi^n)^{-1}M \subset M$, along which ϕ^{n-1} does not vanish. For every one-form $\theta \in \Omega^1(Z)$ satisfying $\theta \wedge \phi|_Z^{n-1} \neq 0$ everywhere, one can construct an open embedding

$$c : C \hookrightarrow M, \quad c^*\phi = \mathrm{d}\left(\frac{t^2}{2}\theta\right) + \phi|_Z, \tag{4}$$

following the recipe in [1, Theorem 1], where $C \subset Z \times \mathbb{R}$ is an open neighborhood of Z.

Fix a Riemannian metric g on M with $c^*g := g_Z + \mathrm{d}t^2$, and denote by π the bivector on M which is g-dual to ϕ; then

$$c^*\pi = t\frac{\partial}{\partial t} \wedge R + \nu,$$

where

$$R = g^\flat(\theta), \quad \nu = g^\flat\left(\phi|_Z + t^2/2\mathrm{d}\theta\right)$$

has no $\frac{\partial}{\partial t}$-component. Therefore, π is a transversally nondegenerate b-bivector, and has singular locus Z. □

Remark 3.1. As explained in [2], the existence of an almost-complex structure on $M \times \mathbb{C}$ is equivalent to the existence of a folded symplectic form on M. In fact, given a b-symplectic form ω on M with compact singular locus $Z(\omega)$, one can obtain a folded symplectic form $\phi \in \Omega^2(M)$ whose folding locus is $Z(\omega)$ and which agrees with ω in the complement of a compact neighborhood of $Z(\omega)$. This can be achieved by merely replacing the normal form (3) by the normal form (4) (see also [10]).

4. Prescribing the singular locus of a *b*-symplectic manifold

The Existence problem (P2) asks for the description of those corank-one Poisson manifolds that appear as singular loci of a closed b-symplectic manifold:

Definition 4.1. A cosymplectic manifold is the singular locus of a b-symplectic manifold (M, ω), if it is diffeomorphic to $Z(\omega)$, endowed with the cosymplectic structure induced by some adapted collar.

Thus, problem (P2) splits into two questions: Firstly, deciding which closed, corank-one Poisson structures come from cosymplectic structures, a matter discussed in [8]. Secondly, describing which (connected) cosymplectic manifolds are singular loci of closed b-symplectic manifolds, which is the question we will tackle.

Note that the question would trivial if either:

- the b-symplectic manifold were not required to have empty boundary (e.g., a closed, adapted collar associated to a cosymplectic manifold (Z, θ, η) would provide such a realization), or
- the cosymplectic manifold were not required to be the entire singular locus of (M, ω) (e.g., the double of the adapted collar is a b-symplectic manifold without boundary, whose singular locus consists of *two* copies of (Z, θ, η)).

We shall therefore assume that all cosymplectic manifolds appearing henceforth are closed and connected. To simplify the exposition, we shall first seek a realization on (M, ω), where M is orientable (or, equivalently, where the singular locus is coorientable). The general case is deferred to Remark 4.2.

Definition 4.2. Let (Z_0, η_0, θ_0), (Z_1, η_1, θ_1) be cosymplectic manifolds. We say that Z_0 is **cosymplectic cobordant** to Z_1 if there exists a compact cosymplectic cobordism (M, ω, θ) and diffeomorphisms of cosymplectic manifolds

$$\varphi_0 : (\partial_{\text{in}} M, \omega|_{\partial_{\text{in}} M}, \theta) \xrightarrow{\simeq} (Z_0, \eta_0, \theta_0),$$
$$\varphi_1 : (\partial_{\text{out}} M, \omega|_{\partial_{\text{out}} M}, \theta) \xrightarrow{\simeq} (Z_1, \eta_1, \theta_1).$$

A cosymplectic manifold (Z, η, θ) will be called **symplectically fillable** if it is cosymplectic cobordant to the empty set. Because of the following Lemma, we will restrict from now on our attention to coorientable singular loci, the general case being deferred to Remark 4.2

Lemma 4.1. *A cosymplectic manifold is the singular locus of a closed, oriented b-symplectic manifold if and only if it is symplectically fillable.*

Proof. If (Z, θ, η) is the singular locus of (M, ω), because Z is coorientable, after removing an open, adapted collar inducing the cosymplectic structure, we obtain a cosymplectic cobordism with two connected components, each of which is a cosymplectic cobordism from (Z, θ, η) to the empty set.

Conversely, if we have a cosymplectic cobordism from (Z, θ, η) to the empty set, then (Z, θ, η) is the singular locus of the double of the cobordism. □

Since cosymplectic cobordisms can be composed, any cosymplectic structure cobordant to a fillable one is also fillable. Thus, it is important to discuss some basic constructions of cosymplectic cobordisms.

Lemma 4.2. *Two cosymplectic structures (η_0, θ_0), (η_1, θ_1) on Z are cobordant if there is a homotopy (η_t, θ_t) of cosymplectic structures joining them, and $[\eta_t]$ is constant.*

In particular, if (θ, η) is a cosymplectic structure on Z which is symplectically fillable, then so are $(\lambda\theta, \eta)$, $(\theta + \alpha, \eta)$ and $(\theta, \eta + \mathrm{d}\beta)$, for any positive real number λ and one-forms α, β for which $\alpha, d\beta$ are sufficiently C^0-small.

Proof. Subdivide $[0, 1]$ into $0 = t_0 < t_1 < \cdots < t_N = 1$ so that $\theta_t|_{\ker \eta_s} > 0$, for all $t, s \in [t_i, t_{i+1}]$. It suffices to show that $(Z, \eta_{t_i}, \theta_{t_i})$ is cobordant to $(Z, \eta_{t_{i+1}}, \theta_{t_{i+1}})$ for each i, so we may as well assume that $N = 1$. Now, $\theta_t \wedge \mathrm{d}t + \eta_0$ is then a symplectic form on $M = Z \times [0, 1]$ defining a cosymplectic cobordism between (Z, η_0, θ_0) and (Z, η_0, θ_1). Hence we may assume without loss of generality that the family of one-forms is constant: $\theta_t = \theta$ for all $t \in [0, 1]$.

Let $\mathcal{F}$ denote the codimension-two foliation on M which is the product of the foliation of the interval by points and the cosymplectic foliation on Z. We employ a suitable adaptation of Thurston's trick for $\mathcal{F}$.

Let $\eta_0 - \eta_t = \mathrm{d}\alpha_t \in \Omega^2(Z)$, and choose $\widetilde{\alpha} \in \Omega^1(M)$ such that $\widetilde{\alpha}_{|Z\times\{t\}} = \mathrm{pr}^*(\alpha_t)|_{Z\times\{t\}}$, where $\mathrm{pr} : M \to Z$ is the canonical projection. Define $\omega' = \mathrm{pr}^*(\eta_0) - \mathrm{d}\widetilde{\alpha}$ which is symplectic on the leaves of $\mathcal{F}$. Since M is compact, for $K > 0$ large enough, the form

$$\omega = \omega' + K\,\mathrm{pr}^*(\theta) \wedge \mathrm{d}t$$

is symplectic and restricts to η_i on $Z \times \{i\}$. Hence (M, ω, θ) is the desired cobordism from (η_0, θ) to (η_1, θ).

Reversing the direction of the path (η_t, θ_t) gives a cobordism in the opposite direction. □

Next, we discuss symplectic fillability for the C^0-dense subset of cosymplectic structures given by 1-forms with rational periods.

Symplectic mapping tori and symplectic fillings

We shall regard a mapping torus as a flat bundle with base $\mathbb{S}^1$. The corresponding holonomy representation is generated by a diffeomorphism $\varphi \in \mathrm{Diff}(F)$. Conversely, the suspension of any such φ defines a mapping torus $Z(\varphi)$ with fiber diffeomorphic to F.

As F is assumed to be compact, mapping tori can be defined as fibrations $Z \to \mathbb{S}^1$ with an Ehresmann connection. We shall also identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$.

A **symplectic mapping torus** is a bundle over the circle, whose total space is endowed with a closed two-form η which is symplectic on each fiber. The kernel of η defines an Ehresmann connection, and its holonomy φ preserves the symplectic structure of the fiber, i.e., $\varphi \in \mathrm{Symp}(F, \sigma)$. Conversely, the suspension of any $\varphi \in \mathrm{Symp}(F, \sigma)$ canonically defines a symplectic mapping torus $(Z(\varphi), \eta_\varphi)$ (which we will simply denote by $Z(\varphi)$).

We shall abuse notation and regard a symplectic mapping torus as a cosymplectic manifold $(Z(\varphi), \eta_\varphi, \theta_\varphi)$ by declaring θ_φ to be the pullback of the standard one-form $\mathrm{d}\theta \in \Omega^1(\mathbb{S}^1)$. Having this convention in mind, Lemma 4.2 implies that a cosymplectic manifold with compact foliation is symplectically fillable if and only if its associated symplectic mapping torus is symplectically fillable.

Moreover, the symplectic fillability of $Z(\varphi)$ only depends on the Hamiltonian isotopy class of φ. The reason is that a Hamiltonian isotopy gives rise to a one-parameter family of cosymplectic structures on the mapping torus of φ, satisfying the hypotheses of Lemma 4.2. (In the case of a symplectic isotopy which is not Hamiltonian, the cohomology class of the two-form is not constant.) Lastly, since changing the identification of the fiber (F, σ) by a symplectomorphism results in conjugating the holonomy by that symplectomorphism, we see whether $Z(\varphi)$ is symplectically fillable or not only depends on the conjugacy class of φ.

In what follows, we discuss a family of Hamiltonian isotopy classes of diffeomorphisms yielding fillable symplectic mapping tori.

Dehn twists

There is a class of symplectomorphisms φ which we can 'cap off': Dehn twists. We briefly recall the construction of those maps, and refer the reader to [18] for further details.

The norm function $\mu : (T^*\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-1}) \to \mathbb{R}$, $\mu(\xi) = \|\xi\|$, associated to the round metric on $\mathbb{S}^{n-1}$, is the moment map of a Hamiltonian $\mathbb{S}^1$-action on $(T^*\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-1})$. Upon identifying $T^*\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$ as $T^*\mathbb{S}^{n-1} = \{(u, v) : \langle u, v\rangle = 0, \|u\| = 1\}$, we can write

$$e^{2\pi it} \cdot (u, v) = \left(\cos(2\pi t)u + \sin(2\pi t)v\|v\|^{-1}, \cos(2\pi t)v - \sin(2\pi t)\|v\|u\right),$$

Then $e^{\pi} \cdot (u, v) = (-u, -v)$ extends by the antipodal map to a symplectomorphism $T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$.

Choose now a function $r : \mathbb{R} \to \mathbb{R}$, satisfying:

1) $r(t) = 0$ for $|t| \geqslant C > 0$;
2) $r(t) - r(-t) = t$ for and $|t| \ll 1$,

and let ϕ^t denote the flow of the Hamiltonian vector field of $r(\mu)$.

Observe that $\phi^{2\pi}$ extends to a symplectomorphism $\psi : T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$, supported on the compact subspace $T(\varepsilon) \subset T^*\mathbb{S}^{n-1}$ the subspace of cotangent vectors of length $\leqslant C$. This is called a **model Dehn twist**.

We can graft this construction onto manifolds using Weinstein's Lagrangian neighborhood theorem. If $l : \mathbb{S}^{n-1} \hookrightarrow (F, \sigma)$ embeds $\mathbb{S}^{n-1}$ as a Lagrangian sphere, there are neighborhoods $\mathbb{S}^{n-1} \subset U \subset T^*\mathbb{S}^{n-1}$ and $l(\mathbb{S}^{n-1}) \subset V \subset F$ and a symplectomorphism $\varphi : (U, \omega_{\text{can}}) \to (V, \omega)$ extending l. If ψ is a model Dehn twist, supported inside U, we produce a symplectomorphism $\tau : (F, \sigma) \to (F, \sigma)$, supported in V, by

$$\tau(x) := \begin{cases} \varphi \circ \psi \circ \varphi^{-1}(x) & \text{if } x \in V; \\ x & \text{if } x \in F \setminus V. \end{cases}$$

Definition 4.3. A symplectomorphism of the form above will be called a **Dehn twist** around $l := l(\mathbb{S}^{n-1})$, and it will be denoted by τ_l.

We also recall that any two Dehn twists around a parametrized Lagrangian sphere l are Hamiltonian isotopic if $n > 2$, and symplectically isotopic if $n = 2$ [18].

We will next make use of the following proposition from [14]:

Proposition 4.1. *Let then $Z(\varphi)$ be the symplectic mapping torus corresponding to a $\varphi \in \mathrm{Sym}(F, \sigma)$, and let $l : \mathbb{S}^{n-1} \hookrightarrow Z(\varphi)$ $(n > 2)$ be a parametrized Lagrangian sphere sitting inside a leaf of $Z(\varphi)$.*

1) *the normal bundle of l in $Z(\varphi)$ has two preferred framings $\pm\tau$;*
2) *the cobordism $M_\pm$ obtained by attaching an n-handle along $l \times 1$ to $Z(\varphi) \times [0, 1]$ using the framing $\pm\tau$ carries the structure of a cosymplectic cobordism;*
3) *The incoming boundary component of both M_- and M_+ is $Z(\varphi)$, and the outgoing components are:*

$$\partial_{\mathrm{out}} M_- = Z(\tau_l \varphi), \quad \partial_{\mathrm{in}} M_+ = Z(\tau_l^{-1} \varphi).$$

Proof. This is a combination of [14, Proposition 19, Remarks 21 and 23 and Theorem 26]. Note that the identification in [14] is $\partial_{\mathrm{out}} M_- = Z(\varphi \tau_l)$, but by conjugating by τ_l we obtain the claimed one. □

Proof of Theorem C. By Lemma 4.1, is it enough to show that $Z(\varphi)$ is symplectically fillable.

By hypothesis, φ is –up to conjugacy by a symplectomorphism– Hamiltonian isotopic to $\tau_{l_1}^{\varepsilon_1} \cdots \tau_{l_m}^{\varepsilon_m}$, where $l_i : \mathbb{S}^{n-1} \hookrightarrow (F, \sigma)$, are parametrized Lagrangian spheres and $\varepsilon_i = \pm 1$, $i = 1, \ldots, m$.

By Proposition 4.1(3),

$$Z(\varphi), \quad Z(\tau_{l_1}^{-\varepsilon_1} \varphi), \quad \ldots, \quad Z(\tau_{l_1}^{-\varepsilon_1} \cdots \tau_{l_m}^{-\varepsilon_m} \varphi) = Z(\mathrm{id}_F)$$

are all cosymplectic cobordant, and $Z(\mathrm{id}_F)$ is symplectically fillable, since it bounds $(F \times \mathbb{D}^2, \sigma + \mathrm{d}y_1 \wedge \mathrm{d}y_2)$.

This shows that $Z(\varphi)$ is symplectically fillable, and hence by Lemma 4.1 it is the singular locus of a closed, oriented b-symplectic manifold (M, ω). □

Proof of Theorem B. Let (Z, η, θ) be a 3-dimensional cosymplectic manifold. By Lemma 4.1 all we must show is that (Z, η, θ) is symplectically fillable.

By Lemma 4.2, we may assume without loss of generality that (Z, η, θ) is a symplectic mapping torus (since we can approximate θ by a C^0-close form with rational periods, and then rescale it to become integral). But, according to Eliashberg [4], all 3-dimensional symplectic mapping tori are symplectically fillable. □

Remark 4.1 (A comment of Y. Mitsumatsu). One can give an alternative proof of Theorem B without using the full strength of Eliashberg's result as follows: every symplectic transformation on a closed surface Σ is symplectically isotopic to a word on Dehn twists [11]. By the proof of Theorem C, our starting mapping torus is cobordant to one whose monodromy is symplectically isotopic to the identity. This means that we arrived at the trivial mapping torus $\Sigma \times \mathbb{S}^1$, but with

$$\eta = \sigma + \theta \wedge \beta,$$

where β is a closed 1-form on Σ. As Y. Mitsumatsu has pointed out to us, given a simple curve l in Σ (a parametrized lagrangian sphere), applying Proposition 4.1 first to l and then to l^{-1}, produces a symplectic cobordism whose outcoming boundary is $\Sigma \times \mathbb{S}^1$ with 2-form $\eta + \lambda\theta \wedge \beta_l$, were β_l is the Poincaré dual of the curve and $\lambda > 0$ is the symplectic volume of the cobordism attached. By applying this operation several times we can get rid of $\theta \wedge \beta$.

Remark 4.2. [The non-orientable case] The normal bundle to the singular locus of a non-orientable b-symplectic manifold is classified by a nontrivial $\tau \in H^1(Z; \mathbb{Z}_2)$, which determines a two-sheeted covering $Z_\tau \to Z$. It is not difficult to see that a cosymplectic manifold (Z, θ, η) is the singular locus of a non-orientable b-symplectic manifold (M, ω), with normal bundle given by $\tau \in H^1(Z; \mathbb{Z}_2)$, if and only if the covering $(Z_\tau, \theta_\tau, \eta_\tau)$ is the singular locus of an orientable b-symplectic manifold.

This allows one to prove version of Theorems B and C when the ambient b-symplectic manifold is non-orientable.

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