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Continuum families of non-displaceable Lagrangian tori in $(\mathbb{C}P^1)^{2m}$

Renato Vianna

We construct a family of Lagrangian tori $\Theta^n_s \subset (\mathbb{C}P^1)^n$, $s \in (0,1)$, where $\Theta^n_{1/2} = \Theta^n$, is the monotone twist Lagrangian torus described in [7]. We show that for $n = 2m$ and $s \geq 1/2$ these tori are non-displaceable. Then by considering $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^2_{\mathrm{eq}})^{n-\sum_i k_i} \subset (\mathbb{C}P^1)^n$, with $s_i \in [1/2,1)$ and $k_i \in 2\mathbb{Z}_{>0}$, $\sum_i k_i \leq n$ we get several l-dimensional families of non-displaceable Lagrangian tori. We also show that there exists partial symplectic quasi-states $\zeta^{\mathfrak{b}_s}_{\mathbf{e}_s}$ and linearly independent homogeneous Calabi quasimorphims $\mu^{\mathfrak{b}_s}_{\mathbf{e}_s}$ [18] for which Θ^{2m}_s are $\zeta^{\mathfrak{b}_s}_{\mathbf{e}_s}$-superheavy and $\mu^{\mathfrak{b}_s}_{\mathbf{e}_s}$-superheavy. We also prove a similar result for $(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \omega_\epsilon)$, where $\{\omega_\epsilon; 0 < \epsilon < 1\}$ is a family of symplectic forms in $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, for which $\omega_{1/2}$ is monotone.

1. Introduction

In [19], Fukaya-Oh-Ohta-Ono construct a one-dimensional family of non-displaceable Lagrangian tori in $(\mathbb{C}P^1)^2$. They arise as fibres of a (informally called) semi-toric moment map [28, Section 3], where the fibres over the interior of the semi-toric moment polytope are Lagrangian tori, but over a special vertex of the polytope lies a Lagrangian S^2 (the anti-diagonal) where the semi-toric moment map is not differentiable.

The weighted barycentre of the semi-toric polytope was proven by Oakley-Usher [22] to be the Chekanov torus [7] in $(\mathbb{C}P^1)^2$. The other regular fibres are Hamiltonian isotopic to so called Chekanov type tori described in [2, Example 3.3.1]. In fact, the semi-toric Lagrangian fibration described in [19] can be seen as a limit of almost toric fibrations, in which 'most of the fibres' are Chekanov type tori, see [27, Section 6.4] and [24, Remark 3.1].

The definition of Chekanov type tori can be easily extended to higher dimensions, see Definition 4.1. In particular, we can get analogues of the

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non-displaceable tori [19]. We can show that these tori are non-displaceable in $(\mathbb{C}P^1)^{2m}$.

1.1. Results

Theorem 1.1. *For a positive even integer $n = 2m$, there is a continuum of non-displaceable Lagrangian tori $\Theta_s^{2m} \subset (\mathbb{C}P^1)^{2m}$, $s \in [1/2, 1)$, for which $\Theta_{1/2}^{2m} = \Theta^{2m}$ is the monotone twist Lagrangian torus described in [7]. More precisely, for any Hamiltonian $\Psi \in \mathrm{Ham}((\mathbb{C}P^1)^{2m})$, we have that $|\Theta_s^{2m} \cap \Psi(\Theta_s^{2m})| \geq 2^{2m}$.*

The case $n = 2$ was proven in [19]. The case $n = 1$ is clearly false, since only the monotone circle is non-displaceable.

Question 1.2. For $n \geq 3$ odd and $s \in [1/2, 1)$, are the tori Θ_s^n from Definition 4.2 (non)-displaceable?

An immediate consequence of the proof of Theorem 1.1 is

Corollary 1.3. *For $s_i \in [1/2, 1)$, and positive even integers k_i, $i = 1, \dots, l$, and $n \geq \sum_i k_i$, the Lagrangian tori*

$$\Theta_{s_1}^{k_1} \times \cdots \times \Theta_{s_l}^{k_l} \times (S^1_{\mathrm{eq}})^{n - \sum_i k_i} \subset (\mathbb{C}P^1)^n$$

are non-displaceable.

Just by looking to the symplectic area spectrum of Maslov index 2 relative homology classes we can conclude:

Proposition 1.4. *The tori Θ_s^n is not symplectomorphic to $\Theta_{s_1}^{k_1} \times \cdots \times \Theta_{s_l}^{k_l} \times (S^1_{\mathrm{eq}})^{n - \sum_i k_i}$, if $n > \sum_i k_i$.*

Consider the counts of holomorphic (for the standard complex structure in $(\mathbb{C}P^1)^n$) Maslov index 2 disks with boundary in Θ_s^n, respectively $\Theta_{s_1}^{k_1} \times \cdots \times \Theta_{s_l}^{k_l}$ ($n = \sum_i k_i$), passing through a fixed point. Among these, look at the count of disks that have minimal area. For $s, s_i \in (1/2, 1)$, this area is $a = 1 - s$, respectively $1 - s_i$ for some $i \in \{1, \dots, l\}$. It follows from Proposition 4.5 that these counts of disks of smaller area are different if $l > 1$. Moreover, we show in Proposition 4.10 that higher Maslov index holomorphic disks with boundary on Θ_s^n must have symplectic area bigger than a. Hence, one expect that in a generic family J_t of almost complex structures, where J_0 is the standard complex structure and J_1 is another regular almost complex structure, J_t-holomorphic disks of positive Maslov index and area smaller than a can only appear in a “birth-death” phenomenon. This should imply that the count of Maslov index 2 disks of symplectic area a with boundary in Θ^n_s is an invariant under generic choice of almost complex structure, and hence under symplectomorphisms (in particular Hamiltonian isotopies) acting on Θ^n_s. This would allow us to prove:

Conjecture 1.5. *The tori Θ^n_s is not symplectomorphic to $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l}$, $n = \sum_i k_i$ — unless $l = 1$ and $s_1 = s$.*

A rigorous statement proving the invariance of the count of the Maslov index 2 disks of minimal area in the above scenario and hence Conjecture 1.5 appears in the preprint [23, Proposition 5.1]. We keep calling the above a Conjecture for chronological reasons and because [23, Remark 5.1] refers to this Conjecture.

Therefore we see that the tori obtained here differ from products of copies of the tori obtained in [19] and copies of the equator in $\mathbb{C}P^1$.

The idea of the proof of Theorem 1.1 is that we are able to find bulk deformations $\mathfrak{b}_s$ for which the bulk deformed Floer Homology of Θ^{2m}_s (decorated with some weakly bounding cochain σ) is non-zero. The invariance property of the bulk deformed Floer Cohomology under the action of Hamiltonian diffeomorphisms [17, Theorem 2.5], allow us to conclude that the above Lagrangian tori are non-displaceable.

Based on the work of Fukaya-Oh-Ohta-Ono [18], regarding spectral invariants with bulk deformations, quasimorphisms and Lagrangian Floer theory, we are able to strengthen our result and find families of homogeneous Calabi quasimorphisms $\mu^{\mathfrak{b}_s}_{\mathbf{e}_s}$ and partial symplectic quasi-states $\zeta^{\mathfrak{b}_s}_{\mathbf{e}_s}$, for which Θ^{2m}_s is $\mu^{\mathfrak{b}_s}_{\mathbf{e}_s}$-superheavy and $\zeta^{\mathfrak{b}_s}_{\mathbf{e}_s}$-superheavy.

For the definition of homogeneous Calabi quasimorphisms, partial symplectic quasi-states and the notion of superheaviness we refer the reader to [11, 12, 18].

Following closely the notation of [18, Lemma 23.3, Theorem 23.4] we summarise the above discussion as:

Theorem 1.6. *For $s \in [1/2, 1)$, there exists a bulk-deformation*

$$\mathfrak{b}_s \in H^2((\mathbb{C}P^1)^{2m}, \Lambda_+),$$

and a weak bounding cochain $b_s \in H^1(\Theta_s^{2m}, \Lambda_0)$ *for which*

$$HF(\Theta_s^{2m}, (\mathfrak{b}_s, b_s); \Lambda_{0,nov}) \cong H^*(\Theta_s^{2m}; \Lambda_{0,nov})$$

Moreover, there are idempotents $\boldsymbol{e}_s$ *in the bulk-deformed quantum-cohomology* $QH^*_{\mathfrak{b}_s}((\mathbb{C}P^1)^{2m}; \Lambda_{0,nov})$*, so that* Θ_s^{2m} *is* $\mu^{\mathfrak{b}_s}_{\boldsymbol{e}_s}$*-superheavy and* $\zeta^{\mathfrak{b}_s}_{\boldsymbol{e}_s}$*-superheavy. Here* $\mu^{\mathfrak{b}_s}_{\boldsymbol{e}_s}$*,* $\zeta^{\mathfrak{b}_s}_{\boldsymbol{e}_s}$ *are respectively the homogeneous Calabi quasimorphism and partial symplectic quasi-states coming from the bulk-deformed spectral invariant associated with* $\boldsymbol{e}_s$ *[18, Section 14].*

Here Λ, Λ_0, Λ_{nov}, $\Lambda_{0,nov}$ and Λ_+ are the Novikov rings:

$$\Lambda = \left\{ \sum_{i\geq 0} a_i T^{\lambda_i} |\ a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

$$\Lambda_0 = \left\{ \sum_{i\geq 0} a_i T^{\lambda_i} |\ a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

$$\Lambda_{nov} = \left\{ \sum_{i\geq 0} a_i q^{n_i} T^{\lambda_i} |\ n_i \in \mathbb{Z}\ a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

$$\Lambda_{0,nov} = \left\{ \sum_{i\geq 0} a_i q^{n_i} T^{\lambda_i} |\ n_i \in \mathbb{Z}\ a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

$$\Lambda_+ = \left\{ \sum_{i\geq 0} a_i T^{\lambda_i} |\ a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{>0}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

The formal parameter T is used to keep track of area of pseudo-holomorphic disks, while the formal parameter $q \in \Lambda_{0,nov}$ is used to keep track of the Maslov index.

The following Corollary follows immediately from [18, Corollary 1.10], see [18, Section 19] for a proof.

Corollary 1.7. *The uncountable set* $\{\mu^{\mathfrak{b}_s}_{\boldsymbol{e}_s}\}$ *of homogeneous Calabi quasimorphisms is linearly independent [18, Definition 1.9].*

To prove linear independency of the above homogeneous Calabi quasimorphisms we use that the tori are disjoint, for different values of s. One could ask:

Question 1.8. Are the tori Θ^n_s Hamiltonian displaceable from $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\text{eq}})^{n-\sum_i k_i}$, for $s, s_i \in (1/2, 1)$?

We note that by construction, these tori intersect for $s, s_i \geq 1/2$. See [24], for non-displaceability in the case $n = 2$, between Θ^n_s (i.e. tori from [19]) $s \geq 3/2$ and the Clifford torus $S^1_{\text{eq}} \times S^1_{\text{eq}}$.

Question 1.9. Are the quasimorphisms arising from (particular choice of bulk-deformation and weak-bounding cochain for) the tori in Corollary 1.3 linearly independent for different partitions $(k_1, \dots, k_l, n - \sum_i k_i)$ of n?

We finish our results by pointing out that the family given in [19] remain non-displaceable after we perform two blowups (of the same size) on the rank zero corners of the singular fibration described in [19], see Figure 1. This follows from applying the same ideas as Fukaya-Oh-Ohta-Ono did for the $\mathbb{C}P^1 \times \mathbb{C}P^1$ case.

Theorem 1.10. *There exists a continuous family of non-displaceable Lagrangian tori* L^ϵ_s *in* $(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega_\epsilon) = (\mathbb{C}P^1 \times \mathbb{C}P^1 \# 2\overline{\mathbb{C}P^2}, \omega_\epsilon)$, *where* $s \in [1/2, 1)$ *and* $\{\omega_\epsilon | 0 < \epsilon < 1\}$ *is a family of symplectic forms for which* $(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega_{1/2})$ *is monotone, containing a monotone Lagrangian* $L^{1/2}_{1/2}$.

Remark 1.11. It is shown in [17, Section 5] and [18, Section 22] a family of non-displaceable Lagrangian tori in $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \geq 2$, endowed with some *non-monotone* symplectic form.

Theorem 1.10 follows, in the same spirit as [18, Theorem 1.11] and Theorem 1.6, from:

Theorem 1.12. *Let* $(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega_\epsilon)$ *and* L^ϵ_s *be as in Theorem 1.10. For* $s \in [1/2, 1)$, *there exists a bulk-deformation* $\mathfrak{b}^\epsilon_s \in H^2(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \Lambda_+)$, *and a weak bounding cochain* $b^\epsilon_s \in H^1(L^\epsilon_s, \Lambda_0)$ *for which*

$$HF(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, (\mathfrak{b}^\epsilon_s, b^\epsilon_s); \Lambda_{0,nov}) \cong H^*(L^\epsilon_s; \Lambda_{0,nov})$$

There are idempotents $\boldsymbol{e}^\epsilon_s$ *in the bulk-deformed quantum-cohomology* $QH(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}; \Lambda)$, *so that* L^ϵ_s *is* $\mu^{\mathfrak{b}^\epsilon_s}_{\boldsymbol{e}^\epsilon_s}$*-superheavy and* $\zeta^{\mathfrak{b}^\epsilon_s}_{\boldsymbol{e}^\epsilon_s}$*-superheavy, where* $\mu^{\mathfrak{b}^\epsilon_s}_{\boldsymbol{e}^\epsilon_s}$, $\zeta^{\mathfrak{b}^\epsilon_s}_{\boldsymbol{e}^\epsilon_s}$ *are the homogeneous Calabi quasimorphism and partial symplectic quasi-states coming from the bulk-deformed spectral invariant associated with* $\boldsymbol{e}^\epsilon_s$ *[18, Section 14]. Moreover, the uncountable set* $\{\mu^{\mathfrak{b}^\epsilon_s}_{\boldsymbol{e}^\epsilon_s}\}$ *of homogeneous Calabi quasimorphisms is linearly independent.*

The rest of the paper is organised as follows:

In Section 2, we make a quick introduction of bulk deformed potential and Floer cohomology for a Lagrangian L satisfying Assumption 2.1. We refer the reader to [17–19] for a complete account. We then prove Lemma 2.5 and Corollary 2.8, to show that, for a Lagrangian torus T, critical points of the potential gives rise to (bulk deformed) Floer cohomology isomorphic to the usual cohomology of T. We believe that 2.5 is known by experts on the field, but we are not aware of it being written.

In Section 3, we define the notion of a pair (X, L) consisting of a Kähler manifold X and a Lagrangian submanifold L being K*-pseudohomogeneous*, for some Lie group K acting holomorphically and Hamiltonianly on X, leaving L invariant. We showed that if (X, L) is K*-pseudohomogeneous*, any Maslov index 2 holomorphic disk with boundary on L such that its boundary is transverse to K-orbits, is regular. We use that to show regularity for the Maslov index 2 disks with boundary in Θ^n_s.

In Section 4, we define the Lagrangian tori Θ^n_s, establish its potential function, essentially computed in [2, 3], and prove it satisfies Assumption 2.1, for some regular almost complex structure J with the same potential function of the standard complex structure. We also prove Proposition 1.4 and show that holomorphic disks of Maslov index bigger than 2 have area bigger than $a = 1 - s$, which we use to argue why Conjecture 1.5 holds once we have [23, Proposition 5.1].

In Section 5, we compute the critical points of the potential bulk deformed by some cocycle in $C^2((\mathbb{C}P^1)^n, \Lambda_+)$. We show that for $n = 2m$, there are bulk deformation $\mathfrak{b}_s$ and a weak bounding cochain b_s which is a critical point of the potential $\mathfrak{PO}^{\Theta^{2m}_s}_{\mathfrak{b}_s}$. It then follows from Corollary 2.8 that the bulk deformed Floer cohomology $HF(\Theta^{2m}_s, (b_s, \mathfrak{b}_s); \Lambda)$ is isomorphic to the cohomology of the torus. Non-displaceability then follows from [16, Theorem G] which is also stated as [17, Theorem 2.5].

In Section 6, we finish the proof of Theorem 1.6.

Finally in Section 7, we describe

$$(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \omega_\epsilon) = (\mathbb{C}P^1 \times \mathbb{C}P^1\#2\overline{\mathbb{C}P^2}, \omega_\epsilon)$$

as two blowups of capacity ϵ on two corners of the moment polytope of $\mathbb{C}P^1 \times \mathbb{C}P^1$. The Lagrangian tori L^ϵ_s on the blowup comes from $\Theta^2_s \in \mathbb{C}P^1 \times \mathbb{C}P^1$. We compute the potential for L^ϵ_s and show the existence of critical points for some bulk deformation. This allow us to prove Theorems 1.10 and 1.12. These tori are equivalent to the fibres of the singular fibration given by blowing up the corners of the "semi-toric polytope" described in [19], see Figure 1.

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2. Floer homology and the potential function

Let X be a symplectic manifold and J a regular and compatible almost complex structure. Let L be a Lagrangian submanifold of X (with a chosen spin structure). We consider a unital canonical A_∞ algebra structure $\{\mathrm{m}_k\}$ on the classical cohomology $H(L;\Lambda_{0,nov})$ [19, Section 6], [16, Corollary 5.4.6, Theorem A]. The potential function is defined from the space of weak bounding cochains $\hat{\mathcal{M}}(L)$ of L to Λ_0. We refer the reader to [16–19] for the definition.

Suppose we are given an compatible almost complex structure J_0 for which (X, L, J_0) satisfy:

Assumption 2.1. Let $\beta \in \pi_2(X, L)$. Assume that:

(A_1) If β is represented by a non-constant J_0-holomorphic disk, then $\mu_L(\beta) \geq 2$,

(A_2) Maslov index 2 J_0-holomorphic disks are regular,

where μ_L is the Maslov index.

Throughout the paper we say an almost complex structure J is regular if it satisfies assumption (A_2).

An almost complex structure satisfying Assumption 2.1, automatically satisfies [19, Condition 6.1], hence by [19, Theorem A.1, Theorem A.2] there is an embedding of $H^1(L, \Lambda_0)$ into $\hat{\mathcal{M}}(L)$ and restricted to $H^1(L, \Lambda_0)$ the potential function $\mathfrak{PO}^L$ is so that

$$\mathrm{m}_0^b(1) = \mathfrak{PO}^L(b)q[L], \tag{2.1}$$

where

$$\text{(2.2)} \qquad \mathrm{m}_0^b(1) = \sum_{k=0}^{\infty} \mathrm{m}_k(b, \ldots, b) = \sum_{\substack{\beta \in \pi_2(X,L), \\ \mu_L(\beta)=2}} q^{\mu_L(\beta)/2} T^{\int_\beta \omega} \exp(b \cap \partial\beta) \mathrm{ev}_{0*}([\mathscr{M}_1(\beta)]).$$

Here $[\mathscr{M}_1(\beta)]$ is the (virtual) fundamental class of the moduli space of J-holomorphic disks in the class β with 1 marked point and $\mathrm{ev}_0 : \mathscr{M}_1(\beta) \to L$ is the evaluation map.

Using a notation closer to [2, 3] we define for $\beta \in \pi_2(X, L)$:

$$\text{(2.3)} \qquad z_\beta(L, b) = T^{\int_\beta \omega} \exp(b \cap \partial\beta).$$

Letting η_β be the degree of $\mathrm{ev}_0 : \mathscr{M}_1(\beta) \to L$, we can write:

$$\text{(2.4)} \qquad \mathfrak{PO}^L(b) = \sum_{\substack{\beta \in \pi_2(X,L), \\ \mu_L(\beta)=2}} \eta_\beta z_\beta(L, b)$$

We want to consider the Floer cohomology of L bulk-deformed by a class $\mathfrak{b} = T^\rho[\mathfrak{s}] \in H^2(X, \Lambda_+)$ [17]. The potential function will depend on the cocycle $\mathfrak{b} \in C^2(X, L; \mathbb{Z})$, even though the Floer cohomology doesn't. Since we use a cocycle in degree 2 (Poincaré dual to a cycle of codimension 2) the degree of the bulked deformed A_∞ maps $\mathrm{m}_k^{\mathfrak{b}}$ [17, (2.6)] is unaffected by the bulk and the bulk deformed potential is given by:

$$\text{(2.5)} \qquad \mathfrak{PO}_{\mathfrak{b}}^L(b) = \sum_{\substack{\beta \in \pi_2(X,L), \\ \mu_L(\beta)=2}} \eta_\beta \exp[(\mathfrak{s} \cap \beta) T^\rho] z_\beta(L, b),$$

where $b \in H^1(L, \Lambda_0)$, is a weak bounding cochain for the curved A_∞ algebra $(H(L, \Lambda_{0,nov}), \{\mathrm{m}_k^{\mathfrak{b}}\})$, with

$$\text{(2.6)} \qquad \mathrm{m}_0^{b,\mathfrak{b}}(1) = \sum_{k=0}^{\infty} \mathrm{m}_k^{\mathfrak{b}}(b, \ldots, b) = \mathfrak{PO}_{\mathfrak{b}}^L(b) q[L].$$

The fact that $b \in H^1(L, \Lambda_0)$ is a weak bounding cochain for

$$(H(L, \Lambda_0), \{\mathrm{m}_k^{\mathfrak{b}}\})$$

implies that we can define a (not curved) A_∞ algebra $(H(L,\Lambda_{0,nov}),\{\mathrm{m}_k^{b,\mathfrak{b}}\})$, where

$$\mathrm{m}_k^{b,\mathfrak{b}}(x_1,\ldots,x_k)=\sum_{j=0}^{\infty}\mathrm{m}_j^{\mathfrak{b}}(b,\ldots,b,x_1,b,\ldots,b,x_2,b,\ldots,b,x_k,b,\ldots,b). \tag{2.7}$$

In particular,

$$(\mathrm{m}_1^{b,\mathfrak{b}})^2=0; \tag{2.8}$$

$$\mathrm{m}_1^{b,\mathfrak{b}}(\mathrm{m}_2^{b,\mathfrak{b}}(x,y))=\pm\mathrm{m}_2^{b,\mathfrak{b}}(\mathrm{m}_1^{b,\mathfrak{b}}(x),y)\pm\mathrm{m}_2^{b,\mathfrak{b}}(x,\mathrm{m}_1^{b,\mathfrak{b}}(y)). \tag{2.9}$$

Definition 2.2. We define the bulk deformed Floer cohomology:

$$HF(L,(b,\mathfrak{b});\Lambda_{0,nov})=\frac{\ker(\mathrm{m}_1^{b,\mathfrak{b}})}{\mathrm{im}(\mathrm{m}_1^{b,\mathfrak{b}})} \tag{2.10}$$

Remark 2.3. Strengthening Assumption 2.1 to assume regularity of holomorphic disks with Maslov index smaller than $n-1$, one should be able to define the Floer cohomology using the Pearl version [4], and analogously define its bulk-deformed version, which should be isomorphic to the one in Definition 2.2. In that framework, the proof of Leibniz rule (2.9) follows the same ideas as [6, Theorem 4].

By the work of Fukaya-Oh-Ohta-Ono, we have:

Theorem 2.4 (Theorem G [16], Theorem 2.5 [17]). *If $\psi:X\to X$ is a Hamiltonian diffeomorphism, then the order of $\psi(L)\cap L$ is not smaller than the rank of $HF(L,(b,\mathfrak{b});\Lambda_{0,nov})\otimes_{\Lambda_{0,nov}}\Lambda_{nov}$.*

We would like to point out that the product $\mathrm{m}_2^{b,\mathfrak{b}}$ can be thought as deformation of the cup product in the sense that for $x,y\in H(L,\Lambda_0)$ of pure degrees $|x|$ and $|y|$,

$$\mathrm{m}_2^{b,\mathfrak{b}}(x,y)=x\cup y+\text{other terms} \tag{2.11}$$

where $x\cup y$ comes from counting constant disks and the other terms is a sum of elements of degree smaller than $|x|+|y|$ in $H(L,\Lambda_{0,nov})$, since it comes from evaluating moduli spaces $\mathscr{M}_{k,l+1}(\beta)$ to a cycle of dimension $|x|+|y|-\mu_L(\beta)$ and (X,L,J) satisfies Assumption (A_1).

The following Lemma is well established for the monotone case in [6], and in the general case in [19].

Lemma 2.5 (Theorem 2.3 of [19]). *Take (X, L) satisfying Assumption 2.1. Also assume that $H(L, \Lambda_0)$ is generated by $H^1(L, \Lambda_0)$ as an algebra with respect to the classical cup product. If* $\mathrm{m}_1^{b,\mathfrak{b}}|_{H^1(L,\Lambda_{0,nov})} = 0$ *then* $\mathrm{m}_1^{b,\mathfrak{b}} \equiv 0$.

Proof. First we point out that $\mathrm{m}_1^{b,\mathfrak{b}}|_{H^0(L,\Lambda_{0,nov})} = 0$. Since $H(L, \Lambda_0)$ is generated by $H^1(L, \Lambda_0)$ with respect to the cup product, we only need to show by induction on the degree that for x and y of pure degree $|x| \geq 1$, $|y| \geq 1$, $\mathrm{m}_1^{b,\mathfrak{b}}(x \cup y) = 0$, if $\mathrm{m}_1^{b,\mathfrak{b}}(z) = 0$ for all z, such that $|z| < |x| + |y|$. Using (2.11),

$$\mathrm{m}_1^{b,\mathfrak{b}}(x \cup y) = \mathrm{m}_1^{b,\mathfrak{b}}(\mathrm{m}_2^{b,\mathfrak{b}}(x, y)) - \mathrm{m}_1^{b,\mathfrak{b}}(\text{other terms}) = 0$$

by induction hypothesis and using the Leibniz rule (2.9). □

Remark 2.6. Lemma 2.5 strengthen the result of [16, Theorem 6.4.35] and [6], showing that the minimal Maslov number M_L of any Lagrangian torus L (or any orientable Lagrangian such that the cohomology ring is generated by H^1) in $\mathbb{C}^n$ is 2, provided T satisfies Assumption 2.1 for some J. That is because the Lagrangian is orientable and $HF(T, (b, \mathfrak{b}); \Lambda) \equiv 0$ (from Theorem 2.4, since T is displaceable), so there must be a Maslov index 2 disk. The inequality $2 \leq M_L \leq n + 1$ was proven in [16, Theorem 6.1.17], for any spin Lagrangian $L \subset \mathbb{C}^n$ satisfying Assumption 2.1, via the use of spectral sequence.

Definition 2.7. Take (X, L) satisfying the assumptions of Lemma 2.5. Assume that $\pi_1(L) \cong H_1(L, \mathbb{Z})$ and $\pi_2(X, L) \cong \pi_2(X) \oplus H_1(L, \mathbb{Z})$. So, we are able to write the Potential function (2.5) in terms of $z_i = z_{\beta_i}$, for some $\beta_1, \ldots, \beta_n \in \pi_2(X, L)$, where $\partial\beta_1, \ldots, \partial\beta_n$ is a basis of $H_1(L, \mathbb{Z})$. We say that b is a *critical point* of $\mathfrak{PO}^L_{\mathfrak{b}}(b)$ if:

$$z_i \frac{\partial \mathfrak{PO}^L_{\mathfrak{b}}(b)}{\partial z_i} = 0.$$

Corollary 2.8 (Theorem 2.3 of [19]). *Take (X, L) satisfying the assumptions of Lemma 2.5 and Definition 2.7. If b is a critical point of* $\mathfrak{PO}^L_{\mathfrak{b}}(b)$ (2.5) *for* $\mathfrak{b} = T^\rho[\mathfrak{s}] \in H^2(X, \Lambda_+)$*, then* $HF(L, (b, \mathfrak{b}); \Lambda) \cong H(L; \Lambda)$.

Proof. Take a basis $x_1, \ldots, x_n$ a basis of $H_1(L, \mathbb{Z})$. Let

$$\beta_1, \ldots, \beta_n \in \pi_2(X, L) \cong \pi_2(X) \oplus H_1(L, \mathbb{Z}),$$

be so that $\partial\beta_i = x_i \in H_1(L, \mathbb{Z})$ and write the Potential $\mathfrak{PO}^L_{\mathfrak{b}}(b)$ (2.5) in terms of $z_i = z_{\beta_i}$.

Since $\mathfrak{s}$ is of degree 2, we have that $\mathrm{m}_1^{b,\mathfrak{b}}(\sigma)$ for $\sigma \in H^1(L,\Lambda)$, only counts contributions of Maslov index 2 disks. A Maslov index 2 J-holomorphic disk in the class $\beta = \gamma + k_1\beta_1 + \cdots + k_n\beta_n$, $\gamma \in \pi_2(X)$ contributes to $\mathrm{m}_1^{b,\mathfrak{b}}(\sigma)$ as

$$\sum_i k_i(\sigma \cap x_i)\eta_\beta \exp[(\mathfrak{s}\cap\beta)T^\rho]T^{\int_\gamma \omega} z_1^{k_1}\cdots z_n^{k_n}$$

Summing all contributions of Maslov index 2 J-holomorphic disks we have:

$$\mathrm{m}_1^{b,\mathfrak{b}}(\sigma) = \sigma \cap \sum_i x_i \left(z_i \frac{\partial \mathfrak{PO}^L_{\mathfrak{b}}(b)}{\partial z_i}\right)$$

Therefore, if b is a critical point of $\mathfrak{PO}^L_{\mathfrak{b}}(b)$, we have that $\mathrm{m}_1^{b,\mathfrak{b}}|_{H^1(L,\Lambda)} = 0$ and by Lemma 2.5, $\mathrm{m}_1^{b,\mathfrak{b}} \equiv 0$, so $HF(L,(b,\mathfrak{b});\Lambda_{0,nov}) \cong H(L;\Lambda_{0,nov})$. □

3. Regularity Lemma

We now move to the Kähler setting and we discuss a Lemma that we will use to prove regularity for Maslov index 2 disks with boundary on Θ^n_s with respect to the standard complex structure in $(\mathbb{C}P^1)^n$. The following definition is inspired in [13, Definition 1.1.1].

Definition 3.1. Let L be a n dimensional Lagrangian in a Kähler manifold X. Assume that K is a Lie group of dimension $n-1$ acting Hamiltonianly and holomorphically on X preserving L. Assume that the action restricted to L is free. Then we say that (X,L) is *K-pseudohomogeneous.*

We get then the following Lemma:

Lemma 3.2. *Let (X,L) be K-pseudohomogeneous, for some Lie group K. If u is a Maslov index 2 holomorphic disk such that ∂u is transverse to the K-orbits, then u is regular.*

The proof of the above Lemma relies on the Lemmas below, very similar to [25, Lemmas 5.19, 5.20].

Lemma 3.3. *Let $u : \mathbb{D} \to X$ be a Maslov index 2 disk in a Kähler manifold X of complex dimension n with boundary on a Lagrangian L. Assume that $u_{|\partial\mathbb{D}}$ is an immersion. Call $W = du(r\partial/\partial\theta)$ a holomorphic vector field along u vanishing at 0 and tangent to the boundary. Assume also that there exists $V_1,\ldots,V_{n-1}$ holomorphic vector fields in u^*TX such that $W \wedge V_1 \wedge \cdots \wedge$*

$V_{n-1} \neq 0$ along the boundary of u. Then u is an immersion and no linear combination of the V_i's is tangent to $u(\mathbb{D})$.

Proof. Up to reparametrization, we may assume $du(0) \neq 0$. The result follows from the fact that the zeros of $\det^2(W \wedge V_1 \wedge \cdots \wedge V_{n-1})$ computes the Maslov index, which is assumed to be 2. So $W \wedge V_1 \wedge \cdots \wedge V_{n-1}$ can only vanish once (with order 1). Since W already vanishes at 0, we cannot have either $du(x) = 0$ or a linear combination of the V_i's being a complex multiple of W. □

Lemma 3.4. *Let $u_{\theta_1,\dots,\theta_{n-1}}$ be an $n-1$ dimensional family of Maslov index 2 holomorphic disks in a Kähler manifold X of complex dimension n, $\theta_i \in (-\epsilon, \epsilon)$. If $u := u_{0,\dots,0}$ and $V_i := \frac{\partial u}{\partial \theta_i}$ satisfy the hypothesis of Lemma 3.3, then u is regular.*

Proof. It follows similar arguments as in [25, Lemma 5.19]. Using Lemma 3.3, we are able to split $u^*TX = T\mathbb{D} \oplus \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$, as holomorphic vector bundles where $\mathfrak{L}_i$ is the trivial line bundle generated by V_i. Also, $u^*_{|\partial\mathbb{D}} TL = T\partial\mathbb{D} \oplus \mathrm{Re}(\mathfrak{L}_1) \oplus \cdots \oplus \mathrm{Re}(\mathfrak{L}_n)$. As in [25, proof of Lemma 5.19], we see that the kernel of the linearised $\bar{\partial}$ operator is isomorphic to

$$T_{\mathrm{Id}}\mathrm{Aut}(\mathbb{D}) \bigoplus_{i=1}^{n-1} \mathrm{hol}((\mathbb{D}, \partial\mathbb{D}), (\mathbb{C}, \mathbb{R}))$$

Hence the kernel has dimension $n + 2 = n + \mu_{\Theta^n_s}(u) = \text{index}$. □

We proceed to:

Proof of Lemma 3.2. Since the K action is holomorphic and ∂u is transverse to the K-orbits, we can build $u_{\theta_1,\dots,\theta_n}$ from a neighbourhood of $\mathrm{Id} \in K$, satisfying all the hypothesis of Lemma 3.4. □

4. The Lagrangian tori Θ^n_s

In this section we give an explicit description of the tori Θ^n_s and of its potential function, which encodes the number of Maslov index 2 disks that Θ^n_s bounds. For a definition of the potential, we refer the reader to [14, Section 4],[16]. See also the definition of superpotential in [3, Section 2.2].

The tori Θ^n_s appears as fibres of a singular Lagrangian fibration analogous to the one described in [3, Example 3.3.1].

4.1. Definition of Θ^n_s

Consider $(\mathbb{C}P^1)^n$ with the standard symplectic form, for which the symplectic area of each $\mathbb{C}P^1$ factor is 1. For $1 \leq i \leq n$, let $[x_i : y_i]$ denote the i-th coordinate of $(\mathbb{C}P^1)^n$. Consider the function $f = \prod_i \frac{x_i}{y_i}$, defined from the complement of $V = \bigcup_{i,j} \{x_i = 0\} \cap \{y_j = 0\}$ to $\mathbb{C}P^1$, whose fibres are preserved by the T^{n-1} action given by

$$
\begin{aligned}
(4.1) \qquad & (\theta_1, \dots, \theta_{n-1}) \cdot ([x_1 : y_1], \dots, [x_{n-1} : y_{n-1}], [x_n : y_n]) \\
& = ([e^{\theta_1} x_1 : y_1], \dots, [e^{i\theta_{n-1}} x_{n-1}, y_{n-1}], [e^{-i\sum_j \theta_j} x_n : y_n]),
\end{aligned}
$$

and $\mathrm{m} : (\mathbb{C}P^1)^n \to \mathbb{R}^{n-1}$ its moment map.

Definition 4.1. Let γ be an embedded circle on $\mathbb{C}^\star$, not enclosing $0 \in \mathbb{C}$, and $\lambda \in \mathbb{R}^{n-1}$. Define the Θ^n-type Lagrangian torus:

$$\Theta^n_{\gamma,\lambda} = \{x \in (\mathbb{C}P^1)^n \setminus V; f(x) \in \gamma, \mathrm{m}(x) = \lambda\}$$

Noting that $\mathrm{m}^{-1}(0) = \{|x_i/y_i| = |x_n/y_n|, \forall i = 1, \dots, n-1\}$, one can see, by using the maximum principle, that $\Theta^n_{\gamma,0}$ bounds only one $(n-1)$-family of holomorphic disks that project injectively to the interior of γ. Call $\beta_\gamma \in \pi_2((\mathbb{C}P^1)^n, \Theta^n_{\gamma,0})$ the class represented by each of the above disk. We note that there are n disjoint holomorphic disks in the class β_γ inside the line $\Delta = \{[x_i : y_i] = [x_n : y_n], \forall i = 1, \dots, n-1\}$. Since $\int_\Delta \omega = n$, we see that $\int_{\beta_\gamma} \omega \in (0, 1)$.

Foliate $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ by curves γ_s, $s \in [0, 1)$ so that γ_0 is a point, say $1 \in \mathbb{C}$, and for $s \in (0, 1)$, γ_s is an embedded circle so that $\int_{\beta_{\gamma_s}} \omega = s$.

Definition 4.2. Define the Lagrangian torus Θ^n_s to be $\Theta^n_{\gamma_s,0}$.

The hamiltonian isotopy class of Θ^n_s, does not depend in the curve γ_s inside $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, but only on $s = \int_{\beta_{\gamma_s}} \omega$.

Consider the divisor $D = f^{-1}(1) \bigcup_i \{y_i = 0\}$ and the holomorphic n-form $\Omega = (\prod_i x_i - 1)^{-1} dx_1 \wedge \dots \wedge dx_n$ defined on $(\mathbb{C}P^1)^n \setminus D$, in coordinates charts $y_i = 1$.

Proposition 4.3 (Auroux). *The tori Θ^n_s are special Lagrangians [2, Definition 2.1] with respect to Ω*

Proof. See [3, Example 3.3.1] and [2, Proposition 5.2]. □

Also, we clearly have:

Proposition 4.4. *We have that* $((\mathbb{C}P^1)^n, \Theta_s^n)$ *is* T^{n-1}*-pseudohomogeneous, for the action* (4.1).

4.2. The Potential of Θ_s^n

We come back to our Lagrangian tori Θ_s^n. We would like to describe the potential $\mathfrak{PO}^L$ in coordinates of the form (2.3) with respect to a nice basis for $\pi_2((\mathbb{C}P^1)^n, \Theta_s^n)$. Fix a point $a_s \in \gamma_s$. Consider the S^1 action given by the i-th coordinate of the T^{n-1} action described in (4.1). Take the orbit lying in $\Theta_s^n \cap f^{-1}(a_s)$ and consider its parallel transport over the segment $[0, a_s]$, formed by orbits of the considered S^1 action that collapse to a point over 0, giving rise to a Lagrangian disk. Define $\alpha_i \in \pi_2(\Theta_s^n, (\mathbb{C}P^1)^n)$ to be the class of the above disk. Also, from now one we write $\beta = \beta_{\gamma_s}$ and $H_i = p_i^*[\mathbb{C}P^1] \in \pi_2((\mathbb{C}P^1)^n)$ the pullback of the class of the line by the i-th projection. Note that $\beta, \alpha_1, \dots, \alpha_{n-1}, H_1, \dots, H_n$ are generators of $\pi_2((\mathbb{C}P^1)^n, \Theta_s^n)$. We assume that our monotone symplectic form is so that $\int_{H_i} \omega = 1$.

Set $u = z_\beta$ and $w_i = z_{\alpha_i}$, $i \in (1, \dots, n-1)$. Note that

$$z_{H_i}(\nabla') = T^{\int_{H_i} \omega} \exp(b \cap \partial H_i) = T.$$

Proposition 4.5 ([2, 3]). *The potential function encoding the count of Maslov index 2 holomorphic disks with boundary on the Lagrangian tori* Θ_s^n *(for some spin structure) is given by*

$$\mathfrak{PO}^{\Theta_s^n} = u + \frac{T}{u}(1 + w_1 + \cdots + w_{n-1}) \left(1 + \frac{1}{w_1} + \cdots + \frac{1}{w_{n-1}}\right) \tag{4.2}$$

Idea of proof. First we consider positivity of intersection of an holomorphic disk with the complex submanifolds $\{x_i = 0\}$, $\{y_i = 0\}$, $\{\prod_i x_i = \prod_i y_i\}$, for all $i \in (1, \dots, n)$, to conclude that Maslov index 2 classes admitting holomorphic representatives must be of the form β, $H_i - \beta - \alpha_i + \alpha_j$, where $i, j = 1, \dots, n$ and $\alpha_n = 0$. Computations of the holomorphic disks and their algebraic count can be done explictly. We omit here since it follows a straightforward procedure as in [2, Proposition 5.12], see final remark after Proposition 3.3 in [3]. See also [25, Section 5] for similar computations.

We can choose a spin structure so that every disk counts positively, i.e., $\mathrm{ev}_0 : \mathscr{M}_1 \to \Theta_s^n$ is orientation preserving, e.g. by choosing a trivialisation of $T\Theta_s^n$ using the boundary of $\{\alpha_1, \dots, \alpha_{n-1}, \beta\}$, as spin structure. See [25, Section 5.5] and [8, Section 8], for a complete discussion in a similar scenario. □

Remark 4.6. The potential of Θ^n_s can be obtained from the known potential for the Clifford torus, $\underset{n}{\times} S^1_{\mathrm{eq}}$. It is given by

$$\mathfrak{PO}^{\mathrm{Clif}} = z_1 + \cdots + z_n + \frac{T}{z_1} + \cdots + \frac{T}{z_n}.$$

We obtain the potential for Θ^n_s via wall-crossing transformation $u = z_n(1 + w_1 + \cdots w_{n-1})$, $w_i = z_i/z_n$. See [3, Example 3.3.1].

Proposition 4.7. *The tori Θ^n_s satisfy Assumption 2.1, with respect to the standard complex structure of $(\mathbb{C}P^1)^n$.*

Proof. To prove Assumption (A_1) we use similar argument as in [2, Example 3.3.1]. First we use that Θ^n_s are special Lagrangians, and hence, by [2, Lemma 3.1], the Maslov index is twice the intersection with the divisor D. This shows that $\mu_{\Theta^n_s}(\beta) \geq 0$, $\forall \beta \in \pi_2((\mathbb{C}P^1)^n, \Theta^n_s)$ represented by an holomorphic disk u. Now, if u is a Maslov index 0 holomorphic disk, then $f \circ u$ is well define and lies in $\mathbb{C} \setminus \{1\}$, hence it is a constant in γ_s. Since the regular fibres of f are diffeomorphic to $(\mathbb{C}^*)^{n-1}$, we have that u is itself is constant.

The proof of Assumption (A_2) follows from $((\mathbb{C}P^1)^n, \Theta^n_s)$ being T^{n-1}-pseudohomogeneous together with Lemma 3.2. We just need to check that since the T^{n-1}-orbit in Θ^n_s is generated by $\partial\alpha_i$, therefore transverse to the boundary of the Maslov index 2 disks with boundary in Θ^n_s, whose relative homotopy classes are β and $H_i - \beta - \alpha_i + \alpha_j$, $i, j = 1, \ldots, n$ and $\alpha_n = 0$. □

4.3. Regarding Proposition 1.4, and Conjecture 1.5

We start noting that Maslov index 2 classes in $H_2((\mathbb{C}P^1)^n, \Theta^n_s; \mathbb{Z})$ are of the form

$$\beta + k_1(H_1 - 2\beta) + \cdots + k_n(H_n - 2\beta) + l_1\alpha_1 + \cdots + l_{n-1}\alpha_{n-1}, \tag{4.3}$$

where β is the Maslov index 2 and α_i the Maslov index 0 classes described in Section 4.2, viewed in $H_2((\mathbb{C}P^1)^n, \Theta^n_s; \mathbb{Z})$ via $\pi_2((\mathbb{C}P^1)^n, \Theta^n_s) \hookrightarrow H_2((\mathbb{C}P^1)^n, \Theta^n_s; \mathbb{Z})$. Recalling that $\int_{H_i} \omega = 1$ and $\int_{\alpha_i} \omega = 0$, we see that area of Maslov index 2 disks belongs to $\{s + (1 - 2s)\mathbb{Z}\} \subset \mathbb{R}$.

Proof of Proposition 1.4. We note that each torus

$$\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\mathrm{eq}})^{n - \sum_i k_i}$$

bounds a disk of Maslov index 2 and symplectic area $1/2$, if $n > \sum_i k_i$, coming from a Maslov index 2 disk in the last $\mathbb{C}P^1$ factor, with boundary in its equator S^1_{eq}. We see that $1/2$ is in $\{s + (1-2s)\mathbb{Z}\}$ if and only if $s = 1/2$. This rules out the possibility of $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\text{eq}})^{n-\sum_i k_i}$ being symplectomorphic to Θ^n_s for $s \neq 1/2$.

For $s = 1/2$ the torus Θ^n_s is monotone, hence the Maslov index 2 J-holomorphic disks becomes an invariant of its symplectomorphism class — this was first pointed out in [10], see also [25, Theorem 6.4]. This invariant allows us to distinguish between (the symplectomorphism classes of) Θ^n_s and $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\text{eq}})^{n-\sum_i k_i}$. For instance, one could look for pairs (σ_1, σ_2) of (relative homotopy classes represented by) Maslov index 2 holomorphic disks with $\partial\sigma_1 = -\partial\sigma_2$. For the torus Θ^n_s, we must have $\partial\sigma_i = \pm\partial\beta$, i.e., only one possibility for $\partial\sigma_i$ modulo sign, see Proposition 4.5. But for each torus $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\text{eq}})^{n-\sum_i k_i}$ we have more than one possibility for $\partial\sigma_i$, modulo sign. □

Remark 4.8. Note that, by Proposition 4.5, the total number of Maslov index 2 holomorphic disks with boundary in Θ^n_s is $1 + n^2$, while for the tori $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l} \times (S^1_{\text{eq}})^{n-\sum_i k_i}$ it is $\sum_{i=1}^l (1 + k_i^2) + 2(n - \sum_{i=1}^l k_i) = 2n + \sum_{i=1}^l (k_i - 1)^2$. Hence they can be equal if $(n-1)^2 = \sum_{i=1}^l (k_i - 1)^2$.

Remark 4.9. The above argument also proves the monotone version ($s = 1/2$) of Conjecture 1.5.

We proceed now to show that holomorphic disks with boundary in Θ^n_s with Maslov index bigger than 2 have area bigger than $a = 1 - s$ — the minimal area of Maslov index 2 holomorphic disks for $s > 1/2$.

Proposition 4.10. *For $k > 0$ and $s \in [1/2, 1)$, the area of holomorphic Maslov index $2k$ disk with boundary on Θ^n_s is least $1 - s$, with respect to the standard complex structure in $(\mathbb{C}P^1)^n$. The minimum only occur if $k = 1$.*

Proof. Maslov index $2k$ disks are in relative classes of the form

$$k\beta + k_1(H_1 - 2\beta) + \cdots + k_n(H_n - 2\beta) + l_1\alpha_1 + \cdots + l_{n-1}\alpha_{n-1}. \tag{4.4}$$

If they are represented by holomorphic disks, their intersection with the divisors $\{y_i = 0\}$ and $\{\prod_{i=1}^n x_i = \prod_{i=1}^n y_i\} = \overline{\{f^{-1}(1)\}}$ is non-negative —

recall from Definitions 4.1, 4.2 that 1 is in the interior of $\gamma \subset \mathbb{C}^*$. Noting that

$$\beta \cdot \{y_i = 0\} = 0, \quad \alpha_j \cdot \{y_i = 0\} = 0, \quad H_j \cdot \{y_i = 0\} = \delta_{ij},$$

and

$$\beta \cdot \overline{\{f^{-1}(1)\}} = 1, \quad \alpha_j \cdot \overline{\{f^{-1}(1)\}} = 0, \quad H_j \cdot \overline{\{f^{-1}(1)\}} = 1,$$

$i, j = 1, \dots, n$, we get that

$$k_i \geq 0 \ \forall i = 1, \dots, n \quad \text{and} \quad k - \sum_{i=1}^{n} k_i \geq 0.$$

The result follows from taking the symplectic area of (4.4), which is

$$ks + \sum_{i=1}^{n} k_i(1 - 2s) = s\left(k - \sum_{i=1}^{n} k_i\right) + (1 - s)\left(\sum_{i=1}^{n} k_i\right)$$

□

As pointed out before the above Proposition allows us to argue why Conjecture 1.5 holds, using [23, Proposition 5.1]. Indeed, for $s > 1/2$, the number of Maslov index 2 holomorphic disks with boundary in Θ^n_s and with minimal area $a = 1 - s$ is n^2, by Proposition 4.5. Hence the number of Maslov index 2 disks with boundary in $\Theta^{k_1}_{s_1} \times \cdots \times \Theta^{k_l}_{s_l}$ and with minimal area is at most $\sum_{i=1}^{l} k_i^2 < (\sum_{i=1}^{l} k_i)^2 = n^2$, for $l > 1$.

5. Proof of Theorem 1.1 — Bulk deformations

In this section we use bulk deformations to prove that the tori Θ^n_s are non-displaceable for n even and $s \in [1/2, 1)$, as done in [19] for the case $n = 2$. In [19], Fukaya-Oh-Ohta-Ono used the cocycle Poincaré dual to the anti-diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$ to bulk-deform Floer-homology. In this section we will bulk-deform Floer-homology by an element of the form $T^\rho[h] \in H^*((\mathbb{C}P^1)^n, \Lambda_+)$, where $[h] \in H^2((\mathbb{C}P^1)^n, \mathbb{Z})$.

For $1 \leq i \leq n$, let h_i be the cocycle Poincaré dual to $\{y_i = 0\} \subset (\mathbb{C}P^1)^n$.

Proposition 5.1. *The potential for the Lagrangian tori Θ^n_s, bulk deformed by the cocycle*

$$\mathfrak{b} = T^\rho[(k_1 + k_n)h_1 + \cdots + (k_{n-1} + k_n)h_{n-1} + k_n h_n] \in C^2((\mathbb{C}P^1)^n, \Lambda_+)$$

is given by

$$\mathfrak{PO}_{\mathfrak{b}}^{\Theta_s^n}(b) = u + \frac{T}{u}(1 + w_1 + \cdots + w_{n-1})\left(1 + \frac{e^{k_1 T^\rho}}{w_1} + \cdots + \frac{e^{k_{n-1}T^\rho}}{w_{n-1}}\right)e^{k_n T^\rho}$$

Proof. The relative classes β, α_j have no intersection with $\{y_k = 0\}$ viewed as a cycle in $(\mathbb{C}P^1)^n \setminus \Theta_s^n$. Therefore the disk in the class $H_i - \beta - \alpha_i + \alpha_j$ intersect $\{y_k = 0\}$ if and only if $k = i$, and with multiplicity 1. Hence, the coefficient of the monomial Tw_j/uw_i is bulk-deformed by $\mathfrak{b}_s$ to $e^{(k_i+k_n)T^\rho}$. □

Lemma 5.2. *The potential for the Lagrangian tori Θ_s^n, bulk deformed by the cocycle*

$$\mathfrak{b} = T^\rho[(k_1 + k_n)h_1 + \cdots + (k_{n-1} + k_n)h_{n-1} + k_n h_n] \in C^2((\mathbb{C}P^1)^n, \Lambda_+)$$

have its critical points given by:

$$w_i = \epsilon_i e^{\frac{k_i}{2}T^\rho}, \quad u = \epsilon_n e^{\frac{k_n}{2}T^\rho} T^{\frac{1}{2}}\left(1 + \sum_{i\geq 1}^{n-1} \epsilon_i e^{\frac{k_i}{2}T^\rho}\right),$$

where $\epsilon_i = \pm 1$.

Proof. For easier notation, let $b_i = e^{k_i T^\rho}$. Taking the differential of the bulk deformed potential $\mathfrak{PO}_{\mathfrak{b}}^{\Theta_s^n}(b)$ with respect to w_i and equating to 0, we get, after multiplying by w_i, equations

$$(i): w_i + \sum_{j\neq i} \frac{b_j w_i}{w_j} - b_i\left(\frac{1}{w_i} + \sum_{j\neq i}\frac{w_j}{w_i}\right) = 0. \tag{5.1}$$

Summing all the equations $(1), \ldots, (n)$, we end up with

$$\sum_{i=1}^{n-1} w_i - \sum_{i=1}^{n-1} \frac{b_i}{w_i} = 0$$

Let

$$L = \sum_{i=1}^{n-1} w_i = \sum_{i=1}^{n-1} \frac{b_i}{w_i}.$$

We have that

$$w_i L - b_i = \sum_{j\neq i} \frac{b_j w_i}{w_j},$$
$$\frac{L}{w_i} - 1 = \sum_{j\neq i} \frac{w_j}{w_i}.$$

Substituting the above into equations (i) (see (5.1)), we get that

$$\left(w_i - \frac{b_i}{w_i}\right)(1+L) = 0 \tag{5.2}$$

So if $u, w_1, \dots w_{n-1}$ are critical points of the bulk deformed potential $\mathfrak{PO}^{\Theta^n_s}_{\mathfrak{b}}(b)$, besides equation (5.2), we must have

$$\partial_u \mathfrak{PO}^{\Theta^n_s}_{\mathfrak{b}} = 1 - \frac{b_n T}{u^2}(1+L)^2 = 0 \tag{5.3}$$

Hence $L \neq -1$, and therefore

$$w_i = \sqrt{b_i} = \epsilon_i e^{\frac{k_i}{2}T^\rho}, \;\; u = \sqrt{b_n} T^{\frac{1}{2}}(1+L) = \epsilon_n e^{\frac{k_n}{2}T^\rho} T^{\frac{1}{2}} \left(1 + \sum_{i\geq 1}^{n-1} \epsilon_i e^{\frac{k_i}{2}T^\rho}\right),$$

□

We call the *valuation* of an element in Λ_+ the smallest exponent with non-zero coefficient. Looking at the expression of the critical points of the previous Lemma, one can see that:

Lemma 5.3. *Looking at the critical points given on Lemma 5.2 we have that, the valuation of u is not $1/2$ if and only if $n = 2m$ and $m-1$ ϵ_i's are equal to 1 while the other m ϵ_i's are equal to -1, where $i = 1, \dots, 2m-1$. In that case, the valuation of u is $T^{1/2+\rho}$, provided $\sum_{i=1}^{2m-1} \epsilon_i k_i \neq 0$.*

Now we recall that

$$u = z_\beta = T^s \exp(b \cap \partial\beta)$$

for the class β defined in the beginning of Section 4.2. By Lemma 5.3, we have:

Corollary 5.4. *Take $s > 1/2$ and consider the cocycle $\mathfrak{b}_s = T^{s-1/2}[(k_1 + k_{2m})h_1 + \cdots + (k_{2m-1} + k_{2m})h_{2m-1} + k_{2m}h_{2m}] \in C^2((\mathbb{C}P^1)^{2m}, \Lambda_+)$. Assume that not all k_i's are 0, for $i = 1, \ldots, 2m-1$, i.e., $[\mathfrak{b}_s]$ is not a multiple of the monotone symplectic form. Then there exists b_s a critical point of $\mathfrak{PO}^{\Theta^{2m}_s}_{\mathfrak{b}_s}$.*

Recalling that Θ^{2m}_s satisfy Assumption 2.1 (Propositions 4.7), for some almost complex structure J, and noting that Θ^{2m}_s is a contractible Lagrangian torus of $(\mathbb{C}P^1)^{2m}$, we have that $((\mathbb{C}P^1)^{2m}, \Theta^{2m}_s)$ satisfy all the hypothesis of Corollary 2.8. Therefore, from Corollaries 2.8 and 5.4, we deduce:

Theorem 5.5. *For $s \geq 1/2$ there exists a bulk $[\mathfrak{b}_s] \in H^2((\mathbb{C}P^1)^{2m}, \Lambda_+)$ and a weak bounding cochain $b_s \in H^1(\Theta^{2m}_s, \Lambda_0)$ such that*

$$HF(\Theta^{2m}_s, (b_s, \mathfrak{b}_s); \Lambda_{0,nov}) \cong H(\Theta^{2m}_s, \Lambda_{0,nov}).$$

This proves the first part of Theorem 1.6. Theorem 1.1 follows from Theorem 2.4 and Theorem 5.5. □

Corollary 1.3 follows from the same arguments as above using that

$$\mathfrak{PO}_{\mathfrak{b}}^{\Theta_{s_1}^{k_1} \times \cdots \times \Theta_{s_l}^{k_l} \times (S^2_{\text{eq}})^{n - \sum_i k_i}} = \mathfrak{PO}_{\mathfrak{b}}^{\Theta_{s_1}^{k_1}} + \cdots + \mathfrak{PO}_{\mathfrak{b}}^{\Theta_{s_l}^{k_l}} + \mathfrak{PO}_{\mathfrak{b}}^{(S^2_{\text{eq}})^{n - \sum_i k_i}}$$

6. Quasi-morphisms and quasi-states

In this section we prove the last part of Theorem 1.6. It follows arguments similar to [18, Theorem 23.4].

Lemma 6.1. *For any*

$$\mathfrak{b} = T^{\rho}[l_1 h_1 + \cdots + l_{n-1}h_{n-1} + l_n h_n] \in C^2((\mathbb{C}P^1)^n, \Lambda_+),$$

the bulk deformed Quantum cohomology [18, Section 5] is semi-simple.

Proof. By [20, Theorem 1.1.1] (see also [14, Theorem 6.1], for the Fano case) we have an isomorphism between the bulk deformed Quantum cohomology of a toric symplectic manifold and the Jacobian Ring of the bulk deformed toric potential. If the bulk deformed toric potential has only non-degenerate critical points, we can split the Quantum cohomology ring into orthogonal algebra summands according to the factors corresponding to the critical points under the isomorphism given in [20, Theorem 1.1.1].

Naming now $z_i = z_{\beta_i}$ (2.3), for β_i the class of Maslov index 2 holomorphic disk intersecting $\{x_i = 0\}$, we have that the bulk deformed potential of a toric fiber is:

$$\mathfrak{PO}_{\mathfrak{b}} = z_1 + \cdots + z_n + \frac{Te^{l_1 T^\rho}}{z_1} + \cdots + \frac{Te^{l_n T^\rho}}{z_n}, \tag{6.1}$$

whose critical points are given by

$$(z_1, \ldots, z_n) = (\epsilon_1 T^{1/2} e^{l_1 T^\rho/2}, \ldots, \epsilon_n T^{1/2} e^{l_n T^\rho/2}).$$

Hence, there are 2^n idempotents of $QH_{\mathfrak{b}}((\mathbb{C}P^1)^n; \Lambda_{0,nov})$, $\mathbf{e}_1^{\mathfrak{b}}, \ldots, \mathbf{e}_{2^n}^{\mathfrak{b}}$ for which

$$QH_{\mathfrak{b}}((\mathbb{C}P^1)^n; \Lambda_{0,nov}) = \bigoplus_{i=1}^{2^n} \Lambda_{0,nov} \mathbf{e}_i^{\mathfrak{b}}.$$

□

In [18, Section 17, (17.18)], given X a symplectic manifold and L a relatively spin Lagrangian submanifold, Fukaya-Oh-Ohta-Ono construct an homomorphism:

$$i^*_{\mathrm{qm},(b,\mathfrak{b})} : QH_{\mathfrak{b}}(X; \Lambda_{0,nov}) \to HF(L, (b, \mathfrak{b}); \Lambda_{0,nov}), \tag{6.2}$$

which is proven to be a ring homomorphism in [1], see [18, Remark 17.16] and [15, Section 4.7].

Applying Lemma 6.1 for $(\mathbb{C}P^1)^{2m}$ and $\mathfrak{b}_s$ given in Theorem 5.5, using that $i^*_{\mathrm{qm},(b_s,\mathfrak{b}_s)}$ is unital and $HF(\Theta_s^{2m}, (b_s, \mathfrak{b}_s); \Lambda_{0,nov}) \neq 0$, we have:

Proposition 6.2. *There exists an idempotent $\boldsymbol{e}_s \in QH_{\mathfrak{b}_s}((\mathbb{C}P^1)^{2m}; \Lambda_{0,nov})$ for which $i^*_{\mathrm{qm},(b_s,\mathfrak{b}_s)}(\boldsymbol{e}_s) \neq 0$ in $HF(\Theta_s^{2m}, (b_s, \mathfrak{b}_s); \Lambda_{0,nov})$.*

Theorem 1.6 follows then from Proposition 6.2 and Theorem 18.8 of [18]. □

7. Tori in $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$

In this section we prove Theorem 1.12. We will describe a model for

$$(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega_\epsilon) = (\mathbb{C}P^1 \times \mathbb{C}P^1 \# 2\overline{\mathbb{C}P^2}, \omega_\epsilon)$$

which is equivalent to performing two blowups of capacities ϵ centred at the rank 0 elliptic singularities (corners) of the singular fibration of $\mathbb{C}P^1 \times \mathbb{C}P^1$ described in [19], see Figure 1.

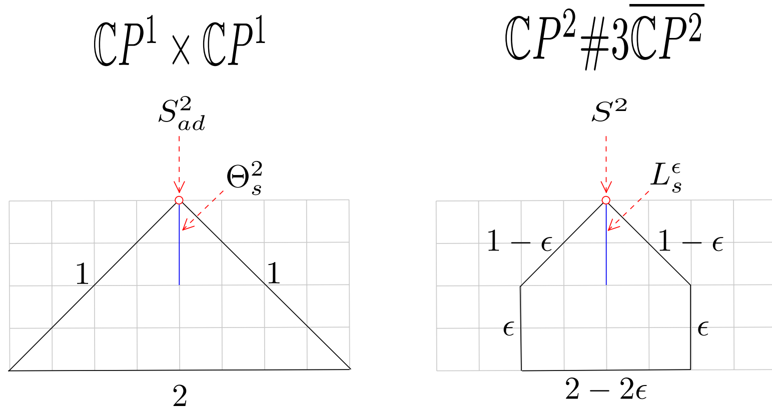

Figure 1: Singular fibrations of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$.

Consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ with coordinates $([x_1 : y_1], [x_2 : y_2])$ as in Section 4.1. Consider also the tori Θ^2_s, the function $f = x_1x_2/y_1y_2$, the relative class β and $\alpha := \alpha_1$ and the divisor $D = f^{-1}(1) \cup \{y_1 = 0\} \cup \{y_2 = 0\}$, as defined in Section 4.2.

From Proposition 4.3 and [2, Lemma 3.1], we have that $2[D] \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \Theta^2_s)$ is Poincaré dual to the Maslov class $\mu_{\Theta^2_s} \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1, \Theta^2_s)$. In particular the Maslov index 2 holomorphic disks, computed in Proposition 4.5 for $n = 2$, do not intersect $\overline{f^{-1}(1)} \cap \{y_1 = 0\} = ([1 : 0], [0 : 1]) = p_1$ and $\overline{f^{-1}(1)} \cap \{y_2 = 0\} = ([0 : 1], [1 : 0]) = p_2$.

Let $B_i(\epsilon)$ be the ball of capacity [21, Section 12] ϵ (radius $\sqrt{\epsilon/\pi}$) centered at p_i, in the coordinate plane $x_i = 1$, $y_j = 1$, $i, j = 1, 2$, $i \neq j$. Denote $S_i(\epsilon) = \partial B_i(\epsilon)$. Let $(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \omega_\epsilon)$ be the result of blowing up [21, Section 7] $\mathbb{C}P^1 \times \mathbb{C}P^1$ with respect to $B_1(\epsilon)$ and $B_2(\epsilon)$, so that the exceptional curves E_i (coming from collapsing the Hopf fibration in $S_i(\epsilon)$) have symplectic area $\omega_\epsilon(E_i) = \epsilon$, $i = 1, 2$. Let j_ϵ be the induced complex structure and L^ϵ_s correspond to Θ^2_s after the blowup. Note that ϵ can take any value in $(0, 1)$, so that $B_1(\epsilon) \cap B_2(\epsilon) = \emptyset$.

Note also that $f = x_1x_2/y_1y_2$ is constant along the fibers of the Hopf fibration of both $S_1(\epsilon)$ and $S_2(\epsilon)$. In particular it give rise to a (j_ϵ, j)-holomorphic function $\tilde{f} : \mathbb{C}P^2\#3\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$.

For computing the potential for L^ϵ_s it is interesting that the disks of Proposition 4.5, remain essentially the same. This can be obtained by stretching the complex structure j_ϵ. So take δ small enough so that $B_1(\delta) \cup B_2(\delta)$ does not intersect any Maslov index 2 holomorphic disk. Consider a diffeomorphism $\varphi : (\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \omega_\epsilon) \to (\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \omega_\delta)$ coming from a finite neck stretch

[5, 9] along $S_i(\epsilon + \delta') \subset (\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega_\epsilon)$ [5, 9], see also [26, Section 3], which sends L_s^ϵ to L_s^δ. The diffeomorphism φ is equivalent to considering an inflation along the exceptional curves E_i, $i = 1, 2$. Set $J_\delta = \varphi^* j_\delta$, an ω_ϵ compatible almost complex structure.

Lemma 7.1. *We have that $(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, L_s^\epsilon, J_\delta)$ satisfy Assumption 2.1. The potential function for L_s^ϵ with respect to J_δ, is given by:*

$$\mathfrak{PO}^{L_s^\epsilon} = u + \frac{T}{u}(1 + w)\left(1 + \frac{1}{w}\right) + T^{1-\epsilon}\left(w + \frac{1}{w}\right) \tag{7.1}$$

Proof. It is enough to compute the j_δ-holomorphic disks with boundary in L_s^δ. The j_δ-holomorphic disks that don't intersect the exceptional divisors E_1, E_2, corresponds to the holomorphic disks in $\mathbb{C}P^1 \times \mathbb{C}P^1$ with boundary in Θ_s^2, which gives the terms

$$u + \frac{T}{u}(1 + w)\left(1 + \frac{1}{w}\right)$$

of $\mathfrak{PO}^{L_s^\delta}$, and are regular.

Let $\tilde{D}$ be the proper transform of the divisor $D \in \mathbb{C}P^1 \times \mathbb{C}P^1$. It can be checked that, twice $\tilde{D} + E_1 + E_2$ is Poincaré dual to the Maslov class $\mu_{L_s^\delta}$. This implies Assumption (A_1), as in the proof of Proposition 4.7. Moreover, Maslov index 2 disks intersects $\tilde{D} + E_1 + E_2$ once. Which means that if a j_δ-holomorphic disk u intersects either E_1 or E_2, by positivity of intersection, it does not intersect $\tilde{D}$ and hence $\tilde{f} \circ u : \mathbb{D} \to \mathbb{C}^*$ must be constant. There are two Maslov index 2 disks in the fiber $\tilde{f}^{-1}(c)$, for $c \in \gamma_s$. Looking at the intersections with E_i, and the proper transform of $\{x_i = 0\}$ and $\{y_i = 0\}$, we can see that the relative classes of these disks are $H_1 - E_1 + \alpha$ and $H_2 - E_2 - \alpha$ (for some orientation of α). Since, $\omega_\epsilon(H_i - E_i \pm \alpha) = 1 - \epsilon$, we get the remaining term

$$T^{1-\epsilon}\left(w + \frac{1}{w}\right).$$

To show regularity of the above disks, one notes that the pre-image under $\tilde{f}$ of a small neighbourhood $\mathcal{N}_s$ of γ_s contain the whole family of the above disks and is actually toric. Moreover, $(\tilde{f}^{-1}(\mathcal{N}_s), L_s^\delta)$ is T^2-homogeneous [13], or if you will, S^1-pseudohomogeneous (Definition 3.1) for a j_δ-holomorphic S^1-action transverse to $\partial\alpha$, which shows Assumption (A_2).

The choice of spin structure is given by trivialising TL_s^ϵ according to $\{\alpha, \beta\}$ and is so that the evaluation map is orientation preserving, as in the proof of Proposition 4.5. See also [25, Section 5.5] and [8, Section 8]. □

Remark 7.2. The above potential can also be computed by a technique similar to the one developed in [19] and also by some gluing procedure similar to the one developed in Section 5.2 of the `arXiv.1002.1660v1` version of [19] and in [28].

Remark 7.3. For each $\delta' > 0$, the family $\{L_s^\epsilon : s \in [1/2, 1-\delta']\}$ can be seen as fibres of an almost toric fibration (ATF) of $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, represented by an almost toric base diagram (ATBD) analogous to the one in Figure 9 (A_3) of [27]. In fact, the singular fibration described by the second diagram in Figure 1 can be thought as a limit of ATFs described by sliding nodes of the ATBD in Figure 9 (A_3) of [27]. Moreover, the potential $\mathfrak{PO}^{L_s^\epsilon}$ can be obtained from the toric potential

$$\mathfrak{PO}^{\mathrm{toric}} = u_1 + u_2 + \frac{T}{u_1} + \frac{T}{u_2} + \frac{T^{1-\epsilon}u_1}{u_2} + \frac{T^{1-\epsilon}u_2}{u_1},$$

via wall-crossing transformation $u = u_1(1+w)$, $w = u_2/u_1$, giving another example where actual computations meet wall-crossing predictions [2, 3, 25].

Let $\mathfrak{s} \in C^2(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2})$ be the cocycle Poincaré dual to $\{y_1 = 0\} \cup E_1$, so $[\mathfrak{s}] = H_1 - E_2 + E_1$. Analogous to Proposition 5.1, we have:

Proposition 7.4. *The potential for L_s^ϵ, bulk deformed by the cocycle $\mathfrak{b} = T^\rho\mathfrak{s} \in C^2(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \Lambda_+)$ is given by:*

$$\mathfrak{PO}_{\mathfrak{b}}^{L_s^\epsilon} = u + \frac{T}{u}(1+w)\left(e^{T^\rho} + \frac{1}{w}\right) + T^{1-\epsilon}\left(e^{T^\rho}w + \frac{1}{w}\right). \tag{7.2}$$

We can then compute the critical points of $\mathfrak{PO}_{\mathfrak{b}}^{L_s^\epsilon}$ and obtain:

Lemma 7.5. *We have that $w = -e^{\frac{-T^\rho}{2}}$ and $u = \pm T^{\frac{1}{2}}(1 - e^{\frac{-T^\rho}{2}})^{\frac{1}{2}}(e^{T^\rho} - e^{\frac{T^\rho}{2}})^{\frac{1}{2}}$ are critical points of $\mathfrak{PO}_{\mathfrak{b}}^{L_s^\epsilon}$. The valuations of w and u are respectively* 0 *and* $1/2 + \rho$.

Since we have that $\int_\beta \omega_\epsilon = s$ and $\int_\alpha \omega_\epsilon = 0$:

Lemma 7.6. *For $s > 1/2$ and $\mathfrak{b}_s^\epsilon = T^{s-1/2}[\mathfrak{s}]$, there exists a weak bounding cochain $b_s^\epsilon \in H^1(L_s^\epsilon, \Lambda_0)$ which is a critical point of $\mathfrak{PO}_{\mathfrak{b}_s^\epsilon}^{L_s^\epsilon}$.*

Following similar arguments as in Sections 5 and 6, we are able to prove Theorem 1.12 and consequently Theorem 1.10. □

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Dept. Matemática, Instituto de Matemática
Universidade Federal do Rio de Janeiro (UFRJ)
Centro de Tecnologia — Bloco C, Cidade Universitária
Av. Athos da Silveira Ramos, 149 — Ilha do Fundão
21941-909 Rio de Janeiro, Brazil
E-mail address: r0387843@gmail.com

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