

Algebraic symplectic analogues of additive quotients

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Motivated by the study of hyperkähler structures in moduli problems and hyperkähler implosion, we initiate the study of non-reductive hyperkähler and algebraic symplectic quotients with an eye towards those naturally tied to projective geometry, like cotangent bundles of blow-ups of linear arrangements of projective space. In the absence of a Kempf-Ness theorem for non-reductive quotients, we focus on constructing algebraic symplectic analogues of additive quotients of affine spaces, and obtain hyperkähler structures on large open subsets of these analogues by comparison with reductive analogues. We show that the additive analogue naturally arises as the central fibre in a one-parameter family of isotrivial but non-symplectomorphic varieties coming from the variation of the level set of the moment map. Interesting phenomena only possible in the non-reductive theory, like non-finite generation of rings, already arise in easy examples, but do not substantially complicate the geometry.

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1. Introduction

Some of the most fascinating moduli spaces arise as hyperkähler or algebraic symplectic analogues of simpler moduli spaces. The classical example is the moduli space of Higgs bundles over a curve, which contains the cotangent bundle to the moduli space of vector bundles as a dense open subset. The theory of Higgs bundles and the Hitchin fibration is extraordinarily fruitful in several areas of mathematics [8, 10]. Another example is Nakajima quiver varieties [18], which are algebraic symplectic analogues of moduli spaces of quiver representations, and play an important role in representation theory. Each of these spaces, along with their analogues, arises via a quotient procedure from a group action on an affine space. These examples suggest that hyperkähler analogues of other moduli problems, particularly those arising as quotients of affine spaces, might bear similar fruit. One could dream that such a picture may emerge in the other archetypical moduli problem, the moduli of curves. Independently, especially given work on hyperkähler analogues of toric varieties, it is worth asking how much of the basic structure of hyperkähler quotients can be extended to nice enough projective varieties and their cotangent bundles. In fact, it has recently been shown that a wide range of projective varieties, including blow-ups of projective space along projective linear arrangements, can be described as a quotient of affine spaces by a solvable group action [6]. The moduli space of stable pointed rational curves $\overline{M}_{0,n}$ admits such a description; so one can begin to explore the dream.

On the one hand, hyperkähler quotients by torus actions have been extensively studied for many years. On the other, hyperkähler implosion, recently developed in a series of papers by Dancer, Kirwan and their co-authors, can be thought of as addressing very special cases of unipotent actions via the study of a universal imploded cross-section [2–4]. Our motivation is to broaden the class of unipotent actions amenable to a hyperkähler quotient-like construction, while staying tied to smooth projective geometry, by virtue of using the full solvable group action. In effect, the unipotent actions to consider are graded in the sense of [1], where the torus action arises via the universal torsor of the projective variety as in [6]. One can then elide the technical difficulties of the stratified singularities that arise in implosion, because these are contained in unstable loci for the torus action, for a suitable ample character. The present paper initiates this general

study by focusing on the base case of linear actions of the additive group which, when suitably modified to non-linear additive actions and put in a family together with a Néron-Severi torus action, suffice to realise $\overline{M}_{0,n}$ as a quotient of affine space.

The theory of algebraic symplectic and hyperkähler analogues for reductive quotients of affine spaces is reasonably well understood following work of Proudfoot [19], which unifies the construction of hypertoric varieties and Nakajima quiver varieties. The construction of these analogues is given by an algebraic symplectic or hyperkähler reduction of the cotangent lift of the action, where one takes a quotient of a level set of a moment map on the cotangent bundle. Over the complex numbers, the algebraic symplectic and hyperkähler analogues of a reductive quotient of an affine space are homeomorphic by the Kempf–Ness Theorem [11], which relies on the fact that a complex reductive group is the complexification of its maximal compact subgroup. More precisely, the algebraic symplectic analogue is constructed by taking a geometric invariant theory (GIT) quotient of the zero level set of an algebraic moment map by the reductive group and the hyperkähler analogue is a topological quotient by the maximal compact group of the intersection of the algebraic moment map zero level set with a smooth moment map level set.

For non-reductive groups, one does not have a Kempf–Ness Theorem relating GIT quotients and symplectic reductions and, although one may expect such a result, it is not clear what shape it should take, as the maximal compact subgroup of the additive group is trivial. Until this hope is realised, we can only focus on the algebraic symplectic side; however, we will see below that the additive algebraic symplectic analogue inherits a hyperkähler structure via a comparison with a reductive algebraic symplectic analogue.

In the simplest cases the additive theory provides new phenomena. We show how to produce examples where the ring of invariant functions on the zero level set of the algebraic moment map is non-finitely generated, and yet still work with the spaces in practice without difficulty. Strikingly, as we show in Theorems 3.13 and 3.14, the additive theory, in contrast to the reductive theory, gives a natural one-parameter family $\mathcal{X} \rightarrow \mathbb{C}^*$ constructed as smooth algebraic symplectic reductions at non-zero values in \mathbb{C}^* , which is isotrivial as a family of varieties, but not as a family of algebraic symplectic varieties. The central fibre of this family is singular and is the extrinsic quotient of the zero level set obtained by using invariant functions on the cotangent bundle (although, there exist invariant functions on the zero level set which do not extend to global invariant functions on the cotangent bundle). The algebraic symplectic analogue should be a quotient of the zero

level set of the algebraic moment map, which is singular. However, we avoid these singularities by presenting an algebraic symplectic reduction which is a quotient of a dense open subset of the zero level set, determined by stability analysis; in Proposition 3.16, we show that this algebraic symplectic reduction is also an open subset of the central fibre of our 1-parameter family and provides an algebraic symplectic analogue of a dense open subset of the additive quotient of the affine space. Conveniently, this also circumvents the need for the ring of invariant functions on this zero level set to be finitely generated.

A fruitful idea of non-reductive geometric invariant theory is to embed a non-reductive group in a reductive group and use techniques of reductive GIT [7]. We can embed \mathbb{G}_a in SL_2 as the upper triangular unipotent subgroup. Over the complex numbers, any linear action of the additive group extends to SL_2 and the ring of \mathbb{G}_a -invariants on the affine space V is naturally isomorphic to the ring of SL_2 -invariants on $W := V \times \mathbb{A}^2$ and so, in particular, is finitely generated (this is Weitzenböck's Theorem [21]). Furthermore, we have an isomorphism

$$V//\mathbb{G}_a := \mathrm{Spec} \mathbb{C}[V]^{\mathbb{G}_a} \cong \mathrm{Spec} \mathbb{C}[W]^{\mathrm{SL}_2} =: W//\mathrm{SL}_2.$$

Theorem 4.4 gives a direct comparison between the algebraic symplectic structures on the additive algebraic symplectic analogue of $V//\mathbb{G}_a$ and the reductive algebraic symplectic analogue of $W//\mathrm{SL}_2$, realised as a birational symplectomorphism differing in codimension at least two. Then, via the Kempf-Ness Theorem for SL_2 , we establish a hyperkähler structure on a dense open subset of the additive algebraic symplectic analogue, in a way similar to the implosion picture. Since every linear algebraic group G over the complex numbers is isomorphic to a semidirect product $G_u \rtimes G_r$ of a reductive group G_r with a unipotent group G_u and G_u admits a subnormal series whose quotients are isomorphic to \mathbb{G}_a , one can perform symplectic reduction in stages by taking reductions by \mathbb{G}_a followed by a symplectic reduction by a reductive group; thus, it suffices to understand additive and reductive symplectic reductions. Therefore, in this paper, we focus symplectic reduction for the additive group.

The outline of this paper is as follows. Section 2 is a summary of the reductive quotient theory, detailing algebraic moment maps for cotangent lifts of the action, symplectic and hyperkähler reduction and the results of Proudfoot on hyperkähler analogues. Section 3 addresses the additive case, describing the moment map and the geometry and stability properties of the level sets. We study the non-zero level set case and the zero level set

case separately. The main results are Theorems 3.13 and 3.14 describing the algebraic symplectic reductions for non-zero values in a one-parameter family, followed by Proposition 3.16 presenting an algebraic symplectic reduction of a dense open subset of the zero level set, determined by stability analysis. Section 4 explicitly compares the additive analogue to the familiar SL_2 -analogue, summarised in Theorem 4.4, and as a corollary of this result, we obtain a hyperkähler structure on a dense open subset of the additive analogue. In Section 5, we show how the whole story relates to projective geometry, by looking at the cotangent bundle to the blow-up of n points on \mathbb{P}^{n-2} and the moduli space of pointed stable rational curves. Section 6 is devoted to examples of computations of rings of additive invariants on the zero level set of the moment map.

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2. Algebraic symplectic analogues of reductive quotients

Let G be a reductive group acting linearly on affine space $V \cong \mathbb{A}^N$; i.e., via a representation $\rho : G \rightarrow GL(V)$. We can construct a GIT quotient of the G -action on V by linearising the action on the structure sheaf of V using a character $\chi : G \rightarrow \mathbb{G}_m$; the corresponding GIT quotient is a morphism of algebraic varieties

$$V^{\chi-ss} \rightarrow V//_{\chi}G$$

where $V^{\chi-ss}$ is the open set of semistable points in V for the G -action linearised by χ in the sense of Mumford [17]. The GIT quotient $V//_{\chi}G$ is the projective spectrum of the ring of χ -semi-invariants on V and is projective over the affine GIT quotient $V//G := \text{Spec } \mathbb{C}[V]^G$.

In this section, we study the lift of the G -action to the cotangent bundle T^*V , which is algebraic symplectic with respect to the Liouville form on T^*V and has an algebraic moment map $\mu : T^*V \rightarrow \mathfrak{g}^*$. An algebraic symplectic reduction of the G -action on T^*V is given by taking a GIT quotient of a level of this moment map at a central value of the co-Lie algebra. Finally, we give a survey of Proudfoot’s construction of an algebraic symplectic analogue of

$V//_{\chi}G$, which, over the complex numbers, is homeomorphic to a hyperkähler analogue [19].

2.1. Algebraic moment maps for cotangent lifts of linear actions

We recall that an algebraic symplectic manifold (M, ω) over \mathbb{C} is a smooth complex algebraic variety with an algebraic symplectic form ω (that is, ω is a non-degenerate closed algebraic 2-form on M). This definition also makes sense over a more general field k . The classical example of an algebraic symplectic manifold is the cotangent bundle T^*X of a smooth algebraic variety X with symplectic form given by the Liouville 2-form. For an affine space $V \cong \mathbb{A}^n$, if we identify $T^*V \cong V \times V^*$ and take coordinates x_1, \dots, x_n on V and $\alpha_1, \dots, \alpha_n$ on V^* , then the Liouville form on T^*V is given by

$$\omega = \sum_{l=1}^n dx_l \wedge d\alpha_l.$$

Let G be an affine algebraic group which acts linearly on an affine space $V \cong \mathbb{A}^n$ via a representation $\rho : G \rightarrow \text{GL}(V)$. The cotangent lift of the G -action to $T^*V \cong V \times V^*$ is given by

$$g \cdot (v, \gamma) = (\rho(g)v, \gamma\rho(g)^{-1})$$

where $g \in G$ and we consider $v \in V$ as a column vector and $\gamma \in V^*$ as a row vector. The following lemma is well-known, but as we were unable to find a suitable reference, we include a proof for the sake of completeness.

Lemma 2.1. *Let G be an algebraic group and $\rho : G \rightarrow \text{GL}(V)$ be a G -representation. Let $\xi \in \mathfrak{g}^*$ be a point which is fixed by the coadjoint action. Then the following statements hold.*

- (i) *The cotangent lift of the G -action on T^*V is algebraic symplectic for the Liouville form ω on T^*V .*
- (ii) *This action has an algebraic moment map $\mu : T^*V \rightarrow \mathfrak{g}^*$ (that is, a G -equivariant algebraic map that lifts the infinitesimal action) given by*

$$\mu(v, \gamma) \cdot A := \gamma(A_v) = \gamma(\rho_*(A)v)$$

for $A \in \mathfrak{g}^* = (\text{Lie } G)^*$ and $(v, \gamma) \in T^*V \cong V \times V^*$.

- (iii) *A point $p \in \mu^{-1}(\xi)$ is a smooth point of the level set $\mu^{-1}(\xi)$ if and only if the G -action on p is locally trivial (that is, the stabiliser group G_p is finite).*
- (iv) *If $p \in \mu^{-1}(\xi)$ is a smooth point, then there is a canonical non-degenerate bilinear form on the vector space*

$$T_p(\mu^{-1}(\xi))/\mathfrak{g}$$

*induced by the non-degenerate bilinear form ω_p on $T_p(T^*V)$.*

Proof. For (i), we need to show $\sigma_g^*\omega = \omega$, where $\sigma_g : T^*V \rightarrow T^*V$ denotes the action of $g \in G$. For $p = (v, \gamma) \in T^*V$, we identify $T_p(T^*V) \cong T^*V \cong V \times V^*$ and write tangent vectors as pairs $p' = (v', \gamma')$. Then

$$\begin{aligned} (\sigma_g^*\omega)_p(p', p'') &= \omega_{g \cdot p}(d_p\sigma_g(p'), d_p\sigma_g(p'')) \\ &= \gamma''(\rho(g)^{-1}\rho(g)v') - \gamma'(\rho(g)^{-1}\rho(g)v'') = \omega_p(p', p''). \end{aligned}$$

For (ii), we first check the equivariance of μ : for $g \in G$, $A \in \mathfrak{g}$ and $(v, \gamma) \in T^*V$, we have

$$\begin{aligned} \mu(g \cdot (v, \gamma)) \cdot A &= \mu((\rho(g)v, \gamma\rho(g)^{-1})) \cdot A = \gamma\rho(g)^{-1}\rho_*(A)\rho(g)v \\ &= \gamma\text{ad}_{\rho(g)^{-1}}(\rho_*(A))v = \text{ad}_g^*(\mu(v, \gamma)) \cdot A. \end{aligned}$$

It remains to check that the moment map lifts the infinitesimal action. The infinitesimal action $A_p \in T_p(T^*V)$ of $A \in \mathfrak{g}$ on $p = (v, \gamma) \in T^*V$ is given by $A_p := (\rho_*(A)v, \gamma\rho_*(-A))$. Then

$$\begin{aligned} (2.1) \quad \omega_p(A_p, p') &= \gamma'\rho_*(A)v - \gamma\rho_*(-A)v' \\ &= \gamma'\rho_*(A)v + \gamma\rho_*(A)v' = d_p\mu(p') \cdot A \end{aligned}$$

as required.

For (iii), p is a smooth point of $\mu^{-1}(\xi)$ if and only if $d_p\mu$ is surjective. By the infinitesimal lifting property (2.1) of the moment map, we see that p is smooth if and only if $\mathfrak{g}_p = 0$ (or, equivalently, if and only if G_p is finite).

Statement (iv) follow from an algebraic version of the Marsden–Weinstein and Meyer Theorem [14, 15]. Let us briefly sketch the proof. By the preimage theorem, we have $T_p\mu^{-1}(\xi) \subset \ker d_p\mu$. Moreover, the vector spaces $\ker d_p\mu$ and $T_p(G \cdot p)$ are orthogonal complements with respect to the bilinear form

ω_p , as $\zeta \in \ker d_p$ if and only if

$$d_p\mu(\zeta) \cdot A = 0 \forall A \in \mathfrak{g} \iff \omega_p(A_p, \zeta) = 0 \forall A \in \mathfrak{g} \iff \zeta \in T_p(G \cdot p)^{\omega_p}.$$

Therefore, $T_p(G \cdot p) \subset T_p(\mu^{-1}(\xi)) = \ker d_p\mu = T_p(G \cdot p)^{\omega_p}$; that is $T_p(G \cdot p)$ is an isotropic subspace of $(T_p(T^*V), \omega_p)$ and there is an induced non-degenerate bilinear form ω'_p on

$$T_p(G \cdot p)^{\omega_p} / T_p(G \cdot p) \cong T_p(\mu^{-1}(\xi)) / \mathfrak{g}$$

defined by $\omega'_p([\zeta_1], [\zeta_2]) := \omega_p(\zeta_1, \zeta_2)$, where the above isomorphism uses the infinitesimal action $\mathfrak{g} \rightarrow T_p(T^*V)$, whose image is $T_p(G \cdot p)$ and whose kernel \mathfrak{g}_p is zero, as p is a smooth point). □

The level sets of an algebraic moment map $\mu : T^*V \rightarrow \mathfrak{g}^*$ are closed algebraic subvarieties of T^*V , and so are affine algebraic varieties, which are smooth if the level set is taken at a regular value of μ (that is, if and only if G acts locally freely on this level set). Moreover, for central elements $\xi \in \mathfrak{g}^*$, the level sets $\mu^{-1}(\xi)$ are invariant under the linear G -action.

2.2. Algebraic symplectic reductions for reductive groups

For the moment, we restrict ourselves to reductive groups G in order to take a reductive GIT quotient of the G -action on the level sets of the moment map for the cotangent lift of this action. In the following section, we will consider non-reductive groups.

Definition 2.2. Let G be a reductive group acting linearly on V and algebraically symplectically on (T^*V, ω) with algebraic moment map $\mu : T^*V \rightarrow \mathfrak{g}^*$ as above. Let χ be a character of G and $\xi \in \mathfrak{g}^*$ be a central element; then we refer to (χ, ξ) as a central pair. The algebraic symplectic reduction of the G -action on (T^*V, ω) at a central pair (χ, ξ) is the GIT quotient of the linear G -action on the affine algebraic variety $\mu^{-1}(\xi)$ with respect to the character defined by χ , we denote this as follows

$$\pi = \pi_{(\chi, \xi)} : \mu^{-1}(\xi)^{\chi-ss} \rightarrow \mu^{-1}(\xi) //_{\chi} G.$$

Remark 2.3. Over the complex numbers, for any $\xi \in \mathfrak{g}^*$, one can construct a holomorphic symplectic reduction of the G -action on T^*V at ξ as the topological quotient $\mu^{-1}(\xi)/G$. If ξ is a regular value of μ and the G -action on $\mu^{-1}(\xi)$ is free and proper, then $\mu^{-1}(\xi)/G$ is a holomorphic symplectic manifold by a holomorphic version of the Marsden–Weinstein–Meyer Theorem.

In a more general setting, one can defined a stratified holomorphic symplectic structure along the lines of [12]. In this case, the holomorphic reduction and the algebraic symplectic reduction at the trivial character are related by the natural map

$$\mu^{-1}(\xi)/G \rightarrow \mu^{-1}(\xi)//G$$

which is surjective, but not necessarily injective. The algebraic and holomorphic reductions coincide if all points of $\mu^{-1}(\xi)$ are stable.

We now state an algebraic version of the Marsden–Weinstein–Meyer Theorem.

Proposition 2.4. *Let G be a reductive group acting linearly on an affine space V and consider the algebraic symplectic reduction $\mu^{-1}(\xi)//_{\chi}G$ of the G -action on T^*V at a central pair (χ, ξ) . If G acts (set theoretically) freely on $\mu^{-1}(\xi)^{\chi-ss}$, then $\mu^{-1}(\xi)//_{\chi}G$ has a unique structure of a algebraic symplectic manifold compatible with that of T^*V .*

Proof. The assumption that G acts on the GIT semistable set $\mu^{-1}(\xi)^{\chi-ss}$ with trivial stabiliser groups, implies that $\mu^{-1}(\xi)^{\chi-s} = \mu^{-1}(\xi)^{\chi-ss}$, as all orbits have top dimension and so must be closed. Hence, the G -action on the GIT semistable is scheme theoretically free (i.e., set theoretically free and proper) and, moreover, the semistable set is contained in the smooth locus of $\mu^{-1}(\xi)$ by Lemma 2.1. The GIT quotient

$$\pi : \mu^{-1}(\xi)^{\chi-ss} \rightarrow \mu^{-1}(\xi)//_{\chi}G$$

is a geometric quotient, which is smooth and, in fact, a principal G -bundle by Luna’s étale slice theorem [13].

To prove the existence and uniqueness of the algebraic symplectic form on $\mu^{-1}(\xi)//_{\chi}G$, we can use an algebraic version of the Marsden–Weinstein–Meyer Theorem: by Lemma 2.1 the Liouville form ω determines a non-degenerate two form on $\mu^{-1}(\xi)//_{\chi}G$, which is closed, as ω is closed. □

For semisimple groups, such as SL_n , we only have the trivial character $\chi = 0$ and the trivial central value $\xi = 0$. For the central pair $(0, 0)$, the hypothesis of this proposition are never satisfied, as all points are semistable and $0 \in \mu^{-1}(0)$ is a fixed point of the linear G -action. In cases when the hypothesis of this proposition fail, we cannot expect to have a stratified symplectic structure analogous to [12], as the GIT quotient can identify orbits of different dimensions and the stratification in loc. cit. is obtained using the conjugacy class of the stabiliser subgroups.

Instead, we show there is an open subvariety of the algebraic symplectic reduction which admits the structure of an algebraic symplectic manifold. Since the GIT quotient restricts to a geometric quotient on the stable locus, one may expect to use the stable locus as this open subset. However, we cannot take the whole stable locus as the presence of non-trivial but finite stabiliser groups can cause singularities in the geometric quotient (for example, the Kleinian surface singularities have a GIT quotient construction given by a finite group acting linearly on \mathbb{A}^2 , for which all points are stable).

Lemma 2.5. *Let G be a reductive group acting linearly on $V = \mathbb{A}^n$ and (χ, ξ) be a central pair. Then there exists an open subvariety of $\mu^{-1}(\xi)//_{\chi}G$ admitting a unique structure of an algebraic symplectic manifold compatible with T^*V .*

Proof. Let $\pi : \mu^{-1}(\xi)^{\chi-ss} \rightarrow \mu^{-1}(\xi)//_{\chi}G$ denote the GIT quotient, which restricts to a geometric quotient $\mu^{-1}(\xi)^{\chi-s} \rightarrow \mu^{-1}(\xi)^{\chi-s}/G$. The smooth locus of $\mu^{-1}(\xi)^{\chi-s}/G$ inherits an algebraic symplectic structure following Lemma 2.1. \square

2.3. The algebraic symplectic analogue

The inclusion $V \subset T^*V$, as the zero section, realises V as a Lagrangian subvariety of T^*V . In fact, the zero section V is contained entirely in the zero level set of the algebraic moment map and, as G is reductive, the G -equivariant closed immersion $j : V \hookrightarrow \mu^{-1}(0)$ induces a closed immersion

$$\bar{j} : V//_{\chi}G \hookrightarrow \mu^{-1}(0)//_{\chi}G.$$

If the G -action on $\mu^{-1}(0)^{\chi-ss}$ is free, then $\mu^{-1}(0)//_{\chi}G$ is a smooth algebraic symplectic variety and \bar{j} is Lagrangian, as $V \subset T^*V$ is Lagrangian.

Definition 2.6. The algebraic symplectic analogue of $V//_{\chi}G$ is the algebraic symplectic reduction of the G -action on T^*V at the central pair $(\chi, 0)$.

The above terminology comes from Proudfoot's notion of a hyperkähler analogue [19]. We rephrase Proposition 2.4 of loc. cit. in terms of the algebraic symplectic structure.

Proposition 2.7 ([19], Proposition 2.4). *Let G be a reductive group acting linearly on an affine space V with respect to a character $\chi : G \rightarrow \mathbb{G}_m$. If the G -action on $\mu^{-1}(0)^{\chi-ss}$ is free and $V^{\chi-ss}$ is non-empty, then*

- i) $V//_{\chi}G$ is an Lagrangian submanifold of $\mu^{-1}(0)//_{\chi}G$, and
- ii) $T^*(V//_{\chi}G)$ is a dense open subset of $\mu^{-1}(0)//_{\chi}G$.

More generally, one has the following result.

Corollary 2.8. *Let G be a reductive group acting linearly on an affine space V and χ be a character of G . Then there exist open subvarieties $U \subset V//_{\chi}G$ and $U' \subset \mu^{-1}(0)//_{\chi}G$ with a commutative square*

$$\begin{array}{ccc}
 U & \xrightarrow{\quad} & U' \\
 \downarrow & & \downarrow \pi \\
 V//_{\chi}G & \xrightarrow{\quad \tilde{\iota} \quad} & \mu^{-1}(0)//_{\chi}G
 \end{array}$$

whose horizontal maps are closed immersions such that

- (i) U' is an algebraic symplectic manifold;
- (ii) U is a Lagrangian subvariety of U' ;
- (iii) T^*U is an open set of U' , which is dense if non-empty.

Proof. Let $U \subset V^{\chi^{-s}}/G$ and $U' \subset \mu^{-1}(0)^{\chi^{-s}}/G$ denote the smooth loci of the geometric quotients of the stable sets. Then U' is an algebraic symplectic manifold by Lemma 2.5 and the remaining parts follow in the same way as in the proof of Proposition 2.7. □

2.4. Hyperkähler reduction

Over $k = \mathbb{C}$, the above algebraic symplectic quotient constructions are related to a hyperkähler quotient construction. The starting point is the observation that the cotangent bundle of a complex affine space $V = \mathbb{A}_{\mathbb{C}}^n$ is not only algebraic symplectic: the holomorphic symplectic form $\omega = \omega_{\mathbb{C}}$ is part of a hyperkähler structure on T^*V coming from an identification $T^*V \cong \mathbb{H}^n$ where \mathbb{H} denotes the quaternions.

Let $1, i, j, k$ denote the standard basis of the quaternions \mathbb{H} ; then we recall that the standard flat hyperkähler structure on \mathbb{H}^n comes from the Quaternionic Hermitian inner product $Q : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{H}$ given by

$$Q(z, w) = \sum_{l=1}^n z_l w_l^{\dagger}$$

where \dagger denotes the Quaternionic conjugate. The triple of complex structures I, J, K on \mathbb{H}^n are given by left multiplication by i, j, k and the hyperkähler metric g and corresponding kähler forms $\omega_I, \omega_J, \omega_K$ are given by the following formula

$$Q = g + i\omega_I + j\omega_J + k\omega_K.$$

We identify $T^*V \cong \mathbb{H}^n$ by $(v, \gamma) \mapsto (v_1 - \gamma_1 j, \dots, v_n - \gamma_n j)$ and write the corresponding hyperkähler structure on T^*V as $(g, \omega_I, \omega_J, \omega_K)$. If we take coordinates x_1, \dots, x_n on V and $\alpha_1, \dots, \alpha_n$ on V^* , then

$$\omega_{\mathbb{R}} = \frac{1}{2i} \left(\sum_{k=1}^n dx_k \wedge d\bar{x}_k + d\alpha_k \wedge d\bar{\alpha}_k \right) \quad \text{and} \quad \omega_{\mathbb{C}} = \sum_{k=1}^n dx_k \wedge d\alpha_k.$$

Let G be a complex reductive group with maximal compact subgroup K ; then G is the complexification of K . Suppose that we have a linear G -action on V such that K acts unitarily. Then we can consider the cotangent lift of the K -action to T^*V . In this case, the triple of Kähler forms are preserved by the action of K on T^*V and, moreover, this K -action on T^*V is hyper-Hamiltonian; that is, it has a hyperkähler moment map given by a pair of moment maps $\mu_{HK} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ for the symplectic forms $(\omega_{\mathbb{R}} := \omega_I, \omega_{\mathbb{C}} := \omega_J + i\omega_K)$. More precisely,

$$\begin{aligned} \mu_{\mathbb{R}} : T^*V &\rightarrow \mathfrak{k}^* & \mu_{\mathbb{R}}(v, \gamma) &:= \mu(v) - \mu(\gamma) \\ \mu_{\mathbb{C}} : T^*V &\rightarrow \mathfrak{g}^* = \mathfrak{k}_{\mathbb{C}}^* & \mu_{\mathbb{C}}(v, \gamma) \cdot A &:= \gamma(A_v). \end{aligned}$$

Remark 2.9. The action of the maximal compact group K is hyper-Hamiltonian, not the action of G . For non-reductive groups, there is no longer a bijective correspondence with compact groups; thus, it is not immediate how one should construct a theory of hyperkähler reductions for non-reductive groups.

Let $(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$ be a regular value of the hyperkähler moment map that is fixed by the coadjoint action. Then the hyperkähler reduction of the K -action on T^*V at $(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$ is defined to be

$$T^*V //_{(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})} K := \mu_{HK}^{-1}((\chi_{\mathbb{R}}, \chi_{\mathbb{C}})) / K = (\mu_{\mathbb{R}}^{-1}(\chi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\chi_{\mathbb{C}})) / K.$$

If the K -action on the level set $\mu_{HK}^{-1}((\chi_{\mathbb{R}}, \chi_{\mathbb{C}}))$ is free, then this hyperkähler reduction has a unique hyperkähler structure compatible with the hyperkähler structure on the cotangent bundle by [9].

The hyperkähler reduction is the (smooth, rather than algebraic) symplectic reduction of the K -action on $\mu_{\mathbb{C}}^{-1}(\chi_{\mathbb{C}})$ with respect to $\chi_{\mathbb{R}}$. Hence, by work of Kempf and Ness [11], the hyperkähler reduction is homeomorphic to the GIT quotient of $G = K_{\mathbb{C}}$ acting on $\mu_{\mathbb{C}}^{-1}(\chi_{\mathbb{C}})$ with respect to the character of G corresponding to $\chi_{\mathbb{R}}$ (that is, the algebraic symplectic reduction of G acting on T^*V with respect to the central pair $(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$).

Proudfoot proves the following result using the Kempf–Ness homeomorphism and the argument outlined in Proposition 2.7.

Proposition 2.10 ([19], Proposition 2.4). *Let G be a complex reductive group acting linearly on $V = \mathbb{A}_{\mathbb{C}}^n$ and let $\chi \in \mathfrak{k}^*$ be fixed by the coadjoint action. If χ is a regular value of the moment map μ and $(\chi, 0)$ is a regular value of the hyperkähler moment map μ_{HK} , then $T^*(V//_{\chi}G)$ is homeomorphic to an open subset of the hyperkähler reduction $T^*V////_{\chi}K$ which is dense if non-empty.*

3. Algebraic symplectic analogues of quotients of \mathbb{G}_a -representations

In this section, we study linear actions of the additive group $\mathbb{G}_a = \mathbb{C}^+$ on an affine space $V := \mathbb{A}^n$ over the complex numbers. We restrict to the field of complex numbers $k = \mathbb{C}$, as in this case, the ring of invariants $\mathbb{C}[V]^{\mathbb{G}_a}$ is finitely generated. In this section, we describe the construction of an algebraic symplectic analogue of an open subset of $V//\mathbb{G}_a$.

3.1. Non-reductive GIT for representations of the additive group

Let \mathbb{G}_a act linearly on $V \cong \mathbb{A}^n$ over the complex numbers. A key idea of non-reductive GIT is to transfer a non-reductive GIT problem to a reductive setting by embedding the non-reductive group in a reductive group; see [7]. We fix a closed embedding $\mathbb{G}_a \hookrightarrow \mathrm{SL}_2$ as the unipotent subgroup in the upper triangular Borel subgroup of SL_2 . In this case, the representation $\rho : \mathbb{G}_a \rightarrow \mathrm{GL}(V)$ factors via the embedding $\mathbb{G}_a \hookrightarrow \mathrm{SL}_2$ by the Jacobson–Morozov theorem (alternatively, we can extend the action to SL_2 by putting the derivation of the \mathbb{G}_a -action in Jordan normal form). Then

$$\mathbb{C}[V]^{\mathbb{G}_a} \cong \mathbb{C}[V \times_{\mathbb{G}_a} \mathrm{SL}_2]^{\mathrm{SL}_2} \cong \mathbb{C}[V \times \mathrm{SL}_2 / \mathbb{G}_a]^{\mathrm{SL}_2}$$

and as, moreover, $\mathrm{SL}_2/\mathbb{G}_a \cong \mathbb{A}^2 - \{0\}$ has a normal affine completion \mathbb{A}^2 with complement of codimension 2, it follows that

$$\mathbb{C}[V]^{\mathbb{G}_a} \cong \mathbb{C}[W]^{\mathrm{SL}_2},$$

where $W := V \times \mathbb{A}^2$. In particular, $\mathbb{C}[V]^{\mathbb{G}_a}$ is finitely generated.

Definition 3.1. For a linear \mathbb{G}_a -action on an affine space $V \cong \mathbb{A}^n$, the non-reductive enveloping quotient is the morphism of affine varieties

$$q : V \rightarrow V//\mathbb{G}_a := \mathrm{Spec}(\mathbb{C}[V]^{\mathbb{G}_a}).$$

The term enveloping quotient was introduced in [7] to emphasise the fact that the morphism q is not surjective in general (for example, the enveloping quotient for the \mathbb{G}_a -representation $V = \mathrm{Sym}^1(\mathbb{A}^2)^{\oplus 2}$ is not surjective). A further difference of non-reductive actions is that taking invariants is not exact and so, for an invariant closed subset Z of V , there may be intrinsic invariants to Z which do not extend to invariants on V . For a detailed comparison of reductive and non-reductive GIT, see [7].

The enveloping quotient restricts to a better behaved quotient on certain stable sets defined in [7]. To define these sets, consider the \mathbb{G}_a -equivariant embeddings

$$\begin{array}{ccccccc} i' : & V & \xrightarrow{i} & V \times_{\mathbb{G}_a} \mathrm{SL}_2 & \rightarrow & W = V \times \mathbb{A}^2 & \rightarrow & \mathbb{P} := \mathbb{P}(\mathbb{A}^1 \times W) \\ & v & \mapsto & [v, I] & \mapsto & (v, (1, 0)) & \mapsto & [1 : v : 1 : 0], \end{array}$$

where SL_2 acts trivially on the copy of \mathbb{A}^1 appearing in \mathbb{P} . Since SL_2 has no non-trivial characters, the line bundles $\mathcal{O}_{V \times_{\mathbb{G}_a} \mathrm{SL}_2}$ and $\mathcal{O}_{\mathbb{P}}(1)$ admit canonical SL_2 -linearisations and we can consider the (semi)stable sets for the reductive group SL_2 in the sense of Mumford. In the terminology of [7] Definition 5.2.5, the pair $(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ is a fine reductive envelope for the \mathbb{G}_a -action on V .

Definition 3.2 ([7], Definitions 5.1.6 and 5.2.11). For $v \in V$, we make the following definitions.

- 1) v is Mumford (semi)stable for the \mathbb{G}_a -action on V if $i(v)$ is SL_2 - (semi)stable in $V \times_{\mathbb{G}_a} \mathrm{SL}_2$.
- 2) v is completely (semi)stable for the \mathbb{G}_a -action on V if $i'(v)$ is SL_2 - (semi)stable in \mathbb{P} .

Let $V^{m(s)s}$ and $V^{\overline{(s)s}}$ denote the Mumford (semi)stable and the completely (semi)stable sets respectively.

Although the definition of the Mumford (semi)stable locus depends a priori on the choice of an embedding of \mathbb{G}_a into a reductive group, it turns out that this notion is intrinsic to the \mathbb{G}_a action (cf. [7] Proposition 5.1.9). The following theorem relates these concepts.

Theorem 3.3 ([7], Theorem 5.3.1). *For \mathbb{G}_a acting linearly on $V \cong \mathbb{A}^n$, there is a commutative diagram*

$$\begin{array}{ccccc}
 V^{\overline{ss}} \hookrightarrow & V^{ms} = V^{mss} \hookrightarrow & V^{\overline{ss}} = V & & \\
 \downarrow & \downarrow & \downarrow & & \\
 V^{\overline{ss}}/\mathbb{G}_a \hookrightarrow & V^{ms}/\mathbb{G}_a \hookrightarrow & V//G & &
 \end{array}$$

where all horizontal maps are open inclusions, the two left most vertical maps are locally trivial geometric quotients, and the right vertical map is the enveloping quotient.

The completely (semi)stable sets can be computed using the Hilbert–Mumford criterion for the SL_2 -action on $\mathbb{P} = \mathbb{P}(\mathbb{A}^1 \times V \times \mathbb{A}^2)$. As the SL_2 -action on the line \mathbb{A}^1 is trivial, we see that $i'(v) = [1 : v : 1 : 0]$ is always semistable and so $V^{\overline{ss}} = V$. Let $T_d \subset SL_2$ be the diagonal torus; then T_d acts with weight zero on \mathbb{A}^1 and, as SL_2 acts on \mathbb{A}^2 by its standard representation, the T_d weights on \mathbb{A}^2 are $(+1, -1)$. Hence, the T_d -weight set of $i'(v)$ already contains the weights 0 and 1. We state and prove a well-known result that describes the completely stable locus.

Proposition 3.4. *Let \mathbb{G}_a act linearly on $V \cong \mathbb{A}^n$; then the complement to the completely stable set is the non-negative weight space for the diagonal torus $T_d \subset SL_2$ acting on V .*

Proof. It is straight forward to verify that SL_2 -stability of $i'(v)$ is equivalent to T_d -stability of $i'(v)$, because $[0 : 0 : 1 : 0]$ has a negative and a positive weight for all tori $T \neq T_d$ which are obtained by conjugating T_d by an element in the upper triangular Borel. By the Hilbert–Mumford criterion, $i'(v)$ is T_d -stable if and only if $v \in V$ has a negative T_d -weight, as $i'(v)$ already has a positive T_d -weight. □

3.2. The moment map for the cotangent lift

Let \mathbb{G}_a act linearly on $V \cong \mathbb{A}^n$ over the complex numbers; then this extends to a linear SL_2 -action $\rho : SL_2 \rightarrow GL(V)$. We can consider the cotangent lifts

of these actions and their associated moments maps, as well as the cotangent lift of the SL_2 -action on $W := V \times \mathbb{A}^2$ using Lemma 2.1. All these moment maps can be described using the moment map $\mu_{SL_2, V} : T^*V \rightarrow \mathfrak{sl}_2^*$ for the SL_2 -action on T^*V .

We identify $T^*V \cong V \times V^*$ and consider points $x \in V$ as column vectors and points $\alpha \in V^*$ as row vectors. The algebraic moment map $\mu_{SL_2, V} : T^*V \rightarrow \mathfrak{sl}_2^*$ is given by

$$(3.1) \quad \mu_{SL_2, V}(x, \alpha) \cdot B = \alpha(\rho_*(B)x)$$

where $\rho_* : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}(V)$ denotes the derivative of the representation ρ . Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

denote the standard basis of \mathfrak{sl}_2 and let H^*, E^*, F^* denote the dual basis of \mathfrak{sl}_2^* . Then, using this basis, we can write

$$\mu_{SL_2, V}(x, \alpha) = \Phi_H(x, \alpha)H^* + \Phi_E(x, \alpha)E^* + \Phi_F(x, \alpha)F^*$$

where Φ_H, Φ_E, Φ_F are algebraic functions on T^*V .

Remark 3.5. It is clear from the expression for the moment map (3.1) that if our SL_2 -representation V decomposes as a sum of SL_2 -representations $V = V' \oplus V''$, then the moment map on T^*V is the sum of the moment maps for each summand; that is, $\mu_{SL_2, V} = \mu_{SL_2, V'} + \mu_{SL_2, V''}$.

Thus, it suffices to compute $\mu_{SL, V}$ for irreducible SL_2 -representations V .

Example 3.6. Let us consider the irreducible SL_2 -representation $V_{k+1} := \text{Sym}^k(\mathbb{C}^2)$ of dimension $k + 1$. The \mathbb{G}_a -action on V_{k+1} is given by the representation $\mathbb{G}_a \rightarrow \text{GL}(V_{k+1})$

$$c \mapsto \begin{pmatrix} 1 & \binom{k}{1}c & \binom{k}{2}c^2 & \dots & c^k \\ 0 & 1 & \binom{k-1}{1}c & & c^{k-1} \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & c \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

The derivative of the SL_2 -representation gives the infinitesimal action $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}_{k+1}$

$$\begin{pmatrix} h & e \\ f & -h \end{pmatrix} \mapsto \begin{pmatrix} kh & ke & 0 & \dots & 0 \\ f & (k-2)h & (k-1)e & \ddots & \vdots \\ 0 & 2f & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & e \\ 0 & \dots & 0 & kf & -kh \end{pmatrix}.$$

We take coordinates $x = (x_1, \dots, x_{k+1})$ on V_{k+1} and $\alpha = (\alpha_1, \dots, \alpha_{k+1})$ on V_{k+1}^* and identify $T^*V_{k+1} \cong V_{k+1} \times V_{k+1}^*$. Then the cotangent lift of the SL_2 -action on V_{k+1} has moment map

$$\mu_{SL_2, V_{k+1}} = \Phi_H^k H^* + \Phi_E^k E^* + \Phi_F^k F^* : T^*V_{k+1} \longrightarrow \mathfrak{sl}_2^*$$

where

$$\begin{aligned} \Phi_H^k(x, \alpha) &= \sum_{j=0}^k (k-2j)x_{j+1}\alpha_{j+1}, \\ \Phi_E^k(x, \alpha) &= \sum_{j=1}^k (k+1-j)x_{j+1}\alpha_j \quad \text{and} \quad \Phi_F^k(x, \alpha) = \sum_{j=1}^k jx_j\alpha_{j+1} \end{aligned}$$

and the \mathbb{G}_a -action has moment map $\mu_{\mathbb{G}_a, V_{k+1}} = \Phi_E^k : T^*V_{k+1} \rightarrow \mathbb{C}$.

Since the subgroup $\mathbb{G}_a \hookrightarrow SL_2$ corresponds to the 1-dimensional Lie subalgebra of \mathfrak{sl}_2 spanned by the basis vector E , the algebraic moment map $\mu_{\mathbb{G}_a, V} : T^*V \rightarrow (\text{Lie } \mathbb{G}_a)^* \cong \mathbb{C}$ for the \mathbb{G}_a -action on T^*V is given by Φ_E .

Remark 3.7. Under the isomorphisms

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} \cong \mathbb{C}[T^*V \times \mathbb{A}^2]^{SL_2} \cong (\mathbb{C}[T^*V][u, v])^{SL_2},$$

the \mathbb{G}_a -invariant function Φ_E corresponds to the SL_2 -invariant function

$$f = u^2\Phi_E - uv\Phi_H - v^2\Phi_F.$$

Indeed, Φ_E is obtained by evaluating f at $(u, v) = (1, 0)$ and f is SL_2 -invariant, as $\mu_{SL_2, V}$ is SL_2 -equivariant.

As the SL_2 -representation W is the direct sum of V and the standard SL_2 -representation \mathbb{A}^2 , the moment map for the SL_2 -action on T^*W is the sum of the moment maps for SL_2 -acting on T^*V and $T^*\mathbb{A}^2$. By Example 3.6, the moment map for the SL_2 -action on $T^*\mathbb{A}^2$ is given by

$$\mu_{SL_2, \mathbb{A}^2}(u, v, \lambda, \eta) = (u\lambda - v\eta)H^* + v\lambda E^* + u\eta F^*$$

where $(u, v, \lambda, \eta) \in \mathbb{A}^2 \times \mathbb{A}^2)^*$. Hence, $\mu_{SL_2, W} : T^*W \rightarrow \mathfrak{sl}_2^*$ is given by

$$(3.2) \quad \begin{aligned} \mu_{SL_2, W}(z, \gamma) = & (\Phi_H(x, \alpha) + u\lambda - v\eta)H^* + (\Phi_E(x, \alpha) + v\lambda)E^* \\ & + (\Phi_F(x, \alpha) + u\eta)F^*, \end{aligned}$$

where $z = (x, (u, v)) \in W = V \times \mathbb{A}^2$ and $\gamma = (\alpha, (\lambda, \eta)) \in W^* \cong V^* \times (\mathbb{A}^2)^*$.

Remark 3.8. Let $\iota : T^*V \hookrightarrow T^*W$ be the \mathbb{G}_a -equivariant inclusion $(x, \alpha) \mapsto (x, (1, 0), \alpha, (0, 1))$. Then there is a commutative diagram

$$\begin{array}{ccc} T^*V & \xrightarrow{\mu_{\mathbb{G}_a, V}} & \mathbb{C} \\ \downarrow \iota & & \uparrow \\ T^*W & \xrightarrow{\mu_{SL_2, W}} & \mathfrak{sl}_2^* \end{array}$$

where $\mathfrak{sl}_2^* \rightarrow \mathbb{C} \cong \text{Lie}(\mathbb{G}_a)^*$ is the projection map induced by the inclusion $\mathbb{G}_a \hookrightarrow SL_2$.

3.3. The geometry of the level sets of the moment map

We continue to use the notation above: let V be a complex \mathbb{G}_a -representation and let $W = V \times \mathbb{A}^2$ be the associated SL_2 -representation. As above, we write the moment map for the SL_2 -action on T^*V as

$$\mu_{SL_2, V} = \Phi_H H^* + \Phi_E E^* + \Phi_F F^*$$

and we write the moment map for the \mathbb{G}_a -action on T^*V (resp. SL_2 -action on T^*W) as

$$\mu_{\mathbb{G}_a} : T^*V \rightarrow \mathbb{C} \quad (\text{resp. } \mu_{SL_2} : T^*W \rightarrow \mathfrak{sl}_2^*).$$

As every element $\xi \in \mathbb{C} \cong (\text{Lie } \mathbb{G}_a)^*$ is central, every level set $H_V(\xi) := \mu_{\mathbb{G}_a}^{-1}(\xi)$ is a \mathbb{G}_a -invariant closed subvariety of T^*V . Henceforth, we exclude a degenerate case: if the \mathbb{G}_a -action on V is trivial, then $\mu_{\mathbb{G}_a} = 0$ and so the

level sets are either empty (for $\xi \neq 0$) or equal to the whole cotangent bundle T^*V (for $\xi = 0$).

We first describe the level sets for the \mathbb{G}_a -representations $V_{k+1} = \text{Sym}^k(\mathbb{C}^2)$.

Lemma 3.9. *For $k \geq 1$, let $V_{k+1} := \text{Sym}^k(\mathbb{C}^2)$ be the $(k + 1)$ -dimensional irreducible SL_2 -representation and let $\mu_{\mathbb{G}_a, V_{k+1}} : T^*V_{k+1} \rightarrow \mathbb{C}$ be the moment map for the cotangent lift of the \mathbb{G}_a -action. For $\xi \in \mathbb{C}$, the level set $H_k(\xi) := \mu_{\mathbb{G}_a, V_{k+1}}^{-1}(\xi)$ is a quadric affine hypersurface in T^*V_{k+1} . Moreover, these hypersurfaces have the following properties.*

- 1) $H_k(\xi)$ is irreducible if and only if $\xi \neq 0$ or $k > 1$. The hypersurface $H_1(0)$ has two irreducible components, which are both isomorphic to \mathbb{A}^3 .
- 2) $H_k(\xi)$ is non-singular if and only if $\xi \neq 0$.
- 3) For $\xi = 0$, the singular locus is the \mathbb{G}_a -fixed locus; that is, $\text{Sing } H_k(0) = (T^*V_{k+1})^{\mathbb{G}_a}$.
- 4) $H_k(0)$ is normal if and only if $k > 1$.

Proof. We use the explicit form of the moment map

$$\mu_{\mathbb{G}_a, V_{k+1}}(x, \alpha) = \sum_{j=1}^k (k - j + 1)x_{j+1}\alpha_j.$$

given by Example 3.6. As this is a homogeneous quadratic function, the corresponding level sets are affine quadric hypersurfaces. We note that each monomial appearing in this hypersurface equation involves distinct variables and, for $k > 1$, we have more than one such monomial; thus, for $k > 1$, the hypersurface $H_k(\xi)$ is irreducible. If $k = 1$, then $\mu_{\mathbb{G}_a, V_2}(x, \alpha) = x_2\alpha_1$ and the hypersurface $H_1(\xi)$ is irreducible if and only if $\xi \neq 0$.

The singular loci of these hypersurfaces are

$$\text{Sing } H_k(\xi) := \{(x, \alpha) \in H_k(\xi) : x_2 = \dots = x_{k+1} = \alpha_1 = \dots = \alpha_k = 0\}.$$

In particular, $\text{Sing } H_k(\xi)$ is empty for $\xi \neq 0$. Conversely, for $\xi = 0$, the singular locus of $H_k(0)$ is isomorphic to \mathbb{A}^2 and coincides with the \mathbb{G}_a -fixed locus

$$(T^*V_{k+1})^{\mathbb{G}_a} = \{(x, \alpha) : x_2 = \dots = x_{k+1} = \alpha_1 = \dots = \alpha_k = 0\} = \text{Sing } H_k(0).$$

Since $\text{codim}(\text{Sing } H_k(0), H_k(0)) = 2k + 1 - 2 = 2k - 1$, we see that $H_k(0)$ is non-singular in codimension 1 if and only if $k > 1$. In particular, $H_1(0)$ is not normal, as it fails to be non-singular in codimension 1. Since normality is equivalent to regular in codimension 1 and Serre's S_2 condition, which automatically holds for irreducible affine hypersurfaces (for example, see [16] III §8 Proposition 2), we conclude that, for $k > 1$, the hypersurfaces $H_k(0)$ are normal. \square

We can then extend this to any \mathbb{G}_a -representation using Remark 3.5.

Proposition 3.10. *Let V be a (non-trivial) \mathbb{G}_a -representation and let $\mu_{\mathbb{G}_a} : T^*V \rightarrow \mathbb{C}$ denote the moment map for the cotangent lift of this action. For $\xi \in \mathbb{C}$, the level set $H_V(\xi) := \mu_{\mathbb{G}_a}^{-1}(\xi)$ is an affine quadric hypersurface in T^*V with the following properties.*

- 1) $H_V(\xi)$ is irreducible if and only if $\xi \neq 0$ or $V \neq \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$, as an SL_2 -representation, for some $n \geq 0$.
- 2) $H_V(\xi)$ is non-singular if and only if $\xi \neq 0$.
- 3) For $\xi = 0$, the singular locus is the \mathbb{G}_a -fixed locus; that is, $\text{Sing } H_V(0) = (T^*V)^{\mathbb{G}_a}$.
- 4) $H_V(0)$ is normal if and only if $V \neq \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$, as an SL_2 -representation, for some $n \geq 0$.

Proof. We extend the \mathbb{G}_a -action on V to SL_2 and decompose V as a sum of irreducible SL_2 -representations

$$V = \text{Sym}^{k_1}(\mathbb{C}^2) \oplus \dots \oplus \text{Sym}^{k_m}(\mathbb{C}^2).$$

On $\text{Sym}^{k_j}(\mathbb{C}^2)$ (resp. $\text{Sym}^{k_j}(\mathbb{C}^2)^*$), we take coordinates $x_1^{(j)}, \dots, x_{k_j+1}^{(j)}$ (resp. $\alpha_1^{(j)}, \dots, \alpha_{k_j+1}^{(j)}$). By Remark 3.5, we have

$$\mu_{\mathbb{G}_a}(x, \alpha) = \sum_{j=1}^m \mu_{\mathbb{G}_a, k_j}(x_i^{(j)}, \alpha_i^{(j)}) = \sum_{j=1}^m \sum_{i=1}^{k_j} (k_j + 1 - i) x_{i+1}^{(j)} \alpha_i^{(j)}.$$

Hence

$$\text{Sing } H_V(\xi) := \{(x, \alpha) \in H_V(\xi) : x_{i+1}^{(j)} = \alpha_i^{(j)} = 0 \text{ for } 1 \leq j \leq m, 1 \leq i \leq k_j\}$$

and we see that 1), 2) and 3) hold. We prove 4) as in Lemma 3.9: we have

$$\text{codim}(\text{Sing } H_V(0), H_V(0)) = \sum_{j=1}^m 2(k_j + 1) - 1 - 2m = 2 \sum_{j=1}^m k_j - 1$$

where $k_j \geq 0$, for $j = 1, \dots, m$, and, as the action is non-trivial, there is at least one j such that $k_j > 0$. Hence, $\text{codim}(\text{Sing } H_V(0), H_V(0)) \geq 1$, with equality if and only if there exists j_0 such that $k_{j_0} = 1$ and $k_j = 0$ for all $j \neq j_0$ (that is, if and only if $V = \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus m-1}$). In exactly the same way as in the proof of Lemma 3.9, $H_V(0)$ is normal if and only if $V \neq \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$ for some $n \geq 0$. □

3.4. (Semi)stability on the level sets of the moment map

Let $(T^*V)^{\bar{s}}$ be the completely stable set for the \mathbb{G}_a -action and write $H_V(\xi)^{\bar{s}} := H_V(\xi) \cap (T^*V)^{\bar{s}}$ (that is, the completely stable set determined with respect to the ambient invariants on T^*V).

Lemma 3.11. *Let V be a (non-trivial) \mathbb{G}_a -representation; then fix an extension to SL_2 and write V as a sum of irreducible SL_2 -representations*

$$V = \text{Sym}^{k_1}(\mathbb{C}^2) \oplus \dots \oplus \text{Sym}^{k_m}(\mathbb{C}^2)$$

and take coordinates $x_1^{(j)}, \dots, x_{k_j+1}^{(j)}$ (resp. $\alpha_1^{(j)}, \dots, \alpha_{k_j+1}^{(j)}$) on $\text{Sym}^{k_j}(\mathbb{C}^2)$ (resp. $\text{Sym}^{k_j}(\mathbb{C}^2)^*$). Then the complement to the completely stable locus $T^*V - (T^*V)^{\bar{s}}$ is given by

$$\{(x, \alpha) : \alpha_i^{(j)} = x_{k_j+2-i}^{(j)} = 0 \text{ for } 1 \leq j \leq m, 1 \leq i \leq \lceil k_j/2 \rceil\}.$$

Proof. This is a direct consequence of Proposition 3.4 and the fact that the weights for the diagonal torus $T_d \subset \text{SL}_2$ acting on $\text{Sym}^k(\mathbb{C}^2)$ are $k, k - 2, \dots, -k + 2, -k$ and, dually, the T_d -weights on $\text{Sym}^k(\mathbb{C}^2)^*$ are $-k, -k + 2, \dots, k - 2, k$. □

An important consequence of this is the following result.

Corollary 3.12. *Let V be a \mathbb{G}_a -representation with moment map $\mu_{\mathbb{G}_a} : T^*V \rightarrow \mathbb{C}$; then*

$$\mu_{\mathbb{G}_a}(T^*V - (T^*V)^{\bar{s}}) = 0.$$

In particular, if $\mu_{\mathbb{G}_a}(x, \alpha) \neq 0$ then (x, α) is completely stable and so, for $\xi \neq 0$, the hypersurfaces $H_V(\xi) := \mu_{\mathbb{G}_a}^{-1}(\xi)$ are completely stable everywhere; that is, $H_V(\xi)^{\bar{s}} = H_V(\xi)$. Furthermore, $H_V(0)^{\bar{s}} \subset H_V(0) - H_V(0)^{\mathbb{G}_a}$.

Proof. From the proof of Proposition 3.10, the moment map is given by

$$\mu_{\mathbb{G}_a}(x, \alpha) = \sum_{j=1}^m \left(\sum_{i=1}^{\lceil k_j/2 \rceil} (k_j + 1 - i)x_{i+1}^{(j)}\alpha_i^{(j)} + \sum_{i=\lceil k_j/2 \rceil + 1}^{k_j} (k_j + 1 - i)x_{i+1}^{(j)}\alpha_i^{(j)} \right).$$

On $T^*V - (T^*V)^{\bar{s}}$, the first part of this expression vanishes, as the $\alpha_i^{(j)}$ s are zero for $1 \leq i \leq \lceil k_j/2 \rceil$, and the second part vanishes, as the $x_{i+1}^{(j)}$ s are zero for $\lceil k_j/2 \rceil + 1 \leq i \leq k$. □

3.5. Algebraic symplectic reduction at non-zero values

In this section, we construct an algebraic symplectic reduction of the linear \mathbb{G}_a -action on T^*V at $\xi \in \mathbb{C}^*$ by taking the \mathbb{G}_a -quotient of the level set $H_V(\xi) = \mu_{\mathbb{G}_a}^{-1}(\xi)$. We first describe this quotient.

Theorem 3.13. *Let \mathbb{G}_a act linearly on an affine space V with moment map $\mu_{\mathbb{G}_a} : T^*V \rightarrow \mathbb{C}$. For $\xi \neq 0$, consider the hypersurface $H_V(\xi) := \mu_{\mathbb{G}_a}^{-1}(\xi)$. Then we have the following:*

- i) $H_V(\xi)^{\bar{s}} = H_V(\xi)^{ms} = H_V(\xi)$,
- ii) $H_V(\xi)$ admits a locally trivial geometric quotient $H_V(\xi)/\mathbb{G}_a$, which is smooth,
- iii) $H_V(\xi)/\mathbb{G}_a$ is affine and so $\mathbb{C}[H_V(\xi)]^{\mathbb{G}_a}$ is finitely generated,
- iv) all invariants in $\mathbb{C}[H_V(\xi)]^{\mathbb{G}_a}$ are restrictions of invariants in $\mathbb{C}[T^*V]^{\mathbb{G}_a}$,
- v) $H_V(\xi) \rightarrow H_V(\xi)/\mathbb{G}_a$ is a trivial \mathbb{G}_a -torsor.

In particular, we have $H_V(\xi)/\mathbb{G}_a = H_V(\xi)//\mathbb{G}_a := \text{Spec } \mathbb{C}[H_V(\xi)]^{\mathbb{G}_a}$ and the non-reductive quotient $\pi : H_V(\xi) \rightarrow H_V(\xi)//\mathbb{G}_a$ is surjective.

Proof. The first statement is Corollary 3.12 and it follows that there is a Zariski locally trivial geometric quotient $H_V(\xi) \rightarrow H_V(\xi)/\mathbb{G}_a$. By Proposition 3.10, $H_V(\xi)$ is smooth and so the quotient $H_V(\xi)/\mathbb{G}_a$ is also smooth

which gives ii). The quotient $H_V(\xi)/\mathbb{G}_a$ is a priori only quasi-affine; however, we will show that $\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)$ is affine and deduce from this that the quotient is affine. Consider the closure

$$\overline{\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)}$$

of $\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)$ in $\mathbb{A}^2 \times T^*V$. By Remark 3.7, this hypersurface is defined by the vanishing of $f - \xi$ where

$$f = u^2\Phi_E - uv\Phi_H - v^2\Phi_F.$$

The boundary, which is given by setting $(u, v) = (0, 0)$, is empty and so $\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)$ is affine. As the reductive GIT-quotient of an affine variety is affine, it follows that

$$H_V(\xi)/\mathbb{G}_a \cong (\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi))/\mathrm{SL}_2$$

is affine. Hence, $H_V(\xi)/\mathbb{G}_a$ is affine and so has finitely generated coordinate ring

$$\mathbb{C}[H_V(\xi)/\mathbb{G}_a] \cong \mathbb{C}[H_V(\xi)]^{\mathbb{G}_a} \cong \mathbb{C}[\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)]^{\mathrm{SL}_2}.$$

Moreover, we have a surjection

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} \cong \mathbb{C}[T^*V \times \mathbb{A}^2]^{\mathrm{SL}_2} \twoheadrightarrow \mathbb{C}[\mathrm{SL}_2 \times_{\mathbb{G}_a} H_V(\xi)]^{\mathrm{SL}_2} \cong \mathbb{C}[H_V(\xi)]^{\mathbb{G}_a}$$

which completes the proof of iii-iv).

Finally, to show $H_V(\xi) \rightarrow H_V(\xi)/\mathbb{G}_a$ is isomorphic to the trivial \mathbb{G}_a -torsor, we show there exists a function $\sigma \in \mathbb{C}[H_V(\xi)]$ whose derivation under the \mathbb{G}_a -action is 1, which gives a section perpendicular to the \mathbb{G}_a -action that enables us to define an isomorphism with the trivial \mathbb{G}_a -torsor over $H_V(\xi)/\mathbb{G}_a$. More precisely, we claim that there exists σ' such that $\mu_{\mathbb{G}_a} = D(\sigma')$, from which it follows that $\sigma := \sigma'/\xi$ has derivation 1 in $\mathbb{C}[H_V(\xi)]$. It suffices to provide σ' for the irreducible SL_2 -representations $V = \mathrm{Sym}^n \mathbb{C}^2$: if we take coordinates x_i, α_i on T^*V as above, then, for appropriate choices of constants λ_i , the image of $\sigma' := \sum_{i=1}^{n+1} \lambda_i x_i \alpha_i$ under the derivation is $\mu_{\mathbb{G}_a}$. \square

The following theorem proves that the algebraic symplectic reductions $H_V(\xi)/\mathbb{G}_a$ are algebraic symplectic manifolds, which fit together to give a 1-parameter family of algebraic symplectic reductions. Moreover, the fibres of this 1-parameter family are isomorphic as algebraic varieties, but not as algebraic symplectic varieties; instead, they are ‘scaled symplectomorphic’ to each other.

Theorem 3.14. *For $\xi \neq 0$, the quotient $\mathcal{X}_\xi := H_V(\xi)/\mathbb{G}_a$ is an algebraic symplectic manifold with symplectic form ω_ξ induced by the Liouville form ω , which form a 1-parameter family*

$$\begin{array}{ccc} \mathcal{X}_\xi & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \xi & \hookrightarrow & \mathbb{C}^* \end{array}$$

where $H_V(\mathbb{C}^*) := \mu_{\mathbb{G}_a}^{-1}(\mathbb{C}^*) \rightarrow \mathcal{X} := H_V(\mathbb{C}^*)/\mathbb{G}_a$ is a trivial \mathbb{G}_a -torsor. Furthermore, for $\xi_1, \xi_2 \in \mathbb{C}^*$, there is an isomorphism

$$\Phi_{\xi_1, \xi_2} : H_V(\xi_1)/\mathbb{G}_a \rightarrow H_V(\xi_2)/\mathbb{G}_a$$

of algebraic varieties which scales the symplectic forms as follows: $\xi_2\omega_{\xi_1} = \xi_1\Phi_{\xi_1, \xi_2}^*\omega_{\xi_2}$.

Proof. By Theorem 3.13, the quotient $H_V(\xi)/\mathbb{G}_a$ is a smooth affine variety for $\xi \neq 0$ and, moreover, it has an algebraic symplectic form ω_ξ induced by the Liouville form ω on T^*V by Lemma 2.1.

The linear action of \mathbb{G}_a on the affine space T^*V has finitely generated ring of invariants and so we can consider the non-reductive enveloping quotient $\pi : T^*V \rightarrow T^*V//\mathbb{G}_a := \text{Spec } \mathbb{C}[T^*V]^{\mathbb{G}_a}$, which restricts to a Zariski locally trivial geometric quotient on the completely stable locus. By Corollary 3.12,

$$H_V(\mathbb{C}^*) := \mu_{\mathbb{G}_a}^{-1}(\mathbb{C}^*) \subset (T^*V)^{\bar{s}}$$

and so $H_V(\mathbb{C}^*) \rightarrow H_V(\mathbb{C}^*)/\mathbb{G}_a$ is a Zariski locally trivial geometric quotient. The morphism $\mathcal{X} := H_V(\mathbb{C}^*)/\mathbb{G}_a \rightarrow \mathbb{C}^*$ defining this 1-parameter family is given by the \mathbb{G}_a -invariant function $\mu_{\mathbb{G}_a}$ and restricts to the quotient $H_V(\xi)/\mathbb{G}_a$ at $\xi \in \mathbb{C}^*$, as every \mathbb{G}_a -invariant function on $H_V(\xi)$ extends to a \mathbb{G}_a -invariant on T^*V . Moreover, $H_V(\mathbb{C}^*) \rightarrow H_V(\mathbb{C}^*)/\mathbb{G}_a$ is a trivial \mathbb{G}_a -torsor, as the derivation of the function $\sigma'/\mu_{\mathbb{G}_a} \in \mathbb{C}[H_V(\mathbb{C}^*)] = \mathbb{C}[T^*V]_{\mu_{\mathbb{G}_a}}$ is 1, where σ' is the function appearing in the proof of Theorem 3.14.

Let $\phi : T^*V \rightarrow T^*V$ be the map given by multiplication by $C = \xi_2/\xi_1$ on V and the identity map on V^* ; this map restricts to a \mathbb{G}_a -equivariant isomorphism

$$\phi : H_V(\xi_1) \rightarrow H_V(\xi_2)$$

and induces an isomorphism of quotients

$$\Phi : H_V(\xi_1)/\mathbb{G}_a \rightarrow H_V(\xi_2)/\mathbb{G}_a.$$

The pullback of the Liouville form ω on T^*V via $\phi : T^*V \rightarrow T^*V$ satisfies $\phi^*\omega = C\omega$. Let $\pi_\xi : H_v(\xi) \rightarrow H_V(\xi)/\mathbb{G}_a$ be the quotient map and $i_\xi : H_V(\xi) \hookrightarrow T^*V$ be the inclusion; then

$$\pi_{\xi_1}^* \Phi^*(\omega_{\xi_2}) = \phi^* \pi_{\xi_2}^*(\omega_{\xi_2}) = \phi^* i_{\xi_2}^*(\omega) = i_{\xi_1}^* \phi^*(\omega) = i_{\xi_1}^*(C\omega) = C\pi_{\xi_1}^*(\omega_\xi),$$

and, as π_{ξ_1} is surjective, it follows that $\Phi^*(\omega_{\xi_2}) = C\omega_{\xi_1}$. □

Following the proof of this result, we are able to describe the central fibre of this 1-parameter family.

Corollary 3.15. *The central fibre of the 1-parameter family $\mathcal{X} = H_V(\mathbb{C}^*)/\mathbb{G}_a$ constructed above from the algebraic symplectic reductions of the \mathbb{G}_a -action on T^*V at non-zero values is the zero fibre of the morphism $\mu_{\mathbb{G}_a} : T^*V//\mathbb{G}_a \rightarrow \mathbb{C}$. More precisely, there is a \mathbb{G}_a -invariant morphism*

$$H_V(0) \rightarrow \mathcal{X}_0 \cong \text{Spec } \mathbb{C}[T^*V]^{\mathbb{G}_a}/(\mu_{\mathbb{G}_a})$$

constructed using the extrinsic \mathbb{G}_a -invariant functions on the ambient space T^*V .

At this point, we should emphasise that in general, the hypersurface $H_V(0) \subset T^*V$ admits \mathbb{G}_a -invariant functions which do not extend to T^*V and so there is a non-surjective homomorphism

$$\mathbb{C}[T^*V]^{\mathbb{G}_a}/(\mu_{\mathbb{G}_a}) \rightarrow \mathbb{C}[H_V(0)]^{\mathbb{G}_a}$$

and, if the latter is finitely generated, then we have a morphism of affine varieties

$$H_V(0)//\mathbb{G}_a := \text{Spec } \mathbb{C}[H_V(0)]^{\mathbb{G}_a} \rightarrow \mathcal{X}_0.$$

We should also point out that the hyperkähler metric on T^*V is not preserved by the \mathbb{G}_a -action and so does not automatically give a hyperkähler metric on these algebraic symplectic reductions.

3.6. Constructing the algebraic symplectic analogue

The non-reductive quotient $V//\mathbb{G}_a$ will often be singular, as not all points in V are stable (for example, the origin has full stabiliser group, so is not stable). Therefore, we can only expect to provide an algebraic symplectic analogue of an open subset of this quotient.

Proposition 3.16. *Let V be a \mathbb{G}_a -representation. Then the open set $U := V^{\bar{s}}/\mathbb{G}_a \subset V//\mathbb{G}_a$ has an algebraic symplectic analogue $U' := H_V(0)^{\bar{s}}/\mathbb{G}_a$; that is, we have the following.*

- (i) *There is a closed immersion $U \hookrightarrow U'$.*
- (ii) *U' is an algebraic symplectic manifold, with algebraic symplectic form ω' induced by the Liouville form ω on T^*V .*
- (iii) *$U \subset U'$ is a Lagrangian subvariety.*
- (iv) *T^*U is an open subset of U' which is dense if non-empty.*
- (v) *U' is an open subset of the central fibre \mathcal{X}_0 of the 1-parameter family \mathcal{X} .*

Proof. As $V^{\bar{s}} \subset V$ is open and the quotient $V^{\bar{s}} \rightarrow U := V^{\bar{s}}/\mathbb{G}_a$ is a Zariski locally trivial geometric quotient, $U \subset V//\mathbb{G}_a$ is a smooth open subvariety. The hypersurface $H_V(0)$ is singular, but the completely stable locus avoids the singularities:

$$H_V(0)^{\bar{s}} \subset H_V(0) - H_V(0)^{\mathbb{G}_a} = H_V(0)_{\text{reg}}$$

by Proposition 3.10 and Corollary 3.12. Hence, as $H_V(0)^{\bar{s}} \rightarrow H_V(0)^{\bar{s}}/\mathbb{G}_a =: U'$ is a Zariski locally trivial geometric quotient, U' is also smooth. Since the completely stable locus is determined by the extrinsic invariants on T^*V , it follows that $V \cap H_V(0)^{\bar{s}} = V^{\bar{s}}$ and we have a pullback square

$$\begin{array}{ccc} V^{\bar{s}} & \longrightarrow & H_V(0)^{\bar{s}}, \\ \downarrow & & \downarrow \\ V & \longrightarrow & H_V(0) \end{array}$$

such that both horizontal morphisms are closed immersions. It follows that the induced map on \mathbb{G}_a -quotients $U \rightarrow U'$ is also a closed immersion.

The algebraic symplectic structure on U' is a consequence of the algebraic version of the Marsden-Weinstein-Meyer Theorem and the fact that $U \subset U'$ is a Lagrangian subvariety follows from the fact that $V \subset T^*V$ is a Lagrangian subvariety. The proof of statements (iii) and (iv) follow exactly as in the proof of Proposition 2.7.

For (v), we recall that the central fibre \mathcal{X}_0 of the 1-parameter family \mathcal{X} is the \mathbb{G}_a -quotient of $H_V(0)$ obtained using only the \mathbb{G}_a -invariant functions on T^*V . Since the completely stable locus $H_V(0)^{\bar{s}}$ is determined using invariant functions on T^*V , the Zariski locally trivial geometric quotient $H_V(0)^{\bar{s}} \rightarrow U'$

is obtained by restricting $H_V(0) \rightarrow \mathcal{X}_0$ and so, in particular, U' is an open subset of \mathcal{X}_0 . □

The completely stable set $H_V(0)^{\bar{s}}$ is determined using only the ambient \mathbb{G}_a -invariants on the affine space T^*V . However, there may be additional intrinsic \mathbb{G}_a -invariants on $H_V(0)$ which do not extend to invariants on T^*V , as the homomorphism

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} \rightarrow \mathbb{C}[H_V(0)]^{\mathbb{G}_a}$$

is not surjective in general. Since the \mathbb{G}_a -action on $H_V(0)$ does not extend to SL_2 , we cannot conclude that $\mathbb{C}[H_V(0)]^{\mathbb{G}_a}$ is finitely generated (see §3.7 below).

The additional intrinsic invariants can be used to construct a potentially larger open subset which admits a Zariski locally trivial geometric quotient, referred to as the ‘locally trivial stable set’ in [7]. The \mathbb{G}_a -quotient of this locally trivial stable set $H_V(0)^{lts}$ would also admit the structure of a smooth algebraic symplectic variety. However, as there is no Hilbert–Mumford style criteria to compute the locally trivial stable set, we would need to compute the invariant ring, which is impractical and thus justifies our use of the potentially smaller, but easily understandable, subset of completely stable points. Furthermore, the following lemma shows that these subsets can differ in at most a codimension 2 subset.

Lemma 3.17. *The complement of $H_V(0)^{\bar{s}} \subset H_V(0)^{lts}$ has codimension ≥ 2 .*

Proof. The codimension of $H_V(0)^{\bar{s}} \subset H_V(0)^{lts}$ is greater than or equal to the codimension of $H_V(0)^{\bar{s}} \subset H_V(0)$, which we can explicitly compute using Lemma 3.11. If we write V as a sum of irreducible SL_2 -representations $V = \mathrm{Sym}^{k_1}(\mathbb{C}^2) \oplus \cdots \oplus \mathrm{Sym}^{k_m}(\mathbb{C}^2)$, then the codimension of $H_V(0)^{\bar{s}} \subset H_V(0)$ is

$$2 \left(\sum_{j=1}^m \lceil k_j/2 \rceil \right) - 1,$$

which is greater than or equal to 2, unless $V = \mathrm{Sym}^1(\mathbb{C}^2)$ or $V = \mathrm{Sym}^2(\mathbb{C}^2)$. If $V = \mathrm{Sym}^1(\mathbb{C}^2)$, then $H_V(0)$ is 3 dimensional and $H_V(0)^{\bar{s}} = H_V(0)^{lts}$, as the complement to the completely stable locus is the \mathbb{G}_a -fixed locus (see Proposition 5.1 below). If $V = \mathrm{Sym}^2(\mathbb{C}^2)$, then we also have $H_V(0)^{\bar{s}} = H_V(0)^{lts}$ (essentially, as on the complement of the completely stable locus the action is not separated, see §6.2). □

3.7. Finite generation of the invariant ring of $H_V(0)$

In the simplest example $V = \text{Sym}^1(\mathbb{C}^2)$, the ring $\mathbb{C}[H_V(0)]^{\mathbb{G}_a}$ is non-finitely generated.

Example 3.18. Let $V = \text{Sym}^1(\mathbb{C}^2)$ be the 2-dimensional \mathbb{G}_a -representation. If we take coordinates x_1, x_2 on V and α_1, α_2 on V^* , then the action of $s \in \mathbb{G}_a$ on $T^*V \cong V \times V^*$ is

$$s \cdot (x_1, x_2, \alpha_1, \alpha_2) = (x_1 + sx_2, x_2, \alpha_1, \alpha_2 - s\alpha_1).$$

By Example 3.6, the moment map $\mu_{\mathbb{G}_a} : T^*V \rightarrow \mathbb{C}$ is given by $\mu_{\mathbb{G}_a}(x, \alpha) = x_2\alpha_1$. The ring of \mathbb{G}_a -invariant functions on the cotangent space is finitely generated

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} = \mathbb{C}[x_2, \alpha_1, x_1\alpha_1 + x_2\alpha_2],$$

but the ring of \mathbb{G}_a -invariant functions on the hypersurface $H_V(0) := \mu_{\mathbb{G}_a}^{-1}(0)$ is not: for $n \geq 0$,

$$x_1^n \alpha_1 \quad \text{and} \quad x_2 \alpha_2^n$$

are \mathbb{G}_a -invariant functions on $\mu_{\mathbb{G}_a}^{-1}(0)$, but the functions x_1 and α_2 are not \mathbb{G}_a -invariant. We see that taking \mathbb{G}_a -invariants is not exact: there are \mathbb{G}_a -invariant functions $x_1^n \alpha_1$ and $x_2 \alpha_2^n$ on $\mu_{\mathbb{G}_a}^{-1}(0)$ which do not extend to \mathbb{G}_a -invariant functions on the ambient affine space T^*V .

There is an immediate extension of this example: we can add on copies of the trivial representation.

Corollary 3.19. *For the \mathbb{G}_a -representation $V = \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$, the ring of \mathbb{G}_a -invariant functions on $H_V(0) = \mu_{\mathbb{G}_a}^{-1}(0)$ is non-finitely generated.*

However, this seemingly negative result is not as discouraging as one may think: by Proposition 3.10, this representation V is the only case for which the hypersurfaces $H_V(0)$ are not normal. If we consider the induced \mathbb{G}_a -action on the normalisation of $H_V(0)$ for $V = \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$, then we see that the ring of \mathbb{G}_a -invariants is finitely generated.

Lemma 3.20. *Let $V = \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$ and $\widetilde{H}_V(0) \rightarrow H_V(0)$ be the normalisation of the zero level set of the moment map. Then $\mathbb{C}[\widetilde{H}_V(0)]^{\mathbb{G}_a}$ is finitely generated.*

Proof. As the non-normal hypersurface $H_V(0)$ has two irreducible components, its normalisation is the disjoint union of the normalisation of each of these irreducible components. However, the irreducible components are both affine spaces of dimension $3 + 2n$ and so are already normal. As the \mathbb{G}_a -action on both these affine spaces is linear, the ring of \mathbb{G}_a -invariants on the normalisation is finitely generated. \square

The next smallest example is given by $V = \text{Sym}^2(\mathbb{C}^2)$, which we study in §6.2 and show that the ring of \mathbb{G}_a -invariants on $H_V(0)$ is finitely generated. If $\mathbb{C}[H_V(0)]^{\mathbb{G}_a}$ is finitely generated, then we can define

$$H_V(0)//\mathbb{G}_a := \text{Spec } \mathbb{C}[H_V(0)]^{\mathbb{G}_a},$$

which contains $H_V(0)^{\bar{s}}/\mathbb{G}_a$ as an open subset. In this case, as the composition

$$V \hookrightarrow H_V(0) \hookrightarrow T^*V \twoheadrightarrow V$$

is the identity, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[H_V(0)]^{\mathbb{G}_a} & \xrightarrow{\phi} & \mathbb{C}[V]^{\mathbb{G}_a} \\ \uparrow & & \uparrow \text{Id} \\ \mathbb{C}[T^*V]^{\mathbb{G}_a} & \longleftarrow & \mathbb{C}[V]^{\mathbb{G}_a}. \end{array}$$

Hence ϕ is surjective and gives a closed immersion $V//\mathbb{G}_a \rightarrow H_V(0)//\mathbb{G}_a$.

As we can only expect to have an algebraic symplectic structure on an open subset of the \mathbb{G}_a -quotient of $H_V(0)$, the question of whether $\mathbb{C}[H_V(0)]^{\mathbb{G}_a}$ is finitely generated is not particularly pertinent. However, we note that the zero level set of the moment map $H_V(0)$ is the only level set which is preserved by the action of the upper triangular Borel B in SL_2 . For representations V which are not equal to a sum of trivial representations with one copy of $\text{Sym}^1(\mathbb{C}^2)$, the hypersurface $H_V(0)$ is normal and irreducible. It would be interesting to study the ring of B -invariants on $H_V(0)$ from the point of view of [1], which in particular gives a geometrically useful finitely generated subring that should still have the desired separation properties.

4. Comparisons of the holomorphic symplectic quotients

Let \mathbb{G}_a act linearly on an affine space $V = \mathbb{A}^n$ over the complex numbers; then

$$V//\mathbb{G}_a \cong W//\text{SL}_2$$

where $W := V \times \mathbb{A}^2$. In the previous section, we studied algebraic symplectic reductions of the \mathbb{G}_a -action on T^*V . We can alternatively consider algebraic symplectic reductions of the SL_2 -action on T^*W . As SL_2 is semisimple, it has trivial character group and, as only $0 \in \mathfrak{sl}_2^*$ is coadjoint fixed, there is only one algebraic symplectic reduction

$$\mu_{SL_2}^{-1}(0)//SL_2$$

where $\mu_{SL_2} : T^*W \rightarrow \mathfrak{sl}_2^*$ denotes the algebraic moment map for the SL_2 -action on T^*W . Although \mathbb{G}_a also has trivial character group, every element in its co-Lie algebra is coadjoint fixed and we have a 1-parameter family of algebraic symplectic reductions by Theorem 3.14. In many ways, this reflects the fact that the representation theory of the reductive group SL_2 is discrete, whereas the representation theory for the non-reductive group \mathbb{G}_a is not.

In this section, we compare the non-reductive algebraic symplectic quotients $\mu_{\mathbb{G}_a}^{-1}(\xi)//\mathbb{G}_a$ studied above with the reductive algebraic symplectic quotient $\mu_{SL_2}^{-1}(0)//SL_2$. We continue to assume the \mathbb{G}_a -representation V is non-trivial.

4.1. Equivariant morphisms between the level sets

In order to relate the non-reductive holomorphic symplectic quotients with the reductive holomorphic symplectic quotient, we want to construct a \mathbb{G}_a -equivariant morphism $\mu_{\mathbb{G}_a}^{-1}(\xi) \rightarrow \mu_{SL_2}^{-1}(0)$. In this section, we assume that as an SL_2 -representation

$$V \neq \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n},$$

so that the hypersurface $H_V(0)$ is normal. The non-normal case will be studied separately in §6.1.

We recall that $\mu_{SL_2}^{-1}(0)$ is the closed subvariety of T^*W defined by the equations

$$\begin{aligned} \Phi_H(x, \alpha) + u\lambda - v\eta &= 0 \\ \Phi_E(x, \alpha) + v\lambda &= 0 \\ \Phi_F(x, \alpha) + u\eta &= 0. \end{aligned}$$

where $\mu_{SL_2, V} = \Phi_H H^* + \Phi_E E^* + \Phi_F F^* : T^*V \rightarrow \mathfrak{sl}_2^*$ is the moment map for the SL_2 -action on T^*V and (x, α) are coordinates on T^*V and (u, v, λ, η) are coordinates on $T^*\mathbb{A}^2$.

Remark 4.1. There are families of natural \mathbb{G}_a -equivariant inclusions $V \hookrightarrow W = V \times \mathbb{A}^2$ (resp. $V^* \hookrightarrow W^* = V^* \times (\mathbb{A}^2)^*$ given by $x \mapsto (x, a, 0)$ (resp. $\alpha \mapsto (\alpha, 0, b)$) for constants $a, b \in \mathbb{C}$, which give rise to inclusions $\iota_{a,b} : T^*V \hookrightarrow T^*W$. However, $\iota_{a,b}(\mu_{\mathbb{G}_a}^{-1}(\xi)) \subset \mu_{\text{SL}_2}^{-1}(0)$ if and only if $\Phi_H = \xi = 0$ and $\Phi_F = -ab$. As the functions Φ_H and Φ_F are not constant (cf. Example 3.6), these natural families of \mathbb{G}_a -equivariant maps $\iota_{a,b}$ do not map one level set to the other.

Fortunately, there are \mathbb{G}_a -equivariant inclusions $\mu_{\mathbb{G}_a}^{-1}(\xi) \rightarrow \mu_{\text{SL}_2}^{-1}(0)$ when $\xi = 0$.

Proposition 4.2. *Let $f : T^*V \rightarrow T^*W$ be a morphism that extends the identity map on T^*V . Then f restricts to a \mathbb{G}_a -equivariant morphism $f : \mu_{\mathbb{G}_a}^{-1}(\xi) \rightarrow \mu_{\text{SL}_2}^{-1}(0)$ only if $\xi = 0$. For $\xi = 0$, the \mathbb{G}_a -equivariant morphisms $f : \mu_{\mathbb{G}_a}^{-1}(0) \rightarrow \mu_{\text{SL}_2}^{-1}(0)$ are given by two 1-parameter families of morphisms $\{i_a\}_{a \neq 0}$ and $\{j_b\}_{b \neq 0}$, where, for $(x, \alpha) \in \mu_{\mathbb{G}_a}^{-1}(0)$,*

$$i_a(x, \alpha) := \left(x, (a, 0), \alpha, -\frac{\Phi_H(x, \alpha)}{a}, -\frac{\Phi_F(x, \alpha)}{a} \right)$$

and

$$j_b(x, \alpha) := \left(x, -\frac{\Phi_F(x, \alpha)}{b}, \frac{\Phi_H(x, \alpha)}{b}, \alpha, (0, b) \right).$$

Proof. Such an extension f is given by functions $f_u, f_v, f_\lambda, f_\eta \in \mathbb{C}[T^*V]$. If $f(\mu_{\mathbb{G}_a}^{-1}(\xi)) \subset \mu_{\text{SL}_2}^{-1}(0)$, then, on $\mu_{\mathbb{G}_a}^{-1}(\xi)$, we have

$$\Phi_H + f_u f_\lambda - f_v f_\eta = \xi + f_v f_\lambda = \Phi_F + f_u f_\eta = 0.$$

If moreover $f : \mu_{\mathbb{G}_a}^{-1}(\xi) \rightarrow \mu_{\text{SL}_2}^{-1}(0)$ is \mathbb{G}_a -equivariant, then $f_v, f_\lambda \in \mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(\xi)]^{\mathbb{G}_a}$ and f_u, f_η satisfy

$$\begin{aligned} f_u(c \cdot (x, \alpha)) &= f_u(x, \alpha) + c f_v(x, \alpha) \\ f_\eta(c \cdot (x, \alpha)) &= f_\eta(x, \alpha) - c f_\lambda(x, \alpha) \end{aligned}$$

for $c \in \mathbb{G}_a$ and $(x, \alpha) \in \mu_{\mathbb{G}_a}^{-1}(0)$.

First suppose that $\xi \neq 0$. Since $\mu_{\mathbb{G}_a} = \xi = -f_v f_\lambda$ on $\mu_{\mathbb{G}_a}^{-1}(\xi)$ and the equation $\mu_{\mathbb{G}_a}$ is irreducible (here we use the assumption that $V \neq \text{Sym}^1(\mathbb{C}^2) \oplus \text{Sym}^0(\mathbb{C}^2)^{\oplus n}$; see also Example 3.6), it follows that both f_v and f_λ are non-zero constant functions. However, in this case, there are no functions f_u and f_η which transform under \mathbb{G}_a as above.

Hence let $\xi = 0$. As $f_v f_\lambda = 0$ on the irreducible hypersurface $\mu_{\mathbb{G}_a}^{-1}(0)$, it follows that either $f_v = 0$ or $f_\lambda = 0$ on $\mu_{\mathbb{G}_a}^{-1}(0)$. We will show that the first case gives rise to the family i_a ; then, similarly, the second case gives rise to the family j_b . If $f_v = 0$, it follows that $f_u \in \mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]^{\mathbb{G}_a}$. Since Φ_H is irreducible but not \mathbb{G}_a -invariant (cf. Example 3.6) and $\Phi_H + f_u f_\lambda = 0$ on $\mu_{\mathbb{G}_a}^{-1}(0)$, we conclude that $f_u = a$ for some non-zero constant a . It then follows that $f = i_a$.

The \mathbb{G}_a -equivariance of i_a and j_b on $\mu_{\mathbb{G}_a}^{-1}(0)$ can be verified using the following transformation properties of Φ_E, Φ_F and Φ_H :

$$\begin{aligned} \Phi_H(c \cdot (x, \alpha)) &= \Phi_H(x, \alpha) + 2c\Phi_E(x, \alpha) \\ \Phi_F(c \cdot (x, \alpha)) &= \Phi_F(x, \alpha) - c\Phi_H(x, \alpha) - c^2\Phi_E(x, \alpha), \end{aligned}$$

for $c \in \mathbb{G}_a$ (which follow from the SL_2 -equivariance of $\mu_{\mathrm{SL}_2, V} : T^*V \rightarrow \mathfrak{sl}_2^*$). □

Henceforth, for notational simplicity, we consider only the morphism $i := i_1$, and we note that the same results hold, with only minor modifications to the proofs, for i_a and j_b .

Lemma 4.3. *The morphism $i : \mu_{\mathbb{G}_a}^{-1}(0) \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0)$ has the following properties.*

- (i) *The stabiliser groups are preserved by i .*
- (ii) *Let $p, p' \in \mu_{\mathbb{G}_a}^{-1}(0)$; then $i(p)$ and $i(p')$ lie in the same SL_2 -orbit if and only if p and p' lie in the same \mathbb{G}_a -orbit.*
- (iii) *For $p \in \mu_{\mathbb{G}_a}^{-1}(0)$, we have $\{g \in \mathrm{SL}_2 : g \cdot i(p) \in i(\mu_{\mathbb{G}_a}^{-1}(0))\} = \mathbb{G}_a$*

Proof. For (i) suppose $p \in \mu_{\mathbb{G}_a}^{-1}(0)$ has \mathbb{G}_a -stabiliser subgroup H . If $g \in \mathrm{SL}_2$ fixes $i(p)$, then

$$g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and thus $g \in \mathbb{G}_a \subset \mathrm{SL}_2$. By the \mathbb{G}_a -equivariance of i , it follows that $i(p)$ has SL_2 -stabiliser group H .

For (ii), if $i(p)$ and $i(p')$ lie in the same SL_2 -orbit, then $g \cdot i(p) = i(p')$ for some $g \in \mathrm{SL}_2$. However, the same argument as above shows that $g \in \mathbb{G}_a$ and so $i(g \cdot p) = i(p')$, but as i is injective, we conclude $g \cdot p = p'$. The converse direction is trivial.

For (iii), the inclusion “ \subseteq ” follows by the argument used in (i), and the opposite inclusion follows by \mathbb{G}_a -equivariance of i . □

Theorem 4.4. *Let V be a \mathbb{G}_a -representation and let $W = V \times \mathbb{A}^2$ be the associated SL_2 -representation; then the non-reductive and reductive algebraic symplectic analogues are birationally symplectomorphic. More precisely, the equivariant inclusion $i : \mu_{\mathbb{G}_a}^{-1}(0) \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0)$ induces a morphism*

$$\mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}}/\mathbb{G}_a \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0)//\mathrm{SL}_2$$

which is an isomorphism on open subsets and, moreover, this isomorphism respects the induced symplectic forms.

Proof. Let $U = \mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}} \cap i^{-1}(\mu_{\mathrm{SL}_2}^{-1}(0)^{\mathrm{SL}_2-s})$; then $U \subset \mu_{\mathbb{G}_a}^{-1}(0)$ is a \mathbb{G}_a -invariant open subset. Let $U' = \mathrm{SL}_2 \cdot i(U)$; that is, U' is the image of the morphism $U \times_{\mathbb{G}_a} \mathrm{SL}_2 \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0)$ given by $[u, g] \mapsto g \cdot i(u)$. By construction, the open subset $U' \subset \mu_{\mathrm{SL}_2}^{-1}(0)^{\mathrm{SL}_2-s}$ is a SL_2 -invariant subset on which SL_2 acts with trivial stabilisers by Lemma 4.3 (i); therefore, U' is smooth by Lemma 2.4 (iii). It follows from Lemma 4.3 (iii) that the morphism $U \times_{\mathbb{G}_a} \mathrm{SL}_2 \rightarrow U'$ is bijective and, moreover, is an isomorphism as U' is smooth.

The \mathbb{G}_a -action on $U \subset \mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}}$ admits a Zariski locally trivial geometric quotient and the SL_2 -action on $U' \subset \mu_{\mathrm{SL}_2}^{-1}(0)^{\mathrm{SL}_2-s}$ admits a geometric quotient, which is a principal SL_2 -bundle. Moreover, we have isomorphisms

$$U/\mathbb{G}_a \cong (U \times_{\mathbb{G}_a} \mathrm{SL}_2)/\mathrm{SL}_2 \cong U'/\mathrm{SL}_2$$

which proves the desired birationality statement.

The Liouville form ω_V on T^*V (resp. ω_W on T^*W) induces algebraic symplectic forms $\omega_{\mathbb{G}_a}$ (resp. ω_{SL_2}) on U/\mathbb{G}_a (resp. U'/SL_2) by Proposition 3.16 (resp. Lemma 2.5). We consider the commutative square

$$\begin{array}{ccc} U & \xrightarrow{i} & U' \\ \pi_{\mathbb{G}_a} \downarrow & & \downarrow \pi_{\mathrm{SL}_2} \\ U/\mathbb{G}_a & \xrightarrow{\bar{i}} & U'/\mathrm{SL}_2. \end{array}$$

From the explicit definition of $i : T^*V \rightarrow T^*W$, we see that $i^*\omega_W = \omega_V$. Therefore

$$\pi_{\mathbb{G}_a}^*(\omega_{\mathbb{G}_a}) = \omega_V = i^*(\omega_W) = i^*\pi_{\mathrm{SL}_2}^*(\omega_{\mathrm{SL}_2}) = \pi_{\mathbb{G}_a}^*\bar{i}^*(\omega_{\mathrm{SL}_2}).$$

Since $\pi_{\mathbb{G}_a} : U \rightarrow U/\mathbb{G}_a$ is surjective, it follows that $\omega_{\mathbb{G}_a} = \bar{i}^*\omega_{\mathrm{SL}_2}$. □

For the reductive algebraic symplectic analogue, we can use the Kempf-Ness Theorem for SL_2 , which gives a homeomorphism between the algebraic

symplectic analogue and the hyperkähler analogue. Then using the above birational symplectomorphism we obtain the following corollary.

Corollary 4.5. *The additive algebraic symplectic analogue $\mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}}/\mathbb{G}_a$ of $V^{\bar{s}}/\mathbb{G}_a$ admits a hyperkähler structure on a dense open subset. Furthermore, assuming a bounded number of trivial representations in V , then for all but finitely many cases, this is a codimension at least 2 complement in $\mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a$.*

Proof. Everything has been established except the codimension at least two condition. This follows from a dimension count. The non-stable locus for such a representation is high codimension, but $\mu_{\mathrm{SL}_2}^{-1}(0)$ is always codimension at most 3, and the image of $\mu_{\mathbb{G}_a}^{-1}(0)$ under the inclusion is codimension at most 5, in T^*W . Thus U in the proof of the theorem is high codimension complement both in $\mu_{\mathbb{G}_a}^{-1}(0)$ and in $\mu_{\mathrm{SL}_2}^{-1}(0)$ for all but finitely many cases. By upper semi-continuity, the codimension of the complement can only increase in the quotients and analogues. \square

Remark 4.6. In fact, the proof can easily be refined to handle all the cases with a computation. It is likely that the codimension at least 2 complement result always holds. One can refine the statement further to get explicit comparisons between the high codimension loci for the SL_2 quotient analogues and the \mathbb{G}_a quotient analogues.

5. Geometric applications

In [6], an algebraic uniformisation for an iterated blow up X of a projective space along a collection of linear subspaces is provided by a description of X as a non-reductive GIT quotient of an affine space V by a solvable group $G = \mathbb{G}_a^t \rtimes \mathbb{G}_m^s$, where s is the Picard rank of X and $t \geq 0$. One can consider the cotangent lift of this action to T^*V and take an algebraic symplectic reduction by G , by extending the ideas of §3, to construct an algebraic symplectic analogue of X . However, the construction of [6], often involves several copies of \mathbb{G}_a (that is, $t \geq 2$) or a non-linearisable action of \mathbb{G}_a (which is the case, for example, if $X = \overline{M}_{0,n}$ and $n \geq 6$).

The simplest geometric example covered by [6], which we can directly apply the techniques of §3, is $X = \mathrm{Bl}_n \mathbb{P}^{n-2}$, which is a non-reductive GIT quotient of $V = \mathbb{A}^{2n}$ by a linear action of $\mathbb{G}_a \times (\mathbb{G}_m)^{n+1}$. Let us briefly explain how this construction works. Let $X' := \mathrm{Bl}_n \mathbb{P}^{n-1}$, which is toric, as we can choose the n points to be the n coordinate points in \mathbb{P}^{n-1} . The Cox ring

of X' is given by

$$\text{Cox}(X') = \mathbb{C}[y_1, x_1, \dots, y_n, x_n],$$

where y_1, \dots, y_n are the homogeneous coordinates on \mathbb{P}^{n-1} before blowing up and x_i represent the exceptional divisors of each point being blown up. Then X' is a GIT quotient of $\mathbb{A}^{2n} = \text{Spec Cox}(X')$ by its Picard torus $T_{X'} \cong T_X \cong \mathbb{G}_m^{n+1}$, which acts as follows: for $t = (t_0, t_1, \dots, t_n) \in T_{X'}$

$$t \cdot (y_i, x_i) = \left(t_0 \prod_{j \neq i} t_j^{-1} y_i, t_i x_i \right).$$

To relate X' and X , consider the linear projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ sending the n coordinate points in \mathbb{P}^{n-1} to the given n points in \mathbb{P}^{n-2} ; this induces a rational map $X' \dashrightarrow X$, which is a Zariski-locally trivial \mathbb{A}^1 -bundle over its image. Moreover, this lifts to a \mathbb{G}_a -action on $\mathbb{A}^{2n} = \text{Spec Cox}(X')$ given by, for $c \in \mathbb{G}_a$,

$$c \cdot (y_i, x_i) = (y_i + cx_i, x_i),$$

which is normalised by the torus action. By [6] Theorem 1.1, $X = \mathbb{A}^n //_{\chi} \mathbb{G}_a \rtimes (\mathbb{G}_m)^{n+1}$ for $\chi \in \text{Hom}(T_X, \mathbb{G}_m) \cong \text{Pic}(X)$ an ample divisor class.

In this class of examples, the \mathbb{G}_a -action on $V = \mathbb{A}^{2n}$ is linear and, as an SL_2 -representation we have $V = \text{Sym}^1(\mathbb{C}^2)^{\oplus n}$. In the next subsection, we study these representations and the algebraic symplectic analogues of $V // \mathbb{G}_a$ in greater depth.

5.1. Representation of the form $V = \text{Sym}^1(\mathbb{C}^2)^{\oplus n}$

Consider the \mathbb{G}_a -representation $V = \text{Sym}^1(\mathbb{C}^2)^{\oplus n}$ and its cotangent lift to T^*V . More precisely, if we write

$$\mathbb{C}[V] = \mathbb{C}[y_i, x_i]_{1 \leq i \leq n} \quad \text{and} \quad \mathbb{C}[T^*V] = \mathbb{C}[y_i, x_i, \beta_i, \alpha_i]_{1 \leq i \leq n},$$

then $c \in \mathbb{G}_a$ acts by

$$c \cdot (y_i, x_i, \beta_i, \alpha_i) = (y_i + cx_i, x_i, \beta_i, \alpha_i - c\beta_i).$$

Hence, the \mathbb{G}_a -moment map $\mu : T^*V \rightarrow \mathbb{C}$ is given by $\sum_{j=1}^n \beta_j x_j$.

Proposition 5.1. *Let $V = \text{Sym}^1(\mathbb{C}^2)^{\oplus n}$ be as above. Then*

- (i) *The complement to the completely stable locus for the \mathbb{G}_a -action on $\mu_{\mathbb{G}_a}^{-1}(0)$ is the \mathbb{G}_a -fixed locus. In particular, the completely stable locus is the largest possible stable set.*
- (ii) *The \mathbb{G}_a -invariants on T^*V give a separating set of invariants for the \mathbb{G}_a -action on $\mu_{\mathbb{G}_a}^{-1}(0)$ and, moreover, the nullcone of these invariants is the \mathbb{G}_a -fixed locus.*

Proof. For (i), by Lemma 3.11,

$$\mu_{\mathbb{G}_a}^{-1}(0) - \mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}} = \{(y_i, x_i, \beta_i, \alpha_i)_{1 \leq i \leq n} : x_i = \beta_i = 0 \text{ for } i = 1, \dots, n\},$$

which is precisely the \mathbb{G}_a -fixed locus.

For (ii), we let $N(T^*V)$ (resp. $N(\mu_{\mathbb{G}_a}^{-1}(0))$) denote the nullcone, which is the common zero set of all non-constant homogeneous \mathbb{G}_a -invariant functions on T^*V (resp. $\mu_{\mathbb{G}_a}^{-1}(0)$). For T^*V , the ring of \mathbb{G}_a -invariant functions is finitely generated:

$$\mathbb{C}[x_i, \beta_i, x_i y_j - x_j y_i, \beta_i \alpha_j - \beta_j \alpha_i, x_i \alpha_j + y_i \beta_j] \twoheadrightarrow \mathbb{C}[T^*V]^{\mathbb{G}_a}$$

and

$$N(T^*V) = \{(y_i, x_i, \beta_i, \alpha_i)_{1 \leq i \leq n} : x_i = \beta_i = 0 \text{ for } i = 1, \dots, n\} = \mu_{\mathbb{G}_a}^{-1}(0)^{\mathbb{G}_a}.$$

As there may be additional invariants on $\mu_{\mathbb{G}_a}^{-1}(0)$, which do not extend to T^*V , we have $N(\mu_{\mathbb{G}_a}^{-1}(0)) \subset N(T^*V)$. However, we claim in this case, that these nullcones coincide, from which Statement (ii) follows. We argue by contradiction: suppose $f \in \mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]^{\mathbb{G}_a}$ and f is not identically zero on N_{T^*V} . We can lift f to a (no longer invariant) function on T^*V , whose derivation under the \mathbb{G}_a -action must lie in the ideal $I := (\sum_{i=1}^n x_i \beta_i)$. Since f is not identically zero on N_{T^*V} , it must contain a monomial of the form $m = \prod_{i=1}^n y_i^{r_i} \alpha_i^{s_i}$. Then the derivation of m is given by

$$D(m) = \sum_{i=1}^n \prod_{j \neq i} y_j^{r_j} \alpha_j^{s_j} (r_i x_i y_i^{r_i-1} \alpha_i^{s_i} - s_i y_i^{r_i} \alpha_i^{s_i-1} \beta_i).$$

However, each monomial in $D(m)$ is not divisible by any of the monomials $x_i \beta_i$ in the equation defining I and can only be ‘integrated’ in one way (as only $x_i = D(y_i)$ and $\beta_i = D(-\alpha_i)$ lie in the image of D). Hence, no such f can exist with $D(f) \in I$. □

This proposition justifies the following definition.

Definition 5.2. For \mathbb{G}_a acting on $\mu_{\mathbb{G}_a}^{-1}(0)$, we define the non-reductive quotient

$$q : \mu_{\mathbb{G}_a}^{-1}(0) \rightarrow \mu_{\mathbb{G}_a}^{-1}(0) // \mathbb{G}_a := \text{Spec}(\mathbb{C}[T^*V]^{\mathbb{G}_a} / I)$$

where I is the ideal defining $\mu_{\mathbb{G}_a}^{-1}(0) \subset T^*V$.

Remark 5.3. Using this definition, we have $\mu_{\mathbb{G}_a}^{-1}(0) // \mathbb{G}_a = \mathcal{X}_0$, where \mathcal{X}_0 is the central fibre of the 1-parameter family of algebraic symplectic reductions given by Corollary 3.15.

By Proposition 5.1, we have $q^{-1}(0) = \mu_{\mathbb{G}_a}^{-1}(0)^{\mathbb{G}_a} = \mu_{\mathbb{G}_a}^{-1}(0) - \mu_{\mathbb{G}_a}^{-1}(0)^s$ and $q(\mu_{\mathbb{G}_a}^{-1}(0)^s) = \mu_{\mathbb{G}_a}^{-1}(0)^s / \mathbb{G}_a$ has the structure of a smooth algebraic symplectic variety by Proposition 3.16.

Let $W = V \times \mathbb{A}^2$; then as SL_2 -representation $W = \text{Sym}^1(\mathbb{C}^2)^{\oplus n+1}$. Let $\mu_{\text{SL}_2}^{-1}(0) \subset T^*W$ be the zero level set for the SL_2 -moment map on T^*W and recall that there is a \mathbb{G}_a -equivariant closed immersion $i : \mu_{\mathbb{G}_a}^{-1}(0) \hookrightarrow \mu_{\text{SL}_2}^{-1}(0)$ given by Proposition 4.2.

Proposition 5.4. *Let V and W be as above. Then*

- (i) $i(\mu_{\mathbb{G}_a}^{-1}(0)^s) \subset \mu_{\text{SL}_2}^{-1}(0)^{\text{SL}_2-s}$,
- (ii) $\mu_{\mathbb{G}_a}^{-1}(0)^s / \mathbb{G}_a$ has a hyperkähler structure.

Proof. For (i), if $p \in \mu_{\mathbb{G}_a}^{-1}(0)^s$, then there exists $f \in \mathbb{C}[T^*V]^{\mathbb{G}_a}$ such that $f(p) \neq 0$ by Proposition 5.1. Let h be the image of f under the homomorphisms

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} \cong \mathbb{C}[T^*V \times \mathbb{A}^2]^{\text{SL}_2} \rightarrow \mathbb{C}[T^*W]^{\text{SL}_2},$$

where the final homomorphism is induced by an SL_2 -equivariant projection $\pi : T^*W \rightarrow T^*V \times \mathbb{A}^2$ such that $\pi \circ i : T^*V \rightarrow T^*V \times \mathbb{A}^2$ is given by the identity map on T^*V and the point $(1, 0) \in \mathbb{A}^2$. Then $h(i(p)) = f(p) \neq 0$. Since $T^*W = \text{Sym}^1(\mathbb{C}^2)^{\oplus 2(n+1)}$, the stable set $(T^*W)^{\text{SL}_2-s}$ is the union of all open subsets given by the non-vanishing locus of an SL_2 -invariant. Hence $i(p)$ is SL_2 -stable and has trivial stabiliser group by Lemma 4.3.

For (ii), let $U = \mu_{\mathbb{G}_a}^{-1}(0)^s$ and $U' = \text{SL}_2 \cdot i(U)$; then by the argument used in Theorem 4.4, we have

$$U / \mathbb{G}_a \cong U' / \text{SL}_2$$

and, moreover, as SL_2 acts freely on $U' \subset \mu_{\text{SL}_2}^{-1}(0)^{\text{SL}_2-s}$, its SL_2 -quotient has a hyperkähler structure coming from the hyperkähler reduction of T^*W by $\text{SU}(2)$ via the Kempf–Ness homeomorphism. □

5.2. The blow up of n points on \mathbb{P}^{n-2}

Let us return to $\text{Bl}_n(\mathbb{P}^{n-2}) = V//_{\chi} \mathbb{G}_a \rtimes T$, where $V = \mathbb{A}^{2n}$ and $T = \mathbb{G}_m^{n+1}$. In the previous section, we described the moment map and algebraic symplectic reduction for the \mathbb{G}_a -action on T^*V . In this section, we also take into account the torus action. The following lemma gives the equations for the zero level set of the algebraic moment map for the torus action.

Lemma 5.5. *For $T = \mathbb{G}_m^{n+1}$ acting on T^*V as above, the algebraic moment map $\mu_T : T^*V \rightarrow \mathbb{C}^{n+1}$ is given by*

$$\mu_T((y_i, x_i, \beta_i, \alpha_i)_{1 \leq i \leq n}) = \left(\sum_{i=1}^n \alpha_i y_i, \beta_1 x_1 - \sum_{j \neq 1} \alpha_j y_j, \dots, \beta_n x_n - \sum_{j \neq n} \alpha_j y_j \right).$$

Hence $\mu_T^{-1}(0)$ is defined by the ideal

$$I = \left(\sum_{i=1}^n \alpha_i y_i, \alpha_j y_j + \beta_j x_j : 1 \leq j \leq n \right).$$

The construction in [6] as applied to this example tells us three facts we will now use. The first is that the rational quotient map from V to $\text{Bl}_n(\mathbb{P}^{n-2})$ is surjective. The second is that the action of $\mathbb{G}_a \rtimes T$ is scheme-theoretically free on an open subset U of V that surjects onto $\text{Bl}_n(\mathbb{P}^{n-2})$ under the quotient map. The third is that this open subset U is exactly the stable locus for the torus action.

These facts allow us to describe a hyperkähler structure on $T^*\text{Bl}_n(\mathbb{P}^{n-2})$ by taking a quotient in two stages: first by the \mathbb{G}_a -action and then by the torus action. Observe that $\mu_{\mathbb{G}_a}^{-1}(0)$ is closed under the torus action and there is an induced torus action on the quotient $\mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a$. We know by Proposition 5.1 and the explicit form of the fixed point locus, that the open subset $\mu_{\mathbb{G}_a}^{-1}(0)^s$, which is the complement of the fixed point locus, must also be closed under the torus action. Furthermore, the complement of $\mu_{\mathbb{G}_a}^{-1}(0)^s$ in $\mu_{\mathbb{G}_a}^{-1}(0)$, that is, the non-stable points for the \mathbb{G}_a -action, are contained in the unstable points for the torus action.

Lemma 5.6. *The complement \mathcal{C} of the \mathbb{G}_a -stable points in T^*V is a subset of the unstable points for the torus action. Furthermore, the image of \mathcal{C} under the \mathbb{G}_a -quotient map is contained in the unstable points for the induced torus action on $\mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a$.*

Proof. By Proposition 5.1, the complement \mathcal{C} is the nullcone of the \mathbb{G}_a -action. Therefore any homogeneous function on the quotient of T^*V pulls-back to a \mathbb{G}_a -invariant function that vanishes on \mathcal{C} . In particular, if f is a homogeneous semi-invariant function for the torus action on the quotient, then the pull-back must vanish on \mathcal{C} , and f itself must vanish on the image of \mathcal{C} . □

By Proposition 5.4 there is a hyperkähler structure on $\mu_{\mathbb{G}_a}^{-1}(0)^s/\mathbb{G}_a$. By the above lemma, the GIT torus quotient of $\mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a$ is equal to the GIT torus quotient of $\mu_{\mathbb{G}_a}^{-1}(0)^s/\mathbb{G}_a$. By direct observation, the solvable group acts freely on $\mu_{\mathbb{G}_a}^{-1}(0)^s$ and hence the torus acts freely on $\mu_{\mathbb{G}_a}^{-1}(0)^s/\mathbb{G}_a$. Therefore one can now apply the standard torus hyperkähler quotient construction to $\mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a$. By our above facts, this gives us a hyperkähler structure on $T^*\text{Bl}_n(\mathbb{P}^{n-2})$ by restriction to an open set: first, Proposition 3.16 tells us $T^*(V^s/\mathbb{G}_a)$ is an open subset of $\mu_{\mathbb{G}_a}^{-1}(0)^s/\mathbb{G}_a$; then, the further torus quotient analogue must contain $T^*(V^s/(\mathbb{G}_a \times T))$, which equals $T^*\text{Bl}_n(\mathbb{P}^{n-2})$, as an open subset.

Proposition 5.7. *The cotangent bundle $T^*\text{Bl}_n(\mathbb{P}^{n-2})$ has a hyperkähler structure realised as an open subset of an algebraic symplectic analogue of a quotient of affine space by a solvable group action.*

5.3. The moduli space of pointed stable rational curves

In this section, we construct an algebraic symplectic analogue of an open subset of the moduli space of pointed stable rational curves $\overline{M}_{0,n}$, using the construction of $\overline{M}_{0,n}$ described in [6] as a quotient of an affine space V by a non-linear $\mathbb{G}_a \times \mathbb{G}_m^\rho$ -action, where ρ is the rank of the Picard group of $\overline{M}_{0,n}$.

Let us outline this solvable quotient construction of $\overline{M}_{0,n}$. The affine space V has coordinate ring

$$\mathbb{C}[V] = \mathbb{C}[y_1, \dots, y_{n-1}, x_I]_{1 \leq |I| \leq n-4}$$

where $I \subset \{1, \dots, n-1\}$ and the action of $s \in \mathbb{G}_a$ is given by

$$s \cdot x_I = x_I \quad \text{and} \quad s \cdot y_i = y_i + s \prod_{I \ni i} x_I.$$

In particular, for $n > 5$, the \mathbb{G}_a -action is non-linear. The torus action comes from the Picard torus: we have

$$\text{Pic}(\overline{M}_{0,n}) = \mathbb{Z}H \oplus \bigoplus_I \mathbb{Z}E_I$$

where the direct sum is over subsets $I \subset \{1, \dots, n-1\}$ such that $1 \leq |I| \leq n-4$, then $t = (t_H, t_I) \in T := \text{Hom}(\text{Pic } X, \mathbb{G}_m)$ acts on V by

$$t \cdot x_I = t_I x_I \quad \text{and} \quad t \cdot y_i = t_H \left(\prod_{I \ni i} t_I \right)^{-1} y_i.$$

The action of the Picard torus on V is linear and it normalises the \mathbb{G}_a -action:

$$t \cdot s \cdot (x_I, y_i) = (t_H s) \cdot t \cdot (x_I, y_i).$$

Hence, there is an action of the semi-direct product $\mathbb{G}_a \rtimes T$ on V and by [6] Corollary 3.9, we have that $\overline{M}_{0,n} = V //_{\chi} \mathbb{G}_a \rtimes T$, where χ is an ample character of T .

Let us consider the cotangent lift of these actions: we identify $T^*V \cong V \times V^*$ and take coordinates (α_I, β_i) on V^* . Then for $s \in \mathbb{G}_a$ and $t = (t_H, t_I) \in T$, we have

$$\begin{aligned} s \cdot (x_I, y_i, \alpha_I, \beta_i) &= \left(x_I, y_i + s \prod_{I \ni i} x_I, \alpha_I - s \sum_{i \in I} \beta_i \prod_{I \neq J \ni i} x_J, \beta_i \right) \\ t \cdot (x_I, y_i, \alpha_I, \beta_i) &= \left(t_I x_I, t_H \left(\prod_{I \ni i} t_I \right)^{-1} y_i, t_I^{-1} \alpha_I, t_H^{-1} \left(\prod_{I \ni i} t_I \right) \beta_i \right). \end{aligned}$$

The torus action normalises the \mathbb{G}_a -action:

$$t \cdot s \cdot (x_I, y_i, \alpha_I, \beta_i) = (t_H s) \cdot t \cdot (x_I, y_i, \alpha_I, \beta_i)$$

and so $\mathbb{G}_a \rtimes T$ acts on T^*V . As usual, we equip T^*V with the Liouville algebraic symplectic form

$$\omega = \sum_I dx_I \wedge d\alpha_I + \sum_{i=1}^{n-1} dy_i \wedge d\beta_i.$$

Lemma 5.8. *The Liouville form ω is preserved by the $\mathbb{G}_a \rtimes T$ -action on T^*V . The \mathbb{G}_a -action has moment map $\mu : T^*V \rightarrow \mathbb{C}$ given by*

$$\mu_{\mathbb{G}_a}(x_I, y_i, \alpha_I, \beta_i) = \sum_{i=1}^{n-1} \beta_i \prod_{I \ni i} x_I$$

and the subtori $\mathbb{G}_m \cong T_H \subset T$ and $\mathbb{G}_m \cong T_I \subset T$ have moment maps $\mu_H : T^*V \rightarrow \mathbb{C}$ and $\mu_I : T^*V \rightarrow \mathbb{C}$ given by

$$\mu_H(x_I, y_i, \alpha_I, \beta_i) = \sum_{i=1}^{n-1} y_i \beta_i \quad \text{and} \quad \mu_I(x_I, y_i, \alpha_I, \beta_i) = \alpha_I x_I - \sum_{i \in I} y_i \beta_i$$

Furthermore, the \mathbb{G}_a -fixed locus is the singular locus of the zero level set of the \mathbb{G}_a -moment map:

$$(T^*V)^{\mathbb{G}_a} = \text{Sing } \mu_{\mathbb{G}_a}^{-1}(0).$$

Proof. As the torus action is linear, we only need to check that the \mathbb{G}_a -action preserves ω . For $s \in \mathbb{G}_a$, let $\sigma_s : T^*V \rightarrow T^*V$ denote the action of s . For $p = (x_I, y_i, \alpha_I, \beta_i) \in T^*V$ and $p' \in T_p(T^*V) \cong T^*V$, we have

$$d_p \sigma_s(p') = \left(x'_I, y'_i + s \sum_{I \in i} x'_I \prod_{I \neq J \ni i} x_J, \tilde{\alpha}_I, \beta'_i \right),$$

where

$$\tilde{\alpha}_I := \alpha'_I - s \left(\sum_{i \in I} \beta'_i \prod_{I \neq J \ni i} x_J + \sum_{i \in I} \sum_{I \neq J \ni i} \beta_i x'_J \prod_{I, J \neq K \ni i} x_K \right).$$

From which one verifies that

$$(\sigma_s^* \omega)_p(p', p'') := \omega_{\sigma_s(p)}(d_p \sigma_s(p'), d_p \sigma_s(p'')) = \omega_p(p', p'').$$

The \mathbb{G}_a -moment map is \mathbb{G}_a -invariant and the infinitesimal action of $1 \in \mathbb{C}$ on $p \in T^*V$ is given by

$$1_p = \left(0, \prod_{I \ni i} x_I, - \sum_{i \in I} \beta_i \prod_{I \neq J \ni i} x_J, 0 \right).$$

Using this, we verify that the infinitesimal lifting property

$$\omega_p(1_p, p') = - \sum_I x'_I \left(- \sum_{i \in I} \beta_i \prod_{I \neq J \ni i} x_J \right) + \sum_i \beta'_i \prod_{I \ni i} x_I = d_p \mu_{\mathbb{G}_a}(p').$$

The moment maps for the torus factors are computed from the infinitesimal action.

Finally, we have $\text{Sing } \mu_{\mathbb{G}_a}^{-1}(0) = (T^*V)^{\mathbb{G}_a}$, because both are given by

$$\left\{ (x_I, y_i, \alpha_I, \beta_i) : \prod_{I \ni i} x_I = 0 \forall i, \quad \sum_{i \in I} \beta_i \prod_{I \neq J \ni i} x_J = 0 \forall I \right\},$$

by direct computation. □

One can construct a symplectic reduction of the $\mathbb{G}_a \times T$ -action on T^*V in stages, by first taking a quotient of the \mathbb{G}_a -action on the zero level set and then taking the symplectic quotient for the torus action with respect to the ample character χ . As the \mathbb{G}_a -action on $\mu_{\mathbb{G}_a}^{-1}(0)$ is not linear, we can't extend the action to SL_2 in order to use the notions of completely or Mumford (semi)stable defined in [7]. However, we can work with the notion of locally trivial stable (Definition 4.2.3 in loc. cit.).

Proposition 5.9. *For the \mathbb{G}_a -action on the zero level set, the locally trivial stable set is the complement of the \mathbb{G}_a -fixed locus:*

$$\mu_{\mathbb{G}_a}^{-1}(0)^{lts} = \mu_{\mathbb{G}_a}^{-1}(0) - \mu_{\mathbb{G}_a}^{-1}(0)^{\mathbb{G}_a}$$

and there is an induced algebraic symplectic structure on the \mathbb{G}_a -quotient of this open set.

Proof. Consider the following \mathbb{G}_a -invariant functions:

$$f_i = \prod_{I \ni i} x_I \quad \text{and} \quad f_I = \sum_{i \in I} \beta_i \prod_{I \neq J \ni i} x_J.$$

The ring of invariant functions in the localised rings at these functions

$$\mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]_{f_i}^{\mathbb{G}_a} \quad \text{and} \quad \mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]_{f_I}^{\mathbb{G}_a}$$

are finitely generated, as the derivations of y_i/f_i and α_I/f_I under the \mathbb{G}_a -action are 1. Moreover, such an element which is sent to 1 by the derivation of

the \mathbb{G}_a -action gives an isomorphism from this localised open subset to a trivial \mathbb{G}_a -torsor, and so there are geometric quotients $\mu_{\mathbb{G}_a}^{-1}(0)_{f_i} \rightarrow \mu_{\mathbb{G}_a}^{-1}(0)_{f_i}/\mathbb{G}_a$ and $\mu_{\mathbb{G}_a}^{-1}(0)_{f_I} \rightarrow \mu_{\mathbb{G}_a}^{-1}(0)_{f_I}/\mathbb{G}_a$. Therefore,

$$\mu_{\mathbb{G}_a}^{-1}(0) - \mu_{\mathbb{G}_a}^{-1}(0)^{\mathbb{G}_a} = \bigcup_i \mu_{\mathbb{G}_a}^{-1}(0)_{f_i} \cup \bigcup_I \mu_{\mathbb{G}_a}^{-1}(0)_{f_I} \subset \mu_{\mathbb{G}_a}^{-1}(0)^{lts}.$$

Since no \mathbb{G}_a -fixed point is locally trivial stable, the above inclusion must be an equality. The final statement follows as $\mu_{\mathbb{G}_a}^{-1}(0)^{lts} \rightarrow \mu_{\mathbb{G}_a}^{-1}(0)^{lts}/\mathbb{G}_a$ is a locally trivial geometric quotient. □

As the T -action preserves $\mu_{\mathbb{G}_a}^{-1}(0)^{lts}$ and normalises the \mathbb{G}_a -action, there is an induced free T -action on $\mu_{\mathbb{G}_a}^{-1}(0)^{lts}/\mathbb{G}_a$ and the T -moment map descends to this quotient. Therefore, we can now construct the algebraic symplectic reduction U' of the T -action on $\mu_{\mathbb{G}_a}^{-1}(0)^{lts}/\mathbb{G}_a$ with respect to the character χ as a GIT quotient. To relate this to $\overline{M}_{0,n}$, consider the open subset $U := V^{lts} //_{\chi} \mathbb{G}_a \rtimes T$ of $\overline{M}_{0,n}$, where we also have $V^{lts} = V - V^{\mathbb{G}_a}$. Then we have the following result, whose proof follows in the same way as that of Proposition 3.16.

Corollary 5.10. *The open subset U of $\overline{M}_{0,n}$ defined above has algebraic symplectic analogue U' equal to the algebraic symplectic reduction of the T -action on $\mu_{\mathbb{G}_a}^{-1}(0)^{lts}/\mathbb{G}_a$ with respect to the character χ . In particular, $U \subset U'$ is Lagrangian and $T^*U \subset U'$.*

The hyperkähler metric on T^*V is not preserved by the \mathbb{G}_a -action and so we cannot say anything about a hyperkähler structure on U' . It is natural to expect that there should be an induced hyperkähler structure on U' , which can perhaps be realised by exhibiting a Kempf–Ness type homeomorphism for additive group quotients or by extending the action on an open subset of T^*V to SL_2 and using the Kempf–Ness theorem for reductive quotients.

6. Examples

We can explicitly study some examples, using an algorithm of van den Essen [20] and Derksen [5] to compute rings of \mathbb{G}_a -invariants on the zero level set of the moment map. Let us quickly explain the algorithm.

For a \mathbb{G}_a -action on an affine variety, the algorithm produces a list of generators for $\mathbb{C}[X]^{\mathbb{G}_a}$, which terminates if and only if the ring is finitely

generated. The first step is to choose $f \in \mathbb{C}[X]^{\mathbb{G}_a}$ such that on X_f the \mathbb{G}_a -action becomes a trivial \mathbb{G}_a -torsor (the existence of such an f is a consequence of the derivation of the \mathbb{G}_a -action being locally nilpotent). Then, as taking \mathbb{G}_a -invariants is left exact, there is an inclusion

$$\mathbb{C}[X]^{\mathbb{G}_a} \hookrightarrow \mathbb{C}[X]_f^{\mathbb{G}_a}$$

and $\mathbb{C}[X]^{\mathbb{G}_a} = \mathbb{C}[X] \cap \mathbb{C}[X]_f^{\mathbb{G}_a}$. The algorithm gives generators for this intersection using colon ideals and can be performed using a computer algebra package, which is adept at computing Groebner bases; for details see [5] Algorithms 2.6 and 2.7.

6.1. The 2 dimensional irreducible SL_2 -representation

Let $V = \text{Sym}^1(\mathbb{C}^2) \cong \mathbb{A}^2$ with \mathbb{G}_a -action given by $c \cdot (x_1, x_2) = (x_1 + cx_2, x_2)$, for $c \in \mathbb{G}_a$. We have the GIT quotient $\pi_V : V \rightarrow V//\mathbb{G}_a = \text{Spec } \mathbb{C}[x_2] \cong \mathbb{A}^1$, which restricts to a geometric quotient on $V^s = \{(x_1, x_2) : x_2 \neq 0\}$.

The moment map for the cotangent lift of the SL_2 -action to T^*V is given by

$$\mu_{SL_2, V}(x, \alpha) = (x_1\alpha_1 - x_2\alpha_2)H^* + (x_2\alpha_1)E^* + (x_1\alpha_2)F^*$$

and the \mathbb{G}_a -moment map is given by $\mu_{\mathbb{G}_a}(x, \alpha) = x_2\alpha_1$. By Example 3.18, $\mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]^{\mathbb{G}_a}$ is non-finitely generated and so we instead work with the normalisation

$$\widetilde{\mu_{\mathbb{G}_a}^{-1}(0)} = \mu_{\mathbb{G}_a}^{-1}(0)_{(1)} \bigsqcup \mu_{\mathbb{G}_a}^{-1}(0)_{(2)}$$

where $\mu_{\mathbb{G}_a}^{-1}(0)_{(1)} = \text{Spec } \mathbb{C}[x_1, \alpha_1, \alpha_2]$ has $\mu_{\mathbb{G}_a}^{-1}(0)_{(1)}^{\bar{s}} = \{(x_1, \alpha_1, \alpha_2) : \alpha_1 \neq 0\}$ and has non-reductive GIT quotient $\mu_{\mathbb{G}_a}^{-1}(0)_{(1)}/\mathbb{G}_a = \text{Spec } \mathbb{C}[x_1, \alpha_1]$, and $\mu_{\mathbb{G}_a}^{-1}(0)_{(2)} = \text{Spec } \mathbb{C}[x_1, x_2, \alpha_2]$ has $\mu_{\mathbb{G}_a}^{-1}(0)_{(2)}^{\bar{s}} = \{(x_1, x_2, \alpha_2) : x_2 \neq 0\}$ and has non-reductive GIT quotient $\mu_{\mathbb{G}_a}^{-1}(0)_{(2)}/\mathbb{G}_a = \text{Spec } \mathbb{C}[x_2, \alpha_2]$.

Let $W = V \times \mathbb{A}^2$ and write $\mathbb{C}[T^*W] = \mathbb{C}[T^*V][u, v, \lambda, \eta]$. Then for the SL_2 -action on T^*W , the zero level set $\mu_{SL_2}^{-1}(0) \subset T^*W$ is defined by the ideal

$$(x_1\alpha_1 - x_2\alpha_2 + \lambda u - \eta v, x_2\alpha_1 + \lambda v, x_1\alpha_2 + \eta u)$$

and $\mathbb{C}[T^*W]^{SL_2} = \mathbb{C}[h_1, \dots, h_6]/(h_1h_4 - h_2h_5 + h_3h_6)$ where

$$\begin{aligned} h_1 &= x_2u - x_1v & h_2 &= \alpha_1u + \alpha_2v & h_3 &= x_1\alpha_1 + x_2\alpha_2 \\ h_4 &= \alpha_1\eta - \alpha_2\lambda & h_5 &= x_1\lambda + x_2\eta & h_6 &= u\lambda + v\eta. \end{aligned}$$

We have $\mu_{SL_2}^{-1}(0)^{SL_2-s} = \cup_{i=1}^6 \mu_{SL_2}^{-1}(0)_{h_i}$.

The \mathbb{G}_a -equivariant closed embedding $i : \mu_{\mathbb{G}_a}^{-1}(0) \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0)$ induces

$$\overline{i_{(j)}} : \mu_{\mathbb{G}_a}^{-1}(0)_{(j)} // \mathbb{G}_a \rightarrow \mu_{\mathrm{SL}_2}^{-1}(0) // \mathrm{SL}_2$$

where

$$\begin{aligned} \overline{i_{(1)}}(x_1, \alpha_1) &= (0, \alpha_1, \alpha_1 x_1, 0, -\alpha_1 x_1^2, -\alpha_1 x_1), \\ \overline{i_{(2)}}(x_2, \alpha_2) &= (x_2, 0, x_2 \alpha_2, -x_2 \alpha_2^2, 0, x_2 \alpha_2). \end{aligned}$$

We have $i_{(j)}(\mu_{\mathbb{G}_a}^{-1}(0)_{(j)}) \subset \mu_{\mathrm{SL}_2}^{-1}(0)_I^{\mathrm{SL}_2^{-s}}$ and the morphisms $\overline{i_{(j)}}$ induce birational symplectomorphisms between the reductive and non-reductive algebraic symplectic analogue. Furthermore, for $V // \mathbb{G}_a \cong \mathbb{A}^1 \cong W // \mathbb{G}_a$, we see that the non-reductive algebraic symplectic analogue $\mu_{\mathbb{G}_a}^{-1}(0)_{(2)} // \mathbb{G}_a \cong \mathbb{A}^2$ provides a better algebraic symplectic analogue that the reductive algebraic symplectic analogue $\mu_{\mathrm{SL}_2}^{-1}(0) // \mathrm{SL}_2$, which is a singular subvariety of \mathbb{A}^6 .

6.2. The 3 dimensional irreducible SL_2 -representation

Let $V = \mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{A}^3$ with the \mathbb{G}_a -action given by, for $c \in \mathbb{G}_a$, we have $c \cdot (x_1, x_2, x_3) = (x_1 + 2cx_2 + c^2x_3, x_2 + cx_3, x_3)$ and the SL_2 -action

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \mapsto \begin{pmatrix} p^2 & 2pq & q^2 \\ pr & ps + qr & qs \\ r^2 & 2rs & s^2 \end{pmatrix}.$$

The non-reductive quotient $V \rightarrow V // \mathbb{G}_a = \mathrm{Spec} \mathbb{C}[x_3, x_1x_3 - x_2^2] \cong \mathbb{A}^2$ restricts to a geometric quotient on $V^s = \{(x_1, x_2, x_3) : x_3 \neq 0\}$.

The algebraic moment map for SL_2 acting on T^*V is given by the holomorphic functions

$$\begin{aligned} \Phi_E(x, \alpha) &= \mu_{\mathbb{G}_a}(x, \alpha) = 2x_2\alpha_1 + \alpha_2x_3, \\ \Phi_H(x, \alpha) &= 2x_1\alpha_1 - 2x_3\alpha_3, \\ \Phi_F(x, \alpha) &= x_1\alpha_2 + 2\alpha_3x_2. \end{aligned}$$

For the \mathbb{G}_a -action on $\mu_{\mathbb{G}_a}^{-1}(0)$, we have

$$\mu_{\mathbb{G}_a}^{-1}(0) - \mu_{\mathbb{G}_a}^{-1}(0)^{\bar{s}} = \{(x, a) \in \mu_{\mathbb{G}_a}^{-1}(0) : x_3 = a_1 = 0\}$$

which strictly contains

$$\mu_{\mathbb{G}_a}^{-1}(0)^{\mathbb{G}_a} = \mathrm{Sing} \mu_{\mathbb{G}_a}^{-1}(0) = \{(x, a) \in \mu_{\mathbb{G}_a}^{-1}(0) : x_2 = x_3 = a_1 = a_2 = 0\}.$$

By the algorithm of van den Essen and Derksen, $\mathbb{C}[\mu_{\mathbb{G}_a}^{-1}(0)]^{\mathbb{G}_a}$ is finitely generated by

$$\begin{aligned} f_1 &= x_3, & f_2 &= x_1x_3 - x_2^2, & f_3 &= a_1, \\ f_4 &= x_1a_1 + x_2a_2 + x_3a_3, & f_5 &= 4a_1a_3 - a_2^2 \\ f_6 &= x_1a_1 - x_3a_3, & f_7 &= 4x_3a_3^2 + 2x_2a_2a_3 - 4x_1a_1a_3 + x_1a_2^2 \\ f_8 &= 2x_3x_1a_3 - 2x_2^2a_3 - 2x_1^2a_1 - x_1x_2a_2. \end{aligned}$$

The invariants f_6, f_7 and f_8 are intrinsic to the hypersurface, but one easily checks that these do not give any additional separation or increase the stable locus; i.e., the locally trivial stable set coincides with the completely stable set in this example.

For the SL_2 -action on T^*W , the zero level set $\mu_{SL_2}^{-1}(0) \subset T^*W$ is defined by the ideal

$$(2x_1\alpha_1 - 2x_3\alpha_3 + \lambda u - \eta v, 2x_2\alpha_1 + \alpha_2x_3 + \lambda v, x_1\alpha_2 + 2\alpha_3x_2 + \eta u)$$

and $\mathbb{C}[\mu_{SL_2}^{-1}(0)]^{SL_2}$ is generated by

$$\begin{aligned} h_1 &= x_1v^2 - 2x_2uv + x_3u^2 & h_2 &= x_1x_3 - x_2^2 \\ h_3 &= \alpha_1u^2 + \alpha_2uv + \alpha_3v^2 & h_4 &= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 \\ h_5 &= 4\alpha_1\alpha_3 - \alpha_2^2 & h_6 &= u\lambda + v\eta \\ h_7 &= \alpha_1\eta^2 - \alpha_2\eta\lambda + \alpha_3\lambda^2 & h_8 &= x_1\lambda^2 + 2x_2\eta\lambda + x_3\eta^2 \\ h_9 &= x_1v\lambda - x_2(u\lambda - v\eta) - x_3u\eta & h_{10} &= 2\alpha_1u\eta - \alpha_2(u\lambda - v\eta) - 2\alpha_3\lambda v. \end{aligned}$$

The birational symplectomorphism $\bar{i} : \mu_{\mathbb{G}_a}^{-1}(0)//\mathbb{G}_a \rightarrow \mu_{SL_2}^{-1}(0)//SL_2$ given by Theorem 4.4 can be explicitly determined:

$$\bar{i}(y_1, \dots, y_8) = (y_1, y_2, y_3, y_4, y_5, -2y_6, -y_6y_7, -2y_6y_8, 0, 0).$$

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