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Symplectic Bott-Chern cohomology of solvmanifolds

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We study the symplectic Bott-Chern cohomology by L.-S. Tseng and S.-T. Yau for solvmanifolds endowed with left-invariant symplectic structures. Our results are applicable to cohomology with values in local systems. Studying symplectic Bott-Chern cohomology of solvmanifolds with values in local systems, we give some remarks on symplectic Hodge theory.

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Introduction

In complex geometry, besides Dolbeault and de Rham cohomologies, also Bott-Chern and Aeppli cohomologies deserve much interest. For a complex manifold X, the *Bott-Chern cohomology* and the *Aeppli cohomology* are defined as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}, \quad \text{respectively} \quad H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}}.$$

In fact, they appear as natural tools in studying the geometry of complex (possibly non-Kähler) manifolds, see, e.g., [1–3, 6, 11, 13, 23, 37, 52, 57, 58].

Let X be a $2n$-dimensional manifold endowed with a symplectic structure ω. We define the co-differential operator $\mathrm{d}^\Lambda := [\mathrm{d}, -\iota_{\omega^{-1}}]$ on the de Rham complex $\wedge^\bullet X$ which was introduced by J.-L. Koszul, [38], and by J.-L. Brylinski, [14]. In [55–57], L.-S. Tseng and S.-T. Yau introduced and studied the cohomologies

$$H^\bullet_{BC}(X;\mathbb{R}) := \frac{\ker\mathrm{d}\cap\ker\mathrm{d}^\Lambda}{\operatorname{im}\mathrm{d}\,\mathrm{d}^\Lambda} \quad \text{and} \quad H^\bullet_{A}(X;\mathbb{R}) := \frac{\ker\mathrm{d}\,\mathrm{d}^\Lambda}{\operatorname{im}\mathrm{d}+\operatorname{im}\mathrm{d}^\Lambda}.$$

Since they are analogous to Bott-Chern and Aeppli cohomologies of complex manifolds respectively, we call $H^\bullet_{BC}(X;\mathbb{R})$ the *symplectic Bott-Chern cohomology* and $H^\bullet_{A}(X;\mathbb{R})$ the *symplectic Aeppli cohomology*. We notice that Tseng and Yau did not use this terminology. But this terminology is very useful in this paper for using the result in [5], see Corollary 1.3 and its proof. Moreover, we remark that $H^\bullet_{BC}(X;\mathbb{R})$ and $H^\bullet_{A}(X;\mathbb{R})$ can be considered as the Bott-Chern and Aeppli cohomologies of a "generalized" complex manifold, (see [8],) where we regard a symplectic structure as a generalized complex structure of type 0, (see [16, 29]).

We recall that *solvmanifolds* are compact quotients $\Gamma\backslash G$ of connected simply-connected solvable Lie groups G by co-compact discrete subgroups Γ. When G is nilpotent, we call $\Gamma\backslash G$ *nilmanifold*. The purpose of this paper is to study the symplectic Bott-Chern and Aeppli cohomologies of symplectic solvmanifolds. Let $\Gamma\backslash G$ be a solvmanifold and $\mathfrak{g}$ the Lie algebra of G. Consider the cochain complex $\wedge^\bullet\mathfrak{g}^*$ of left-invariant differential forms and the inclusion

$$\iota : \wedge^\bullet\mathfrak{g}^* \hookrightarrow \wedge^\bullet \Gamma\backslash G\,.$$

This inclusion induces the map $H^\bullet(\mathfrak{g};\mathbb{R}) \to H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{R})$ where $H^\bullet(\mathfrak{g};\mathbb{R})$ is the cohomology of $\wedge^\bullet\mathfrak{g}^*$ and $H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{R})$ is the de Rham cohomology of

$\Gamma\backslash G$. Suppose that $\Gamma\backslash G$ admits a left-invariant symplectic structure. Then the co-differential d^Λ is defined on $\wedge^\bullet\mathfrak{g}^*$ and we can define the symplectic Bott-Chern and Aeppli cohomologies $H^\bullet_{BC}(\mathfrak{g};\mathbb{R})$ and $H^\bullet_A(\mathfrak{g};\mathbb{R})$ of the Lie algebra $\mathfrak{g}$. We consider the maps

$$H^\bullet_{BC}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{BC}(X;\mathbb{R}) \quad \text{and} \quad H^\bullet_A(\mathfrak{g};\mathbb{R}) \to H^\bullet_A(X;\mathbb{R})$$

induced by the inclusion $\iota : \wedge^\bullet\mathfrak{g}^* \hookrightarrow^\bullet \Gamma\backslash G$. In [40], M. Macrì prove that if the map $H^\bullet(\mathfrak{g};\mathbb{R}) \to H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{R})$ is an isomorphism, then the maps

$$H^\bullet_{BC}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{BC}(X;\mathbb{R}) \quad \text{and} \quad H^\bullet_A(\mathfrak{g};\mathbb{R}) \to H^\bullet_A(X;\mathbb{R})$$

are also isomorphisms. Hence, by Hattori's theorem in [34], if G is completely solvable, then the maps

$$H^\bullet_{BC}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{BC}(X;\mathbb{R}) \quad \text{and} \quad H^\bullet_A(\mathfrak{g};\mathbb{R}) \to H^\bullet_A(X;\mathbb{R})$$

are isomorphisms. In this paper, we give a new proof of Macrì's result (Theorem 3.2). Moreover, in this paper, we will reach a more general case. In fact, on a general solvmanifold $\Gamma\backslash G$, the map $H^\bullet(\mathfrak{g};\mathbb{R}) \to H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{R})$ is not an isomorphism and we can not apply Macrì's result to general solvmanifolds. In [36], for any solvmanifold $\Gamma\backslash G$, the second author construct an explicit finite-dimensional sub-complex $A^\bullet_\Gamma \subseteq \wedge^\bullet \Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ so that the inclusion induces a cohomology isomorphism

$$H^\bullet(A^\bullet_\Gamma) \cong H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

In this paper, we show the following result.

Theorem (see Theorem 6.8). *Suppose that a left-invariant symplectic structure ω on a solvmanifold $\Gamma\backslash G$ satisfies $\omega \in A^2_\Gamma$. Then, the finite-dimensional sub-complex $A^\bullet_\Gamma$ allows to compute the symplectic Bott-Chern and Aeppli cohomologies of $\Gamma\backslash G$.*

For a compact $2n$-dimensional symplectic manifold X with a symplectic form ω, the cohomologies $H^\bullet_{BC}(X;\mathbb{R})$ and $H^\bullet_A(X;\mathbb{R})$ relate to the *Hard Lefschetz Condition*, namely, for every $k \in \mathbb{Z}$ with $0 \le k \le n$, the map

$$\left[\omega^k\right] \smile \cdot : H^{n-k}_{dR}(X;\mathbb{R}) \to H^{n+k}_{dR}(X;\mathbb{R})$$

is an isomorphism. The Hard Lefschetz Condition is equivalent to the natural map $H^\bullet_{BC}(X;\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})$ being injective: in such a case, we say that X satisfies the $\mathrm{d}\,\mathrm{d}^\Lambda$-*Lemma*.

As similar to the complex case, [6], there is an inequality *à la* Frölicher relating the dimensions of the symplectic Bott-Chern and Aeppli cohomologies and the Betti numbers, [8]: for every $k \in \mathbb{Z}$,

$$\dim_{\mathbb{R}} H^k_{BC}(X;\mathbb{R}) + \dim_{\mathbb{R}} H^k_{A}(X;\mathbb{R}) - 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) \geq 0;$$

furthermore, the equality holds for every $k \in \mathbb{Z}$ if and only if X satisfies the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma.

It is known that a non-toral nilmanifold does not satisfy the Hard Lefschetz Condition, [10]. In Section 4, we compute the symplectic cohomologies for 4-dimensional solvmanifolds, for 6-dimensional nilmanifolds, and for the Nakamura manifold. In this sense, the numbers

$$\left\{\dim_{\mathbb{R}} H^k_{BC}(X;\mathbb{R}) + \dim_{\mathbb{R}} H^k_{A}(X;\mathbb{R}) - 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})\right\}_{k\in\mathbb{Z}}$$

provide a measure of the non-Kählerianity of the nilmanifold X.

We further consider symplectic cohomologies with values in a local system in Section 5. By using $\mathfrak{sl}_2(\mathbb{C})$-representations on bi-differential $\mathbb{Z}$-graded complexes, we study twisted Hard Lefschetz Condition and $D_\phi D^{\Lambda}_{\phi}$-Lemma. Finally, in Section 6, we investigate twisted symplectic cohomologies on solvmanifolds. As similar to the non-twisted case, by the Hattori theorem, [34], and the Mostow theorem, [47], we compute the symplectic cohomologies of special solvmanifolds with values in a local system by using Lie algebras. Moreover, considering the spaces of differential forms on the solvmanifolds with values in certain local systems so that these spaces have structures of differential graded algebras given in Hain's paper [31], we compute the symplectic cohomologies of these differential graded algebras by using the Sullivan minimal models constructed by the second author in [36], see Theorem 6.6, Theorem 6.7. In fact, the cohomology computation by $A^{\bullet}_{\Gamma}$ as above is a consequence of such results. By these results, we compute the twisted symplectic cohomologies of Sawai's examples of symplectic solvmanifolds which satisfy the Hard Lefschetz Condition but do not satisfy the twisted Hard Lefschetz Condition.

In general, twisted cohomologies do not have self-duality and so, when considering the Lefschetz operators on twisted cohomologies, surjectivity and injectivity are not equivalent. By this, as regards twisted Hard Lefschetz Conditions and $D_\phi D^{\Lambda}_{\phi}$-Lemma, we have differences between the twisted case and the non-twisted case. We explain such a difference by using Sawai's examples.

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1. Preliminaries and notations on cohomology computations

In this section, we briefly recall the results in [5], about several cohomologies associated to a bounded double complex, respectively bi-differential $\mathbb{Z}$-graded complex, of $\mathbb{C}$-vector spaces. In particular, we are concerned with studying when such cohomologies can be recovered by means of a suitable (possibly finite-dimensional) sub-complex. We also consider Lefschetz conditions and $\partial\overline{\partial}$-Lemma for bi-differential $\mathbb{Z}$-graded complexes.

Remark 1.1. For a graded vector space $V^\bullet$ and endomorphisms $A \in \mathrm{End}^p(V^\bullet)$ and $B \in \mathrm{End}^q(V^\bullet)$ of degrees p and q respectively, we denote $[A, B] = AB - (-1)^{pq}BA$.

1.1. Cohomologies of double complexes

Consider a bounded double complex $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of $\mathbb{C}$-vector spaces. Namely, $\partial \in \mathrm{End}^{1,0}\left(A^{\bullet,\bullet}\right)$ and $\overline{\partial} \in \mathrm{End}^{0,1}\left(A^{\bullet,\bullet}\right)$ are such that $\partial^2 = \overline{\partial}^2 = \left[\partial, \overline{\partial}\right] = 0$, and $A^{p,q} = \{0\}$ but for finitely-many $(p, q) \in \mathbb{Z}^2$.

One can consider several cohomologies associated to $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$. More precisely, for $p \in \mathbb{Z}$ and for $q \in \mathbb{Z}$, one has the cohomologies

$$H^{\bullet,q}_{\partial}(A^{\bullet,\bullet}) := H^{\bullet,q}\left(A^{\bullet,q},\, \partial\right) \quad \text{and} \quad H^{p,\bullet}_{\overline{\partial}}(A^{\bullet,\bullet}) := H^{p,\bullet}\left(A^{p,\bullet},\, \overline{\partial}\right).$$

By denoting the total complex associated to $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ by

$$\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right) := \bigoplus_{p+q=\bullet} A^{p,q}, \mathrm{d} := \partial + \overline{\partial}\right),$$

one has the cohomology

$$H^\bullet_{dR}(A^{\bullet,\bullet}) := H^\bullet\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right), \mathrm{d}\right).$$

Furthermore, for $(p,q) \in \mathbb{Z}^2$, one can consider the *Bott-Chern cohomology*, [13],

$$H^{p,q}_{BC}(A^{\bullet,\bullet}) := H\left(A^{p-1,q-1} \xrightarrow{\partial\overline{\partial}} A^{p,q} \xrightarrow{\partial+\overline{\partial}} A^{p+1,q} \oplus A^{p,q+1}\right),$$

and the *Aeppli cohomology*, [1],

$$H^{p,q}_{A}(A^{\bullet,\bullet}) := H\left(A^{p-1,q} \oplus A^{p,q-1} \xrightarrow{(\partial,\overline{\partial})} A^{p,q} \xrightarrow{\partial\overline{\partial}} A^{p+1,q+1}\right).$$

1.2. Frölicher inequalities

Let $\left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces.

We recall that the natural filtrations induce naturally two spectral sequences such that

$${}'E_1^{\bullet_1,\bullet_2}\left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right) \simeq H^{\bullet_2}\left(A^{\bullet_1,\bullet}, \overline{\partial}\right) \Rightarrow H^{\bullet_1+\bullet_2}\left(\mathrm{Tot}^{\bullet}\left(A^{\bullet,\bullet}\right), \mathrm{d}\right),$$

and

$${}''E_1^{\bullet_1,\bullet_2}\left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right) \simeq H^{\bullet_1}\left(A^{\bullet,\bullet_2}, \partial\right) \Rightarrow H^{\bullet_1+\bullet_2}\left(\mathrm{Tot}^{\bullet}\left(A^{\bullet,\bullet}\right), \mathrm{d}\right),$$

see, e.g., [44, §2.4], see also [26, §3.5], [22, Theorem 1, Theorem 3]. In particular, one gets the Frölicher inequalities:

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Tot}^{\bullet} H^{\bullet,\bullet}_{\overline{\partial}}\left(A^{\bullet,\bullet}\right) \geq \dim_{\mathbb{C}} H^{\bullet}_{dR}\left(A^{\bullet,\bullet}\right) \\ \text{and} \quad &\dim_{\mathbb{C}} \mathrm{Tot}^{\bullet} H^{\bullet,\bullet}_{\partial}\left(A^{\bullet,\bullet}\right) \geq \dim_{\mathbb{C}} H^{\bullet}_{dR}\left(A^{\bullet,\bullet}\right). \end{aligned}$$

Furthermore, the first author and A. Tomassini proved in [8] the following inequality *à la* Frölicher for the Bott-Chern cohomology: assuming that $\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\partial}\left(A^{\bullet,\bullet}\right) < +\infty$ and $\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}}\left(A^{\bullet,\bullet}\right) < +\infty$, it holds that

$$\dim_{\mathbb{C}} \mathrm{Tot}^{\bullet} H^{\bullet,\bullet}_{BC}\left(A^{\bullet,\bullet}\right) + \dim_{\mathbb{C}} \mathrm{Tot}^{\bullet} H^{\bullet,\bullet}_{A}\left(A^{\bullet,\bullet}\right) \geq 2 \dim_{\mathbb{C}} H^{\bullet}_{dR}\left(A^{\bullet,\bullet}\right),$$

and the equality holds if and only if $\left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right)$ satisfies the $\partial\overline{\partial}$-Lemma, namely, the natural map $\mathrm{Tot}^{\bullet} H^{\bullet,\bullet}_{BC}\left(A^{\bullet,\bullet}\right) \to H^{\bullet}_{dR}\left(A^{\bullet,\bullet}\right)$ is injective.

1.3. Bi-differential $\mathbb{Z}$-graded complexes

In the symplectic case, the space of differential forms has just a $\mathbb{Z}$-graduation. Hence we consider the case of a bounded bi-differential $\mathbb{Z}$-graded complex

$\left(A^\bullet,\, \partial,\, \overline{\partial}\right)$ of $\mathbb{K}$-vector spaces, where $\mathbb{K} \in \{\mathbb{R},\, \mathbb{C}\}$. Namely, $A^\bullet$ is a $\mathbb{Z}$-graded $\mathbb{K}$-vector space endowed with $\partial \in \mathrm{End}^1\left(A^\bullet\right)$ and $\overline{\partial} \in \mathrm{End}^{-1}\left(A^\bullet\right)$ such that $\partial^2 = \overline{\partial}^2 = \left[\partial, \overline{\partial}\right] = 0$, and $A^k = \{0\}$ but for finitely-many $k \in \mathbb{Z}$.

Define

$$H^\bullet_{dR}(A^\bullet) := \frac{\ker(\partial + \overline{\partial})}{\mathrm{im}(\partial + \overline{\partial})}, \quad H^\bullet_{\partial}(A^\bullet) := \frac{\ker \partial}{\mathrm{im}\, \partial}, \quad H^\bullet_{\overline{\partial}}(A^\bullet) := \frac{\ker \overline{\partial}}{\mathrm{im}\, \overline{\partial}},$$

and

$$H^\bullet_{BC}(A^\bullet) := \frac{\ker \partial \cap \ker \overline{\partial}}{\mathrm{im}\, \partial\overline{\partial}}, \quad H^\bullet_{A}(A^\bullet) := \frac{\ker \partial\overline{\partial}}{\mathrm{im}\, \partial + \mathrm{im}\, \overline{\partial}}.$$

Starting with a bi-differential $\mathbb{Z}$-graded complex, one can construct a double complex. Following [14, §1.3], [16, §4.2], see [27, §II.2], [21, §II], take an infinite cyclic multiplicative group $\{\beta^m : m \in \mathbb{Z}\}$, and define the $\mathbb{Z}^2$-graded $\mathbb{K}$-vector space

$$\mathrm{Doub}^{\bullet_1,\bullet_2} A^\bullet := A^{\bullet_1 - \bullet_2} \otimes_{\mathbb{K}} \mathbb{K}\beta^{\bullet_2}$$

endowed with

$$\partial \otimes_{\mathbb{K}} \mathrm{id} \in \mathrm{End}^{1,0}\left(\mathrm{Doub}^{\bullet,\bullet} A^\bullet\right) \quad \text{and} \quad \overline{\partial} \otimes_{\mathbb{K}} \beta \in \mathrm{End}^{0,1}\left(\mathrm{Doub}^{\bullet,\bullet} A^\bullet\right),$$

which satisfy $(\partial \otimes_{\mathbb{K}} \mathrm{id})^2 = \left(\overline{\partial} \otimes_{\mathbb{K}} \beta\right)^2 = \left[\partial \otimes_{\mathbb{K}} \mathrm{id},\, \overline{\partial} \otimes_{\mathbb{R}} \beta\right] = 0$.

For $\sharp \in \left\{dR,\, \partial,\, \overline{\partial},\, BC,\, A\right\}$, there are natural isomorphisms of $\mathbb{K}$-vector spaces, [8, Lemma 1.5]:

$$H^{\bullet_1,\bullet_2}_{\sharp}\left(\mathrm{Doub}^{\bullet,\bullet} A^\bullet\right) \simeq H^{\bullet_1 - \bullet_2}_{\sharp}\left(A^\bullet\right) \otimes_{\mathbb{K}} \mathbb{K}\beta^{\bullet_2}.$$

1.4. Cohomology computations

We recall the following result in [5], concerning the relations between the Bott-Chern cohomology of a double complex and the Bott-Chern cohomology of a suitable sub-complex.

Theorem 1.2 ([5, Theorem 1.3]). *Let $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, and let $\left(C^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a sub-complex. Suppose that*

(i) the induced map $H^{\bullet,\bullet}_{\overline{\partial}}\left(C^{\bullet,\bullet}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(A^{\bullet,\bullet}\right)$ is an isomorphism,

(ii) the induced map $H^{\bullet,\bullet}_{\partial}\left(C^{\bullet,\bullet}\right) \to H^{\bullet,\bullet}_{\partial}\left(A^{\bullet,\bullet}\right)$ is an isomorphism, and

(iii) for any $(p,q) \in \mathbb{Z}^2$, the induced map

$$\begin{aligned} &\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}\left(C^{\bullet,\bullet}\right) \to \mathrm{Tot}^{p+q+1}\left(C^{\bullet,\bullet}\right)\right) \cap C^{p,q}}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}\left(C^{\bullet,\bullet}\right) \to \mathrm{Tot}^{p+q}\left(C^{\bullet,\bullet}\right)\right)} \\ \to &\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}\left(A^{\bullet,\bullet}\right) \to \mathrm{Tot}^{p+q+1}\left(A^{\bullet,\bullet}\right)\right) \cap A^{p,q}}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}\left(A^{\bullet,\bullet}\right) \to \mathrm{Tot}^{p+q}\left(A^{\bullet,\bullet}\right)\right)} \end{aligned}$$

is surjective.

Then the induced map $H^{\bullet,\bullet}_{BC}\left(C^{\bullet,\bullet}\right) \to H^{\bullet,\bullet}_{BC}\left(A^{\bullet,\bullet}\right)$ is surjective.

As a corollary, we get the following result concerning cohomology computations for bi-differential $\mathbb{Z}$-graded complexes.

Corollary 1.3. *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex and $B^\bullet \subseteq A^\bullet$ be a bi-differential $\mathbb{Z}$-graded sub-complex. Suppose that:*

(i) the cohomologies $H^\bullet_\partial(A^\bullet)$ and $H^\bullet_{\overline{\partial}}(A^\bullet)$, and $H^\bullet_\partial(B^\bullet)$ and $H^\bullet_{\overline{\partial}}(B^\bullet)$ are finite-dimensional;

(ii) the inclusion $\iota\colon B^\bullet \hookrightarrow A^\bullet$ induces the isomorphisms $H^\bullet_\partial(B^\bullet) \xrightarrow{\simeq} H^\bullet_\partial(A^\bullet)$ and $H^\bullet_{\overline{\partial}}(B^\bullet) \xrightarrow{\simeq} H^\bullet_{\overline{\partial}}(A^\bullet)$;

(iii) there exists a map $\mu\colon A^\bullet \to B^\bullet$ of bi-differential $\mathbb{Z}$-graded complexes such that $\mu \circ \iota = \mathrm{id}_{B^\bullet}$.

Then the inclusion $\iota\colon B^\bullet \hookrightarrow A^\bullet$ induces the isomorphism

$$H^\bullet_{BC}(B^\bullet) \xrightarrow{\simeq} H^\bullet_{BC}(A^\bullet).$$

Proof. We have the induced map $\iota\colon H^\bullet_{BC}(B^\bullet) \to H^\bullet_{BC}(A^\bullet)$ and $\mu\colon H^\bullet_{BC}(A^\bullet) \to H^\bullet_{BC}(B^\bullet)$ such that $\mu \circ \iota = \mathrm{id}$. Hence $\iota\colon H^\bullet_{BC}(B^\bullet) \to H^\bullet_{BC}(A^\bullet)$ is injective.

We prove the surjectivity of the map $\iota\colon H^\bullet_{BC}(B^\bullet) \to H^\bullet_{BC}(A^\bullet)$ by using Theorem 1.2. Consider the double complex $(\mathrm{Doub}^{\bullet_1,\bullet_2} A^\bullet := A^{\bullet_1-\bullet_2} \otimes_{\mathbb{K}} \mathbb{K}\beta^{\bullet_2}, \partial \otimes_{\mathbb{K}} \mathrm{id}, \overline{\partial} \otimes_{\mathbb{K}} \beta)$, as in Section 1.3, and the double sub-complex $\mathrm{Doub}^{\bullet,\bullet} B^\bullet$. Then, by the assumption, the inclusion $\iota \otimes \mathrm{id}\colon \mathrm{Doub}^{\bullet,\bullet} B^\bullet \hookrightarrow \mathrm{Doub}^{\bullet,\bullet} A^\bullet$ induces the isomorphisms

$$H^{\bullet,\bullet}_{\partial\otimes_{\mathbb{K}}\mathrm{id}}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \xrightarrow{\simeq} H^{\bullet,\bullet}_{\partial\otimes_{\mathbb{K}}\mathrm{id}}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet)$$

and

$$H^{\bullet,\bullet}_{\overline{\partial}\otimes_{\mathbb{K}}\beta}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \xrightarrow{\simeq} H^{\bullet,\bullet}_{\overline{\partial}\otimes_{\mathbb{K}}\beta}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet).$$

Considering the spectral sequences of double complexes, by [44, Theorem 3.5], the inclusion $\iota \otimes \mathrm{id}\colon \mathrm{Doub}^{\bullet,\bullet} B^\bullet \hookrightarrow \mathrm{Doub}^{\bullet,\bullet} A^\bullet$ induces the isomorphism

$$H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} B^\bullet) \xrightarrow{\simeq} H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} A^\bullet).$$

We prove that, for any $(p,q) \in \mathbb{Z}^2$, the induced map

$$\iota \otimes \mathrm{id}\colon \frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \to \mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet} \wedge B^\bullet)\right) \cap \mathrm{Doub}^{p,q} B^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \to \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet)\right)}$$
$$\to \frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to \mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet)\right) \cap \mathrm{Doub}^{p,q} A^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet)\right)}$$

is surjective.

Consider the map $\mu \otimes \mathrm{id}\colon \mathrm{Doub}^{\bullet,\bullet} B^\bullet \to \mathrm{Doub}^{\bullet,\bullet} A^\bullet$. This map is a homomorphism of double complexes. Then, by the assumption that $\mu \circ \iota = \mathrm{id}$, for the induced map

$$\mu \otimes \mathrm{id}\colon H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} B^\bullet),$$

we have $(\mu \otimes \mathrm{id}) \circ (\iota \otimes \mathrm{id}) = \mathrm{id}$. Since

$$\iota \otimes \mathrm{id}\colon H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} B^\bullet) \xrightarrow{\simeq} H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} A^\bullet)$$

is an isomorphism, $\mu \otimes \mathrm{id}\colon H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet,\bullet} B^\bullet)$ is its inverse map. Consider

$$\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \to \mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet)\right) \cap \mathrm{Doub}^{p,q} B^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet) \to \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} B^\bullet)\right)}$$
$$\subseteq H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet_1,\bullet_2} B^\bullet)$$

and

$$\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to \mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet)\right) \cap \mathrm{Doub}^{p,q} A^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet) \to \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet} A^\bullet)\right)}$$
$$\subseteq H^\bullet_{dR}(\mathrm{Tot}^\bullet \mathrm{Doub}^{\bullet_1,\bullet_2} A^\bullet).$$

By $\mu\otimes\mathrm{id}(\mathrm{Doub}^{p,q}A^\bullet)\subseteq\mathrm{Doub}^{p,q}B^\bullet$, the induced map

$$\iota\otimes\mathrm{id}\colon \frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\to\mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet}\wedge B^\bullet)\right)\cap\mathrm{Doub}^{p,q}B^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\to\mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\right)}$$
$$\to\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\to\mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\right)\cap\mathrm{Doub}^{p,q}A^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\to\mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\right)}$$

is an isomorphism with the inverse map

$$\mu\otimes\mathrm{id}\colon \frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\to\mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet}\wedge A^\bullet)\right)\cap\mathrm{Doub}^{p,q}A^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\to\mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\right)}$$
$$\to\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\to\mathrm{Tot}^{p+q+1}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\right)\cap\mathrm{Doub}^{p,q}B^\bullet}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\to\mathrm{Tot}^{p+q}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\right)}.$$

By Theorem 1.2, for any $(p,q)\in\mathbb{Z}^2$, the induced map

$$\begin{aligned}\iota\otimes\mathrm{id}\colon H^{p-q}_{BC}(B^\bullet)\otimes_{\mathbb{K}}\mathbb{K}\beta^q\simeq H^{p,q}_{BC}(\mathrm{Doub}^{\bullet,\bullet}B^\bullet)\\ \to H^{p,q}_{BC}(\mathrm{Doub}^{\bullet,\bullet}A^\bullet)\simeq H^{p-q}_{BC}(A^\bullet)\otimes_{\mathbb{K}}\mathbb{K}\beta^q\end{aligned}$$

is surjective and so $H^{p,q}_{BC}(B^\bullet)\to H^{p,q}_{BC}(A^\bullet)$ is surjective. Hence the corollary follows. □

1.5. Hard Lefschetz condition for bi-differential $\mathbb{Z}$-graded complexes

We recall here some definitions and results concerning $\mathfrak{sl}_2(\mathbb{C})$-representations, and Hard Lefschetz Condition for bi-differential $\mathbb{Z}$-graded complexes; we refer to [60].

Consider a (possibly non-finite-dimensional) $\mathbb{C}$-vector space V. Let $\phi:\mathfrak{sl}_2(\mathbb{C})\to\mathrm{End}(V)$ be a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Take the basis

$$X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad\text{and}\quad Z=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{C})$. We recall that ϕ is said of *finite Z-spectrum*, [60, Definition 2.2], if

(i) V can be decomposed as the direct sum of the eigenspaces of $\phi(Z)$, and

(ii) $\phi(Z)$ has only finitely many distinct eigenvalues.

For $\mathfrak{sl}_2(\mathbb{C})$-representations of finite Z-spectrum, one has the following results, [60, Corollary 2.4, Corollary 2.5, Corollary 2.6]:

(i) all eigenvalues of $\phi(Z)$ are integers;

(ii) consider, for $k \in \mathbb{Z}$, the eigenspace V_k of $\phi(Z)$ with respect to the eigenvalue k; for any $k \in \mathbb{N}$, the maps $\phi(X)^k \colon V_k \to V_{-k}$ and $\phi(Y)^k \colon V_{-k} \to V_k$ are isomorphisms;

(iii) consider, for $k \in \mathbb{Z}$, the set

$$P_k = \{v \in V_k : \phi(X)v = 0\} = \left\{v \in V_k : \phi(Y)^{k+1}v = 0\right\}$$

of primitive elements; then one has the following decompositions:
a) $V_k = P_k \oplus \operatorname{im}\phi(X)$ for any $k \in \mathbb{Z}$;
b) $V_k = \bigoplus_{j\in\mathbb{N}} \phi(X)^j\,(P_{k+2j})$ for any $k \in \mathbb{N}$;
c) $V_{-k} = \bigoplus_{j\in\mathbb{N}} \phi(X)^{k+j}\,(P_{k+2j})$ for any $k \in \mathbb{N}\setminus\{0\}$.

Now let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N}\setminus\{0\}$. We define $H \in \operatorname{End}^0(A^\bullet)$ as $H = \sum_{k\in\mathbb{Z}}(n-k)\pi_{A^k}$ where $\pi_{A^k} : A^\bullet \to A^k$ denotes the projection. We suppose that we have operators $L \in \operatorname{End}^2(A^\bullet)$ and $\Lambda \in \operatorname{End}^{-2}(A^\bullet)$ satisfying the following relations:

$$[\partial, L] = 0, \quad \left[\overline{\partial}, L\right] = -\partial, \quad \left[\overline{\partial}, \Lambda\right] = 0, \quad [\partial, \Lambda] = \overline{\partial}, \tag{1}$$

and

$$[\Lambda, L] = H, \quad [L, H] = 2L, \quad [\Lambda, H] = -2\Lambda. \tag{2}$$

Then we have the representation $\phi \colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{End}(A^\bullet)$ of finite Z-spectrum given by

$$\phi(X) = \Lambda, \quad \phi(Y) = L, \quad \text{and } \phi(Z) = H.$$

Hence, in particular, $L^k \colon A^{n-k} \to A^{n+k}$ is an isomorphism, for any $k \in \{0, \dots, n\}$.

An element $\alpha \in A^\bullet$ is called *harmonic* if $\partial\alpha = 0$ and $\overline{\partial}\alpha = 0$. Let $H^\bullet_{\mathrm{hr}}(A^\bullet)$ be the space of all the harmonic elements. Then, by the above relations, $H^\bullet_{\mathrm{hr}}(A^\bullet)$ is a $\mathfrak{sl}_2(\mathbb{C})$-submodule of finite Z-spectrum. Hence, in particular, $L^k : H^{n-k}_{\mathrm{hr}}(A^\bullet) \to H^{n+k}_{\mathrm{hr}}(A^\bullet)$ is an isomorphism, for any $k \in \{0, \dots, n\}$.

The same argument in [60, Section 3] still works in yielding the following result.

Theorem 1.4 (see [60, Theorem 0.1]). *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$, and that (1) and (2) hold. The following conditions are equivalent:*

(i) the inclusion $H^\bullet_{hr}(A^\bullet) \hookrightarrow A^\bullet$ induces the surjection $H^\bullet_{hr}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$;

(ii) for any $k \in \{0, \dots, n\}$, the induced map $L^k \colon H^{n-k}_{\partial}(A^\bullet) \to H^{n+k}_{\partial}(A^\bullet)$ is surjective.

Remark 1.5. Note that the surjectivity of $H^\bullet_{\mathrm{hr}}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$ does not imply the injectivity of $L^k \colon H^{n-k}_{\partial}(A^\bullet) \to H^{n+k}_{\partial}(A^\bullet)$. In the second part of the above theorem, we can not state that $L^k \colon H^{n-k}_{\partial}(A^\bullet) \to H^{n+k}_{\partial}(A^\bullet)$ is an isomorphism, unless the cohomology $H^\bullet_{\partial}(A^\bullet)$ is self-dual and finite-dimensional.

In fact, the following result holds.

Corollary 1.6. *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$, and that (1) and (2) hold. Furthermore, assume that the cohomology $H^\bullet_{\partial}(A^\bullet)$ is finite-dimensional and, for each $k \in \{0, \dots, n\}$, we have $\dim H^{n-k}_{\partial}(A^\bullet) = \dim H^{n+k}_{\partial}(A^\bullet)$. Then the following conditions are equivalent:*

(i) the inclusion $H^\bullet_{hr}(A^\bullet) \hookrightarrow A^\bullet$ induces the surjection $H^\bullet_{hr}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$;

(ii) for any $k \in \{0, \dots, n\}$, the induced map $L^k \colon H^{n-k}_{\partial}(A^\bullet) \to H^{n+k}_{\partial}(A^\bullet)$ is an isomorphism.

Arguing by induction as in [16, Proposition 5.4], thanks to the above relations for ∂, $\overline{\partial}$, L, and Λ, we have the following result.

Proposition 1.7 (see [16, Proposition 5.4]). *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$, and that (1) and (2) hold. Suppose that for any $k \in \{0, \dots, n\}$, the induced map $L^k \colon H^{n-k}_{\partial}(A^\bullet) \to H^{n+k}_{\partial}(A^\bullet)$ is an isomorphism. Moreover, suppose that, for $\alpha \in A^1$, if $\partial\overline{\partial}\alpha = 0$, then $\overline{\partial}\alpha$ is ∂-exact. Then we have*

$$\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial \cap \operatorname{im}\overline{\partial}.$$

1.6. $\partial\overline{\partial}$-Lemma for bi-differential $\mathbb{Z}$-graded complexes

In this section, we prove some results concerning $\partial\overline{\partial}$-Lemma for bi-differential $\mathbb{Z}$-graded complexes, in relation with the Hard Lefschetz Condition.

As for the following result, compare also [16, Theorem 4.3] and [25, Proposition 2].

Proposition 1.8. *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$. Suppose that:*

(i) the inclusion $H^\bullet_{hr}(A^\bullet) \subseteq A^\bullet$ induces the surjection $H^\bullet_{hr}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$;

(ii) $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial \cap \ker\overline{\partial}$.

Then we have $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}$.

Proof. For $s \in \mathbb{N} \setminus \{0\}$, we show that $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}$ on A^{2n-s}. We will prove this by induction on $s \in \mathbb{N} \setminus \{0\}$.

First we consider the case $s = 0$. By $\operatorname{im}\overline{\partial} = \{0\}$, we have $\operatorname{im}\overline{\partial} \cap \ker\partial = \{0\} = \operatorname{im}\partial \cap \ker\overline{\partial}$, and $\operatorname{im}\partial\overline{\partial} = \operatorname{im}(-\overline{\partial}\partial) = \{0\}$.

Then we consider the case $s = 1$. Let $\beta \in \operatorname{im}\overline{\partial} \cap \ker\partial \cap A^{2n-1} = \operatorname{im}\partial \cap \ker\overline{\partial} \cap A^{2n-1}$ such that $\beta = \overline{\partial}\alpha$ for some $\alpha \in A^{2n}$. Then we have $\partial\overline{\partial}\alpha = \partial\beta = 0$. By the assumption and by $\partial A^{2n} = 0$, we have $\alpha_{\mathrm{hr}} \in H^{2n}_{\mathrm{hr}}(A^\bullet)$ such that $\alpha = \alpha_{\mathrm{hr}} + \partial\gamma$ for some $\gamma \in A^{2n-1}$. Hence we have $\beta = \overline{\partial}\alpha_{\mathrm{hr}} + \overline{\partial}\partial\gamma = \overline{\partial}\partial\gamma = \partial\overline{\partial}(-\gamma)$.

Assume now that $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}$ on A^{2n-r} with $r \leq s$. We need to prove $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}$ on A^{2n-s-1}. Let

$$\alpha_{2n-s-1} \in \operatorname{im}\overline{\partial} \cap \ker\partial \cap A^{2n-s-1} = \operatorname{im}\partial \cap \ker\overline{\partial} \cap A^{2n-s-1}$$

such that $\alpha_{2n-s-1} = \overline{\partial}\alpha_{2n-s}$ for some $\alpha_{2n-s} \in A^{2n-s}$. Then we have $0 = \partial\alpha_{2n-s-1} = \partial\overline{\partial}\alpha_{2n-s} = -\overline{\partial}\partial\alpha_{2n-s}$. Hence $\partial\alpha_{2n-s} \in \operatorname{im}\partial\overline{\partial} \cap A^{2n-s+1}$ follows from induction hypothesis and we have $\partial\alpha_{2n-s} = \partial\overline{\partial}\alpha_{2n-s+1}$ for some $\alpha_{2n-s+1} \in A^{2n-s+1}$. By the assumption that $H^{2n-s}_{\mathrm{hr}}(A^\bullet) \to H^{2n-s}_{\partial}(A^\bullet)$ is surjective, we have that there exist $\beta_{2n-s} \in A^{2n-s}$ harmonic and $\gamma_{2n-s-1} \in A^{2n-s-1}$ such that

$$\alpha_{2n-s} - \overline{\partial}\alpha_{2n-s+1} - \beta_{2n-s} = \partial\gamma_{2n-s-1}.$$

Hence we have

$$\alpha_{2n-s-1} = \overline{\partial}\alpha_{2n-s} = \overline{\partial}\partial\gamma_{2n-s-1} = \partial\overline{\partial}(-\gamma_{2n-s-1}) \in \operatorname{im}\partial\overline{\partial}.$$

Thus the proposition follows. □

As for the following result, compare also [16, Theorem 4.2] and [25, Theorem 2].

Proposition 1.9. *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$. Suppose that*

$$\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}.$$

Then the inclusion $H^\bullet_{hr}(A^\bullet) \subseteq A^\bullet$ induces the surjection $H^\bullet_{hr}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$.

Proof. For a ∂-closed k-element $\alpha \in A^k$, we have $\partial\overline{\partial}\alpha = -\overline{\partial}\partial\alpha = 0$. By $\overline{\partial}\alpha \in \operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial}$, we have $\beta \in A^{k-1}$ such that $\overline{\partial}\alpha = \partial\overline{\partial}\beta$. Hence $\alpha - \partial\beta$ is a harmonic element which is cohomologous to α. □

We summarize the contents of Proposition 1.8 and Proposition 1.9 in the following corollary.

Corollary 1.10. *Let $\left(A^\bullet, \partial, \overline{\partial}\right)$ be a bounded bi-differential $\mathbb{Z}$-graded complex such that $A^k = \{0\}$ for $k < 0$ or $k > 2n$, for some $n \in \mathbb{N} \setminus \{0\}$. Suppose that $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial \cap \ker\overline{\partial}$. The following conditions are equivalent:*

(i) the inclusion $H^\bullet_{hr}(A^\bullet) \subseteq A^\bullet$ induces the surjection $H^\bullet_{hr}(A^\bullet) \to H^\bullet_{\partial}(A^\bullet)$;

(ii) $\operatorname{im}\overline{\partial} \cap \ker\partial = \operatorname{im}\partial\overline{\partial} = \operatorname{im}\partial \cap \ker\overline{\partial}$

2. Preliminaries and notations on symplectic structures

In this section, we set the notations concerning symplectic structures on manifolds and symplectic Hodge theory, referring to, e.g., [14–16, 42, 55, 60] for more details.

2.1. Symplectic structures

Let X be a $2n$-dimensional compact manifold endowed with a *symplectic structure* ω, namely, a non-degenerate d-closed 2-form on X. The symplectic

form ω yields the natural isomorphism

$$I\colon TX \to T^*X, \quad I(\cdot)(\cdot\cdot) := \omega(\cdot, \cdot\cdot).$$

Define the *canonical Poisson bi-vector* associated to ω as

$$\Pi := \omega^{-1} := \omega\left(I^{-1}\cdot, I^{-1}\cdot\cdot\right) \in \wedge^2 TX,$$

and, for $k \in \mathbb{N}$, let $\left(\omega^{-1}\right)^k$ be the bi-linear form on $\wedge^k X$ defined, on the simple elements $\alpha^1 \wedge \cdots \wedge \alpha^k \in \wedge^k X$ and $\beta^1 \wedge \cdots \wedge \beta^k \in \wedge^k X$, as

$$\left(\omega^{-1}\right)^k \left(\alpha^1 \wedge \cdots \wedge \alpha^k,\, \beta^1 \wedge \cdots \wedge \beta^k\right) := \det\left(\omega^{-1}\left(\alpha^i,\, \beta^j\right)\right)_{i,j\in\{1,\ldots,k\}}.$$

Define the *symplectic-$\star$-operator*, [14, §2],

$$\star_\omega\colon \wedge^\bullet X \to \wedge^{2n-\bullet} X$$

by requiring that, for every $\alpha, \beta \in \wedge^k X$,

$$\alpha \wedge \star_\omega \beta = \left(\omega^{-1}\right)^k (\alpha, \beta)\, \omega^n.$$

The operators

$$\begin{aligned} L \in \mathrm{End}^2\left(\wedge^\bullet X\right), &\qquad L(\alpha) := \omega \wedge \alpha, \\ \Lambda \in \mathrm{End}^{-2}\left(\wedge^\bullet X\right), &\qquad \Lambda(\alpha) := -\iota_\Pi \alpha, \\ H \in \mathrm{End}^0\left(\wedge^\bullet X\right), &\qquad H(\alpha) := \sum_{k\in\mathbb{Z}} (n-k)\, \pi_{\wedge^k X}\alpha, \end{aligned}$$

yield an $\mathfrak{sl}_2(\mathbb{C})$-representation on $\wedge^\bullet X \otimes \mathbb{C}$, see, e.g., [60, Corollary 1.6] (where $\iota_\xi\colon \wedge^\bullet X \to \wedge^{\bullet-2} X$ denotes the interior product with $\xi \in \wedge^2(TX)$, and $\pi_{\wedge^k X}\colon \wedge^\bullet X \to \wedge^k X$ denotes the natural projection onto $\wedge^k X$, for $k \in \mathbb{Z}$).

Define the *symplectic co-differential operator* as

$$\mathrm{d}^\Lambda := [\mathrm{d}, \Lambda] \in \mathrm{End}^{-1}\left(\wedge^\bullet X\right).$$

One has that

$$\mathrm{d}^2 = \left(\mathrm{d}^\Lambda\right)^2 = \left[\mathrm{d}, \mathrm{d}^\Lambda\right] = 0,$$

see, e.g., [38, page 266, page 265], [14, Proposition 1.2.3, Theorem 1.3.1].

2.2. Symplectic cohomologies

Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω.

By considering the bi-differential $\mathbb{Z}$-graded complex $\left(\wedge^\bullet X, \mathrm{d}, \mathrm{d}^\Lambda\right)$, one has the symplectic cohomologies

$$H^\bullet_\sharp(X;\mathbb{R}) := H^\bullet_\sharp(\wedge^\bullet X), \quad \text{for } \sharp \in \left\{dR, \mathrm{d}^\Lambda, BC, A\right\}.$$

More precisely, other than the cohomologies

$$H^\bullet_{dR}(X;\mathbb{R}) := \frac{\ker \mathrm{d}}{\operatorname{im} \mathrm{d}} \quad \text{and} \quad H^\bullet_{\mathrm{d}^\Lambda}(X;\mathbb{R}) := \frac{\ker \mathrm{d}^\Lambda}{\operatorname{im} \mathrm{d}^\Lambda},$$

one can define, following L.-S. Tseng and S.-T. Yau, [55, §3.2, §3.3], see also [56, 57],

$$H^\bullet_{BC}(X;\mathbb{R}) := \frac{\ker \mathrm{d} \cap \ker \mathrm{d}^\Lambda}{\operatorname{im} \mathrm{d}\,\mathrm{d}^\Lambda} \quad \text{and} \quad H^\bullet_{A}(X;\mathbb{R}) := \frac{\ker \mathrm{d}\,\mathrm{d}^\Lambda}{\operatorname{im} \mathrm{d} + \operatorname{im} \mathrm{d}^\Lambda}.$$

In view of generalized complex geometry, [16, 17, 29], these cohomologies are the symplectic counterpart of the Bott-Chern and Aeppli cohomologies for complex manifolds, [57].

Note that $\star_\omega^2 = \mathrm{id}_{\wedge^\bullet X}$, [14, Lemma 2.1.2], and that $\mathrm{d}^\Lambda\lfloor_{\wedge^k X} = (-1)^{k+1} \star_\omega \mathrm{d} \star_\omega$ for any $k \in \mathbb{N}$, [14, Theorem 2.2.1]. In particular, it follows that the symplectic Hodge-$\star$-operator induces the isomorphism, [14, Corollary 2.2.2],

$$\star_\omega \colon H^\bullet_{dR}(X;\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_{\mathrm{d}^\Lambda}(X;\mathbb{R}).$$

In [55], L.-S. Tseng and S.-T. Yau developed a Hodge theory for the symplectic cohomologies. More precisely, fixed an almost-Kähler structure $(J, \omega, g := \omega(\cdot, J\cdot\cdot))$ on X, they defined self-adjoint elliptic differential operators whose kernel is isomorphic to the above cohomologies, [55, Proposition 3.3, Theorem 3.5, Theorem 3.16]. In particular, X being compact, it follows that $\dim_{\mathbb{R}} H^\bullet_\sharp(X;\mathbb{R}) < +\infty$ for $\sharp \in \left\{dR, \mathrm{d}^\Lambda, BC, A\right\}$, [55, Corollary 3.6, Corollary 3.17]. As another consequence, the Hodge-$*$-operator $*_g \colon \wedge^\bullet X \to \wedge^{2n-\bullet} X$ induces the isomorphism, [55, Corollary 3.25],

$$*_g \colon H^\bullet_{BC}(X;\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_{A}(X;\mathbb{R}).$$

2.3. Hard Lefschetz Condition

Several special cohomological properties can be defined on symplectic manifolds. More precisely, a compact $2n$-dimensional manifold X endowed with a symplectic structure ω is said to satisfy:

- the *Hard Lefschetz Condition* if, for any $k \in \mathbb{N}$, the map $\left[\omega^k\right] \smile \cdot : H^{n-k}_{dR}(X;\mathbb{R}) \to H^{n+k}_{dR}(X;\mathbb{R})$ is an isomorphism;
- the *Brylinski conjecture*, [14, Conjecture 2.2.7], if every de Rham cohomology class admits a d-closed d^Λ-closed representative, namely, the natural map $H^\bullet_{BC}(X;\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})$ induced by the identity is surjective;
- the $\mathrm{d}\,\mathrm{d}^\Lambda$*-Lemma* if every d-exact d^Λ-closed form is also $\mathrm{d}\,\mathrm{d}^\Lambda$-exact, namely, if the natural map $H^\bullet_{BC}(X;\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})$ induced by the identity is injective.

By [42, Corollary 2], [60, Theorem 0.1], [43, Proposition 1.4], [30], [55, Proposition 3.13], [16, Theorem 5.4], it follows that, for compact symplectic manifolds, the Hard Lefschetz Condition, the Brylinski conjecture, and the $\mathrm{d}\,\mathrm{d}^\Lambda$-Lemma are equivalent properties. In this case, it follows that the natural maps

$$\begin{array}{ccccc} & & H^\bullet_{BC}(X;\mathbb{R}) & & \\ & \swarrow & & \searrow & \\ H^\bullet_{dR}(X;\mathbb{R}) & & & & H^\bullet_{\mathrm{d}^\Lambda}(X;\mathbb{R}) \\ & \searrow & & \swarrow & \\ & & H^\bullet_A(X;\mathbb{R}) & & \end{array}$$

induced by the identity are actually isomorphisms.

(See also, e.g., [55, Proposition 3.13], [12], [40, §3, Theorem 4], [7, Remark 2.3], [8, Theorem 4.4], [36, §8], [35, §5].)

3. Symplectic cohomologies for solvmanifolds

In this section, we apply the results in [5] in order to provide tools for the computations of the symplectic cohomologies for solvmanifolds. In particular, we recover a theorem by M. Macrì, [40], for completely-solvable solvmanifolds, see Theorem 3.2, and we extend the result to the general case in

Theorem 6.8. Such results will be used in Section 4 to investigate explicit examples.

3.1. Notations

In order to fix notations, let $X = \Gamma\backslash G$ be a *solvmanifold* (that is, a compact quotient of a connected simply-connected solvable Lie group by a co-compact discrete subgroup) endowed with a G-left-invariant symplectic structure ω. Denote the Lie algebra associated to G by $\mathfrak{g}$: it is endowed with the linear symplectic structure $\omega \in \wedge^2\mathfrak{g}^*$. Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Given a bi-differential $\mathbb{Z}$-graded sub-complex $\left(A^\bullet, \mathrm{d}, \mathrm{d}^\Lambda\right) \hookrightarrow \left(\wedge^\bullet X, \mathrm{d}, \mathrm{d}^\Lambda\right)$, consider, for $\sharp \in \left\{dR, \mathrm{d}^\Lambda, BC, A\right\}$,

$$\iota\colon H^\bullet_\sharp(A^\bullet) \to H^\bullet_\sharp(X;\mathbb{R}).$$

In particular, by means of left-translations, one has the $\mathbb{Z}$-graded $\mathbb{R}$-vector sub-space $\iota\colon \wedge^\bullet \mathfrak{g}^* \hookrightarrow \wedge^\bullet X$. Since ω is G-left-invariant, the space $\wedge^\bullet\mathfrak{g}^*$ is endowed with the (restrictions of the) differentials d and d^Λ. In particular, $\left(\wedge^\bullet\mathfrak{g}^*, \mathrm{d}, \mathrm{d}^\Lambda\right)$ is a bi-differential $\mathbb{Z}$-graded sub-complex of $\left(\wedge^\bullet X, \mathrm{d}, \mathrm{d}^\Lambda\right)$. For $\sharp \in \left\{dR, \mathrm{d}^\Lambda, BC, A\right\}$, denote

$$\iota\colon H^\bullet_\sharp(\mathfrak{g};\mathbb{R}) := H^\bullet_\sharp(\wedge^\bullet\mathfrak{g}^*) \to H^\bullet_\sharp(X;\mathbb{R}).$$

3.2. Subgroups of symplectic cohomologies

We firstly note the following result, as a consequence of the symplectic Hodge theory developed by L.-S. Tseng and S.-T. Yau in [55].

Corollary 3.1. *Let $\Gamma\backslash G$ be a $2n$-dimensional solvmanifold endowed with a G-left-invariant symplectic structure ω. Let $(A^\bullet, \mathrm{d})$ be a sub-complex of $(\wedge^\bullet X, \mathrm{d})$ such that $A^2 \ni \omega$, and suppose that there exists an almost-Kähler structure (J, ω, g) on X such that the Hodge-$*$-operator associated to g satisfies $*_g\lfloor_{A^\bullet} : A^\bullet \to A^{2n-\bullet}$. Then, for $\sharp \in \left\{dR, \mathrm{d}^\Lambda, BC, A\right\}$, the natural map*

$$\iota\colon H^\bullet_\sharp(A^\bullet) \to H^\bullet_\sharp(X;\mathbb{R})$$

is injective.

Proof. Take an almost-Kähler structure (J, ω, g) as in the statement. In particular, J is an almost-complex structure being compatible with ω, that is,

$g := \omega(\cdot, J\cdot\cdot)$ is a J-Hermitian metric with associated fundamental form ω. Consider the Hodge-$*$-operator $*_g \colon \wedge^\bullet X \to \wedge^{2n-\bullet} X$ associated to g. By [55, Theorem 3.5, Corollary 3.6], the 4^{th}-order self-adjoint differential operator

$$\begin{aligned} D_{\mathrm{d}+\mathrm{d}^\wedge} := {} & \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* + \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}\,\mathrm{d}^\Lambda\right) + \left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)^* \\ & + \left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}^*\,\mathrm{d}^\Lambda\right) + \mathrm{d}^*\,\mathrm{d} + \left(\mathrm{d}^\Lambda\right)^*\mathrm{d}^\Lambda \end{aligned}$$

is elliptic, and it induces an orthogonal decomposition

$$\wedge^\bullet X = \ker D_{\mathrm{d}+\mathrm{d}^\wedge} \overset{\perp}{\oplus} \mathrm{d}\,\mathrm{d}^\Lambda \wedge^\bullet X \overset{\perp}{\oplus} \left(\mathrm{d}^* \wedge^{\bullet+1} X + \left(\mathrm{d}^\Lambda\right)^* \wedge^{\bullet-1} X\right),$$

and hence the isomorphism

$$H^\bullet_{BC}(X;\mathbb{R}) \simeq \ker D_{\mathrm{d}+\mathrm{d}^\wedge},$$

By the hypotheses, one has that d and d^Λ, and $*_g$, restricts to $A^\bullet$. In particular, $D_{\mathrm{d}+\mathrm{d}^\wedge}\lfloor_{A^\bullet} \colon A^\bullet \to A^\bullet$ induces an isomorphism

$$H^\bullet_{BC}(A^\bullet) \simeq \ker D_{\mathrm{d}+\mathrm{d}^\wedge}.$$

Hence, (as in [5, Theorem 1.6],) one gets the commutative diagram

$$\begin{array}{ccc} \ker D_{\mathrm{d}+\mathrm{d}^\wedge}\lfloor_{A^\bullet} & \xrightarrow{\simeq} & H^\bullet_{BC}(A^\bullet) \\ \downarrow & & \downarrow \\ \ker D_{\mathrm{d}+\mathrm{d}^\wedge} & \xrightarrow{\simeq} & H^\bullet_{BC}(X;\mathbb{R}) \end{array}$$

from which it follows that the natural map $H^\bullet_{BC}(A^\bullet) \to H^\bullet_{BC}(X;\mathbb{R})$ is injective.

The theorem follows, by considering the differential elliptic operators $[\mathrm{d}, \mathrm{d}^*]$ and $\left[\mathrm{d}^\Lambda, \left(\mathrm{d}^\Lambda\right)^*\right]$ and, [55, Theorem 3.16],

$$\begin{aligned} D_{\mathrm{d}\,\mathrm{d}^\wedge} := {} & \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)\left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^* + \left(\mathrm{d}\,\mathrm{d}^\Lambda\right)^*\left(\mathrm{d}\,\mathrm{d}^\Lambda\right) + \left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)\left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)^* \\ & + \left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right)^*\left(\mathrm{d}\left(\mathrm{d}^\Lambda\right)^*\right) + \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^\Lambda\left(\mathrm{d}^\Lambda\right)^*, \end{aligned}$$

such that $H^\bullet_A(X;\mathbb{R}) \simeq \ker D_{\mathrm{d}\,\mathrm{d}^\wedge}$, [55, Corollary 3.17], or by noting that $*_g D_{\mathrm{d}+\mathrm{d}^\wedge} = D_{\mathrm{d}\,\mathrm{d}^\wedge} *_g$, from which one has the isomorphism $*_g \colon H^\bullet_{BC}(X;\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_A(X;\mathbb{R})$, [55, Lemma 3.23, Proposition 3.24, Corollary 3.25]. □

3.3. Symplectic cohomologies for completely-solvable solvmanifolds

By A. Hattori's theorem [34, Corollary 4.2], if G is *completely-solvable*, (that is, for any $g \in G$, all the eigen-values of $\operatorname{Ad} g$ are real,) then the natural map $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})$ is an isomorphism.

The following result states that, for a completely-solvable solvmanifold, the L.-S. Tseng and S.-T. Yau symplectic cohomologies can be computed using just left-invariant forms; we refer to [40, Theorem 3] by M. Macrì for a different proof of the same result.

Theorem 3.2 (see [40, Theorem 3]). *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant symplectic structure ω. Suppose that the natural map $H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{dR}(X;\mathbb{R})$ is an isomorphism. Then, for $\sharp \in \{\mathrm{d}^\Lambda, BC, A\}$, the natural map*

$$\iota\colon H^\bullet_\sharp(\mathfrak{g};\mathbb{R}) \to H^\bullet_\sharp(X;\mathbb{R})$$

is an isomorphism.

Proof. We split the proof in the following steps.

`Step 1` – *The symplectic d^Λ-cohomology.*

Since ω is G-left-invariant, then the symplectic-$\star$-operator $\star_\omega\colon \wedge^\bullet X \to \wedge^{\dim X-\bullet}X$ induces the isomorphism $\star_\omega\lfloor_{\wedge^\bullet\mathfrak{g}^*}\colon \wedge^\bullet\mathfrak{g}^* \to \wedge^{\dim_{\mathbb{R}}\mathfrak{g}-\bullet}\mathfrak{g}^*$. Hence, since

$$\begin{aligned} &(\star_\omega\lfloor_{\wedge^\bullet\mathfrak{g}^*})^2 = \star_\omega^2\lfloor_{\wedge^\bullet\mathfrak{g}^*} = \mathrm{id}_{\wedge^\bullet\mathfrak{g}^*} \\ \text{and}\quad &\mathrm{d}^\Lambda\lfloor_{\wedge^\bullet\mathfrak{g}^*} = (-1)^{k+1}\ \star_\omega\lfloor_{\wedge^\bullet\mathfrak{g}^*}\mathrm{d}\lfloor_{\wedge^\bullet\mathfrak{g}^*}\star_\omega\lfloor_{\wedge^\bullet\mathfrak{g}^*} \end{aligned}$$

for any $k \in \mathbb{N}$, one has the isomorphism $\star_\omega\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \xrightarrow{\simeq} H^{\dim_{\mathbb{R}}\mathfrak{g}-\bullet}_{\mathrm{d}^\Lambda}(\mathfrak{g};\mathbb{R})$. Therefore one gets the commutative diagram

$$\begin{array}{ccc} H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) & \xrightarrow[\iota]{\simeq} & H^\bullet_{dR}(X;\mathbb{R}) \\ {\scriptstyle\star_\omega}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\star_\omega}\downarrow{\scriptstyle\simeq} \\ H^{\dim_{\mathbb{R}}\mathfrak{g}-\bullet}_{\mathrm{d}^\Lambda}(\mathfrak{g};\mathbb{R}) & \xrightarrow[\iota]{} & H^{\dim X-\bullet}_{\mathrm{d}^\Lambda}(X;\mathbb{R}), \end{array}$$

from which it follows that the natural map $\iota\colon H^\bullet_{\mathrm{d}^\Lambda}(\mathfrak{g};\mathbb{R}) \to H^\bullet_{\mathrm{d}^\Lambda}(X;\mathbb{R})$ is an isomorphism.

`Step 2` – *The symplectic Bott-Chern cohomology.*

Apply Corollary 1.3 to $\iota\colon \left(\wedge^\bullet \mathfrak{g}^*,\, \mathrm{d},\, \mathrm{d}^\Lambda\right) \hookrightarrow \left(\wedge^\bullet X,\, \mathrm{d},\, \mathrm{d}^\Lambda\right)$, where the map $\mu\colon \wedge^\bullet X \to \wedge^\bullet \mathfrak{g}^*$ is the F. A. Belgun symmetrization map, [9, Theorem 7]: namely, by [45, Lemma 6.2], consider a G-bi-invariant volume form η on G such that $\int_X \eta = 1$, and define

$$\mu\colon \wedge^\bullet X \otimes_{\mathbb{R}} \mathbb{C} \to \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}}, \quad \mu(\alpha) := \int_X \alpha \lfloor_x \eta(x).$$

`Step 3` – *The symplectic Aeppli cohomology.*

Let J be a G-left-invariant ω-compatible almost-complex structure on X, (see, e.g., [15, Proposition 12.6],) and consider the G-left-invariant J-Hermitian metric $g := \omega\,(\cdot,\, J\cdot)$. By [55, Corollary 3.25], the Hodge-$*$-operator $*_g\colon \wedge^\bullet X \to \wedge^{2n-\bullet} X$ induces the isomorphism

$$*_g\colon H^\bullet_{BC}(X;\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_A(X;\mathbb{R}),$$

and, since g is G-left-invariant, also the isomorphism

$$*_g\colon H^\bullet_{BC}(\mathfrak{g};\mathbb{R}) \xrightarrow{\simeq} H^{2n-\bullet}_A(\mathfrak{g};\mathbb{R}).$$

Hence one has the commutative diagram

$$\begin{array}{ccc} H^\bullet_{BC}(\mathfrak{g};\mathbb{R}) & \xrightarrow[\iota]{\simeq} & H^\bullet_{BC}(X;\mathbb{R}) \\ {\scriptstyle *_g}\downarrow{\scriptstyle \simeq} & & {\scriptstyle *_g}\downarrow{\scriptstyle \simeq} \\ H^{\dim_{\mathbb{R}}\mathfrak{g}-\bullet}_A(\mathfrak{g};\mathbb{R}) & \xrightarrow[\iota]{} & H^{\dim X-\bullet}_A(X;\mathbb{R}), \end{array}$$

from which it follows that also the natural map $\iota\colon H^\bullet_A(\mathfrak{g};\mathbb{R}) \to H^\bullet_A(X;\mathbb{R})$ is an isomorphism. □

4. Applications

In this section, as an application of Theorem 3.2, we explicitly compute the symplectic cohomologies of some low-dimensional nilmanifolds and solvmanifolds.

We recall that, by [8, Theorem 4.4], for a compact manifold X endowed with a symplectic structure, for any $k \in \mathbb{Z}$, the inequality

$$\Delta^k := \dim_{\mathbb{R}} H^k_{BC}(X;\mathbb{R}) + \dim_{\mathbb{R}} H^k_A(X;\mathbb{R}) - 2\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) \geq 0$$

holds. Furthermore, the equality holds for any $k \in \mathbb{Z}$ if and only if X satisfies the Hard Lefschetz Condition. Non-tori nilmanifolds never satisfy the Hard Lefschetz Condition by [10, Theorem A]. Hence, for nilmanifolds, the numbers $\{\Delta^k\}_{k\in\mathbb{Z}}$ provide a degree of non-Kählerianity. As regards Hard Lefschetz Condition for solvmanifolds, we refer to [35, 36]. Finally, recall that, by [54, Theorem 4.3], for compact symplectic manifolds, it holds always $\Delta^1 = 0$.

(As a matter of notations, by writing the structure equations of the Lie algebra $\mathfrak{g}$ associated to a solvmanifold, we write, e.g., $(0,0,0,12)$: we mean that there exists a basis $\{e^1,e^2,e^3,e^4\}$ of $\mathfrak{g}^*$ such that $\mathrm{d}\,e^1 = \mathrm{d}\,e^2 = \mathrm{d}\,e^3 = 0$ and $\mathrm{d}\,e^4 = e^1\wedge e^2$; furthermore, we shorten $e^1\wedge e^2 =: e^{12}$; we follow the notations in [12, 19, 50].)

4.1. 4-dimensional solvmanifolds

According to [12, Theorem 6.2, Table 2], the compact 4-dimensional manifolds that are diffeomorphic to a solvmanifold admitting a left-invariant symplectic structure are the following:

(a) the *torus* $4\,\mathfrak{g}_1 = (0,\,0,\,0,\,0)$ endowed with the left-invariant symplectic structure $\omega := e^{12} + e^{34}$: the Lie algebra is Abelian, and the symplectic structure satisfies the Hard Lefschetz Condition;

(b) the differentiable manifold underlying the *primary Kodaira surface* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1 = (0,\,0,\,0,\,23)$ endowed with the left-invariant symplectic structure $\omega := e^{12} + e^{34}$: the Lie algebra is nilpotent, and hence no left-invariant symplectic structure on such manifold satisfies the Hard Lefschetz Condition;

(c) the solvmanifold associated to $\mathfrak{g}_1 \oplus \mathfrak{g}^{-1}_{3.4} = (0,\,0,\,-23,\,24)$ endowed with the left-invariant symplectic structure $\omega := e^{12} + e^{34}$: the Lie algebra is completely-solvable, and the symplectic structure satisfies the Hard Lefschetz Condition;

(d) the differentiable manifold underlying the *hyper-elliptic surface*, whose associated Lie algebra is $\mathfrak{g}_1 \oplus \mathfrak{g}^0_{3.5} = (0,\,0,\,-24,\,23)$, endowed with the

left-invariant symplectic structure $\omega := e^{12}+e^{34}$: it yields a Kähler structure on a solvmanifold, and hence the symplectic structure satisfies the Hard Lefschetz Condition;

(e) the manifold associated to $\mathfrak{g}_{4.1} = (0,\, 0,\, 12,\, 13)$ endowed with the left-invariant symplectic structure $\omega := e^{14} + e^{23}$: the Lie algebra is nilpotent, and hence no left-invariant symplectic structure on such manifold satisfies the Hard Lefschetz Condition.

We compute the symplectic cohomologies of the above manifolds endowed with the indicated left-invariant symplectic structures. Note that, in case of the nilmanifolds *(a)*, *(b)*, and *(e)*, by Theorem 6.8, see also [40, Theorem 3], it suffices to consider the left-invariant forms. On the other hand, since the symplectic structures on the manifolds *(c)* and *(d)* satisfy the Hard Lefschetz Condition, we know that the Bott-Chern, Aeppli, and de Rham cohomologies are all isomorphic.

In Table 1 we list the harmonic representatives with respect to the left-invariant metric $g := \sum_{j=1}^{4} e^j \odot e^j$, and in Table 2 we summarize the dimensions of the symplectic cohomologies and the non-Kählerianity degrees $\left\{\Delta^k\right\}_{k\in\{1,2,3\}}$.

$H^\bullet_\sharp$		$H^1_\sharp$	$H^2_\sharp$	$H^3_\sharp$
$4\,\mathfrak{g}_1$	$H^\bullet_{BC}$	$\mathbb{R}\left\langle e^1, e^2, e^3, e^4\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{123}, e^{124}, e^{134}, e^{234}\right\rangle$
	$H^\bullet_A$	$\mathbb{R}\left\langle e^1, e^2, e^3, e^4\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{123}, e^{124}, e^{134}, e^{234}\right\rangle$
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	$H^\bullet_{BC}$	$\mathbb{R}\left\langle e^1, e^2, e^3\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{13}, e^{23}, e^{24}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{123}, e^{134}, e^{234}\right\rangle$
	$H^\bullet_A$	$\mathbb{R}\left\langle e^1, e^2, e^4\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{13}, e^{14}, e^{24}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{124}, e^{134}, e^{234}\right\rangle$
$\mathfrak{g}^{-1}_{3.4} \oplus \mathfrak{g}_1$	$H^\bullet_{BC}$	$\mathbb{R}\left\langle e^1, e^2\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{134}, e^{234}\right\rangle$
	$H^\bullet_A$	$\mathbb{R}\left\langle e^1, e^2\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{134}, e^{234}\right\rangle$
$\mathfrak{g}^{0}_{3.5} \oplus \mathfrak{g}_1$	$H^\bullet_{BC}$	$\mathbb{R}\left\langle e^1, e^2\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{134}, e^{234}\right\rangle$
	$H^\bullet_A$	$\mathbb{R}\left\langle e^1, e^2\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{134}, e^{234}\right\rangle$
$\mathfrak{g}_{4.1}$	$H^\bullet_{BC}$	$\mathbb{R}\left\langle e^1, e^2\right\rangle$	$\mathbb{R}\left\langle e^{12}, e^{13}, e^{14}, e^{23}\right\rangle$	$\mathbb{R}\left\langle e^{123}, e^{124}\right\rangle$
	$H^\bullet_A$	$\mathbb{R}\left\langle e^3, e^4\right\rangle$	$\mathbb{R}\left\langle e^{14}, e^{23}, e^{24}, e^{34}\right\rangle$	$\mathbb{R}\left\langle e^{134}, e^{234}\right\rangle$

Table 1: The symplectic Bott-Chern and Aeppli cohomologies for 4-dimensional solvmanifolds.

$\dim_{\mathbb{R}} H^\bullet_\sharp$		$4\,\mathfrak{g}_1$ (0, 0, 0, 0)	$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ (0, 0, 0, 23)	$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_1$ (0, 0, −23, 24)	$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1$ (0, 0, −24, 23)	$\mathfrak{g}_{4.1}$ (0, 0, 12, 13)
k=1	$\dim_{\mathbb{R}} H^1_{dR}$	4	3	2	2	2
	$\dim_{\mathbb{R}} H^1_{BC}$	4	3	2	2	2
	$\dim_{\mathbb{R}} H^1_{A}$	4	3	2	2	2
	Δ^1	0	0	0	0	0
k=2	$\dim_{\mathbb{R}} H^2_{dR}$	6	4	2	2	2
	$\dim_{\mathbb{R}} H^2_{BC}$	6	5	2	2	4
	$\dim_{\mathbb{R}} H^2_{A}$	6	5	2	2	4
	Δ^2	0	2	0	0	4
k=3	$\dim_{\mathbb{R}} H^3_{dR}$	4	3	2	2	2
	$\dim_{\mathbb{R}} H^3_{BC}$	4	3	2	2	2
	$\dim_{\mathbb{R}} H^3_{A}$	4	3	2	2	2
	Δ^3	0	0	0	0	0
HLC?		✓	×	✓	✓	×

Table 2: Summary of the dimensions of symplectic Bott-Chern and Aeppli cohomologies for 4-dimensional solvmanifolds.

4.2. 6-dimensional nilmanifolds

The 6-dimensional nilmanifolds can be classified in terms of their Lie algebra, up to isomorphisms, in 34 classes, according to V. V. Morozov's classification, [46], see also [41].

As regards the complex geometry of 6-dimensional nilmanifolds, S. Salamon proved in [50] that just 18 of these 34 classes admit a complex structure. A complete classification, up to equivalence, of the left-invariant complex structures on 6-dimensional nilmanifolds follows from the works by several authors, and was completed in [20]. In [4, 39], the Bott-Chern cohomology for each of such structures is computed. In view of [6], such computations provide a measure of the non-Kählerianity of 6-dimensional nilmanifolds, but for nilmanifolds associated to the Lie algebra $\mathfrak{h}_7 = (0, 0, 0, 12, 13, 23)$ possibly.

As regards the symplectic geometry of 6-dimensional nilmanifolds, M. Goze and Y. Khakimdjanov proved that 26 of the 34 classes of 6-dimensional Lie algebras admit a symplectic structure, [28].

Finally, G. R. Cavalcanti and M. Gualtieri proved in [19] that every 6-dimensional nilmanifolds admit a generalized-complex structure.

By applying Theorem 3.2, see also [40, Theorem 3], one can compute the symplectic cohomologies of the 6-dimensional nilmanifolds endowed with a

left-invariant symplectic structure. The results of the computations, which we performed with the aid of Sage [49], are summarized in Table 3.

5. Twisted symplectic cohomologies and twisted Hard Lefschetz Condition

In this section, we study twisted symplectic cohomologies, and in particular twisted Hard Lefschetz Condition and $D_\phi D_\phi^\Lambda$-Lemma.

Let X be be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. For a real or complex vector space V, consider a trivial vector bundle $E_\phi = X \times V$ with a connection form $\phi \in \wedge^1 X \times \mathrm{End}(V)$.

5.1. Twisted symplectic cohomologies

Define the operator

$$D_\phi := \mathrm{d} + \phi$$

on the space $\wedge^\bullet (X; E_\phi)$ of the differential forms with values in the vector bundle E_ϕ.

Define the operators

$$*_\omega \colon \wedge^\bullet (X; E_\phi) \to \wedge^{2n-\bullet} (X; E_\phi),$$

and

$$\begin{aligned} L &\colon \wedge^\bullet (X; E_\phi) \to \wedge^{\bullet+2} (X; E_\phi), \\ \Lambda &\colon \wedge^\bullet (X; E_\phi) \to \wedge^{\bullet-2} (X; E_\phi), \\ H &\colon \wedge^\bullet (X; E_\phi) \to \wedge^\bullet (X; E_\phi), \end{aligned}$$

as the natural extensions of the operators in Section 2.1.

Define

$$D_\phi^\Lambda := [D_\phi, \Lambda].$$

By the same way as in [14, Theorem 2.2.1], (see also [16, Proposition 5.1],) on $\wedge^k (X; E_\phi)$, we have

$$D_\phi^\Lambda \lfloor_{\wedge^k (X; E_\phi)} = (-1)^{k+1} \star_\omega D_\phi \star_\omega .$$

We suppose now that ϕ is flat, i.e., $\mathrm{d}\phi + \phi \wedge \phi = 0$. Then the pair $(\wedge^\bullet (X; E_\phi), D_\phi)$ is a differential graded module over the differential graded

			$\dim_{\mathbb{R}} H^\bullet_\tau$			$k=1$				$k=2$				$k=3$			
	[12]	[19]		ω	$\omega=12+34+56$	$\dim_{\mathbb{R}} H^1_{dR}$	$\dim_{\mathbb{R}} H^1_{BC}$	$\dim_{\mathbb{R}} H^1_A$	Δ^1	$\dim_{\mathbb{R}} H^2_{dR}$	$\dim_{\mathbb{R}} H^2_{BC}$	$\dim_{\mathbb{R}} H^2_A$	Δ^2	$\dim_{\mathbb{R}} H^3_{dR}$	$\dim_{\mathbb{R}} H^3_{BC}$	$\dim_{\mathbb{R}} H^3_A$	Δ^3
$\mathfrak{g}_{6.N2}$	30	1	$(0,0,12,13,14,15)$	$16+34-25$	$(0,15,16,13,14,0)$	2	2	2	0	3	5	5	4	4	6	6	4
$\mathfrak{g}_{6.N19}$	31	3	$(0,0,12,13,14,23+15)$	$16+24+34-25$	$(0,63+15,16,13-16,14,0)$	2	2	2	0	3	5	5	4	4	6	6	4
$\mathfrak{g}_{6.N11}$	27	4	$(0,0,12,13,23,14)$	$15+24+34-26$	$(0,63,16,13-16,14,0)$	2	2	2	0	4	6	6	4	6	6	6	0
$\mathfrak{g}^1_{6.N18}$	29	5	$(0,0,12,13,23,14-25)$	$15+24-35+16$	$(35,35+13,0,15,0,54+23)$	2	2	2	0	4	6	6	4	6	6	6	0
$\mathfrak{g}^{-1}_{6.N18}$	28	6	$(0,0,12,13,23,14+25)$	$15+24+35+16$	$(53,31-35,0,51,0,54+32)$	2	2	2	0	4	6	6	4	6	6	6	0
$\mathfrak{g}_{6.N20}$	32	8	$(0,0,12,13,14+23,24+15)$	$16+2\times34-25$	$(0,64+15,2\times16,\frac{1}{2}\times13,14+\frac{1}{2}\times63,0)$	2	2	2	0	3	5	5	4	4	6	6	4
$\mathfrak{g}_{6.N6}$	13	10	$(0,0,0,12,13,14+23)$	$16-2\times34-25$	$(0,13+\frac{1}{2}\times64,16,0,\frac{1}{2}\times14,0)$	3	3	3	0	6	9	9	6	8	11	11	6
$\mathfrak{g}_{6.N7}$	14	11	$(0,0,0,12,13,24)$	$26+14+35$	$(0,14,0,31,0,35)$	3	3	3	0	6	9	9	6	8	9	9	2
$\mathfrak{g}_{6.N1}$	12	12	$(0,0,0,12,13,14)$	$16+24+35$	$(0,14,0,13,0,15)$	3	3	3	0	6	9	9	6	8	10	10	4
$\mathfrak{g}_{6.N3}$	11	13	$(0,0,0,12,13,23)$	$15+24+36$	$(0,15,0,13,0,35)$	3	3	3	0	8	11	11	6	12	13	13	2
$\mathfrak{g}_{6.N17}$	25	14	$(0,0,0,12,14,15+23)$	$13+26-45$	$(0,0,0,15+32,16,13)$	3	3	3	0	5	6	6	2	6	7	7	2
$\mathfrak{g}_{6.N15}$	24	15	$(0,0,0,12,14,15+23+24)$	$13+26-45$	$(0,0,0,15+32+36,16,13)$	3	3	3	0	5	6	6	2	6	7	7	2
$\mathfrak{g}_{5.6}\oplus\mathfrak{g}_1$	17	16	$(0,0,0,12,14,15+24)$	$13+26-45$	$(0,0,0,15+36,16,13)$	3	3	3	0	5	6	6	2	6	7	7	2
$\mathfrak{g}_{5.2}\oplus\mathfrak{g}_1$	15	17	$(0,0,0,12,14,15)$	$13+26-45$	$(0,0,0,15,16,13)$	3	3	3	0	5	6	6	2	6	7	7	2
$\mathfrak{g}_{6.N9}$	19	19	$(0,0,0,12,14,13+42)$	$15+26+34$	$(0,16,0,15+63,0,13)$	3	3	3	0	5	8	8	6	6	10	10	8
$\mathfrak{g}_{6.N8}$	18	20	$(0,0,0,12,14,23+24)$	$16-34+25$	$(0,54+53,15,0,0,13)$	3	3	3	0	5	8	8	6	6	10	10	8
$\mathfrak{g}_{6.N16}$	26	23	$(0,0,0,12,14-23,15+34)$	$16+35+24$	$(0,14+36,0,16+35,0,15)$	3	3	3	0	4	7	7	6	4	7	7	6
$\mathfrak{g}_{6.N10}$	20	24	$(0,0,0,12,14+23,13+42)$	$15+2\times26+34$	$(0,16+\frac{1}{2}\times35,0,15+\frac{1}{2}\times63,0,\frac{1}{2}\times13)$	3	3	3	0	5	8	8	6	6	10	10	8
$\mathfrak{g}_{4.1}\oplus2\mathfrak{g}_1$	9	26	$(0,0,0,0,12,15)$	$16+25+34$	$(0,14,0,13,0,0)$	4	4	4	0	7	9	9	4	8	12	12	8
$\mathfrak{g}_{5.5}\oplus\mathfrak{g}_1$	8	27	$(0,0,0,0,12,14+25)$	$13+26+45$	$(0,0,0,15+36,0,13)$	4	4	4	0	7	9	9	4	8	12	12	8
$\mathfrak{g}_{6.N.4}$	6	28	$(0,0,0,0,12,14+23)$	$13+26+45$	$(0,0,0,15+32,0,13)$	4	4	4	0	8	10	10	4	10	12	12	4
$2\mathfrak{g}_{3.1}$	5	29	$(0,0,0,0,12,34)$	$15+36+24$	$(0,15,0,36,0,0)$	4	4	4	0	8	10	10	4	10	11	11	2
$\mathfrak{g}_{5.1}\oplus\mathfrak{g}_1$	4	30	$(0,0,0,0,12,13)$	$16+25+34$	$(0,15,0,13,0,0)$	4	4	4	0	9	11	11	4	12	14	14	4
$\mathfrak{g}_{6.N.5}$	7	31	$(0,0,0,0,13+42,14+23)$	$16+25+34$	$(0,16+35,0,15+63,0,0)$	4	4	4	0	8	10	10	4	10	11	11	2
$\mathfrak{g}_{3.1}\oplus3\mathfrak{g}_1$	2	33	$(0,0,0,0,0,12)$	$16+23+45$	$(0,13,0,0,0,0)$	5	5	5	0	11	12	12	2	14	16	16	4
$6\mathfrak{g}_1$	1	34	$(0,0,0,0,0,0)$	$12+34+56$	$(0,0,0,0,0,0)$	6	6	6	0	15	15	15	0	20	20	20	0

Table 3: Summary of the dimensions of the symplectic Bott-Chern and Aeppli cohomologies for 6-dimensional nilmanifolds.

algebra $\wedge^\bullet X$. Moreover we have

$$\left(D_\phi^\Lambda\right)^2 = 0.$$

Now we define the *twisted symplectic cohomologies*

$$H^\bullet_{dR}(X;E_\phi) := \frac{\ker D_\phi}{\operatorname{im} D_\phi} \quad \text{and} \quad H^\bullet_{D^\Lambda_\phi}(X;E_\phi) := \frac{\ker D^\Lambda_\phi}{\operatorname{im} D^\Lambda_\phi},$$

and

$$H^\bullet_{BC}(X;E_\phi) := \frac{\ker D_\phi \cap \ker D^\Lambda_\phi}{\operatorname{im} D_\phi D^\Lambda_\phi} \quad \text{and} \quad H^\bullet_{A}(X;E_\phi) := \frac{\ker D_\phi D^\Lambda_\phi}{\operatorname{im} D_\phi + \operatorname{im} D^\Lambda_\phi}.$$

By the same way as in [60, Section 1], (see also [16, Proposition 5.2],) we have the relations

$$\begin{array}{lll} [D_\phi, L] = 0, & \left[D^\Lambda_\phi, L\right] = -D_\phi, & L = -\star_\omega \Lambda \star_\omega, \\ \left[D^\Lambda_\phi, \Lambda\right] = 0, & [D_\phi, \Lambda] = D^\Lambda_\phi, & \Lambda = -\star_\omega L \star_\omega \\ [\Lambda, L] = H, & [L, H] = 2L, & [\Lambda, H] = -2\Lambda. \end{array}$$

By $D^\Lambda_\phi = (-1)^{k+1} *_\omega D_\phi \star_\omega$, we get that the operator $\star_\omega \colon \wedge^\bullet(X;E_\phi) \to \wedge^{2n-\bullet}(X;E_\phi)$ induces the isomorphism

$$H^\bullet_{dR}(X;E_\phi) \xrightarrow{\simeq} H^{2n-\bullet}_{D^\Lambda_\phi}(X;E_\phi).$$

Take an almost-complex structure J being compatible with ω, that is, $g := \omega(\cdot, J\cdot)$ is a J-Hermitian metric with associated fundamental form ω. Recall that there is a canonical way to construct J, see, e.g., [15, Proposition 12.6]. Consider the Hodge-$*$-operator $*_g \colon \wedge^\bullet(X;E_\phi) \to \wedge^{2n-\bullet}\left(X;E^*_\phi\right)$ associated to g where E^*_ϕ is the dual of E_ϕ. Consider the 4^{th}-order self-adjoint differential operators

$$\begin{aligned} D_{D_\phi + D^\Lambda_\phi} := {} & \left(D_\phi D^\Lambda_\phi\right)\left(D_\phi D^\Lambda_\phi\right)^* + \left(D_\phi D^\Lambda_\phi\right)^*\left(D_\phi D^\Lambda_\phi\right) + \left(D^*_\phi D^\Lambda_\phi\right)\left(D^*_\phi D^\Lambda_\phi\right)^* \\ & + \left(D^*_\phi D^\Lambda_\phi\right)^*\left(D^*_\phi D^\Lambda_\phi\right) + D^*_\phi D_\phi + \left(D^\Lambda_\phi\right)^* D^\Lambda_\phi \end{aligned}$$

and

$$\begin{aligned} D_{D_\phi D^\Lambda_\phi} := {} & \left(D_\phi D^\Lambda_\phi\right)\left(D_\phi D^\Lambda_\phi\right)^* + \left(D_\phi D^\Lambda_\phi\right)^*\left(D_\phi D^\Lambda_\phi\right) \\ & + \left(D_\phi\left(D^\Lambda_\phi\right)^*\right)\left(D_\phi\left(D^\Lambda_\phi\right)^*\right)^* + \left(D_\phi\left(D^\Lambda_\phi\right)^*\right)^*\left(D_\phi\left(D^\Lambda_\phi\right)^*\right) \\ & + D_\phi D^*_\phi + D^\Lambda_\phi\left(D^\Lambda_\phi\right)^*. \end{aligned}$$

Then, as similar to [55], these operators are elliptic, and hence induce orthogonal decompositions

$$\begin{aligned}\wedge^\bullet(X;E_\phi) &= \ker D_{D_\phi+D_\phi^\Lambda} \overset{\perp}{\oplus} D_\phi D_\phi^\Lambda \wedge^\bullet(X;E_\phi) \\ &\overset{\perp}{\oplus} \left(D_\phi^* \wedge^{\bullet+1}(X;E_\phi) + \left(D_\phi^\Lambda\right)^* \wedge^{\bullet-1}(X;E_\phi)\right)\end{aligned}$$

and

$$\begin{aligned}\wedge^\bullet(X;E_\phi) &= \ker D_{D_\phi D_\phi^\Lambda} \overset{\perp}{\oplus} \left(D_\phi \wedge^{\bullet-1}(X;E_\phi) + D_\phi^\Lambda \wedge^{\bullet+1}(X;E_\phi)\right) \\ &\overset{\perp}{\oplus} \left(D_\phi D_\phi^\Lambda\right)^* \wedge^\bullet(X;E_\phi).\end{aligned}$$

Therefore we have the isomorphisms

$$H^\bullet_{BC}(X;E_\phi) \simeq \ker D_{D_\phi+D_\phi^\Lambda} \quad \text{and} \quad H^\bullet_A(X;E_\phi) \simeq \ker D_{D_\phi D_\phi^\Lambda}.$$

By noting that $*_g D_{D_\phi+D_\phi^\Lambda} = D_{D_\phi D_\phi^\Lambda} *_g$, we have the isomorphism

$$*_g \colon H^\bullet_{BC}(X;E_\phi) \overset{\simeq}{\to} H^{2n-\bullet}_A(X;E_\phi^*).$$

5.2. Twisted Hard Lefschetz Condition

As in [14, page 102], we define $c \in \wedge^\bullet(X;E_\phi)$ to be *symplectically-harmonic* if $D_\phi c = D_\phi^\Lambda c = 0$.

We say that (X, ω) satisfies the *E_ϕ-twisted Hard Lefschetz Condition* if, for each $1 \leq k \leq n$, the linear map $[\omega^k] \smile \cdot \colon H^{n-k}_{dR}(X;E_\phi) \to H^{n+k}_{dR}(X;E_\phi)$ is an isomorphism.

By the above relations, by using the $\mathfrak{sl}_2(\mathbb{C})$-representation theory, we have the following result as an application of Theorem 1.4.

Theorem 5.1. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $E_\phi = X \times V$ be a trivial vector bundle on X with a connection form $\phi \in \wedge^1 X \times \mathrm{End}(V)$. Suppose that ϕ is flat.*

The following two conditions are equivalent:

(i) for each $1 \leq k \leq n$, the linear map

$$[\omega^k] \smile \cdot \colon H^{n-k}_{dR}(X;E_\phi) \to H^{n+k}_{dR}(X;E_\phi)$$

is surjective;

(ii) there is a symplectically-harmonic representative in each cohomology class in $H^{\bullet}_{dR}(X; E_\phi)$.

By the Poincaré duality for local systems and Corollary 1.6, we have the following result.

Theorem 5.2. *Let* X *be a* $2n$*-dimensional compact manifold endowed with a symplectic structure* ω*. Let* $E_\phi = X \times V$ *be a trivial vector bundle on* X *with a connection form* $\phi \in \wedge^1 X \times \mathrm{End}(V)$*. Suppose that* ϕ *is flat, and that the dual flat bundle of* E_ϕ *is isomorphic to* E_ϕ *itself.*

Then the following two conditions are equivalent:

(i) (X, ω) *satisfies the* E_ϕ*-twisted Hard Lefschetz Condition;*

(ii) there is a symplectically-harmonic representative in each cohomology class in $H^{\bullet}(X; E_\phi)$.

5.3. $D_\phi D^\Lambda_\phi$-Lemma

We say that (X, ω) satisfies the $D_\phi D^\Lambda_\phi$*-Lemma* if we have

$$\mathrm{im}\, D^\Lambda_\phi \cap \ker D_\phi = \mathrm{im}\, D_\phi D^\Lambda_\phi = \mathrm{im}\, D_\phi \cap \ker D^\Lambda_\phi,$$

namely, the natural maps $H^{\bullet}_{BC}(X; E_\phi) \to H^{\bullet}_{dR}(X; E_\phi)$ and $H^{\bullet}_{BC}(X; E_\phi) \to H^{\bullet}_{\mathrm{d}^\Lambda}(X; E_\phi)$ induced by the identity are isomorphisms.

In studying the relations between the twisted Hard Lefschetz Condition and the $D_\phi D^\Lambda_\phi$-Lemma, we need the following result.

Proposition 5.3. *Let* X *be a* $2n$*-dimensional compact manifold endowed with a symplectic structure* ω*. Let* $E_\phi = X \times V$ *be a trivial vector bundle on* X *with a connection form* $\phi \in \wedge^1 X \times \mathrm{End}(V)$*. Suppose that* ϕ *is flat.*

Suppose that

- *either* E_ϕ *is isomorphic to trivial flat bundle* E_0*,*
- *or* $H^0_{dR}(X; E_\phi) = \{0\}$.

Suppose also that (X, ω) *satisfies the* E_ϕ*-twisted Hard Lefschetz Condition.*

Then we have

$$\mathrm{im}\, D^\Lambda_\phi \cap \ker D_\phi = \mathrm{im}\, D_\phi \cap \ker D^\Lambda_\phi.$$

Proof. In the case $E_\phi \simeq E_0$, we have $(\wedge^\bullet(X;E_\phi), D_\phi) \simeq (\wedge^\bullet(X)\otimes\mathbb{C}^n, \mathrm{d})$. Hence we can prove as in the ordinary case, see [16, Proposition 5.4].

Suppose $H^0_{dR}(X;E_\phi) = \{0\}$. Let $\alpha \in \wedge^1(X;E_\phi)$ with $D_\phi D^\Lambda_\phi \alpha = 0$. Then, by the assumption, we have $D^\Lambda_\phi \alpha = 0$. Hence by Proposition 1.7, we have

$$\operatorname{im} D^\Lambda_\phi \cap \ker D_\phi = \operatorname{im} D_\phi \cap \operatorname{im} D^\Lambda_\phi.$$

By this equation and the relation $D^\Lambda_\phi\lfloor_{\wedge^k(X;E_\phi)} = (-1)^{k+1} \star_\omega D_\phi \star_\omega$, we have also

$$\operatorname{im} D_\phi \cap \ker D^\Lambda_\phi = \operatorname{im} D_\phi \cap \operatorname{im} D^\Lambda_\phi.$$

Hence the proposition follows. □

By this proposition and Proposition 1.8 and Proposition 1.9, we have the following result.

Corollary 5.4. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $E_\phi = X\times V$ be a trivial vector bundle on X with a connection form $\phi \in \wedge^1 X \times \mathrm{End}(V)$. Suppose that ϕ is flat. Suppose that the monodromy representation of E_ϕ is semi-simple.*

Consider the following two conditions:

(i) (X,ω) satisfies the E_ϕ-twisted Hard Lefschetz Condition;

(ii) (X,ω) satisfies the $D_\phi D^\Lambda_\phi$-Lemma.

Then the first condition implies the second one. Moreover if the dual flat bundle of E_ϕ is isomorphic to E_ϕ itself, then the two conditions are equivalent.

Arguing as in [55, Proposition 3.13], one has the following result.

Corollary 5.5. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $E_\phi = X\times V$ be a trivial vector bundle on X with a connection form $\phi \in \wedge^1 X \times \mathrm{End}(V)$. Suppose that ϕ is flat. Suppose that the monodromy representation of E_ϕ is semi-simple.*

Consider the following two conditions :

(i) (X,ω) satisfies the E_ϕ-twisted Hard Lefschetz Condition;

(ii) the map $H^\bullet_{BC}(X;E_\phi) \to H^\bullet_{dR}(X;E_\phi)$ is an isomorphism.

Then the first condition imply the second one. Moreover if the dual flat bundle of E_ϕ is isomorphic to E_ϕ itself, then the two conditions are equivalent.

The following result is a straightforward corollary.

Corollary 5.6. *Let X be a $2n$-dimensional compact manifold endowed with a symplectic structure ω. Let $E_\phi = X \times V$ be a trivial vector bundle on X with a connection form $\phi \in \wedge^1 X \times \mathrm{End}(V)$. Suppose that ϕ is flat. Suppose that the monodromy representation of E_ϕ is semi-simple. We assume that (X, ω) satisfies the E_ϕ-twisted Hard Lefschetz Condition.*

Then for each $k \in \mathbb{Z}$, we have $\dim_{\mathbb{R}} H^k_{BC}(X; E_\phi) = \dim_{\mathbb{R}} H^k_{dR}(X; E_\phi)$.

In [53], Simpson showed the following result.

Theorem 5.7 ([53, Lemma 2.6]). *Let (X, ω) be a compact Kähler manifold and E_ϕ a flat bundle over X whose monodromy representation is semi-simple. Then (X, ω) satisfies the E_ϕ-twisted Hard Lefschetz Condition.*

6. Twisted cohomologies on solvmanifolds

In this section, we study twisted symplectic cohomologies for special solvmanifolds.

Let G be a connected simply-connected solvable Lie group. Denote by $\mathfrak{g}$ its associated Lie algebra, and by $\rho\colon G \to \mathrm{GL}(V_\rho)$ a representation on a real or complex vector space V_ρ.

We consider the cochain complex $\wedge^\bullet \mathfrak{g}^*$ with the derivation d which is the dual to the Lie bracket of $\mathfrak{g}$. Then the pair

$$(\wedge^\bullet \mathfrak{g}^* \otimes V_\rho,\, \mathrm{d}_\rho := \mathrm{d} + \rho_*)$$

is a differential graded module over the differential graded algebra $\wedge^\bullet \mathfrak{g}^*$. Here $\rho_* \in \mathfrak{g}^* \otimes \mathfrak{gl}(V_\rho)$ is the derivation of ρ. We can consider the cochain complex $(\wedge^\bullet \mathfrak{g}^* \otimes V_\rho,\, \mathrm{d}_\rho)$ given by the twisted G-invariant differential forms on G.

Suppose that G has a lattice Γ. Since $\pi_1(\Gamma\backslash G) = \Gamma$, we have a flat vector bundle E_{ρ_*} with flat connection D_{ρ_*} on $\Gamma\backslash G$ whose monodromy is $\rho\lfloor_\Gamma$. We can regard E_{ρ_*} as the flat bundle $\Gamma\backslash G \times V_\rho$ with the connection form ρ_*, and we have the inclusion

$$\iota\colon\ \wedge^\bullet \mathfrak{g}^* \otimes V_\rho \hookrightarrow \wedge^\bullet (\Gamma\backslash G\,; E_{\rho_*})$$

of cochain complexes. Consider the natural extension

$$\mu\colon\ \wedge^\bullet (\Gamma\backslash G\,; E_{\rho_*}) \to \wedge^\bullet \mathfrak{g}^* \otimes V_\rho$$

of the F. A. Belgun symmetrization map, [9, Theorem 7]. Then this map also satisfies

$$D_{\rho_*} \circ \mu = \mu \circ D_{\rho_*} \quad \text{and} \quad \mu \circ \iota = \mathrm{id}\,.$$

6.1. Twisted symplectic cohomologies of special solvmanifolds

In some special cases, the inclusion $\iota\colon \wedge^\bullet \mathfrak{g}^* \otimes V_\rho \hookrightarrow \wedge^\bullet (\Gamma\backslash G\,; E_{\rho_*})$ is a quasi-isomorphism.

Theorem 6.1 ([34, Corollary 4.2], [47, Theorem 8.2, Corollary 8.1]). *Let G be a connected simply-connected solvable Lie group with a lattice Γ.*

If:

- **(H):** *either the representation $\rho \oplus \mathrm{Ad}$ is $\mathbb{R}$-triangular, [34],*
- **(M):** *or the two images $(\rho \oplus \mathrm{Ad})(G)$ and $(\rho \oplus \mathrm{Ad})(\Gamma)$ have the same Zariski-closure in $GL(V_\rho) \times \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$,*

then the inclusion $\iota\colon \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho \hookrightarrow \wedge^\bullet (\Gamma\backslash G\,; E_{\rho_*})$ induces the isomorphism

$$H^\bullet_{dR}(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_{dR}(\Gamma\backslash G\,; E_{\rho_*}).$$

We suppose that $\Gamma\backslash G$ admits a G-left-invariant symplectic structure $\omega \in \wedge^2\mathfrak{g}^*$. Then the operators L, Λ, $\star_\omega$, and D^Λ_ϕ are defined on $\wedge^\bullet\mathfrak{g}^* \otimes V_\rho$. We consider the cohomologies $H^\bullet_{D_\rho}(\mathfrak{g}; V_\rho)$, $H^\bullet_{D^\Lambda_\rho}(\mathfrak{g}; V_\rho)$ and $H^\bullet_{BC}(\mathfrak{g}; V_\rho)$ of $\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho$.

We have the following result.

Theorem 6.2. *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant symplectic structure $\omega \in \wedge^2\mathfrak{g}^*$. We suppose that either condition (H) or condition (M) in Theorem 6.1 holds.*

Then the inclusion $\iota\colon \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho \subset \wedge^\bullet (\Gamma\backslash G\,; E_{\rho_*})$ induces the isomorphism

$$H^\bullet_{BC}(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_{BC}(\Gamma\backslash G\,; E_{\rho_*})\,.$$

Proof. By Theorem 6.1, we have that ι induces the isomorphism $H^\bullet_{D_\rho}(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_{D_\rho}(\Gamma\backslash G\,; E_{\rho_*})$. By using the symplectic-$\star$-operator

$$\star_\omega\colon \wedge^\bullet \mathfrak{g}^* \otimes V_\rho \to \wedge^{\dim X - \bullet}\mathfrak{g}^* \otimes V_\rho,$$

we have that ι induces the isomorphism $H^\bullet_{D^\Lambda_\rho}(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_{D^\Lambda_\rho}(\Gamma\backslash G\,; E_{\rho_*})$ as in the proof of Theorem 3.2.

Consider the F. A. Belgun symmetrization map $\mu\colon \wedge^\bullet(\Gamma\backslash G\,; E_{\rho_*}) \to \wedge^\bullet\mathfrak{g}^*\otimes V_\rho$ as above. Then, since ω is G-left-invariant, μ commutes with the operators L, Λ, $\star_\omega$, and D^Λ_ϕ. Hence we get that ι induces the isomorphism

$$H^\bullet_{BC}(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_{BC}(\Gamma\backslash G\,; E_{\rho_*})$$

from Corollary 1.3. □

Corollary 6.3. *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant symplectic structure $\omega\in\wedge^2\mathfrak{g}^*$. We suppose that either condition (H) or condition (M) in Theorem 6.1 holds.*

Then the inclusion $\iota\colon \wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\otimes V_\rho\subset\wedge^\bullet(\Gamma\backslash G\,; E_{\rho_*})$ induces the isomorphism

$$H^\bullet_A(\mathfrak{g}; V_\rho) \xrightarrow{\simeq} H^\bullet_A(\Gamma\backslash G\,; E_{\rho_*}).$$

Proof. By the F. A. Belgun symmetrization map $\mu : \wedge^\bullet(\Gamma\backslash G\,; E_{\rho_*})\to\wedge^\bullet\mathfrak{g}^*\otimes V_\rho$ as above, the induced map $\iota\colon H^\bullet_A(\mathfrak{g}; V_\rho)\to H^\bullet_A(\Gamma\backslash G\,; E_{\rho_*})$ is injective. Hence it is sufficient to show that there exists an isomorphism $H^\bullet_A(\mathfrak{g}; V_\rho)\xrightarrow{\simeq} H^\bullet_A(\Gamma\backslash G\,; E_{\rho_*})$.

Let J be a G-left-invariant ω-compatible almost-complex structure on X, (see, e.g., [15, Proposition 12.6],) and consider the G-left-invariant J-Hermitian metric $g := \omega(\cdot, J\cdot)$. Consider the Hodge-$*$-operator $*_g\colon \wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\otimes V_\rho\to\wedge^{2n-\bullet}\mathfrak{g}^*_{\mathbb{C}}\otimes V_{\check\rho}$ on left-invariant forms where $\check\rho$ is the dual representation of ρ. Like the duality between Bott-Chern and Aeppli cohomologies of compact symplectic manifolds, we have the isomorphism $H^\bullet_A(\mathfrak{g}; V_\rho)\xrightarrow{\simeq} H^{2n-\bullet}_{BC}(\mathfrak{g}; V_{\check\rho})$ induced by $*_g$. If ρ satisfies either condition *(H)* or condition *(M)* in Theorem 6.1, the dual representation $\check\rho$ also does. Hence by Theorem 6.2, we have the isomorphism

$$H^\bullet_{BC}(\mathfrak{g}; V_{\check\rho}) \xrightarrow{\simeq} H^\bullet_{BC}\left(\Gamma\backslash G\,; E^*_{\rho_*}\right).$$

By the duality between Bott-Chern and Aeppli cohomologies on $\Gamma\backslash G$, we have $H^\bullet_A(\Gamma\backslash G\,; E_{\rho_*})\xrightarrow{\simeq} H^{2n-\bullet}_{BC}(\Gamma\backslash G\,; E^*_{\rho_*})$ and hence we have $H^\bullet_A(\mathfrak{g}; V_\rho)\xrightarrow{\simeq} H^\bullet_A(\Gamma\backslash G\,; E_{\rho_*})$. □

6.2. Twisted minimal model of solvmanifolds

Consider a connected simply-connected solvable Lie group G with a lattice Γ. Consider the adjoint action $\mathrm{ad}\colon \mathfrak{g}\ni X\mapsto \mathrm{ad}_X := [X,\cdot]\in\mathrm{Der}(\mathfrak{g})$, and, for any $X\in\mathfrak{g}$, consider its unique Jordan decomposition $\mathrm{ad}_X = (\mathrm{ad}_X)_{\mathrm{s}} +$

$(\mathrm{ad}_X)_{\mathrm{n}}$, where $(\mathrm{ad}_X)_{\mathrm{s}} \in \mathfrak{gl}(\mathfrak{g})$ is semi-simple and $(\mathrm{ad}_X)_{\mathrm{n}} \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent, see, e.g., [24, II.1.10].

Denote by $\mathfrak{n}$ the nilradical of $\mathfrak{g}$, and let V be an $\mathbb{R}$-vector sub-space of $\mathfrak{g}$ such that *(i)* $\mathfrak{g} = V \oplus \mathfrak{n}$ in the category of $\mathbb{R}$-vector spaces, and, *(ii)* for any $A, B \in V$, it holds that $(\mathrm{ad}_A)_{\mathrm{s}}(B) = 0$, see, e.g., [24, Proposition III.1.1]. Hence, define the map

$$\mathrm{ad}_{\mathrm{s}} \colon \mathfrak{g} = V \oplus \mathfrak{n} \ni (A, X) \mapsto (\mathrm{ad}_{\mathrm{s}})_{A+X} := (\mathrm{ad}_A)_{\mathrm{s}} \in \mathrm{Der}(\mathfrak{g}).$$

Moreover, one has that *(iii)* $[\mathrm{ad}_{\mathrm{s}}(\mathfrak{g}), \mathrm{ad}_{\mathrm{s}}(\mathfrak{g})] = \{0\}$, and *(iv)* $\mathrm{ad}_{\mathrm{s}} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is $\mathbb{R}$-linear, see, e.g., [24, Proposition III.1.1].

The map $\mathrm{ad}_{\mathrm{s}} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is actually a representation of $\mathfrak{g}$ such that its image $\mathrm{ad}_{\mathrm{s}}(\mathfrak{g})$ is Abelian and consists of semi-simple elements. Hence denote by $\mathrm{Ad}_{\mathrm{s}} \colon G \to \mathrm{Aut}(\mathfrak{g})$ the unique representation which lifts $\mathrm{ad}_{\mathrm{s}} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, see, e.g., [59, Theorem 3.27], and by $\mathrm{Ad}_{\mathrm{s}} \colon G \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ its natural $\mathbb{C}$-linear extension.

Let T be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. Let

$$\mathcal{C} := \{\beta \circ \mathrm{Ad}_{\mathrm{s}} \in \mathrm{Hom}(G; \mathbb{C}^*) : \beta \in \mathrm{Char}(T)\}.$$

For $\alpha \in \mathcal{C}$, consider $\alpha \colon G \to \mathrm{GL}(V_\alpha) \simeq \mathbb{C}^*$. We consider the differential graded algebra

$$\bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha$$

with the T-action. Denote by

$$\left(\bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \right)^T$$

the sub-differential graded algebra which consists of T-invariant elements.

Since $\mathrm{Ad}_{\mathrm{s}}(G) \subseteq \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ consists of simultaneously diagonalizable elements, let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ with respect to which $\mathrm{Ad}_{\mathrm{s}} = \mathrm{diag}(\alpha_1, \dots, \alpha_n) : G \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ for some characters $\alpha_1 \in \mathrm{Hom}(G; \mathbb{C}^*)$, …, $\alpha_n \in \mathrm{Hom}(G; \mathbb{C}^*)$, and let $\{x_1, \dots, x_n\}$ be its dual basis of $\mathfrak{g}^*_{\mathbb{C}}$. Then we have

$$\left(\bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \right)^T = \wedge^\bullet \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle$$

where $\{v_{\alpha_j}\}$ is a basis of V_{α_j} for each $j \in \{1, \dots, n\}$. In [36, Section 5], the second author showed that we have a differential graded algebra isomorphism

$$\wedge^\bullet \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle \simeq \wedge^\bullet \mathfrak{u}^*$$

where $\mathfrak{u}$ is the Lie algebra of the unipotent hull of G, which is the unipotent algebraic group determined by G. Since $\mathfrak{u}$ is nilpotent, $\wedge^\bullet \mathfrak{u}^*$ is a minimal differential graded algebra.

We also consider the Zariski-closure S of $\mathrm{Ad}_s(\Gamma)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. For $\beta \in \mathrm{Char}(S)$, we denote by E_β the rank one flat bundle with the monodromy representation $\beta \circ \mathrm{Ad}_s \lfloor_\Gamma$. Consider the differential algebra

$$\bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta) \, .$$

Differential graded algebras of this kind were considered by Hain in [31] for rational homotopy on non-nilpotent spaces. In [36], the second author constructed the minimal model of $\bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta)$. By $S \subseteq T$, for $\beta \in \mathrm{Char}(S)$, we have characters $\alpha \in \mathcal{C}$ such that $E_{\alpha_*} = E_\beta$. (Since $S \neq T$ in general, such α is not unique.) Hence we have the map

$$\bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \to \bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta) \, .$$

Now we consider the map $\iota \colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta)$ given by the composition

$$\wedge^\bullet \mathfrak{u}^* \simeq \left(\bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \right)^T \subseteq \bigoplus_{\alpha \in \mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \to \bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta) \, .$$

The second author proved the following result in [36].

Theorem 6.4 ([36, Theorem 1.1, Theorem 5.4]). *Let G be a connected simply-connected solvable Lie group with a lattice Γ. The above map $\iota \colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta)$ induces a cohomology isomorphism. Hence $\wedge^\bullet \mathfrak{u}^*$ is the minimal model of $\bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^\bullet (\Gamma \backslash G \, ; E_\beta)$.*

Note 6.5. We consider the maps $\iota\colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$ given by the composition

$$\wedge^\bullet \mathfrak{u}^* \simeq \left(\bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha\right)^T \subseteq \bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}^* \otimes V_\alpha \to \bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\,.$$

as above. Let $A^\bullet_\Gamma = \iota^{-1}(\wedge^\bullet \Gamma\backslash G \otimes \mathbb{C})$ where we consider $\wedge^\bullet \Gamma\backslash G \otimes \mathbb{C} = \wedge^\bullet (\Gamma\backslash G\,; E_{1_S})$ for the trivial character 1_S. Then the map $\iota\colon A^\bullet_\Gamma \subset \wedge^\bullet \Gamma\backslash G \otimes \mathbb{C}$ induces a cohomology isomorphism. Set

$$\mathcal{C}_\Gamma := \{\beta\circ \mathrm{Ad}_{\mathrm{s}} \in \mathrm{Hom}\,(G;\mathbb{C}^*) : \beta\in \mathrm{Char}(T), (\beta\circ \mathrm{Ad}_{\mathrm{s}})\lfloor_\Gamma = 1\}\,.$$

As [36, Corollary 7.6], we have

$$A^\bullet_\Gamma = \left(\bigoplus_{\alpha\in\mathcal{C}_\Gamma} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha\right)^T. \tag{3}$$

By using the basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}^*_{\mathbb{C}}$ such that

$$\left(\bigoplus_{\alpha\in\mathcal{C}_\Gamma} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha\right)^T \cong \wedge^\bullet \langle x_1\otimes v_{\alpha_1},\dots,x_n\otimes v_{\alpha_n}\rangle$$

as above, since we have $\wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha = \alpha\cdot \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}}$ in $\wedge^\bullet \Gamma\backslash G\otimes\mathbb{C}$ for $\alpha\in\mathcal{A}_\Gamma$, the differential graded algebra $A^\bullet_\Gamma$ can be written as

$$A^p_\Gamma = \mathbb{C}\big\langle \alpha_{i_1\cdots i_p}\, x_{i_1}\wedge\cdots\wedge x_{i_p} \,\big|\, 1\le i_1<i_2<\cdots<i_p\le n \text{ such that } \alpha_{i_1\cdots i_p}\lfloor_\Gamma = 1\big\rangle, \tag{4}$$

where we have shortened $\alpha_{i_1\cdots i_p} := \alpha_{i_1}\cdot\cdots\cdot\alpha_{i_p} \in \mathrm{Hom}\,(G;\mathbb{C}^*)$.

We suppose now that $\Gamma\backslash G$ admits a G-left-invariant symplectic structure $\omega\in\wedge^2\mathfrak{g}$. We assume that ω is T-invariant (equivalently $\omega\in A^2_\Gamma$).

Then the operators L and Λ on $\bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}\otimes V_\alpha$ commute with the T-action. Hence L and Λ and the differential $D^\Lambda = D\Lambda - \Lambda D$ are defined on $\wedge^\bullet \mathfrak{u}^* \simeq \big(\bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}^*\otimes V_\alpha\big)^T$, where D is the differential on the differential graded algebra $\wedge^\bullet \mathfrak{u}^*$. Now we can regard ω as a symplectic form on the Lie algebra $\mathfrak{u}$. The symplectic-$\star$-operator $\star_\omega$ is defined on $\wedge^\bullet \mathfrak{u}^*$. We consider the

cohomologies

$$H_D^\bullet\left(\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)\right),$$
$$H_{D^\wedge}^\bullet\left(\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)\right),$$
$$\text{and}\quad H_{BC}^\bullet\left(\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)\right)$$

of $\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)$ and the cohomologies $H_D^\bullet(\mathfrak{u})$, $H_{D^\wedge}^\bullet(\mathfrak{u})$, and $H_{BC}^\bullet(\mathfrak{u})$ of $\wedge^\bullet\mathfrak{u}^*$.

Theorem 6.6. *Let G be a connected simply-connected solvable Lie group with a lattice Γ and endowed with a G-left-invariant T-invariant symplectic structure ω.*

The above map $\iota\colon\ \wedge^\bullet\mathfrak{u}^*\to\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)$ induces the Bott-Chern cohomology isomorphism

$$H_{BC}^\bullet(\mathfrak{u})\xrightarrow{\simeq}H_{BC}^\bullet\left(\bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)\right).$$

Proof. Set

$$\mathcal{A}:=\left\{\alpha_{i_1\dots i_p}\in\mathrm{Hom}(G;\mathbb{C}^*):1\le i_1\le\cdots\le i_p\le n\right\}$$

and

$$\mathcal{A}':=\{\beta\in\mathrm{Char}(S):\ \text{there exists }\alpha\in\mathcal{A}\text{ such that }E_\beta=E_{\alpha_*}\}.$$

We consider the projection

$$p\colon\ \bigoplus_{\beta\in\mathrm{Char}(S)}\wedge^\bullet(\Gamma\backslash G\,;E_\beta)\to\bigoplus_{\beta\in\mathcal{A}'}\wedge^\bullet(\Gamma\backslash G\,;E_\beta).$$

Let $\beta\in\mathcal{A}'$. Take $\alpha\in\mathcal{A}$ such that $E_\beta=E_{\alpha_*}$. Then we have the inclusion

$$\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\subseteq\wedge^\bullet(\Gamma\backslash G\,;E_\beta),$$

and we consider the F. A. Belgun symmetrization map, [9, Theorem 7],

$$\mu_\alpha \colon \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha.$$

We define the map

$$\Phi_\beta := \sum_{\alpha\in\mathcal{A} \text{ s.t. } E_\beta = E_{\alpha_*}} \mu_\alpha : \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \bigoplus_{\alpha\in\mathcal{A} \text{ s.t. } E_\beta = E_{\alpha_*}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha.$$

Then for distinct characters α and α' with $E_{\alpha_*} = E_\beta = E_{\alpha'_*}$, for the inclusion $\iota_\alpha \colon \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \to \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$, we have $\mu_{\alpha'} \circ \iota_\alpha = 0$ (see the proof of [36, Proposition 6.1]). Hence, for

$$\iota_\beta \colon \bigoplus_{\alpha\in\mathcal{A} \text{ s.t. } E_\beta = E_{\alpha_*}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \to \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\,,$$

we have $\Phi_\beta \circ i_\beta = \mathrm{id}$. We define the map

$$\Phi := \sum_{\beta\in\mathcal{A}'} \Phi_\beta \colon \bigoplus_{\beta\in\mathcal{A}'} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha.$$

Then for $\iota \colon \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \to \bigoplus_{\beta\in\mathcal{A}'} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$, we have $\Phi \circ \iota = \mathrm{id}$.

Since ω is G-left-invariant, the map Φ commutes with the operators L and Λ and the operator D^Λ. Since the T-action on $\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ is diagonalizable, we can take the direct sum

$$\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha = \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \right)^T \oplus D^\bullet$$

of cochain complexes, where $\left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha\right)^T$ is the sub-complex that consists of the elements of $\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g} \otimes V_\alpha$ fixed by the action of T and $D^\bullet$ is its complement for the action. Hence the projection

$$q \colon \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \to \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \right)^T$$

is a cochain complex map. Since ω is T-invariant, the map $q \colon \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \to \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha\right)^T$ commutes with the operators L and Λ and the operator D^Λ.

Since we have

$$\left(\bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha\right)^T = \wedge^\bullet \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle,$$

we have

$$\left(\bigoplus_{\alpha\in\mathcal{C}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha\right)^T = \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha\right)^T$$

and so we have

$$\wedge^\bullet \mathfrak{u}^* \simeq \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha\right)^T.$$

We consider the maps $\iota\colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$ given by the composition

$$\begin{aligned}\wedge^\bullet \mathfrak{u}^* \simeq \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha\right)^T &\to \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha \\ &\to \bigoplus_{\beta\in\mathcal{A}'} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\end{aligned}$$

and $\Psi\colon \bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \wedge^\bullet \mathfrak{u}^*$ given by the composition

$$\begin{aligned}\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \xrightarrow{p} \bigoplus_{\beta\in\mathcal{A}'} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \xrightarrow{\Phi} \bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha \\ \xrightarrow{q} \left(\bigoplus_{\alpha\in\mathcal{A}} \wedge^\bullet \mathfrak{g} \otimes V_\alpha\right)^T \simeq \wedge^\bullet \mathfrak{u}^*.\end{aligned}$$

Then by the above arguments, ι and Φ commute with the differentials D and D^Λ and satisfy $\Psi \circ \iota = \mathrm{id}$.

By Theorem 6.4 the injection $\iota\colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$ induces the isomorphism

$$H_D^\bullet(\mathfrak{u}) \xrightarrow{\simeq} H_D^\bullet \left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right).$$

By this isomorphism and the symplectic-$\star$-operator $\star_\omega$, the injection $\iota\colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$ induces the isomorphism

$$H^\bullet_{D^\wedge}(\mathfrak{u}) \xrightarrow{\simeq} H^\bullet_{D^\wedge}\left(\bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right).$$

Hence by using the map $\Psi\colon \bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \wedge^\bullet \mathfrak{u}^*$ as above, the theorem follows from Corollary 1.3. □

Corollary 6.7. *Let G be a connected simply-connected solvable Lie group with a lattice Γ and endowed with a G-left-invariant T-invariant symplectic structure ω.*

The above map $\iota\colon \wedge^\bullet \mathfrak{u}^* \to \bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)$ induces the Aeppli cohomology isomorphism

$$H^\bullet_A(\mathfrak{u}) \xrightarrow{\simeq} H^\bullet_A\left(\bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right).$$

Proof. By the map $\Psi\colon \bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta) \to \wedge^\bullet \mathfrak{u}^*$ constructed in the proof of Theorem 6.6, the induced map

$$H^\bullet_A(\mathfrak{u}) \to H^\bullet_A\left(\bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right)$$

is injective. Hence it is sufficient to show that there exists an isomorphism

$$H^\bullet_A(\mathfrak{u}) \cong H^\bullet_A\left(\bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right).$$

Since $\mathfrak{u}$ is a nilpotent Lie algebra and ω can be regard as a symplectic form on $\mathfrak{u}$, like the duality between Bott-Chern and Aeppli cohomologies of compact symplectic manifolds, we have the isomorphism $H^\bullet_A(\mathfrak{u}) \xrightarrow{\simeq} H^{2n-\bullet}_{BC}(\mathfrak{u})$ induced by Hodge-$*$-operator. By Theorem 6.6, we have an isomorphism $H^{2n-\bullet}_{BC}(\mathfrak{u}) \cong H^{2n-\bullet}_{BC}\left(\bigoplus_{\beta\in \mathrm{Char}(S)} \wedge^\bullet (\Gamma\backslash G\,; E_\beta)\right)$. Now for $\beta \in \mathrm{Char}(S)$, we

have $\beta^{-1} \in \mathrm{Char}(S)$ and hence by the duality between Bott-Chern and Aeppli cohomologies of $\Gamma\backslash G$, we have

$$\begin{aligned} H_{BC}^{2n-\bullet}\left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^{\bullet}(\Gamma\backslash G\,;E_{\beta})\right) &\cong H_{A}^{\bullet}\left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^{\bullet}\left(\Gamma\backslash G\,;E_{\beta}^{*}\right)\right) \\ &= H_{A}^{\bullet}\left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^{\bullet}\left(\Gamma\backslash G\,;E_{\beta^{-1}}\right)\right) \\ &= H_{A}^{\bullet}\left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^{\bullet}(\Gamma\backslash G\,;E_{\beta})\right). \end{aligned}$$

Hence we have

$$H_{A}^{\bullet}(\mathfrak{u}) \cong H_{A}^{\bullet}\left(\bigoplus_{\beta\in\mathrm{Char}(S)} \wedge^{\bullet}(\Gamma\backslash G\,;E_{\beta})\right).$$

□

We apply these results to untwisted symplectic cohomologies. Consider $A_{\Gamma}^{\bullet} = \iota^{-1}(\wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C})$ as in Note 6.5. Then $A_{\Gamma}^{\bullet}$ is a sub-complex of the bi-differential $\mathbb{Z}$-graded $\wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C}$ and by Theorem 6.6 and Corollary 6.7, the inclusion $A_{\Gamma}^{\bullet} \subset \wedge^{\bullet}\Gamma\backslash G \otimes \mathbb{C}$ induces isomorphisms

$$H_{BC}^{\bullet}(A_{\Gamma}^{\bullet}) \xrightarrow{\simeq} H_{BC}^{\bullet}(\Gamma\backslash G\,;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

and

$$H_{A}^{\bullet}(A_{\Gamma}^{\bullet}) \xrightarrow{\simeq} H_{A}^{\bullet}(\Gamma\backslash G\,;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Hence we have the result for symplectic Bott-Chern and Aeppli cohomologies on general solvmanifolds as we give in Introduction.

Theorem 6.8. *Let $\Gamma\backslash G$ be a $2n$-dimensional solvmanifold endowed with a G-left-invariant symplectic structure ω. Consider the sub-complex $A_{\Gamma}^{\bullet} \subseteq \wedge^{\bullet}\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ defined in* (3) *(or* (4)*) as in Note 6.5. Suppose that $\omega \in A_{\Gamma}^{2}$. Then, the inclusion $A_{\Gamma}^{\bullet} \hookrightarrow \wedge^{\bullet}\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ induces the isomorphisms*

$$H_{BC}^{\bullet}(A_{\Gamma}^{\bullet}) \xrightarrow{\simeq} H_{BC}^{\bullet}(\Gamma\backslash G\,;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

and

$$H_{A}^{\bullet}(A_{\Gamma}^{\bullet}) \xrightarrow{\simeq} H_{A}^{\bullet}(\Gamma\backslash G\,;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

6.3. Examples

As an explicit application of Theorem 6.8, we compute the symplectic cohomologies of the (non-completely-solvable) Nakamura manifold.

Example 6.9 (The complex parallelizable Nakamura manifold). Consider the Lie group

$$G := \mathbb{C} \ltimes_\phi \mathbb{C}^2 \quad \text{where} \quad \phi(z) := \begin{pmatrix} \mathrm{e}^z & 0 \\ 0 & \mathrm{e}^{-z} \end{pmatrix}.$$

There exist $a + \sqrt{-1}\, b \in \mathbb{C}$ and $c + \sqrt{-1}\, d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}\, b) + \mathbb{Z}(c + \sqrt{-1}\, d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}\, b)$ and $\phi(c + \sqrt{-1}\, d)$ are conjugate to elements of $\mathrm{SL}(4;\mathbb{Z})$, where we regard $\mathrm{SL}(2;\mathbb{C}) \subset \mathrm{SL}(4;\mathbb{R})$, see [33]. Hence there exists a lattice $\Gamma := \left(\mathbb{Z}\left(a + \sqrt{-1}\, b\right) + \mathbb{Z}\left(c + \sqrt{-1}\, d\right)\right) \ltimes_\phi \Gamma''$ of G such that Γ'' is a lattice of $\mathbb{C}^2$. Let $X := \Gamma\backslash G$ be the *complex parallelizable Nakamura manifold*, [48, §2].

For a coordinate set (z_1, z_2, z_3) of $\mathbb{C} \ltimes_\phi \mathbb{C}^2$, we have the basis $\left\{ \frac{\partial}{\partial z_1}, \mathrm{e}^{z_1} \frac{\partial}{\partial z_2}, \mathrm{e}^{-z_1} \frac{\partial}{\partial z_3} \right\}$ of the Lie algebra $\mathfrak{g}_+$ of the G-left-invariant holomorphic vector fields on G such that

$$\left(\mathrm{Ad}_{(z_1,z_2,z_3)}\right)_{\mathrm{s}} = \mathrm{diag}\left(1,\, \mathrm{e}^{z_1},\, \mathrm{e}^{-z_1}\right) \in \mathrm{Aut}(\mathfrak{g}_+).$$

case *(a)*	$A^\bullet_\Gamma$
0	$\mathbb{C}\langle 1 \rangle$
1	$\mathbb{C}\langle \mathrm{d}\, z_1, \mathrm{d}\, z_{\bar 1} \rangle$
2	$\mathbb{C}\langle \mathrm{d}\, z_{1\bar 1}, \mathrm{d}\, z_{23}, \mathrm{d}\, z_{2\bar 3}, \mathrm{d}\, z_{3\bar 2}, \mathrm{d}\, z_{\bar 2\bar 3} \rangle$
3	$\mathbb{C}\langle \mathrm{d}\, z_{123}, \mathrm{d}\, z_{12\bar 3}, \mathrm{d}\, z_{13\bar 2}, \mathrm{d}\, z_{3\bar 1\bar 2}, \mathrm{d}\, z_{2\bar 1\bar 3}, \mathrm{d}\, z_{\bar 1\bar 2\bar 3}, \mathrm{d}\, z_{\bar 1 23}, \mathrm{d}\, z_{1\bar 2\bar 3} \rangle$
4	$\mathbb{C}\langle \mathrm{d}\, z_{123\bar 1}, \mathrm{d}\, z_{13\bar 1\bar 2}, \mathrm{d}\, z_{23\bar 2\bar 3}, \mathrm{d}\, z_{12\bar 1\bar 3}, \mathrm{d}\, z_{1\bar 1\bar 2\bar 3} \rangle$
5	$\mathbb{C}\langle \mathrm{d}\, z_{23\bar 1\bar 2\bar 3}, \mathrm{d}\, z_{123\bar 2\bar 3} \rangle$
6	$\mathbb{C}\langle \mathrm{d}\, z_{123\bar 1\bar 2\bar 3} \rangle$

Table 4: The cochain complex $A^\bullet_\Gamma$ in (3) for the complex parallelizable Nakamura manifold in case *(a)*.

(a) If $b \in \pi\,\mathbb{Z}$ and $d \in \pi\,\mathbb{Z}$, then, for $z \in \left(a + \sqrt{-1}\,b\right)\,\mathbb{Z} + \left(c + \sqrt{-1}\,d\right)\,\mathbb{Z}$, we have $\phi(z) \in \mathrm{SL}(2;\mathbb{R})$. In this case, the sub-complex $A^\bullet_\Gamma \subseteq \wedge^\bullet\,\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ defined in (3) is summarized in Table 4. (In order to shorten the notations, we write, for example, $\mathrm{d}\,z_{12\bar{2}} := \mathrm{d}\,z_1 \wedge \mathrm{d}\,z_2 \wedge \mathrm{d}\,\bar{z}_2$.)

(b) If $b \notin \pi\,\mathbb{Z}$ or $d \notin \pi\,\mathbb{Z}$, then the sub-complex $A^\bullet_\Gamma \subseteq \wedge^\bullet\,\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ defined in (3) is given in Table 5.

case *(b)*	$A^\bullet_\Gamma$
0	$\mathbb{C}\,\langle 1\rangle$
1	$\mathbb{C}\,\langle \mathrm{d}\,z_1, \mathrm{d}\,z_{\bar{1}}\rangle$
2	$\mathbb{C}\,\langle \mathrm{d}\,z_{1\bar{1}}, \mathrm{d}\,z_{23}, \mathrm{d}\,z_{\bar{2}\bar{3}}\rangle$
3	$\mathbb{C}\,\langle \mathrm{d}\,z_{123}, \mathrm{d}\,z_{\bar{1}\bar{2}\bar{3}}, \mathrm{d}\,z_{\bar{1}23}, \mathrm{d}\,z_{1\bar{2}\bar{3}}\rangle$
4	$\mathbb{C}\,\langle \mathrm{d}\,z_{123\bar{1}}, \mathrm{d}\,z_{23\bar{2}\bar{3}}, \mathrm{d}\,z_{1\bar{1}\bar{2}\bar{3}}\rangle$
5	$\mathbb{C}\,\langle \mathrm{d}\,z_{23\bar{1}\bar{2}\bar{3}}, \mathrm{d}\,z_{123\bar{2}\bar{3}}\rangle$
6	$\mathbb{C}\,\langle \mathrm{d}\,z_{123\bar{1}\bar{2}\bar{3}}\rangle$

Table 5: The cochain complex $A^\bullet_\Gamma$ in (3) for the complex parallelizable Nakamura manifold in case *(b)*.

Consider the G-left-invariant symplectic structure

$$\omega := \mathrm{d}\,z_1 \wedge \mathrm{d}\,\bar{z}_1 + \mathrm{d}\,z_2 \wedge \mathrm{d}\,z_3 + \mathrm{d}\,\bar{z}_2 \wedge \mathrm{d}\,\bar{z}_3.$$

Note that, in both case *(a)* and case *(b)*, we have $\omega \in A^2_\Gamma$. The operator d on $A^\bullet_\Gamma$ is trivial and so also d^Λ is. Hence we have the natural isomorphism $A^\bullet_\Gamma \xrightarrow{\simeq} H^\bullet_{BC}(\Gamma\backslash G) \otimes_{\mathbb{R}} \mathbb{C}$. Since we have also the natural isomorphism $A^\bullet_\Gamma \xrightarrow{\simeq} H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{C})$, the natural map $H^\bullet_{BC}(\Gamma\backslash G) \otimes_{\mathbb{R}} \mathbb{C} \to H^\bullet_{dR}(\Gamma\backslash G\,;\mathbb{C})$ is an isomorphism. Hence the $\mathrm{d}\,\mathrm{d}^\Lambda$-lemma holds, equivalently, the Hard Lefschetz Condition holds.

Remark 6.10 ([35]). In particular, by the direct computations of Lefschetz operators, we can show that solvmanifolds $\Gamma\backslash G$ such that $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ with a semi-simple action $\phi\colon \mathbb{R}^n \to \mathrm{GL}(\mathbb{R}^m)$ satisfy the Hard Lefschetz Condition, [35, Corollary 1.5]. In particular, the completely-solvable Nakamura manifold satisfies the Hard Lefschetz Condition, [35].

We investigate explicitly the Sawai manifold, [51], as an example of a symplectic solvmanifold satisfying the Hard Lefschetz Condition but not the twisted Hard Lefschetz Condition and not the DD^Λ-Lemma, see also [36]. We compute the twisted symplectic Bott-Chern cohomology and the twisted minimal model.

Example 6.11 (The Sawai manifold [51], [36, Example 1]). We consider the 8-dimensional solvable Lie group

$$G := G_1 \times \mathbb{R}$$

where G_1 is the matrix group defined as

$$G_1 := \left\{ \begin{pmatrix} \mathrm{e}^{a_1 t} & 0 & 0 & 0 & 0 & \mathrm{e}^{-a_3 t}\, x_2 & y_1 \\ 0 & \mathrm{e}^{a_2 t} & 0 & \mathrm{e}^{-a_1 t}\, x_3 & 0 & 0 & y_2 \\ 0 & 0 & \mathrm{e}^{a_3 t} & 0 & \mathrm{e}^{-a_2 t}\, x_1 & 0 & y_3 \\ 0 & 0 & 0 & \mathrm{e}^{-a_1 t} & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-a_2 t} & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-a_3 t} & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t, x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \right\},$$

where a_1, a_2, a_3 are distinct real numbers such that $a_1 + a_2 + a_3 = 0$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. The cochain complex $(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ is generated by a basis $\{\alpha, \beta, \zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3\}$ of $\mathfrak{g}^*$ such that

$$\begin{aligned} \mathrm{d}\,\alpha &= 0, \\ \mathrm{d}\,\beta &= 0, \\ \mathrm{d}\,\zeta_j &= a_j\, \alpha \wedge \zeta_j \quad \text{for } j \in \{1,2,3\}, \\ \mathrm{d}\,\eta_1 &= -a_1\, \alpha \wedge \eta_1 - \zeta_2 \wedge \zeta_3, \\ \mathrm{d}\,\eta_2 &= -a_2\, \alpha \wedge \eta_2 - \zeta_3 \wedge \zeta_1, \\ \mathrm{d}\,\eta_3 &= -a_3\, \alpha \wedge \eta_3 - \zeta_1 \wedge \zeta_2. \end{aligned}$$

In [51, Theorem 1], H. Sawai showed that, for some $a_1, a_2, a_3 \in \mathbb{R}$, the group G has a lattice Γ and $\Gamma\backslash G$ satisfies formality and has a G-invariant symplectic form,

$$\omega := \alpha \wedge \beta + \zeta_1 \wedge \eta_1 - 2\,\zeta_2 \wedge \eta_2 + \zeta_3 \wedge \eta_3,$$

satisfying the Hard Lefschetz Condition.

We have

$$\mathrm{Ad}_s(G) = \left\{ \begin{pmatrix} \mathrm{e}^{a_1 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{a_2 t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{a_3 t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-a_1 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-a_2 t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-a_3 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Consider the 1-dimensional representation

$$\alpha_1 := \mathrm{e}^{a_1 t}\,.$$

Then, in [36, Theorem 9.1], the second author showed that $(\Gamma\backslash G\,,\,\omega)$ does not satisfy the E_{α_1}-twisted Hard Lefschetz Condition.

We compute the two cohomologies $H^{\bullet}_{dR}\,(\Gamma\backslash G\,;E_{\alpha_1})$ and $H^{\bullet}_{BC}\,(\Gamma\backslash G\,;E_{\alpha_1})$ by using Theorem 6.2. The results of the computations are summarized in Table 6, respectively Table 7.

k	$H^k_{dR}\,(\Gamma\backslash G\,;E_{\alpha_1})$	$\dim_{\mathbb{C}} H^k_{dR}\,(\Gamma\backslash G\,;E_{\alpha_1})$
0	0	0
1	$\mathbb{C}\,\langle[\zeta_1]_{\alpha_1}\rangle$	1
2	$\mathbb{C}\,\langle[\alpha\wedge\zeta_1]_{\alpha_1}, [\beta\wedge\zeta_1]_{\alpha_1}\rangle$	2
3	$\mathbb{C}\,\langle[\alpha\wedge\beta\wedge\zeta_1]_{\alpha_1}, [\zeta_1\wedge\zeta_2\wedge\eta_2+\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	2
4	$\mathbb{C}\,\langle[\alpha\wedge\zeta_1\wedge\zeta_2\wedge\eta_2+\alpha\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}, [\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2+\beta\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	2
5	$\mathbb{C}\,\langle[\alpha\wedge\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2+\alpha\wedge\beta\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	1
6	0	0
7	0	0
8	0	0

Table 6: E_{α_1}-twisted de Rham cohomology $H^{\bullet}_{dR}\,(\Gamma\backslash G\,;E_{\alpha_1})$ of the Sawai manifold.

k	$H^k_{BC}(\Gamma\backslash G\,;E_{\alpha_1})$	$\dim_{\mathbb{C}} H^k_{BC}(\Gamma\backslash G\,;E_{\alpha_1})$
0	0	0
1	$\mathbb{C}\langle[\zeta_1]_{\alpha_1}\rangle$	1
2	$\mathbb{C}\langle[\alpha\wedge\zeta_1]_{\alpha_1},[\beta\wedge\zeta_1]_{\alpha_1}\rangle$	2
3	$\mathbb{C}\langle[\alpha\wedge\beta\wedge\zeta_1]_{\alpha_1},[\zeta_1\wedge\zeta_2\wedge\eta_2]_{\alpha_1},[\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	3
4	$\mathbb{C}\langle[\alpha\wedge\zeta_1\wedge\zeta_2\wedge\eta_2]_{\alpha_1},[\alpha\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1},[\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2]_{\alpha_1},[\beta\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	4
5	$\mathbb{C}\langle[\zeta_1\wedge\zeta_2\wedge\eta_2\wedge\zeta_3\wedge\eta_3]_{\alpha_1},[\alpha\wedge\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2]_{\alpha_1},[\alpha\wedge\beta\wedge\zeta_1\wedge\zeta_3\wedge\eta_3]_{\alpha_1}\rangle$	3
6	$[\alpha\wedge\zeta_1\wedge\zeta_2\wedge\eta_2\wedge\zeta_3\wedge\eta_3]_{\alpha_1},[\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2\wedge\zeta_3\wedge\eta_3]_{\alpha_1}$	2
7	$[\alpha\wedge\beta\wedge\zeta_1\wedge\zeta_2\wedge\eta_2\wedge\zeta_3\wedge\eta_3]_{\alpha_1}$	1
8	0	0

Table 7: E_{α_1}-twisted Bott-Chern cohomology $H^\bullet_{BC}(\Gamma\backslash G\,;E_{\alpha_1})$ of the Sawai manifold.

By these computations, the natural map

$$H^\bullet_{BC}(\Gamma\backslash G\,;E_{\alpha_1})\to H^\bullet_{dR}(\Gamma\backslash G\,;E_{\alpha_1})$$

induced by the identity is surjective, and hence there is a symplectically-harmonic representative in each de Rham cohomology class with values in E_{α_1}. On the other hand, the natural map

$$H^\bullet_{BC}(\Gamma\backslash G\,;E_{\alpha_1})\to H^\bullet_{dR}(\Gamma\backslash G\,;E_{\alpha_1})$$

induced by the identity is not injective, and hence the DD^Λ-Lemma does not hold.

Next we consider the twisted minimal model $\wedge^\bullet\mathfrak{u}$. Consider

$$\alpha_2:=\mathrm{e}^{a_2t},\quad \alpha_3:=\mathrm{e}^{a_3t},$$

and define, for $j\in\{1,2,3\}$,

$$\breve{\alpha}:=\alpha,\quad \breve{\beta}:=\beta,\quad \breve{\zeta}_j:=\zeta_j\otimes v_{\alpha_j},\qquad\text{and}\qquad \breve{\eta}_j:=\eta_j\otimes v_{\alpha_j^{-1}}.$$

Then we have

$$\wedge^\bullet\mathfrak{u}=\wedge^\bullet\left\langle\breve{\alpha},\,\breve{\beta},\,\breve{\zeta}_1,\,\breve{\zeta}_2,\,\breve{\zeta}_3,\,\breve{\eta}_1,\,\breve{\eta}_2,\,\breve{\eta}_3\right\rangle$$

such that

$$\begin{aligned} \mathrm{d}\breve{\alpha} = \mathrm{d}\breve{\beta} = \mathrm{d}\breve{\zeta}_1 = \mathrm{d}\breve{\zeta}_2 = \mathrm{d}\breve{\zeta}_3 = 0, \\ \mathrm{d}\breve{\eta}_1 = \breve{\zeta}_2 \wedge \breve{\zeta}_3, \\ \mathrm{d}\breve{\eta}_2 = \breve{\zeta}_3 \wedge \breve{\zeta}_1, \\ \mathrm{d}\breve{\eta}_3 = \breve{\zeta}_1 \wedge \breve{\zeta}_2. \end{aligned}$$

We have

$$\omega = \breve{\alpha} \wedge \breve{\beta} + \breve{\zeta}_1 \wedge \breve{\eta}_1 - 2\,\breve{\zeta}_2 \wedge \breve{\eta}_2 + \breve{\zeta}_3 \wedge \breve{\eta}_3.$$

We have

$$H^{\bullet}_{BC}(\mathfrak{u}) \simeq H^{\bullet}_{BC}\left(\bigoplus_{\beta \in \mathrm{Char}(S)} \wedge^{\bullet}(\Gamma\backslash G\,; E_{\beta})\right).$$

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