Dirac groupoids and Dirac bialgebroids

M. Jotz Lean

We describe infinitesimally Dirac groupoids via geometric objects that we call Dirac bialgebroids. In the two well-understood special cases of Poisson and presymplectic groupoids, the Dirac bialgebroids are equivalent to the Lie bialgebroids and IM-2-forms, respectively. In the case of multiplicative involutive distributions on Lie groupoids, we find new properties of infinitesimal ideal systems.

1. Introduction

Courant and Weinstein [7], and independently Dorfman [11], introduced the notion of Dirac structure in the late 1980's. Courant and Weinstein were studying work of Littlejohn on the "guiding centre of motion" of a particle in a magnetic field [26]. They introduced Dirac structures as a unified approach to presymplectic and Poisson manifolds. Dorfman showed Dirac

structures to be useful in the theory of infinite dimensional Hamiltonian systems. Because the theory of Hamiltonian systems with constraints can be well formulated using these objects, they were named after Dirac and his theory of constraints [10]. Poisson and presymplectic manifolds and involutive distributions are the corner cases of Dirac manifolds.

To be more explicit, let M be a smooth manifold. Then the direct sum $TM \oplus T^*M$ of vector bundles over M is endowed with a bracket $[\![\cdot, \cdot]\!]$ and a pairing $\langle \cdot, \cdot \rangle$ given by

$$[\![(X_1, \theta_1), (X_2, \theta_2)]\!] = ([X_1, X_2], \mathcal{L}_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1)$$

and

$$\langle (X_1, \theta_1), (X_2, \theta_2) \rangle = \theta_1(X_2) + \theta_2(X_1)$$

for all vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and one-forms $\theta_1, \theta_2 \in \Omega^1(M)$. A Dirac structure on M is a subbundle $\mathsf{D} \subseteq TM \oplus T^*M$ that is maximally isotropic relative to the pairing, and the sections of which are closed under the bracket. The graphs of the vector bundle morphisms $\pi^\sharp \colon T^*M \to TM$ and $\omega^\flat \colon TM \to T^*M$ associated to a Poisson bivector π and, respectively, a closed 2-form ω on M are two examples of Dirac structures.

Here we are interested in Dirac groupoids, i.e. Lie groupoids with Dirac structures that are compatible with the multiplication. As in the Lie group case, a geometric structure on a Lie groupoid, that is compatible with the multiplication, has a counterpart on the Lie algebroid of the groupoid. This counterpart can simply be understood as the differential of the structure, which is a structure of the same type but on the Lie algebroid. For instance, the counterpart of a multiplicative Poisson structure on a Lie groupoid is a Poisson structure on the Lie algebroid of the groupoid, together with some compatibility conditions between the Poisson and the Lie algebroid structures [35]. More generally, Ortiz has proved in his thesis that the Lie algebroid of a Dirac groupoid is a Dirac algebroid, i.e. a Lie algebroid with a Dirac structure that is compatible with the Lie algebroid structure (an LA-Dirac structure). This defines a bijection between multiplicative Dirac structures on a source simply connected Lie groupoid and LA-Dirac structures on its Lie algebroid.

On the other hand, we observe in the literature that the infinitesimal counterpart of a multiplicative geometric object is often described as a geometric object of completely different, to some extent more simple, nature. A multiplicative distribution on a Lie group corresponds for instance to an ideal in its Lie algebra [19, 37]. Knowing this, it would be rather cumbersome

to infinitesimally describe a multiplicative distribution on a Lie group as a distribution on the Lie algebra, together with some compatibility condition in terms of the tangent Lie algebra.

To illustrate this, let us recall how Poisson and presymplectic groupoids, and involutive multiplicative distributions on Lie groupoids can be described infinitesimally. Weinstein introduced Poisson groupoids as a simultaneous generalisation of Poisson Lie groups and symplectic groupoids [43]. He proved that a multiplicative Poisson structure on a Lie groupoid induces a Lie algebroid structure on the dual of the Lie algebroid of the Lie groupoid. Mackenzie and Xu then introduced the notion of *Lie bialgebroid* to encode the compatibility condition of this pair of Lie algebroids in duality [34]. Then they showed that there is a one-to-one correspondence between source simply connected Poisson groupoids and integrable Lie bialgebroids[1] [35]. Lie bialgebroids describe hence infinitesimally Poisson groupoids. Liu, Weinstein and Xu introduced at the same time the notion of *Courant algebroid*. With this new notion, a Lie bialgebroid could be defined as a pair of Lie algebroids which are two transverse Dirac structures in a Courant algebroid [27].

Presymplectic groupoids were originally introduced by Xu [44] under the name "quasi-symplectic groupoids" and then used by Bursztyn, Crainic, Weinstein and Zhu in a work motivated by the integration of Dirac structures twisted by closed 3-forms [4]. There a presymplectic groupoid is a Lie groupoid with a multiplicative closed 2-form satisfying some additional regularity conditions. Afterwards the terminology slightly changed and now one often calls presymplectic a Lie groupoid with any multiplicative closed 2-form. The infinitesimal description of a presymplectic groupoid is an IM-2-form, i.e. a very particular vector bundle morphism from the Lie algebroid of the groupoid to the cotangent bundle of the manifold of units [3, 4].

Infinitesimal ideal systems, the "ideals" in Lie algebroids [23], are the infinitesimal counterpart of multiplicative distributions [18, 23]. Multiplicative distributions have appeared to be useful as the polarisations in Hawkins' groupoid approach to quantisation [18], and wide multiplicative distributions on Lie groupoids were also studied in a modern approach to Cartan's work on pseudogroups [9].

Lie bialgebroids, IM-2-forms and infinitesimal ideal systems seem by nature very different, although Poisson groupoids, presymplectic groupoids and multiplicative distributions on Lie groupoids are three basic classes of Dirac groupoids.

---

[1]A Lie bialgebroid (A, A^*) is integrable if A is an integrable Lie algebroid.

Our goal is to understand how Lie bialgebroids, IM-2-forms and infinitesimal ideal systems are in fact the corner cases of an infinitesimal description of Dirac groupoids. In [21] we constructed a Courant algebroid B over M associated to a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$, such that the space of units U of the Dirac structure seen as a groupoid is naturally embedded as a Dirac structure in B. We classified[2] the Dirac homogeneous spaces of $(G \rightrightarrows M, \mathsf{D})$ via Dirac structures in B, and we conjectured consequently that the whole information about D was contained in the Manin pair (B, U). In [20] we studied linear splittings of the double vector bundle $TA \oplus T^*A$ over a Lie algebroid A, and we proved that these correspond to a certain class of *Dorfman connections*, which that paper introduced. We studied linear splittings adapted to LA-Dirac structures (the linearisations of multiplicative Dirac structures on Lie groupoids), and completely understood LA-Dirac structures in terms of the corresponding Dorfman connections, showing how to reconstruct an LA-Dirac structure from some simple data.

Here we build on the results in those two papers to prove how the Manin pair (B, U) associated to a Dirac groupoid is in fact equivalent to this Dirac groupoid, and which Manin pairs appear in this manner as the infinitesimal version of Dirac groupoids. The examples that we discuss show that our theorem provides a unified proof for the results in [3, 35].

Let us summarise our main result in a more detailed manner. Let G be a Lie groupoid over M, with Lie algebroid A. Then the bundle $TG \oplus T^*G$ has the structure of a Lie groupoid over $TM \oplus A^*$. A Dirac structure D on G is multiplicative if it is a subgroupoid of $TG \oplus T^*G$. The set of units of D seen as a groupoid is a subbundle $\iota \colon U \hookrightarrow TM \oplus A^*$. We proved in [21] that U inherits a natural Lie algebroid structure with anchor the restriction to U of the projection onto TM. The Lie algebroid U is compatible with the Lie algebroid A in a sense that this paper identifies. The triple (A, U, ι) is then a Dirac bialgebroid, a new geometrical notion that we now define.

Let $(A, \rho, [\cdot, \cdot])$ and $(U, \rho_U, \llbracket \cdot, \cdot \rrbracket)$ be two Lie algebroids over M with an injective morphism $\iota \colon U \to TM \oplus A^*$ of vector bundles such that $\rho_U = \mathrm{pr}_{TM} \circ \iota$. The triple (A, U, ι) is called a *Dirac bialgebroid* (over A) if there exists a Courant algebroid C over M such that

  1) (C, U) is a Manin pair [5], i.e. U is a Dirac structure in C,

---

2) there exists a (degenerate) Courant morphism[3] $\Phi\colon A \oplus T^*M \to C$ such that

$$\Phi(A \oplus T^*M) + U = C$$

and $\langle u, \Phi(\tau) \rangle_C = \langle \iota(u), \tau \rangle$ for all $(u, \tau) \in U \times_M (A \oplus T^*M)$.

Two Dirac bialgebroids (A, U, ι) and (A, U', ι') over A are *equivalent* they define the same Lie algebroid $\iota(U) = \iota'(U') \subseteq TM \oplus A^*$ with the restriction of pr_{TM} as anchor. Our main theorem is the following:

Theorem 1.1. *Let $(G \rightrightarrows M, \mathsf{D})$ be a source simply connected Dirac groupoid. Let A be the Lie algebroid of $G \rightrightarrows M$ and $\iota\colon U \hookrightarrow TM \oplus A^*$ the set of units of D. The triple (A, U, ι) is a Dirac bialgebroid and the map*

$$(G, \mathsf{D}) \mapsto (A, U, \iota)$$

defines a bijection between multiplicative Dirac structures on $G \rightrightarrows M$ and equivalence classes of Dirac bialgebroids over A.

Recall for instance that a Lie bialgebroid (A, A^*) is a pair of Lie algebroids in duality that are two transverse Dirac structures in a Courant algebroid; which we can hence write $A \oplus A^*$. Set $\iota := (\rho_\star, \mathrm{id}_{A^*})\colon A^* \to TM \oplus A^*$, where $\rho_\star$ is the anchor of the Lie algebroid A^*. Section 7.2 shows that (A, A^*, ι) is a Dirac bialgebroid, with ambient Courant algebroid $C = A \oplus A^*$.

Take now an IM-2-form $\sigma\colon A \to T^*M$, i.e. with

$$\langle \rho(a), \sigma(b) \rangle = -\langle \rho(b), \sigma(a) \rangle \quad \text{and} \quad \sigma[a, b] = \mathcal{L}_{\rho(a)}\sigma(b) - \mathbf{i}_{\rho(b)}\mathbf{d}\sigma(a)$$

for all $a, b \in \Gamma(A)$. Define $\iota = (\mathrm{id}_{TM}, \sigma^t)\colon TM \to TM \oplus A^*$. Then (A, TM, ι) is a Dirac bialgebroid with ambient Courant algebroid C the standard Courant-Dorfman structure on $TM \oplus T^*M$ (see Section 7.3).

Finally consider an infinitesimal ideal system (A, F_M, J, ∇), i.e. $F_M \subseteq TM$ is an involutive subbundle, $J \subseteq A$ is a subbundle such that $\rho(J) \subseteq F_M$ and ∇ is a flat F_M-connection on A/J with the following properties:

1) If $\bar{a} \in \Gamma(A/J)$ is ∇-parallel, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.

2) If $\bar{a}, \bar{b} \in \Gamma(A/J)$ are ∇-parallel, then $\overline{[a, b]}$ is also ∇-parallel.

---

[3]If A is a Lie algebroid over M, then the vector bundle $A \oplus T^*M$ inherits the structure of a *degenerate Courant algebroid*, see Example 2.2.

3) If $\bar{a} \in \Gamma(A/J)$ is ∇-parallel, then $\overline{\rho(a)} \in \Gamma(TM/F_M)$ is ∇^{F_M}-parallel, where

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$$

is the Bott connection associated to F_M.

Section 7.4 shows that (A, F_M, J, ∇) is an infinitesimal ideal system if and only if $(A, F_M \oplus J^\circ, \iota)$ is a Dirac bialgebroid with $\iota \colon F_M \oplus J^\circ \to TM \oplus A^*$ the inclusion, and the Lie algebroid bracket $[\cdot, \cdot]_\nabla$ on $F_M \oplus J^\circ$ defined by $[(X_1, \alpha_1), (X_2, \alpha_2)]_\nabla = ([X_1, X_2], \nabla^*_{X_1}\alpha_2 - \nabla^*_{X_2}\alpha_1)$. In order to prove this, we construct a natural Lie algebroid structure on

$$\bar{A} := (F_M \oplus A)/\operatorname{graph}(-\rho|_J : J \to F_M),$$

which exists if and only if the quadruple (A, F_M, J, ∇) is an infinitesimal ideal system. We find this result interesting in its own right because it is a new manner of forming the quotient of an algebroid by an infinitesimal ideal system, as a generalisation of the quotient of a Lie algebra by an ideal. The ambient Courant algebroid C is in this example the Courant algebroid $\bar{A} \oplus \bar{A}^*$ defined by the Lie bialgebroid $(\bar{A}, \bar{A}^*)$ with trivial Lie algebroid structure on $\bar{A}^*$.

Note that the two families $(A, A^*, (\rho_\star, \operatorname{id}_{A^*}))$ and $(A, TM, (\operatorname{id}_{TM}, \sigma^t))$ of Dirac bialgebroids can be seen as extreme cases of Dirac bialgebroids, like Poisson and presymplectic groupoids are the extreme cases of Dirac groupoids. In the first family, the interesting information is in the pair (A, A^*) and in the second one, it is rather in the embedding

$$(\operatorname{id}_{TM}, \sigma^t) \colon TM \to TM \oplus A^*.$$

Let us finally point out that a Dirac groupoid is an *LA-groupoid*; the space D has a Lie groupoid structure over U and a Lie algebroid structure over G, and the structures are compatible [30]. If the Lie algebroid structure is integrable, the Dirac groupoid should integrate to a presymplectic double Lie groupoid ("presymplectic" in the sense of [4]). Hence, this paper gives a class of examples of Courant algebroids (including the ones defined by Lie bialgebroids) that could be seen as integrating to "presymplectic 2-groupoids", following Mehta and Tang's proposal to integrate Courant algebroids defined by Lie bialgebroids to "symplectic 2-groupoids" [36].

Methodology and outline of the paper

First Section 2 recalls some background on Lie groupoids and their tangent and cotangent prolongations, on Courant algebroids and Dirac structures, on Dorfman connections and dull algebroids, and on splittings of VB-algebroids and 2-term representations up to homotopy.

Section 3 introduces the notions of A-Manin pairs and Dirac bialgebroids, before showing how they are equivalent. Then Section 4 proves that the Manin pair associated in [21] to a Dirac groupoid (G, D) is an A-Manin pair, and so that the triple (A, U, ι) is a Dirac bialgebroid.

Section 5 studies LA-Dirac structures on Lie algebroids. First we summarise the results in [20] on linear splittings of LA-Dirac structures and we describe the two 2-term representations up to homotopy describing in a linear splitting the two VB-algebroid sides. Then we describe how this data allows us to construct an A-Manin pair associated to each LA-Dirac structure, and how this defines a bijection between A-Manin pairs and LA-Dirac structures over A.

Section 6 recalls a result of [38] about the one-to-one correspondence between Dirac groupoids and integrable Dirac algebroids, i.e. integrable Lie algebroids with compatible Dirac structures. Then we describe explicitly the Lie algebroid $A(\mathsf{D}) \to U$ of the multiplicative Dirac structure in terms of the Dorfman connection $\pounds^U \colon \Gamma(U) \times \Gamma(U^*) \to \Gamma(U^*)$ that is dual to the Lie algebroid bracket on sections of U.

Section 6.3 finally shows that the Dirac bialgebroid associated to the Dirac algebroid (A, D_A) that is canonically isomorphic to $(A, A(\mathsf{D}))$ is the same as the Dirac bialgebroid constructed in Section 4. Section 6.4 gives as a summary a sketch of the strategy to integrate an integrable Dirac bialgebroid to a Dirac groupoid.

Section 7 shows how the notion of Dirac bialgebroid encompasses the notions of Lie bialgebroids, IM-2-forms and infinitesimal ideal systems.

Let us repeat the different steps that lead us to our main theorem:

§ 3 Equivalence classes of A-Manin pairs are in one-to-one correspondence with equivalence classes of Dirac bialgebroids.

§ 4 A Dirac groupoid defines an equivalence class of A-Manin pairs and so of Dirac bialgebroids.

§ 5 There is a bijection between LA-Dirac structures and equivalence classes of Dirac bialgebroids.

§ 6.1 The Lie algebroid of a Dirac groupoid is a Dirac algebroid. This defines a bijection between source simply connected Dirac groupoids and integrable Dirac algebroids [38].

§ 6.2 Explicit computation of the LA-Dirac structure defined as in Section 6.1 by a multiplicative Dirac structure.

§ 6.3 The Dirac bialgebroids defined by a Dirac groupoid and by the corresponding LA-Dirac structure coincide.
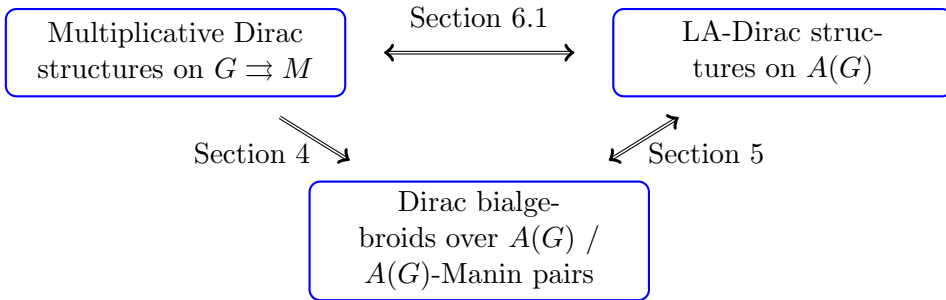
Notation and conventions

In the following, $G \rightrightarrows M$ is always a (Hausdorff) Lie groupoid over the base M. The space of units M is always considered to be a subset of the space G of arrows. The structure maps of a Lie groupoid is written as $(\epsilon, \mathsf{s}, \mathsf{t}, \mathsf{m}, \mathsf{i})$, unless specified otherwise. The Lie algebroid of a Lie groupoid is constructed using right-invariant vector fields on the groupoid, and the anchor is the restriction of $T\mathsf{t}$ to $T^{\mathsf{s}}G|_M$. To simplify the notation, the Lie algebroid $(A(G), [\cdot, \cdot]_{A(G)}, \rho_{A(G)})$ of a Lie groupoid G is always written as $(A, [\cdot, \cdot], \rho)$.

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the space of sections of E is written as $\Gamma(E)$. We write in general $q_E \colon E \to M$ for vector bundle projections, except for $p_M = q_{TM} \colon TM \to M$, $c_M = q_{T^*M} \colon T^*M \to M$ and $\pi_M = q_{TM \oplus T^*M} \colon TM \oplus T^*M \to M$.

Given a section ε of E^*, we always write $\ell_\varepsilon \colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$. We write $\phi^t \colon B^* \to A^*$ for the dual morphism to a morphism $\phi \colon A \to B$ of vector bundles over the identity, and we write $F^*\omega$ for the pullback of a form $\omega \in \Omega(N)$ under a smooth map $F \colon M \to N$ of manifolds.

Let A be a Lie algebroid. For each $a \in \Gamma(A)$, we have two derivations over $\rho(a)$:

(1.1)
$$\mathcal{L}_a \colon \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M), \quad \mathcal{L}_a(a', \theta) = ([a, a'], \mathcal{L}_{\rho(a)}\theta)$$
$$\mathcal{L}_a \colon \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*), \quad \mathcal{L}_a(X, \alpha) = ([\rho(a), X], \mathcal{L}_a\alpha).$$

Note also that the anchor ρ of a Lie algebroid A defines a vector bundle morphism $(\rho, \rho^t) \colon A \oplus T^*M \to TM \oplus A^*$.

Acknowledgements

2. Preliminaries

This section collects necessary background on the Lie groupoid structure on the Pontryagin bundle $TG \oplus T^*G$ of a Lie groupoid $G \rightrightarrows M$, on Courant algebroids, Dirac structures and Dorfman connections, and on double vector bundles, VB-algebroids and representations up to homotopy.

2.1. Tangent and cotangent Lie groupoids

Let $G \rightrightarrows M$ be a Lie groupoid. Applying the tangent functor to each of the maps defining G yields a Lie groupoid structure on TG with base TM, source $T\mathsf{s}$, target $T\mathsf{t}$ and multiplication $T\mathsf{m} \colon T(G \times_M G) \to TG$. The identity at $v_p \in T_pM$ is $1_{v_p} = T_p\epsilon v_p$. This defines the *tangent prolongation of $G \rightrightarrows M$*.

There is also an induced Lie groupoid structure on $T^*G \rightrightarrows A^* = (TM)^\circ$ [6] (see also [40]). The source map $\hat{\mathsf{s}} \colon T^*G \to A^*$ is given by

$$\hat{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}G \text{ for } \alpha_g \in T^*_gG,$$
$$\hat{\mathsf{s}}(\alpha_g)(a_{\mathsf{s}(g)}) = \alpha_g(T_{\mathsf{s}(g)}L_g(a_{\mathsf{s}(g)} - T_{\mathsf{s}(g)}\mathsf{t}a_{\mathsf{s}(g)}))$$

for all $a_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)}G$, and the target map $\hat{\mathsf{t}} \colon T^*G \to A^*$ is given by

$$\hat{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}G, \qquad \hat{\mathsf{t}}(\alpha_g)(a_{\mathsf{t}(g)}) = \alpha_g(T_{\mathsf{t}(g)}R_g(a_{\mathsf{t}(g)}))$$

for all $a_{\mathsf{t}(g)} \in A_{\mathsf{t}(g)}G$. If $\hat{\mathsf{s}}(\alpha_g) = \hat{\mathsf{t}}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by $(\alpha_g \star \alpha_h)(T\mathsf{m}(v_g, v_h)) = \alpha_g(v_g) + \alpha_h(v_h)$ for composable pairs $(v_g, v_h) \in T_{(g,h)}(G \times_M G)$.

The fibred product $TG \times_G T^*G$ inherits an induced Lie groupoid structure over $TM \times_M A^*$. We write $\mathbb{T}\mathsf{t}$ for the target map $TG \times_G T^*G \to TM \times_M A^*$, and $\mathbb{T}\mathsf{s} \colon TG \times_G T^*G \to TM \times_M A^*$ for the source map. Here again, we write $p_g \star p_h$ for the product of compatible $p_g, p_h \in TG \oplus T^*G$.

Note that the restriction to M of the kernel of $\mathbb{T}\mathsf{s}$ can be identified as follows with $A \oplus T^*M$. The elements of $T_gG \times T_g^*G$ that are sent by $\mathbb{T}\mathsf{s}$ to $0_{\mathsf{s}(g)} \in T_{\mathsf{s}(g)}M \times A_{\mathsf{s}(g)}^*$ are of the form $\bigl((R_g)_* a_{\mathsf{t}(g)}, (T_g\mathsf{t})^t \theta_{\mathsf{t}(g)}\bigr)$ with $a_{\mathsf{t}(g)} \in A_{\mathsf{t}(g)} = T_{\mathsf{t}(g)}^{\mathsf{s}}M$ and $\theta_{\mathsf{t}(g)} \in T_{\mathsf{t}(g)}^*M$. We identify $\bigl(a_m, (T_m\mathsf{t})^t \theta_m\bigr) \in \ker(\mathbb{T}\mathsf{s})_m$ with (a_m, θ_m) in $A_m \times T_m^*M$. Given a section $\tau = (a, \theta)$ of $A \oplus T^*M$, the *right invariant section* $\tau^r \in \Gamma(\ker \mathbb{T}\mathsf{s})$ defined by τ is given by $\tau^r(g) = \tau_{\mathsf{t}(g)} \times (0_g, 0_g) = (a^r(g), T_g\mathsf{t}^t\theta(\mathsf{t}(g))) \in \ker(\mathbb{T}\mathsf{s})_g$ for all $g \in G$. Furthermore, the restriction of $\mathbb{T}\mathsf{t}$ to $\ker(\mathbb{T}\mathsf{s})|_M$ is, via the identification above, the map

$$(\rho, \rho^t) \colon A \oplus T^*M \to TM \oplus A^*.$$

2.2. Courant algebroids and Dirac structures

A **Courant algebroid** [27, 41] over a manifold M is a vector bundle $C \to M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(C)$, and a vector bundle map $\rho \colon C \to TM$ over the identity called the anchor, which satisfy the following conditions[4]

1) $[\![c_1, [\![c_2, c_3]\!]]\!] = [\![[\![c_1, c_2]\!], c_3]\!] + [\![c_2, [\![c_1, c_3]\!]]\!]$,

2) $\rho(c_1)\langle c_2, c_3 \rangle = \langle [\![c_1, c_2]\!], c_3 \rangle + \langle c_2, [\![c_1, c_3]\!] \rangle$,

3) $[\![c_1, c_2]\!] + [\![c_2, c_1]\!] = \mathcal{D}\langle c_1, c_2 \rangle$

---

[4]The following conditions

4) $\rho([\![c_1, c_2]\!]) = [\rho(c_1), \rho(c_2)]$,
5) $[\![c_1, fc_2]\!] = f[\![c_1, c_2]\!] + (\rho(c_1)f)c_2$

are then also satisfied. They are often part of the definition in the literature, but [42] observed that they follow from (1)-(3). We quickly give here a simple manner to get (4)-(5) from (1)-(3). To get (5), replace c_2 by fc_2 in (2). Then replace c_2 by fc_2 in (1) in order to get (4).

for all $c_1, c_2, c_3 \in \Gamma(C)$ and $f \in C^\infty(M)$. Here, we use the notation $\mathcal{D} := \rho^t \circ \mathbf{d} \colon C^\infty(M) \to \Gamma(C)$, using $\langle \cdot, \cdot \rangle$ to identify C with C^*: $\langle \mathcal{D}f, c \rangle = \rho(c)(f)$ for all $f \in C^\infty(M)$ and $c \in \Gamma(C)$.

If C_1 and C_2 are Courant algebroids over the same base M, then a **Courant morphism** is a vector bundle morphism $\Phi \colon C_1 \to C_2$ over the identity on M, such that $\rho_2 \circ \Phi = \rho_1$, $\langle c, c' \rangle_{C_1} = \langle \Phi(c), \Phi(c') \rangle_{C_2}$ and $\Phi[\![c, c']\!]_{C_1} = [\![\Phi(c), \Phi(c')]\!]_{C_2}$ for all $c, c' \in \Gamma(C_1)$.

Example 2.1. The direct sum $TM \oplus T^*M$ with the projection on TM as anchor map, $\rho = \mathrm{pr}_{TM}$, the symmetric bracket $\langle \cdot, \cdot \rangle$ given by

(2.1) $$\langle (v_m, \theta_m), (w_m, \omega_m) \rangle = \theta_m(w_m) + \omega_m(v_m)$$

for all $m \in M$, $v_m, w_m \in T_m M$ and $\theta_m, \omega_m \in T_m^* M$ and the **Courant-Dorfman bracket** given by

$$[\![(X_1, \theta_1), (X_2, \theta_2)]\!] = ([X_1, X_2], \pounds_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1)$$

for all $(X_1, \theta_1), (X_2, \theta_2) \in \Gamma(TM \oplus T^*M)$, yield the standard example of a Courant algebroid (often called the *standard Courant algebroid over M*). The map $\mathcal{D} \colon C^\infty(M) \to \Gamma(TM \oplus T^*M)$ is given by $\mathcal{D}f = (0, \mathbf{d}f)$.

Example 2.2. Let $G \rightrightarrows M$ be a Lie groupoid. The Courant-Dorfman bracket and the pairing on sections of $TG \oplus T^*G \to G$ restrict to right-invariant sections (defined in the previous section) and give rise to the following structure on $A \oplus T^*M$.

The direct sum $A \oplus T^*M \to M$, with the anchor $\rho \circ \mathrm{pr}_A \colon A \oplus T^*M \to TM$, the pairing $\langle (a_1, \theta_1), (a_2, \theta_2) \rangle_d = \theta_2(\rho(a_1)) + \theta_1(\rho(a_2))$ and the bracket[5] $[\cdot, \cdot]_d$ on $\Gamma(A \oplus T^*M)$,

$$[(a_1, \theta_1), (a_2, \theta_2)]_d = ([a_1, a_2], \pounds_{\rho(a_1)}\theta_2 - \mathbf{i}_{\rho(a_2)}\mathbf{d}\theta_1)$$

is a *degenerate Courant algebroid*. That is, $(A \oplus T^*M, \rho \circ \mathrm{pr}_A, \langle \cdot, \cdot \rangle_d, [\cdot, \cdot]_d)$ satisfies the axioms (1)–(5) of a Courant algebroid, except the non-degeneracy of the pairing (see [22]).

A **Dirac structure**[7] $\mathsf{D} \subseteq C$ is a subbundle satisfying

1) $\mathsf{D}^\perp = \mathsf{D}$ relative to the pairing on C,

2) $[\Gamma(\mathsf{D}), \Gamma(\mathsf{D})] \subseteq \Gamma(\mathsf{D})$.

---

[5] "d" stands for degenerate.

The rank of the vector bundle D is then half the rank of C, and the triple $(\mathsf{D} \to M, \rho|_\mathsf{D}, [\cdot, \cdot]|_{\Gamma(\mathsf{D}) \times \Gamma(\mathsf{D})})$ is a Lie algebroid over M. Dirac structures appear naturally in several contexts in geometry and geometric mechanics (see for instance [2] for an introduction to the geometry and applications of Dirac structures). Three classes of great interest in this paper are collected in the following example.

Example 2.3. Let M be a smooth manifold and $F \subseteq TM$ an involutive subbundle. Then $\mathsf{D}_F = F \oplus F^\circ \subseteq TM \oplus T^*M$ is a Dirac structure on M.

Now let $\pi \in \Gamma\left(\bigwedge^2 TM\right)$ be a Poisson bivector field. Then $\mathsf{D}_\pi \subseteq TM \oplus T^*M$ defined by

$$\mathsf{D}_\pi = \operatorname{graph}(\pi^\sharp \colon T^*M \to TM, \pi^\sharp(\theta_m) = \pi(\theta_m, \cdot))$$

is a Dirac structure on M.

Finally let $\omega \in \Omega^2(M)$ be a closed 2-form on M. Then $\mathsf{D}_\omega \subseteq TM \oplus T^*M$ defined by

$$\mathsf{D}_\omega = \operatorname{graph}(\omega^\flat \colon TM \to T^*M, \omega^\flat(v_m) = \omega(v_m, \cdot))$$

is a Dirac structure on M.

2.3. Dorfman connections and dull algebroids

In the following, an *anchored* vector bundle is a vector bundle $Q \to M$ with a vector bundle morphism $\rho_Q \colon Q \to TM$ over the identity. An anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$ are said to be *paired* if there exists a fibrewise pairing $\langle \cdot, \cdot \rangle \colon Q \times_M B \to \mathbb{R}$ and a map $\mathbf{d}_B \colon C^\infty(M) \to \Gamma(B)$ such that

$$(2.2) \qquad\qquad \langle q, \mathbf{d}_B f \rangle = \rho_Q(q)(f)$$

for all $q \in \Gamma(Q)$ and $f \in C^\infty(M)$. Then $(Q \to M, \rho)$ and $(B, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$ are said to be **paired by** $\langle \cdot, \cdot \rangle$.

Definition 2.4. Let $(Q \to M, \rho_Q)$ be an anchored vector bundle that is paired with $(B \to M, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$. A **Dorfman ($Q$-)connection on** B is an

$\mathbb{R}$-linear map

$$\Delta\colon \Gamma(Q) \to \mathrm{Der}(B)$$

such that[6]

1) Δ_q is a derivation over $\rho_Q(q) \in \mathfrak{X}(M)$,

2) $\Delta_{fq}b = f\Delta_q b + \langle q, b\rangle \cdot \mathbf{d}_B f$ and

3) $\Delta_q(\mathbf{d}_B f) = \mathbf{d}_B(\rho_Q(q)f)$

for all $f \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$.

Remark 2.5. If the pairing $\langle \cdot, \cdot\rangle\colon Q \times_M B \to \mathbb{R}$ is nondegenerate, then $B \simeq Q^*$ and the map $\mathbf{d}_B = \mathbf{d}_{Q^*}\colon C^\infty(M) \to \Gamma(Q^*)$ is *defined* by (2.2): we have then $\mathbf{d}_{Q^*}f = \rho_Q^t \mathbf{d}f$ for all $f \in C^\infty(M)$.

The map $\Delta^*\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ that is dual to Δ in the sense of dual derivations, i.e. $\langle \Delta_{q_1}^* q_2, \tau\rangle = \rho_Q(q_1)\langle q_2, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle$ for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ is then a dull bracket on $\Gamma(Q)$ as in the following definition.

Definition 2.6. A **dull algebroid** is an anchored vector bundle $(Q \to M, \rho_Q)$ with a bracket $[\cdot, \cdot]_Q$ on $\Gamma(Q)$ such that

(2.3) $$\rho_Q[q_1, q_2]_Q = [\rho_Q(q_1), \rho_Q(q_2)]$$

and (the Leibniz identity)

$$[f_1 q_1, f_2 q_2]_Q = f_1 f_2 [q_1, q_2]_Q + f_1 \rho_Q(q_1)(f_2)q_2 - f_2 \rho_Q(q_2)(f_1)q_1$$

for all $f_1, f_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$.

That is, a dull algebroid is a **Lie algebroid** if its bracket is in addition skew-symmetric and satisfies the Jacobi identity. The following examples are crucial in this paper.

Example 2.7.   1) Let $(A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid. The Dorfman connection that is dual to the bracket on sections of A is the Lie derivative

$$\pounds^A : \Gamma(A) \times \Gamma(A^*) \to \Gamma(A^*).$$

---

[6]Note that the map $\langle q, \cdot\rangle \cdot \mathbf{d}_B f$ can be seen as a section of $\mathrm{Hom}(B, B)$, i.e. as a derivation over $0 \in \mathfrak{X}(M)$.

2) Let C be a Courant algebroid and D a Dirac structure in C. The **Bott-Dorfman connection**

$$\Delta^D \colon \Gamma(D) \times \Gamma(C/D) \to \Gamma(C/D),$$

is defined by

$$\Delta^D_d \bar{c} = \overline{[\![d, c]\!]}, \quad \text{for } d \in \Gamma(D), c \in \Gamma(C).$$

Modulo the identification $D^* \simeq C/D$, this Dorfman connection is the Lie derivative $\Delta^D = \pounds^D \colon \Gamma(D) \times \Gamma(D^*) \to \Gamma(D^*)$ that is dual to the Lie algebroid structure on D.

Let $A \to M$ be a vector bundle. The vector bundle $TM \oplus A^*$ is always anchored by the projection pr_{TM} to TM, and paired with $A \oplus T^*M$ via the obvious pairing and the map $\mathbf{d}_{A \oplus T^*M} \colon C^\infty(M) \to \Gamma(A \oplus T^*M)$, $f \mapsto (0, \mathbf{d}f)$. We are particularly interested in Dorfman connections

$$\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M).$$

Assume that a subbundle $U \subseteq TM \oplus A^*$ has a Lie algebroid structure with the anchor pr_{TM}. The Lie algebroid bracket on $\Gamma(U)$ can be extended to a dull bracket on $TM \oplus A^*$. The Dorfman connection $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ that is dual to this bracket satisfies $\Delta_u \tau \in \Gamma(U^\circ)$ for all $u \in \Gamma(U)$ and $\tau \in \Gamma(U^\circ)$ and quotients hence to a Dorfman connection

$$\bar{\Delta} \colon \Gamma(U) \times \Gamma(A \oplus T^*M/U^\circ) \to \Gamma(A \oplus T^*M/U^\circ).$$

Note that $U^* \simeq A \oplus T^*M/U^\circ$ and the Dorfman connection $\bar{\Delta}$ is equal to $\pounds^U \colon \Gamma(U) \times \Gamma(U^*) \to \Gamma(U^*)$, the Lie derivative associated to the Lie algebroid U. We say that the Dorfman connection Δ is an **extension** of $\pounds^U$.

2.4. VB-algebroids and representations up to homotopy

We briefly recall the definitions of double vector bundles, of their *linear* and *core* sections, and of their *linear splittings* and *lifts*. We refer to [16, 32, 39] for more detailed treatments. A **double vector bundle** is a commutative

square

$$
\begin{array}{ccc}
D & \xrightarrow{\ \pi_B\ } & B \\[2pt]
{\scriptstyle \pi_A}\big\downarrow & & \big\downarrow{\scriptstyle q_B} \\[2pt]
A & \xrightarrow[\ q_A\ ]{} & M
\end{array}
$$

of vector bundles such that

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4)$$

for $d_1, d_2, d_3, d_4 \in D$ with $\pi_A(d_1) = \pi_A(d_2)$, $\pi_A(d_3) = \pi_A(d_4)$ and $\pi_B(d_1) = \pi_B(d_3)$, $\pi_B(d_2) = \pi_B(d_4)$. Here, $+_A$ and $+_B$ are the additions in $D \to A$ and $D \to B$, respectively. The vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and of π_B, which has a natural vector bundle structure on C over M. The inclusion $C \hookrightarrow D$ is denoted by $C_m \ni c \longmapsto \overline{c} \in \pi_A^{-1}(0_m^A) \cap \pi_B^{-1}(0_m^B)$.

The space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by two distinguished classes of sections (see [33]), the *linear* and the *core sections* which we now describe. For a section $c \colon M \to C$, the corresponding **core section** $c^\dagger \colon B \to D$ is defined as $c^\dagger(b_m) = \tilde{0}_{b_m} +_A \overline{c(m)}$, $m \in M$, $b_m \in B_m$. If not specified otherwise we denote the corresponding core section $A \to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B is written $\Gamma_B^c(D)$.

A section $\xi \in \Gamma_B(D)$ is called **linear** if $\xi \colon B \to D$ is a bundle morphism from $B \to M$ to $D \to A$ over a section $a \in \Gamma(A)$. The space of linear sections of D over B is denoted by $\Gamma_B^\ell(D)$. Given $\psi \in \Gamma(B^* \otimes C)$, there is a linear section $\widetilde{\psi} \colon B \to D$ over the zero section $0^A \colon M \to A$ given by $\widetilde{\psi}(b_m) = \tilde{0}_{b_m} +_A \overline{\psi(b_m)}$. We call $\widetilde{\psi}$ a **core-linear section**.

Example 2.8. Let A, B, C be vector bundles over M and consider $D = A \times_M B \times_M C$. With the vector bundle structures $D = q_A^!(B \oplus C) \to A$ and $D = q_B^!(A \oplus C) \to B$, one finds that $(D; A, B; M)$ is a double vector bundle called the **decomposed double vector bundle with core** C. The core sections are given by

$$c^\dagger \colon b_m \mapsto (0_m^A, b_m, c(m)), \ \text{where } m \in M, \ b_m \in B_m, \ c \in \Gamma(C),$$

and similarly for $c^\dagger \colon A \to D$. The space of linear sections $\Gamma_B^\ell(D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, \psi) \colon b_m \mapsto (a(m), b_m, \psi(b_m)), \ \text{where } \psi \in \Gamma(B^* \otimes C), \ a \in \Gamma(A).$$

In particular, the fibred product $A \times_M B$ is a double vector bundle over the sides A and B, with core $M \times 0$.

A **linear splitting** of $(D; A, B; M)$ is an injective morphism of double vector bundles $\Sigma \colon A \times_M B \hookrightarrow D$ over the identity on the sides A and B. Any double vector bundle admits a linear splitting [14, 39]. A linear splitting Σ of D is also equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$-modules

$$(2.4) \qquad 0 \longrightarrow \Gamma(B^* \otimes C) \hookrightarrow \Gamma_B^\ell(D) \longrightarrow \Gamma(A) \longrightarrow 0,$$

where the third map is the map that sends a linear section (ξ, a) to its base section $a \in \Gamma(A)$. The splitting σ_A is called a **horizontal lift**. Given Σ, the horizontal lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$ is given by $\sigma_A(a)(b_m) = \Sigma(a(m), b_m)$ for all $a \in \Gamma(A)$ and $b_m \in B$. By the symmetry of a linear splitting, we find that a lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$ is equivalent to a lift $\sigma_B \colon \Gamma(B) \to \Gamma_A^\ell(D)$, $\sigma_B(b)(a(m)) = \sigma_A(a)(b(m))$.

Example 2.9. Let $q_E \colon E \to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE \to TM$ are the derivatives of the structure maps of $E \to M$.

$$
\begin{array}{ccc}
TE & \xrightarrow{p_E} & E \\
\downarrow{\scriptstyle Tq_E} & & \downarrow{\scriptstyle q_E} \\
TM & \xrightarrow{p_M} & M
\end{array}
$$

The space TE is a double vector bundle with core bundle $E \to M$. The map $\bar{} \colon E \to p_E^{-1}(0^E) \cap (Tq_E)^{-1}(0^{TM})$ sends $e_m \in E_m$ to $\bar{e}_m = \frac{d}{dt}\big|_{t=0} te_m \in T_{0_m^E} E$. Hence the core vector field corresponding to $e \in \Gamma(E)$ is the vertical lift $e^\uparrow \colon E \to TE$, i.e. the vector field with flow $\phi \colon E \times \mathbb{R} \to E$, $\phi_t(e_m') = e_m' + te(m)$. An element of $\Gamma_E^\ell(TE) = \mathfrak{X}^\ell(E)$ is called a **linear vector field**. It is well-known (see e.g. [32]) that a linear vector field $\xi \in \mathfrak{X}^l(E)$ covering $X \in \mathfrak{X}(M)$ corresponds to a derivation $D^* \colon \Gamma(E^*) \to \Gamma(E^*)$ over $X \in \mathfrak{X}(M)$. The precise correspondence is given by[7]

$$(2.5) \qquad \xi(\ell_\varepsilon) = \ell_{D^*(\varepsilon)} \quad \text{and} \quad \xi(q_E^* f) = q_E^*(X(f))$$

---

[7]Since its flow is a flow of vector bundle morphisms, a linear vector field sends linear functions to linear functions and pullbacks to pullbacks.

for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$. We write $\widehat{D}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation D of $\Gamma(E)$. The choice of a linear splitting Σ for $(TE; TM, E; M)$ is equivalent to the choice of a connection on E: we can define $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by $\sigma_{TM}(X) = \widehat{\nabla_X}$ for all $X \in \mathfrak{X}(M)$. Conversely, a connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defines a lift $\sigma_{TM}^\nabla \colon \mathfrak{X}(M) \to \mathfrak{X}^l(E)$ and a linear splitting $\Sigma^\nabla \colon TM \times_M E \to TE$. Given ∇, it is easy to see using the equalities in (2.5) that the Lie bracket of vector fields on E is given by $\left[\sigma_{TM}^\nabla(X), \sigma_{TM}^\nabla(Y)\right] = \sigma_{TM}^\nabla[X, Y] - R_\nabla(X, Y)^\uparrow$, $\left[\sigma_{TM}^\nabla(X), e^\uparrow\right] = (\nabla_X e)^\uparrow$, $\left[e_1^\uparrow, e_2^\uparrow\right] = 0$, for all $X, Y \in \mathfrak{X}(M)$ and $e, e_1, e_2 \in \Gamma(E)$.

A double vector bundle $(D; A, B; M)$ is a **VB-algebroid** ([31]; see also [16]) if there are Lie algebroid structures on $D \to B$ and $A \to M$, such that the anchor $\Theta \colon D \to TB$ is a morphism of double vector bundles over $\rho_A \colon A \to TM$ on one side and if the Lie bracket is linear:

$$\begin{aligned}
[\Gamma_B^\ell(D), \Gamma_B^\ell(D)] &\subset \Gamma_B^\ell(D), \\
[\Gamma_B^\ell(D), \Gamma_B^c(D)] &\subset \Gamma_B^c(D), \\
[\Gamma_B^c(D), \Gamma_B^c(D)] &= 0.
\end{aligned}$$

The vector bundle $A \to M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1, \xi_2 \in \Gamma_B^\ell(D)$ are linear over $a_1, a_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]$ is linear over $[a_1, a_2]$.

Example 2.10. Consider a Lie algebroid A. For $a \in \Gamma(A)$, we have two particular types of sections of $TA \to TM$: the linear section $Ta \colon TM \to TA$, and the core section $a^\dagger \colon TM \to TA$, $a^\dagger(x_m) = T_m 0^A x_m +_{p_A} \frac{d}{dt}\big|_{t=0} t \cdot a(m)$. The identities $[Ta, Tb] = T[a, b]$, $[Ta, b^\dagger] = [a, b]^\dagger$, $[a^\dagger, b^\dagger] = 0$, $\rho_{TA}(Ta) = \widehat{[\rho(a), \cdot]} \in \mathfrak{X}^l(TM)$ and $\rho_{TA}(a^\dagger) = (\rho(a))^\uparrow \in \mathfrak{X}^c(TM)$ define a VB-algebroid structure $(TA \to TM, A \to M)$.

The cotangent space $(T^*A \to A^*, A \to M)$ also is a VB-algebroid. The projection $r_A \colon T^*A \to A^*$ is given by $r_A(\alpha_{a_m})(b_m) = \alpha_{a_m}\left(\frac{d}{dt}\big|_{t=0} a_m + t b_m\right)$ for all $b_m \in A_m$. For $\theta \in \Omega^1(M)$ we have

$$\theta^\dagger \in \Gamma_{A^*}^c(T^*A), \quad \theta^\dagger(\alpha(m)) = \mathbf{d}_{0_m^A} \ell_\alpha - q_A^* \theta(0_m^A)$$

and for $a \in \Gamma(A)$, the section $a^R \in \Gamma_{A^*}^l(T^*A)$ is defined by $a^R(\alpha(m)) = \mathbf{d}_{a(m)}(\ell_\alpha - q_A^*\langle \alpha, a \rangle)$ for $\alpha \in \Gamma(A^*)$. The Lie algebroid structure on $T^*A \to A^*$ is given by $[a_1^R, a_2^R] = [a_1, a_2]^R$, $[a^R, \theta^\dagger] = (\mathscr{L}_{\rho(a)}\theta)^\dagger$, $[\theta_1^\dagger, \theta_2^\dagger] = 0$,

$\rho_{T^*A}(a^R) = \widehat{\pounds_a} \in \mathfrak{X}(A^*)$ and $\rho_{T^*A}(\theta^\dagger) = (\rho^*\theta)^\uparrow \in \mathfrak{X}(A^*)$ for $a, a_i \in \Gamma(A)$ and $\theta, \theta_i \in \Omega^1(M)$.

The space $TA \times_A T^*A$ has the structure of a double vector bundle:

$$\begin{array}{ccc} TA \oplus T^*A & \xrightarrow{\Pi_A} & TM \oplus A^* \\ \scriptstyle{c_A}\downarrow & & \downarrow\scriptstyle{p_M} \\ A & \xrightarrow{q} & M \end{array}$$

The vertical vector bundle is the Pontryagin bundle $TA \oplus T^*A$ of A seen as a manifold, and the horizontal projection is defined by $\Pi_A(v_{a_m}, \alpha_{a_m}) = (q_*(v_{a_m}), r_A(\alpha_{a_m}))$.

The Lie algebroid $TA \oplus T^*A \to TM \oplus A^*$ is defined as the pullback to the diagonals $\Delta_A \to \Delta_M$ of the Lie algebroid $TA \times T^*A \to TM \times A^*$. We have the special linear sections $a^l := (Ta, a^R)\colon TM \oplus A^* \to TA \oplus T^*A$ for $a \in \Gamma(A)$ and $(b,\theta)^\dagger := (b^\dagger, \theta^\dagger)\colon TM \oplus A^* \to TA \oplus T^*A$ for $(b,\theta) \in \Gamma(A \oplus T^*M)$. We write $\Theta\colon TA \oplus T^*A \to T(TM \oplus A^*)$ for the anchor of $TA \oplus T^*A \to TM \oplus A^*$. The Lie algebroid $(TA \oplus T^*A, \Theta, [\cdot,\cdot])$ is described by the following identities

$$[a_1^l, a_2^l] = [a_1, a_2]^l, \quad [a^l, \tau^\dagger] = (\pounds_a\tau)^\dagger, \quad [\tau_1^\dagger, \tau_2^\dagger] = 0$$
$$\Theta(a^l) = \widehat{\pounds_a}, \quad \Theta(\tau^\dagger) = ((\rho,\rho^t)\tau)^\uparrow.$$

for $a, a_1, a_2 \in \Gamma(A)$, $\tau, \tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$.

Let E_0, E_1 be two vector bundles over the same base M as A, and $\partial\colon E_0 \to E_1$ a vector bundle morphism. A **2-term representation up to homotopy** of A on $\partial\colon E_0 \to E_1$ [1, 16] is the collection of two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial \circ \nabla^0 = \nabla^1 \circ \partial$, and an element $R \in \Omega^2(A, \mathrm{Hom}(E_1, E_0))$ such that $R_{\nabla^0} = R \circ \partial$, $R_{\nabla^1} = \partial \circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}}R = 0$, where ∇^{Hom} is the connection induced on $\mathrm{Hom}(E_1, E_0)$ by ∇^0 and ∇^1 and the operator $\mathbf{d}_{\nabla^{\mathrm{Hom}}}$ on $\Omega^\bullet(A, \mathrm{Hom}(E_1, E_0))$ is given by the Koszul formula:

$$\mathbf{d}_{\nabla^{\mathrm{Hom}}}\omega(a_1, \dots, a_{k+1}) = \sum_{i<j}(-1)^{i+j}\omega([a_i, a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{k+1})$$
$$+ \sum_i (-1)^{i+1}\nabla^{\mathrm{Hom}}_{a_i}(\omega(a_1, \dots, \hat{a}_i, \dots, a_{k+1}))$$

for all $\omega \in \Omega^k(A, \mathrm{Hom}(E_1, E_0))$ and $a_1, \dots, a_{k+1} \in \Gamma(A)$.

Consider again a VB-algebroid $(D \to B, A \to M)$. The anchor $\Theta(c^\dagger)$ of a core section $c^\dagger \in \Gamma_B^c(D)$ is given by $\Theta(c^\dagger) = (\partial_B c)^\uparrow$, defining a vector bundle morphism $\partial_B \colon C \to B$. Choose a linear splitting $\Sigma \colon A \times_M B \to D$. Since the anchor Θ of a linear section is linear, for each $a \in \Gamma(A)$ the vector field $\Theta(\sigma_A(a)) \in \mathfrak{X}^l(B)$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$. This defines a linear connection $\nabla^B \colon \Gamma(A) \times \Gamma(B) \to \Gamma(B) \colon \Theta(\sigma_A(a)) = \widehat{\nabla_a^B}$ for all $a \in \Gamma(A)$. Since the bracket of a linear section with a core section is again a core section, we find a linear connection $\nabla^C \colon \Gamma(A) \times \Gamma(C) \to \Gamma(C)$ such that $[\sigma_A(a), c^\dagger] = (\nabla_a^C c)^\dagger$ for all $c \in \Gamma(C)$ and $a \in \Gamma(A)$. The difference $\sigma_A[a_1, a_2] - [\sigma_A(a_1), \sigma_A(a_2)]$ is a core-linear section for all $a_1, a_2 \in \Gamma(A)$. This defines a vector valued form $R \in \Omega^2(A, \operatorname{Hom}(B, C))$ by $[\sigma_A(a_1), \sigma_A(a_2)]$ $= \sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)}$, for all $a_1, a_2 \in \Gamma(A)$. For more details on these constructions, see [16], where the following result is proved.

Theorem 2.11. *Let $(D \to B; A \to M)$ be a VB-algebroid and choose a linear splitting $\Sigma \colon A \times_M B \to D$. The triple (∇^B, ∇^C, R) defined as above is a 2-term representation up to homotopy of A on the complex $\partial_B \colon C \to B$.*

Conversely, let $(D; A, B; M)$ be a double vector bundle with core C such that A has a Lie algebroid structure, and choose a linear splitting $\Sigma \colon A \times_M B \to D$. Then if (∇^B, ∇^C, R) is a 2-term representation up to homotopy of A on a complex $\partial_B \colon C \to B$, then the equations above define a VB-algebroid structure on $(D \to B; A \to M)$.

Example 2.12. Let $E \to M$ be a vector bundle. By Example 2.9 the tangent double TE has a VB-algebroid structure $(TE \to E, TM \to M)$. Consider a linear splitting $\Sigma \colon E \times_M TM \to TE$ and the corresponding linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. The representation up to homotopy corresponding to this splitting is given by $\partial_E = \operatorname{id}_E \colon E \to E$, $(\nabla, \nabla, R_\nabla)$.

Example 2.13. Let A be a Lie algebroid and consider the VB-algebroid $(TA \to TM, A \to M)$; see Example 2.10. A linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ defines a horizontal lift $\sigma_A \colon \Gamma(A) \to \Gamma_{TM}^l(TA)$. The corresponding 2-term representation up to homotopy is on $\partial_{TM} = \rho \colon A \to TM$ and given by $(\nabla^{\mathrm{bas}}, \nabla^{\mathrm{bas}}, R_\nabla^{\mathrm{bas}})$, where $\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and $\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ are the basic connections associated to ∇:

$$\nabla_a^{\mathrm{bas}} a' = [a, a'] + \nabla_{\rho(a')} a, \qquad \nabla_a^{\mathrm{bas}} X = [\rho(a), X] + \rho(\nabla_X a)$$

for all $a, a' \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$ [1, 13]. The tensor

$$R_\nabla^{\mathrm{bas}} \in \Omega^2(A, \operatorname{Hom}(TM, A))$$

is defined by

$$R_\nabla^{\text{bas}}(a_1, a_2)X = -\nabla_X[a_1, a_2] + [\nabla_X a_1, a_2] + [a_1, \nabla_X a_2]$$
$$- \nabla_{\nabla_{a_1}^{\text{bas}} X} a_2 + \nabla_{\nabla_{a_2}^{\text{bas}} X} a_1$$

for all $a_1, a_2 \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$.

Example 2.14. We now consider the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ with core $A \oplus T^*M$. Dorfman connections[8] $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ are equivalent to linear splittings $\Sigma^\Delta \colon (TM \oplus A^*) \times_M A \to TA \oplus T^*A$;

$$\Sigma^\Delta(X, \alpha)(a_m) = (T_m a X(m), \mathbf{d}\ell_\alpha(a_m)) - \Delta_{(X,\alpha)}(a, 0)^\uparrow(a_m),$$

where for $b \in \Gamma(A)$ and $\theta \in \Omega^1(M)$, the pair $(b, \theta)^\uparrow(a_m) \in T_{a_m}A \times T_{a_m}^*A$ is defined by $(b, \theta)^\uparrow(a_m) = \big(b^\uparrow(a_m), (q_A^*\theta)(a_m)\big)$ [20].

The horizontal lift

$$\sigma_A^\Delta \colon \Gamma(A) \to \Gamma_{TM \oplus A^*}^l(TA \oplus_A T^*A)$$

is given by $\sigma_A^\Delta(a)(v_m, \alpha_m) = (T_m a v_m, \mathbf{d}_{a_m}\ell_\alpha) - \Delta_{(X,\alpha)}(a, 0)^\uparrow(a_m)$ for any choice of section $(X, \alpha) \in \Gamma(TM \oplus A^*)$ such that $(X, \alpha)(m) = (v_m, \alpha_m)$, or in other words by

$$\sigma_A^\Delta(a) = (Ta, a^R) - \widetilde{\Omega.a} = a^l - \widetilde{\Omega.a}$$

for all $a \in \Gamma(A)$, where $\Omega \colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M)$ is defined by

$$(2.6) \qquad\qquad \Omega_{(X,\alpha)}a = \Delta_{(X,\alpha)}(a, 0) - (0, \mathbf{d}\langle \alpha, a\rangle).$$

The two maps $\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*)$,

$$(2.7) \qquad\qquad \nabla_a^{\text{bas}}(X, \alpha) = (\rho, \rho^t)(\Omega_{(X,\alpha)}a) + \pounds_a(X, \alpha)$$

and $\nabla^{\text{bas}} \colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$,

$$\nabla_a^{\text{bas}}(a', \theta) = \Omega_{(\rho, \rho^t)(a', \theta)}a + \pounds_a(a', \theta)$$

---

[8]$TM \oplus A^*$ is always anchored by pr_{TM}.

are ordinary linear A-connections. The formula

$$R_\Delta^{\text{bas}}(a_1, a_2)(X, \alpha) = -\Omega_{(X,\alpha)}[a_1, a_2] + \pounds_{a_1}\left(\Omega_{(X,\alpha)}a_2\right) - \pounds_{a_2}\left(\Omega_{(X,\alpha)}a_1\right)$$
$$+ \Omega_{\nabla_{a_2}^{\text{bas}}(X,\alpha)}a_1 - \Omega_{\nabla_{a_1}^{\text{bas}}(X,\alpha)}a_2$$

defines a tensor $R_\Delta^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM \oplus A^*, A \oplus T^*M))$. We prove in [20] that the 2-term representation up to homotopy defined by the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ and the splitting Σ is the 2-term representation of A on $(\rho, \rho^t)\colon A \oplus T^*M \to TM \oplus A^*$ given by $(\nabla^{\text{bas}}, \nabla^{\text{bas}}, R_\Delta^{\text{bas}})$. Compare this with the 2-term representation up to homotopy given by a linear splitting of $(TA \to TM, A \to M)$ as in Example 2.12.

3. A-Manin pairs and Dirac bialgebroids

A Manin pair over a manifold M is a pair (C, U) of vector bundles over M, where C has the structure of a Courant algebroid and U is a Dirac structure in C [5].

Definition 3.1. Let $(A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid.

1) An A-**Manin pair over** M is a Manin pair (C, U) over M, together with
   a) an injective morphism $\iota\colon U \to TM \oplus A^*$ of vector bundles, such that $\rho_U := \rho_C|_U = \text{pr}_{TM} \circ \iota$ and
   b) a (degenerate) Courant morphism $\Phi\colon A \oplus T^*M \to C$ such that

   $$\Phi(A \oplus T^*M) + U = C$$

   and $\langle u, \Phi(\tau)\rangle_C = \langle \iota(u), \tau\rangle$ for all $(u, \tau) \in U \times_M (A \oplus T^*M)$.

2) Let $(U \to M, \rho_U, [\cdot, \cdot]_U)$ be a Lie algebroid and $\iota\colon U \to TM \oplus A^*$ an injective vector bundle morphism that is compatible with the anchors: $\text{pr}_{TM} \circ \iota = \rho_U$. The triple (A, U, ι) is a **Dirac bialgebroid** (over A) if U is the Dirac structure and ι is the injective morphism of an A-Manin pair.

3) Two A-Manin pairs (C, U) and (C', U'), and respectively two Dirac bialgebroids (A, U, ι) and (A, U', ι'), are **equivalent** if they define the same Lie algebroid $\iota(U) = \iota'(U')$ in the pr_{TM}-anchored vector bundle $TM \oplus A^*$.

By definition, an A-Manin pair determines an unique Dirac bialgebroid (A, U, ι). Conversely, we show that the equivalence class of an A-Manin pair (C, U) with structure maps ι, Φ is completely determined by the equivalence class of the corresponding Dirac bialgebroid (A, U, ι). More precisely, given A, U and ι, we can reconstruct C and Φ up to isomorphism. Consider a Dirac bialgebroid (A, U, ι) and identify U with $\iota(U) \subseteq A \oplus T^*M$. If $\tau \in U^\circ \subseteq TM \oplus A^*$, then $\Phi(\tau)$ satisfies

$$\langle u, \Phi(\tau) \rangle_C = \langle \tau, u \rangle = 0$$

for all $u \in U$. Since U is a Dirac structure, we find that Φ restricts to a map $U^\circ \to U$. (Conversely, we find easily that $\Phi(\tau) \in U$ if and only if $\tau \in U^\circ$.) Next choose $\tau_1 \in U^\circ$ and $\tau_2 \in A \oplus T^*M$. Then

$$\langle \Phi(\tau_1), \tau_2 \rangle = \langle \Phi(\tau_1), \Phi(\tau_2) \rangle_C = \langle \tau_1, \tau_2 \rangle_d = \langle (\rho, \rho^t)\tau_1, \tau_2 \rangle,$$

which shows that $\Phi|_{U^\circ} = (\rho, \rho^t)|_{U^\circ} \colon U^\circ \to U$. In particular, (ρ, ρ^t) sends U° to U, and U° is isotropic in $A \oplus T^*M$. Consider the vector bundle map $U \oplus A \oplus T^*M \to C$, $(u, \tau) \mapsto u + \Phi(\tau)$. By hypothesis, this map is surjective. Its kernel is the set of pairs (u, τ) with $u = -\Phi(\tau)$, i.e. the graph of $-(\rho, \rho^t)|_{U^\circ} \colon U^\circ \to U$. It follows that C can be identified with

$$(3.1) \qquad \frac{U \oplus A \oplus T^*M}{\operatorname{graph}(-(\rho, \rho^t)|_{U^\circ} \colon U^\circ \to U)}.$$

We use the notation $u \oplus \tau$ for $u + \Phi(\tau) \in C$. The anchor of C is then $c \colon C \to TM$, $c(u \oplus \tau) = \operatorname{pr}_{TM}(u) + \rho \circ \operatorname{pr}_A(\tau)$, and the bracket is given by $\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rangle_C = \langle u_1 + \Phi(\tau_1), u_2 + \Phi(\tau_2) \rangle_C = \langle u_1, \tau_2 \rangle + \langle u_2, \tau_1 \rangle + \langle \tau_1, (\rho, \rho^t)\tau_2 \rangle$. The map $\mathcal{D} \colon C^\infty(M) \to \Gamma(C)$ is given by $\langle\langle u \oplus \tau, \mathcal{D}f \rangle\rangle_C = (\operatorname{pr}_{TM}(u) + \rho \circ \operatorname{pr}_A(\tau))f$ for all $u \oplus \tau \in \Gamma(C)$, i.e. $\mathcal{D}f = 0 \oplus (0, \mathbf{d}f)$.

We show that the Courant algebroid bracket is uniquely determined by the Lie algebroid structure on U. Take an extension $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ of $\pounds^U$. First note that since the dual dull bracket on $\Gamma(TM \oplus A^*)$ is anchored by pr_{TM}, we have

$$(3.2) \qquad \Delta_{(X, \alpha)}(a, \theta) = \Delta_u(a, 0) + (0, \pounds_X \theta)$$

for $X \in \mathfrak{X}(M)$, $\alpha \in \Gamma(A^*)$, $a \in \Gamma(A)$ and $\theta \in \Omega^1(M)$. We define the connection $\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*)$ associated to Δ as in (2.7), i.e. $\nabla_a^{\mathrm{bas}} \nu = (\rho, \rho^t)\big(\Delta_\nu(a, 0) - (0, \mathbf{d}\langle \nu, (a, 0)\rangle)\big) + \pounds_a \nu$ for $a \in \Gamma(A)$ and $\nu \in$

$\Gamma(TM \oplus A^*)$. We prove that the bracket on $\Gamma(C)$ is given by

$$(3.3) \quad [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]_C = ([u_1, u_2]_U + \nabla^{\mathrm{bas}}_{\mathrm{pr}_A \tau_1} u_2 - \nabla^{\mathrm{bas}}_{\mathrm{pr}_A \tau_2} u_1)$$
$$\oplus ([\tau_1, \tau_2]_d + \Delta_{u_1} \tau_2 - \Delta_{u_2} \tau_1 + (0, \mathbf{d}\langle \tau_1, u_2 \rangle)).$$

We know that

$$[\![u_1 \oplus 0, u_2 \oplus 0]\!]_C = [u_1, u_2]_U \oplus 0, \quad [\![0 \oplus \tau_1, 0 \oplus \tau_2]\!]_C = 0 \oplus [\tau_1, \tau_2]_d$$

for $u_1, u_2 \in \Gamma(U)$ and $\tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$.

Note that C/U is isomorphic to $A \oplus T^*M/U^\circ$ via the map $\Psi \colon C/U \to A \oplus T^*M/U^\circ$, $\overline{u \oplus \tau} \to \overline{\tau}$. These two vector bundles are isomorphic to U^* and $\pounds^U$ is given by

$$\pounds^U_u \overline{\tau} = \overline{\Delta_u \tau} \quad \text{in} \quad A \oplus T^*M/U^\circ$$

and by

$$\pounds^U_u \overline{u' \oplus \tau} = \overline{[\![u \oplus 0, u' \oplus \tau]\!]_C} \quad \text{in} \quad C/U.$$

Hence $\pounds^U_u \overline{0 \oplus \tau} = (\Psi^{-1} \circ \pounds^U_u \circ \Psi) \overline{0 \oplus \tau} = \overline{0 \oplus \Delta_u \tau}$ and so

$$\overline{[\![u \oplus 0, 0 \oplus \tau]\!]_C} = \overline{0 \oplus \Delta_u \tau}.$$

We want to compute $v = v(\tau, u) \in \Gamma(U)$ such that $[\![u \oplus 0, 0 \oplus \tau]\!]_C = v \oplus \Delta_u \tau$. First note that

$$[\![u \oplus 0, 0 \oplus \tau]\!]_C + [\![0 \oplus \tau, u \oplus 0]\!]_C$$
$$= \mathcal{D}\langle\langle u \oplus 0, 0 \oplus \tau \rangle\rangle_C = \mathcal{D}\langle \tau, u \rangle = (0, \mathbf{d}\langle \tau, u \rangle).$$

We write $u = (X, \alpha)$ and $\tau = (a, \theta)$. Then, by the Leibniz property of the Courant algebroid bracket on C, we find for $\tau' = (b, \omega) \in \Gamma(A \oplus T^*M)$:

$$\rho(a)\langle u, \tau' \rangle = c(0 \oplus \tau)\langle\langle u \oplus 0, 0 \oplus \tau' \rangle\rangle_C$$
$$= \langle\langle [\![0 \oplus \tau, u \oplus 0]\!], 0 \oplus \tau' \rangle\rangle_C + \langle\langle u \oplus 0, [\![0 \oplus \tau, 0 \oplus \tau']\!]\rangle\rangle_C$$
$$= \langle\langle (-v) \oplus (-\Delta_u \tau + (0, \mathbf{d}\langle \tau, u \rangle)), 0 \oplus \tau' \rangle\rangle_C$$
$$+ \langle\langle u \oplus 0, 0 \oplus (\pounds_a \tau' + (0, -\mathbf{i}_{\rho(b)} \mathbf{d}\theta)) \rangle\rangle_C$$
$$\overset{(3.2)}{=} -\langle v, \tau' \rangle - \langle (\rho, \rho^t)(\Delta_u(a, 0) - (0, \mathbf{d}\langle a, \alpha \rangle)), \tau' \rangle + \cancel{\rho(b)\langle \theta, X \rangle}$$
$$- \cancel{\langle \pounds_X \theta, \rho(b) \rangle} + \langle u, \pounds_a \tau' \rangle - \cancel{\mathbf{d}\theta(\rho(b), X)}$$

This leads to $-\langle v, \tau' \rangle = \langle (\rho, \rho^t)(\Delta_u(a, 0) - (0, \mathbf{d}\langle a, \alpha \rangle)) + \pounds_a u, \tau' \rangle$ and, since τ' was arbitrary, we have shown that $[\![u \oplus 0, 0 \oplus \tau]\!] = (-\nabla^{\mathrm{bas}}_a u) \oplus \Delta_u \tau$. Note

that (3.3) does not depend on the choice of the extension Δ of $\pounds^U$. We have proved the following proposition:

Proposition 3.2. *The map*

$$\left\{ \begin{array}{c} \textit{Equivalence classes} \\ \textit{of A-Manin pairs} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \textit{Equivalence classes of} \\ \textit{Dirac bialgebroids over } A \end{array} \right\},$$

that sends the class of (C, U) (with structure maps ι, Φ) to the class of (A, U, ι), is a bijection.

4. The Dirac bialgebroid associated to a Dirac groupoid

First we recall the definition of a Dirac groupoid, as well as some of their properties.

Definition 4.1 ([38]). A Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$ is a Lie groupoid $G \rightrightarrows M$ with a Dirac structure D such that $\mathsf{D} \subseteq TG \oplus T^*G$ is a Lie subgroupoid.

The Dirac structure D is then said to be **multiplicative**. We write U for the set of units of D seen as a groupoid, i.e. for the vector bundle $U = \mathsf{D} \cap (TM \oplus A^*)$. The inclusion of U in $TM \oplus A^*$ is always called $\iota \colon U \to TM \oplus A^*$. We write K for the vector bundle $K = \mathsf{D} \cap (A \oplus T^*M)$. We have $K = U^\circ$ and also $(\rho, \rho^t)(K) \subseteq U$ since $\mathbb{T}\mathsf{t}(\mathsf{D}) \subseteq U$.

Let u be a section of U. Then there exists a smooth section d of D such that $d|_M = u$ and $\mathbb{T}\mathsf{t} \circ d = u \circ \mathsf{t}$ [21]. We then write $u \sim_\epsilon d$ and $d \sim_\mathsf{t} u$. A section d of D satisfying these two conditions is called a **star section**. In [21] we proved the following two results.

Theorem 4.2. *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid, $d \sim_\mathsf{t} u$ a star section of D and $a \in \Gamma(A)$. Then the Lie derivative $\pounds_{a^r} d$ can be written as a sum*

$$\pounds_{a^r} d = \mathcal{L}_a d + (\tau_{d,a})^r$$

with $\mathcal{L}_a d$ a star section and $\tau_{d,a}$ a section of $A \oplus T^*M$.

Theorem 4.3. *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid. Then there is an induced bracket $[\cdot, \cdot]_U \colon \Gamma(U) \times \Gamma(U) \to \Gamma(U)$ defined by $[u, u']_U = [\![d, d']\!]|_M$ for any choice of star sections $d \sim_\mathsf{t} u$, $d' \sim_\mathsf{t} u'$ of D. The triple $(U, [\cdot, \cdot]_U, \rho_U := \mathrm{pr}_{TM})$ is a Lie algebroid over M.*

We shall prove that (A, U, ι) is a Dirac bialgebroid. In order to do this, we describe how an A-Manin pair is naturally associated to a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$. The Lie algebroid U is the Dirac structure in this Manin pair.

Set $\mathsf{B} =: (U \oplus A \oplus T^*M)/K^{\mathsf{t}}$, where we see $K^{\mathsf{t}} = (\ker \mathbb{T}\mathsf{t} \cap \mathsf{D})|_M$ as a subbundle of $U \oplus (A \oplus T^*M) = \mathsf{D}|_M + \ker \mathbb{T}\mathsf{s}|_M$. We write sections of B as pairs $u \oplus \tau := u + \tau + K^{\mathsf{t}}$, with $u \in \Gamma(U)$ and $\tau \in \Gamma(A \oplus T^*M)$. Since $(\ker \mathbb{T}\mathsf{s})^{\perp} = \ker \mathbb{T}\mathsf{t}$ relative to the canonical symmetric pairing on $TG \oplus T^*G$, we have $(U \oplus (A \oplus T^*M))^{\perp} = (\ker \mathbb{T}\mathsf{s}|_M + \mathsf{D}|_M)^{\perp} = K^{\mathsf{t}}$, and so the pairing $\langle \cdot, \cdot \rangle$ on $TG \oplus T^*G$ restricts and projects to a nondegenerate symmetric pairing on B:

$$\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rangle_{\mathsf{B}} = \alpha_2(a_1) + \alpha_1(a_2) + \theta_1(\bar{X}_2 + \rho(a_2)) + \theta_2(\bar{X}_1 + \rho(a_1))$$
$$= \langle u_1, \tau_2 \rangle + \langle u_2, \tau_1 \rangle + \langle \tau_1, \tau_2 \rangle_d$$

where $u_i = (\bar{X}_i, \alpha_i)$ and $\tau_i = (a_i, \theta_i)$. We then define $\mathsf{b} \colon \mathsf{B} \to TM$ by $\mathsf{b}(u \oplus \tau) = \mathrm{pr}_{TM}(u) + \rho \circ \mathrm{pr}_A(\tau)$ and, finally, $[\![\cdot, \cdot]\!]_{\mathsf{B}} \colon \Gamma(\mathsf{B}) \times \Gamma(\mathsf{B}) \to \Gamma(\mathsf{B})$ by

$$[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]_{\mathsf{B}} = [\![d_1 + \tau_1^r, d_2 + \tau_2^r]\!]|_M + K^{\mathsf{t}}$$

for all $\tau_1, \tau_2 \in \Gamma(\ker \mathbb{T}\mathsf{s}|_M)$, $u_1, u_2 \in \Gamma(U)$ and star sections $d_i \sim_{\mathsf{t}} u_i$ of D. Here, $[\![\cdot, \cdot]\!]$ is the Courant-Dorfman bracket on sections of $TG \oplus T^*G$. We have proved in [21] that $(\mathsf{B}, \mathsf{b}, [\![\cdot, \cdot]\!]_{\mathsf{B}}, \langle \cdot, \cdot \rangle_{\mathsf{B}})$ is a Courant algebroid.

Theorem 4.4. *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid, and let A be the Lie algebroid of $G \rightrightarrows M$. Then the pair (B, U) defined as above is naturally an A-Manin pair.*

Proof. By construction, U is a subbundle of $TM \oplus A^*$. We first show that U is a Dirac structure in B, when we identify U with its image under the injective map $U \to \mathsf{B}$, $u \mapsto u \oplus 0$. A dimension count shows that $2 \operatorname{rank}(U) = \operatorname{rank}(\mathsf{B})$, and U is obviously isotropic relative to $\langle \cdot, \cdot \rangle_{\mathsf{B}}$. By Theorem 4.3, $\Gamma(U)$ is closed under the bracket on $\Gamma(\mathsf{B})$.

By construction again, we find immediately that the map $\Phi \colon A \oplus T^*M \to \mathsf{B}$, $\tau \mapsto 0 \oplus \tau$ is a morphism of (degenerate) Courant algebroids: it is easy to check that $[\![0 \oplus \tau_1, 0 \oplus \tau_2]\!]_{\mathsf{B}} = 0 \oplus [\tau_1, \tau_2]_d$, $\langle 0 \oplus \tau_1, 0 \oplus \tau_2 \rangle_{\mathsf{B}} = \langle \tau_1, \tau_2 \rangle_d$ and $\mathsf{b}(0 \oplus \tau) = \rho \circ \mathrm{pr}_A(\tau)$. The sum $\iota_U + \Phi \colon U \oplus (A \oplus T^*M) \to \mathsf{B}$, $(u, \tau) \mapsto u \oplus \tau$ is surjective, and we have $\langle u, \Phi(\tau) \rangle = \langle u \oplus 0, 0 \oplus \tau \rangle_{\mathsf{B}} = \langle u, \tau \rangle$. $\square$

Corollary 4.5. *The triple (A, U, ι), with U endowed with the Lie algebroid structure defined in Theorem 4.3, is a Dirac bialgebroid.*

Our goal is to show that this defines a bijection between source simply connected Dirac groupoids and integrable Dirac bialgebroids.

The following proposition shows that the Lie derivative $\mathcal{L}^{\mathsf{D}}$ is right-invariant and projects to the Lie derivative $\mathcal{L}^U$. Note that this is in fact the key feature of a multiplicative Dirac structure, and that it resembles the right-invariance of the Bott connection corresponding to a multiplicative involutive distribution on a Lie groupoid [23].

Proposition 4.6. *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid.*

1) *The Dorfman connection*

$$\mathcal{L}^{\mathsf{D}} = \Delta^{\mathsf{D}} \colon \Gamma(\mathsf{D}) \times \Gamma(TG \oplus T^*G/\mathsf{D}) \to \Gamma(TG \oplus T^*G/\mathsf{D})$$

induces as follows a Dorfman connection $\Delta \colon \Gamma(U) \times \Gamma(A \oplus T^*M/K) \to \Gamma(A \oplus T^*M/K)$. For a star section $d \sim_{\mathsf{t}} u$ of D and a right-invariant section τ^r of $TG \oplus T^*G$, we have

$$\Delta_d^{\mathsf{D}} \overline{\tau^r} = (\Delta_u \bar{\tau})^r.$$

2) *Note that $A \oplus T^*M/K \simeq U^*$ and the Dorfman connection Δ is exactly the Dorfman connection defined by U in B, or in other words the Lie derivative $\mathcal{L}^U$ of $\Gamma(U^*)$ by sections of U.*

Proof of Proposition 4.6. For the first claim write $\tau = (a, \theta)$, with $a \in \Gamma(A)$ and $\theta \in \Omega^1(M)$, and $\mathrm{pr}_{TG}(d) = X$, $\mathrm{pr}_{TM}(u) = \bar{X}$. Using Theorem 4.2, we get the equality

$$[\![d, \tau^r]\!] = -[\![\tau^r, d]\!] + (0, \mathbf{d}\langle d, \tau^r \rangle) = -\pounds_{a^r} d + (0, \mathbf{i}_X \mathbf{dt}^*\theta) + (0, \mathbf{t}^* \mathbf{d}\langle u, \tau \rangle)$$
$$= -\mathcal{L}_a d - (\tau_{d,a} - (0, \mathbf{i}_{\bar{X}} \mathbf{d}\theta + \mathbf{d}\langle u, \tau \rangle))^r,$$

which implies

$$\overline{[\![d, \tau^r]\!]} = \overline{-(\tau_{d,a} - (0, \mathbf{i}_{\bar{X}} \mathbf{d}\theta + \mathbf{d}\langle u, \tau \rangle))^r}$$

in $TG \oplus T^*G/\mathsf{D}$. This is 0 if and only if $\tau_{d,a} - (0, \mathbf{i}_{\bar{X}} \mathbf{d}\theta + \mathbf{d}\langle u, \tau \rangle) \in \Gamma(K)$.

We show that this does not depend on the choice of d over the section $u \in \Gamma(U)$. If $d' \sim_{\mathsf{t}} u$ is an other star section of D over u, then we have $d - d' \in \Gamma(\mathsf{D} \cap \ker \mathbb{T}\mathsf{t})$ and $(d - d')|_M = 0$. Choose a (local) basis of sections $\tau_1, \ldots, \tau_r$ of K^{t}. Then $d - d'$ can be written as $d - d' = \sum_{i=1}^r f_i \tau_i^l$ with $f_1, \ldots, f_r \in C^\infty(G)$ vanishing on M. Using this, it is easy to check that $[\![d - d', \tau^r]\!] = -\sum_{i=1}^r a^r(f_i)\tau_i^l \in \Gamma(\mathsf{D})$ and so that $\overline{[\![d - d', \tau^r]\!]} = 0$. Set

$\Delta_u\bar\tau = \overline{-\tau_{d,a} + (0, \mathbf{i}_{\bar X}\mathbf{d}\theta + \mathbf{d}\langle u, \tau\rangle)}$. Checking that Δ is a Dorfman connection is straightforward.

For the second claim let u' be a section of U and d' a star section of D over u'. We compute

$$\begin{aligned}
\langle\Delta_u\bar\tau, u'\rangle &= \langle\Delta_d^{\mathsf{D}}\overline{\tau^r}, d'\rangle|_M = \langle[\![d, \tau^r]\!], d'\rangle|_M \\
&= \left(\mathrm{pr}_{TG}(d)\langle\tau^r, d'\rangle - \langle\tau^r, [\![d, d']\!]\rangle\right)|_M \\
&= \mathrm{pr}_{TM}(u)\langle\tau, u'\rangle - \langle\tau, [u, u']_U\rangle = \langle\pounds_u\bar\tau, u'\rangle.
\end{aligned}$$

$\square$

5. The Dirac bialgebroid associated to a Dirac algebroid

Let A be a Lie algebroid. Recall from Example 2.10 that the space $TA \oplus T^*A$ has the structure of a double vector bundle with sides $TM \oplus A^*$ and A and with core $A \oplus T^*M$. The vector bundle $TA \oplus T^*A \to A$ is equipped with the standard Courant algebroid structure, and the vector bundle $TA \oplus T^*A \to TM \oplus A^*$ is equipped with a Lie algebroid structure, which was described in Example 2.10. These two structures are compatible and $TA \oplus T^*A$ is an LA-Courant algebroid. (LA-Courant algebroids are defined in [25]. An alternative, perhaps more handy definition in terms of representations up to homotopy and Dorfman 2-representations is given in [22].)

Definition 5.1. A Dirac algebroid is a Lie algebroid A with a Dirac structure $\mathsf{D}_A \subseteq TA \oplus T^*A$ that is also a sub-algebroid of $TA \oplus T^*A \to TM \oplus A^*$ over a subbundle $U \subseteq TM \oplus A^*$.

We also say that D_A is an LA-Dirac structure on A. It is in fact a sub-double Lie algebroid of the LA-Courant algebroid $(TA \oplus T^*A; A, TM \oplus A^*; M)$.

This section shows how LA-Dirac structures are equivalent to A-Manin pairs.

5.1. Decompositions of LA-Dirac structures

Let A be a Lie algebroid. Recall that in §2.10 we have described linear splittings of the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$. Now we consider sub-double vector bundles

$$\begin{array}{ccc}
\begin{array}{ccc}
D & \longrightarrow & U \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
&
\text{of}
&
\begin{array}{ccc}
TA \oplus T^*A & \longrightarrow & TM \oplus A^* \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
\end{array}.$$

We denote the core of D by $K \subseteq A \oplus T^*M$. There exists a Dorfman connection Δ such that $D \to A$ is spanned by the sections $k^\dagger$, for all $k \in \Gamma(K)$ and $\sigma_{TM \oplus A^*}^\Delta(u)$ for all $u \in \Gamma(U)$ (see [20]). The Dorfman connection Δ is then said to be **adapted** to D and the double vector subbundle D is completely determined by the triple (U, K, Δ). Conversely we write $D_{(U,K,\Delta)}$ for the double vector subbundle defined in this manner by a Dorfman connection Δ and two subbundles $U \subseteq TM \oplus A^*$, $K \subseteq A \oplus T^*M$. The set of sections of $D \to U$ is spanned as a $C^\infty(U)$-module by the sections $\sigma_A^\Delta(a)|_U$ and $k^\dagger|_U$ for all $a \in \Gamma(A)$ and $k \in \Gamma(K)$. We have $D_{(U,K,\Delta)} = D_{(U,K,\Delta')}$ if and only if $(\Delta - \Delta')_u \tau \in \Gamma(K)$ for all $u \in \Gamma(U)$ and $\tau \in \Gamma(A \oplus T^*M)$.

By the results in [20], D is a Dirac structure D_A over A, if and only if $K = U^\circ$, any adapted Dorfman connection Δ satisfies $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$ and $k \in \Gamma(K)$ and the induced quotient Dorfman connection $\bar{\Delta} : \Gamma(U) \times \Gamma(A \oplus T^*M/K) \to \Gamma(A \oplus T^*M/K)$, $\bar{\Delta}_u \bar{\tau} = \overline{\Delta_u \tau}$ is flat. That is, since $A \oplus T^*M/U^\circ \simeq U^*$, the Dorfman connection $\bar{\Delta}$ is in fact dual to a Lie algebroid structure on U with anchor pr_{TM}. The Lie algebroid bracket is the restriction to $\Gamma(U) \times \Gamma(U)$ of the dull bracket $[\![\cdot, \cdot]\!]_\Delta$ that is dual to Δ. The quotient Dorfman connection and so also the obtained Lie algebroid structure on U do not depend on the choice of the adapted Dorfman connection to D_A.

Next, a double vector subbundle as above defines a VB-subalgebroid $(D \to A, U \to M)$ of $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ if and only if $(\rho, \rho^t)(K) \subseteq U$, $\nabla_a^{\mathrm{bas}} k \in \Gamma(K)$ for all $a \in \Gamma(A)$ and $k \in \Gamma(K)$, $\nabla_a^{\mathrm{bas}} u \in \Gamma(U)$ for all $u \in \Gamma(U)$ and $R_\Delta^{\mathrm{bas}}(a, b) u \in \Gamma(K)$ for all $a, b \in \Gamma(A)$ and $u \in \Gamma(U)$. We get the following theorem.

Theorem 5.2. *[20] Consider a Lie algebroid A and a Dorfman connection*

$$\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M).$$

Let $U \subseteq TM \oplus A^*$ and $K \subseteq A \oplus T^*M$ be subbundles. Then $D_{(U,K,\Delta)}$ is a Dirac structure in $TA \oplus T^*A \to A$ and a subalgebroid of $TA \oplus T^*A \to TM \oplus A^*$ over U if and only if:

1) $K = U^\circ$

2) $(\rho, \rho^t)(K) \subseteq U$,

3) $(U, \mathrm{pr}_{TM}, [\![\cdot, \cdot]\!]_\Delta|_{\Gamma(U) \times \Gamma(U)})$ *is a Lie algebroid,*

4) $\nabla_a^{\mathrm{bas}} u \in \Gamma(U)$ *for all $a \in \Gamma(A)$ and $u \in \Gamma(U)$,*

5) $R_\Delta^{\mathrm{bas}}(a, b) u \in \Gamma(K)$ *for all $u \in \Gamma(U)$, $a, b \in \Gamma(A)$.*

In the next subsection we use Theorem 5.2 to construct the Manin pair associated to the Dirac bialgebroid.

Remark 5.3. The equation

$$(5.1) \qquad \langle \nabla_{a'}^{\mathrm{bas}} \nu, \tau \rangle + \langle \nu, \nabla_{a'}^{\mathrm{bas}} \tau \rangle$$
$$= \rho(a') \langle \nu, \tau \rangle - \langle [\![\nu, (\rho, \rho^t) \tau]\!]_\Delta + [\![(\rho, \rho^t) \tau, \nu]\!]_\Delta, \tau' \rangle$$

for $\tau' = (a', \theta') \in \Gamma(A \oplus T^*M)$, $\nu \in \Gamma(TM \oplus A^*)$ and $\tau \in \Gamma(A \oplus T^*M)$, is easily checked.

Hence the two A-connections ∇^{bas} are not in duality, but if $K = U^\circ$ and the restriction of the dull bracket to $\Gamma(U) \times \Gamma(U)$ is skew-symmetric, then for $a \in \Gamma(A)$, $\nabla_a^{\mathrm{bas}} k \in \Gamma(K)$ for all $k \in \Gamma(K)$ if and only if $\nabla_a^{\mathrm{bas}} u \in \Gamma(U)$ for all $u \in \Gamma(U)$; see [20].

Given the linear splitting given by the Dorfman connection adapted to D_A, the Lie algebroid structure $\mathsf{D}_A \to A$ is described by the 2-term representation up to homotopy defined by the vector bundle morphism $\mathrm{pr}_A \colon K \to A$, the connections $(\mathrm{pr}_A \circ \Omega) \colon \Gamma(U) \times \Gamma(A) \to \Gamma(A)$ and $\Delta \colon \Gamma(U) \times \Gamma(K) \to \Gamma(K)$, and the curvature $R_\Delta \in \Omega^2(U, \mathrm{Hom}(A, K))$.

The Lie algebroid structure of the other side $\mathsf{D}_A \to U$ is described by the complex $(\rho, \rho^t) \colon K \to U$ with the basic connections $\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \Gamma(U) \to \Gamma(U)$ and $\nabla^{\mathrm{bas}} \colon \Gamma(A) \times \Gamma(K) \to \Gamma(K)$ and the basic curvature $R_\Delta^{\mathrm{bas}} \in \Omega^2(A, \mathrm{Hom}(U, K))$.

Remark 5.4. A long but straightforward computation shows that these two 2-term representations up to homotopy form a matched pair [15]. One of the seven equations defining the matched pair is exactly (A.4) in Lemma A.4, which is also the key to the most complicated equation. (A.5) is a useful tool to check that equation as well. Corollary A.3 is another one, and most of the remaining equations follow immediately from the definition of the involved objects.

As a consequence, (D_A, U, A, M) is a double Lie algebroid [15], and the core K has an induced Lie algebroid structure, which is given by the restriction of $[\cdot, \cdot]_d$ to $\Gamma(K)$, with Lie algebroid morphisms to A and U. To see this, use [15, Remark 3.5] and Lemmas A.1 and A.3 below.

5.2. The A-Manin pair associated to an LA-Dirac structure

Consider a triple (U, U°, Δ) as in Theorem 5.2 and define the vector bundle C with base M as in (3.1). We write $u \oplus \tau$ for the class in C of a pair

$(u, \tau) \in \Gamma(U \oplus (A \oplus T^*M))$. It is easy to check that the pairing

$$\langle\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rangle\rangle_C := \langle u_1, \tau_2 \rangle + \langle u_2, \tau_1 \rangle + \langle \tau_1, (\rho, \rho^t)\tau_2 \rangle$$

defines a symmetric fibrewise pairing $\langle\langle \cdot, \cdot \rangle\rangle_C$ on C. It is easy to see that this pairing $\langle\langle \cdot, \cdot \rangle\rangle_C$ is nondegenerate, and that the vector bundle C is isomorphic to $U \oplus U^*$. Now set

$$c \colon C \to TM, \qquad c(u \oplus \tau) = \mathrm{pr}_{TM}(u) + \rho \circ \mathrm{pr}_A(\tau).$$

Theorem 5.5. *Let A be a Lie algebroid and consider an LA-Dirac structure (D_A, U, A, M) over A. Choose a Dirac structure $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ that is adapted to D_A. Then C in (3.1) is a Courant algebroid with anchor c, pairing $\langle\langle \cdot, \cdot \rangle\rangle_C$ and bracket $[\![\cdot, \cdot]\!] \colon \Gamma(C) \times \Gamma(C) \to \Gamma(C)$,*

$$[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!] = \left([\![u_1, u_2]\!]_\Delta + \nabla^{\mathrm{bas}}_{\tau_1} u_2 - \nabla^{\mathrm{bas}}_{\tau_2} u_1 \right)$$
$$\oplus \left([\tau_1, \tau_2]_D + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle \tau_1, u_2 \rangle) \right).$$

The map $\mathcal{D} = c^* \circ \mathbf{d} \colon C^\infty(M) \to \Gamma(C)$ is given by $f \mapsto 0 \oplus (0, \mathbf{d}f)$. The bracket does not depend on the choice of Δ.

The proof of this theorem can be found in the appendix.

Remark 5.6.   1) It is easy to check that the Courant algebroid structure does not depend on the choice of Δ adapted to D_A.

2) This construction has some similarities with the one of matched pairs of Courant algebroids in [17]. It would be interesting to understand the relation between the two constructions.

It is easy to see that the Lie algebroid $(U, \mathrm{pr}_{TM}|_U, [\![\cdot, \cdot]\!]_\Delta)$ is a Dirac structure in C, and that (C, U) is an A-Manin pair over M. Hence, we get the following corollary.

Corollary 5.7. *Let A be a Lie algebroid and consider an LA-Dirac structure (D_A, U, A, M) over A. Then the triple (A, U, ι), with ι the inclusion of U in $TM \oplus A^*$, is a Dirac bialgebroid.*

Now we can prove the following theorem.

Theorem 5.8. *Let A be a Lie algebroid over a manifold M. Then the construction above defines a one–one correspondence between LA-Dirac structures on A and equivalence classes of Dirac bialgebroids on A.*

Proof. We show here that there is a one-one correspondence between pairs $(U, K = U^\circ, \Delta)$ as in Theorem 5.2 and Dirac bialgebroids (A, U, ι). From the definition of the bracket, anchor and pairing in Theorem 5.5, we find immediately that (C, U) is an A-Manin pair and so that (A, U, ι) is a Dirac bialgebroid.

Conversely, choose an A-Manin pair (C, U). Recall from the proof of Proposition 3.2 that C can be written as in (3.1) and that the anchor, pairing and brackets are defined as in Theorem 5.5, with $\Delta\colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ any extension of the Lie derivative $\mathcal{L}^U$ dual to the Lie algebroid structure on U. The proof of Theorem 5.5 shows that all the conditions in Theorem 5.2 are then satisfied for the pair U and Δ. □

6. The Lie algebroid of a multiplicative Dirac structure

In this section, we recall how the Lie algebroid of a Dirac groupoid is equipped with an LA-Dirac structure [38]. Then we compute explicitly the Dirac algebroid of a Dirac groupoid and we prove that they induce equivalent Dirac bialgebroids.

6.1. Dirac groupoids correspond to Dirac algebroids

Ortiz shows in [38] that modulo the canonical isomorphism (see [24])

the Lie algebroid $A(\mathsf{D}) \to U$ of a multiplicative Dirac structure D on $G \rightrightarrows M$ is an LA-Dirac structure

$$
\begin{array}{ccc}
\mathsf{D}_A & \longrightarrow & U \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
$$

i.e. (A, D_A) is a Dirac algebroid. Assume that $G \rightrightarrows M$ is source simply connected. Ortiz proves that this defines a one-to-one correspondence between LA-Dirac structures over $A = A(G)$ and multiplicative Dirac structures on $G \rightrightarrows M$ [38]:

(6.1)
$$
\left\{ \begin{array}{c} \mathsf{D}_A \text{ LA-Dirac structure} \\ \text{on } A \end{array} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{array}{c} \mathsf{D} \text{ multiplicative Dirac structure} \\ \text{on } G \rightrightarrows M \end{array} \right\}.
$$

6.2. The Lie algebroid of a multiplicative Dirac structure

Now consider again a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$ and consider an extension $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ of the induced Dorfman connection $\pounds^U$ as in Proposition 4.6. (Recall that $U^* \simeq (A \oplus T^*M)/U^\circ$.) Define then $\Omega \colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M)$ as in (2.6).

Lemma 6.1. *For $a \in \Gamma(A)$, the map $\phi_a \colon \Gamma(U) \to \Gamma(U^*)$ defined by*

$$
\phi_a(u) = \overline{\Omega_u a}, \qquad a \in \Gamma(A), u \in \Gamma(U)
$$

satisfies $\phi_a^t = -\phi_a$.

Proof. The proof is just a computation:

$$
\begin{aligned}
\langle \phi_a(u_1), u_2 \rangle &= \langle \Delta_{u_1} a - (0, \mathbf{d}\langle u_1, a \rangle), u_2 \rangle = \langle \pounds^U_{u_1} \overline{(a, 0)}, u_2 \rangle - \mathrm{pr}_{TM}(u_2)\langle u_1, a \rangle \\
&= \langle [\![u_1 \oplus 0, 0 \oplus a]\!]_{\mathsf{B}}, u_2 \oplus 0 \rangle - \mathrm{pr}_{TM}(u_2)\langle u_1, a \rangle \\
&= -\langle [\![0 \oplus a, u_1 \oplus 0]\!]_{\mathsf{B}}, u_2 \oplus 0 \rangle \\
&= -\rho(a)\langle u_1 \oplus 0, u_2 \oplus 0 \rangle + \langle u_1 \oplus 0, [\![0 \oplus a, u_2 \oplus 0]\!]_{\mathsf{B}} \rangle \\
&= -\langle u_1, \phi_a(u_2) \rangle
\end{aligned}
$$

for $u_1, u_2 \in \Gamma(U)$ and $a \in \Gamma(A)$. $\square$

Remark 6.2. By the proof of Proposition 4.6, we have $\phi_a(u) = \overline{-\tau_{a,d}}$ for any star section $d \in \Gamma(\mathsf{D})$ over $u \in \Gamma(U)$.

We compute the linear and core sections of the Lie algebroid $A(TG \oplus T^*G) \to TM \oplus A^*$. Let a be a section of A, $m \in M$, $(v_m, \alpha_m) \in U_m$ and consider the curve $\hat{a}(v_m, \alpha_m) \colon (-\varepsilon, \varepsilon) \to TG \oplus T^*G$,

$$\hat{a}(v_m, \alpha_m)(t) = \big(T_m \operatorname{Exp}(ta)v_m, (T_{\operatorname{Exp}(ta)(m)} L_{\operatorname{Exp}(-ta)})^t \alpha_m\big)$$

where $\varepsilon > 0$ is so that $\operatorname{Exp}(ta)(m)$ is defined for all $t \in (-\varepsilon, \varepsilon)$. Here, $L_{\operatorname{Exp}(ta)}$ is the left multiplication by the bisection $\operatorname{Exp}(ta)$ of $G \rightrightarrows M$. In the same manner, given $\tau \in \Gamma(A \oplus T^*M)$, consider the curve $\tau^c(v_m, \alpha_m) \colon \mathbb{R} \to TG \oplus T^*G$, $\tau^c(v_m, \alpha_m)(t) = (v_m, \alpha_m) + t\tau(m)$. We define the linear and core sections $a^l, \tau^\times \in \Gamma(A(TG \oplus T^*G))$ by

$$a^l(v_m, \alpha_m) = \frac{d}{dt}\bigg|_{t=0} \hat{a}(v_m, \alpha_m)(t) \quad \text{and}$$

$$\tau^\times(v_m, \alpha_m) = \frac{d}{dt}\bigg|_{t=0} \sigma^c(v_m, \alpha_m)(t)$$

for all $(v_m, \alpha_m) \in TM \oplus A^*$. More generally, if $\phi \in \Gamma(\operatorname{Hom}(TM \oplus A^*, A \oplus T^*M))$ and $a \in \Gamma(A)$, then the linear section $\alpha_{a,\phi} \in \Gamma(A(TG \oplus T^*G))$ is given by

$$\alpha_{a,\phi}(v_m, \alpha_m) := a^l(v_m, \alpha_m) + \frac{d}{dt}\bigg|_{t=0} (v_m, \alpha_m) + t\phi(v_m, \alpha_m)$$

for all $(v_m, \alpha_m) \in TM \oplus A^*$, where the sum is taken in the fibre over (v_m, α_m) of $A(TG \oplus T^*G) \to TM \oplus A^*$.

We find out which $a \in \Gamma(A)$ and $\phi \in \Gamma(\operatorname{Hom}(TM \oplus A^*, A \oplus T^*M))$ lead to a section $\alpha_{a,\phi}$ that restricts to a section of $A(\mathsf{D})$ on U.

Proposition 6.3. *For all $a \in \Gamma(A)$, we have $\alpha_{a,\phi}|_U \in \Gamma(\mathsf{A}(\mathsf{D}))$ if and only if $\overline{\phi|_U} = \phi_a$.*

Proof. Since $\mathsf{D} = \mathsf{D}^\perp$, a vector field $X \in \mathfrak{X}(TG \oplus T^*G)$ restricts to a section of $T\mathsf{D}$ if and only if $\mathbf{d}\ell_d(X) = 0$ on D for all $d \in \Gamma(\mathsf{D})$, where $\ell_d \colon TG \oplus T^*G \to \mathbb{R}$ is the linear function associated to d.

Choose $u_m \in U$, a star-section $d \in \Gamma(\mathsf{D})$, $d \sim_{\mathsf{t}} u'$, defined in a neighbourhood of $m \in M$ and compute:

$$
\begin{aligned}
\mathbf{d}l_d(u_m)(\alpha_{a,\phi}(u_m)) &= \left.\frac{d}{dt}\right|_{t=0} \langle \hat{a}(u_m)(t), d(\mathrm{Exp}(ta)(m)) \rangle \\
&\quad + \langle u_m + t\phi(u_m), d(m) \rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \left\langle u_m, (L^*_{\mathrm{Exp}(ta)}d)(m) \right\rangle + t \cdot \langle \phi(u_m), \mathbb{T}t(d(m)) \rangle \\
&= \langle u_m, (\pounds_{a^r}d)(m) \rangle + \langle \phi(u_m), u'(m) \rangle \\
&= \langle u_m, \overline{\tau_{a,d}}(m) \rangle + \langle \phi(u_m), u'(m) \rangle.
\end{aligned}
$$

Since $\phi_a(u') = -\overline{\tau_{d,a}}$ (Remark 6.2) we find that this vanishes for all star sections $d \in \Gamma(D)$ if and only if $\overline{\phi|_U}^t = -\phi_a$. By Lemma 6.1, this is equivalent to $\overline{\phi|_U} = \phi_a$. □

Corollary 6.4. *The subbundle* $A(\mathsf{D}) \to U$ *of* $A(TG \oplus T^*G) \to TM \oplus A^*$ *is spanned by the sections* $\alpha_{a,\Omega.a}|_U$ *and* $\tau^\times|_U$, *for all* $a \in \Gamma(A)$ *and* $\tau \in \Gamma(U^\circ)$.

This corollary allows us to describe in the next section $\mathsf{D}_A = I(A(\mathsf{D}))$ in terms of the Lie algebroid U.

6.3. Conclusion and main result

Consider again a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$ and an extension $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ of the induced Dorfman connection $\pounds^U$ as in Proposition 4.6. The Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$ induces an A-Manin pair (B, U) as in Theorem 4.4 and hence a Dirac bialgebroid (A, U, ι). By Theorem 5.8 there is then an LA-Dirac structure on A corresponding to (A, U, ι). Our goal is to show how this LA-Dirac structure is, via the canonical isomorphism $A(TG \oplus T^*G) \simeq TA \oplus T^*A$, the Lie algebroid $A(\mathsf{D}) \to U$ of the multiplicative Dirac structure $\mathsf{D} \rightrightarrows U$.

Note that given a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$, we have found two Lie algebroid structures on the bundle U of units of D. The first one is $(\mathrm{pr}_{TM}, [\cdot, \cdot]_U)$, which was defined in Theorem 4.3, the Lie algebroid structure of U as a Dirac structure in B. The second Lie algebroid structure on U, $(\mathrm{pr}_{TM}, [\![\cdot, \cdot]\!]_{\Delta'}|_{\Gamma(U) \times \Gamma(U)})$ was defined by any Dorfman connection Δ' that is adapted to the LA-Dirac structure $\mathsf{D}_A = I(A(\mathsf{D}))$ in $TA \oplus T^*A \to A$. In order to prove our main theorem, it remains to show that given a Dirac groupoid $(G \rightrightarrows M, \mathsf{D})$, we have $[\cdot, \cdot]_U = [\![\cdot, \cdot]\!]_{\Delta'}|_{\Gamma(U) \times \Gamma(U)}$ for any Dorfman connection Δ' that is adapted to $\mathsf{D}_A = I(A(\mathsf{D}))$. For this, it suffices to show that Δ is adapted to D_A.

The canonical isomorphism $I\colon A(TG \oplus T^*G) \to TA \oplus T^*A$ of double vector bundles can quickly be described as follows. The linear section $\alpha_{a,\phi}$ for $a \in \Gamma(A)$ and $\phi \in \Gamma(\mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$ is sent by I to $(Ta, a^R)) - \widetilde{\phi}$. In particular, the image of $\alpha_{a,\Omega(\cdot,a)}$ is $\sigma_A^\Delta(a)$. The image under I of the core section $\tau^\times$ is $\tau^\dagger$. As a consequence, any extension Δ of the Dorfman connection $\pounds^U\colon \Gamma(U) \times \Gamma(A \oplus T^*M/U^\circ) \to \Gamma(A \oplus T^*M/U^\circ)$ that is dual to $[\,\cdot\,,\cdot\,]_U$ is adapted to $\mathsf{D}_A = A(\mathsf{D})$. In other words, the double vector subbundle $(\mathsf{D}_A := I(A(\mathsf{D})); U, A; M)$ of $(TA \oplus T^*A; TM \oplus A^*, A; M)$ corresponds to the triple (U, U°, Δ).

This yields the following proposition.

Proposition 6.5. *Let $(G \rightrightarrows M, \mathsf{D})$ be a Dirac groupoid and let A be the Lie algebroid of $G \rightrightarrows M$. Then the bijection defined by Theorem 5.8 associates the LA-Dirac structure $\mathsf{D}_A := I(A(\mathsf{D}))$ to the class of the Dirac bialgebroid (A, U, ι), with U endowed with the Lie algebroid structure $(\mathrm{pr}_{TM}, [\,\cdot\,,\cdot\,]_U)$.*

Corollary 4.5, Propositions 6.5 and 5.8 together with (6.1) now yield our main theorem.

Theorem 6.6. *Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. Then the construction in Theorem 4.4 defines a one-to-one correspondence between multiplicative Dirac structures on $G \rightrightarrows M$ and equivalence classes of Dirac bialgebroids over A.*

Hence, it does make sense to say that a Dirac bialgebroid (A, U, ι) is *integrable* if the Lie algebroid A integrates to a source-simply connected Lie groupoid.

Section 7 shows how the infinitesimal descriptions of Poisson groupoids, presymplectic groupoids and multiplicative distributions on Lie groupoids are special cases of Theorem 6.6.

6.4. Concrete integration recipe

For the convenience of the reader, let us quickly sketch how a Dirac groupoid is reconstructed from an integrable Dirac bialgebroid (A, U, ι).

Step 1: Let $(G \rightrightarrows, M)$ be the source-simply connected Lie groupoid integrating the Lie algebroid A [8] and identify U with the Lie algebroid $U \simeq \iota(U) \subseteq TM \oplus A^*$ with anchor pr_{TM}.

Step 2: Extend the bracket on $\Gamma(U)$ to a dull bracket on $TM \oplus A^*$, i.e. using the Leibniz identity, but possibly loosing the skew-symmetry and the

Jacobi identity. This dull bracket is dual to a Dorfman connection
$\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$.

Step 3: Take the double vector subbundle $(\mathsf{D}_A; U, A; M) \subseteq (TA \oplus T^*A; TM \oplus A^*, A; M)$ that is defined by the triple (U, U°, Δ), hence with core $U^\circ \subseteq A \oplus T^*M$.

Step 4: The space D_A is an LA-Dirac structure on A, which integrates by the results in [38] to a multiplicative Dirac structure D on $G \rightrightarrows M$.

7. Examples

This section describes the A-Manin pairs and Dirac bialgebroids in the special cases of Dirac Lie groups, Poisson groupoids, presymplectic groupoids, and multiplicative distributions on Lie groupoids. Of course, the obtained infinitesimal descriptions show some redundancy, as Lie bialgebroids, IM-2-forms and infinitesimal ideal systems are very special cases of our general notion of Dirac bialgebroids. For instance, the Lie bialgebroid encoding a Poisson groupoid is now replaced by the pair of dual algebroids together with an inclusion (not the trivial one) of A^* in $TM \oplus A^*$. On the other hand, the examples show that Lie bialgebroids, IM-2-forms and infinitesimal ideal systems are the corner cases of Dirac bialgebroids, just as Poisson structures, presymplectic structures and distributions are the corner cases of Dirac structures.

7.1. Dirac Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The papers [19, 37] show that multiplicative Dirac structures on G correspond to pairs $(\mathfrak{i}, \delta)$ where $\mathfrak{i} \subseteq \mathfrak{g}$ is an ideal and $\delta \colon \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{i}$ is a Lie algebra 1-cocycle[9], such that the dual $\delta^t \colon \mathfrak{i}^\circ \times \mathfrak{i}^\circ \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{i}^\circ$. Equivalently, the pair $(\mathfrak{g}/\mathfrak{i}, \mathfrak{i}^\circ)$ is a Lie bialgebra. There is then a quadratic Lie algebra structure on $\mathfrak{g}/\mathfrak{i} \times \mathfrak{i}^\circ$ [29, Theorem 1.12], or in other words $\mathfrak{g}/\mathfrak{i} \times \mathfrak{i}^\circ$ is a Courant algebroid over a point. The pair $(\mathfrak{g}/\mathfrak{i} \times \mathfrak{i}^\circ, \mathfrak{i}^\circ)$ is a $\mathfrak{g}$-Manin pair and $(\mathfrak{g}, \mathfrak{i}^\circ, \iota : \mathfrak{i}^\circ \hookrightarrow \mathfrak{g}^*)$ is then a Dirac bialgebroid.

---

[9]The space $\bigwedge^2 \mathfrak{g}/\mathfrak{i}$ is a $\mathfrak{g}$-module via the action
$$x \cdot ((y + \mathfrak{i}) \wedge (z + \mathfrak{i})) = ([x, y] + \mathfrak{i}) \wedge (z + \mathfrak{i}) + (y + \mathfrak{i}) \wedge ([x, z] + \mathfrak{i}).$$

Conversely, note that a Dirac bialgebroid over a point is a triple $(\mathfrak{g}, \mathfrak{p}, \iota \colon \mathfrak{p} \hookrightarrow \mathfrak{g}^*)$ with $\mathfrak{g}, \mathfrak{p}$ Lie algebras and ι an injective vector space morphism such that there exists a quadratic Lie algebra $\mathfrak{m}$ with a Lie algebra morphism $\Phi \colon \mathfrak{g} \to \mathfrak{m}$ such that

1) Φ has isotropic image,

2) $\mathfrak{p}$ is a Lagrangian subalgebra of $\mathfrak{m}$,

3) $\langle \Phi(x), \xi \rangle = \langle \iota(x), \xi \rangle$ for $x \in \mathfrak{g}$ and $\xi \in \mathfrak{p}$,

4) $\Phi(\mathfrak{g}) + \mathfrak{p} = \mathfrak{m}$.

We let the reader verify that $\mathfrak{p}^\circ \subseteq \mathfrak{g}$ must be an ideal and $(\mathfrak{g}/\mathfrak{p}^\circ, \mathfrak{p})$ a Lie bialgebra. Hence, simply connected Dirac Lie groups are in one-to-one correspondence with Dirac bialgebroids over points, which we call **Dirac bialgebras**.

7.2. Poisson groupoids

The infinitesimal description of a Poisson groupoid $(G \rightrightarrows M, \pi)$ is known to be its Lie bialgebroid (A, A^*) [34, 35]. In that case, our Dirac bialgebroid is $(A, A^*, (\rho_\star, \mathrm{id}_{A^*}) \colon A^* \to TM \oplus A^*)$. That is, A^* is identified with the graph of its anchor $\rho_\star$. The A-Manin pair defining this Dirac bialgebroid is $(A \oplus A^*, A^*)$, where $A \oplus A^* \to M$ is the Courant algebroid defined by the Lie bialgebroid [27]. (Note that we have already shown in [21] that the Courant algebroid B defined by the Dirac groupoid $(G \rightrightarrows M, \pi)$ is, up to an isomorphism, $A \oplus A^*$.) The map $\Phi \colon A \oplus T^*M \to A \oplus A^*$ is $(a, \theta) \mapsto (a + \rho_\star^t \theta, \rho^t \theta)$.

The double $A \oplus A^*$ being a Courant algebroid is equivalent to (A, A^*) being a Lie bialgebroid. By definition of the Courant bracket, the subbundle A^* is then automatically a Dirac structure in $A \oplus A^*$. The surjectivity and pairing conditions for the A-Manin pair are then obvious. The compatibility of Φ with pairings and anchors follows easily from the equality $\rho \circ \rho_\star^t = -\rho_\star \circ \rho^t$, which is always satisfied for Lie bialgebroids. We only have to check that the map Φ preserves the brackets. First, for $a_1, a_2 \in \Gamma(A)$,

$$[\![\Phi(a_1, 0), \Phi(a_2, 0)]\!]_{A \oplus A^*} = [\![(a_1, 0), (a_2, 0)]\!]_{A \oplus A^*} = ([a_1, a_2]_A, 0)$$
$$= \Phi([(a_1, 0), (a_2, 0)]_d).$$

We then have on the one hand for $\theta \in \Omega^1(M)$:

$$\llbracket \Phi(a,0), \Phi(0,\theta) \rrbracket_{A \oplus A^*} = \llbracket (a,0), (\rho_\star^t \theta, \rho^t \theta) \rrbracket_{A \oplus A^*}$$
$$= ([a, \rho_\star^t \theta]_A - \mathbf{i}_{\rho^t \theta} \mathbf{d}_{A^*} a, \pounds_a \rho^t \theta).$$

On the other hand, we have

$$\Phi([(a,0),(0,\theta)]_d) = \Phi(0, \pounds_{\rho(a)} \theta) = (\rho_\star^t \pounds_a \theta, \rho^t \pounds_{\rho(a)} \theta).$$

The equation $\rho^t \pounds_{\rho(a)} \theta = \pounds_a \rho^t \theta$ is verified using the definitions of $\pounds_a$ and $\pounds_{\rho(a)}$ and the compatibility of the bracket with the anchor. Then $\rho_\star^t \pounds_a \theta = [a, \rho_\star^t \theta]_A - \mathbf{i}_{\rho^t \theta} \mathbf{d}_{A^*} a$ for all $\theta \in \Omega^1(M)$ and $a \in \Gamma(A)$ is equivalent to Equation (11) in [32, Lemma 12.1.8], hence always satisfied if (A, A^*) is a Lie bialgebroid. Finally we have

$$\llbracket \Phi(0,\theta), \Phi(0,\omega) \rrbracket_{A \oplus A^*} = \llbracket (\rho_\star^t \theta, \rho^t \theta), (\rho_\star^t \omega, \rho^t \omega) \rrbracket_{A \oplus A^*}$$
$$= \left([\rho_\star^t \theta, \rho_\star^t \omega]_A + \pounds_{\rho^t \theta} \rho_\star^t \omega - \mathbf{i}_{\rho^t \omega} \mathbf{d}_{A^*} \rho_\star^t \theta, [\rho^t \theta, \rho^t \omega]_{A^*} + \pounds_{\rho_\star^t \theta} \rho^t \omega - \mathbf{i}_{\rho_\star^t \omega} \mathbf{d}_A \rho^t \theta \right)$$

and on the other hand $\Phi([(0,\theta),(0,\omega)]_d) = \Phi(0) = 0$. Hence, by symmetry of the Lie bialgebroid (A, A^*), we only have to check that

$$[\rho_\star^t \theta, \rho_\star^t \omega]_A = -\pounds_{\rho^t \theta} \rho_\star^t \omega + \mathbf{i}_{\rho^t \omega} \mathbf{d}_{A^*} \rho_\star^t \theta$$

for all $\theta, \omega \in \Omega^1(M)$. If $\omega = \mathbf{d}f$ with some $f \in C^\infty(M)$, this is [32, Lemma 12.1.5]. To see this, note that

$$\rho_\star^t \mathbf{d}((\rho \circ \rho_\star)^t \theta(f)) = -\pounds_{\rho^t \theta}(\mathbf{d}_\star f).$$

The general case (with ω not necessarily exact) follows easily.

Let us now quickly describe a Dorfman connection adapted to the Dirac algebroid $(A, A(D_\pi))$. Via the canonical isomorphism $TA \oplus T^*A \simeq A(TG \oplus T^*G)$, the Lie algebroid of the Dirac structure D_π is D_{π_A}, where π_A is the linear Poisson structure on A that is equivalent to the Lie algebroid structure on A^* [35]. The sides of D_{π_A} are A and $U = \mathrm{graph}(\rho_\star : A^* \to TM)$ and its core is $K = \mathrm{graph}(-\rho_\star^t : T^*M \to A)$. Choose any connection $\nabla : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$, denote by $\nabla^* : \mathfrak{X}(M) \times \Gamma(A^*) \to \Gamma(A^*)$ the dual connection and construct the basic A^*-connection on A^*: $\nabla^{*\mathrm{bas}} : \Gamma(A^*) \times \Gamma(A^*) \to \Gamma(A^*)$, $\nabla^{*\mathrm{bas}}_{\alpha_1} \alpha_2 = \nabla_{\rho_\star \alpha_2} \alpha_1 + [\alpha_1, \alpha_2]_{A^*}$ for all $\alpha_1, \alpha_2 \in \Gamma(A^*)$. The Dorfman connection $\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$

defined by

$$\Delta_{(X,\alpha)}(a,\theta) = \Big(\langle a, \nabla_{\cdot}^{*\mathrm{bas}}\alpha \rangle + \nabla_X a - \rho_{\star}^t \langle \nabla_{\cdot}^*\alpha, a \rangle, \pounds_X\theta + \langle \nabla_{\cdot}^*\alpha, a \rangle \Big)$$

is adapted to D_{π_A}. The corresponding dull bracket $[\![\cdot,\cdot]\!]_\Delta$ on sections of $TM \oplus A^*$ is given by

$$(7.1) \quad [\![(X_1,\alpha_1),(X_2,\alpha_2)]\!]_\Delta = \big([X_1,X_2], \nabla_{X_1}\alpha_2 - \nabla_{X_2}\alpha_1$$
$$+ \nabla_{\rho_\star(\alpha_2)}\alpha_1 - \nabla_{\rho_\star(\alpha_1)}\alpha_2 + [\alpha_1,\alpha_2]_{A^*},\big)$$

which restricts to $[\![(\rho_\star(\alpha_1),\alpha_1),(\rho_\star(\alpha_2),\alpha_2)]\!]_\Delta = (\rho_\star[\alpha_1,\alpha_2]_{A^*},[\alpha_1,\alpha_2]_{A^*})$ on sections of U. For more details, see [20].

7.3. Presymplectic groupoids

Presymplectic groupoids are described infinitesimally in [3, 4] by IM-2-forms: Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. A vector bundle map $\sigma\colon A \to T^*M$ is an IM-2-form if

1) $\langle \rho(a_1), \sigma(a_2) \rangle = -\langle \rho(a_2), \sigma(a_1) \rangle$ and

2) $\sigma[a_1,a_2] = \pounds_{\rho(a_1)}\sigma(a_2) - \mathbf{i}_{\rho(a_2)}\mathbf{d}\sigma(a_1)$

for all $a_1, a_2 \in \Gamma(A)$.

We show that the Dirac bialgebroid corresponding to this example is $(A, TM, (\mathrm{id}_{TM}, \sigma^t)\colon TM \to TM \oplus A^*)$. Consider a vector bundle morphism $\sigma\colon A \to T^*M$ on a Lie algebroid $A \to M$. We want to define an A Manin-pair $(TM \oplus T^*M, TM)$, where $TM \oplus T^*M$ is the standard Courant algebroid (see also [21], where we check that B is in this case isomorphic to $TM \oplus T^*M$), TM is seen as isomorphic to $\mathrm{graph}(\sigma^t\colon TM \to A^*) \subseteq TM \oplus A^*$ and the map $\Phi\colon A \oplus T^*M \to TM \oplus T^*M$ is $(a,\theta) \mapsto (\rho(a), \sigma(a) + \theta)$. We quickly check that Φ is a morphism of degenerate Courant algebroids if and only if σ is an IM-2-form.

The map Φ is obviously compatible with the anchors. Choose (a_1,θ_1), $(a_2,\theta_2) \in \Gamma(A \oplus T^*M)$. Then

$$\langle \Phi(a_1,\theta_1), \Phi(a_2,\theta_2) \rangle = \langle (\rho(a_1), \sigma(a_1) + \theta_1), (\rho(a_2), \sigma(a_2) + \theta_2) \rangle$$
$$= \langle (a_1,\theta_1), (a_2,\theta_2) \rangle_d + \langle \sigma(a_1), \rho(a_2) \rangle + \langle \sigma(a_2), \rho(a_1) \rangle$$

and

$$\llbracket \Phi(a_1, \theta_1), \Phi(a_2, \theta_2) \rrbracket = \llbracket (\rho(a_1), \sigma(a_1) + \theta_1), (\rho(a_2), \sigma(a_2) + \theta_2) \rrbracket$$
$$= (\rho[a_1, a_2], \pounds_{\rho(a_1)}(\sigma(a_2) + \theta_2) - \mathbf{i}_{\rho(a_2)}\mathbf{d}(\sigma(a_1) + \theta_1))$$
$$= \Phi[(a_1, \theta_1), (a_2, \theta_2)]_d + (0, \pounds_{\rho(a_1)}\sigma(a_2) - \mathbf{i}_{\rho(a_2)}\mathbf{d}\sigma(a_1) - \sigma[a_1, a_2]).$$

Hence, Φ is a morphism of Courant algebroids if and only if $\langle \rho(a_1), \sigma(a_2) \rangle = -\langle \rho(a_2), \sigma(a_1) \rangle$ and $\sigma[a_1, a_2] = \pounds_{\rho(a_1)}\sigma(a_2) - \mathbf{i}_{\rho(a_2)}\mathbf{d}\sigma(a_1)$ for all $a_1, a_2 \in \Gamma(A)$.

The Dirac algebroid corresponding to a presymplectic groupoid $(G \rightrightarrows M, \omega)$ is $(A, A(D_\omega)) \simeq (A, D_{\sigma^*\omega_{\text{can}}})$, where ω_{can} is the canonical symplectic structure on T^*M. The vector bundles U and K over M are

$$U = \text{graph}(-\sigma^t \colon TM \to A^*) \quad \text{and} \quad K = \text{graph}(\sigma \colon A \to T^*M).$$

The Dorfman connection $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ defined by

$$\Delta_{(X,\alpha)}(a, \theta) = (\nabla_X a, \pounds_X(\theta - \sigma(a)) + \langle \nabla_\cdot^*(\sigma^t X + \alpha), a \rangle + \sigma(\nabla_X a)).$$

The dual dull bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ on sections of $TM \oplus A^*$ is given by

$$\llbracket (X_1, \alpha_1), (X_2, \alpha_2) \rrbracket_\Delta$$
$$= ([X_1, X_2], \nabla_{X_1}^*(\alpha_2 + \sigma^t X_2) - \nabla_{X_2}^*(\alpha_1 + \sigma^t X_1) - \sigma^t[X_1, X_2])$$

and restricts to $\llbracket (X_1, -\sigma^t X_1), (X_2, -\sigma^t X_2) \rrbracket_\Delta = ([X_1, X_2], -\sigma^t[X_1, X_2])$ on sections of U.

7.4. Multiplicative distributions

Multiplicative distributions on Lie groupoids are described infinitesimally in [18, 23] via infinitesimal ideal systems:

Definition 7.1. [23] Let $(q \colon A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $J \subseteq A$ a subbundle such that $\rho(J) \subseteq F_M$ and ∇ a flat partial F_M-connection on A/J with the following properties[10]:

1) If $a \in \Gamma(A)$ is ∇-parallel, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.

2) If $a_1, a_2 \in \Gamma(A)$ are ∇-parallel, then $[a_1, a_2]$ is also ∇-parallel.

---

[10]We say by abuse of notation that a section $a \in \Gamma(A)$ is ∇-parallel if its class $\bar{a}$ in $\Gamma(A/J)$ is ∇-parallel.

3) If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a)$ is ∇^{F_M}-parallel, where

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \to \Gamma(TM/F_M)$$

is the Bott connection associated to F_M.

The triple (F_M, J, ∇) is an infinitesimal ideal system in A.

We prove the following theorem.

Theorem 7.2. *Let* $(q \colon A \to M, \rho, [\cdot, \cdot])$ *be a Lie algebroid,* $F_M \subseteq TM$ *an involutive subbundle,* $J \subseteq A$ *a subbundle such that*

$$\rho(J) \subseteq F_M \quad and \quad \tilde{\nabla} \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$$

be a connection such that $\tilde{\nabla}_X j \in \Gamma(J)$ *for all* $j \in \Gamma(J)$ *and* $X \in \Gamma(F_M)$, *and which thus defines a connection* $\nabla \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$. *Then* (A, F_M, J, ∇) *is an infinitesimal ideal system if and only if* $(A, F_M \oplus J^\circ, \iota)$ *is a Dirac bialgebroid, where* $\iota \colon F_M \oplus J^\circ \hookrightarrow TM \oplus A^*$ *is the inclusion and* $F_M \oplus J^\circ$ *has the anchor* pr_{TM} *and the bracket*

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_{F_M \oplus J^\circ} = ([X_1, X_2], \tilde{\nabla}^*_{X_1} \alpha_2 - \tilde{\nabla}^*_{X_2} \alpha_1),$$

$X_1, X_2 \in \Gamma(F_M)$, $\alpha_1, \alpha_2 \in \Gamma(J^\circ)$.

First note that $F_M \oplus J^\circ$ is a Lie algebroid with this structure if and only if ∇ is flat. To prove this theorem, we construct a Manin pair associated to the infinitesimal ideal system.

We have shown in [23] that if the quotients are smooth and ∇ has no holonomy, there is an induced Lie algebroid structure on $(A/J)/\nabla \to M/F_M$ such that the canonical projection $A \to (A/J)/\nabla$ over $M \to M/F_M$ is a fibration of Lie algebroids. In the case of a Lie algebra $\mathfrak{g}$, an ideal system is just an ideal $\mathfrak{i}$, and this quotient is the usual quotient of the Lie algebra by its ideal $\mathfrak{i}$. We show that there is an alternative construction of a quotient Lie algebroid, that simplifies as well to the usual quotient $\mathfrak{g}/\mathfrak{i}$ in the Lie algebra case.

The paper [12] shows that an infinitesimal ideal system can alternatively be defined as follows. Let A be a Lie algebroid over the base M, J a subbundle of A and F_M an involutive subbundle of TM such that $\rho(J) \subseteq F_M$. Let $\tilde{\nabla} \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ be a connection such that $\tilde{\nabla}_X j \in \Gamma(J)$ for all $j \in \Gamma(J)$ and $X \in \Gamma(F_M)$, and which thus defines a connection

$\nabla\colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$.[11] The triple (F_M, J, ∇) is an infinitesimal ideal system if

1) ∇ is flat,

2) $\nabla_a^{\mathrm{bas}} X \in \Gamma(F_M)$,

3) $\nabla_a^{\mathrm{bas}} j \in \Gamma(J)$ and

4) $R_{\tilde\nabla}^{\mathrm{bas}}(a_1, a_2)(X) \in \Gamma(F_M)$

for all $X \in \Gamma(F_M)$, $a_1, a_2 \in \Gamma(A)$ and $j \in \Gamma(J)$, where the two connections ∇^{bas} and the tensor $R_{\tilde\nabla}^{\mathrm{bas}}$ are defined by $\tilde\nabla$ as in Example 2.12.

Now consider the direct sum $F_M \oplus A$ of vector bundles over M. Since the anchor ρ restricts to a map $\rho|_J\colon J \to F_M$, the vector bundle

$$\bar{A} := \frac{F_M \oplus A}{\mathrm{graph}(-\rho|_J)} \to M,$$

inherits the anchor $\bar\rho(X \oplus a) = X + \rho(a)$. (Given $X \in \Gamma(F_M)$ and $a \in \Gamma(A)$, we write $X \oplus a$ for the class of (X, a) in $\bar{A}$.) Define $[\cdot\,,\cdot]_{\bar{A}}\colon \Gamma(\bar{A}) \times \Gamma(\bar{A}) \to \Gamma(\bar{A})$ by

$$[X_1 \oplus a_1, X_2 \oplus a_2]_{\bar{A}} = ([X_1, X_2] + \nabla_{a_1}^{\mathrm{bas}} X_2 - \nabla_{a_2}^{\mathrm{bas}} X_1)$$
$$\oplus ([a_1, a_2] + \tilde\nabla_{X_1} a_2 - \tilde\nabla_{X_2} a_1).$$

This is skew-symmetric and well-defined because $[X \oplus a, (-\rho(j)) \oplus j]_{\bar{A}} = (-\rho(\tilde\nabla_X j + \nabla_a^{\mathrm{bas}} j)) \oplus (\tilde\nabla_X j + \nabla_a^{\mathrm{bas}} j)$ and $\tilde\nabla_X j + \nabla_a^{\mathrm{bas}} j \in \Gamma(J)$ for all $j \in \Gamma(J)$. It is easy to check that it is compatible with the anchor, i.e. that is satisfies the Leibniz identity. To see that the Jacobi identity is satisfied, one can check that

$$[[X_1 \oplus a_1, X_2 \oplus a_2]_{\bar{A}}, X_3 \oplus a_3]_{\bar{A}}$$
$$+ [[X_2 \oplus a_2, X_3 \oplus a_3]_{\bar{A}}, X_1 \oplus a_1]_{\bar{A}}$$
$$+ [[X_3 \oplus a_3, X_1 \oplus a_1]_{\bar{A}}, X_2 \oplus a_2]_{\bar{A}} = (-\rho(j)) \oplus j,$$

with

$$j := \left[R_{\tilde\nabla}^{\mathrm{bas}}(a_1, a_2) X_3 - R_{\tilde\nabla}(X_1, X_2) a_3 \right] + \mathrm{c.p.} \in \Gamma(\mathrm{J}).$$

---

[11]Conversely, given ∇, there always exists a connection $\tilde\nabla$ projecting in this manner to ∇. We say that $\tilde\nabla$ is an *extension* of ∇.

The maps $A \to \bar{A}$, $a \mapsto 0 \oplus a$ and $F_M \to \bar{A}$, $X \mapsto X \oplus 0$ are easily seen to be morphisms of Lie algebroids over the identity. Note that the second map is injective. We let the reader complete the proof of the following theorem.

Theorem 7.3. *Let $(q \colon A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $J \subseteq A$ a subbundle and ∇ a partial F_M-connection on A/J. Consider any extension $\tilde{\nabla} \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ of ∇. Then $(A, F_M, J^\circ, \nabla)$ is an infinitesimal ideal system if and only if*

$$\bar{A} = \frac{F_M \oplus A}{\operatorname{graph}(-\rho|_J)} \to M$$

with the anchor $\bar{\rho}(X \oplus a) = X + \rho(a)$ and the bracket $[X_1 \oplus a_1, X_2 \oplus a_2]_{\bar{A}} = ([X_1, X_2] + \nabla^{\mathrm{bas}}_{a_1} X_2 - \nabla^{\mathrm{bas}}_{a_2} X_1) \oplus ([a_1, a_2] + \tilde{\nabla}_{X_1} a_2 - \tilde{\nabla}_{X_2} a_1)$ is a Lie algebroid, which does not depend on the choice of the extension $\tilde{\nabla}$ of the connection ∇.

Consider now the Courant algebroid $\bar{A} \oplus \bar{A}^*$ defined by the trivial Lie bialgebroid $(\bar{A}, \bar{A}^*)$, i.e. with the trivial Lie algebroid structure on $\bar{A}^*$. The vector bundle $\bar{A}^*$ can be described as follows: sections of $\bar{A}^*$ are pairs $(\bar{\theta}, \rho^t \theta + \alpha) \in \Gamma(F_M^* \oplus A^*)$ with $\theta \in \Omega^1(M)$ and $\alpha \in \Gamma(J^\circ)$. Here, $\bar{\theta}$ is the class of θ in $T^*M/F_M^\circ \simeq F_M^*$. Hence, there is a natural inclusion of J° in $\bar{A}^*$, and so a natural inclusion i of $F_M \oplus J^\circ$ in $\bar{A} \oplus \bar{A}^*$. Elements of $i(F_M \oplus J^\circ) \subseteq \bar{A} \oplus \bar{A}^*$ can be written $(X \oplus 0, (0, \alpha))$ with $X \in F_M$ and $\alpha \in J^\circ$. It is easy to see that $F_M \oplus J^\circ$ is isotropic in $\bar{A} \oplus \bar{A}^*$. To see that it is maximal isotropic, take $(X' \oplus a, (\bar{\theta}, \rho^t \theta + \alpha')) \in \bar{A} \oplus \bar{A}^*$ such that $\langle (X' \oplus a, (\bar{\theta}, \rho^t \theta + \alpha')), (X \oplus 0, (0, \alpha)) \rangle = 0$ for all $(X \oplus 0, (0, \alpha)) \in F_M \oplus J^\circ$. Since this is $\theta(X) + \alpha(a)$, we find that $a \in J$ and $\theta \in F_M^\circ$. Hence, $(X' \oplus a, (\bar{\theta}, \rho^t \theta + \alpha')) = ((X' + \rho(a)) \oplus 0, (0, \rho^t \theta + \alpha')) \in F_M \oplus J^\circ$ (recall that $\rho(J) \subseteq F_M$ and so $\rho^t(F_M^\circ) \subseteq J^\circ$). The Courant bracket of $(X_1 \oplus 0, (0, \alpha_1))$ and $(X_2 \oplus 0, (0, \alpha_2)) \in F_M \oplus J^\circ$ equals

$$[\![(X_1 \oplus 0, (0, \alpha_1)), (X_2 \oplus 0, (0, \alpha_2))]\!]_{\bar{A} \oplus \bar{A}^*}$$
$$= ([X_1, X_2] \oplus 0, (0, \tilde{\nabla}^*_{X_1} \alpha_2 - \tilde{\nabla}^*_{X_2} \alpha_1)),$$

which is again an element of $F_M \oplus J^\circ$. This shows that $F_M \oplus J^\circ$ is a Dirac structure in $\bar{A} \oplus \bar{A}^*$.

Next we define the map $\Phi \colon A \oplus T^*M \to \bar{A} \oplus \bar{A}^*$, $\Phi(a, \theta) = (0 \oplus a, (\bar{\theta}, \rho^t \theta))$. The equality $i(F_M \oplus J^\circ) + \Phi(A \oplus T^*M) = \bar{A} \oplus \bar{A}^*$ is immediate, as well as the equality $\langle i(X, \alpha), \Phi(a, \theta) \rangle = \langle (X \oplus 0, (0, \alpha)), (0 \oplus a, (\bar{\theta}, \rho^t \theta)) \rangle = \theta(X) +$

$\alpha(a)$ for all $(X, \alpha) \in F_M \oplus J^\circ$ and $(a, \theta) \in A \oplus T^*M$. To see that Φ is a morphism of (degenerate) Courant algebroids, note first that

$$\pounds_{0 \oplus a}(\bar{\theta}, \rho^t \theta) = \left(\overline{\pounds_{\rho(a)}\theta}, \rho^t(\pounds_{\rho(a)}\theta)\right)$$

and

$$\mathbf{i}_{0 \oplus a}\mathbf{d}_{\bar{A}}(\bar{\theta}, \rho^t \theta) = \left(\overline{\mathbf{i}_{\rho(a)}\mathbf{d}\theta}, \rho^t(\mathbf{i}_{\rho(a)}\mathbf{d}\theta)\right)$$

for $a \in \Gamma(A)$ and $\theta \in \Omega^1(M)$. Then we can compute

$$
\begin{aligned}
[\![\Phi(a, \theta), \Phi(b, \omega)]\!]_{\bar{A} \oplus \bar{A}^*} &= [\![(0 \oplus a, (\bar{\theta}, \rho^t \theta)), (0 \oplus b, (\bar{\omega}, \rho^t \omega))]\!]_{\bar{A} \oplus \bar{A}^*} \\
&= \left(0 \oplus [a, b], (\overline{\pounds_{\rho(a)}\omega - \mathbf{i}_{\rho(b)}\mathbf{d}\theta}, \rho^t(\pounds_{\rho(a)}\omega - \mathbf{i}_{\rho(b)}\mathbf{d}\theta))\right) \\
&= \Phi([(a, \theta), (b, \omega)]_d).
\end{aligned}
$$

The compatibility of Φ with the anchors is immediate and compatibility with the pairing is checked as follows:

$$
\begin{aligned}
\langle \Phi(a, \theta), \Phi(b, \omega) \rangle &= \langle (0 \oplus a, (\rho^t \theta, \bar{\theta})), (0 \oplus b, (\rho^t \omega, \bar{\omega})) \rangle = \theta(\rho(b)) + \omega(\rho(a)) \\
&= \langle (a, \theta), (\rho, \rho^t)(b, \omega) \rangle = \langle (a, \theta), (b, \omega) \rangle_d.
\end{aligned}
$$

We have hence shown that $(\bar{A} \oplus \bar{A}^*, i(F_M \oplus J^\circ))$ is an A-Manin pair.

The Dirac algebroid corresponding to a Lie groupoid with a multiplicative distribution $(G \rightrightarrows M, F \oplus F^\circ)$ is $(A, A(F \oplus F^\circ)) \simeq (A, F_A \oplus F_A^\circ)$, with $F_A \simeq A(F)$. This double vector subbundle of TA has sides A and $F_M \subseteq TM$, and core $J \subseteq A$. Choose any connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ that is adapted to F_A, i.e. $T_m a(v_m) - \frac{d}{dt}\big|_{t=0} a(m) + t \nabla_{v_m} a \in F_A(a(m))$ for all $v_m \in F_M$ and $a \in \Gamma(A)$ (see [23]), and consequently $\nabla_X j \in \Gamma(J)$ for all $X \in \Gamma(F_M)$ and $j \in \Gamma(J)$. Then the Dorfman connection $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$,

$$\Delta_{(X, \alpha)}(a, \theta) = (\nabla_X a, \pounds_X \theta + \langle \nabla^*_\cdot \alpha, a \rangle)$$

is adapted to $F_A \oplus F_A^\circ$ and the dual dull bracket given by

$$[\![(X_1, \alpha_1), (X_2, \alpha_2)]\!] = ([X_1, X_2], \nabla^*_{X_1} \alpha_2 - \nabla^*_{X_2} \alpha_1)$$

restricts to the Lie algebroid bracket found above on sections of $F_M \oplus J^\circ$.

Appendix A. Proof of Theorem 5.5

Note that the equality

$$\text{(A.1)} \qquad \nabla_a^{\text{bas}} \circ (\rho, \rho^t) = (\rho, \rho^t) \circ \nabla_a^{\text{bas}}$$

is easily verified. The connection $\nabla_a^{\text{bas}} \colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ defines a connection $\nabla_a^{\text{bas}} \colon \Gamma(A \oplus T^*M) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$, $\nabla_\tau^{\text{bas}} \tau' = \nabla_{\text{pr}_A(\tau)}^{\text{bas}} \tau'$. Recall also (3.2) $\Delta_{(X,\alpha)}(a,\theta) = \Delta_u(a,0) + (0, \mathcal{L}_X \theta)$ for all $(X,\alpha) \in \Gamma(TM \oplus A^*)$ and $(a,\theta) \in \Gamma(A \oplus T^*M)$. First we prove a few lemmas.

Lemma A.1. *The equality*

$$\text{(A.2)} \qquad [\tau_1, \tau_2]_d = \Delta_{(\rho,\rho^t)(\tau_1)} \tau_2 - \nabla_{\tau_2}^{\text{bas}} \tau_1$$

holds for all $\tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$.

Proof. Write $\tau_1 = (a_1, \theta_1)$ and $\tau_2 = (a_2, \theta_2) \in \Gamma(A \oplus T^*M)$. Then:

$$\begin{aligned}
\nabla_{\tau_2}^{\text{bas}} \tau_1 &= \nabla_{a_2}^{\text{bas}} \tau_1 = \Omega_{(\rho,\rho^t)\tau_1} a_2 + \mathcal{L}_{a_2} \tau_1 \\
&= \Delta_{(\rho,\rho^t)\tau_1} \tau_2 - (0, \mathbf{d}\langle \theta_1, \rho(a_2) \rangle) - (0, \mathcal{L}_{\rho(a_1)} \theta_2) + ([a_2, a_1], \mathcal{L}_{\rho(a_2)} \theta_1) \\
&= \Delta_{(\rho,\rho^t)\tau_1} \tau_2 - [\tau_1, \tau_2]_d.
\end{aligned}$$
$\qquad \square$

Lemma A.2. *Let A be a Lie algebroid and choose a Dorfman connection $\Delta \colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$. Then, for all $\nu \in \Gamma(TM \oplus A^*)$ and $\tau, \tau' \in \Gamma(A \oplus T^*M)$:*

$$\text{(A.3)} \qquad \langle (\rho, \rho^t) \Delta_\nu \tau - [\![\nu, (\rho, \rho^t) \tau]\!]_\Delta - \nabla_\tau^{\text{bas}} \nu, \tau' \rangle = \langle \nabla_{\tau'}^{\text{bas}} \nu, \tau \rangle.$$

This yields the following corollary.

Corollary A.3. *Assume that (U, K, Δ) defines an LA-Dirac structure in $TA \oplus T^*A$. Then, for all $u \in \Gamma(U)$ and $k \in \Gamma(K)$:*

$$(\rho, \rho^t) \Delta_u k = [\![u, (\rho, \rho^t) k]\!]_\Delta + \nabla_k^{\text{bas}} u.$$

Proof. By Lemma A.2, we have

$$\langle (\rho, \rho^t) \Delta_u k - [\![u, (\rho, \rho^t) k]\!]_\Delta - \nabla_k^{\text{bas}} u, \tau \rangle = \langle \nabla_\tau^{\text{bas}} u, k \rangle$$

for all $\tau \in \Gamma(A \oplus T^*M)$. Since ∇_τ^{bas} preserves $\Gamma(U)$ by Theorem 5.2, the right-hand side vanishes. $\qquad \square$

Proof of Lemma A.2. We write $\tau = (a, \theta)$ and $\nu = (X, \alpha)$. Then for any $\tau' = (a', \theta') \in \Gamma(A \oplus T^*M)$ we have

$$\langle (\rho, \rho^t) \Delta_\nu \tau - [\![\nu, (\rho, \rho^t) \tau]\!]_\Delta - \nabla^{\mathrm{bas}}_a \nu, \tau' \rangle - \langle \nabla^{\mathrm{bas}}_{a'} \nu, \tau \rangle$$

$$\overset{(5.1)}{=} \langle \Delta_\nu \tau - \Omega_\nu a, (\rho, \rho^t) \tau' \rangle - \langle \mathcal{L}_a \nu, \tau' \rangle + \langle \nu, \nabla^{\mathrm{bas}}_{a'} \tau \rangle$$

$$\quad - \rho(a') \langle \nu, \tau \rangle + [\![(\rho, \rho^t) \tau, \nu]\!]_\Delta, \tau' \rangle$$

$$= \langle (0, \mathcal{L}_X \theta + \mathbf{d}\langle \alpha, a \rangle), (\rho, \rho^t) \tau' \rangle - \langle \mathcal{L}_a \nu, \tau' \rangle - \rho(a') \langle \nu, \tau \rangle + \rho(a) \langle \nu, \tau' \rangle$$

$$\quad - \langle \nu, \Delta_{(\rho, \rho^t) \tau} \tau' - \nabla^{\mathrm{bas}}_{a'} \tau \rangle$$

$$\overset{(A.2)}{=} \langle \mathcal{L}_X \theta, \rho(a') \rangle + \rho(a') \langle \alpha, a \rangle + \langle \nu, \mathcal{L}_a \tau' \rangle - \rho(a') \langle \nu, \tau \rangle - \langle \nu, [\tau, \tau']_d \rangle$$

$$= \langle \mathcal{L}_X \theta, \rho(a') \rangle - \rho(a') \langle \theta, X \rangle + \overline{\langle \nu, \mathcal{L}_a(a', \theta') \rangle}$$

$$\quad - \langle \nu, (\overline{[a, a'], \mathcal{L}_{\rho(a)} \theta'} - \mathbf{i}_{\rho(a')} \mathbf{d}\theta) \rangle$$

$$= \langle \mathbf{i}_X \mathbf{d}\theta, \rho(a') \rangle + \langle \mathbf{i}_{\rho(a')} \mathbf{d}\theta, X \rangle = 0.$$

$\square$

We also need the following lemma, which is proved at the end of this section.

Lemma A.4. *Let $A \to M$ be a Lie algebroid and (K, U, Δ) an LA-Dirac triple. Then Condition (5) of Theorem 5.2 is equivalent to:*

$$(A.4) \qquad \nabla^{\mathrm{bas}}_\tau [\![u, v]\!]_\Delta - [\![\nabla^{\mathrm{bas}}_\tau u, v]\!]_\Delta - [\![u, \nabla^{\mathrm{bas}}_\tau v]\!]_\Delta + \nabla^{\mathrm{bas}}_{\Delta_u \tau} v - \nabla^{\mathrm{bas}}_{\Delta_v \tau} u$$
$$= -(\rho, \rho^t) R_\Delta(u, v) \tau$$

for all $u, v \in \Gamma(U)$ and $\tau \in \Gamma(A \oplus T^*M)$.
   The equality

$$(A.5) \qquad \Delta_u [\tau_1, \tau_2]_d - [\Delta_u \tau_1, \tau_2]_d - [\tau_1, \Delta_u \tau_2]_d + \Delta_{\nabla^{\mathrm{bas}}_{a_1} u} \tau_2 - \Delta_{\nabla^{\mathrm{bas}}_{a_2} u} \tau_1$$
$$+ (0, \mathbf{d}\langle \tau_1, \nabla^{\mathrm{bas}}_{a_2} u \rangle)$$
$$= -R^{\mathrm{bas}}_\Delta(a_1, a_2) u$$

holds for all $\tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$ with $\mathrm{pr}_A(\tau_i) =: a_i$ and all $u \in \Gamma(U)$.

Proof of Theorem 5.5. We start by checking that $[\![\cdot, \cdot]\!]$ is well-defined. Choose $k \in \Gamma(K)$, $\tau \in \Gamma(A \oplus T^*M)$ with $\mathrm{pr}_A(k) = a$, $\mathrm{pr}_A(\tau) = a'$, and $u \in \Gamma(U)$. We

have then

$$\llbracket u \oplus \tau, (\rho, \rho^t)(-k) \oplus k \rrbracket = \left(-\llbracket u, (\rho, \rho^t)k \rrbracket_\Delta - \nabla_{a'}^{\mathrm{bas}}(\rho, \rho^t)k - \nabla_a^{\mathrm{bas}} u \right)$$
$$\oplus \left([\tau, k]_d + \Delta_u k + \Delta_{(\rho, \rho^t)k}\tau - (0, \mathbf{d}\langle \tau, (\rho, \rho^t)k \rangle) \right).$$

Using Lemma A.1, we see immediately that the second term of this sum equals $\nabla_{a'}^{\mathrm{bas}}k + \Delta_u k$. We verify that this is a section of K. By Remark 5.3 or the considerations before Theorem 5.2, we know that $\nabla_{a'}^{\mathrm{bas}}k \in \Gamma(K)$, and $\Delta_u k \in \Gamma(K)$ follows from $K = U^\circ$ and the fact that $\llbracket \cdot, \cdot \rrbracket_\Delta$, which is dual to Δ, preserves sections of U.

Recall (A.1) and $(\rho, \rho^t)\Delta_u k = \llbracket u, (\rho, \rho^t)k \rrbracket_\Delta + \nabla_{a'}^{\mathrm{bas}}u$ by Lemma A.3. These yield

$$\llbracket u \oplus \tau, (\rho, \rho^t)(-k) \oplus k \rrbracket = (-(\rho, \rho^t)(\nabla_a^{\mathrm{bas}}k + \Delta_u k)) \oplus (\nabla_a^{\mathrm{bas}}k + \Delta_u k),$$

which is 0 in C. The equality $\llbracket (\rho, \rho^t)(-k) \oplus k, u \oplus \tau \rrbracket = 0$ follows with

$$\llbracket (\rho, \rho^t)(-k) \oplus k, u \oplus \tau \rrbracket_C + \llbracket u \oplus \tau, (\rho, \rho^t)(-k) \oplus k \rrbracket_C$$
$$= \mathcal{D}\langle\langle u \oplus \tau, (\rho, \rho^t)(-k) \rangle\rangle_C$$

proved below and $\langle\langle u \oplus \tau, (\rho, \rho^t)(-k) \oplus k \rangle\rangle_C = \langle u, k \rangle = 0$ since $U = K^\circ$.

Choose u, $u_i \in \Gamma(U)$ and $\tau = (a, \theta)$, $\tau_i = (a_i, \theta_i) \in \Gamma(A \oplus T^*M)$, $i = 1, 2, 3$. We check $(1) - (3)$ in the definition of a Courant algebroid, in reverse order. First we have

$$\llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rrbracket_C + \llbracket u_2 \oplus \tau_2, u_1 \oplus \tau_1 \rrbracket_C$$
$$= (\llbracket u_1, u_2 \rrbracket_\Delta + \llbracket u_2, u_1 \rrbracket_\Delta) \oplus ([\tau_1, \tau_2]_d + [\tau_2, \tau_1]_d + (0, \mathbf{d}\langle \tau_1, u_2 \rangle + \mathbf{d}\langle \tau_2, u_1 \rangle)$$
$$= 0 \oplus (0, \mathbf{d}(\langle \tau_1, \tau_2 \rangle_d + \langle \tau_1, u_2 \rangle + \langle \tau_2, u_1 \rangle))$$
$$= \mathcal{D}\langle\langle u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rangle\rangle_C.$$

We denote by $\mathrm{Skew}_\Delta(\nu_1, \nu_2)$ the anti-commutator $\llbracket \nu_1, \nu_2 \rrbracket_\Delta + \llbracket \nu_2, \nu_1 \rrbracket_\Delta$, $\nu_1, \nu_2 \in \Gamma(TM \oplus A^*)$. We check the equality

$$c(u_1 \oplus \tau_1)\langle\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3 \rangle\rangle_C$$
$$= \langle\langle \llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rrbracket_C, u_3 \oplus \tau_3 \rangle\rangle_C + \langle\langle u_2 \oplus \tau_2, \llbracket u_1 \oplus \tau_1, u_3 \oplus \tau_3 \rrbracket_C \rangle\rangle_C.$$

The bracket $\llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2 \rrbracket$ equals

$$(\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\tau_1}^{\mathrm{bas}}u_2 - \nabla_{\tau_2}^{\mathrm{bas}}u_1) \oplus ([\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle \tau_1, u_2 \rangle)).$$

Using (A.2), we can replace $[\tau_1, \tau_2]_d$ by

$$\nabla^{\mathrm{bas}}_{\tau_1}\tau_2 - \Delta_{(\rho,\rho^t)\tau_2}\tau_1 + (0, \mathbf{d}\langle \tau_1, (\rho, \rho^t)\tau_2 \rangle)$$

in this expression and we find that $\langle\!\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3 \rangle\!\rangle_C$ equals

$$\begin{aligned}
&\langle [\![u_1, u_2]\!]_\Delta + \nabla^{\mathrm{bas}}_{\tau_1}u_2 - \nabla^{\mathrm{bas}}_{\tau_2}u_1, \tau_3 \rangle + X_3\langle \tau_1, u_2 \rangle + X_3\langle \tau_1, (\rho, \rho^t)\tau_2 \rangle \\
&+ \langle \nabla^{\mathrm{bas}}_{\tau_1}\tau_2 - \Delta_{(\rho,\rho^t)\tau_2}\tau_1 + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, u_3 \rangle + \rho(a_3)\langle \tau_1, u_2 \rangle \\
&+ \rho(a_3)\langle \tau_1, (\rho, \rho^t)\tau_2 \rangle + \langle \nabla^{\mathrm{bas}}_{\tau_1}\tau_2 - \Delta_{(\rho,\rho^t)\tau_2}\tau_1 + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, (\rho, \rho^t)\tau_3 \rangle.
\end{aligned}$$

Using (A.3) we replace $\langle -\nabla^{\mathrm{bas}}_{\tau_2}u_1 + (\rho, \rho^t)\Delta_{u_1}\tau_2, \tau_3 \rangle$ in this equation by

$$\langle \nabla^{\mathrm{bas}}_{\tau_3}u_1, \tau_2 \rangle + \langle [\![u_1, (\rho, \rho^t)\tau_2]\!]_\Delta, \tau_3 \rangle.$$

Now we sum $\langle\!\langle [\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3 \rangle\!\rangle_C$ with $\langle\!\langle [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!], u_2 \oplus \tau_2 \rangle\!\rangle_C$. The terms $\langle [\![u_1, u_2]\!]_\Delta, \tau_3 \rangle$, $\langle [\![u_1, u_3]\!]_\Delta, \tau_2 \rangle$, $\langle \Delta_{u_1}\tau_2, u_3 \rangle$ and $\langle \Delta_{u_1}\tau_3, u_2 \rangle$ add up to

$$X_1\langle\!\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3 \rangle\!\rangle_C - X_1\langle \tau_2, (\rho, \rho^t)\tau_3 \rangle.$$

The terms $X_3\langle \tau_1, u_2 \rangle$ and $-\langle \Delta_{u_3}\tau_1, u_2 \rangle$ add up to $\langle \tau_1, [\![u_2, u_3]\!]_\Delta \rangle$ and the terms $X_2\langle \tau_1, u_3 \rangle$ and $-\langle \Delta_{u_2}\tau_1, u_3 \rangle$ add up to $\langle \tau_1, [\![u_3, u_2]\!]_\Delta \rangle$, which cancels out $\langle \tau_1, [\![u_2, u_3]\!]_\Delta \rangle$ since $[\![\cdot, \cdot]\!]_\Delta$ is skew-symmetric on sections of U. In a similar manner,

$$\begin{aligned}
\text{(A.6)} \qquad &X_3\langle \tau_1, (\rho, \rho^t)\tau_2 \rangle - \langle \Delta_{u_2}\tau_1, (\rho, \rho^t)\tau_3 \rangle \\
&\qquad + X_2\langle \tau_1, (\rho, \rho^t)\tau_3 \rangle - \langle \Delta_{u_3}\tau_1, (\rho, \rho^t)\tau_2 \rangle
\end{aligned}$$

is $\langle \tau_1, [\![u_3, (\rho, \rho^t)\tau_2]\!]_\Delta + [\![u_2, (\rho, \rho^t)\tau_3]\!]_\Delta \rangle$, and

$$\text{(A.7)} \quad -\langle \Delta_{(\rho,\rho^t)\tau_2}\tau_1, u_3 \rangle + \rho(a_3)\langle \tau_1, u_2 \rangle - \langle \Delta_{(\rho,\rho^t)\tau_3}\tau_1, u_2 \rangle + \rho(a_2)\langle \tau_1, u_3 \rangle$$

is $\langle \tau_1, [\![(\rho, \rho^t)\tau_2, u_3]\!]_\Delta + [\![(\rho, \rho^t)\tau_3, u_2]\!]_\Delta \rangle$. As a consequence, (A.6) and (A.7) add up to

$$\langle \tau_1, \mathrm{Skew}_\Delta(u_2, (\rho, \rho^t)\tau_3) + \mathrm{Skew}_\Delta(u_3, (\rho, \rho^t)\tau_2) \rangle.$$

By (5.1), this adds up with

$$\langle \nabla^{\mathrm{bas}}_{\tau_1}u_2, \tau_3 \rangle + \langle \nabla^{\mathrm{bas}}_{\tau_1}\tau_2, u_3 \rangle + \langle \nabla^{\mathrm{bas}}_{\tau_1}u_3, \tau_2 \rangle + \langle \nabla^{\mathrm{bas}}_{\tau_1}\tau_3, u_2 \rangle$$

to $\rho(a_1)\langle u_2, \tau_3\rangle + \rho(a_1)\langle u_3, \tau_2\rangle$. Similarly, using (5.1) and (A.1), the terms $\langle \nabla_{\tau_1}^{\mathrm{bas}}\tau_2, (\rho, \rho^t)\tau_3\rangle$ and $\langle \nabla_{\tau_1}^{\mathrm{bas}}\tau_3, (\rho, \rho^t)\tau_2\rangle$ add up to

$$\rho(a_1)\langle \tau_2, (\rho, \rho^t)\tau_3\rangle - \langle \tau_1, \mathrm{Skew}_\Delta((\rho, \rho^t)\tau_2, (\rho, \rho^t)\tau_3)\rangle.$$

The terms $\rho(a_3)\langle \tau_1, (\rho, \rho^t)\tau_2\rangle$, $\rho(a_2)\langle \tau_1, (\rho, \rho^t)\tau_3\rangle$, $-\langle \Delta_{(\rho,\rho^t)\tau_2}\tau_1, (\rho, \rho^t)\tau_3\rangle$ and $-\langle \Delta_{(\rho,\rho^t)\tau_3}\tau_1, (\rho, \rho^t)\tau_2\rangle$ sum up to $\langle \tau_1, \mathrm{Skew}_\Delta((\rho, \rho^t)\tau_2, (\rho, \rho^t)\tau_3)\rangle$.

Putting everything together, we get

$$c(u_1 \oplus \tau_1)\langle\langle u_2 \oplus \tau_2, u_3 \oplus \tau_3\rangle\rangle_C - X_1\langle \tau_2, (\rho, \rho^t)\tau_3\rangle$$
$$+ \langle \nabla_{\tau_3}^{\mathrm{bas}} u_1, \tau_2\rangle + \langle [\![u_1, (\rho, \rho^t)\tau_2]\!]_\Delta, \tau_3\rangle + \langle \nabla_{\tau_2}^{\mathrm{bas}} u_1, \tau_3\rangle + \langle [\![u_1, (\rho, \rho^t)\tau_3]\!]_\Delta, \tau_2\rangle.$$

By (A.3) the third, fourth and fifth term add up to $\langle \Delta_{u_1}\tau_2, (\rho, \rho^t)\tau_3\rangle$, which cancels out the second and the sixth.

For the Jacobi identity, we check that

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]_\Delta}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$$
$$:= [\![[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3]\!] + [\![u_2 \oplus \tau_2, [\![u_1 \oplus \tau_1, u_3 \oplus \tau_3]\!]]\!]$$
$$- [\![u_1 \oplus \tau_1, [\![u_2 \oplus \tau_2, u_3 \oplus \tau_3]\!]]\!]$$
$$= -(\rho, \rho^t)(k) \oplus k,$$

with $k := [R_\Delta^{\mathrm{bas}}(a_1, a_2)u_3 - R_\Delta(u_1, u_2)\tau_3] + \text{c.p.}$

To see that k is a section of K, recall from Theorem 5.2 that since $u_i = (X_i, \alpha_i) \in \Gamma(U)$, $i = 1, 2, 3$, we have $R_\Delta(u_1, u_2)\tau_3 + \text{c.p.} \in \Gamma(K)$. By the same theorem, we find $R_\Delta^{\mathrm{bas}}(a_1, a_2)u_3 + \text{c.p.} \in \Gamma(K)$. We write $\tau_i = (a_i, \theta_i)$ for $i = 1, 2, 3$. Since $[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!]$ is

$$\left([\![u_1, u_2]\!]_\Delta + \nabla_{\tau_1}^{\mathrm{bas}} u_2 - \nabla_{\tau_2}^{\mathrm{bas}} u_1\right) \oplus \left([\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle \tau_1, u_2\rangle)\right),$$

its bracket $[\![[\![u_1 \oplus \tau_1, u_2 \oplus \tau_2]\!], u_3 \oplus \tau_3]\!]$ with $u_3 \oplus \tau_3$ is

$$\left([\![[\![u_1, u_2]\!]_\Delta, u_3]\!]_\Delta + [\![\nabla_{\tau_1}^{\mathrm{bas}} u_2 - \nabla_{\tau_2}^{\mathrm{bas}} u_1, u_3]\!]_\Delta \right.$$
$$+ \nabla_{[\tau_1,\tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle \tau_1, u_2\rangle)}^{\mathrm{bas}} u_3$$
$$\left. - \nabla_{\tau_3}^{\mathrm{bas}}\left([\![u_1, u_2]\!]_\Delta + \nabla_{\tau_1}^{\mathrm{bas}} u_2 - \nabla_{\tau_2}^{\mathrm{bas}} u_1\right)\right)$$
$$\oplus \left([[\tau_1, \tau_2]_d, \tau_3]_d + [\Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, \tau_3]_d + [(0, \mathbf{d}\langle \tau_1, u_2\rangle), \tau_3]_d\right.$$
$$+ \Delta_{[\![u_1, u_2]\!]_\Delta + \nabla_{\tau_1}^{\mathrm{bas}} u_2 - \nabla_{\tau_2}^{\mathrm{bas}} u_1}\tau_3 - \Delta_{u_3}([\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1$$
$$+ (0, \mathbf{d}\langle \tau_1, u_2\rangle)) + (0, \mathbf{d}\langle [\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1$$
$$\left. + (0, \mathbf{d}\langle \tau_1, u_2\rangle), u_3\rangle)\right).$$

Since $[(0, \mathbf{d}\langle\tau_1, u_2\rangle), \tau_3]_d = 0$ and

$$\mathrm{pr}_A\left([\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle\tau_1, u_2\rangle)\right)$$
$$= [a_1, a_2] + \mathrm{pr}_A\left(\Omega_{u_1}a_2 - \Omega_{u_2}a_1\right),$$

this is

$$\llbracket\llbracket u_1 \oplus \tau_1, u_2 \oplus \tau_2\rrbracket, u_3 \oplus \tau_3\rrbracket$$
$$= \left(\llbracket\llbracket u_1, u_2\rrbracket_\Delta, u_3\rrbracket_\Delta + \left\llbracket\nabla^{\mathrm{bas}}_{\tau_1}u_2 - \nabla^{\mathrm{bas}}_{a_2}u_1, u_3\right\rrbracket_\Delta\right.$$
$$+ \nabla^{\mathrm{bas}}_{[a_1,a_2]+\mathrm{pr}_A\left(\Omega_{u_1}a_2-\Omega_{u_2}a_1\right)}u_3$$
$$\left. - \nabla^{\mathrm{bas}}_{a_3}\left(\llbracket u_1, u_2\rrbracket_\Delta + \nabla^{\mathrm{bas}}_{a_1}u_2 - \nabla^{\mathrm{bas}}_{a_2}u_1\right)\right)$$
$$\oplus\left([[\tau_1, \tau_2]_d, \tau_3]_d + [\Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1, \tau_3]_d + \Delta_{\llbracket u_1,u_2\rrbracket_\Delta+\nabla^{\mathrm{bas}}_{a_1}u_2-\nabla^{\mathrm{bas}}_{a_2}u_1}\tau_3\right.$$
$$- \Delta_{u_3}([\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle\tau_1, u_2\rangle))$$
$$\left. + (0, \mathbf{d}\langle[\tau_1, \tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0, \mathbf{d}\langle\tau_1, u_2\rangle), u_3\rangle)\right).$$

In the same manner, we get

$$\llbracket u_2 \oplus \tau_2, \llbracket u_1 \oplus \tau_1, u_3 \oplus \tau_3\rrbracket\rrbracket$$
$$= \left(\llbracket u_2, \llbracket u_1, u_3\rrbracket_\Delta\rrbracket_\Delta + \left\llbracket u_2, \nabla^{\mathrm{bas}}_{a_1}u_3 - \nabla^{\mathrm{bas}}_{a_3}u_1\right\rrbracket_\Delta\right.$$
$$\left. + \nabla^{\mathrm{bas}}_{a_2}\left(\llbracket u_1, u_3\rrbracket_\Delta + \nabla^{\mathrm{bas}}_{a_1}u_3 - \nabla^{\mathrm{bas}}_{a_3}u_1\right) - \nabla^{\mathrm{bas}}_{[a_1,a_3]+\mathrm{pr}_A\left(\Omega_{u_1}a_3-\Omega_{u_3}a_1\right)}u_2\right)$$
$$\oplus\left([\tau_2, [\tau_1, \tau_3]_d]_d + [\tau_2, \Delta_{u_1}\tau_3 - \Delta_{u_3}\tau_1]_d + [\tau_2, (0, \mathbf{d}\langle\tau_1, u_3\rangle)]_d\right.$$
$$+ \Delta_{u_2}([\tau_1, \tau_3]_d + \Delta_{u_1}\tau_3 - \Delta_{u_3}\tau_1 + (0, \mathbf{d}\langle\tau_1, u_3\rangle))$$
$$\left. - \Delta_{\llbracket u_1,u_3\rrbracket_\Delta+\nabla^{\mathrm{bas}}_{a_1}u_3-\nabla^{\mathrm{bas}}_{a_3}u_1}\tau_2 + (0, \mathbf{d}\langle\tau_2, \llbracket u_1, u_3\rrbracket_\Delta + \nabla^{\mathrm{bas}}_{a_1}u_3 - \nabla^{\mathrm{bas}}_{a_3}u_1\rangle)\right).$$

We start by studying the U-part of $\mathrm{Jac}_{\llbracket\cdot,\cdot\rrbracket_\Delta}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$. By the computations above, this equals

$$\cancel{[\![[\![u_1,u_2]\!]_\Delta,u_3]\!]_\Delta} + \cancel{[\![u_2,[\![u_1,u_3]\!]_\Delta]\!]_\Delta} - \cancel{[\![u_1,[\![u_2,u_3]\!]_\Delta]\!]_\Delta}$$
$$- R_{\nabla^{\mathrm{bas}}}(a_1,a_2)u_3 + R_{\nabla^{\mathrm{bas}}}(a_1,a_3)u_2 - R_{\nabla^{\mathrm{bas}}}(a_2,a_3)u_1$$
$$+ \left[\!\left[\nabla^{\mathrm{bas}}_{a_1}u_2 - \nabla^{\mathrm{bas}}_{a_2}u_1, u_3\right]\!\right]_\Delta + \nabla^{\mathrm{bas}}_{\mathrm{pr}_A(\Omega_{u_1}a_2 - \Omega_{u_2}a_1)}u_3 - \nabla^{\mathrm{bas}}_{a_3}[\![u_1,u_2]\!]_\Delta$$
$$+ \left[\!\left[u_2, \nabla^{\mathrm{bas}}_{a_1}u_3 - \nabla^{\mathrm{bas}}_{a_3}u_1\right]\!\right]_\Delta + \nabla^{\mathrm{bas}}_{a_2}[\![u_1,u_3]\!]_\Delta - \nabla^{\mathrm{bas}}_{\mathrm{pr}_A(\Omega_{u_1}a_3 - \Omega_{u_3}a_1)}u_2$$
$$- \left[\!\left[u_1, \nabla^{\mathrm{bas}}_{a_2}u_3 - \nabla^{\mathrm{bas}}_{a_3}u_2\right]\!\right]_\Delta - \nabla^{\mathrm{bas}}_{a_1}[\![u_2,u_3]\!]_\Delta + \nabla^{\mathrm{bas}}_{\mathrm{pr}_A(\Omega_{u_2}a_3 - \Omega_{u_3}a_2)}u_1$$

Since $(U, \mathrm{pr}_{TM}, [\![\cdot,\cdot]\!]_\Delta)$ is a Lie algebroid, the first three terms vanish together. Next note that since $(\nabla^{\mathrm{bas}}, \nabla^{\mathrm{bas}}, R^{\mathrm{bas}}_\Delta)$ is a representation up to homotopy of A on $(\rho, \rho^t)\colon A \oplus T^*M \to TM \oplus A^*$, the basic curvature satisfies

$$(A.8) \qquad R_{\nabla^{\mathrm{bas}}}(a_1,a_2)\nu = (\rho,\rho^t) \circ R^{\mathrm{bas}}_\Delta(a_1,a_2)\nu$$

for all $a_1, a_2 \in \Gamma(A)$, $\nu \in \Gamma(TM \oplus A^*)$. Using this, (A.4), (4) in Theorem 5.2, the skew-symmetry of $[\![\cdot,\cdot]\!]_\Delta$ on $\Gamma(U)$, and the equality $\mathrm{pr}_A\,\Omega_{u_1}a_2 = \mathrm{pr}_A\,\Delta_{u_1}\tau_2$, we find easily that the U-part of $\mathrm{Jac}_{[\![\cdot,\cdot]\!]_\Delta}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$ is

$$- (\rho,\rho^t)(R^{\mathrm{bas}}_\Delta(a_1,a_2)u_3 + R^{\mathrm{bas}}_\Delta(a_3,a_1)u_2 + R^{\mathrm{bas}}_\Delta(a_2,a_3)u_1)$$
$$+ (\rho,\rho^t)(R_\Delta(u_1,u_2)\tau_3 + R_\Delta(u_2,u_3)\tau_1 + R_\Delta(u_3,u_1)\tau_2).$$

Now we look more carefully at the $A \oplus T^*M$-part of $\mathrm{Jac}_{[\![\cdot,\cdot]\!]_\Delta}(u_1 \oplus \tau_1, u_2 \oplus \tau_2, u_3 \oplus \tau_3)$. Sorting out the terms yields that this is

$$\cancel{[\![[\tau_1,\tau_2]_d,\tau_3]_d} + \cancel{[\tau_2,[\tau_1,\tau_3]_d]_d} - \cancel{[\tau_1,[\tau_2,\tau_3]_d]_d}$$
$$- \Delta_{u_1}[\tau_2,\tau_3]_d + [\Delta_{u_1}\tau_2,\tau_3]_d + [\tau_2,\Delta_{u_1}\tau_3]_d + \Delta_{\nabla^{\mathrm{bas}}_{a_3}u_1}\tau_2 - \Delta_{\nabla^{\mathrm{bas}}_{a_2}u_1}\tau_3$$
$$- (0,\mathbf{d}\langle\tau_2, \nabla^{\mathrm{bas}}_{a_3}u_1\rangle) + \Delta_{u_2}[\tau_1,\tau_3]_d - [\Delta_{u_2}\tau_1,\tau_3]_d - [\tau_1,\Delta_{u_2}\tau_3]_d - \Delta_{\nabla^{\mathrm{bas}}_{a_3}u_2}\tau_1$$
$$+ \Delta_{\nabla^{\mathrm{bas}}_{a_1}u_2}\tau_3 + (0,\mathbf{d}\langle\tau_1, \nabla^{\mathrm{bas}}_{a_3}u_2\rangle) - \Delta_{u_3}[\tau_1,\tau_2]_d - [\tau_2,\Delta_{u_3}\tau_1]_d + [\tau_1,\Delta_{u_3}\tau_2]_d$$
$$+ \Delta_{\nabla^{\mathrm{bas}}_{a_2}u_3}\tau_1 - \Delta_{\nabla^{\mathrm{bas}}_{a_1}u_3}\tau_2 - (0,\mathbf{d}\langle\tau_1, \nabla^{\mathrm{bas}}_{a_2}u_3\rangle)$$
$$- R_\Delta(u_1,u_2)\tau_3 - R_\Delta(u_2,u_3)\tau_1 - R_\Delta(u_3,u_1)\tau_2$$
$$- \Delta_{u_3}(0,\mathbf{d}\langle\tau_1,u_2\rangle) + (0,\mathbf{d}\langle[\tau_1,\tau_2]_d + \Delta_{u_1}\tau_2 - \Delta_{u_2}\tau_1 + (0,\mathbf{d}\langle\tau_1,u_2\rangle), u_3\rangle)$$
$$+ [\tau_2,(0,\mathbf{d}\langle\tau_1,u_3\rangle)]_d + \Delta_{u_2}(0,\mathbf{d}\langle\tau_1,u_3\rangle) + (0,\mathbf{d}\langle\tau_2,[\![u_1,u_3]\!]_\Delta + \nabla^{\mathrm{bas}}_{a_1}u_3\rangle)$$
$$- [\tau_1,(0,\mathbf{d}\langle\tau_2,u_3\rangle)]_d - \Delta_{u_1}(0,\mathbf{d}\langle\tau_2,u_3\rangle) - (0,\mathbf{d}\langle\tau_1,[\![u_2,u_3]\!]_\Delta\rangle).$$

By the Jacobi identity for $[\cdot,\cdot]_d$ and three times (A.5) together with $-[\tau_2,\Delta_{u_3}\tau_1]_d = [\Delta_{u_3}\tau_1,\tau_2]_d - (0,\mathbf{d}\langle(\rho,\rho^t)\Delta_{u_3}\tau_1,\tau_2\rangle)$, we get

$$
\begin{aligned}
& R_\Delta^{\mathrm{bas}}(a_2,a_3)u_1 - R_\Delta^{\mathrm{bas}}(a_1,a_3)u_2 + R_\Delta^{\mathrm{bas}}(a_1,a_2)u_3 \\
& \quad - R_\Delta(u_1,u_2)\tau_3 - R_\Delta(u_2,u_3)\tau_1 - R_\Delta(u_3,u_1)\tau_2 \\
& \quad - \underline{(0,\mathbf{d}X_3\langle\tau_1,u_2\rangle)} + (0,\mathbf{d}\langle[\tau_1,\tau_2]_d + \Delta_{a_1}\tau_2 - \Delta_{a_2}\tau_1,u_3\rangle) + \underline{(0,\mathbf{d}X_3\langle\tau_1,u_2\rangle)} \\
& \quad + (0,\mathbf{d}\rho(a_2)\langle\tau_1,u_3\rangle) + \underline{(0,\mathbf{d}X_2\langle\tau_1,u_3\rangle)} + (0,\mathbf{d}\langle\tau_2,[\![u_1,u_3]\!]_\Delta + \nabla_{a_1}^{\mathrm{bas}}u_3\rangle) \\
& \quad - (0,\mathbf{d}\rho(a_1)\langle\tau_2,u_3\rangle) - \underline{(0,\mathbf{d}X_1\langle\tau_2,u_3\rangle)} - \underline{(0,\mathbf{d}\langle\tau_1,[\![u_2,u_3]\!]_\Delta\rangle)} \\
& \quad - (0,\mathbf{d}\langle(\rho,\rho^t)\Delta_{u_3}\tau_1,\tau_2\rangle) \\
&= R_\Delta^{\mathrm{bas}}(a_2,a_3)u_1 + R_\Delta^{\mathrm{bas}}(a_3,a_1)u_2 + R_\Delta^{\mathrm{bas}}(a_1,a_2)u_3 \\
& \quad - R_\Delta(u_1,u_2)\tau_3 - R_\Delta(u_2,u_3)\tau_1 - R_\Delta(u_3,u_1)\tau_2 + (0,\mathbf{d}f),
\end{aligned}
$$

with (using (A.2))

$$
\begin{aligned}
f &= \langle\Delta_{(\rho,\rho^t)\tau_1}\tau_2 - \nabla_{\tau_2}^{\mathrm{bas}}\tau_1,u_3\rangle + \rho(a_2)\langle\tau_1,u_3\rangle \\
& \quad + \langle\tau_2,\nabla_{a_1}^{\mathrm{bas}}u_3 - (\rho,\rho^t)\Delta_{u_3}\tau_1\rangle - \rho(a_1)\langle\tau_2,u_3\rangle \\
&= -\langle\nabla_{\tau_2}^{\mathrm{bas}}\tau_1,u_3\rangle + \rho(a_2)\langle\tau_1,u_3\rangle \\
& \quad + \langle\tau_2,\nabla_{a_1}^{\mathrm{bas}}u_3 - (\rho,\rho^t)\Delta_{u_3}\tau_1 - [\![(\rho,\rho^t)\tau_1,u_3]\!]_\Delta\rangle \\
&\overset{(\mathrm{A}.3)}{=} -\langle\nabla_{\tau_2}^{\mathrm{bas}}\tau_1,u_3\rangle + \rho(a_2)\langle\tau_1,u_3\rangle - \langle\nabla_{\tau_2}^{\mathrm{bas}}u_3,\tau_1\rangle \\
& \quad - \langle\mathrm{Skew}_\Delta(u_3,(\rho,\rho^t)\tau_1),a_2\rangle \\
&\overset{(5.1)}{=} 0. \tag*{$\square$}
\end{aligned}
$$

We have used Theorem A.4 for the Jacobi identity. For completeness, we prove this theorem here.

Proof of Theorem A.4. We start with the first equation. For simplicity, we write here a for $(a,0)$, $\Delta_u a := \Delta_u(a,0)$ and $\mathbf{d}f := (0,\mathbf{d}f)$ for $a \in \Gamma(A)$, $f \in C^\infty(M)$ and $u \in \Gamma(U)$. First, we find that for $a_1,a_2 \in \Gamma(A)$ and $u_1 = (X_1,\alpha_1), u_2 = (X_2,\alpha_2) \in \Gamma(U)$, the equation $0 = \langle R_\Delta^{\mathrm{bas}}(a_1,a_2)u_1,u_2\rangle$ can be written

$$
\begin{aligned}
0 = & -\langle\Delta_{u_1}[a_1,a_2] - \mathbf{d}\langle\alpha_1,[a_1,a_2]\rangle,u_2\rangle \\
& + \rho(a_1)\langle\Delta_{u_1}a_2 - \mathbf{d}\langle\alpha_1,a_2\rangle,u_2\rangle - \langle\Delta_{u_1}a_2 - \mathbf{d}\langle\alpha_1,a_2\rangle,\pounds_{a_2}u_2\rangle \\
& - \rho(a_2)\langle\Delta_{u_1}a_1 - \mathbf{d}\langle\alpha_1,a_1\rangle,u_2\rangle + \langle\Delta_{u_1}a_1 - \mathbf{d}\langle\alpha_1,a_1\rangle,\pounds_{a_2}u_2\rangle \\
& + \langle\Delta_{(\rho,\rho^t)\Omega_{u_1}a_2+\pounds_{a_2}u_1}a_1 - \mathbf{d}\langle(\rho,\rho^t)\Omega_{u_1}a_2 + \pounds_{a_2}u_1,a_1\rangle,u_2\rangle \\
& - \langle\Delta_{(\rho,\rho^t)\Omega_{u_1}a_1+\pounds_{a_1}u_1}a_2 - \mathbf{d}\langle(\rho,\rho^t)\Omega_{u_1}a_1 + \pounds_{a_1}u_1,a_2\rangle,u_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\langle \Delta_{u_1}[a_1, a_2], u_2 \rangle + X_2 \langle \alpha_1, [a_1, a_2] \rangle \\
&\quad + \rho(a_1)\langle \Delta_{u_1} a_2, u_2 \rangle - \rho(a_1)X_2\langle \alpha_1, a_2 \rangle \\
&\quad - \langle \Delta_{u_1} a_2, \pounds_{a_1} u_2 \rangle + [\rho(a_1), X_2]\langle \alpha_1, a_2 \rangle \\
&\quad - \rho(a_2)\langle \Delta_{u_1} a_1, u_2 \rangle + \rho(a_2)X_2\langle \alpha_1, a_1 \rangle \\
&\quad + \langle \Delta_{u_1} a_1, \pounds_{a_2} u_2 \rangle - [\rho(a_2), X_2]\langle \alpha_1, a_1 \rangle \\
&\quad + \langle \Delta_{(\rho,\rho^t)\Omega_{u_1} a_2} a_1, u_2 \rangle + [\rho(a_2), X_1]\langle a_1, \alpha_2 \rangle - \langle a_1, [\![\pounds_{a_2} u_1, u_2]\!]_\Delta \rangle \\
&\quad - X_2 \langle (\rho,\rho^t)\Omega_{u_1} a_2 + \pounds_{a_2} u_1, a_1 \rangle \\
&\quad - \langle \Delta_{(\rho,\rho^t)\Omega_{u_1} a_1} a_2, u_2 \rangle - [\rho(a_1), X_1]\langle a_2, \alpha_2 \rangle + \langle a_2, [\![\pounds_{a_1} u_1, u_2]\!]_\Delta \rangle \\
&\quad + X_2 \langle (\rho,\rho^t)\Omega_{u_1} a_1 + \pounds_{a_1} u_1, a_2 \rangle.
\end{aligned}
$$

We use

$$
\begin{aligned}
&X_2 \langle \alpha_1, [a_1, a_2] \rangle + \rho(a_2)X_2\langle \alpha_1, a_1 \rangle - [\rho(a_2), X_2]\langle \alpha_1, a_1 \rangle - X_2\langle \pounds_{a_2} u_1, a_1 \rangle \\
&= X_2(-\langle \alpha_1, [a_2, a_1] \rangle + \rho(a_2)\langle \alpha_1, a_1 \rangle - \langle \pounds_{a_2}\alpha_1, a_1 \rangle) = 0,
\end{aligned}
$$

similarly

$$
-\rho(a_1)X_2\langle \alpha_1, a_2 \rangle + [\rho(a_1), X_2]\langle \alpha_1, a_2 \rangle + X_2\langle \pounds_{a_1} u_1, a_2 \rangle = -X_2\langle \alpha_1, [a_1, a_2] \rangle
$$

and, by (A.2) and (5.1),

$$
\begin{aligned}
&\langle \Delta_{(\rho,\rho^t)\Omega_{u_1} a_2} a_1, u_2 \rangle - X_2\langle (\rho,\rho^t)\Omega_{u_1} a_2, a_1 \rangle \\
&= \langle \nabla^{\mathrm{bas}}_{a_1}\Omega_{u_1} a_2 + [\Omega_{u_1} a_2, a_1]_d, u_2 \rangle - X_2\langle (\rho,\rho^t)\Omega_{u_1} a_2, a_1 \rangle \\
&= \rho(a_1)\langle \Omega_{u_1} a_2, u_2 \rangle - \langle \Omega_{u_1} a_2, \nabla^{\mathrm{bas}}_{a_1} u_2 \rangle \\
&\quad - \langle \mathrm{Skew}_\Delta(u_2, (\rho,\rho^t)\Omega_{u_1} a_2), a_1 \rangle - \langle \pounds_{a_1}\Omega_{u_1} a_2, u_2 \rangle \\
&= -\langle \Omega_{u_1} a_2, (\rho,\rho^t)\Omega_{u_2} a_1 \rangle - \langle \mathrm{Skew}_\Delta(u_2, (\rho,\rho^t)\Omega_{u_1} a_2), a_1 \rangle
\end{aligned}
$$

to get

$$
\begin{aligned}
0 = &-X_2 \langle \alpha_1, [a_1, a_2] \rangle - X_1\langle \alpha_2, [a_1, a_2] \rangle + \langle [a_1, a_2], [\![u_1, u_2]\!]_\Delta \rangle \\
&- \langle a_1, [\![\pounds_{a_2} u_1, u_2]\!]_\Delta \rangle + \langle a_2, [\![\pounds_{a_1} u_1, u_2]\!]_\Delta \rangle \\
&+ \rho(a_1)\langle \Delta_{u_1} a_2, u_2 \rangle - X_1\langle a_2, \pounds_{a_1} u_2 \rangle + \langle a_2, [\![u_1, \pounds_{a_1} u_2]\!]_\Delta \rangle \\
&- \rho(a_2)\langle \Delta_{u_1} a_1, u_2 \rangle + X_1\langle a_1, \pounds_{a_2} u_2 \rangle - \langle a_1, [\![u_1, \pounds_{a_2} u_2]\!]_\Delta \rangle \\
&+ [\rho(a_2), X_1]\langle a_1, \alpha_1 \rangle - [\rho(a_1), X_1]\langle a_2, \alpha_2 \rangle \\
&- \langle \Omega_{u_1} a_2, (\rho,\rho^t)\Omega_{u_2} a_1 \rangle - \langle \mathrm{Skew}_\Delta(u_2, (\rho,\rho^t)\Omega_{u_1} a_2), a_1 \rangle \\
&+ \langle \Omega_{u_1} a_1, (\rho,\rho^t)\Omega_{u_2} a_2 \rangle + \langle \mathrm{Skew}_\Delta(u_2, (\rho,\rho^t)\Omega_{u_1} a_1), a_2 \rangle.
\end{aligned}
$$

But since

$$\rho(a_1)\langle \Delta_{u_1} a_2, u_2 \rangle - [\rho(a_1), X_1]\langle a_2, \alpha_2 \rangle - X_1 \langle a_2, \pounds_{a_1} u_2 \rangle$$
$$= -\rho(a_1)\langle a_2, [\![u_1, u_2]\!]_\Delta \rangle + X_1 \langle [a_1, a_2], \alpha_2 \rangle$$
$$= -\langle [a_1, a_2], [\![u_1, u_2]\!]_\Delta \rangle - \langle a_2, \pounds_{a_1}[\![u_1, u_2]\!]_\Delta \rangle + X_1 \langle [a_1, a_2], \alpha_2 \rangle,$$

we find

$$\begin{aligned}
\text{(A.9)} \quad 0 = {}& -X_2 \langle \alpha_1, [a_1, a_2] \rangle + X_1 \langle \alpha_1, [a_1, a_2] \rangle - \langle [a_1, a_2], [\![u_1, u_2]\!]_\Delta \rangle \\
& - \langle a_1, [\![\pounds_{a_2} u_1, u_2]\!]_\Delta \rangle + \langle a_2, [\![\pounds_{a_1} u_1, u_2]\!]_\Delta \rangle + \langle a_2, [\![u_1, \pounds_{a_1} u_2]\!]_\Delta \rangle \\
& - \langle a_1, [\![u_1, \pounds_{a_2} u_2]\!]_\Delta \rangle - \langle a_2, \pounds_{a_1}[\![u_1, u_2]\!]_\Delta \rangle + \langle a_1, \pounds_{a_2}[\![u_1, u_2]\!]_\Delta \rangle \\
& - \langle \Omega_{u_1} a_2, (\rho, \rho^t)\Omega_{u_2} a_1 \rangle - \langle \mathrm{Skew}_\Delta(u_2, (\rho, \rho^t)\Omega_{u_1} a_2), a_1 \rangle \\
& + \langle \Omega_{u_1} a_1, (\rho, \rho^t)\Omega_{u_2} a_2 \rangle + \langle \mathrm{Skew}_\Delta(u_2, (\rho, \rho^t)\Omega_{u_1} a_1), a_2 \rangle.
\end{aligned}$$

Now, writing $\tau = (a, \theta)$, we compute:

$$\begin{aligned}
& \nabla_\tau^{\mathrm{bas}}[\![u_1, u_2]\!]_\Delta - [\![\nabla_\tau^{\mathrm{bas}} u_1, u_2]\!]_\Delta - [\![u_1, \nabla_\tau^{\mathrm{bas}} u_2]\!]_\Delta + \nabla_{\Delta_{u_1}\tau}^{\mathrm{bas}} u_2 - \nabla_{\Delta_{u_2}\tau}^{\mathrm{bas}} u_1 \\
={}& (\rho, \rho^t)\Omega_{[\![u_1, u_2]\!]_\Delta} a + \pounds_a[\![u_1, u_2]\!]_\Delta - [\![(\rho, \rho^t)\Omega_{u_1} a + \pounds_a u_1, u_2]\!]_\Delta \\
& - [\![u_1, (\rho, \rho^t)\Omega_{u_2} a + \pounds_a u_2]\!]_\Delta + \nabla_{\mathrm{pr}_A \Omega_{u_1} a}^{\mathrm{bas}} u_2 - \nabla_{\mathrm{pr}_A \Omega_{u_2} a}^{\mathrm{bas}} u_1 \\
\overset{\text{(A.3)}}{=}{}& (\rho, \rho^t)\big(\Delta_{[\![u_1, u_2]\!]_\Delta} a - \mathbf{d}\langle a, [\![u_1, u_2]\!]_\Delta \rangle\big) + \pounds_a[\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta \\
& - [\![u_1, \pounds_a u_2]\!]_\Delta + (\rho, \rho^t)\Delta_{u_2}\Omega_{u_1} a - \langle \nabla_\cdot^{\mathrm{bas}} u_2, \Omega_{u_1} a \rangle - (\rho, \rho^t)\Delta_{u_1}\Omega_{u_2} a \\
& + \langle \nabla_\cdot^{\mathrm{bas}} u_1, \Omega_{u_2} a \rangle - \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2) \\
={}& -(\rho, \rho^t)R_\Delta(u_1, u_2)a - \rho^t \mathbf{d}\langle a, [\![u_1, u_2]\!]_\Delta \rangle + \pounds_a[\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta \\
& - [\![u_1, \pounds_a u_2]\!]_\Delta - \rho^t \mathbf{d}(X_2\langle \alpha_1, a \rangle) - \langle \nabla_\cdot^{\mathrm{bas}} u_2, \Omega_{u_1} a \rangle + \rho^t \mathbf{d}(X_1\langle \alpha_2, a \rangle) \\
& + \langle \nabla_\cdot^{\mathrm{bas}} u_1, \Omega_{u_2} a \rangle - \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2).
\end{aligned}$$

We write the right-hand side of this equation

$$-(\rho, \rho^t)R_\Delta(u_1, u_2)a + \tau'$$

with $\tau' \in \Gamma(TM \oplus A^*)$. Since

$$\mathrm{pr}_{TM}\big(\pounds_a[\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta - [\![u_1, \pounds_a u_2]\!]_\Delta\big)$$

is $[\rho(a), [X_1, X_2]] - [[\rho(a), X_1], X_2] - [X_1, [\rho(a), X_2]]$, which vanishes, we observe that $\mathrm{pr}_{TM} \tau' = 0$. Hence, we just have to show that $\langle \tau', a' \rangle = 0$ for any

section $a' \in \Gamma(A)$. We have

$$
\begin{aligned}
\langle \tau', a' \rangle ={}& -\rho(a')\langle a, [\![u_1, u_2]\!]_\Delta \rangle + \langle \pounds_a [\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta \\
& - [\![u_1, \pounds_a u_2]\!]_\Delta, a' \rangle - \rho(a')X_2\langle \alpha_1, a \rangle - \langle \nabla^{\mathrm{bas}}_{a'} u_2, \Omega_{u_1} a \rangle \\
& + \rho(a')X_1\langle \alpha_2, a \rangle + \langle \nabla^{\mathrm{bas}}_{a'} u_1, \Omega_{u_2} a \rangle \\
& - \langle \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2), a' \rangle \\
={}& \langle [a, a'], [\![u_1, u_2]\!]_\Delta \rangle - \langle a, \pounds_{a'} [\![u_1, u_2]\!]_\Delta \rangle \\
& + \langle \pounds_a [\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta - [\![u_1, \pounds_a u_2]\!]_\Delta, a' \rangle \\
& - \rho(a')X_2\langle \alpha_1, a \rangle - \langle (\rho, \rho^t)\Omega_{u_2} a' + \pounds_{a'} u_2, \Omega_{u_1} a \rangle \\
& + \rho(a')X_1\langle \alpha_2, a \rangle + \langle (\rho, \rho^t)\Omega_{u_1} a' + \pounds_{a'} u_1, \Omega_{u_2} a \rangle \\
& - \langle \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2), a' \rangle.
\end{aligned}
$$

Since the definition of Ω and the duality of Δ and $[\![\cdot, \cdot]\!]_\Delta$ yield

$$
\begin{aligned}
& \langle \pounds_{a'} u_1, \Omega_{u_2} a \rangle + \rho(a')X_1\langle \alpha_2, a \rangle \\
={}& \langle \pounds_{a'} u_1, \Delta_{u_2} a \rangle + X_1\rho(a')\langle \alpha_2, a \rangle \\
={}& X_2\langle \pounds_{a'} u_1, a \rangle - \langle [\![u_2, \pounds_{a'} u_1]\!]_\Delta, a \rangle + X_1\rho(a')\langle \alpha_2, a \rangle,
\end{aligned}
$$

we get

$$
\begin{aligned}
\langle \tau', a' \rangle ={}& \langle [a, a'], [\![u_1, u_2]\!]_\Delta \rangle - \langle a, \pounds_{a'} [\![u_1, u_2]\!]_\Delta \rangle \\
& + \langle \pounds_a [\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta - [\![u_1, \pounds_a u_2]\!]_\Delta, a' \rangle \\
& + \langle (\rho, \rho^t)\Omega_{u_1} a', \Omega_{u_2} a \rangle - \langle (\rho, \rho^t)\Omega_{u_2} a', \Omega_{u_1} a \rangle \\
& - \langle \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2), a' \rangle \\
& - X_1\langle \pounds_{a'} \alpha_2, a \rangle + \langle [\![u_1, \pounds_{a'} u_2]\!]_\Delta, a \rangle - X_2\rho(a')\langle \alpha_1, a \rangle \\
& + X_2\langle \pounds_{a'} \alpha_1, a \rangle - \langle [\![u_2, \pounds_{a'} u_1]\!]_\Delta, a \rangle + X_1\rho(a')\langle \alpha_2, a \rangle \\
={}& \langle [a, a'], [\![u_1, u_2]\!]_\Delta \rangle - \langle a, \pounds_{a'} [\![u_1, u_2]\!]_\Delta - [\![u_1, \pounds_{a'} u_2]\!]_\Delta + [\![u_2, \pounds_{a'} u_1]\!]_\Delta \rangle \\
& + \langle \pounds_a [\![u_1, u_2]\!]_\Delta - [\![\pounds_a u_1, u_2]\!]_\Delta - [\![u_1, \pounds_a u_2]\!]_\Delta, a' \rangle \\
& + \langle (\rho, \rho^t)\Omega_{u_1} a', \Omega_{u_2} a \rangle - \langle (\rho, \rho^t)\Omega_{u_2} a', \Omega_{u_1} a \rangle \\
& - \langle \mathrm{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a, u_2), a' \rangle - X_1\langle \alpha_2, [a, a'] \rangle + X_2\langle \alpha_1, [a, a'] \rangle.
\end{aligned}
$$

We have

$$
-\langle a, [\![u_2, \pounds_{a'} u_1]\!]_\Delta \rangle = \langle a, [\![\pounds_{a'} u_1, u_2]\!]_\Delta \rangle - \langle \mathrm{Skew}_\Delta(\pounds_{a'} u_1, u_2), a \rangle
$$

and since $\nabla^{\mathrm{bas}}_{a'} u_1 \in \Gamma(U)$ by hypothesis, we find

$$
\begin{aligned}
-\operatorname{Skew}_\Delta(\pounds_{a'} u_1, u_2) &= -\operatorname{Skew}_\Delta(\pounds_{a'} u_1 - \nabla^{\mathrm{bas}}_{a'} u_1, u_2) \\
&= \operatorname{Skew}_\Delta((\rho, \rho^t)\Omega_{u_1} a', u_2).
\end{aligned}
$$

This shows that $\langle \tau', a' \rangle$ is equal to the right-hand side of (A.9) (with $a = a_1$, $a' = a_2$), which vanishes.

For the second equation we write $u = (X, \alpha)$ and $\tau_i = (a_i, \theta_i)$ for $i = 1, 2$:

$$
\begin{aligned}
&\Delta_u[\tau_1, \tau_2]_d - [\Delta_u \tau_1, \tau_2]_d - [\tau_1, \Delta_u \tau_2]_d \\
&\quad + \Delta_{\nabla^{\mathrm{bas}}_{a_1} u} \tau_2 - \Delta_{\nabla^{\mathrm{bas}}_{a_2} u} \tau_1 + (0, \mathbf{d}\langle \tau_1, \nabla^{\mathrm{bas}}_{a_2} u \rangle) \\
&= \Omega_u[a_1, a_2] + (0, \pounds_X(\pounds_{\rho(a_1)} \theta_2 - \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1) + \mathbf{d}\langle \alpha, [a_1, a_2]\rangle) \\
&\quad - [\Omega_u a_1 + (0, \pounds_X \theta_1 + \mathbf{d}\langle \alpha, a_1 \rangle), \tau_2]_d \\
&\quad - [\tau_1, \Omega_u a_2 + (0, \pounds_X \theta_2 + \mathbf{d}\langle \alpha, a_2 \rangle)]_d \\
&\quad + \Omega_{\nabla^{\mathrm{bas}}_{a_1} u} a_2 + (0, \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_1} u)} \theta_2 + \mathbf{d}\langle \nabla^{\mathrm{bas}}_{a_1} u, (a_2, 0)\rangle) \\
&\quad - \Omega_{\nabla^{\mathrm{bas}}_{a_2} u} a_1 - (0, \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_2} u)} \theta_1 + \mathbf{d}\langle \nabla^{\mathrm{bas}}_{a_2} u, (a_1, 0)\rangle) + (0, \mathbf{d}\langle \tau_1, \nabla^{\mathrm{bas}}_{a_2} u \rangle) \\
&= -R^{\mathrm{bas}}_\Delta(a_1, a_2) u + (0, \pounds_X(\pounds_{\rho(a_1)} \theta_2 - \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1) + \mathbf{d}\langle \alpha, [a_1, a_2]\rangle) \\
&\quad + (0, \mathbf{i}_{\rho(a_2)} \mathbf{d}\pounds_X \theta_1 - \pounds_{\rho \circ \mathrm{pr}_A \Omega_u a_1} \theta_2 - \mathbf{d}\langle (\rho, \rho^t)\Omega_u a_1, (a_2, 0)\rangle) \\
&\quad - (0, \pounds_{\rho(a_1)} \pounds_X \theta_2 + \pounds_{\rho(a_1)} \mathbf{d}\langle \alpha, a_2 \rangle + \mathbf{i}_{\rho \circ \mathrm{pr}_A \Omega_u a_2} \mathbf{d}\theta_1) \\
&\quad + (0, \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_1} u)} \theta_2 + \mathbf{d}\langle \nabla^{\mathrm{bas}}_{a_1} u, (a_2, 0)\rangle) \\
&\quad - (0, \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_2} u)} \theta_1 + \mathbf{d}\langle \nabla^{\mathrm{bas}}_{a_2} u, (a_1, 0)\rangle) + (0, \mathbf{d}\langle \tau_1, \nabla^{\mathrm{bas}}_{a_2} u \rangle).
\end{aligned}
$$

But we have

$$
\begin{aligned}
&\pounds_X \pounds_{\rho(a_1)} \theta_2 - \pounds_{\rho \circ \mathrm{pr}_A \Omega_u a_1} \theta_2 - \pounds_{\rho(a_1)} \pounds_X \theta_2 + \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_1} u)} \theta_2 \\
&= \pounds_{[X, \rho(a_1)] - \rho \circ \mathrm{pr}_A \Omega_u a_1 + \mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_1} u)} \theta_2 = 0
\end{aligned}
$$

since $\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_1} u) = [\rho(a_1), X] + \rho \circ \mathrm{pr}_A \Omega_u a_1$, and in the same manner

$$
\begin{aligned}
&- \pounds_X \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1 + \mathbf{i}_{\rho(a_2)} \mathbf{d}\pounds_X \theta_1 + \mathbf{i}_{\rho \circ \mathrm{pr}_A \Omega_u a_2} \mathbf{d}\theta_1 - \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_2} u)} \theta_1 \\
&\quad + \mathbf{d}\langle \nabla^{\mathrm{bas}}_{a_2} u, (a_1, 0)\rangle - \mathbf{d}\langle \tau_1, \nabla^{\mathrm{bas}}_{a_2} u \rangle \\
&= \mathbf{i}_{[\rho(a_2), X] + \rho \circ \mathrm{pr}_A \Omega_u a_2} \mathbf{d}\theta_1 - \pounds_{\mathrm{pr}_{TM}(\nabla^{\mathrm{bas}}_{a_2} u)} \theta_1 - \mathbf{d}\langle (0, \theta_1), \nabla^{\mathrm{bas}}_{a_2} u \rangle = 0.
\end{aligned}
$$

Since

$$
-\langle (\rho, \rho^t)\Omega_u a_1, (a_2, 0)\rangle + \langle \nabla^{\mathrm{bas}}_{a_1} u, (a_2, 0)\rangle = \langle \pounds_{a_1} u, (a_2, 0)\rangle = \langle \pounds_{a_1} \alpha, a_2 \rangle
$$

and

$$\pounds_{\rho(a_1)}\mathbf{d}\langle\alpha,a_2\rangle = \mathbf{d}(\rho(a_1)\langle\alpha,a_2\rangle) = \mathbf{d}(\langle\pounds_{a_1}\alpha,a_2\rangle + \langle\alpha,[a_1,a_2]\rangle),$$

the remaining sum

$$\mathbf{d}\langle\alpha,[a_1,a_2]\rangle - \mathbf{d}\langle(\rho,\rho^t)\Omega_u a_1,(a_2,0)\rangle - \pounds_{\rho(a_1)}\mathbf{d}\langle\alpha,a_2\rangle + \mathbf{d}\langle\nabla^{\mathrm{bas}}_{a_1}u,(a_2,0)\rangle$$

vanishes as well.   □

References

[1] C. Arias Abad and M. Crainic, *Representations up to homotopy of Lie algebroids*, J. Reine Angew. Math. **663** (2012), 91–126.

[2] H. Bursztyn, *A brief introduction to Dirac manifolds*, Geometric and Topological Methods for Quantum Field Theory, Cambridge Univ. Press, Cambridge, 2013, pp. 4–38.

[3] H. Bursztyn, A. Cabrera, and C. Ortiz, *Linear and multiplicative 2-forms*, Lett. Math. Phys. **90** (2009), no. 1-3, 59–83.

[4] H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu, *Integration of twisted Dirac brackets*, Duke Math. J. **123** (2004), no. 3, 549–607.

[5] H. Bursztyn, D. Iglesias Ponte, and P. Ševera, *Courant morphisms and moment maps*, Math. Res. Lett. **16** (2009), no. 2, 215–232.

[6] A. Coste, P. Dazord, and A. Weinstein, *Groupoïdes symplectiques*, Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, Publ. Dép. Math. Nouvelle Sér. A, Vol. 87, Univ. Claude-Bernard, Lyon, 1987, pp. i–ii, 1–62.

[7] T. J. Courant and A. Weinstein, *Beyond Poisson structures*, Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), Travaux en Cours, Vol. 27, Hermann, Paris, 1988, pp. 39–49.

[8] M. Crainic and R. L. Fernandes, *Integrability of Lie brackets*, Ann. of Math. (2) **157** (2003), no. 2, 575–620.

[9] M. Crainic, M. A. Salazar, and I. Struchiner, *Multiplicative forms and Spencer operators*, Math. Z. **279** (2015), no. 3-4, 939–979.

[10] P. A. M. Dirac, *Lectures on quantum mechanics*, Belfer Graduate School of Science Monographs Series, Vol. 2, Belfer Graduate School of Science, New York, 1967, Second printing of the 1964 original.

[11] I. Ya. Dorfman, *Dirac structures of integrable evolution equations*, Phys. Lett. A **125** (1987), no. 5, 240–246.

[12] T. Drummond, M. Jotz, and C. Ortiz, *VB-algebroid morphisms and representations up to homotopy*, Differential Geometry and its Applications **40** (2015), 332–357.

[13] R. L. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. **170** (2002), no. 1, 119–179.

[14] J. Grabowski and M. Rotkiewicz, *Higher vector bundles and multi-graded symplectic manifolds*, J. Geom. Phys. **59** (2009), no. 9, 1285–1305.

[15] A. Gracia-Saz, M. Jotz Lean, K. Mackenzie, and R. Mehta, *Double Lie algebroids and representations up to homotopy*, J. Homotopy Relat. Struct. **13** (2018), no. 2, 287–319.

[16] A. Gracia-Saz and R. A. Mehta, *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math. **223** (2010), no. 4, 1236–1275.

[17] M. Grützmann and M. Stiénon, *Matched pairs of Courant algebroids*, Indag. Math. (N.S.) **25** (2014), no. 5, 977–991.

[18] E. Hawkins, *A groupoid approach to quantization*, J. Symplectic Geom. **6** (2008), no. 1, 61–125.

[19] M. Jotz, *Dirac Lie groups, Dirac homogeneous spaces and the theorem of Drinfeld*, Indiana Univ. Math. J. **60** (2011), 319–366.

[20] M. Jotz Lean, *Dorfman connections and Courant algebroids*, J. Math. Pures Appl. **116** (2018), 1–39.

[21] M. Jotz Lean, *Homogeneous spaces of Dirac groupoids*, J. Geom. Phys. **104** (2016), 89–111.

[22] M. Jotz Lean, *On LA-Courant algebroids and Poisson Lie 2-algebroids*, `arXiv:1811.04842`, (2018).

[23] M. Jotz Lean and C. Ortiz, *Foliated groupoids and infinitesimal ideal systems*, Indag. Math. (N.S.) **25** (2014), no. 5, 1019–1053.

[24] M. Jotz Lean, M. Stiénon, and P. Xu, *Glanon groupoids*, Math. Ann. **364** (2016), no. 1-2, 485–518.

[25] D. Li-Bland, *Phd thesis: LA-Courant algebroids and their applications*, `arXiv:1204.2796`, (2012).

[26] R. G. Littlejohn, *Hamiltonian theory of guiding center motion*, Ph.D. thesis, University of California at Berkeley, 1980.

[27] Z.-J. Liu, A. Weinstein, and P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), no. 3, 547–574.

[28] Z.-J. Liu, A. Weinstein, and P. Xu, *Dirac structures and Poisson homogeneous spaces*, Comm. Math. Phys. **192** (1998), no. 1, 121–144.

[29] J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526.

[30] K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry. I*, Adv. Math. **94** (1992), no. 2, 180–239.

[31] K. C. H. Mackenzie, *Double Lie algebroids and iterated tangent bundles*, `arXiv:math/9808081`, (1998).

[32] K. C. H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, London Mathematical Society Lecture Note Series, Vol. 213, Cambridge University Press, Cambridge, 2005.

[33] K. C. H. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids*, J. Reine Angew. Math. **658** (2011), 193–245.

[34] K. C. H. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. **73** (1994), no. 2, 415–452.

[35] K. C. H. Mackenzie and P. Xu, *Integration of Lie bialgebroids*, Topology **39** (2000), no. 3, 445–467.

[36] R. A. Mehta and X. Tang, *From double Lie groupoids to local Lie 2-groupoids*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 4, 651–681.

[37] C. Ortiz, *Multiplicative Dirac structures on Lie groups.*, C. R., Math., Acad. Sci. Paris **346** (2008), no. 23-24, 1279–1282.

[38] C. Ortiz, *Multiplicative Dirac structures*, Pacific J. Math. **266** (2013), no. 2, 329–365.

[39] J. Pradines, *Fibrés vectoriels doubles et calcul des jets non holonomes*, Esquisses Mathématiques [Mathematical Sketches], Vol. 29, Université d'Amiens U.E.R. de Mathématiques, Amiens, 1977.

[40] J. Pradines, *Remarque sur le groupoïde cotangent de Weinstein-Dazord*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 13, 557–560.

[41] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, ProQuest LLC, Ann Arbor, MI, 1999, Thesis (Ph.D.)–University of California, Berkeley.

[42] K. Uchino, *Remarks on the definition of a Courant algebroid*, Lett. Math. Phys. **60** (2002), no. 2, 171–175 (English).

[43] A. Weinstein, *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan **40** (1988), no. 4, 705–727.

[44] P. Xu, *Momentum maps and Morita equivalence*, J. Differential Geom. **67** (2004), no. 2, 289–333.

MATHEMATISCHES INSTITUT, GEORG-AUGUST UNIVERSITÄT GÖTTINGEN
BUNSENSTRASSE 3-5, 37073 GÖTTINGEN, GERMANY
E-mail address: `madeleine.jotz-lean@mathematik.uni-goettingen.de`