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New examples of compact special Lagrangian submanifolds embedded in hyper-Kähler manifolds

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We construct smooth families of compact special Lagrangian submanifolds embedded in some toric hyper-Kähler manifolds, which never become holomorphic Lagrangian submanifolds via any hyper-Kähler rotations. These families converge to special Lagrangian immersions with self-intersection points in the sense of currents. To construct them, we apply the desingularization method developed by Joyce.

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1. Introduction

In 1982, Harvey and Lawson have introduced in [5] the notion of calibrated submanifolds in Riemannian manifold. The calibrated submanifolds are special classes of minimal submanifolds, and they had already been well-studied by many researchers. One of the importances of calibrated submanifolds is the volume minimizing property, that is, every compact calibrated submanifold minimizes the volume functional in its homology class.

The several kinds of calibrated submanifolds are defined in the Riemannian manifolds with special holonomy. For example, special Lagrangian submanifolds are middle dimensional calibrated submanifolds embedded in Riemannian manifolds with $SU(n)$ holonomy, so called Calabi-Yau manifolds. In hyper-Kähler manifolds, which are Riemannian manifolds with $Sp(n)$ holonomy, there is a notion of holomorphic Lagrangian submanifolds those are calibrated by the n-th power of the Kähler form. At the same time, hyper-Kähler manifolds are naturally regarded as Calabi-Yau manifolds, special Lagrangian submanifolds also make sense in these manifolds. Hence there are two kinds of calibrated submanifolds in hyper-Kähler manifolds, and it is well-known that every holomorphic Lagrangian submanifold becomes special Lagrangian by the hyper-Kähler rotations. The converse may not hold although compact counterexamples have not been found.

Another importance of calibrated geometry is that some of the calibrated submanifolds have the moduli spaces with good structure. For instance, McLean has shown that the moduli space of compact special Lagrangian submanifolds becomes a smooth manifold, whose dimension is equal to the first betti number of the special Lagrangian submanifold [13].

Although the construction of compact special Lagrangian submanifolds embedded in Calabi-Yau manifolds is not easy in general, Y-I. Lee [12], Joyce [8][9] and D. A. Lee [11] developed the gluing method for the construction of families of compact special Lagrangian submanifolds converging to special Lagrangian immersions with self-intersection points in the sense of currents. Moreover D. A. Lee construct a non-totally geodesic special Lagrangian submanifold in the flat torus by applying his gluing method. After these works, several concrete examples of special Lagrangian submanifolds are constructed by gluing method. See [6][3][4], for example.

In this paper we apply the result in [8][9] to the construction of new examples of compact special Lagrangian submanifolds embedded in toric hyper-Kähler manifolds. Moreover, these examples never become holomorphic Lagrangian submanifolds with respect to any complex structures given by the hyper-Kähler rotations.

A hyper-Kähler manifold is a Riemannian manifold (M^{4n}, g) equipped with an integrable hypercomplex structure (I_1, I_2, I_3), so that g is hermitian with respect to every I_α, and $\omega_\alpha := g(I_\alpha \cdot, \cdot)$ are closed. For any $\theta \in \mathbb{R}$, note that $e^{\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)$ becomes a holomorphic symplectic 2-form with respect to I_1. If the holomorphic symplectic form vanishes on a submanifold $L^{2n} \subset M$, L is called a holomorphic Lagrangian submanifold. Clearly, this definition does not depend on θ.

Similarly, we can define the notion of holomorphic Lagrangian submanifold with respect to a complex structure $aI_1 + bI_2 + cI_3$ for every unit vector (a, b, c) in $\mathbb{R}^3$. The new complex structure $aI_1 + bI_2 + cI_3$ is called a hyper-Kähler rotation of (M, g, I_1, I_2, I_3).

The hyper-Kähler manifold M is naturally regarded as the Calabi-Yau manifold by the complex structure I_1, the Kähler form ω_1 and the holomorphic volume form $(\omega_2 + \sqrt{-1}\omega_3)^n$. Then we can easily see that holomorphic Lagrangian submanifolds with respect to $\cos(\alpha\pi/n)I_2 + \sin(\alpha\pi/n)I_3$ are special Lagrangian for every $\alpha = 1, \ldots, 2n$. Conversely, it has been unknown whether there exist special Lagrangian submanifolds embedded in hyper-Kähler manifolds never coming from holomorphic Lagrangian submanifolds with respect to any complex structure given by the hyper-Kähler rotations. The main result of this paper is described as follows.

Theorem 1.1. *Let $n \geq 2$. There exist smooth compact special Lagrangian submanifolds $\{\tilde{L}_t\}_{0<t<\delta}$ and $\{L_\alpha\}_{\alpha=1,\ldots,2n}$ embedded in a hyper-Kähler manifold M^{4n}, which satisfy $\lim_{t\to 0} \tilde{L}_t = \bigcup_\alpha L_\alpha$ in the sense of currents, and $\tilde{L}_t$ is diffeomorphic to $2n(\mathbb{P}^1)^n \# (S^1 \times S^{2n-1})$. Moreover, each L_α is the holomorphic Lagrangian submanifold of M with respect to $\cos(\alpha\pi/n)I_2 + \sin(\alpha\pi/n)I_3$, although $\tilde{L}_t$ never become holomorphic Lagrangian submanifolds with respect to any complex structure given by the hyper-Kähler rotations whichever we choose the orientation of $\tilde{L}_t$.*

This is one of examples which we obtain in this article. Furthermore, we obtain special Lagrangian $2\mathbb{P}^2 \# 2\overline{\mathbb{P}^2} \# (S^1 \times S^3)$ embedded in a hyper-Kähler manifold of dimension 8 and special Lagrangian $(3N + 1)(\mathbb{P}^1)^2 \# N(S^1 \times S^3)$ embedded in another 8-dimensional hyper-Kähler manifold, both of which never become holomorphic Lagrangian submanifolds with respect to any complex structure given by the hyper-Kähler rotations.

Theorem 1.1 has another significance from the point of the view of the compactification of the moduli spaces of compact special Lagrangian submanifolds. In general, the moduli space $\mathcal{M}(L)$ of the deformations of compact special Lagrangian submanifolds $L \subset M$ is not necessarily compact, consequently the study of its compactification is important problem. It is known that a compactification of $\mathcal{M}(L)$ is given by the geometric measure theory. The special Lagrangian immersion $\bigcup_\alpha L_\alpha$ appeared in Theorem 1.1 is the concrete example of an element of $\overline{\mathcal{M}(\tilde{L}_t)}\backslash\mathcal{M}(\tilde{L}_t)$. D. A. Lee also considered the similar situation, however the Calabi-Yau structures of ambient space of $\tilde{L}_t$ is deformed by the parameter t in [11].

Here, we describe the outline of the proof. Let (M, J, ω, Ω) be a Kähler manifold of complex dimension $m \geq 3$ with holomorphic volume form $\Omega \in H^0(K_M)$, and $L_\alpha \subset M$ be connected special Lagrangian submanifolds, where $\alpha = 1, \ldots, A$. Put $\mathcal{V} = \{1, \ldots, A\}$, and suppose we have a quiver $(\mathcal{V}, \mathcal{E}, s, t)$, namely, $\mathcal{V}$ consists of finite vertices, $\mathcal{E}$ consists of finite directed edges, and s, t are maps $\mathcal{E} \to \mathcal{V}$ so that $s(h)$ is the source of $h \in \mathcal{E}$ and $t(h)$ is the target.

A subset $S \subset \mathcal{E}$ is called a cycle if it is written as $S = \{h_1, h_2, \ldots, h_l\}$ and $t(h_k) = s(h_{k+1})$, $t(h_l) = s(h_1)$ hold for all $k = 1, \ldots, l-1$. Then $\mathcal{E}$ is said to be *covered by cycles* if every edge $h \in \mathcal{E}$ is contained in some cycles of $\mathcal{E}$.

If there are two special Lagrangian submanifolds $L_0, L_1 \subset M$ intersecting transversely at $p \in L_0 \cap L_1$, then we can define a type at the intersection point p, which is a positive integer less than m. Then we have the next result, which follows from Theorem 9.7 of [8] by some additional arguments.

Theorem 1.2. *Let $(\mathcal{V}, \mathcal{E}, s, t)$ be a quiver, and L_α be connected compact special Lagrangian submanifolds embedded in a Calabi-Yau manifold M of dimension $m \geq 3$ for every $\alpha \in \mathcal{V}$. Assume that $L_{s(h)}$ and $L_{t(h)}$ intersects transversely at only one point p if $h \in \mathcal{E}$, and p is the intersection point of type 1, and $L_\alpha \cap L_\beta$ is empty if $\alpha \neq \beta$ and there are no edges connecting α and β. Then, if $\mathcal{E}$ is covered by cycles, there exist $\delta > 0$ and a family of compact special Lagrangian submanifolds $\{\tilde{L}_{t_1,\ldots,t_N}\}_{0<t_1,\ldots,t_N<\delta}$ embedded in M which satisfies $\lim_{t_1,\ldots,t_N \to 0} \tilde{L}_{t_1,\ldots,t_N} = \bigcup_{\alpha\in\mathcal{V}} L_\alpha$ in the sense of currents. Here, N is the first betti number of $(\mathcal{V}, \mathcal{E}, s, t)$.*

To obtain Theorem 1.1, we apply Theorem 1.2 to the case that M is a toric hyper-Kähler manifold and L_α is a holomorphic Lagrangian submanifold with respect to $\cos(\alpha\pi/n)I_2 + \sin(\alpha\pi/n)I_3$. Accordingly, the proof is reduced to looking for toric hyper-Kähler manifolds M and their holomorphic Lagrangian submanifolds $L_1, \ldots, L_{2n}$ satisfying the assumption of Theorem 1.2. In particular, to find L_α's so that $\mathcal{E}$ is covered by cycles is not so easy. The author cannot develop the systematic way to find such examples in toric hyper-Kähler manifolds, however, we can raise some concrete examples in this article.

In toric hyper-Kähler manifolds, many holomorphic Lagrangian submanifolds are obtained as the inverse image of some special polytopes by the hyper-Kähler moment maps, where the polytopes are naturally given by the hyperplane arrangements which determine the toric hyper-Kähler manifolds. We can compute the type at the intersection point of two holomorphic Lagrangian submanifolds, if the intersection point is the fixed point of the torus action. Finally, we can find examples of toric hyper-Kähler manifolds and such polytopes, which satisfy the assumption Theorem 1.2.

Next we have to show that these examples of special Lagrangian submanifolds never become holomorphic Lagrangian submanifolds. Since $\tilde{L}_t$ is contained in the homology class $\sum_\alpha (-1)^\alpha [L_\alpha]$, we obtain the volume of $\tilde{L}_t$ by integrating the real part of the holomorphic volume form over $\sum_\alpha (-1)^\alpha [L_\alpha]$. On the other hand, if $\tilde{L}_t$ is holomorphic Lagrangian submanifold with respect to some $aI_1 + bI_2 + cI_3$, then the volume can be also computed by integrating $(a\omega_1 + b\omega_2 + c\omega_3)^n$ over $\sum_\alpha (-1)^\alpha [L_\alpha]$, since $a\omega_1 + b\omega_2 + c\omega_3$ should be the Kähler form on $\tilde{L}_t$. These two values of the volume do not coincide, we have a contradiction. At the same time, we have another simpler proof if the first betti number $\tilde{L}_t$ is odd, since any holomorphic Lagrangian submanifolds become Kähler manifolds which always have even first betti number. The example constructed in Theorem 1.1 satisfies $b_1 = 1$, hence we can use this proof. However, we have other examples in Section 6 whose first betti number may be even.

This article is organized as follows. First of all we define σ-holomorphic Lagrangian submanifolds in Section 2 and review the constructions of them in toric hyper-Kähler manifolds in Section 3. Next we review the definition of the type at the intersection point of two special Lagrangian submanifolds, and then compute them in the case of toric hyper-Kähler manifolds in Section 4. In Section 5, we prove Theorem 1.2 by using Theorem 9.7 of [8]. In Section 6, we find toric hyper-Kähler manifolds and their holomorphic Lagrangian submanifolds which satisfy the assumption of Theorem 1.2, and obtain compact special Lagrangian submanifolds embedded in some toric hyper-Kähler manifolds. In Section 7, we show the examples obtained in Section 6 never become σ-holomorphic Lagrangian submanifolds for any $\sigma \in S^2$.

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2. Holomorphic Lagrangian submanifolds

Definition 2.1. A Riemannian manifold (M, g) equipped with integrable complex structures (I_1, I_2, I_3) is a *hyper-Kähler manifold* if each I_α is orthogonal with respect to g, they satisfy the quaternionic relation $I_1 I_2 I_3 = -1$ and fundamental 2-forms $\omega_\alpha := g(I_\alpha \cdot, \cdot)$ are closed.

We put $\omega = (\omega_1, \omega_2, \omega_3)$ and call it the hyper-Kähler structure. For each

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^2 = \{(a, b, c) \in \mathbb{R}^3;\ a^2 + b^2 + c^2 = 1\},$$

we have another Kähler structure

$$(M, I^\sigma, \omega^\sigma) := \left(M, \sum_{i=1}^{3} \sigma_i I_i, \sum_{i=1}^{3} \sigma_i \omega_i \right).$$

Take $\sigma', \sigma'' \in S^2$ so that $(\sigma, \sigma', \sigma'')$ forms an orthonormal basis in $\mathbb{R}^3$. Suppose it has the positive orientation, that is,

$$\sigma \wedge \sigma' \wedge \sigma'' = (1, 0, 0) \wedge (0, 1, 0) \wedge (0, 0, 1)$$

holds. Then we have another hyper-Kähler structure $(\omega^\sigma, \omega^{\sigma'}, \omega^{\sigma''})$ called the hyper-Kähler rotation of ω.

Definition 2.2. Let (M, g, I_1, I_2, I_3) be a hyper-Kähler manifold of real dimension $4n$, and $L \subset M$ be a $2n$-dimensional oriented submanifold. Fix $\sigma \in S^2$ arbitrarily. Then L is a *σ-holomorphic Lagrangian submanifold* if $\omega^{\sigma'}|_L = \omega^{\sigma''}|_L = 0$ and the orientation of L is given by $(\omega^\sigma)^n|_L$.

It is easy to see that the above definition does not depend on the choice of σ', σ''.

Any hyper-Kähler manifolds can be regarded as Calabi-Yau manifolds by considering the pair of a Kähler manifold (M, I_1, ω_1) and a holomorphic volume form $(\omega_2 + \sqrt{-1}\omega_3)^n \in H^0(M, K_M)$, where K_M is the canonical line bundle of the complex manifold (M, I_1). Therefore, we can consider the notion of special Lagrangian submanifolds in M as follows.

Definition 2.3. Let (M, g, I_1, I_2, I_3) be a hyper-Kähler manifold of real dimension $4n$, and $L \subset M$ be a $2n$-dimensional oriented submanifold. Then L is a *special Lagrangian submanifold* if $\omega_1|_L = \mathrm{Im}(\omega_2 + \sqrt{-1}\omega_3)^n|_L = 0$ holds and the orientation of L is given by $\mathrm{Re}(\omega_2 + \sqrt{-1}\omega_3)^n|_L$.

Remark 2.4. For $\theta \in \mathbb{R}$, $L \subset M$ is often called a *special Lagrangian submanifold of phase* $e^{\sqrt{-1}\theta}$ if $\omega_1|_L = \mathrm{Im}\{e^{-\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)^n\}|_L = 0$ and the orientation is given by $\mathrm{Re}\{e^{-\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)^n\}|_L$. In this paper we only consider the special Lagrangian submanifolds of phase 1.

Example 1. Let (M, g, I_1, I_2, I_3) be a hyper-Kähler manifold and suppose a compact Lie group K acts on M preserving g, I_1, I_2, I_3, and there exists a hyper-Kähler moment map $\mu_K : M \to \mathrm{Im}\mathbb{H} \otimes \mathbf{k}^*$, where $\mathbf{k}$ is the Lie algebra of K. For $\zeta \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{k}^*)^K$, suppose that K acts on $\mu_K^{-1}(\zeta)$ freely, where $(\mathbf{k}^*)^K \subset \mathbf{k}^*$ is the subset of fixed points under the coadjoint action. Then by [7], the quotient space $\mu_K^{-1}(\zeta)/K$ inherits the natural hyper-Kähler structure from g, I_1, I_2, I_3 and becomes a smooth hyper-Kähler manifold which is called a hyper-Kähler quotient.

Next we assume that a σ-holomorphic Lagrangian submanifold $\hat{L} \subset M$ which is closed under the K-action is given. We put $\mu_K = (\mu_{K,1}, \mu_{K,2}, \mu_{K,3})$ and $\mu_K^\sigma := \sigma_1\mu_{K,1} + \sigma_2\mu_{K,2} + \sigma_3\mu_{K,3}$ for $\sigma \in S^2$. We define $\zeta^\sigma \in (\mathbf{k}^*)^K$ similarly. If $\sigma, \sigma', \sigma''$ is an orthonormal basis, then $\mu_K^{\sigma'}$ and $\mu_K^{\sigma''}$ is locally constant on $\hat{L}$. Here, we assume

$$\mu_K^{\sigma'}|_{\hat{L}} \equiv \zeta^{\sigma'}, \quad \mu_K^{\sigma''}|_{\hat{L}} \equiv \zeta^{\sigma''},$$

then

$$\mu_K^{-1}(\zeta) \cap \hat{L} = (\mu_K^\sigma)^{-1}(\zeta^\sigma) \cap \hat{L}$$

holds. Now $(\hat{L}, I^\sigma, \omega^\sigma)$ is a Kähler manifold and $\mu_K^\sigma|_{\hat{L}} : \hat{L} \to \mathbf{k}^*$ is a Kähler moment map. Since we have supposed that K acts on $\mu_K^{-1}(\zeta)$ freely, then $(\mu_K^\sigma)^{-1}(\zeta^\sigma) \cap \hat{L}$ is a smooth submanifold of $\hat{L}$, hence $\mu_K^{-1}(\zeta) \cap \hat{L}$ is a smooth submanifold of $\mu_K^{-1}(\zeta)$. By taking quotients, we obtain a smooth submanifold

$$L := (\mu_K^{-1}(\zeta) \cap \hat{L})/K \subset \mu_K^{-1}(\zeta)/K.$$

It is easy to check that L is a σ-holomorphic Lagrangian submanifold of $\mu_K^{-1}(\zeta)/K$.

Example 2. Let $\mu_1(x) = xi\overline{x} \in \mathrm{Im}\mathbb{H}$ for $x \in \mathbb{H}$, and $\sigma, \sigma', \sigma'' \in S^2$ is an orthonormal basis. Then each level set of $(\mu_1^{\sigma'}, \mu_1^{\sigma''})$ is a σ-holomorphic Lagrangian submanifold of $\mathbb{H}$ if it is smooth. Let

$$L(\sigma, q, \delta) := \{x \in \mathbb{H};\ \mu_1(x) = q + t\sigma,\ -\delta < t < \delta\} \tag{1}$$

for $q \in \mathrm{Im}\mathbb{H}$ and $\delta > 0$. Since the only critical point of $(\mu_1^{\sigma'}, \mu_1^{\sigma''})$ is the origin of $\mathbb{H}$, therefore $L(\sigma, q, \delta)$ is a smooth holomorphic Lagrangian submanifold of $\mathbb{H}$ if $q \neq 0$ and δ is sufficiently small.

3. Toric hyper-Kähler manifolds

3.1. Construction

In this subsection we review the construction of toric hyper-Kähler manifolds briefly. Let $u_{\mathbb{Z}} : \mathbb{Z}^d \to \mathbb{Z}^n$ be a surjective $\mathbb{Z}$ linear map which induces a homomorphisms between tori and their Lie algebras, denoted by $\hat{u} : T^d \to T^n$ and $u : \mathbf{t}^d \to \mathbf{t}^n$, respectively. Throughout of this article we identify the Lie algebra $\mathbf{t}^d$ of the torus with $\mathbb{R}^d$. We put $K := \mathrm{Ker}\ \hat{u} \in T^d$ and $\mathbf{k} := \mathrm{Ker}\ u \in \mathbf{t}^d$, where $\mathbf{k}$ is the Lie algebra of the subtorus K. The adjoint map of u is denoted by $u^* : (\mathbf{t}^n)^* \to (\mathbf{t}^d)^*$ and it induces $u^* : V \otimes (\mathbf{t}^n)^* \to V \otimes (\mathbf{t}^d)^*$ naturally for any vector space V, which is also denoted by the same symbol.

Next we consider the action of T^d on the quaternionic vector space $\mathbb{H}^d$ given by $(x_1, \dots, x_d) \cdot (g_1, \dots, g_d) := (x_1 g_1, \dots, x_d g_d)$ for $x_k \in \mathbb{H}$ and $g_k \in S^1$. Then this action preserves the standard hyper-Kähler structure on $\mathbb{H}^d$, and the hyper-Kähler moment map $\mu_d : \mathbb{H}^d \to \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^d)^*$ is given by

$$\mu_d(x_1, \dots, x_d) = (x_1 i\overline{x_1}, \dots, x_d i\overline{x_d}).$$

Here, $\mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$ is the pure imaginary part of $\mathbb{H}$.

Let $\hat{\iota} : K \to T^d$ and $\iota : \mathbf{k} \to \mathbf{t}^d$ be the inclusion maps and put $\mu_K := \iota^* \circ \mu_d : \mathbb{H}^d \to \mathrm{Im}\mathbb{H} \otimes \mathbf{k}^*$ be the hyper-Kähler moment map with respect to K-action on $\mathbb{H}^d$. Then we obtain the hyper-Kähler quotient

$$X(u, \lambda) := \mu_K^{-1}(\iota^*(\lambda))/K$$

for every $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^d)^*$, called toric hyper-Kähler varieties. The complex structures on $X(u, \lambda)$ are denoted by $I_{\lambda,1}, I_{\lambda,2}$ and $I_{\lambda,3}$, and the corresponding Kähler forms are denoted by

$$\omega_\lambda = (\omega_{\lambda,1}, \omega_{\lambda,2}, \omega_{\lambda,3}).$$

Although $X(u,\lambda)$ is not necessarily a smooth manifold, the equivalent condition for the smoothness was obtained by Bielawski-Dancer in [1]. Let $e_1, \dots, e_d \in \mathbb{R}^d$ be the standard basis and $u_k := u(e_k) \in \mathbf{t}^n$. Put

$$H_k = H_k(\lambda) := \{y \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*;\ \langle y, u_k\rangle + \lambda_k = 0\},$$

where

$$\langle y, u_k\rangle = (\langle y_1, u_k\rangle,\ \langle y_2, u_k\rangle,\ \langle y_3, u_k\rangle) \in \mathbb{R}^3 = \mathrm{Im}\mathbb{H}$$

for $y = (y_1, y_2, y_3)$.

Theorem 3.1 ([1]). *The hyper-Kähler quotient $X(u,\lambda)$ is a smooth manifold if and only if both of the following conditions $(*1)(*2)$ are satisfied. $(*1)$ For any $\tau \subset \{1, 2, \dots, d\}$ with $\#\tau = n+1$, the intersection $\bigcap_{k\in\tau} H_k$ is empty. $(*2)$ For every $\tau \subset \{1, 2, \dots, d\}$ with $\#\tau = n$, the intersection $\bigcap_{k\in\tau} H_k$ is nonempty if and only if $\{u_k;\ k \in \tau\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

The T^d action on $\mathbb{H}^d$ induces a $T^n = T^d/K$ action on $X(u,\lambda)$ preserving the hyper-Kähler structure of $X(u,\lambda)$, and the hyper-Kähler moment map $\mu_\lambda = (\mu_{\lambda,1},\ \mu_{\lambda,2},\ \mu_{\lambda,3}) : X(u,\lambda) \to \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$ is defined by

$$u^*(\mu_\lambda([x])) := \mu_d(x) - \lambda,$$

where $[x] \in X(u,\lambda)$ is the equivalence class represented by $x \in \mu_K^{-1}(\iota^*(\lambda))$.

Let $\sigma \in S^2$. A T^n-invariant submanifold $L \subset X(u,\lambda)$ is a σ-holomorphic Lagrangian submanifold if $\mu_\lambda(L)$ is contained in $q + \sigma \otimes (\mathbf{t}^n)^*$ for some $q \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$.

3.2. Local model of the neighborhood of a fixed point

Let $X = X(u,\lambda)$ be a smooth toric hyper-Kähler manifold of real dimension $4n$, $\omega = \omega_\lambda$ and $\mu = \mu_\lambda$. Denote by X^* the maximal subset of X on whom T^n acts freely. Let $p \in X$ be a fixed point of the T^n-action. Then we can see that

$$H_{k_1} \cap H_{k_2} \cap \cdots \cap H_{k_n} = \{\mu(p)\}$$

for some $k_1, \ldots, k_n$. In this subsection we consider the local structure around p, then we may suppose without loss of generality that

$$k_i = i, \quad \mu(p) = 0, \quad u = (I_n \ u') \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z}^n),$$

where I_n is the identity matrix and $u' \in \mathrm{Hom}(\mathbb{Z}^{d-n}, \mathbb{Z}^n)$. Moreover, recall that the hyper-Kähler structure on $X(u, \lambda)$ only depends on $\iota^*(\lambda)$. Since the projection to the first n components Ker $\iota^* \to \mathbf{t}^n$ is surjective, λ can be taken such that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, then $\mu(p) = 0$ implies $p = [\hat{p}]$ for some $\hat{p} = (0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots, a_d)$, with $a_k i \overline{a_k} = \lambda_k$. Here, $a_k, \lambda_k \neq 0$ hold for all $k = n+1, \ldots, d$ by the smoothness of $X(u, \lambda)$ and Theorem 3.1 $(*1)$. Then we have

$$\mu([x_1, \ldots, x_n, x_{n+1}, \ldots, x_d]) = (x_1 i \overline{x_1}, \ldots, x_n i \overline{x_n}). \tag{2}$$

The tangent space $T_p X$ is identified with the subspace $W_{\hat{p}} \subset \mathbb{H}^d$ which is the orthogonal complement of $T_{\hat{p}}(K\hat{p})$ in $T_{\hat{p}} \mu_K^{-1}(\iota^*(\lambda))$, where $K\hat{p} = \{g \cdot \hat{p};\ g \in K\}$. Then one can see that

$$W_{\hat{p}} = \mathbb{H}^n \times \{0\} = \{(v_1, \ldots, v_n, 0, \ldots, 0) \in \mathbb{H}^d\},$$

hence we obtain the canonical isomorphism

$$T_p X \cong \mathbb{H}^n.$$

Now we put

$$V(\sigma) := \mu_n^{-1}(\sigma \otimes \{(t_1, \ldots, t_n) \in (\mathbf{t}^n)^*;\ t_i \geq 0\})$$

for $\sigma \in S^2$, where $\mu_n(x_1, \ldots, x_n) = (x_1 i \overline{x_1}, \ldots, x_n i \overline{x_n})$. If we take

$$x = (z, w) \in \mathbb{C}^2 = \mathbb{H}$$

such that $x i \overline{x} = \sigma$, then we may write

$$V(\sigma) = \{(\alpha_1 z, \overline{\alpha_1} w, \ldots, \alpha_n z, \overline{\alpha_n} w) \in \mathbb{H}^n;\ (\alpha_1 \ldots, \alpha_n) \in \mathbb{C}^n\}. \tag{3}$$

Then $V(\sigma)$ is a T^n invariant σ-holomorphic Lagrangian subspace of $\mathbb{H}^n$. Put

$$\hat{L}_\sigma := V(\sigma) \times \prod_{k=n+1}^{d} L(\sigma, \lambda_k, \delta),$$
$$L_\sigma := (\mu_K^{-1}(\iota^*(\lambda)) \cap \hat{L}_\sigma)/K.$$

for sufficiently small δ. Then L_σ is a smooth holomorphic Lagrangian submanifold by the argument in Examples 1 and 2, and we have the following proposition.

Proposition 3.2. *Let $(X(u, \lambda), \omega, \mu)$, p and L_σ be as above. Then there is an open neighborhood $U \subset L_\sigma$ of p and $\varepsilon > 0$ such that*

$$\mu(U) = \sigma \otimes \{t \in (\mathbf{t}^n)^*;\ \|t\| < \varepsilon, t_i \geq 0\}$$

and

$$T_pU = V(\sigma)$$

holds under the identification $T_pX \cong \mathbb{H}^n$.

Proof. The first assertion follows from (2). The second assertion follows by $T_pU = T_pX \cap (V(\sigma) \times \mathbb{H}^{d-n}) = V(\sigma)$. □

4. Characterizing angles

4.1. Calabi-Yau manifolds

For the desingularization of special Lagrangian immersions which intersect transversely on a point, one should consider the characterizing angles, introduced by Lawlor [10].

Let (M, J, ω) be a Kähler manifold, where J is a complex structure, ω is a Kähler form. Suppose that there is a Lagrangian immersion $\iota : L \to M$, where ι is embedding on $L \backslash \{p_+, p_-\}$ and $\iota(L)$ intersects at $\iota(p_+) = \iota(p_-) = p \in M$ transversely. We suppose L is not necessarily to be connected, and the orientation of L is fixed.

Theorem 4.1 (Proposition 9.1 of [8]). *Let (J_0, ω_0) be the standard Kähler structure on $\mathbb{C}^m$. There exists a linear map $v : T_pM \to \mathbb{C}^m$ satisfying the following conditions; (i) v is a $\mathbb{C}$-linear isomorphism preserving the Kähler forms, (ii) there is $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbb{R}^m$ which satisfies*

$0 < \varphi_1 \le \cdots \le \varphi_m < \pi$ *and*

$$v \circ \iota_*(T_{p_+}L) = \mathbb{R}^m = \{(t_1, \dots, t_m) \in \mathbb{C}^m;\ t_i \in \mathbb{R}\},$$
$$v \circ \iota_*(T_{p_-}L) = \mathbb{R}^m_\varphi = \{(t_1 e^{\sqrt{-1}\varphi_1}, \dots, t_m e^{\sqrt{-1}\varphi_m}) \in \mathbb{C}^m;\ t_i \in \mathbb{R}\}.$$

(iii) v maps the orientation of $\iota_*(T_{p_+}L)$ to the standard orientation of $\mathbb{R}^m$. Moreover, $\varphi_1, \dots, \varphi_m$ and the induced orientation of $\mathbb{R}^m_\varphi$ by $v, \iota_*(T_{p_-}L)$ do not depend on the choice of v.

Here, we give an explanation for the reader's convenience how to determine the characterizing angles $\varphi_1, \dots, \varphi_m$ in Proposition 4.1. Choose a $\mathbb{C}$-linear isomorphism $v_0 : T_pM \to \mathbb{C}^m$ which preserves the Kähler metrics. Then $V_\pm := v_0 \circ \iota_*(T_{p_\pm}L)$ are Lagrangian subspaces of $\mathbb{C}^m$, therefore we can take $g_\pm \in U(m)$ such that $g_+ \cdot V_+ = g_- \cdot V_- = \mathbb{R}^m$. We may choose g_+ so that it preserves the orientations of V_+ and $\mathbb{R}^m$. Put $P = g_+ g_-^{-1} \in U(m)$. The eigenvalues of ${}^tPP$ are written as $e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_m}$ for some $0 \le \theta_1 \le \cdots \le \theta_m < 2\pi$. Then φ_i are given by $\varphi_i = \theta_i/2$. Here φ_i can never be 0 since V_+ and V_- intersect transversely.

Here, $\varphi = (\varphi_1, \dots, \varphi_m)$ is called the characterizing angles between (L, p_+) and (L, p_-). Under the above situation, assume that there is a holomorphic volume form Ω on M satisfying

$$\omega^m/m! = (-1)^{m(m-1)/2}(\sqrt{-1}/2)^m \Omega \wedge \overline{\Omega},$$

where m is the complex dimension of M. Let $\Omega_0 := dz_1 \wedge \cdots \wedge dz_m$ be the standard holomorphic volume form on $\mathbb{C}^m$, and assume that $\iota : L \to M$ is a special Lagrangian immersion. Then there exists $v : T_pM \to \mathbb{C}^m$ satisfying Theorem 4.1. In this case we can see $v^*\Omega_0 = e^{\sqrt{-1}\theta}\Omega_p$ for some θ. Since both of $\iota_*(T_{p_+}L) \subset T_pM$ and $\mathbb{R}^m \subset \mathbb{C}^m$ are special Lagrangian, we have $(v^{-1})^*\Omega_p|_{v \circ \iota_*(T_{p_+}L)} = \Omega_0|_{\mathbb{R}^m} = dt_1 \wedge \cdots \wedge dt_m$, therefore we can see $e^{\sqrt{-1}\theta} = 1$.

Since both of $\iota_*(T_{p_\pm}L)$ are special Lagrangian subspaces, there is a positive integer $k = 1, 2, \dots m-1$ and $\varphi_1 + \cdots + \varphi_m = k\pi$ holds. Then the intersection point $p \in M$ is said to be of type k. Note that the type depends on the order of p_+, p_-. If we take the opposite order, the characterizing angles become $\pi - \varphi_m, \dots, \pi - \varphi_1$ and the type becomes $m - k$.

4.2. Hyper-Kähler manifolds

For an oriented manifold L, we denote by $\overline{L}$ the oriented manifold diffeomorphic to L with the opposite orientation.

Let

$$\sigma(\theta) = (0, \cos\theta, \sin\theta) \in S^2.$$

Proposition 4.2. *Suppose L is a $\sigma(\theta)$-holomorphic Lagrangian submanifold in a hyper-Kähler manifold $(M^{4n}, g, I_1, I_2, I_3)$. Then L is a special Lagrangian submanifold if $\theta = \frac{k\pi}{n}$ for even $k \in \mathbb{Z}$, and $\overline{L}$ is special Lagrangian if $\theta = \frac{k\pi}{n}$ for odd $k \in \mathbb{Z}$.*

Proof. Put $\sigma' = (0, -\sin\theta, \cos\theta)$ and $\sigma'' = (1, 0, 0)$. By the assumptions, we have $\omega^{\sigma'}|_L = \omega^{\sigma''}|_L = 0$. Here, $\omega^{\sigma''}|_L = 0$ implies L is Lagrangian. Since we have

$$\begin{aligned} e^{-n\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)^n|_L &= (\omega^{\sigma(\theta)} + \sqrt{-1}\omega^{\sigma'})^n|_L \\ &= (\omega^{\sigma(\theta)})^n|_L, \end{aligned}$$

then if we put $\theta = \frac{k\pi}{n}$, we obtain

$$\mathrm{Re}(\omega_2 + \sqrt{-1}\omega_3)^n|_L = (-1)^k(\omega^{\sigma(\theta)})^n|_L, \quad \mathrm{Im}(\omega_2 + \sqrt{-1}\omega_3)^n|_L = 0.$$

□

Proposition 4.3. *Suppose $n\theta_\pm \in \pi\mathbb{Z}$ and let $V_\pm$ be T^n-invariant $\sigma(\theta_\pm)$-holomorphic Lagrangian subspaces of $\mathbb{H}^n$ given by*

$$V_+ := V(\sigma(\theta_+)), \quad V_- := V(\sigma(\theta_-)).$$

Then the characterizing angles between V_+ and V_- are given by $(\theta_- - \theta_+)/2$ with multiplicity $2n$.

Proof. By (3), we have

$$V_\pm = \{(\sqrt{-1}z_1, e^{\sqrt{-1}\theta_\pm}\overline{z_1}, \ldots, \sqrt{-1}z_n, e^{\sqrt{-1}\theta_\pm}\overline{z_n}) \in \mathbb{H}^n;\ z_1, \ldots, z_n \in \mathbb{C}\}$$

respectively. Put

$$A(\theta) := \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{-1} & e^{-\sqrt{-1}\theta} \\ -1 & \sqrt{-1}e^{-\sqrt{-1}\theta} \end{pmatrix},$$

and

$$g_+ := \begin{pmatrix} A(\theta_+) & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & A(\theta_+) \end{pmatrix}, \quad g_- := \begin{pmatrix} A(\theta_-) & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & A(\theta_-) \end{pmatrix}.$$

Since $g_+V_+ = g_-V_- = \mathbb{R}^{2n}$ holds, then the characterizing angles are the argument of the square root of the eigenvalues of ${}^tPP$, where $P = g_+g_-^{-1}$, by the proof of Theorem 4.1. Since

$${}^t(A(\theta_+)A(\theta_-)^{-1})A(\theta_+)A(\theta_-)^{-1} = e^{\sqrt{-1}(\theta_- - \theta_+)}\mathrm{Id},$$

the characterizing angles turn out to be $(\theta_- - \theta_+)/2$ with multiplicity $2n$. □

Now we consider the case that

$$(M, J, \omega, \Omega) = (X(u, \lambda), I_1, (\omega_{\lambda,2} + \sqrt{-1}\omega_{\lambda,3})^n)$$

and $L = L_+ \sqcup L_-$, where $L_\pm$ is embedded as $\sigma(\theta_\pm)$-holomorphic Lagrangian submanifolds respectively, for some $\theta_\pm \in \mathbb{R}$. Denote by $\iota : L \to X(u, \lambda)$ the immersion. Assume that the image of L is a T^n invariant subset of $X(u, \lambda)$, and $p \in X(u, \lambda)$ is the fixed point of the torus action. In this subsection, we see the characterizing angles between (L_+, p_+) and (L_-, p_-) in this situation, where $\iota^{-1}(p) = \{p_+, p_-\}$.

Proposition 4.4. *Under the above setting, assume that there is a sufficiently small $r > 0$ and*

$$(\mu(L_\pm) - \mu(p)) \cap B(r) = \sigma_\pm \otimes \{x \in (\mathbf{t}^n)^*;\ \|x\| < r, x_i \geq 0\}$$

holds. Then the characterizing angles between (L_+, p_+) and (L_-, p_-) are given by $(\theta_- - \theta_+)/2$ with multiplicity $2n$.

Proof. By Proposition 3.2, we can see that

$$T_{p_\pm}L_\pm = V(\sigma(\theta_\pm))$$

respectively. Thus we have the assertion by Proposition 4.3. □

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Although Theorem 1.2 follows from Theorem 9.7 of [8] essentially, we need some additional argument about the quivers. Let $Q = (\mathcal{V}, \mathcal{E}, s, t)$ be a quiver, that is, $\mathcal{V}$ consists of finite vertices, $\mathcal{E}$ consists of finite directed edges, and $s, t : \mathcal{E} \to \mathcal{V}$ are maps. Here, $s(h)$ and $t(h)$ means the source and the target of $h \in \mathcal{E}$ respectively. The quiver is said to be connected if any two vertices are connected by some edges. Given the quiver, we have operators

$$\partial : \mathbb{R}^{\mathcal{E}} \to \mathbb{R}^{\mathcal{V}},$$
$$\partial^* : \mathbb{R}^{\mathcal{V}} \to \mathbb{R}^{\mathcal{E}}$$

defined by

$$\partial\left(\sum_{h\in\mathcal{E}} A_h \cdot h\right) := \sum_{h\in\mathcal{E}} A_h \cdot (s(h) - t(h)),$$
$$\partial^*\left(\sum_{k\in\mathcal{V}} x_k \cdot k\right) := \sum_{h\in\mathcal{E}} (x_{s(h)} - x_{t(h)}) \cdot h.$$

Here, $\mathbb{R}^{\mathcal{E}}$ and $\mathbb{R}^{\mathcal{V}}$ are the free $\mathbb{R}$-modules generated by elements of $\mathcal{E}$ and $\mathcal{V}$ respectively. Since ∂^* is the adjoint of ∂, we have

$$\mathbf{h}_0(Q) - \mathbf{h}_1(Q) = \#\mathcal{V} - \#\mathcal{E}, \tag{4}$$

where $\mathbf{h}_0(Q) = \dim \mathrm{Ker}\partial^*$ and $\mathbf{h}_1(Q) = \dim \mathrm{Ker}\partial$. Note that $\mathbf{h}_0(Q)$ is equal to the number of the connected components of Q.

We need the following lemmas for the proof of Theorem 1.2.

Lemma 5.1. *Let Q be as above. The set $(\mathbb{R}_{>0})^{\mathcal{E}} \cap \mathrm{Ker}(\partial)$ is nonempty if and only if $\mathcal{E}$ is covered by cycles.*

Proof. Suppose that $\mathcal{E} = \bigcup_\alpha S_k$ holds for some cycles $S_1, \ldots, S_N$. For a subset $S \subset \mathcal{E}$, define $\chi_S \in \mathbb{R}^{\mathcal{E}}$ by

$$(\chi_S)_h := \begin{cases} 1 & (h \in S), \\ 0 & (h \notin S). \end{cases}$$

Then $\sum_{k=1}^N \chi_{S_k}$ is contained in $(\mathbb{R}_{>0})^{\mathcal{E}} \cap \mathrm{Ker}(\partial)$.

Conversely, assume that there exists $A = \sum_{h\in\mathcal{E}} A_h \cdot h \in \mathrm{Ker}(\partial)$ with $A_h > 0$ for every h, and take $h_0 \in \mathcal{E}$ arbitrarily. Since $\partial(A) = 0$, we have

$$\sum_{h\in s^{-1}(t(h_0))} A_h = \sum_{h\in t^{-1}(t(h_0))} A_h \geq A_{h_0} > 0.$$

Hence $s^{-1}(t(h_0))$ is nonempty, we can take $h_1 \in s^{-1}(t(h_0))$. By repeating this procedure, we obtain $h_0, h_1, \dots, h_l$ so that $t(h_k) = s(h_{k+1})$ for $k = 0, \dots, l-1$. Stop this procedure when $t(h_l) = s(h_k)$ holds for some $k = 0, \dots, l$. Since $\mathcal{V}$ is finite, this procedure always stops for some $l < +\infty$. Then we have an nonempty cycle $S_0 = \{h_k, h_{k+1}, \dots, h_l\}$. If h_0 is contained in S_0, then we have the assertion, hence suppose $h_0 \notin S_0$. Put $A_0 := \min_{h\in S_0} A_h > 0$,

$$P_0 := \{h \in \mathcal{E};\ A_h = A_0\},$$
$$\mathcal{E}_1 := \mathcal{E}\backslash P_0.$$

Then we have a new quiver $((\mathcal{V}, \mathcal{E}_1, s, t))$ and the boundary operator $\partial_1 : \mathbb{R}^{\mathcal{E}_1} \to \mathbb{R}^{\mathcal{V}}$. Now, put $A^{(1)} := A - A_0\chi_{S_0} \in \mathbb{R}^{\mathcal{E}_1}$. Then each component of $A^{(1)}$ is positive. Moreover we can see that

$$\begin{aligned}
\partial_1(A^{(1)}) &= \sum_{h\in\mathcal{E}\backslash S_0} A_h(s(h) - t(h)) + \sum_{h\in S_0\backslash P_0} (A_h - A_0)(s(h) - t(h)) \\
&= \sum_{h\in\mathcal{E}} A_h(s(h) - t(h)) - \sum_{h\in S_0} A_h(s(h) - t(h)) \\
&\quad + \sum_{h\in S_0} (A_h - A_0)(s(h) - t(h)) \\
&= \partial(A) - \sum_{h\in S_0} A_0(s(h) - t(h)) \\
&= -A_0\partial(\chi_{S_0}) = 0,
\end{aligned}$$

thus $A^{(1)}$ is contained in $(\mathbb{R}_{>0})^{\mathcal{E}_1} \cap \mathrm{Ker}(\partial_1)$. Then we can apply the above procedure for $h_0 \in \mathcal{E}_1$ and we can construct S_k inductively. Since $\mathcal{E}$ is finite and $\#\mathcal{E} > \#\mathcal{E}_1 > \cdots$ holds, there is k_0 such that $h_0 \in S_{k_0}$. $\square$

Lemma 5.2. *Let $Q = (\mathcal{V}, \mathcal{E}, s, t)$ be as above and $\mathcal{E}' = \mathcal{E}\backslash\{h\}$ for $h \in \mathcal{E}$. Then $Q' = (\mathcal{V}, \mathcal{E}', s, t)$ satisfies either $(\mathbf{h}_0(Q'), \mathbf{h}_1(Q')) = (\mathbf{h}_0(Q) + 1, \mathbf{h}_1(Q))$ or $(\mathbf{h}_0(Q'), \mathbf{h}_1(Q')) = (\mathbf{h}_0(Q), \mathbf{h}_1(Q) - 1)$ for any $h \in \mathcal{E}$.*

Proof. First of all we can check that

$$\operatorname{Ker}(\partial_Q^*) \subset \operatorname{Ker}(\partial_{Q'}^*), \quad \operatorname{Im}(\partial_{Q'}) \subset \operatorname{Im}(\partial_Q). \tag{5}$$

It suffices to show $\mathbf{h}_0(Q') = \mathbf{h}_0(Q)$ or $\mathbf{h}_0(Q)+1$ by (4). Put

$$\mathcal{E}_1 = \{h \in \mathcal{E};\ A_h = 0 \text{ for any } A \in \operatorname{Ker}(\partial)\}, \quad \mathcal{E}_2 = \mathcal{E}\backslash\mathcal{E}_1.$$

Let $h \in \mathcal{E}_1$. Then we have $\operatorname{Ker}(\partial_Q) = \operatorname{Ker}(\partial_{Q'})$ which implies

$$\operatorname{rk}(\partial_Q) = \operatorname{Ker}(\partial_{Q'}) + 1.$$

Since there are orthogonal decompositions

$$\mathbb{R}^{\mathcal{V}} = \operatorname{Ker}(\partial_Q^*) \oplus \operatorname{Im}(\partial_Q) = \operatorname{Ker}(\partial_{Q'}^*) \oplus \operatorname{Im}(\partial_{Q'}),$$

we obtain $\mathbf{h}_0(Q') = \mathbf{h}_0(Q) + 1$.

Next assume $h \in \mathcal{E}_2$. By (5), it suffices to show $\operatorname{Ker}(\partial_{Q'}^*) \subset \operatorname{Ker}(\partial_Q^*)$. Let $x \in \operatorname{Ker}(\partial_{Q'}^*)$ and $A \in \operatorname{Ker}(\partial_Q)$. Then $\langle \partial_Q^* x, A\rangle = \langle x, \partial_Q A\rangle = 0$ and

$$\langle \partial_Q^* x, A\rangle = \sum_{h' \in \mathcal{E}} (x_{s(h')} - x_{t(h')})A_{h'} = (x_{s(h)} - x_{t(h)})A_h$$

hold. Since $h \in \mathcal{E}_2$, there exists $A \in \operatorname{Ker}(\partial_Q)$ such that $A_h \neq 0$, hence $x_{s(h)} - x_{t(h)}$ should be 0. Consequently we have shown that if $h \in \mathcal{E}_2$ then $\mathbf{h}_0(Q') = \mathbf{h}_0(Q)$. □

Let L_α be a compact connected smooth special Lagrangian submanifold of the Calabi-Yau manifold (M, J, ω, Ω) of $\dim_{\mathbb{C}} M = m$ for every $\alpha \in \mathcal{V}$. For every $h \in \mathcal{E}$, suppose $L_{s(h)}$ and $L_{t(h)}$ intersects transversely at $p_h \in L_{s(h)} \cap L_{t(h)}$, where p_h is the intersection point of type 1. Assume that $p_h \neq p_{h'}$ if $h \neq h'$, and assume that $\bigcup_{\alpha\in\mathcal{V}} L_\alpha \backslash \{p_h; h \in \mathcal{E}\}$ is embedded in M. Let L_Q be a differential manifold obtained by taking the connected sum of $L_{s(h)}$ and $L_{t(h)}$ at p_h for every $h \in \mathcal{E}$. By Theorem 9.7 of [8], if $(\mathbb{R}_{>0})^{\mathcal{E}} \cap \operatorname{Ker}(\partial)$ is nonempty, there exists a compact smooth special Lagrangian submanifolds $\tilde{L}_t$ for every sufficiently small $t \in (\mathbb{R}_{>0})^{\mathcal{E}} \cap \operatorname{Ker}(\partial)$, which converges to $\bigcup_{\alpha\in\mathcal{V}} L_\alpha$ as $|t| \to 0$ in the sense of currents. Here, $\tilde{L}_t$ is diffeomorphic to L_Q.

Now the assumption that $(\mathbb{R}_{>0})^{\mathcal{E}} \cap \operatorname{Ker}(\partial)$ is nonempty can be replaced by the assumption that $\mathcal{E}$ is covered by cycles, hence the proof of Theorem 1.2 is completed.

Proposition 5.3. *If $Q=(\mathcal{V},\mathcal{E},s,t)$ is a connected quiver, then L_Q is diffeomorphic to*

$$L_1\#L_2\#\cdots\#L_A\#N(S^1\times S^{m-1}),$$

where $\mathcal{V}=\{1,\dots,A\}$ and $N=\dim \mathrm{Ker}(\partial)$, and the orientation of each L_α is determined by $\mathrm{Re}\Omega|_{L_\alpha}$.

Proof. Let $Q=(\mathcal{V},\mathcal{E},s,t)$ be a connected quiver and $Q'=(\mathcal{V},\mathcal{E}',s|_{\mathcal{E}'},t|_{\mathcal{E}'})$, where $\mathcal{E}'=\mathcal{E}\backslash\{h\}$. Let $\mathcal{E}_1,\mathcal{E}_2$ be as in the proof of Lemma 5.2.

If $h\in\mathcal{E}_1$, then the quiver Q' consists of two connected components $Q_1=(\mathcal{W}_1,\mathcal{F}_1,s|_{\mathcal{F}_1},t|_{\mathcal{F}_1})$ and $Q_2=(\mathcal{W}_2,\mathcal{F}_2,s|_{\mathcal{F}_2},t|_{\mathcal{F}_2})$, where $\mathcal{V}=\mathcal{W}_1\sqcup\mathcal{W}_2$ and $\mathcal{F}_i=\mathcal{E}'\cap(s^{-1}(\mathcal{W}_i)\cup t^{-1}(\mathcal{W}_i))$. Then we can see that $L_Q=L_{Q_1}\#L_{Q_2}$.

If $h\in\mathcal{E}_2$, then $Q'=(\mathcal{V},\mathcal{E}',s|_{\mathcal{E}'},t|_{\mathcal{E}'})$ is also connected, hence L_Q is constructed from $L_{Q'}$ in the following way. Take any distinct points $p_+,p_-\in L_{Q'}$ and their neighborhood $B_{p_\pm}\subset L_{Q'}$ so that $B_{p_+}\cap B_{p_-}$ is empty and $B_{p_\pm}$ are diffeomorphic to the Euclidean unit ball. Now we have a polar coordinate $(r_\pm,\Theta_\pm)\in B_{p_\pm}\backslash\{p_\pm\}$, where $r_\pm\in(0,1)$ is the distance from $p_\pm$, and $\Theta_\pm\in S^{m-1}$. By taking a diffeomorphism $\psi:(r,\Theta)\mapsto(1-r,\varphi(\Theta))$, we can glue $B_{p_+}\backslash\{p_+\}$ and $B_{p_-}\backslash\{p_-\}$, then obtain L_Q. Here, $\varphi:S^{m-1}\to S^{m-1}$ is a diffeomorphism which reverse the orientation. Note that the differentiable structure of L_Q is independent of the choice of $p_\pm$, $B_{p_\pm}$ and φ. Therefore we may suppose p_+ and p_- is contained in an open subset $U\subset L_Q$, where $U=B(0,10)$ and $B_{p_\pm}=B(\pm x_0,1)$, respectively. Here $B(x,r)=\{x'\in\mathbb{R}^m;\ \|x'-x\|<r\}$ and $x_0=(5,0,\dots,0)\in\mathbb{R}^m$. Then $(U\backslash\{x_0,-x_0\})/\psi$ is diffeomorphic to $S^1\times S^{m-1}\backslash\{\mathrm{pt.}\}$, hence L_Q is diffeomorphic to $L_{Q'}\#S^1\times S^{m-1}$.

By repeating these two types of procedures, we finally obtain a quiver $Q''=(\mathcal{V},\emptyset,s,t)$, and we have $(\mathbf{h}_0(Q''),\mathbf{h}_1(Q''))=(\#\mathcal{V},0)$. By counting $\mathbf{h}_0$ and $\mathbf{h}_1$ on each step, it turns out that we have to follow the former procedures $\#\mathcal{V}-1$ times and the latter procedures $\mathbf{h}_1(Q)$ times until we reach Q''. Therefore we obtain the assertion by considering the procedures inductively. □

6. The construction of compact special Lagrangian submanifolds in $X(u,\lambda)$

Here we construct examples of compact special Lagrangian submanifolds in $X(u,\lambda)$, using Theorem 1.2. We construct a one parameter family of compact special Lagrangian submanifolds which degenerates to the union $\bigcup_i L_i$ of some σ_i-holomorphic Lagrangian submanifolds L_i in Subsection 6.1.

Let $X(u,\lambda)$ be a smooth toric hyper-Kähler manifold, and

$$V(q,\sigma) := q + \sigma \otimes (\mathbf{t}^n)^* \subset \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$$

for $q \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$. If the intersection of $V(q,\sigma)$ and H_k is not empty, then one can see that $V(q,\sigma) \cap H_k$ is a hyperplane of $V(q,\sigma)$, and it yields two half-spaces, namely, the closures of the connected components of $V(q,\sigma)\backslash H_k$. We call them half spaces in $V(q,\sigma)$ induced by H_k.

Definition 6.1. We call $\triangle \subset \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$ *a σ-Delzant polytope* if it is a compact subset of the form $\triangle = \bigcap_{k=1}^d V_k$, where V_k is one of the half spaces in $V(q,\sigma)$ induced by H_k for some q independent of k.

For a σ-Delzant polytope $\triangle$, $L_\triangle := \mu_\lambda^{-1}(\triangle)$ is σ-holomorphic Lagrangian if it is smooth, since $\triangle$ is contained in $q + \sigma \otimes (\mathbf{t}^n)^*$ for some q. Since T^n-action is closed on $L_\triangle$, we may regard $(L_\triangle, I^\sigma_{\lambda,1}|_{L_\triangle})$ as a toric variety, equipped with a Kähler form $\omega^\sigma_{\lambda,1}|_{L_\triangle}$ and a Kähler moment map $\mu^\sigma_{\lambda,1} : L_\triangle \to (\mathbf{t}^n)^*$. In particular, $L_\triangle$ is an oriented manifold whose orientation is induced naturally from $I^\sigma_{\lambda,1}$. We denote by $\overline{L}_\triangle$ the oriented manifold diffeomorphic to $L_\triangle$ with the opposite orientation. By the assumption $X(u,\lambda)$ is smooth, u and λ satisfies $(*1)(*2)$ of Theorem 3.1, then it is easy to see that $\triangle$ is a Delzant polytope in the ordinary sense, consequently $L_\triangle$ turns out to be a smooth toric variety. For the definition of Delzant polytopes, see [2] for example.

Take

$$\tau \in \mathcal{T} := \left\{ \tau \subset \{1,\dots,d\};\ \#\tau = n,\ \bigcap_{k\in\tau} H_k \neq \emptyset \right\}.$$

Then by Theorem 3.1, $\bigcap_{k\in\tau} H_k$ consists of one point and we denote it by q_τ. Supposing $q_\tau \in \triangle$, it is a vertex of $\triangle$. Now, let $u|_\tau := (u_k)_{k\in\tau} \in \mathrm{Hom}(\mathbb{Z}^\tau, \mathbb{Z}^n)$, where $\mathbb{Z}^\tau = \{(n_k)_{k\in\tau};\ n_k \in \mathbb{Z}\} \cong \mathbb{Z}^n$. Note that $u|_\tau$ extends to $\mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^\tau)^* \to \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$ naturally. Then we may write

$$u|_\tau^{-1}(\triangle - q_\tau) \subset \sigma \otimes \{(r_k)_{k\in\tau} \in \mathbb{R}^\tau;\ \varepsilon_k r_k \geq 0\}$$

for some $(\varepsilon_k)_{k\in\tau} \in \{1,-1\}^\tau$.

Definition 6.2. For $\alpha = 0,1$, let $\triangle_\alpha$ be a $\sigma(\theta_\alpha)$-Delzant polytope. Then we write $\angle \triangle_0 \triangle_1 = \theta_1 - \theta_0$ if there is $q_\tau \in \triangle_0 \cap \triangle_1$ for some $\tau \in \mathcal{T}$ and we have

$$u|_\tau^{-1}(\triangle_0 - q_\tau) \subset \sigma(\theta_0) \otimes \{(r_k)_{k\in\tau} \in \mathbb{R}^\tau;\ \varepsilon_k r_k \geq 0\},$$
$$u|_\tau^{-1}(\triangle_1 - q_\tau) \subset \sigma(\theta_1) \otimes \{(r_k)_{k\in\tau} \in \mathbb{R}^\tau;\ \varepsilon_k r_k \geq 0\},$$

for the same $(\varepsilon_k)_{k\in\tau} \in \{1,-1\}^\tau$.

Remark 6.3. Suppose $q_\tau \in \triangle_0 \cap \triangle_1$ for some $\tau \in \mathcal{T}$ and $\angle\triangle_0\triangle_1 = \theta_1 - \theta_0 \notin \pi\mathbb{Z}$ holds. Then q_τ is the only point in $\triangle_0 \cap \triangle_1$, since

$$(q_\tau + \sigma(\theta_0) \otimes (\mathbf{t}^n)^*) \cap (q_\tau + \sigma(\theta_1) \otimes (\mathbf{t}^n)^*) = \{q_\tau\}.$$

For $m \in \mathbb{Z}_{>0}$, let

$$d_m(l_1, l_2) := \min\{|l_1 - l_2 + mk|;\ k \in \mathbb{Z}\},$$

for $l_1, l_2 \in \mathbb{Z}$, which induces a distance function on $\mathbb{Z}/m\mathbb{Z}$.

The main result of this article is described as follows.

Theorem 6.4. *Let $X(u,\lambda)$ be a smooth toric hyper-Kähler manifold, and $\triangle_k$ be a $\sigma(k\pi/n)$-Delzant polytope for each $k \equiv 1,\dots,2n \bmod 2n$. Assume that $\triangle_k \cap \triangle_l = \emptyset$ if $d_{2n}(k,l) > 1$, and $\angle\triangle_k\triangle_{k+1} = \pi/n$. Then there exists a family of compact special Lagrangian submanifolds $\{\tilde{L}_t\}_{0<t<\delta}$ which converges to $\bigcup_{k=1}^{2n} L_{\triangle_k}$ as $t \to 0$ in the sense of currents. Moreover, $\tilde{L}_t$ is diffeomorphic to $L_{\triangle_1}\#\overline{L}_{\triangle_2}\#\cdots L_{\triangle_{2n-1}}\#\overline{L}_{\triangle_{2n}}\#(S^1 \times S^{2n-1})$.*

Proof. We apply Theorem 1.2. By Propositions 4.4, we can see that the characterizing angles between $L_{\triangle_k}$ and $L_{\triangle_{k+1}}$ are $\frac{\pi}{2n}$ with multiplicity $2n$. Then the intersection point $L_{\triangle_k} \cap L_{\triangle_{k+1}}$ is of type 1.

In this case, we may put $\mathcal{E} = \mathcal{V} = \mathbb{Z}/2n\mathbb{Z}$ and ∂ is given by

$$\partial(A_0,\dots,A_{2n-1}) = (A_0 - A_1, A_1 - A_2, \dots, A_{2n-1} - A_0),$$

hence we can see $\dim \mathrm{Ker}(\partial) = 1$, which implies we obtain a 1-parameter family of special Lagrangian submanifolds $\{\tilde{L}_t\}$.

Next we consider the topology of $\tilde{L}_t$. When we take a connected sum, we should determine the orientation of $L_{\triangle_k}$ uniformly by the calibration $\mathrm{Re}\Omega$, where $\Omega = (\omega_{\lambda,2} + \sqrt{-1}\omega_{\lambda,3})^n$. Now

$$\Omega|_{L_{\triangle_k}} = (-1)^k (\omega_1^{\sigma(k\pi/n)})^n|_{L_{\triangle_k}}$$

holds, therefore $\tilde{L}_t$ is diffeomorphic to

$$L_{\triangle_1} \# \overline{L}_{\triangle_2} \# \cdots L_{\triangle_{2n-1}} \# \overline{L}_{\triangle_{2n}} \# (S^1 \times S^{2n-1}).$$

□

We will see some examples in the following subsections. To show the given $X(u, \lambda)$ satisfies the assumption of Theorem 6.4, the essential part is to check the condition $\triangle_k \cap \triangle_l = \emptyset$ if $d_{2n}(k, l) > 1$.

Lemma 6.5. *For $y = (y_1, \ldots, y_n) \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$ and $\alpha = 1, \ldots, n$, let $\pi_\alpha : \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^* \to \mathrm{Im}\mathbb{H}$ be the projection defined by $\pi_\alpha(y) = y_\alpha$. Denote by*

$$\mathrm{Conv}(\triangle_0) \subset \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$$

the convex hull of a subset $\triangle_0 \subset \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$.

(1) *Let $\triangle := \mathrm{Conv}(\triangle_0)$ and $\triangle' := \mathrm{Conv}(\triangle_0')$. If*

$$\mathrm{Conv}(\pi_\alpha(\triangle_0)) \cap \mathrm{Conv}(\pi_\alpha(\triangle_0')) = \emptyset$$

holds for some α, then $\triangle \cap \triangle'$ is empty.

(2) *Let $H := \{y \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*;\ y_\alpha + \lambda = 0\}$ for some $\lambda \in \mathrm{Im}\mathbb{H}$ and α. If $-\lambda \notin \mathrm{Conv}(\pi_\alpha(\triangle_0))$, then $\triangle \cap H$ is empty.*

Proof. Let V, W be vector spaces over $\mathbb{R}$, and $f : V \to W$ be a linear map. For $\triangle, \triangle' \subset V$, $\triangle \cap \triangle'$ is empty if $f(\triangle) \cap f(\triangle')$ is empty. Moreover, if $\triangle$ is the convex hull of $\triangle_0 \subset V$, then $f(\triangle) = \mathrm{Conv}(f(\triangle_0))$ holds. Combining these, we obtain (1). (2) also follows from the same argument since $\pi_\alpha(H) = \{-\lambda\}$. □

Under the identification $\mathrm{Im}\mathbb{H} \cong \mathbb{R} \oplus \mathbb{C}$ given by $(a, b, c) \mapsto (a, b + \sqrt{-1}c)$, we can identify $\mathbb{C} \otimes V$ with $(\{0\} \oplus \mathbb{C}) \otimes V \subset \mathrm{Im}\mathbb{H} \otimes V$ for any real vector space V.

In the following subsections, we always suppose $\lambda \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$ is contained in $\mathbb{C} \otimes (\mathbf{t}^n)^*$ for the simplicity. Then all of the $\sigma(\theta)$-Delzant polytopes are contained in $e^{\sqrt{-1}\theta}(\mathbf{t}^n)^*$, accordingly we often discuss in $\mathbb{C} \otimes (\mathbf{t}^n)^*$ instead of $\mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$.

6.1. Example (1)

Let

$$u = (I_n\ I_n\ \cdots\ I_n) \in \mathrm{Hom}(\mathbb{Z}^{2n^2}, \mathbb{Z}^n)$$

and $\lambda = (\lambda_{1,1}, \ldots, \lambda_{1,n}, \lambda_{2,1}, \ldots, \lambda_{2,n}, \ldots, \lambda_{2n,1}, \ldots, \lambda_{2n,n})$, where I_n is the identity matrix. Suppose that $\lambda_{k,\alpha} = \lambda_{l,\alpha}$ holds only if $k = l$, then we can show that $X(u,\lambda)$ is smooth by Theorem 3.1 as follows. Since $H_{k,\alpha} \cap H_{l,\alpha}$ is empty for $k \neq l$, $k_1, \ldots, k_N$ should be taken without overlapping if $H_{k_1,\alpha_1} \cap \cdots \cap H_{k_N,\alpha_N}$ is nonempty, hence $(*1)$ of Theorem 3.1 holds. By the same reason, if $H_{k_1,\alpha_1} \cap \cdots \cap H_{k_n,\alpha_n}$ is nonempty, then we may suppose $k_1 = 1, k_2 = 2, \ldots, k_n = n$. Since $(u_{1,\alpha_1} \cdots u_{n,\alpha_n}) = I_n$, we obtain $(*2)$.

We also assume that $\lambda_{k,\alpha} \in \mathbb{C} \cong \{0\} \oplus \mathbb{C}$ holds for every k, α, as mentioned above. Moreover we suppose that

$$\arg(-\lambda_{k+1,\alpha} + \lambda_{k,\alpha}) = \theta_0 + \frac{n+1}{n} k\pi \tag{6}$$

for some $\theta_0 \in \mathbb{R}$. Note that $X(u,\lambda)$ is a direct product of multi Eguchi-Hanson spaces.

Next we put $q_k := -(\lambda_{k,1}, \ldots, \lambda_{k,n}) \in \mathbb{C} \otimes (\mathbf{t}^n)^*$, and

$$\begin{aligned}\Box_k &:= q_k + e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n} k\pi)} \otimes \Box(r_{k,1}, \ldots, r_{k,n}) \\ &\subset V\left(q_k, \sigma\left(\theta_0 + \frac{n+1}{n} k\pi\right)\right),\end{aligned}$$

where $r_{k,\alpha} = |\lambda_{k+1,\alpha} - \lambda_{k,\alpha}|$, and a hyperrectangle $\Box(r_1, \ldots, r_n) \subset (\mathbf{t}^n)^* \cong \mathbb{R}^n$ is defined by

$$\Box(r_1, \ldots, r_n) := \{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ 0 \le t_1 \le r_1,\ \ldots,\ 0 \le t_n \le r_n\}.$$

By combining (6), we have $\lambda_{k,\alpha} - \lambda_{k+1,\alpha} = r_{k,\alpha} e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n} k\pi)}$.

Next we study the intersection of $\Box_{k-1}$ and $\Box_k$. We can check that $\Box_{k-1} \cap \Box_k = \{q_k\}$ and every element in $\Box_{k-1}$ satisfies

$$\begin{aligned}
&q_{k-1} + e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n}(k-1)\pi)}(t_1, \dots, t_n) \\
&= q_{k-1} + e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n}(k-1)\pi)}(r_{k-1,1}, \dots, r_{k-1,n}) \\
&\quad - e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n}(k-1)\pi)}(r_{k-1,1} - t_1, \dots, r_{k-1,n} - t_n) \\
&= q_{k-1} + (\lambda_{k-1,1} - \lambda_{k,1}, \dots, \lambda_{k-1,n} - \lambda_{k,n}) \\
&\quad + e^{\sqrt{-1}(\theta_0 + \frac{n+1}{n}(k-1)+1)\pi}(r_{k-1,1} - t_1, \dots, r_{k-1,n} - t_n) \\
&= q_k + e^{\sqrt{-1}\theta_0 + \sqrt{-1}\frac{(n+1)k-1}{n}\pi}(r_{k-1,1} - t_1, \dots, r_{k-1,n} - t_n).
\end{aligned}$$

Therefore, $\angle \Box_{k-1} \Box_k = \pi/n$. Of course, the same argument goes well for $\Box_{2n}$ and $\Box_1$.

To apply Theorem 6.4, $\Box_k \cap \Box_l$ should be empty if $d_{2n}(k, l) > 1$. However, this condition does not hold in general, accordingly we need to choose $\lambda_{k,\alpha}$ carefully. Unfortunately, the author cannot find the good criterion for $\lambda_{k,\alpha}$ satisfying the above condition. Here we show one example of $\lambda_{k,\alpha}$ which satisfies the assumption of Theorem 6.4.

First of all, take $a_1, \dots, a_n \in \mathbb{R}$ so that every a_m is larger than 1, and put

$$\begin{aligned}
-\rho_{2m-1} &:= e^{\sqrt{-1}\frac{2(m-1)}{n}\pi} + a_m(e^{\sqrt{-1}\frac{2m}{n}\pi} - e^{\sqrt{-1}\frac{2(m-1)}{n}\pi}), \\
-\rho_{2m} &:= e^{\sqrt{-1}\frac{2(m+1)}{n}\pi} + a_m(e^{\sqrt{-1}\frac{2m}{n}\pi} - e^{\sqrt{-1}\frac{2(m+1)}{n}\pi})
\end{aligned}$$

for each $m = 1, \dots, n$. Denote by $\mathbf{l}_k \subset \mathbb{C}$ the segment connecting $-\rho_k$ and $-\rho_{k+1}$. Then we can easily see that $\mathbf{l}_{k-1} \cap \mathbf{l}_k = \{-\rho_k\}$ and

$$\arg(-\rho_{k+1} + \rho_k) = \frac{n+2}{2n}\pi + \frac{n+1}{n}k\pi.$$

Note that we can regard $k \in \mathbb{Z}/2n\mathbb{Z}$ and $m \in \mathbb{Z}/n\mathbb{Z}$.

Proposition 6.6. *Let $\rho_1, \dots, \rho_{2n}$ be as above. If every $a_k - 1$ is sufficiently small, then $\mathbf{l}_{2m-1} \cap \mathbf{l}_k$ are empty for all $m = 1, \dots, n$ and $k = 1, \dots, 2n$ with $d_{2n}(k, 2m-1) > 1$.*

Proof. Let $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ be the projection given by taking the real part. It suffices to show that $\mathrm{Re}(\mathbf{l}_{2m-1} e^{-\sqrt{-1}\frac{2m}{n}\pi}) \cap \mathrm{Re}(\mathbf{l}_k e^{-\sqrt{-1}\frac{2m}{n}\pi})$ is empty under the given assumptions. Let $-\rho_{2m-1} + t(-\rho_{2m} + \rho_{2m-1}) \in \mathbf{l}_{2m-1}$. Then we can check that

$$\begin{aligned}&\mathrm{Re}(-\rho_{2m-1}e^{-\sqrt{-1}\frac{2m}{n}\pi}+t(-\rho_{2m}+\rho_{2m-1})e^{-\sqrt{-1}\frac{2m}{n}\pi})\\&=(1-a_m)\cos\frac{2\pi}{n}+a_m,\end{aligned}$$

which implies $\mathrm{Re}(\mathbf{l}_{2m-1}e^{-\sqrt{-1}\frac{2m}{n}\pi})=\{-(a_m-1)\cos\frac{2\pi}{n}+a_m\}$. If we can see that

$$\mathrm{Re}(-\rho_k e^{-\sqrt{-1}\frac{2m}{n}\pi})<-(a_m-1)\cos\frac{2\pi}{n}+a_m \tag{7}$$

for all $k\neq 2m-1, 2m$, we have the assertion. Since

$$\begin{aligned}\mathrm{Re}(-\rho_{2l}e^{-\sqrt{-1}\frac{2m}{n}\pi})&=-(a_l-1)\cos(\frac{2(1+l-m)}{n}\pi)\\&\quad+a_l\cos(\frac{2(l-m)}{n}\pi),\\\mathrm{Re}(-\rho_{2l'-1}e^{-\sqrt{-1}\frac{2m}{n}\pi})&=-(a_{l'}-1)\cos(\frac{2(1-l'+m)}{n}\pi)\\&\quad+a_{l'}\cos(\frac{2(l'-m)}{n}\pi)\end{aligned}$$

and $d_{2n}(2l,2m)>1$, $d_{2n}(2l'-1,2m-1)>1$ holds, we have

$$\cos(2(l-m)\pi/n)\leq\cos(2\pi/n),\quad \cos(2(l'-m)\pi/n)\leq\cos(2\pi/n).$$

By $\cos(\frac{2(1+l-m)}{n}\pi)\geq -1$ and $\cos(\frac{2(1-l'+m)}{n}\pi)\geq -1$, we obtain

$$\begin{aligned}\mathrm{Re}(-\rho_{2l}e^{-\sqrt{-1}\frac{2m}{n}\pi})&\leq(a_l-1)+a_l\cos\frac{2\pi}{n}\\&=(a_l-1)(1+\cos\frac{2\pi}{n})+\cos\frac{2\pi}{n},\\\mathrm{Re}(-\rho_{2l'-1}e^{-\sqrt{-1}\frac{2m}{n}\pi})&\leq(a_{l'}-1)+a_{l'}\cos\frac{2\pi}{n}\\&=(a_{l'}-1)(1+\cos\frac{2\pi}{n})+\cos\frac{2\pi}{n}\end{aligned}$$

Now, if we assume $a_l-1<(1-\cos\frac{2\pi}{n})/(1+\cos\frac{2\pi}{n})$, then the left-hand-side of (7) is less than 1. Since

$$-(a_m-1)\cos\frac{2\pi}{n}+a_m=(a_m-1)(1-\cos\frac{2\pi}{n})+1,$$

the right-hand-side of (7) is always larger than 1 and we obtain the inequality (7). □

Now, divide $\{1,\dots,n\}$ into two nonempty sets

$$\{1,\dots,n\} = A_+ \sqcup A_-,$$

and define $\lambda_{k,\alpha}$ by $\lambda_{k,\alpha} = \rho_k$ if $\alpha \in A_+$, and $\lambda_{k,\alpha} = \rho_{k-1}e^{\sqrt{-1}(n+1)\pi/n}$ if $\alpha \in A_-$. Then $\{\lambda_{k,\alpha}\}$ satisfies (6) for $\theta_0 = \frac{n+2}{2n}\pi$. Here, we suppose $a_k - 1$ are sufficiently small so that Proposition 6.6 holds.

Proposition 6.7. *Let $\square_1,\dots,\square_{2n}$ be as above. Then $\square_k \cap \square_l$ is empty if $d_{2n}(k,l) > 1$. Moreover, $\square_k$ is a $\sigma(\theta_0 + \frac{n+1}{n}k\pi)$-Delzant polytope.*

Proof. It suffices to show $\pi_\alpha(\square_k) \cap \pi_\alpha(\square_l)$ is empty for some α by Lemma 6.5 (1). If $\alpha \in A_+$, then $\pi_\alpha(\square_k) = \mathbf{l}_k$, and if $\alpha \in A_-$, then $\pi_\alpha(\square_k)$ is equal to $e^{\sqrt{-1}(n+1)\pi/n}\mathbf{l}_{k-1}$. Therefore $\pi_\alpha(\square_k) \cap \pi_\alpha(\square_l)$ is empty for some $\alpha \in A_+$ if k is odd, and $\pi_\alpha(\square_k) \cap \pi_\alpha(\square_l)$ is empty for some $\alpha \in A_-$ if k is even by Proposition 6.6.

Let $H_{k,\alpha} = \{y \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*;\ y_\alpha + \lambda_{k,\alpha} = 0\}$. Then $\square_k$ is a convex set defined as the intersection of half spaces in $V(q_k, \sigma(\theta_0 + \frac{n+1}{n}k\pi))$ induced by $H_{k,1},\dots,H_{k,n},H_{k+1,1},\dots,H_{k+1,n}$. Consequently, it suffices to show that $\square_k \cap H_{l,\alpha}$ is empty for $l \neq k, k+1$ and all α. Applying Lemma 6.5 (2), it suffices to see that $-\lambda_{l,\alpha} \notin \pi_\alpha(\square_k)$. Suppose $-\lambda_{l,\alpha} \in \pi_\alpha(\square_k)$. Since one can see that $-\lambda_{l,\alpha}$ is contained in both of $\pi_\alpha(\square_l)$ and $\pi_\alpha(\square_{l-1})$, then $\pi_\alpha(\square_l) \cap \pi_\alpha(\square_k)$ and $\pi_\alpha(\square_{l-1}) \cap \pi_\alpha(\square_k)$ are nonempty, which implies that $\mathbf{l}_l \cap \mathbf{l}_k$ and $\mathbf{l}_{l-1} \cap \mathbf{l}_k$ are nonempty. Since either $d_{2n}(k,l-1) > 1$ or $d_{2n}(k,l) > 1$ holds, and either l or $l-1$ is odd, it contradicts to Proposition 6.6. □

Since $L_{\square_k} = (\mathbb{P}^1)^n$, and there is an orientation preserving diffeomorphism between $(\mathbb{P}^1)^n$ and $(\overline{\mathbb{P}^1})^n$, we obtain the following example.

Theorem 6.8. *Let $X(u,\lambda)$ be as above. Then there exists a 1-parameter family of compact smooth special Lagrangian submanifolds $\{\tilde{L}_t\}_{0<t<\delta}$ embedded in $X(u,\lambda)$, all of which are diffeomorphic to*

$$2n(\mathbb{P}^1)^n \# (S^1 \times S^{2n-1}),$$

and converges to $\bigcup_{k=1}^{2n} L_{\square_k}$ as $t \to 0$ in the sense of currents.

6.2. Example (2)

Here we construct one more example in an 8 dimensional toric hyper-Kähler manifolds.

Let

$$u := \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \in \mathrm{Hom}(\mathbb{Z}^5, \mathbb{Z}^2),$$

and $\lambda = (\lambda_0, \dots, \lambda_4) \in \mathbb{C} \otimes (\mathbf{t}^5)^*$. Put

$$q_1 := -(\lambda_1, \lambda_2),\ q_2 := -(\lambda_3, \lambda_2),\ q_3 := -(\lambda_3, \lambda_4),\ q_4 := -(\lambda_1, \lambda_4)$$

and $\triangle_k := q_k + \tau_k \otimes \triangle$ for $k = 1, \dots, 4$, where

$$\begin{aligned} \tau_1 &:= \lambda_1 + \lambda_2 - \lambda_0, \\ \tau_2 &:= \lambda_3 + \lambda_2 - \lambda_0, \\ \tau_3 &:= \lambda_3 + \lambda_4 - \lambda_0, \\ \tau_4 &:= \lambda_1 + \lambda_4 - \lambda_0, \end{aligned}$$

and

$$\triangle := \{(t_1, t_2) \in (\mathbf{t}^2)^* \cong \mathbb{R}^2;\ t_1 \geq 0,\ t_2 \geq 0,\ t_1 + t_2 \leq 1\}.$$

Here, we take $r_1, r_2 > 0$ and put

$$\lambda_0 = \lambda_1 = 0, \quad \lambda_2 = \sqrt{-1}r_1, \quad \lambda_3 = -r_2 - \sqrt{-1}r_1, \quad \lambda_4 = r_2,$$

then we have $\tau_1 = \sqrt{-1}r_1$, $\tau_2 = -r_2$, $\tau_3 = -\sqrt{-1}r_1$, $\tau_4 = r_2$. Since $H_1 \cap H_3$ and $H_2 \cap H_4$ are empty, $X(u, \lambda)$ becomes smooth. We also have $\triangle_k \subset V(q_k, \sigma(\frac{\pi}{2}k))$

Proposition 6.9. *Under the above setting, $\angle \triangle_k \triangle_{k+1} = \pi/2$ for every $k \equiv 1, \dots, 4$ mod 4.*

Proof. We check the case of $k = 1$, because other cases can be shown similarly. Let $q_k + \tau_k \otimes (t_1, t_2) \in \triangle_k$. Then we have

$$\begin{aligned} q_1 + \tau_1 \otimes (t_1, t_2) &= q_1 + \tau_1 \otimes (1, 0) + \tau_1 \otimes (t_1 - 1, t_2) \\ &= (\lambda_2 - \lambda_0, -\lambda_2) + \sigma(\frac{\pi}{2}) \otimes r_1(t_1 - 1, t_2), \\ q_2 + \tau_2 \otimes (t_1, t_2) &= q_2 + \tau_2 \otimes (1, 0) + \tau_2 \otimes (t_1 - 1, t_2) \\ &= (\lambda_2 - \lambda_0, -\lambda_2) + \sigma(\pi) \otimes r_2(t_1 - 1, t_2), \end{aligned}$$

therefore $\angle \triangle_1 \triangle_2 = \pi/2$. Note that we have to take $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$ in Definition 6.2, since $t_1 - 1$ is nonpositive in this case. $\square$

Proposition 6.10. *Under the above setting, $\triangle_1 \cap \triangle_3$ and $\triangle_2 \cap \triangle_4$ are empty.*

Proof. Since we have

$$\pi_1(\triangle_1) = \{\sqrt{-1}r_1 t;\ 0 \le t \le 1\} \subset \sqrt{-1}\mathbb{R},$$
$$\pi_1(\triangle_3) = \{r_2 + \sqrt{-1}r_1 - t\sqrt{-1}r_1;\ 0 \le t \le 1\} \subset r_2 + \sqrt{-1}\mathbb{R},$$

$\triangle_1 \cap \triangle_3$ is empty by Lemma 6.5. Similarly, $\triangle_2 \cap \triangle_4 = \emptyset$ also holds since

$$\pi_1(\triangle_2) = \{r_2 + \sqrt{-1}r_1 - r_2 t;\ 0 \le t \le 1\} \subset \sqrt{-1}r_1 + \mathbb{R},$$
$$\pi_1(\triangle_4) = \{r_2 t;\ 0 \le t \le 1\} \subset \mathbb{R}.$$

□

Proposition 6.11. *$\triangle_k$ is a $\sigma(\frac{k\pi}{2})$-Delzant polytope.*

Proof. We show the case of $k = 1$, and the other cases are shown similarly. One can check that $\triangle_1$ is the intersection of half spaces in $V(q_1, \frac{k\pi}{2})$ induced by H_0, H_1, H_2. Then it suffices to show that $\triangle_1 \cap H_k$ is empty where $k = 3, 4$. By Lemma 6.5, it is reduced to show $-\lambda_3 \notin \pi_1(\triangle_1)$ and $-\lambda_4 \notin \pi_2(\triangle_1)$. Now we have $-\lambda_3 = r_2 + \sqrt{-1}r_1$ and $\pi_1(\triangle_1) \subset \sqrt{-1}\mathbb{R}$, hence $-\lambda_3 \notin \pi_1(\triangle_1)$ holds. Since $-\lambda_4 = -r_2$ and $\pi_2(\triangle_1) \subset \sqrt{-1}\mathbb{R}$, we have $-\lambda_4 \notin \pi_2(\triangle_1)$. □

By the above arguments, we obtain the following example.

Theorem 6.12. *Let $X(u,\lambda)$ be as above. Then there exists a 1-parameter family of compact smooth special Lagrangian submanifolds $\{\tilde{L}_t\}_{0<t<\delta}$ embedded in $X(u,\lambda)$, all of which are diffeomorphic to*

$$2\mathbb{P}^2 \# 2\overline{\mathbb{P}^2} \# (S^1 \times S^3),$$

and converges to $\bigcup_{k=1}^4 L_{\triangle_k}$ as $t \to 0$ in the sense of currents.

6.3. Example (3)

We can describe a generalization of Theorem 6.4 in the more complicated situation.

Theorem 6.13. *Let $(\mathcal{V}, \mathcal{E}, s, t)$ be a quiver, $X(u,\lambda)$ be a smooth toric hyper-Kähler manifold, and $\{\triangle_k\}_{k\in\mathcal{V}}$ be a family of subsets of* $\mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^n)^*$. *We assume that every $\triangle_k$ is a $\sigma(\theta_k)$-Delzant polytope for some $\theta_k \in \mathbb{R}$, and* *assume* $\angle\triangle_{s(h)}\triangle_{t(h)} = \pi/n$ *if* $h \in \mathcal{E}$*, otherwise* $\triangle_{k_1} \cap \triangle_{k_2} = \emptyset$ *or* $k_1 = k_2$*. Moreover, suppose that* $\mathcal{E}$ *is covered by cycles. Then there exists a family of compact special Lagrangian submanifolds* $\{\tilde{L}_t\}_{0<t<\delta}$ *which converges to* $\bigcup_{k\in\mathcal{V}} L_{\triangle_k}$ *in the sense of currents.*

Proof. The proof is same as that of Theorem 6.4. □

Fix positive real numbers a, b, c, a_m for $m = 1, \dots, N$ so that $0 < a_1 < a_2 < \cdots < a_N$. Let

$$u = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathrm{Hom}(\mathbb{Z}^{2N+6}, \mathbb{Z}^2)$$

and

$$\lambda = (\lambda_{-3}, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_{2N+2}) \in \mathbb{C} \otimes (\mathbf{t}^{2N+6})^*,$$

where $-\lambda_0 = 0$, $-\lambda_{-1} = \sqrt{-1}b$, $-\lambda_{-2} = a + \sqrt{-1}b$, $-\lambda_{-3} = a$, $-\lambda_{2m+1} = a_m + \sqrt{-1}c$ and $-\lambda_{2m+2} = a_m$ for $m = 0, 1, \dots, N$. Here, we put $a_0 = 0$. Then $X(u, \lambda)$ is smooth and becomes the direct product $X(u', \lambda') \times X(u'', \lambda'')$ where $u' = (1, 1, 1, 1) \in \mathrm{Hom}(\mathbb{Z}^4, \mathbb{Z})$, $u'' = (1, \dots, 1) \in \mathrm{Hom}(\mathbb{Z}^{2N+2}, \mathbb{Z})$, $\lambda' = (\lambda_{-3}, \lambda_{-2}, \lambda_{-1}, \lambda_0)$ and $\lambda'' = (\lambda_1, \dots, \lambda_{2N+2})$. Denote by $[p, q] \subset \mathbb{C}$ the segment connecting $p, q \in \mathbb{C}$, and put $\mathbf{A}_- := [-\lambda_0, -\lambda_{-1}]$, $\mathbf{A}_+ := [-\lambda_{-2}, -\lambda_{-3}]$, $\mathbf{B}_+ := [-\lambda_{-1}, -\lambda_{-2}]$, $\mathbf{B}_- := [-\lambda_{-3}, -\lambda_0]$, $\mathbf{A}_m := [-\lambda_{2m+1}, -\lambda_{2m+2}]$ for all m, $\mathbf{B}_{+,m} := [-\lambda_{2m-1}, -\lambda_{2m+1}]$ and $\mathbf{B}_{-,m} := [-\lambda_{2m}, -\lambda_{2m+2}]$ for all $m \geq 1$.

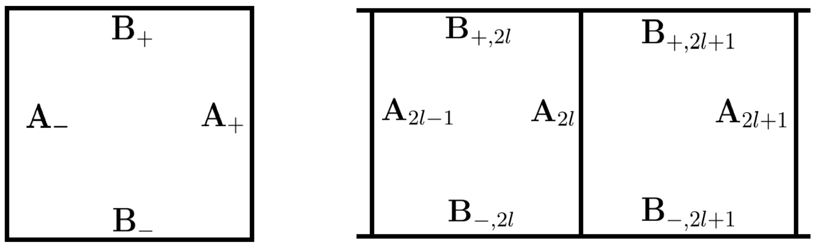
Let

$$\begin{aligned} \square_{2l,1} &:= \mathbf{A}_- \times \mathbf{A}_{2l}, \\ \square_{2l,2} &:= \mathbf{B}_+ \times \mathbf{B}_{+,2l+1}, \\ \square_{2l,3} &:= \mathbf{A}_+ \times \mathbf{A}_{2l+1}, \\ \square_{2l,4} &:= \mathbf{B}_- \times \mathbf{B}_{-,2l+1} \end{aligned}$$

for $l = 0, 1, \dots, [(N-1)/2]$, and

$$\begin{aligned} \square_{2l-1,1} &:= \mathbf{A}_- \times \mathbf{A}_{2l}, \\ \square_{2l-1,2} &:= \mathbf{B}_+ \times \mathbf{B}_{-,2l}, \\ \square_{2l-1,3} &:= \mathbf{A}_+ \times \mathbf{A}_{2l-1}, \\ \square_{2l-1,4} &:= \mathbf{B}_- \times \mathbf{B}_{+,2l} \end{aligned}$$

for $l = 1, \dots, [N/2]$.



Then $\square_{m,j}$ is a $\sigma(j\pi/2)$-holomorphic Lagrangian submanifold and we have $\square_{2l-1,1} = \square_{2l,1}$ and $\square_{2l,3} = \square_{2l+1,3}$.

Lemma 6.14. *We have* $\angle\square_{m,j}\square_{m,j+1} = \frac{\pi}{2}$ *for* $j = 1, 2, 3, 4$*, where we put* $\square_{m,5} = \square_{m,1}$.

Proof. Let $m = 2l$. Then $\square_{2l,1} \cap \square_{2l,2} = \{(-\lambda_{-1}, -\lambda_{4l+1})\}$ holds and we have

$$\square_{2l,1} = (-\lambda_{-1}, -\lambda_{4l+1}) - \sqrt{-1} \otimes \{(r_1, r_2);\ r_1 > 0, r_2 > 0\}$$
$$\square_{2l,2} = (-\lambda_{-1}, -\lambda_{4l+1}) + 1 \otimes \{(r_1, r_2);\ r_1 > 0, r_2 > 0\}.$$

Since $-\sqrt{-1} = \sigma(-\frac{\pi}{2})$ and $1 = \sigma(0)$, we obtain $\angle\square_{m,j}\square_{m,j+1} = \frac{\pi}{2}$. The other cases can be shown in the same way. $\square$

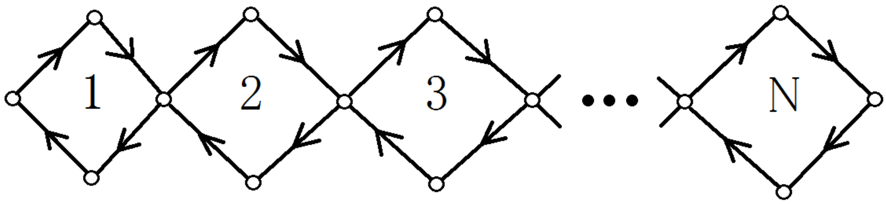
Now let

$$\mathcal{V} := (\{1, \dots, N\} \times \{1, 2, 3, 4\})/\sim,$$

where $\sim$ is defined by $(2l-1, 1) \sim (2l, 1)$ and $(2l, 3) \sim (2l+1, 3)$. We denote by $[m, j] \in \mathcal{V}$ the equivalence class represented by (m, j). Since we have $\square_{m,j} = \square_{m',j'}$ iff $[m, j] = [m', j']$, we put $\square_{[m,j]} := \square_{m,j}$. Put

$$\mathcal{E} := \{[m, j] \to [m, j+1], [m, 4] \to [m, 1];\ m = 1, \dots, N,\ j = 1, 2, 3\},$$

where $x \to y$ means the directed edge whose source is x and the target is y. Then we obtain a quiver $(\mathcal{V}, \mathcal{E}, s, t)$ like the following picture. Lemma 6.14 implies that $\angle\square_{[m,j]}\square_{[m',j']} = \frac{\pi}{2}$ holds if $[m, j] \to [m', j']$.



Lemma 6.15. *For $\square_{[m,j]}$ and $\square_{[m',j']}$, one of the following holds.*

(1) $[m,j] = [m',j']$.
(2) $[m,j] \to [m',j'] \in \mathcal{E}$ *or* $[m',j'] \to [m,j] \in \mathcal{E}$.
(3) $\square_{[m,j]} \cap \square_{[m',j']}$ *is empty.*

Proof. First of all, let $m = m'$. If (1)(2) do not hold, then $(j,j') = (1,3)$, $(3,1)$, $(2,4)$ or $(4,2)$. Since $\pi_1(\square_{[m,1]}) = \mathbf{A}_-$, $\pi_1(\square_{[m,3]}) = \mathbf{A}_+$ and $\mathbf{A}_+ \cap \mathbf{A}_-$ is empty, then Lemma 6.5 gives $\square_{[m,1]} \cap \square_{[m,3]} = \emptyset$. Similarly one can see $\square_{[m,2]} \cap \square_{[m,4]} = \emptyset$. If $|m - m'| \geq 2$, then $\pi_2(\square_{[m,j]}) \cap \pi_2(\square_{[m',j']})$ is empty for any (m,j) and (m',j'), hence $\square_{[m,j]} \cap \square_{[m',j']}$ is empty. Next we consider the intersection of $\square_{[2l-1,j]}$ and $\square_{[2l,j']}$. Since $\square_{[2l-1,1]} = \square_{[2l,1]}$, it suffices to consider the case of $j, j' = 2,3,4$. By seeing the image of π_1, one can see

$$\square_{[2l-1,2]} \cap \square_{[2l,4]} = \square_{[2l-1,4]} \cap \square_{[2l,2]} = \emptyset.$$

By seeing the image of π_2, one can see

$$\square_{[2l-1,3]} \cap \square_{[2l,j']},\ \square_{[2l-1,j]} \cap \square_{[2l,3]},\ \square_{[2l-1,2]} \cap \square_{[2l,2]},\ \square_{[2l-1,4]} \cap \square_{[2l,4]}$$

are empty for any l, j and j'. The case of $\square_{[2l,j]}$ and $\square_{[2l+1,j']}$ can be shown in the same way. $\square$

Now,

$$\{[m,1] \to [m,2], [m,2] \to [m,3], [m,3] \to [m,4], [m,4] \to [m,1]\}$$

are cycles of $\mathcal{E}$ for all m, therefore $\mathcal{E}$ is covered by cycles. Moreover, the above cycles generate $\mathrm{Ker}(\partial)$, hence $\dim \mathrm{Ker}(\partial) = N$.

By Lemmas 6.14 and 6.15, one can see that $\{\square_{[m,j]}\}_{[m,j]\in\mathcal{V}}$ satisfies the assumption of Theorem 6.13, and obtain the following result.

Theorem 6.16. *Let $X(u,\lambda)$ be as above. Then there exists an N-parameter family of compact smooth special Lagrangian submanifolds* $\{\tilde{L}_{(t_1,\ldots,t_N)}\}_{0<t_i<\delta}$

embedded in $X(u,\lambda)$*, all of which are diffeomorphic to*

$$(3N+1)(\mathbb{P}^1)^2 \# N(S^1 \times S^3),$$

and converges to $\bigcup_{[m,j]\in\mathcal{V}} L_{\square_{[m,j]}}$ *as* $(t_1,\dots,t_N) \to 0$ *in the sense of currents.*

7. Obstruction

Here we introduce obstructions for the existence of holomorphic Lagrangian and special Lagrangian submanifolds in hyper-Kähler manifolds. Throughout this section, let $(M^{4n}, g, I_1, I_2, I_3)$ be a hyper-Kähler manifold.

Proposition 7.1. *Let* $L \subset M$ *be a middle dimensional submanifold. If* L *is a special Lagrangian submanifold and a* σ*-holomorphic Lagrangian submanifold for some* $\sigma \in S^2$*, then* $\sigma = \sigma(k\pi/n)$ *for even* k*. If* $\overline{L}$ *is a special Lagrangian submanifold and* L *is a* σ*-holomorphic Lagrangian submanifold for some* $\sigma \in S^2$*, then* $\sigma = \sigma(k\pi/n)$ *for odd* k*.*

Proof. Suppose L or $\overline{L}$ is a special Lagrangian submanifold and L is a σ-holomorphic Lagrangian submanifold. By decomposing $\mathbb{R}^3$ into $\mathbb{R}\sigma$ and its orthogonal complement, we have

$$(1,0,0) = p\sigma + q\tau$$

for some $p, q \in \mathbb{R}$ and $\tau \in S^2$, where τ is orthogonal to σ. Then we have $\omega_1 = p\omega^\sigma + q\omega^\tau$ and

$$0 = \omega_1|_L = p\omega^\sigma|_L + q\omega^\tau|_L = p\omega^\sigma|_L,$$

since L is Lagrangian and σ-holomorphic Lagrangian submanifold. Hence p should be 0 since ω^σ is non-degenerate on L, which means that σ is orthogonal to $(1,0,0)$. Then we may write $\sigma = \sigma(\theta)$ for some $\theta \in \mathbb{R}$. By the condition $\mathrm{Im}(\omega_2 + \sqrt{-1}\omega_3)^n|_L = 0$, we obtain $\theta = k\pi/n$ for some $k = 1, \dots, 2n$. By considering the orientation, we have the assertion. □

Proposition 7.2. *Let* L *be a compact* $\sigma(\theta)$*-holomorphic Lagrangian submanifold in* M *for some* θ*. Then the pairing of the de Rham cohomology class* $[\omega_2 + \sqrt{-1}\omega_3]^n$ *and the homology class* $[L] \in H_{2n}(M,\mathbb{Z})$ *is given by*

$$\langle [\omega_2 + \sqrt{-1}\omega_3]^n, [L] \rangle = e^{\sqrt{-1}n\theta} V,$$

where $V(L) > 0$ *is the volume of* L*.*

Proof. Since L is $\sigma(\theta)$-holomorphic Lagrangian, we have

$$\omega_1|_L = \omega^{\hat{\sigma}(\theta)}|_L = 0,$$

where $\hat{\sigma}(\theta) = (0, -\sin\theta, \cos\theta) \in S^2$. Then we obtain

$$\begin{aligned}\langle[\omega_2 + \sqrt{-1}\omega_3]^n, [L]\rangle &= \int_L (\omega_2 + \sqrt{-1}\omega_3)^n \\ &= \int_L e^{\sqrt{-1}n\theta}\{e^{-\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)\}^n.\end{aligned}$$

Here, we have

$$\begin{aligned}e^{-\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3) &= \cos\theta\omega_2 + \sin\theta\omega_3 + \sqrt{-1}(-\sin\theta\omega_2 + \cos\theta\omega_3) \\ &= \omega^{\sigma(\theta)} + \sqrt{-1}\omega^{\hat{\sigma}(\theta)},\end{aligned}$$

therefore we obtain

$$\begin{aligned}\langle[\omega_2 + \sqrt{-1}\omega_3]^n, [L]\rangle &= e^{\sqrt{-1}n\theta}\int_L (\omega^{\sigma(\theta)} + \sqrt{-1}\omega^{\hat{\sigma}(\theta)})^n \\ &= e^{\sqrt{-1}n\theta}\int_L (\omega^{\sigma(\theta)})^n = e^{\sqrt{-1}n\theta}V(L).\end{aligned}$$

□

Proposition 7.3. *Let $L_1, \dots, L_A$ be compact smooth submanifolds embedded in M, and $\theta \in \mathbb{R}$. Let L_α be a $\sigma(\frac{k_\alpha\pi}{n})$-holomorphic Lagrangian submanifold for $k_\alpha \in \mathbb{Z}$. If the homology class of a compact smooth $\sigma(\theta)$-holomorphic Lagrangian submanifold L is $\sum_{\alpha=1}^A (-1)^{k_\alpha}[L_\alpha]$ or $\sum_{\alpha=1}^A (-1)^{k_\alpha+1}[L_\alpha]$, then $\{\frac{k_1\pi}{n}, \frac{k_2\pi}{n}, \dots, \frac{k_A\pi}{n}\}$ is contained in $\theta + \pi\mathbb{Z}$.*

Proof. Put $\theta_\alpha = \frac{k_\alpha\pi}{n}$. Since L_α is a $\sigma(\theta_\alpha)$-holomorphic Lagrangian submanifold, Proposition 7.2 gives

$$\begin{aligned}\left\langle [\omega_2 + \sqrt{-1}\omega_3]^n, \sum_{\alpha=1}^A (-1)^{k_\alpha}[L_\alpha] \right\rangle &= \sum_{\alpha=1}^A (-1)^{k_\alpha} e^{\sqrt{-1}n\theta_\alpha} V(L_\alpha) \\ &= \sum_{\alpha=1}^A V(L_\alpha). \qquad (8)\end{aligned}$$

Since L is a compact smooth $\sigma(\theta)$-holomorphic Lagrangian submanifold, $\omega_1|_L = \omega^{\hat\sigma(\theta)}|_L = 0$ holds, where we put $\hat\sigma(\theta)$ as in the proof of Proposition 7.2. Therefore we obtain

$$\langle [\omega_2 + \sqrt{-1}\omega_3]^n, [L]\rangle = e^{\sqrt{-1}n\theta} V(L) = e^{\sqrt{-1}n\theta} \langle [\omega^{\sigma(\theta)}]^n, [L]\rangle \tag{9}$$

by Proposition 7.2. Then by combining (8)(9) and the assumption $[L] = \pm \sum_{\alpha=1}^{A} (-1)^{k_\alpha} [L_\alpha]$, θ is given by $\theta = k\pi/n$ for an integer $k = 1, \dots, 2n$. Note that $[L] = (-1)^k \sum_{\alpha=1}^{A} (-1)^{k_\alpha} [L_\alpha]$ holds. Now we have

$$\begin{aligned} \omega^{\sigma(\theta)} &= \mathrm{Re}(e^{-\sqrt{-1}\theta}(\omega_2 + \sqrt{-1}\omega_3)) \\ &= \mathrm{Re}(e^{-\sqrt{-1}(\theta-\theta_\alpha)} e^{-\sqrt{-1}\theta_\alpha}(\omega_2 + \sqrt{-1}\omega_3)) \\ &= \mathrm{Re}(e^{-\sqrt{-1}(\theta-\theta_\alpha)}(\omega^{\sigma(\theta_\alpha)} + \sqrt{-1}\omega^{\hat\sigma(\theta_\alpha)})) \\ &= \cos(\theta-\theta_\alpha)\omega^{\sigma(\theta_\alpha)} + \sin(\theta-\theta_\alpha)\omega^{\hat\sigma(\theta_\alpha)} \end{aligned}$$

and $\omega^{\hat\sigma(\theta_\alpha)}|_{L_\alpha} = 0$, we obtain

$$\begin{aligned} \langle [\omega^{\sigma(\theta)}]^n, [L]\rangle &= (-1)^k \sum_{\alpha=1}^{A} (-1)^{k_\alpha} \langle [\omega^{\sigma(\theta)}]^n, [L_\alpha]\rangle \\ &= (-1)^k \sum_{\alpha=1}^{A} (-1)^{k_\alpha} \cos^n(\theta-\theta_\alpha) \langle [\omega^{\sigma(\theta_\alpha)}]^n, [L_\alpha]\rangle \\ &= (-1)^k \sum_{\alpha=1}^{A} (-1)^{k_\alpha} \cos^n(\theta-\theta_\alpha) V(L_\alpha) \qquad (10) \end{aligned}$$

By combining (8)(9)(10) and putting $\theta = k\pi/n$, we obtain

$$\sum_{\alpha=1}^{A} V(L_\alpha) = (-1)^k \sum_{\alpha=1}^{A} (-1)^{k_\alpha} \cos^n\left(\frac{k\pi}{n} - \theta_\alpha\right) V(L_\alpha).$$

By substituting $\theta_\alpha = k_\alpha\pi/n$, we have

$$\sum_{\alpha=1}^{A} V(L_\alpha) = \sum_{\alpha=1}^{A} (-1)^{k-k_\alpha} \cos^n\left(\frac{k-k_\alpha}{n}\pi\right) V(L_\alpha).$$

Since every $V(L_\alpha)$ is positive, we obtain

$$(-1)^{k-k_\alpha} \cos^n\left(\frac{k-k_\alpha}{n}\pi\right) = 1, \tag{11}$$

then $k - k_\alpha$ should be contained in $n\mathbb{Z}$. If $k - k_\alpha = nl$ for some $l \in \mathbb{Z}$, then $\cos(\frac{k-k_\alpha}{n}\pi) = \cos l\pi = (-1)^l$ holds, which gives

$$(-1)^{k-k_\alpha} \cos^n \left(\frac{k-k_\alpha}{n}\pi\right) = (-1)^{nl}(-1)^{nl} = 1.$$

Thus the assertion follows since (11) holds if and only if $k - k_\alpha \in n\mathbb{Z}$. □

Corollary 7.4. *The special Lagrangian submanifolds $\tilde{L}_t$ obtained in Theorems 6.8, 6.12 and 6.16 are not σ-holomorphic Lagrangian submanifolds for any $\sigma \in S^2$, and $\tilde{L}_t$ with the opposite orientations are not σ-holomorphic Lagrangian submanifolds for any $\sigma \in S^2$.*

Proof. Note that all of special Lagrangian submanifolds $\tilde{L}_t$ obtained in Theorems 6.8, 6.12 and 6.16 are given by applying Theorem 6.13. Accordingly, it suffices to show the assertion for $\tilde{L}_t$ obtained in Theorem 6.13. We apply Proposition 7.3 to $\tilde{L}_t$ with its standard orientation and with the other. Take $k_1 \to k_2 \in \mathcal{E}$ arbitrarily. Then $\angle\triangle_{k_1}\triangle_{k_2} = \pi/n$, hence we have $\theta_{k_2} = \theta_{k_1} + \pi/n$, which implies that $\{\theta_k;\ k \in \mathcal{V}\}$ contains θ_{k_1} and $\theta_{k_1} + \pi/n$. Thus $\{\theta_k;\ k \in \mathcal{V}\}$ is never contained in $\theta + \pi\mathbb{Z}$ for any θ since $n > 1$. By Propositions 7.1 and 7.3, $\tilde{L}_t$ never becomes σ-holomorphic Lagrangian submanifold for any $\sigma \in S^2$. The case of opposite orientation is shown in the same way. □

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