

Local middle dimensional symplectic non-squeezing in the analytic setting

LORENZO RIGOLLI

We prove the following middle dimensional non-squeezing result for analytic symplectic embeddings of domains in \mathbb{R}^{2n} .

Let $\varphi : D \hookrightarrow \mathbb{R}^{2n}$ be an analytic symplectic embedding of a domain $D \subset \mathbb{R}^{2n}$ and P be a symplectic projector onto a linear $2k$ -dimensional symplectic subspace $V \subset \mathbb{R}^{2n}$. Then there exists a positive function $r_0 : D \rightarrow (0, +\infty)$, bounded away from 0 on compact subsets $K \subset D$, such that the inequality $\text{Vol}_{2k}(P\varphi(B_r(x)), \omega_{0|V}^k) \geq \pi^k r^{2k}$ holds for every $x \in D$ and for every $r < r_0(x)$. This claim will be deduced from an analytic middle dimensional non-squeezing result (stated by considering paths of symplectic embeddings) whose proof will be carried on by taking advantage of a work by Álvarez Paiva and Balacheff.

Introduction

Let $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ be the standard symplectic form on \mathbb{R}^{2n} , B_R the ball of radius R

$$B_R = \left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 < R^2 \right\},$$

and Z_r the cylinder

$$Z_r = \{(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}.$$

The Gromov's non-squeezing theorem (see [Gro85] or [HZ94]) states that if $\varphi(B_R) \subset Z_r$, where φ is a symplectic (open) embedding, then $r \geq R$. Symplectic diffeomorphisms are volume preserving, due to the fact that they preserve the multiple of the volume form ω_0^n , but the non-squeezing theorem shows that, unlike volume preserving diffeomorphisms, they also present two-dimensional rigidity phenomena.

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Since symplectic diffeomorphisms preserve the $2k$ -form ω_0^k for every integer $1 \leq k \leq n$, after Gromov's pioneering result one may ask if there are also middle dimensional rigidity phenomena.

Some work in this direction, concerning symplectic embeddings of polydisks, has been done by Guth. In [Gut08] he considers symplectic embeddings of a polydisk $\Gamma := B_2(R_1) \times \cdots \times B_2(R_n)$ with $R_1 \leq \cdots \leq R_n$ into a polydisk $\Gamma' := B_2(R'_1) \times \cdots \times B_2(R'_n)$ with $R'_1 \leq \cdots \leq R'_n$. By the Gromov's non-squeezing theorem and the conservation of the volume under symplectic diffeomorphism, such symplectic embeddings may exist only if $R_1 \leq R'_1$ and $R_1 \cdots R_n \leq R'_1 \cdots R'_n$. On the other hand Guth proved that such embeddings exist if and only if for a certain constant $C(n)$, the inequalities $C(n)R_1 \leq R'_1$ and $C(n)R_1 \cdots R_n \leq R'_1 \cdots R'_n$ hold. In particular this result exclude every middle dimensional non-squeezing phenomena in the case of polydisks embeddings.

In this paper we proceed in a different way, namely we keep the ball as domain of symplectic embeddings and in order to search for middle dimensional non-squeezing phenomena we follow the strategy pursued in [AM13].

First, as in [EG91], we introduce an alternative formulation of Gromov's theorem, which is that the two-dimensional shadow of the image of a radius R ball in \mathbb{R}^{2n} under a symplectic diffeomorphism has area at least πR^2 . More precisely the claim is that every symplectic embedding $\varphi : B_R \hookrightarrow \mathbb{R}^{2n}$ satisfies the inequality

$$(1) \quad \text{area}(P\varphi(B_R), \omega_{0|V}) \geq \pi R^2,$$

where P denotes the symplectic projector onto a symplectic plane V , i.e. the projector along the symplectic orthogonal complement of V .

This second formulation easily implies the classic one. On the other hand, if it were $\text{area}(P\varphi(B_R), \omega_{0|V}) < \pi R^2$, then, by a theorem of Moser's (see [Mos65] or [HZ94]), there would exist a smooth area preserving diffeomorphism ϕ mapping $P\varphi(B_R)$ into a subset of $B_r^2 \cap V$ for some $r < R$, and then the symplectic embedding $(\phi \times id_{V^\perp}) \circ \varphi$ mapping B_R into Z_r would violate the classic formulation of Gromov's theorem.

The alternative formulation of Gromov's theorem has a natural generalization to higher dimensional shadows of a symplectic ball.

In other words, if V is a $2k$ -dimensional symplectic subspace of \mathbb{R}^{2n} and P is the symplectic projector onto V , we may ask whether it is true that

$$(2) \quad \text{Vol}_{2k}(P\varphi(B_R), \omega_{0|V}^k) \geq \pi^k R^{2k},$$

for every symplectic embedding $\varphi : B_R \hookrightarrow \mathbb{R}^{2n}$.

Notice that in the inequality above and throughout all the paper we always consider the symplectic volume induced by the volume form $\omega_{0|V}^k$, which is $k!$ times the standard Euclidean volume.

If $k = 1$ or $k = n$ the inequality holds respectively by the non-squeezing theorem and by the volume preserving property of symplectic diffeomorphisms. So we are interested only in the middle dimensional case when $2 \leq k \leq n - 1$.

If the symplectic diffeomorphism φ is a linear map an affirmative answer to the middle dimensional non-squeezing question has been given in [AM13], nevertheless in the same paper Abbondandolo and Matveyev show that if P is the symplectic projector onto a $2k$ -dimensional symplectic subspace with $2 \leq k \leq n - 1$, then for every $\epsilon > 0$ there exists an open symplectic embedding $\varphi : B_1 \hookrightarrow \mathbb{R}^{2n}$ such that $Vol_{2k}(P\varphi(B_1), \omega_{0|V}^k) < \epsilon$. Since this counterexample deforms very strongly the unitary ball, one may ask how far can the ball be deformed before the middle non-squeezing ends his validity and whether the middle dimensional non-squeezing holds locally. In [AM13] the authors give two different formulations of the local question.

The first one asks whether, fixed a symplectic embedding $\varphi : D \hookrightarrow \mathbb{R}^{2n}$ of a domain $D \subset \mathbb{R}^{2n}$, the inequality

$$(3) \quad Vol_{2k}(P\varphi(B_R(x)), \omega_{0|V}^k) \geq \pi^k R^{2k}$$

holds for any $x \in D$ and for R positive and small enough.

The second formulation is the following.

Let us fix a path of symplectic embeddings of the unit $2n$ -dimensional ball

$$\varphi_t : B_1 \hookrightarrow \mathbb{R}^{2n} \quad t \in [0, 1],$$

such that φ_0 is linear (i.e. it is the restriction to B_1 of a linear symplectomorphism).

We would like to know whether there exists a positive number $t_0 \leq 1$ such that

$$(4) \quad Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) \geq \pi^k, \quad \text{for every } 0 \leq t \leq t_0.$$

The second formulation implies the first one by taking the path of symplectic embeddings

$$(5) \quad \varphi_t(y) := \begin{cases} \frac{1}{t}(\varphi(x + ty) - \varphi(x)) & \text{if } t \in]0, 1], \\ D\varphi(x)[y] & \text{if } t = 0, \end{cases}$$

in fact

$$\text{Vol}_{2k}(P\varphi_t(B_1(0))) = \text{Vol}_{2k}\left(P\frac{1}{t}\varphi(B_t(x)), \omega_{0|V}^k\right) = \frac{\text{Vol}_{2k}(P\varphi(B_t(x)), \omega_{0|V}^k)}{t^{2k}}.$$

In this setting Abbondandolo, Bramham, Hryniewicz and Salomão [ABH+18] have recently proved that if the symplectic projection is onto a 4-dimensional symplectic subspace V , then both these local non-squeezing results hold (in the first formulation the diffeomorphism is required to be C^3).

In this paper we address the same question but we do not impose for any assumption on the dimension of V , instead we require an analyticity hypothesis. First we focus on the second local formulation of the middle dimensional non-squeezing and we prove its validity under the additional assumption that the path of embeddings $t \mapsto \varphi_t$ is analytic in t , i.e. $\forall x \in \overline{B_1}$ the function $t \mapsto \varphi_t(x)$ is analytic.

Theorem 1 (Analytic local non-squeezing). *Let $[0, 1] \ni t \mapsto \varphi_t$ be an analytic path of symplectic embeddings $\varphi_t : \overline{B_1} \hookrightarrow \mathbb{R}^{2n}$, such that φ_0 is linear. Then the middle dimensional non-squeezing inequality*

$$\text{Vol}_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) \geq \pi^k$$

holds for t small enough.

To prove this theorem we need some ingredients.

In Section 1 we recall some facts about contact geometry together with some results about the minimal action of a Reeb orbit in a contact manifold and we introduce Zoll contact manifolds (also known as *regular contact type manifolds*), namely manifolds with the property that all Reeb orbits are periodic with the same period.

In Section 2 we prove a weaker version of the main theorem in [AB14], which says that if a constant volume deformation of the unit ball does not start tangent to all orders to a deformation by convex domains with Zoll boundaries (i.e. the deformation is *not formally trivial*), then the minimal action A_{\min} on the ball is strictly larger than the one of its small deformations.

In Section 4 we will see that this result implies the validity of the non-squeezing inequality (4) for not formally trivial deformations of the unit ball.

On the other hand in case of a *formally trivial deformation* we will have to proceed in a different way. Namely, using some results from Section 3, we will prove that if the deformation of the ball is analytic and trivial then the function $t \mapsto \text{Vol}(P\varphi_t(B_1), \omega_{0|V}^k)$ is analytic and has all vanishing derivatives in $t = 0$. This will enable us to deduce that the function above is constant and consequently that the equality in (4) holds.

Using Theorem 1 we will deduce the local non-squeezing formulation for any fixed analytic symplectic embedding. Moreover, in this latter setting we will prove that, on compact subsets of \mathbb{R}^{2n} , the minimal radius R for which the estimate (3) holds is bounded away from 0. More precisely we shall prove the following result.

Theorem 2. *Let $\varphi : D \hookrightarrow \mathbb{R}^{2n}$ be an analytic symplectic embedding of a domain $D \subset \mathbb{R}^{2n}$. Then there exists a function $r_0 : D \rightarrow (0, +\infty)$ such that the inequality $\text{Vol}_{2k}(P\varphi(B_r(x)), \omega_{0|V}^k) \geq \pi^k r^{2k}$ holds, for every $x \in D$ and for every $r < r_0(x)$. Moreover r_0 is bounded away from 0 on compact subsets $K \subset D$.*

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1. Zoll contact manifolds and minimal action

Let us start recalling some basic facts in contact geometry.

A 1-form α on the $(2n - 1)$ -dimensional manifold M is a *contact form* if $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. In this case (M, α) is called *contact manifold* and the volume of M with respect to the volume form induced by α is denoted by $\text{Vol}(M, \alpha)$.

Moreover the contact form α induces a vector field R_α on M , which is called the Reeb vector field of α and is determined by the requirements:

$$i_{R_\alpha} d\alpha = 0 \quad \& \quad \alpha(R_\alpha) = 1.$$

The *action* $A(\gamma)$ of a periodic Reeb orbit γ on a contact manifold (M, α) is defined as

$$A(\gamma) := \int_\gamma \alpha \in \mathbb{R}.$$

and coincides with the period of γ .

Definition 3. Given any contact manifold (M, α) with at least a closed Reeb orbit we define a function as follows

$$A_{\min}(M, \alpha) := \min_{\gamma} \{A(\gamma) \mid \gamma \text{ is a closed Reeb orbit on } (M, \alpha)\}.$$

Both the volume and the function A_{\min} are invariant under strict contactomorphism. Indeed we have the following simple result.

Proposition 4. *Let (M, α) and (N, β) be two $2n - 1$ dimensional contact manifolds and $\phi : M \rightarrow N$ a strict contactomorphism (i.e. $\phi^*\beta = \alpha$). Then*

- 1) *to each closed Reeb orbit γ of (M, α) corresponds a closed Reeb orbit $\phi \circ \gamma$ of (N, β) ,*
- 2) $A(\gamma) = A(\phi \circ \gamma)$,
- 3) $A_{\min}(M, \alpha) = A_{\min}(N, \beta)$,
- 4) $Vol(M, \alpha) = Vol(N, \beta)$.

Another straightforward fact we will use is the following.

Remark 5. Let $\lambda_0 := \sum_{i=1}^n x_i dy_i$ be the standard Liouville 1-form on \mathbb{R}^{2n} , which is a primitive of ω_0 , let $f : S^{2n-1} \rightarrow (0, +\infty)$ be a C^1 function and define the set $M_f := \{f(x)x \mid x \in S^{2n-1}\} \subset \mathbb{R}^{2n}$. Then $(S^{2n-1}, f^2\lambda_{0|S^{2n-1}})$ and $(M_f, \lambda_{0|M_f})$ are strictly contactomorphic.

Indeed the radial projection $\theta : (S^{2n-1}, f^2\lambda_{0|S^{2n-1}}) \rightarrow (M_f, \lambda_{0|M_f})$ defined by $\theta(x) = f(x)x$ is a strict contactomorphism:

$$\begin{aligned} \theta^*(\lambda_{0|M_f})(x)[v] &= \lambda_{0|M_f}(\theta(x))[d\theta(x)[v]] = \lambda_{0|M_f}(f(x)x)[f(x)[v] + df(x)[v]x] \\ &= f(x)\lambda_{0|S^{2n-1}}(x)[f(x)[v]] + f(x)df(x)[v]\lambda_{0|S^{2n-1}}(x)[x] \\ &= f(x)^2\lambda_{0|S^{2n-1}}(x)[v]. \end{aligned}$$

Now we introduce a special type of contact form.

Definition 6. A contact form on a manifold M is *Zoll* (or *regular*) if its Reeb flow is periodic and all the Reeb orbits have the same period, and hence the same action.

For example the contact form $\lambda_{0|S^{2n-1}}$ induced on the unit sphere S^{2n-1} in \mathbb{R}^{2n} by the standard Liouville 1-form is Zoll.

Later on we will consider two different kinds of deformations of a contact form: *formally trivial* and *not formally trivial*.

Definition 7. A smooth deformation α_t , $t \in [0, t_0)$, of a contact form α_0 is *trivial* if there exists a smooth real valued function $r(t)$ and an isotopy ϕ_t such that $\alpha_t = r(t)\phi_t^*\alpha_0$. A deformation α_t is *formally trivial* if for every non negative m there exists a trivial deformation $\alpha_t^{(m)}$ that has order of contact m with α_t at $t = 0$. Otherwise the deformation is *not formally trivial*.

If instead of deformations of contact forms we choose to consider deformations of convex domains, we give the following definition.

Definition 8. Consider a smooth convex domain $C_0 \subset \mathbb{R}^{2n}$ with the standard Liouville 1-form $\lambda_0|_{\partial C_0}$. A smooth deformation C_t of C_0 is *trivial* (resp. *formally trivial*, resp. *not formally trivial*) if $\theta_t^*(\lambda_0|_{\partial C_t})$ is trivial (resp. formally trivial, resp. not formally trivial), where $\theta_t : S^{2n-1} \rightarrow \partial C_t$ is the radial projection.

It is a result by Weinstein that trivial deformations of a Zoll contact form can be characterized in the following way.

Proposition 9. [Wei74] *Let α_t , $t \in [0, t_0)$, be a smooth deformation of a Zoll contact form α_0 . The deformation is trivial if and only if α_t is a Zoll contact form for every $t \in [0, t_0)$.*

In our case it will turn out that every contact deformation can be reduced to a normal form.

Definition 10. Let $\alpha_t = \rho_t\alpha_0$ be a smooth deformation of the Zoll contact form α_0 , where ρ_t is a smooth family of positive functions on M , and let m be a non negative integer. The deformation α_t is in *normal form up to order m* if

$$\alpha_t = (1 + t\mu^{(1)} + \dots + t^m\mu^{(m)} + t^{m+1}r_t)\alpha_0$$

where, for $i = 1, \dots, m$, the functions $\mu^{(i)}$ are integrals of motion for the Reeb flow of α_0 (i.e. they are constant on the orbit of that flow) and r_t is a smooth function on M depending smoothly on the parameter t .

Using a technique known as the *method of Dragt and Finn* (see [DF76] and [Fin86]), which consist of constructing the required isotopy as composition of isotopies $\phi_t^{(m)}$ which are flows of some particular vector fields, Álvarez Paiva and Balacheff proved the following result.

Theorem 11. [AB14] *Let $\alpha_t = \rho_t \alpha_0$ be a smooth deformation of a Zoll contact form α_0 , where ρ_t is a smooth family of real valued functions on M with $\rho_0 = 1$. Given a non negative integer m , there exists a contact isotopy $\phi_t^{(m)}$ such that $\phi_t^{(m)*} \alpha_t$ is in normal form up to order m .*

Proof. (Idea) The theorem is proved by induction. The case $m = 0$ follows from the Taylor expansion $\rho_t = 1 + tr_t$ around $t = 0$. Supposing that α_t is already in normal form up to order $m - 1$, the point is to find a function h_m in such a way that the flow ϕ_{t,h_m} of the Hamiltonian vector field X_{h_m} determines an isotopy $t \mapsto \phi_{t,h_m}$ for which $\phi_{t,h_m}^* \alpha_t$ is in normal form up to order m . \square

The following proposition will be crucial in the next section.

Proposition 12. [AB14] *Let (M, α) be a Zoll contact manifold and let $\rho : M \rightarrow \mathbb{R}^+$ be a smooth positive function invariant under the Reeb flow of α . Then $A_{\min}(M, \rho\alpha) \leq A_{\min}(M, \alpha) \min \rho$.*

Proof. This essentially follows from the fact that if $u \in M$ is a minimum point for ρ , then the Reeb orbit γ of (M, α) passing through u is also a Reeb orbit for $(M, \rho\alpha)$. Once one checks this by a straightforward computation, we have that the action of γ in $(M, \rho\alpha)$ is

$$\int_{\gamma} \rho\alpha = \min \rho \int_{\gamma} \alpha = A_{\min}(M, \alpha) \min \rho$$

and the proof is complete. \square

We shall make use of the following classical result.

Theorem 13. [Rab78] [Wei78] *Let C be a smooth convex bounded domain of \mathbb{R}^{2n} . The contact manifold $(\partial C, \lambda_0|_{\partial C})$ admits at least one periodic Reeb orbit.*

An important fact is that the function A_{\min} coincides with some well known symplectic capacities such as Hofer-Zehnder and Ekeland-Hofer capacities.

Theorem 14. [EH89] [Vit89] *Let C be a smooth convex bounded domain of \mathbb{R}^{2n} . There exists a distinguished closed characteristic $\bar{\gamma} \subset \partial C$ such that $A_{\min}(\partial C, \lambda_0) = A(\bar{\gamma})$. Moreover, restricting to smooth convex domains of \mathbb{R}^{2n} , the function A_{\min} is a symplectic capacity that we will denote with c .*

Due to the two theorems above, the function $A_{\min}(\partial C, \lambda_{0|\partial C})$ is well defined.

Besides the usual property of a capacity, choosing in a carefully way one among the equivalent definitions of c , the following result can be proved.

Proposition 15. [AM15] *Let $C \subset \mathbb{R}^{2n}$ be a smooth convex bounded domain and P the symplectic projector onto a symplectic linear subspace $V \subset \mathbb{R}^{2n}$. Then $c(PC, \omega_{0|V}) \geq c(C, \omega_0)$.*

2. Deformations of S^{2n-1}

In this section we would like to get some information on how A_{\min} behaves in case of a contact deformation on the unit sphere. The results we are going to state hold in the case of an arbitrary Zoll contact manifold, but in this paper we are interested just in deformations of the standard contact form on the sphere S^{2n-1} , so we can simplify the proof about the Lipschitz continuity of A_{\min} that relies on a result from [Gin87].

Lemma 16. *Fix two real numbers $0 < \delta < \Delta < \infty$ and consider the family $\mathcal{C}_{\delta, \Delta}$ of the convex domains $C \subset \mathbb{R}^{2n}$ which satisfy the (δ, Δ) -pinching condition $B_\delta \subset C \subset B_\Delta$. Every symplectic capacity $c : \mathcal{C}_{\delta, \Delta} \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to the Hausdorff distance.*

Proof. Let us take two elements $C, D \in \mathcal{C}_{\delta, \Delta}$ and let d be their Hausdorff distance.

By assumption

$$\delta B = B_\delta \subset C, D \subset B_\Delta = \Delta B,$$

hence

$$C \subset D + dB \subset D + \frac{d}{\delta} D = \left(1 + \frac{d}{\delta}\right) D,$$

and by the monotonicity and conformality property of symplectic capacities

$$c(C) \leq \left(1 + \frac{d}{\delta}\right)^2 c(D) = \left(1 + 2\frac{d}{\delta} + \frac{d^2}{\delta^2}\right) c(D),$$

therefore

$$c(C) - c(D) \leq d \left(\frac{2}{\delta} + \frac{d}{\delta^2} \right) c(D).$$

Because of the pinching condition we have $c(D) \leq c(\Delta B) = \Delta^2\pi$ and $d \leq \Delta$, thus there exists a fixed real number $M > 0$ such that

$$c(C) - c(D) \leq dM.$$

If $c(C) \geq c(D)$ we have $c(C) - c(D) = |c(C) - c(D)|$ and the claim follows, otherwise if $c(C) < c(D)$ we repeat the same proof switching the role of sets C and D . □

Lemma 17. *There exists a small open neighbourhood U of zero in the Banach space $C^2(S^{2n-1})$ such that if $f \in U$, the map $f \mapsto A_{\min}(S^{2n-1}, (1 + f)\lambda_{0|S^{2n-1}})$ is Lipschitz continuous on U with respect to the C^0 -topology.*

Proof. Let us set $M_{\sqrt{1+f}} := \{\sqrt{1+f(x)}x \mid x \in S^{2n-1}\} \subset \mathbb{R}^{2n}$.

The map $f \mapsto A_{\min}(S^{2n-1}, (1 + f)\lambda_{0|S^{2n-1}})$ is well defined because, as observed in Remark 5, looking for a periodic orbit of $(S^{2n-1}, (1 + f)\lambda_{0|S^{2n-1}})$ is the same as looking for one of $(M_{\sqrt{1+f}}, \lambda_{0|M_{\sqrt{1+f}}})$, and that exists because a C^2 -small deformation of S^{2n-1} still bounds a convex domain of \mathbb{R}^{2n} .

The map $f \mapsto A_{\min}(S^{2n-1}, (1 + f)\lambda_{0|S^{2n-1}})$ is the composition of the maps $f \mapsto M_{\sqrt{1+f}}$ and $M_{\sqrt{1+f}} \mapsto A_{\min}(M_{\sqrt{1+f}}, \lambda_{0|M_{\sqrt{1+f}}})$, the first of which is clearly Lipschitz from the C^0 -distance to the Hausdorff distance. So in order to prove the Lipschitz regularity result we need to show that the minimal action (which is a capacity) of a periodic orbit on a convex domain whose boundary is close to S^{2n-1} , is Lipschitz continuous with respect to the Hausdorff distance. But this follows from Lemma 16. □

The next theorem is a weaker version, which suits in our case, of the one in [AB14] which holds for every Zoll contact manifold. The proof is the same except that in our setting we do not need to use a stronger result about the Lipschitz continuity that generalizes Lemma 17.

Theorem 18. [AB14] *Let (S^{2n-1}, μ_t) , with $t \in [0, t_0)$, be a smooth constant volume deformation of the Zoll contact manifold $(S^{2n-1}, \mu_0 := \lambda_{0|S^{2n-1}})$. If it is not formally trivial, then the function $t \mapsto A_{\min}(S^{2n-1}, \mu_t)$ attains a strict local maximum at $t = 0$.*

Proof. The proof is carried out in four steps.

- 1) First we consider the form $(1 + t\nu_t + t^m r_t)\mu_0$, where $m > 1$ and both ν_t and r_t are smooth function on S^{2n-1} depending smoothly on t . By Lemma 17, the function $f \mapsto A_{\min}(S^{2n-1}, (1 + f)\mu_0)$ is Lipschitz if f is in a small enough C^2 -neighbourhood of zero in $C^\infty(S^{2n-1})$, then, for $t \rightarrow 0$ we have

$$A_{\min}(S^{2n-1}, (1 + t\nu_t + t^m r_t)\mu_0) = A_{\min}(S^{2n-1}, (1 + t\nu_t)\mu_0) + O(t^m).$$

- 2) Let $(1 + t^m \rho + t^{m+1} r_t)\mu_0$ be a deformation of μ_0 and $\bar{\rho}$ the function obtained by averaging ρ along the orbits of the Reeb vector field of μ_0

$$\bar{\rho}(x) := \frac{1}{T} \int_0^T \rho(\varphi_t(x)) dt,$$

where T is the common period of the periodic orbits of the Reeb flow φ_t . According to the induction step of the proof of Theorem 11, there exists a contact isotopy $\phi_t^{(m)} : S^{2n-1} \rightarrow S^{2n-1}$ such that

$$\phi_t^{(m)*} (1 + t^m \rho + t^{m+1} r_t)\mu_0 = (1 + t^m \bar{\rho} + t^{m+1} r'_t)\mu_0,$$

where r'_t is a smooth function depending smoothly on t .

- 3) If $(1 + t^m \rho + t^{m+1} r_t)\mu_0$ is a smooth constant volume deformation and $\bar{\rho}$ is not identically zero, then

$$A_{\min}(S^{2n-1}, (1 + t^m \rho + t^{m+1} r_t)\mu_0) < A_{\min}(S^{2n-1}, \mu_0)$$

for $t \neq 0$ small enough.

To prove this claim, first note that by 2) and 1) follows

$$\begin{aligned} (6) \quad & A_{\min}(S^{2n-1}, (1 + t^m \rho + t^{m+1} r_t)\mu_0) \\ &= A_{\min}(S^{2n-1}, (1 + t^m \bar{\rho} + t^{m+1} r'_t)\mu_0) \\ &= A_{\min}(S^{2n-1}, (1 + t^m \bar{\rho})\mu_0) + O(t^{m+1}). \end{aligned}$$

Since $\bar{\rho}$ is an integral of motion for the Reeb flow of μ_0 and m is a positive integer, we have that $(1 + t^m \bar{\rho})$ is a positive (for small t) integral of motion of μ_0 , thus Proposition 12 implies that

$$(7) \quad A_{\min}(S^{2n-1}, (1 + t^m \bar{\rho})\mu_0) \leq (1 + t^m \min \bar{\rho}) A_{\min}(S^{2n-1}, \mu_0).$$

The deformation $(1 + t^m \bar{\rho} + t^{m+1} r'_t) \mu_0$ is constant volume because contact isotopies preserve the volume. By the property of the exterior derivative

$$\begin{aligned} Vol(S^{2n-1}, \mu_0) &= Vol(S^{2n-1}, (1 + t^m \bar{\rho} + t^{m+1} r'_t) \mu_0) \\ &= \int_{S^{2n-1}} (1 + t^m \bar{\rho} + t^{m+1} r'_t)^n \mu_0 \wedge d\mu_0^{n-1} \\ &= Vol(S^{2n-1}, \mu_0) + nt^m \int_{S^{2n-1}} \bar{\rho} \mu_0 \wedge d\mu_0^{n-1} + O(t^{m+1}), \end{aligned}$$

and thus the integral of $\bar{\rho}$ over S^{2n-1} is zero. Therefore, if in addition $t \neq 0$ and $\bar{\rho}$ is not identically zero, the extrema of $\bar{\rho}$ must have opposite signs and hence its minimum must be negative. Putting together this fact with (6) and (7), we deduce that the function $t \mapsto A_{\min}(S^{2n-1}, (1 + t^m \bar{\rho} + t^{m+1} r'_t) \mu_0)$ attains a strict maximum at $t = 0$, namely

$$A_{\min}(S^{2n-1}, (1 + t^m \bar{\rho} + t^{m+1} r'_t) \mu_0) < A_{\min}(S^{2n-1}, \mu_0), \quad \text{for } t > 0.$$

- 4) We are finally ready to prove the theorem. Let us consider a constant volume deformation μ_t of the Zoll contact form μ_0 . By Gray's stability theorem we can assume that the contact deformation has the form $\mu_t = \rho_t \mu_0$. Expanding ρ_t around $t = 0$, we obtain

$$\mu_t = (1 + t\rho_{(1)} + t^2 r_t) \mu_0,$$

where $\rho_{(1)} = \frac{d\rho_t}{dt}|_{t=0}$ and r_t is a smooth function depending on t .

By 3), if the average $\overline{\rho_{(1)}}$ is not identically zero, then

$$t \mapsto A_{\min}(S^{2n-1}, \mu_t)$$

attains a strict maximum at $t = 0$.

Otherwise, if $\overline{\rho_{(1)}}$ is identically zero, by 2) there exists a contact isotopy $\phi_t^{(2)}$ such that $\phi_t^{(2)*} \mu_t = (1 + t^2 r'_t) \mu_0$. Since $\phi_t^{(2)}$ is a contact isotopy, then $(1 + t^2 r'_t) \mu_0$ is also a constant volume smooth deformation of μ_0 and $A_{\min}(S^{2n-1}, (1 + t^2 r'_t) \mu_0) = A_{\min}(S^{2n-1}, \mu_t)$, so we can rewrite $\mu_t = (1 + t^2 r'_t) \mu_0$ and start anew.

If we repeat this process an arbitrary number of times, we see that either $t \mapsto A_{\min}(S^{2n-1}, \mu_t)$ attains a strict maximum at $t = 0$ or that for any positive integer m , there exist a contact isotopy $\phi_t^{(m)}$ and a smooth function $\nu_t^{(m)}$ on S^{2n-1} depending smoothly on the parameter t , such that

$\phi_t^{(m)*} \mu_t = (1 + t^m \nu_t^{(m)}) \mu_0$. In other words, either $t \mapsto A_{\min}(S^{2n-1}, \mu_t)$ attains a strict maximum at $t = 0$ or the deformation μ_t is formally trivial. \square

3. Analyticity of the volume of a projection

Our next goal is to prove that the fixed domain formulation of the local middle dimensional non-squeezing theorem holds if we consider an analytic path of symplectic embeddings.

To do this we need a result, whose proof relies on calculations made in order to prove Theorem 3 of [AM13].

Proposition 19. *Let $U \ni y_0$ be a domain of \mathbb{R}^n and $[0, 1] \times U \ni (t, y) \mapsto \varphi_{t,y}$ an analytic map such that $\varphi_{t,y}$ are embeddings of the unit n -dimensional ball $\varphi_{t,y} : \overline{B_1} \hookrightarrow \mathbb{R}^n$, with φ_{0,y_0} linear. Moreover, let $P : \mathbb{R}^n \rightarrow V$ be the orthogonal projector onto a k -dimensional linear subspace $V \subset \mathbb{R}^n$ and ρ a constant k -volume form on V . Then the function $(t, y) \mapsto \text{Vol}_k(P\varphi_{t,y}(B_1), \rho)$ is analytic in a neighbourhood of $(0, y_0)$ small enough.*

In the proof we will use the following lemma.

Lemma 20. *Take the hypothesis of the proposition above. The set $S_{t,y} \subset \partial B_1$ defined as*

$$(8) \quad S_{t,y} := \{x \in \partial B_1 \mid P|_{T_{\varphi_{t,y}(x)}\varphi_{t,y}(\partial B_1)} \text{ is not surjective}\}$$

has the property that

$$(9) \quad \partial P\varphi_{t,y}(B_1) = P\varphi_{t,y}(S_{t,y})$$

and can be written as

$$(10) \quad S_{t,y} = \{x \in \partial B_1 \mid F_{t,y}(x) = 0\},$$

where $F_{t,y}(x) := (I - P)(D\varphi_{t,y}(x)^*)^{-1}[x]$. If (t, y) is in a small enough neighbourhood of $(0, y_0)$, $S_{t,y}$ is a submanifold of ∂B_1 such that $S_{t,y} = \phi_{t,y}(S^{k-1})$, where $\phi_{t,y}$ is an analytic path of diffeomorphisms.

Proof. First observe that (9) is an immediate consequence of the definition of $S_{t,y}$. The function $P|_{T_{\varphi_{t,y}(x)}\varphi_{t,y}(\partial B_1)}$ is not surjective if and only if $PD\varphi_{t,y}(x)|_{T_x\partial B_1} : T_x\partial B_1 \rightarrow T_{P\varphi_{t,y}(x)}V \cong \mathbb{R}^k$ is not surjective.

This is true iff

$$\exists u \in \mathbb{R}^k, u \neq 0, \text{ such that } \langle PD\varphi_{t,y}(x)[\xi], u \rangle = 0$$

$\forall \xi \in T_x \partial B_1$, i.e. $\forall \xi$ such that $\langle \xi, x \rangle = 0$.

Since $u = Pu$ and $P = P^*$

$$\langle PD\varphi_{t,y}(x)[\xi], u \rangle = \langle \xi, (PD\varphi_{t,y}(x))^*[u] \rangle = \langle \xi, D\varphi_{t,y}(x)^*[u] \rangle$$

and thus the non surjectivity holds iff

$$D\varphi_{t,y}(x)^*[u] = \lambda x, \text{ where } \lambda \neq 0 \text{ is a real number.}$$

Equivalently

$$(D\varphi_{t,y}(x)^*)^{-1}[x] \in \mathbb{R}^k$$

which is the same as

$$F_{t,y}(x) := (I - P)(D\varphi_{t,y}(x)^*)^{-1}[x] = 0 \in \mathbb{R}^{n-k}.$$

Now, consider the analytic function $G(t, y, x) := (I - P)(\varphi_{t,y}(x)^*)^{-1}[x]$. We have that $\varphi_{0,y_0} = D\varphi_{0,y_0}$ because φ_{0,y_0} is linear, hence $G(0, y, z) = 0$ if $z \in S_{0,y_0}$. Applying the analytic implicit function theorem we deduce that, for (t, y) close to $(0, y_0)$, $S_{t,y}$ is a submanifold of ∂B_1 and $S_{t,y} = \phi'_{t,y}(S_{0,y_0})$ where $\phi'_{t,y}$ is an analytic path of diffeomorphisms. There is a diffeomorphism given by $(D\varphi_{t,y}(x)^*)^{-1}$ between S^{k-1} and S_{0,y_0} , therefore by composition with $\phi'_{t,y}$ we get an analytic path of diffeomorphisms $\phi_{t,y}$ such that $\phi_{t,y}(S^{k-1}) = S_{t,y}$. \square

Now we are ready to prove Proposition 19.

Proof. Take a primitive $\alpha \in \Omega^{k-1}(V)$ of the volume form $\rho \in \Omega^k(V)$, i.e. $d\alpha = \rho$.

As observed in the former lemma $\partial P\varphi_{t,y}(B_1) = P\varphi_{t,y}(S_{t,y})$ and applying Stokes' theorem we get

$$\begin{aligned} Vol_k(P\varphi_{t,y}(B_1), \rho) &= \int_{P\varphi_{t,y}(B_1)} d\alpha = \int_{\partial P\varphi_{t,y}(B_1)} \alpha \\ &= \int_{P\varphi_{t,y}(S_{t,y})} \alpha = \int_{S_{t,y}} (P\varphi_{t,y})^* \alpha \\ &= \int_{S^{k-1}} (P\varphi_{t,y}\phi_{t,y})^* \alpha. \end{aligned}$$

where $\phi_{t,y} : S^{k-1} \rightarrow S_{t,y}$ is the diffeomorphism introduced in the proof of the lemma above. For (t, y) close to $(0, y_0)$, the function $(t, y) \mapsto P\varphi_{t,y}\phi_{t,y}$ is analytic and this implies the analyticity of $(t, y) \mapsto \int_{S^{k-1}} (P\varphi_{t,y}\phi_{t,y})^* \alpha$.

In fact, we can write $\int_{S^{k-1}} (P\varphi_{t,y}\phi_{t,y})^* \alpha = \int_{S^{k-1}} a_{t,y}(x) \nu$ where $a_{t,y}$ is analytic.

Differentiating under integral sign, from the Taylor expansion of $a_{t,y}$ we get a local series expansion of the function $(t, y) \mapsto \int_{S^{k-1}} (P\varphi_{t,y}\phi_{t,y})^* \alpha = Vol_k(P\varphi_{t,y}(B_1), \rho)$, which is therefore analytic. \square

4. Local non-squeezing

In the following B_1 indicates the unit ball in \mathbb{R}^{2n} and $P : \mathbb{R}^{2n} \rightarrow V$ the symplectic projection onto a $2k$ -dimensional symplectic linear subspace $V \subset \mathbb{R}^{2n}$. At first we are interested in proving the local non squeezing formulation for a path of symplectic embeddings starting from a linear one and to do so we will use the following middle dimensional linear non-squeezing result.

Theorem 21. [AM13], [AM15] *Let P be the symplectic projector onto a $2k$ -dimensional symplectic linear subspace $V \subset \mathbb{R}^{2n}$. Then for every linear symplectic isomorphism $L : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ there holds*

$$Vol_{2k}(PL(B_1), \omega_{0|V}^k) \geq \pi^k.$$

The equality holds if and only if the linear subspace $L^{-1}V$ is J -invariant, where J is the standard complex structure on \mathbb{R}^{2n} .

We complete the above result by the following statement.

Addendum 22. *The equality holds if and only if $(B_1 \cap L^{-1}V, \omega_{0|L^{-1}V})$ is linearly symplectomorphic to $(PL(B_1), \omega_{0|V})$ via the symplectic map $L|_{L^{-1}V}$.*

The following result is useful to prove Theorem 21 and the addendum as well.

Lemma 23. [Fed69, Section 1.8.1]

Let $1 \leq k \leq n$, then

$$|\omega^k[u_1, \dots, u_{2k}]| \leq |u_1 \wedge \dots \wedge u_{2k}| \quad \forall u_1, \dots, u_{2k} \in \mathbb{R}^{2n}.$$

Proof. (Addendum) If a symplectomorphism exists, by Lemma 23 we have $Vol_{2k}(PL(B_1), \omega_{0|V}^k) = Vol_{2k}(B_1 \cap L^{-1}V, \omega_{0|L^{-1}V}^k) \leq \pi^k$. But at the same

time Theorem 21 yields $Vol_{2k}(PL(B_1), \omega_{0|V}^k) \geq \pi^k$, hence the equality holds. On the other hand $Vol_{2k}(PL(B_1), \omega_{0|V}^k) = \pi^k$ iff $L^{-1}V$ is J -invariant; and if the claim that

$$PL(B_1 \cap L^{-1}V) = PL(B_1)$$

is true, then $(B_1 \cap L^{-1}V, \omega_{0|L^{-1} \cap V})$ is symplectomorphic to

$$(PL(B_1), \omega_{0|V}) = (PL(B_1 \cap L^{-1}V), \omega_{0|V})$$

via the linear symplectic isomorphism $L : L^{-1}V \rightarrow V$.

To prove the claim we reduce it to the easier case in which P is orthogonal. First we take an ω -compatible inner product $(\cdot, \cdot)'$ on \mathbb{R}^{2n} such that P is orthogonal and we denote with B'_1 and J' the corresponding unit ball and complex structure. In particular V is J' -invariant. Let $\psi : (V, \omega, J') \rightarrow (V, \omega, J)$ be a complex and linear isomorphism. It follows that ψ is an isometry from $(V, (\cdot, \cdot)')$ to $(V, (\cdot, \cdot))$, hence $\psi(B'_1) = B_1$. The image of the unit ball under a linear surjection M is given by

$$M(B_1) = M(B_1 \cap \text{rank} M^*).$$

If we take $N = L\psi$, $M = PN$ and we denote with $*$ ' the adjoint of a matrix with respect to $(\cdot, \cdot)'$, we get

$$\begin{aligned} PL(B_1) &= PL\psi(B'_1) = PN(B'_1) = PN(B'_1 \cap \text{rank}(PN)^{*'}) \\ &= PN(B'_1 \cap \text{rank}(N^{*'}P^{*'})) = PN(B'_1 \cap \text{rank}(N^{*'}P)) \\ &= PN(B'_1 \cap N^{*'}V). \end{aligned}$$

The identity $\psi J' = J\psi$ implies $J' = \psi^{-1}J\psi$ and the fact that $L^{-1}V$ is J -invariant is equivalent to $J\psi N^{-1}V = \psi N^{-1}V$, hence

$$\begin{aligned} N^{*'}V &= N^{*'}J'V = N^{*'}J'NN^{-1}V = J'N^{-1}V \\ &= \psi^{-1}J\psi N^{-1}V = \psi^{-1}\psi N^{-1}V \\ &= N^{-1}V, \end{aligned}$$

thus we obtain

$$\begin{aligned} PL(B_1) &= PN(B'_1 \cap N^{*'}V) = PN(B'_1 \cap N^{-1}V) \\ &= PL\psi(B'_1 \cap \psi^{-1}L^{-1}V) \\ &= PL(B_1 \cap L^{-1}V), \end{aligned}$$

and the claim is proved. □

In order to gain some information about the strong formulation of the local non-squeezing inequality we study the function $t \mapsto Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k)$.

Proposition 24. *Let $[0, 1] \ni t \mapsto \varphi_t$ be a smooth path of symplectic embeddings $\varphi_t : \overline{B_1} \hookrightarrow \mathbb{R}^{2n}$, such that φ_0 is linear and $\varphi_0^{-1}V$ is J -invariant. The deformation of $P\varphi_0(B_1)$ given by $P\varphi_t(B_1)$ can be either formally or not formally trivial:*

- if the deformation is formally trivial, then every order $m \in \mathbb{Z}^+$ derivative of $t \mapsto Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k)$ vanishes in 0;
- if the deformation is not formally trivial, then the strict middle dimensional non-squeezing inequality $Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) > \pi^k$ holds for $t > 0$ small enough.

Proof. By the previous addendum we have that $\psi := \varphi_{0|\varphi_0^{-1}V}$ is a linear symplectomorphism between $(B_1 \cap \varphi_0^{-1}V, \omega_{0|\varphi_0^{-1}V})$ and $(P\varphi_0(B_1), \omega_{0|V})$. Let us call $M_t := \partial P\varphi_t(B_1)$ and consider two 1-forms: the Liouville form $\lambda_{0|\psi^{-1}M_t}$ and its pullback $\mu_t := \theta_t^*(\lambda_{0|\psi^{-1}M_t})$, where $\theta_t : S^{2k-1} \rightarrow \psi^{-1}M_t$ is the radial diffeomorphism such that $\theta_t^{-1}(x) = \frac{x}{\|x\|}$.

Later we will use the capacity c , which is defined only for convex domains, so let us notice once for all that, for small deformations, $\varphi_t(B_1)$ is still convex and that the projection of a convex domain is still convex.

Now we compute the relations between the volume of the deformations. Using Stokes' theorem we get

$$\begin{aligned} Vol_{2k-1}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t}) &= \int_{\psi^{-1}\partial P\varphi_t(B_1)} \lambda_{0|\psi^{-1}M_t} \wedge (d\lambda_{0|\psi^{-1}M_t})^{k-1} \\ &= \int_{\psi^{-1}P\varphi_t(B_1)} \omega_0^k \\ &= Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k). \end{aligned}$$

On the other hand, since $(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t})$ and (S^{2k-1}, μ_t) are strictly contactomorphic, $Vol_{2k-1}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t}) = Vol_{2k-1}(S^{2k-1}, \mu_t)$.

So, if $\mu'_t := \mu_t \rho(t)$, where $\rho(t) := \frac{1}{\sqrt[k]{Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k)}}$, it follows that $Vol(S^{2k-1}, \mu'_t) = 1$ and in particular that μ'_t is a constant volume deformation.

Observing that closed characteristics in (S^{2k-1}, μ'_t) are the same as in (S^{2k-1}, μ_t) we can establish the relations between the minimal action of their

closed Reeb orbits

$$\begin{aligned}
 A_{\min}(S^{2k-1}, \mu'_t) &= \min_{\gamma} \{A(\gamma) \mid \gamma \text{ closed characteristic in } (S^{2k-1}, \mu'_t)\} \\
 &= \min_{\gamma} \left\{ \int_{\gamma} \mu'_t \mid \gamma \text{ closed characteristic in } (S^{2k-1}, \mu'_t) \right\} \\
 &= \min_{\gamma} \left\{ \rho(t) \int_{\gamma} \mu_t \mid \gamma \text{ closed characteristic in } (S^{2k-1}, \mu'_t) \right\} \\
 &= \min_{\gamma} \left\{ \rho(t) \int_{\gamma} \mu_t \mid \gamma \text{ closed characteristic in } (S^{2k-1}, \mu_t) \right\} \\
 &= \rho(t) A_{\min}(S^{2k-1}, \mu_t).
 \end{aligned}$$

Since θ_t is a strict contactomorphism between $(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t})$ and (S^{2k-1}, μ_t) , we also get

$$A_{\min}(S^{2k-1}, \mu_t) = A_{\min}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t}) = c(\psi^{-1}P\varphi_t(B_1)),$$

where ψ is a symplectomorphism.

Thus the quantities $A_{\min}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t})$ and $c(\psi^{-1}P\varphi_t(B_1))$ are equal respectively to $A_{\min}(M_t, \lambda_{0|M_t})$ and $c(P\varphi_t(B_1))$. Notice that the Weinstein conjecture holds in the convex case (Theorem 13), hence a closed characteristic for $(M_t, \lambda_{0|M_t})$ always exists, moreover by Theorem 14 the quantities above are well defined.

Now let us take a deformation (S^{2k-1}, μ'_t) of the standard Zoll contact form $\mu_0 = \lambda_{0|S^{2k-1}}$ on S^{2k-1} , that could be either formally trivial or not formally trivial.

Suppose the former to be true, which is equivalent to say that the deformation $P\varphi_t(B_1)$ is formally trivial.

In this case, in the last part of the proof of Theorem 18 we deduced that for every $m \in \mathbb{Z}^+$ there is a contact isotopy ϕ_t such that

$$\phi_t^* \mu_t = (1 + O(t^m))\mu_0.$$

The volume function is invariant by contact isotopy, so

$$\begin{aligned}
 &Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k) \\
 &= Vol_{2k-1}(S^{2k-1}, \mu_t) \\
 &= \rho(t) Vol_{2k-1}(S^{2k-1}, \mu'_t) \\
 &= \rho(t) Vol_{2k-1}(S^{2k-1}, (1 + O(t^m))\mu_0), \quad \forall m \in \mathbb{Z}^+.
 \end{aligned}$$

By the definition of $\rho(t)$ the above equality is equivalent to

$$\begin{aligned} Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k)^{\frac{k+1}{k}} &= Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k)\rho(t) \\ &= Vol_{2k}(S^{2k-1}, (1 + O(t^m))\mu_0), \forall m \in \mathbb{Z}^+. \end{aligned}$$

Therefore each of m -order derivatives of $Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k)^{\frac{k+1}{k}}$, and hence of $Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k) = Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k)$, vanishes in 0.

Now we suppose that (S^{2k-1}, μ'_t) (equivalently $P\varphi_t(B_1)$) is not formally trivial. By Theorem 18 and the previous calculations, if t is small enough the following inequality holds

$$\begin{aligned} 1 &= \frac{\pi}{\sqrt[k]{\pi^k}} = \frac{A_{\min}(\psi^{-1}M_0, \lambda_{0|\psi^{-1}M_0})}{\sqrt[k]{\pi^k}} = \rho(0)A_{\min}(\psi^{-1}M_0, \lambda_{0|\psi^{-1}M_0}) \\ &= A_{\min}(S^{2k-1}, \mu'_0) > A_{\min}(S^{2k-1}, \mu'_t) \\ &= \frac{A_{\min}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t})}{\sqrt[k]{Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k)}}. \end{aligned}$$

So, recalling that

$$A_{\min}(M_t, \lambda_{0|M_t}) = A_{\min}(\psi^{-1}M_t, \lambda_{0|\psi^{-1}M_t})$$

and

$$Vol_{2k}(\psi^{-1}P\varphi_t(B_1), \omega_{0|\varphi_0^{-1}V}^k) = Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k),$$

if we prove that $A_{\min}(M_t, \lambda_{0|M_t}) \geq \pi$, then $Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k)^{-\frac{1}{k}} < \frac{1}{\pi}$ and the strict local non-squeezing inequality $Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) > \pi^k$ holds. But from the behaviour of the capacity c respect to symplectic projections (Proposition 15), we deduce

$$A_{\min}(M_t, \lambda_{0|M_t}) = c(P\varphi_t(B_1)) \geq c(\varphi_t(B_1)) = c(B_1) = \pi,$$

and hence the result. □

From this result we cannot deduce the general local non-squeezing inequality (4) because in the general case we cannot say much if a trivial deformation occurs. Nevertheless, if the deformation is analytic, the local non-squeezing inequality follows easily as consequence of the proposition above.

Theorem 1 (Analytic local non-squeezing). *Let $[0, 1] \ni t \mapsto \varphi_t$ be an analytic path of symplectic embeddings $\varphi_t : \overline{B_1} \hookrightarrow \mathbb{R}^{2n}$ such that φ_0 is linear. Then the middle dimensional non-squeezing inequality*

$$Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) \geq \pi^k$$

holds for t small enough.

Proof. By Theorem 21 we have that $Vol_{2k}(P\varphi_0(B_1), \omega_{0|V}^k) \geq \pi^k$ and the equality holds if and only if $\varphi_0^{-1}V$ is J -invariant. If the equality does not hold the theorem is trivially true by the continuity of the volume. On the other hand, if the equality holds, Theorem 21 implies that $\varphi_0^{-1}V$ is J -invariant and thus we are under the hypothesis of Proposition 24.

Therefore, in the case of a not formally trivial deformation $P\varphi_t(B_1)$ there is nothing to prove. Otherwise, if the deformation is formally trivial, the function $t \mapsto Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k)$ has vanishing derivatives in 0, but we know by Proposition 19 that if t is small enough this function is analytic and hence constant. Thus we get $Vol_{2k}(P\varphi_t(B_1), \omega_{0|V}^k) = Vol_{2k}(B_1^{2k}, \omega_{0|V}^k) = \pi^k$ for t small enough. \square

We now state a couple of lemmata in order to say something more about the fixed symplectic embedding formulation of the local non-squeezing.

First a result on the local structure of the zero set of an analytic function.

Theorem 25. (*Lojasiewicz’s Structure Theorem*) [KP92, Theorem 5.2.3] *Let $f(x_1, \dots, x_n)$ be a real analytic function in a neighbourhood of a point $y = (y_1, \dots, y_n)$ in \mathbb{R}^n and assume that $x_n \mapsto f(y_1, \dots, y_{n-1}, x_n)$ is not identically zero. After a suitable rotation of the coordinates x_1, \dots, x_{n-1} , there exist numbers $\delta_j > 0$, $j = 1, \dots, n$, and a neighbourhood Q_n (where we define $Q_k := \{(x_1, \dots, x_k) \mid |y_j - x_j| < \delta_j, 1 \leq j \leq k\}$) such that the zero set*

$$Z := \{x \in Q_n \mid f(x) = 0\}$$

has a decomposition

$$Z = V^0 \cup \dots \cup V^{n-1},$$

where the set V^0 is either empty or consists of the point y alone, while for $1 \leq k \leq n - 1$ we may write V^k as a finite disjoint union $V^k = \cup_\lambda \Gamma_\lambda^k$ of k -dimensional subvarieties Γ_λ^k . Each Γ_λ^k is defined by a system of $n - k$

equations:

$$\begin{aligned} x_{k+1} &= {}^\lambda\eta_{k+1}^k(x_1, \dots, x_k), \\ &\dots \\ x_n &= {}^\lambda\eta_n^k(x_1, \dots, x_k), \end{aligned}$$

where each function ${}^\lambda\eta_{k+1}^k$ is real analytic on an open subset $\Omega_\lambda^k \subseteq Q_k \subseteq \mathbb{R}^k$.

Lemma 26. *Let $\varphi : D \rightarrow \mathbb{R}^{2n}$ be an analytic symplectic embedding and $x \in D$. As long as $x + ry \in D$, the map*

$$\varphi_{r,x}(y) := \begin{cases} \frac{1}{r}(\varphi(x + ry) - \varphi(x)) & \text{if } r > 0, \\ D\varphi(x)[y] & \text{if } r = 0, \end{cases}$$

is analytic.

Proof. The function $\varphi(x + ry)$ is analytic in r because it is a composition of analytic maps, thus the map $\frac{1}{r}(\varphi(x + ry) - \varphi(x))$ is analytic in $r > 0$. Since $\varphi(x + ry) - \varphi(x)$ is analytic in $r = 0$, we can express it as a convergent Taylor series centred in 0. But the 0-th coefficient of this expansion must vanish since $\varphi(x + 0y) - \varphi(x) = 0$, hence we can divide by r and we obtain a convergent Taylor expansion for $\frac{1}{r}(\varphi(x + ry) - \varphi(x))$ in $r = 0$. \square

Theorem 2. *Let $\varphi : D \hookrightarrow \mathbb{R}^{2n}$ be an analytic symplectic embedding, with D domain of \mathbb{R}^{2n} . Then there exists a function $r_0 : D \rightarrow (0, +\infty)$ such that the inequality $\text{Vol}_{2k}(P\varphi(B_r(x)), \omega_{0|V}^k) \geq r^{2k}\pi^k$ holds, for every $x \in D$ and for every $r < r_0(x)$. Moreover r_0 is bounded away from 0 on compact subsets $K \subset D$.*

Proof. Let $\varphi_{r,x}$ be the map defined in the lemma above. Observing that

$$\begin{aligned} \text{Vol}_{2k}(P\varphi_{r,x}(B_1(0)), \omega_{0|V}^k) &= \text{Vol}_{2k}\left(P\frac{1}{r}\varphi(B_r(x)), \omega_{0|V}^k\right) \\ &= \frac{\text{Vol}_{2k}(P\varphi(B_r(x)), \omega_{0|V}^k)}{r^{2k}}, \end{aligned}$$

for every fixed $x \in D$ we can apply Theorem 1 to the path $r \mapsto \varphi_{r,x}$ and we deduce the first part of the theorem.

Now we prove the estimate on compact sets.

Define a function

$$f(x, r) := \text{Vol}_{2k}(P\varphi_{r,x}(B_1(0)), \omega_0^k|_V) - \pi^k.$$

There is a subset $\mathcal{D} = \{(x, r) \in D \times [0, +\infty) \mid 0 \leq r < R(x)\}$ on which f is analytic, indeed to see this it is enough to apply Proposition 19 to the analytic map $(r, x) \mapsto \varphi_{r,x}$.

Now, take an arbitrary point $x_0 \in D$. If $f(x_0, 0) > 0$, then by continuity there exists a small neighbourhood $B_{\epsilon_{x_0}} \times [0, r_{x_0})$ of $(x_0, 0)$ in \mathcal{D} , on which f is positive. On the other hand, if $f(x_0, 0) = 0$, we denote with $\gamma_A^{x_0} : [0, 1] \rightarrow \mathcal{D}$ a simple analytic curve such that $\gamma_A^{x_0}(0) = (x_0, 0)$. A consequence of Theorem 1 is that $f(\gamma_A^{x_0}(r))$ must be non negative in a neighbourhood of $r = 0$, i.e. $(x_0, 0)$ is a local minimum for the restriction of f to every analytic curve $\gamma_A^{x_0}$. From this we can deduce that $(x_0, 0)$ is a minimum for f in \mathcal{D} .

To see it, we first extend f to an analytic function in a neighbourhood of $(x_0, 0)$ in \mathbb{R}^{2n+1} . By Theorem 25, there is a small ball $B_\delta(x_0, 0) \subset \mathbb{R}^{2n+1}$ in which we know how the zeros are distributed, in particular $\mathcal{D} \cap (B_\delta(x_0, 0) \setminus f^{-1}(0))$ has at most a finite number N of different connected components $A_i \subset \mathcal{D} \cap B_\delta(x_0, 0)$ such that $(x_0, 0) \in \overline{A_i}$.

The set $(f^{-1}(0) \cup_{i=1}^N A_i) \cap (\mathcal{D} \cap B_\delta(x_0, 0))$ contains a neighbourhood of $(x_0, 0)$ in \mathcal{D} , hence if we prove that $f|_{A_i} > 0$ for every $i \in \{1, \dots, N\}$, we get the desired result. But if it were $f|_{A_i} < 0$, by Theorem 25 we would be able to conclude that there exists an analytic curve $\gamma_A^{x_0}$ laying in the connected component A_i and this would imply that 0 is not a minimum for $\gamma_A^{x_0}$, hence a contradiction.

Therefore $(x_0, 0)$ is a minimum for f in \mathcal{D} and hence there exists a small neighbourhood $B_{\epsilon_{x_0}} \times [0, r_{x_0})$ of $(x_0, 0)$ in \mathcal{D} on which f is positive.

Now we consider an arbitrary compact set $K \subset D$. As we have just seen, to every $x_0 \in D$ we can associate two positive real numbers r_{x_0} and ϵ_{x_0} . The balls of radius ϵ_{x_0} centred in an arbitrary $x_0 \in K$ produce an open cover of K . From this cover we can extract a finite subcover of balls of radius ϵ_{x_i} and if we define r_0 as the minimum in the set of the corresponding r_{x_i} we get the result. \square

References

- [AM13] A. Abbondandolo and R. Matveyev, *How large is the shadow of a symplectic ball?* J. Topol. Anal. **5** (2013), 87–119.

- [ABH+18] A. Abbondandolo, B. Bramham, U. Hryniewicz, and P. Salomão, *Sharp systolic inequalities for Reeb flows on the three sphere*, *Invent. Math.* **211** (2018), no. 2, 687–778.
- [AM15] A. Abbondandolo and P. Majer, *A non-squeezing theorem for symplectic images of the Hilbert ball*, *Calc. Var. Partial Diff. Equ.* **54** (2015), 1469–1506.
- [AB14] J. C. Álvarez Paiva and F. Balacheff, *Contact geometry and isosystolic inequalities*, *Geom. Funct. Anal.* **24** (2014), no. 2, 648–669.
- [DF76] A. J. Dragt and J. M. Finn, *Lie series and invariant functions for analytic symplectic maps*, *J. Math. Phys.* **17** (1976), no. 12, 2215–2227.
- [EG91] Y. Eliashberg and M. Gromov, *Convex symplectic manifolds, Several complex variables and complex geometry, Part 2*, *P. Symp. Pure Math.* **52** (1991), part 2, 135–162.
- [EH89] I. Ekeland and H. Hofer, *Symplectic topology and Hamiltonian dynamics*, *Math Z.* **200** (1989), 355–378.
- [Fed69] H. Federer, *Geometric Measure Theory*, Springer (1969).
- [Fin86] J. M. Finn, *Lie transforms: a perspective*, *Lect. Notes Phys.* **252**, Springer, Berlin (1986), 63–86.
- [Gin87] V. L. Ginzburg, *New generalizations of Poincaré’s geometric theorem*, *Funktsional. Anal. i Prilozehn.* **21** (1987), no. 2, 16–22.
- [Gro85] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, *Invent. Math.* **82** (1985), 307–347.
- [Gut08] L. Guth, *Symplectic embeddings of polydisks*, *Invent. Math.* **172** (2008), 477–489.
- [HZ94] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, (1994).
- [KP92] S. G. Krantz and H. R. Parks, *A Primer on Real Analytic Functions*, Birkhäuser, (1992).
- [Mos65] J. Moser, *On the volume elements of a manifold*, *Trans. Amer. Math. Soc.* **120** (1965), 286–294.

- [Rab78] P. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Commun. Pure Appl. Ana. **31** (1978), 157–184.
- [Vit89] C. Viterbo, *Capacité symplectiques et applications*, Astérisque **177-178** (1989), no. 714, Séminaire Bourbaki 41^{ème} année, 345–362.
- [Wei74] A. Weinstein, *Fourier integral operators, quantization, and the spectra of Riemannian manifolds*, Géométrie symplectique et physique mathématique, Éditions Centre Nat. Recherche Sci., Paris (1975), 289–298.
- [Wei78] A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. Math. **108** (1978), 507–518.

DEPARTMENT OF MATHEMATICS, RUHR UNIVERSITY OF BOCHUM
44801 BOCHUM, GERMANY
E-mail address: LORENZO.RIGOLLI@RUB.DE

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