JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 17, Number 2, 443–476, 2019

Strong deformation retraction of the space of Zoll Finsler projective planes

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We show that the infinite-dimensional space of reversible Zoll Finsler metrics on the projective plane strongly deformation retracts to the canonical round metric. In particular, this space of reversible Zoll Finsler metrics is connected. Moreover, the strong deformation retraction arises from a deformation of the geodesic flow of every reversible Zoll Finsler projective plane to the geodesic flow of the round metric through a family of smooth free circle actions induced by the curvature flow of the canonical round projective plane. This construction provides a description of the geodesics of the reversible Zoll Finsler metrics along the retraction.

Partially supported by the ANR grants Finsler and Min-Max.

1. Introduction

A *Zoll metric* on a closed manifold M is a Riemannian or Finsler metric all of whose geodesics are simple closed curves of the same length. (By assumption, all Finsler metrics are reversible and quadratically convex, see Definition 2.1.) We refer to the classical reference [Be78] for an introduction to the subject and historical comments, see also [B03]. Zoll manifolds have finite fundamental groups. Thus, in the two-dimensional case, they are diffeomorphic to either the sphere or the projective plane. Note that the orientable double cover of a Zoll Finsler projective plane is a Zoll Finsler two-sphere, see Proposition 3.1 or Remark 3.2.

The canonical round metric on the two-sphere or the projective plane is a Zoll Riemannian metric. However, there exist Zoll Riemannian two-spheres which are not round; some are rotationally symmetric, see [Z03], [Be78, §4], while others have no symmetry at all, see [Be78, Corollary 4.71]. Actually, Zoll Riemannian metrics on the two-sphere (modulo isometries and rescaling) form an infinite-dimensional space. Contrariwise, a Riemannian metric on the projective plane is a Zoll metric if and only if it has constant curvature, which follows from Green's theorem, see [Be78, Theorem 5.59], since the orientable double cover of a Zoll projective plane is a Blaschke sphere. This also follows from the fact that round metrics are the only minimizers of the systolic area on the projective plane, see [P52], and that all Zoll metrics share the same systolic area, see [W75], [Be78, §2.C]. This result also holds true in higher dimension, see [Be78, Appendix D]. However, this rigidity result fails in the Finsler case. Indeed, Zoll Finsler metrics on the projective plane (modulo isometries and rescaling) form an infinite-dimensional space, see Appendix.

The goal of this article is to study the space of Zoll Finsler metrics on the projective plane and the dynamics of their geodesic flow. More specifically, one can ask the following question about the topology of such space:

Is the space of all Zoll Finsler metrics on any closed manifold connected (when nonempty)?

In the Riemannian case, this is a famous question whose answer is only known for the projective plane: the canonical round metric is the only Zoll Riemannian metric on the projective plane modulo isometries and rescaling. Even on the two-sphere, the question is wide open, see [B03, Question 200]. (Observe that an approach through the Ricci flow does not work as shown in [J17].) Now, in the Finsler case, Zoll metrics are much more flexible (as forementionned, their moduli space — if nonempty — is always infinite-dimensional, see Appendix) and the question still makes sense.

Our main result shows that the topology of the space of Zoll Finsler metrics on the projective plane is homotopically trivial. This provides the first (positive) answer to the question above for Zoll Finsler metrics on the projective plane.

Theorem 1.1. *The space of Zoll Finsler metrics on the projective plane whose geodesic length is equal to π strongly deformation retracts to the canonical round metric on the projective plane.*

The strong deformation retraction is not given by some abstract existence theorem but proceeds from a natural geometric construction relying on the curvature flow (also referred to as the curve-shortening flow) of the canonical round metric contracting simple closed curves to points or great circles. The construction of the deformation retraction is fairly concrete. It relies on Theorem 1.3 below (and the material developed in the first part of the article) and a construction of Finsler metrics through the Crofton formula due to Álvarez Paiva and Berck [AB]. It follows from this construction that the geodesics of the Zoll Finsler metric F_τ along the deformation (F_τ) of a given Zoll Finsler metric F are obtained by applying the curvature flow of the canonical round metric to the geodesics of the given Zoll Finsler metric F. This approach would carry over to the case of Zoll Finsler metrics on the two-sphere if one could deform the geodesics of these metrics to the equators of the canonical round sphere while preserving their intersection pattern.

The following result is a straightforward consequence of Theorem 1.1. It immediately follows from a construction of [W75] (see also [G76, Appendix B] for a more explicit statement) relying on Moser's trick.

Corollary 1.2. *Let (F_τ) be the family of Zoll Finsler metrics on the projective plane $\mathbb{RP}^2$ given by applying the retraction constructed in Theorem 1.1 to a Zoll Finsler metric F with geodesic length π. There exists a natural one-parameter family of (homogeneous) symplectomorphisms*

$$\phi_\tau : T^*\mathbb{RP}^2 \setminus \{0\} \to T^*\mathbb{RP}^2 \setminus \{0\}$$

with $F_\tau^* \circ \phi_\tau = g_0^*$. In particular, the cogeodesic flows of F and g_0 are symplectically conjugate.

The symplectic conjugacy of the cogeodesic flows of Zoll Finsler two-spheres (and Zoll Finsler projective planes by taking their quotient) has recently been established in [ABHS17] by other means. Therefore, the statement about the symplectic conjugacy in Corollary 1.2 is not new, but our approach yields an alternative proof. Furthermore, it provides extra information on the symplectomorphism ϕ_τ. Indeed, by construction, the symplectomorphism ϕ_τ takes the cogeodesics of g_0 to the curves obtained from the (co)-geodesics of F by applying the curvature flow of the canonical round metric.

In the proof of our main theorem, we will need the following theorem connecting the geodesic flows of Zoll projective planes $\mathbb{RP}^2$. In this result, we identify the unit tangent bundle $U\mathbb{RP}^2$ of any Finsler metric on $\mathbb{RP}^2$ with the unit tangent bundle $U_0\mathbb{RP}^2$ of the canonical round metric g_0 by radial projection.

Theorem 1.3. *Let F be a Zoll Finsler metric on the projective plane $\mathbb{RP}^2$. There exists a natural one-parameter family of smooth free S^1-actions $(\rho_\tau)_{0\leq\tau\leq 1}$ on $U_0\mathbb{RP}^2$*

$$\rho_\tau : S^1 \times U_0\mathbb{RP}^2 \to U_0\mathbb{RP}^2$$

between the geodesic flows of F and g_0 such that every ρ_τ-orbit projects to an embedding of S^1 into $\mathbb{RP}^2$ under the canonical projection $U_0\mathbb{RP}^2 \to \mathbb{RP}^2$. Here, the convergence with respect to τ is in the C^k-topology for any given k.

Here again, the family of circle actions (ρ_τ) connecting the two geodesic flows proceeds in a natural way from the curvature flow: the ρ_τ-orbits correspond to the curves obtained from the F-geodesics by applying the curvature flow of the canonical round metric. This construction makes the family (ρ_τ) more trackable.

Actually, Theorem 1.3 directly follows from [H] once the intersection pattern of closed geodesics on Zoll Finsler surfaces is established, see Section 3. More precisely, the construction of the family of actions (ρ_t) follows from Theorem 4.7, which is a particular case of a result of [H] on the curvature flow. Still, we decided to present a proof of Theorem 4.7, since the estimates required in our case are weaker than those established in [H].

Specifically, the construction of the family of actions (ρ_τ) proceeds as follows. First, we examine the infinitesimal and non-infinitesimal intersection properties of the closed geodesics of Zoll Finsler two-spheres, see Section 3.

Then, we apply the curvature flow of the canonical round sphere to simultaneously deform these simple closed curves into equators of the round sphere. Here, we need to assume that the geodesics of the Zoll Finsler sphere divide the sphere into two domains of the same g_0-area, otherwise the curves shrink to points under the curvature flow of the canonical round sphere, see Theorem 2.3. (This is the case on the orientable double cover of a Zoll Finsler projective plane.) Note also that for arbitrary metrics on the two-sphere, the curvature flow may not converge as it may oscillate between closed geodesics. However, it does converge to equators of the round sphere when applied to simple curves dividing the round sphere into domains of the same area, see Theorem 2.3 for a discussion about the convergence of the curvature flow. Lifting the curve deformations given by the curvature flow to the unit tangent bundle, we connect the geodesic flow of the Zoll Finsler metric to the geodesic flow of the canonical round metric g_0.

We do not know whether our results extend to non-reversible Finsler metrics as several arguments only work in the reversible case. It would be interesting to clarify this point.

Acknowledgment. We are indebted to Juan-Carlos Álvarez Paiva for bringing the preprint [H] to our attention after we sent him a first version of our paper at the end of the summer 2015. The author would like to thank the referee for his or her careful reading and suggestions, and for pointing out some inaccuracies.

2. Preliminaries

In this preliminary section, we go over constructions related to the geodesic flow of a Zoll Finsler metric and review the main features of the curvature flow on the canonical round two-sphere.

Definition 2.1. A (reversible) *Finsler metric* on a closed manifold M is a continuous function $F : TM \to [0, \infty)$ on the tangent bundle TM of M satisfying the following properties (here, $F_x := F_{|T_xM}$ for short):

1) Smoothness: F is smooth outside the zero section;
2) Homogeneity: $F_x(tv) = |t|\, F_x(v)$ for every $v \in T_xM$ and $t \in \mathbb{R}$;
3) Quadratic convexity: for every $x \in M$, the function F_x^2 has positive definite second derivatives on $T_xM \setminus \{0\}$, that is, for every $p, u, v \in$

T_xM, the symmetric bilinear for

$$g_p(u,v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F_x^2(p+tu+sv)_{|t=s=0}$$

is an inner product.

The metric F induces a Minkowski norm F_x on each tangent space T_xM. We will denote by $F^* : T^*M \to \mathbb{R}$ the function whose restriction to each cotangent space T_x^*M is given by the dual norm F_x^*. The function F^* is the Fenchel-dual of F and satisfies the same properties (1), (2) and (3) as F.

The quadratically convex condition (as opposed to a mere convex condition) allows us to define a *geodesic flow* for F acting on the unit tangent bundle UM of M, see [Be78, §1]. The geodesic flow of a Zoll Finsler metric F on M of geodesic length 2π is periodic and defines a smooth free action of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on UM

$$\rho_F : S^1 \to \mathrm{Diff}(UM)$$

given by

$$\rho_F(\theta)(v) = \gamma_v'(\theta)$$

where γ_v is the (arclength parametrized) F-geodesic induced by v.

Recall that the *quotient manifold theorem* asserts that if G is a Lie group acting smoothly, freely and properly on a smooth manifold N, then the quotient space N/G is a topological manifold with a unique smooth structure such that the quotient map $N \to N/G$ is a smooth submersion. This result applies to the S^1-action ρ_F of the geodesic flow of F on UM.

Denote by

$$\Gamma_F = UM/\rho_F$$

the quotient manifold and by

$$q_F : SM \to \Gamma_F \tag{2.1}$$

the quotient submersion. The quotient manifold Γ_F represents the space of unparametrized oriented geodesics of the manifold M with the Zoll Finsler F. When M is a two-sphere, the space Γ_F is diffeomorphic to S^2 as it follows from the homotopy exact sequence of the fibration q_F, [Be78, §2.10].

Let us review the main features of the curvature flow on the canonical round two-sphere.

Definition 2.2. Let $\gamma : S^1 \to S^2$ be a smooth embedded curve on the canonical round sphere S^2. There exists a homotopy $\gamma_t : S^1 \to S^2$ evolving according to the equation

$$\frac{\partial \gamma_t}{\partial t} = \kappa \, \mathfrak{n}$$

where κ is the curvature of γ in S^2 and $\mathfrak{n}$ is its unit normal vector.

This flow, referred to as the *curvature flow* on the canonical round two-sphere, is defined for a maximal time interval $[0, T)$, where T is finite if and only if (γ_t) converges to a point when t tends to T, see [Gr89].

We summarize the properties of the curvature flow on the canonical round two-sphere that we will need in this article as follows.

Theorem 2.3 ([A91], [Ga90], [Gr89]). *The curvature flow (γ_t) of an embedded closed curve γ on the canonical round two-sphere satisfies the following properties:*

1) *the length of γ_t decreases unless γ is a geodesic, in which case the flow is constant;*
2) *the curves γ_t remain embedded, cf. [Ga90, Theorem 3.1] (see also [A91, Theorem 1.3]);*
3) *two disjoint smooth simple curves γ_1 and γ_2 remain disjoint through the curvature flow, that is, $\gamma_{1,t}$ and $\gamma_{2,t}$ are disjoint, unless one of them shrinks to a point, cf. [A91, §1].*
4) *the curvature of γ_t converges to zero in the C^∞-norm unless γ_t converges to a point, cf. [Gr89];*
5) *if γ divides the round sphere into two domains with the same area then the curves γ_t also divide the round sphere into two domains with the same area, cf. [Ga90, Proof of Theorem 5.1], and converge to an equator as unparametrized curves.*
6) *if γ does not divide the round sphere into two domains with the same area then the curvature flow (γ_t) converges to a point.*

Proof. The second part of the point (5) on the convergence of the curvature flow to an equator follows by combining the works of Gage [Ga90] and Grayson [Gr89]. Indeed, from the first part of the point (5), the curves γ_t divide the round sphere into two domains with the same area. In particular, the curvature flow (γ_t) of γ does not converge to a point. From the point (4), its curvature converges to zero in the C^∞-norm and, since the loops γ_t are simple, its length tends to 2π. In particular, its total geodesic curvature $\int_{\gamma_t} \sqrt{\kappa^2+1}\, ds$ tends to 2π, and so, is less than 3π for t large enough. This ensures that the two conditions of Theorem 5.1 in [Ga90] are satisfied. Therefore, we conclude that the curvature flow (γ_t) converges to an equator as unparametrized curves.

For the point (6), let D_t be the domain of the round two-sphere bounded by the simple closed curve γ_t such that the orientation of D_t induces the same orientation as γ_t on its boundary. By the Gauss-Bonnet formula, the area of D_t satisfies

$$|D_t| = 2\pi - \int_{\gamma_t} \kappa_t \, ds.$$

From [Ga90, Lemma 1.3], we have

$$\frac{d}{dt}|D_t| = -\int_{\gamma_t} \kappa_t \, ds = |D_t| - 2\pi.$$

Therefore, $|D_t| = (|D_0| - 2\pi)e^t + 2\pi$. Since $|D_0| \neq 2\pi$, it follows that the curvature flow of γ is only defined on a finite time interval. Hence, the result. □

3. Geodesic intersections on Zoll Finsler two-spheres

In this section, we examine some features satisfied by the geodesics of Zoll Finsler two-spheres.

The following result is established in [LM02] for Riemannian metrics but the proof carries over to Finsler metrics.

Proposition 3.1 ([LM02, Proposition 2.21]). *Let F be a Finsler metric on S^2. The following assertions are equivalent:*

(i) all the geodesics of F are simple closed curves;

(ii) all the geodesics of F are simple closed curves of the same length.

In particular, the orientable double cover of a Zoll Finsler projective plane of geodesic length π is a Zoll Finsler two-sphere of geodesic length 2π.

Remark 3.2. One could directly prove the last statement of Proposition 3.1. Simply observe that a noncontractible geodesic on a Finsler projective plane cannot be approached by a contractible one of the same length.

Thus, all the simple closed geodesics on a Zoll Finsler projective plane lift to simple closed geodesics of twice their length.

The following result clarifies the intersection pattern of geodesics on Zoll Finsler surfaces. Although the result is not surprising, we were unable to find a reference for it in the literature.

Theorem 3.3. *Let F be a Finsler metric on S^2. Every pair of distinct (closed) geodesics has exactly two (transverse) intersection points.*

Proof. Let $\mathcal{G} = \Gamma_F/\pm$ be the space of unparametrized geodesics of F. From [Be78, §2.10], the space $\mathcal{G}$ is diffeomorphic to $\mathbb{RP}^2$. Two distinct (unparametrized) closed geodesics have only transverse intersection points. Thus, the function defined on $\mathcal{G} \times \mathcal{G} \setminus \Delta$, where Δ is the diagonal, which gives the number of intersection points of a pair of distinct (unparameterized) closed geodesics is locally constant. Since $\mathcal{G} \times \mathcal{G} \setminus \Delta$ is connected, this function is constant.

Thus, every pair of distinct closed geodesics has exactly k (transverse) intersection points, where the integer k only depends on the dynamics of the geodesic flow of F. This integer is at least two for topological reasons.

Let γ_1 and γ_2 be two distinct closed geodesics. Since the closed geodesics γ_1 and γ_2 are simple, there exists a connected component D of $S^2 \setminus (\gamma_1 \cup \gamma_2)$ bounded by exactly two geodesic arcs (one lying in γ_1 and the other lying in γ_2). This connected component forms a bigon with endpoints p and q. Changing the parametrization of γ_1 and γ_2 if necessary, we can assume that the tangent vectors $v_1 = \gamma_1'(0)$ and $v_2 = \gamma_2'(0)$ based at p span a sector in T_pS^2 pointing inside D, see Figure 1.

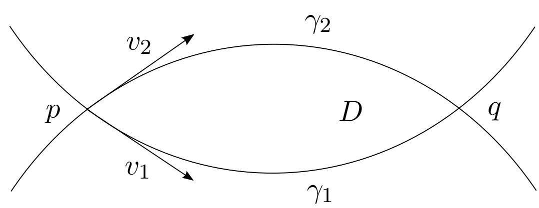

Figure 1: The digon D.

Observe that the connected component D continuously varies with γ_1 and γ_2 as long as v_1 and v_2 are not collinear. In particular, by rotating v_1 to v_2 and v_2 to $-v_1$, we deform γ_1 to γ_2 and γ_2 to $-\gamma_1$ through two homotopies of simple closed geodesics γ_1^t and γ_2^t. Through this process, the digon D bounded by the two arcs of γ_1 and γ_2 joining p to q and directed by v_1 and v_2 deforms through a family of digons D_t bounded by the two arcs of γ_1^t and γ_2^t joining p to some point $q_t \in \gamma_1^t \cap \gamma_2^t$ and directed by v_1^t and v_2^t. The digon D_t is a connected component of $M \setminus (\gamma_1^t \cup \gamma_2^t)$. By construction, the point q_t is the first point of intersection of γ_1^t and γ_2^t when travelling along these two geodesics from p in the directions of v_1^t and v_2^t. At the final time $t = 1$, when the geodesics γ_1^1 and γ_2^1 agree with γ_2 and $-\gamma_1$, the first point of intersection between the geodesics when travelling from p agree with q. That is, $q_1 = q$. As the first point of intersection between γ_2 and $-\gamma_1$ agrees with the last point of intersection between γ_2 and γ_1 along γ_1, it follows that γ_1 and γ_2 have exactly two intersection points, namely p and q. Hence, $k = 2$. □

In the rest of this section, we introduce the (non-metric) notion of normal vector fields along simple loops on a surface. We also determine the number of zeros of nontrivial normal vector fields defined by geodesic variations on a Zoll Finsler two-sphere.

Definition 3.4. Given a closed surface M, let $c : S^1 \times (-\varepsilon, \varepsilon) \to M$ be a smooth map inducing a smooth variation of embedded curves $c_\lambda = c(., \lambda)$ with $\lambda \in (-\varepsilon, \varepsilon)$. Here, the curves c_λ are not necessarily geodesics. Define the following vector field $Y \in \Gamma(c_0^* TM)$ along c_0 as

$$Y(\theta) = \frac{\partial c}{\partial \lambda}(\theta, 0)$$

for every $\theta \in S^1$. When the curves c_λ are geodesics for some Finsler metric F on M, the vector field Y represents the Jacobi field along c_0 generated by the geodesic variation (c_λ), see [S01, §11.2]. The vector field Y induces a *normal vector field* $Y_\perp$ along c_0 defined as

$$Y_\perp(\theta) \equiv Y(\theta) \bmod \mathbb{R}.c_0'(\theta)$$

for every $\theta \in S^1$, where $Y_\perp(\theta)$ lies in the quotient of the plane $T_{c_0(\theta)}M$ by the vector line $\mathbb{R}.c_0'(\theta)$ generated by $c_0'(\theta)$. On an orientable surface, a normal vector field along c_0 is merely a function.

We start with the following observation showing that the notion of normal vector field extends to variations of unparametrized (oriented or unoriented) embedded curves.

Lemma 3.5. *Let c_0 be a curve in a closed manifold M. The normal vector field $Y_\perp$ along c_0 induced by a curve variation (c_λ) does not depend on the parametrization of the curves c_λ.*

Proof. Consider a variation of curves $\bar{c}_\lambda(\cdot) = c_\lambda(\theta(\cdot, \lambda))$, where $\theta(\cdot, \lambda)$ represents a regular change of parameter. At $\theta = \theta(\bar{\theta}, 0)$, the points $\bar{c}_0(\bar{\theta})$ and $c_0(\theta)$ agree as well as the lines generated by $\bar{c}_0'(\bar{\theta})$ and $c_0'(\theta)$. Now, the vector field $\bar{Y}$ induced by the curve variation $(\bar{c}_\lambda)$ satisfies

$$\begin{aligned} \bar{Y}(\bar{\theta}) &= \frac{\partial \bar{c}}{\partial \lambda}(\bar{\theta}, 0) \\ &= \frac{\partial c}{\partial \lambda}(\theta, 0) + \frac{\partial \theta}{\partial \lambda}(\bar{\theta}, 0)\, c_0'(\theta) \\ &\equiv Y(\theta) \bmod \mathbb{R}.c_0'(\theta) \end{aligned}$$

Hence, $\bar{Y}_\perp(\bar{\theta}) = Y_\perp(\theta)$ at the point $\bar{c}_0(\bar{\theta}) = c_0(\theta)$. □

The following property satisfied by every normal Jacobi vector field $Y_\perp$ of a Zoll Finsler two-sphere can be seen as an infinitesimal version of Theorem 3.3.

Theorem 3.6 ([LM02, Theorem 2.15]). *Let F be a Zoll Finsler metric on S^2. Every nontrivial normal Jacobi vector field $Y_\perp$ induced by a variation of an unparametrized (oriented or unoriented) geodesic γ has exactly two zeros.*

Furthermore, the zeros of $Y_\perp$ are simple, that is, the vector fields $Y_\perp$ and $(Y_\perp)'$ do not simultaneously vanish.

Proof. The first statement of the proposition is established in [LM02, Theorem 2.15] for Zoll torsion-free affine connexions on the two-sphere. The arguments carry over in our setting. For the sake of the reader and since the arguments are so elegant, we briefly reproduce them.

Without loss of generality, we can assume that γ is an unparametrized oriented geodesic. Consider the quotient submersion $q : US^2 \to \Gamma$ induced by the geodesic flow of F, where $\Gamma = \Gamma_F$ represents the space of unparametrized oriented geodesics of F, see (2.1). Denote by $PT\Gamma$ the projectivized tangent space of Γ. The submersion $q : US^2 \to \Gamma$ factors through a map $\varphi : US^2 \to PT\Gamma$ under the canonical projection $PT\Gamma \to \Gamma$. That is, the following diagram is commutative

$$\begin{array}{ccc} & & PT\Gamma \\ & \overset{\varphi}{\nearrow} & \downarrow \\ US^2 & \xrightarrow[q]{} & \Gamma \end{array}$$

The map φ can be defined as follows. Identify the tangent plane of Γ at γ with the space of normal Jacobi fields along γ, see [Be78, Proposition 2.13]. With this identification, the map φ takes a unit vector $v \in U_xS^2$ with basepoint $x \in S^2$ to the class of normal Jacobi vector fields along γ_v vanishing at x. Note that the map φ takes every orbit of the geodesic flow of F to a different fiber of the projection $PT\Gamma \to \Gamma$. Observe also that the map φ is a local diffeomorphism and so a covering since US^2 is compact.

The index of the covering is given by

$$\frac{|\pi_1(PT\Gamma)|}{|\pi_1(US^2)|} = \frac{4}{2} = 2$$

since $\Gamma \simeq S^2$ and $US^2 \simeq \mathbb{RP}^3$.

Now, two vectors u and v of US^2 are sent by φ to the same class $[Y_\perp]$ of a nontrivial normal Jacobi field $Y_\perp$ along a geodesic γ if and only if γ_u and γ_v represent the same unparametrized oriented geodesic γ, and their basepoints are zeros of $Y_\perp$.

By definition of the index of a covering, every class of a nontrivial normal Jacobi field along γ has two preimages by φ. It follows that $Y_\perp$ has exactly two zeros.

For the second statement, recall that every Jacobi field Y along c_0 satisfies a second-order linear differential equation, and so does the normal vector field $Y_\perp$, see [LM02, Equation (3)] or [S01, §11.2], namely

$$Y_\perp'' + \kappa Y_\perp = 0$$

where κ is a smooth function. Hence, the zeros of $Y_\perp$ are simple unless $Y_\perp$ is trivial. $\square$

4. Curvature flow and circle action deformations on the unit tangent bundles of Zoll Finsler two-spheres

By analyzing the parabolic partial differential equation satisfied by the curvature flow of the canonical round two-sphere, we show that this flow induces an isotopy of diffeomorphisms of the unit tangent bundles of balanced Zoll

Finsler spheres. As mentioned in the introduction, several results of this section can be derived from [H].

Let F be a Zoll Finsler metric on S^2. The unit tangent bundle US^2 of S^2 with the metric F naturally identifies with the unit tangent bundle U_0S^2 of the canonical metric g_0 on S^2 by radial projection on each tangent plane. With this identification, the smooth free action of S^1 on US^2 given by the geodesic flow F induces a smooth free action on U_0S^2 by conjugacy. This S^1-action is denoted by

$$\rho : S^1 \to \mathrm{Diff}(U_0S^2)$$

and defined as

$$\rho(\theta)(v) = \gamma_v'(\theta) \tag{4.1}$$

for every $\theta \in S^1$ and $v \in U_0S^2$. In this expression, the vector v of U_0S^2 is identified with a vector of US^2 by radial projection. Similarly, the vector $\gamma_v'(\theta)$ of US^2 is identified with a vector of U_0S^2 by radial projection. Observe that the orbits of the actions of S^1 on U_0S^2 project down to embedded closed curves in S^2, namely the F-geodesics γ_v.

Definition 4.1. A Zoll Finsler metric F on S^2 is *balanced* if every F-geodesic of S^2 divides the round sphere into two domains D_1 and D_2 with the same g_0-area, where g_0 is the canonical round metric. This property is satisfied if F is invariant under the antipodal map, that is, if it is the orientable double cover of a Zoll Finsler projective plane.

Remark 4.2. We introduce the notion of balanced Zoll Finsler metrics for the following reason. For a balanced Zoll Finsler metric F on S^2, the simple closed geodesics γ_v induced by the vectors $v \in US^2$ converge to the equators of S^2 with the canonical round metric g_0 through the curvature flow γ_v^t of g_0, see Theorem 2.3.(5). While if F is not balanced, the convergence does not hold anymore since a simple closed curve not dividing the round sphere into two domains with the same area shrinks to a point through the curvature flow of the round sphere, see Theorem 2.3.(6).

For every balanced Zoll Finsler metric F on S^2, consider the map

$$\Psi_t : U_0S^2 \to U_0S^2$$

defined as

$$\Psi_t(v) = (\gamma_v^t)'(0)$$

for every $v \in U_0S^2$ and $t \in [0, \infty)$. Here, γ_v is the F-geodesic induced by v and (γ_v^t) is the curvature flow of γ_v on the canonical round sphere, see Definition 2.2. As previously, we identify the vector v of U_0S^2 with a vector of US^2 and the vector $(\gamma_v^t)'(0)$ of $TS^2 \setminus \{0\}$ with a vector of U_0S^2 by radial projection. Note that Ψ_0 is the identity map on U_0S^2.

We will need the following classical result about the number of zeros of a parabolic partial differential equation.

Theorem 4.3 (see [A88, Theorem C]). *Let $u : S^1 \times [0, T] \to \mathbb{R}$ be a bounded solution of the equation*

$$u_t = a(x, t)\, u_{xx} + b(x, t)\, u_x + c(x, t)\, u$$

where a, a^{-1}, a_t, a_x, a_{xx}, b, b_t, b_x and c are bounded functions. Then, for every $t \in (0, T)$, the number $z(t)$ of zeros of $u(., t)$ is finite.

Furthermore, if both u and u_x vanish at (x_0, t_0) then $z(t_-) > z(t_+)$ for every $t_- < t_0 < t_+$. That is, the number of zeros decreases whenever a multiple zero occurs.

We can now show the following result.

Proposition 4.4. *Let F be a balanced Zoll Finsler metric on S^2. For every $t \in [0, \infty)$, the map $\Psi_t : U_0S^2 \to U_0S^2$ induced by the curvature flow of the canonical round sphere is a local diffeomorphism.*

Proof. Let us show first that the map

$$\Xi_t : US^2 \to TS^2 \setminus \{0\}$$

defined as $\Xi_t(v) = (\gamma_v^t)'(0)$ is an immersion for every $t \geq 0$.

For $t = 0$, this clearly holds true. Indeed, by construction,

$$\Xi_0(v) = (\gamma_v^0)'(0) = v$$

for every $v \in US^2$. That is, the map Ξ_0 is the inclusion map and so is an immersion.

Fix $v \in US^2$ and $\tau \in (0, \infty)$. Let $w = w(\lambda)$ be a smooth curve in US^2 with $w(0) = v$. Denote $\nu = w'(0)$. For the sake of simplicity, we will sometimes write γ_λ for $\gamma_{w(\lambda)}$. Note that $\gamma_0 = \gamma_v$. We want to show that the differential $d\,\Xi_\tau(v)$ of Ξ_τ at v is injective. That is, if the derivative $d\,\Xi_\tau(v)(\nu)$

of $\Xi_\tau(w(\lambda))$ vanishes at $\lambda = 0$ then the vector $\nu = w'(0)$ of TUS^2 is zero. The idea is to write down in local coordinates the partial differential equation satisfied by $\Xi_\tau(w)$ and to study the evolution of the normal Jacobi field given by the geodesic variation (γ_w).

By Theorem 2.3.(2), the curve γ_v^τ defines an embedding of S^1 into S^2. This embedding extends to an embedding $h : S^1 \times (-1, 1) \to S^2$ of a cylinder onto a collar neighborhood of γ_v^τ, which gives rise to a normal coordinate system with $h(., 0) = \gamma_v^\tau$ in the canonical round sphere.

In this normal coordinate system around γ_v^τ, every curve γ_λ^t with (λ, t) close enough to $(0, \tau)$ can be represented in a nonparametric way as the graph

$$\{(x, \mathfrak{u}(x, t, \lambda)) \in S^1 \times (-1, 1) \mid x \in S^1\}$$

of a function $\mathfrak{u}(., t, \lambda)$ over S^1. Observe that $\mathfrak{u}(x, \tau, 0) = 0$ for every $x \in S^1$. From [A90, Eq. (3.2)] or [Ga90, Appendix], the function

$$\mathfrak{u} : S^1 \times (\tau - \delta, \tau + \delta) \times (-\varepsilon, \varepsilon) \to (-1, 1)$$

satisfies the following parabolic partial differential equation of the curvature flow:

$$\mathfrak{u}_t = \mathcal{F}(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})$$

where $\mathcal{F}$ is a smooth function defined on $S^1 \times (-1, 1) \times \mathbb{R}^2$ with

$$\mathcal{F}_q(x, u, p, q) > 0$$

which can be expressed in terms of the coefficients of the canonical round metric in the normal coordinate system. Here, the subscript notations refer to partial differentiations.

In a parametric representation, the abscisse of γ_λ^t is a function of the parameter θ, that is, $x = x(\theta, t, \lambda)$ with $x(\theta, \tau, 0) = \theta$. Thus,

$$\gamma_\lambda^t(\theta) = (x(\theta, t, \lambda), \mathfrak{u}(x(\theta, t, \lambda), t, \lambda)).$$

Differentiating this expression with respect to θ yields the tangent vector $(\gamma_\lambda^t)'(\theta)$ which can be represented as

$$(\gamma_\lambda^t)'(\theta) = (x(\theta, t, \lambda), \mathfrak{u}(x(\theta, t, \lambda), t, \lambda), x_\theta(\theta, t, \lambda), \mathfrak{u}_x(x(\theta, t, \lambda), t, \lambda)\, x_\theta(\theta, t, \lambda))$$

or

$$(\gamma_\lambda^t)'(\theta) = (x, \mathfrak{u}, x_\theta, \mathfrak{u}_x\, x_\theta) \tag{4.2}$$

for short.

Note that $\Xi_t(w) = (\gamma_w^t)'(0)$. Thus, the differential of Ξ_t at v in the direction $\nu = w'(0)$ is obtained by differentiating the relation (4.2) with respect to λ at $\lambda = 0$. That is,

$$d\,\Xi_t(v)(\nu) = (x_\lambda, \mathfrak{u}_x\, x_\lambda + \mathfrak{u}_\lambda, x_{\theta\lambda}, \mathfrak{u}_{xx}\, x_\theta\, x_\lambda + \mathfrak{u}_{x\lambda}\, x_\theta + \mathfrak{u}_x\, x_{\theta\lambda})$$

evaluated at $(0, t, 0)$. Observing that $x_\theta(0, \tau, 0) = 1$, we simplify this expression as follows

$$d\,\Xi_\tau(v)(\nu) = (x_\lambda, \mathfrak{u}_x\, x_\lambda + \mathfrak{u}_\lambda, x_{\theta\lambda}, \mathfrak{u}_{xx}\, x_\lambda + \mathfrak{u}_{x\lambda} + \mathfrak{u}_x\, x_{\theta\lambda}).$$

Now, suppose that ν lies in the kernel of the differential $d\,\Xi_\tau(v)$ of Ξ_τ at v, that is, $d\,\Xi_\tau(v)(\nu) = 0$. In this case, the functions x_λ, $\mathfrak{u}_\lambda$, $x_{\theta\lambda}$ and $\mathfrak{u}_{x\lambda}$ vanish at $(0, \tau, 0)$. Hence, both $\mathfrak{v}$ and $\mathfrak{v}_x$ vanish at $(0, \tau, 0)$, where $\mathfrak{v} = \mathfrak{u}_\lambda$. That is, the function $\mathfrak{v}$ has a multiple zero at $(0, \tau, 0)$.

Now, in a more intrinsic way, the zeros of $\mathfrak{v}$ can be related to the zeros of the normal vector field induced by the curve variation (γ_λ^t) as follows. The vector field along γ_v^t induced by the curve variation (γ_λ^t), see Definition 3.4, is given by

$$Y^t(\theta) = \frac{\partial}{\partial\lambda}\gamma_\lambda^t(\theta)_{|\lambda=0} = (x, \mathfrak{u}, x_\lambda, \mathfrak{u}_x\, x_\lambda + \mathfrak{u}_\lambda) \tag{4.3}$$

evaluated at $(\theta, t, 0)$. As $x_\theta(\theta, t, 0) \neq 0$ for t close enough to τ, it follows from the expression of $(\gamma_v^t)'$ and Y^t, see (4.2) and (4.3), that the normal vector field $Y_\perp^t$ defined in Definition 3.4 vanishes if and only if $\mathfrak{v} = \mathfrak{u}_\lambda$ vanishes. More precisely,

$$Y_\perp^t(\theta) = 0 \Leftrightarrow \mathfrak{v}(x, t) = 0 \tag{4.4}$$

where $\mathfrak{v}(x, t) = \mathfrak{u}_\lambda(x, t, 0)$ and $x = x(\theta, t, 0)$.

The number of zeros of $Y_\perp^t$ is given by the following result.

Lemma 4.5. *The normal vector field $Y_\perp^t$ has exactly two zeros along γ_v^t for every $t \geq 0$.*

Proof. At $t = 0$, the curves γ_λ^0 are geodesic for the Zoll Finsler metric F. It follows from Theorem 3.6 that $Y_\perp^0$ has exactly two zeros along γ_v. Moreover, these zeros are simple. By the implicit function theorem, we deduce that $Y_\perp^t$ has exactly two zeros along γ_v^t for every $t > 0$ small enough.

Let us examine how the number of zeros of $Y_\perp^t$ evolves with t for every $t > 0$. Differentiating the partial differential equation

$$\mathfrak{u}_t = \mathcal{F}(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})$$

with respect to λ yields the following expression

$$\mathfrak{u}_{\lambda t} = \mathcal{F}_u(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})\, \mathfrak{u}_\lambda + \mathcal{F}_p(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})\, \mathfrak{u}_{\lambda x} + \mathcal{F}_q(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})\, \mathfrak{u}_{\lambda xx}.$$

Thus, the function $\mathfrak{v} = \mathfrak{u}_\lambda$ satisfies the parabolic partial differential equation

$$\mathfrak{v}_t = a(x, t, \lambda)\, \mathfrak{v}_{xx} + b(x, t, \lambda)\, \mathfrak{v}_x + c(x, t, \lambda)\, \mathfrak{v} \tag{4.5}$$

where $a = \mathcal{F}_q(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})$, $b = \mathcal{F}_p(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})$ and $c = \mathcal{F}_u(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx})$.

By Theorem 4.3, the number of zeros of $\mathfrak{v}(., t)$ is nonincreasing with t. Therefore, the number of zeros of the normal vector field $Y_\perp^t$ along γ_v^t is nonincreasing too from the relation (4.4). Since $Y_\perp^t$ has exactly two zeros for t small enough, it follows that $Y_\perp^t$ has at most two zeros for every $t > 0$.

Now, if $Y_\perp^t$ had less than two zeros, then all the curves (γ_λ^t) would be on one side of the simple loop γ_v^t for every $\lambda > 0$ small enough (at least to the first order). This is impossible since γ_v^t and γ_w^t divide the round sphere into two domains of the same area. Therefore, the vector field $Y_\perp^t$ has exactly two zeros along γ_v^t for every t. □

Let us continue the proof of Proposition 4.4. Combined with (4.4), Lemma 4.5 shows that the function $\mathfrak{v}(., t)$ has a constant number of zeros, namely two, for every $t \geq 0$. Since $\mathfrak{v}$ satisfies the parabolic partial differential equation (4.5), we deduce from Theorem 4.3 that the functions $\mathfrak{v}$ and $\mathfrak{v}_x$ do not simultaneously vanish. Thus, the differential of Ξ_τ at v is injective. Hence the map $\Xi_t : US^2 \to TS^2$ is an immersion.

Let us show now that the map $\Xi_\tau : US^2 \to TS^2 \setminus \{0\}$ is transverse to the rays $\mathbb{R}_+^* u = \{su \mid s > 0\}$, where the vector u runs over $TS^2 \setminus \{0\}$. We argue by contradiction and assume that there exists $v \in US^2$ and $\nu \in T_v US^2$ such that the vector $\vec{u}$ tangent to the ray passing through $\Xi_\tau(v)$ and the image vector $d\Xi_\tau(v)(\nu)$ are colinear in the tangent space to $TS^2 \setminus \{0\}$ at $\Xi_\tau(v)$. In the previous normal coordinate system, the two vectors can be written as

$$\vec{u} = (0, 0, x_\theta, \mathfrak{u}_x\, x_\theta)$$

and

$$d\Xi_\tau(v)(\nu) = (x_\lambda, \mathfrak{u}_x\, x_\lambda + \mathfrak{u}_\lambda, x_{\theta\lambda}, \mathfrak{u}_{xx}\, x_\lambda + \mathfrak{u}_{x\lambda} + \mathfrak{u}_x\, x_{\theta\lambda})$$

where the expressions are evaluated at $(0,\tau,0)$. Since the two vectors are colinear, the functions x_λ and $\mathfrak{u}_\lambda$, and the determinant

$$\begin{vmatrix} x_\theta & x_{\theta\lambda} \\ \mathfrak{u}_x\, x_\theta & \mathfrak{u}_{xx}\, x_\lambda + \mathfrak{u}_{x\lambda} + \mathfrak{u}_x\, x_{\theta\lambda} \end{vmatrix} = (\mathfrak{u}_{xx}\, x_\lambda + \mathfrak{u}_{x\lambda})\, x_\theta$$

vanish at $(0,\tau,0)$. So does the function $\mathfrak{u}_{x\lambda}$ (recall that $x_\theta(0,\tau,0)=1$). Hence, both $\mathfrak{v}$ and $\mathfrak{v}_x$ vanish at $(0,\tau,0)$, where $\mathfrak{v}=\mathfrak{u}_\lambda$. That is, the function $\mathfrak{v}$ has a multiple zero at $(0,\tau,0)$, which is impossible from the previous argument. Thus, the map $\Xi_\tau : US^2 \to TS^2 \setminus \{0\}$ is transverse to the rays of $TS^2 \setminus \{0\}$.

Therefore, the map $\Psi_t : U_0S^2 \to U_0S^2$ defined from Ξ_t by identifying $U_0S^2 \simeq US^2$ and taking the radial projection $TS^2 \setminus \{0\} \to U_0S^2$ is a local diffeomorphism. □

The previous propositions yield the following result.

Theorem 4.6. *Let F be a balanced Zoll Finsler metric on S^2. For every $t \in [0,\infty)$, the map $\Psi_t : U_0S^2 \to U_0S^2$ induced by the curvature flow of the canonical round sphere is a diffeomorphism.*

Proof. From Proposition 4.4 and since U_0S^2 is compact, the map Ψ_t is a proper local diffeomorphism. Therefore, it is a covering map. Now, the map Ψ_t is π_1-injective (this is clearly the case for Ψ_0 and this property is preserved under homotopy). Hence, the covering Ψ_t is a diffeomorphism for every $t \geq 0$. □

This isotopy of diffeomorphisms allows us to define a deformation ρ_t of the geodesic flow $\rho_0 = \rho$ of balanced Zoll Finsler spheres, *cf.* (4.1), to the geodesic flow of the canonical round sphere as follows.

Let F be a balanced Zoll Finsler metric on S^2. For every $v \in U_0S^2$, consider the unique curve γ_u^t tangent to v at $\theta = 0$ and pointing in the same direction as v. That is, $u = \Psi_t^{-1}(v)$ under the identification $U_0S^2 \simeq US^2$. Reparametrize this curve proportionally to its g_0-arclength preserving both its initial point and its orientation. Define the S^1-action

$$\rho_t : S^1 \to \mathrm{Diff}(U_0S^2)$$

such that $\rho_t(\theta)$ takes v to the tangent vector of this new curve at the point of parameter θ. Since Ψ_t is a diffeomorphism, the map $\rho_t(\theta)$ is also a diffeomorphism of U_0S^2. Clearly, the S^1-action ρ_t on U_0S^2 is free and satisfies the *symmetry property*:

$$\rho_t(\theta)(-v) = -\rho_t(-\theta)(v) \tag{4.6}$$

for every $t \in [0, \infty)$, $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $v \in U_0 S^2$. Moreover, every ρ_t-orbit projects to an embedding of S^1 into S^2 by the canonical projection $U_0 S^2 \to S^2$. It is also worth pointing out that the expression of $\rho_t(\theta)(v)$ vary smoothly with respect to $t \in [0, \infty)$, $\theta \in S^1$ and $v \in U_0 S^2$.

Thus defined, the actions ρ_t satisfy the following convergence result which implies Theorem 1.3 by passing to the quotient.

Theorem 4.7. *Let F be a balanced Zoll Finsler metric on S^2. Then the smooth free S^1-actions*

$$\rho_t : S^1 \times U_0 S^2 \to U_0 S^2$$

C^2-converge to the action ρ_∞ induced by the geodesic flow of the canonical round sphere.

Furthermore, for $t \in [0, \infty]$, every ρ_t-orbit projects to an embedding of S^1 into S^2 under the canonical projection $U_0 S^2 \to S^2$.

Proof. It follows from Theorem 2.3 that for every $\varepsilon > 0$ and every $t \geq 0$ large enough, the unparametrized loops γ_u^t have curvature at most ε on the canonical round sphere. In particular, these loops are uniformly close to the equators to which they are tangent (for the C^2-Fréchet topology). By construction, this implies that the action ρ_t is C^2-close to the action ρ_∞ induced by the g_0-geodesic flow for t large enough.

The last statement about the orbits of ρ_t is also satisfied since these orbits are transverse to the fibers of $U_0 S^2 \to S^2$ and project to the images of the F-geodesics under the curvature flow (which do not self-intersect). □

Remark 4.8. The convergence result of Theorem 4.7 shows that the one-parameter family of S^1-actions (ρ_t) is defined for $t \in [0, \infty]$. The proof of Theorem 4.7 only yields a C^2-convergence. However, the more precise estimates of [H] give rise to a uniformly exponential C^k-convergence, see [H, Theorem 3.1]. Thus, the reparametrization (ρ_τ) of this family with $\tau = 1 - e^{-t}$ provides a C^k-convergence of (ρ_τ) with respect to τ. We will consider this reparametrization in the rest of the article.

Passing to the quotient, we derive a similar result on $\mathbb{RP}^2$, namely Theorem 1.3.

Proof of Theorem 1.3. Apply Theorem 4.7 to the lift of the Zoll Finsler metric of $\mathbb{RP}^2$ to the orientable double cover S^2 (which is balanced). By construction, the simple closed curves γ_u^t on S^2 are equivariant with respect to the antipodal maps on S^1 and S^2, that is, $\gamma_u^t(\theta + u) = -\gamma_u^t(\theta)$. As a result, the free S^1-action ρ_t on U_0S^2 passes to the quotient and gives rise to a free S^1-action on $U\mathbb{RP}^2$ whose orbits project to simple closed curves in $\mathbb{RP}^2$. From Remark 4.8, the convergence of the induced S^1-action on $U\mathbb{RP}^2$ with respect to τ is in the C^k-topology. □

5. Crofton formula on Zoll Finsler two-spheres

We review some constructions on the sphere S^2 equipped with a Zoll Finsler metric F all of whose geodesics are of length 2π, including a general Crofton formula.

Consider the Legendre transform

$$\mathcal{L} : TS^2 \to T^*S^2$$

of the Lagrangian $\frac{1}{2}F^2$. Since F is quadratically convex, the Legendre transform is a diffeomorphism between $TS^2 \setminus \{0\}$ and $T^*S^2 \setminus \{0\}$. By homogeneity of F, it preserves the norm on each fiber of the bundle vectors TS^2 and T^*S^2. In particular, it induces a diffeomorphism between the unit sphere bundle and the unit co-sphere bundle US^2 and U^*S^2. Geometrically, this diffeomorphism is defined as follows: for every vector $v \in U_xS^2$, the image $\mathcal{L}(v)$ of v is the unique covector of $U_x^*S^2$ such that $\mathcal{L}(v)(v) = 1$.

From now on, we will identify TS^2 with T^*S^2 and US^2 with U^*S^2 via the Legendre transform. With these identifications, the action ρ_F of S^1 on US^2 given by the geodesic flow of F induces an action on U^*S^2 by conjugacy by the Legendre transform, namely the co-geodesic flow of F. Despite the risk of confusion, both S^1-actions will be denoted by ρ_F.

Let α be the tautological one-form on T^*S^2. By definition,

$$\alpha_\xi(V) = \xi(d\pi_\xi(V))$$

for every $\xi \in T^*S^2$ and $V \in T_\xi T^*S^2$, where $\pi : T^*S^2 \to S^2$ is the canonical surjection. From the Liouville theorem, the tautological one-form α (and so the symplectic form $\omega_0 = d\alpha$) is invariant under the co-geodesic flow of any Finsler metric. Observe also that the S^1-orbits of ρ_F on U^*S^2 are transverse to the contact structure given by the kernel of α. Now, the pull-back of the 3-form $\alpha \wedge d\alpha$ under the inclusion map $U^*S^2 \hookrightarrow T^*S^2$ defines a volume form on U^*S^2. (More generally, the pull-back of α is a contact one-form on U^*S^2.) Since U^*S^2 is the unit cotangent bundle of a Finsler sphere all of whose geodesics are closed of lengh 2π, the integral of this volume form on U^*S^2 does not depend on the Finsler metric and is equal to $\pm 8\pi^2$ by a result of A. Weinstein, *cf.* [Be78, §2.C]. That is,

$$\int_{U^*S^2} \alpha \wedge d\alpha = \pm 8\pi^2. \tag{5.1}$$

By the quotient manifold theorem, the S^1-action on U^*S^2 given by the co-geodesic flow ρ_F gives rise to a quotient manifold

$$\Gamma_F = U^*S^2/\rho_F$$

diffeomorphic to S^2, representing the space of unparametrized oriented geodesics of the Zoll Finsler metric F on S^2, and a quotient submersion

$$q_F : U^*S^2 \to \Gamma_F. \tag{5.2}$$

By construction, the map q_F takes a unit cotangent vector of S^2 to the unparametrized oriented F-geodesic of S^2 with the Legendre transform of this unit cotangent vector as initial condition. Thus, the projection $\pi(q_F^{-1}(\gamma))$ of a fiber over γ represents the unparametrized closed geodesic of F on S^2 given by $\gamma \in \Gamma_F$. We will sometimes identify γ with $\pi(q_F^{-1}(\gamma))$.

Consider the double fibration

$$\begin{array}{ccccc} & & U^*S^2 & \xrightarrow{\;i\;} & T^*S^2 \\ & \overset{\pi}{\swarrow} & & \overset{q_F}{\searrow} & \\ S^2 & & & & \Gamma_F \end{array}$$

where $i : U^*S^2 \hookrightarrow T^*S^2$ is the canonical injection and $\pi : U^*S^2 \to S^2$ is the canonical surjection. Note that the product map $\pi \times q_F : U^*S^2 \to S^2 \times \Gamma_F$ is an embedding. From [Be78], there exists a unique symplectic form λ_F on Γ_F such that

$$q_F^*\,\lambda_F = i^*\omega_0. \tag{5.3}$$

The general Crofton formula on Finsler surfaces can be stated as follows.

Theorem 5.1 ([AB06, Theorem 5.2]). *With the previous notations, the length of every smooth curve c on S^2 with the Zoll Finsler metric F satisfies*

$$\text{length}_F(c) = \frac{1}{4} \int_{\gamma \in \Gamma_F} \#(\gamma \cap c) \, |\lambda_F| \tag{5.4}$$

where $|\lambda_F|$ is the smooth positive area density on Γ_F induced by the symplectic form λ_F.

Remark 5.2. Strictly speaking, the integrand in the formula (5.4) should be $\#(\pi(q_F^{-1}(\gamma)) \cap c)$ instead of $\#(\gamma \cap c)$, but as aforementioned, we identify the elements γ in Γ_F with the unparametrized geodesics $\pi(q_F^{-1}(\gamma))$ they represent.

Remark 5.3. The Crofton formula (5.4) shows that the Zoll Finsler metric F is uniquely determined by the submersion $q_F : U^*S^2 \to \Gamma_F$ (and the symplectic form λ_F on Γ_F derived from q_F).

6. Deforming Zoll Finsler two-spheres

In this section, we construct a natural deformation of Zoll Finsler metrics on S^2 to the canonical round metric by applying the Crofton formula to the orbits of the converging family of the circle actions given by the curvature flow, see Theorem 4.7.

Consider a Zoll Finsler metric F on S^2 all of whose geodesics are of length 2π. Let ρ be a smooth free S^1-action on U^*S^2 whose orbits are transverse to the contact structure given by the kernel of the tautological one-form α on U^*S^2 and project to embeddings of S^1 into S^2. The action ρ induces a Legendrian action $\bar{\rho}$ defined as follows. Consider the map

$$\mathcal{R} : US^2 \to U^*S^2$$

sending every vector $v \in U_xS^2$ to the unique covector $\xi \in U_x^*S^2$ such that $\xi(v) = 0$ with $(v, \mathcal{L}^{-1}(\xi))$ positively oriented. The map $\mathcal{R} : US^2 \to U^*S^2$ and its restrictions $\mathcal{R}_x : U_xS^2 \to U_x^*S^2$ are diffeomorphisms. Thus, the free S^1-action ρ on U^*S^2 induces a free S^1-action $\bar{\rho}$ on U^*S^2 by conjugation by

$$\Upsilon = \mathcal{L} \circ \mathcal{R}^{-1} : U^*S^2 \to U^*S^2$$

namely

$$\bar{\rho}(\theta) = \Upsilon^{-1} \circ \rho(\theta) \circ \Upsilon$$

for every $\theta \in S^1$. Denote by

$$q_{\bar{\rho}} : U^*S^2 \to \Gamma_{\bar{\rho}}$$

the submersion induced by the free S^1-action $\bar{\rho}$, where $\Gamma_{\bar{\rho}} = U^*S^2/\bar{\rho}$. Since the actions ρ and $\bar{\rho}$ are conjugate, we have $\Gamma_{\bar{\rho}} \simeq \Gamma_\rho$, where $\Gamma_\rho = U^*S^2/\rho$.

In order to apply the results of [AB], we will need the following basic results regarding the submersion $q_{\bar{\rho}}$.

Lemma 6.1. *The fibers of the map $q_{\bar{\rho}} : U^*S^2 \to \Gamma_{\bar{\rho}}$ are Legendrian with respect to the contact structure induced by α on U^*S^2.*

Proof. For every $\gamma \in \Gamma_{\bar{\rho}}$, let $\xi \in q_{\bar{\rho}}^{-1}(\gamma)$ and $V \in T_\xi q_{\bar{\rho}}^{-1}(\gamma)$. The vector $d\pi_\xi(V)$, based at $\pi(\xi)$, is tangent to γ. By definition of $q_{\bar{\rho}}$, the vectors tangent to γ at $\pi(\xi)$ lie in the kernel of ξ. Hence, $\xi(d\pi_\xi(V)) = 0$, that is, $\alpha_\xi(V) = 0$. □

Lemma 6.2. *The product map $\phi_{\bar{\rho}} = \pi \times q_{\bar{\rho}} : U_0^*S^2 \to S^2 \times \Gamma_{\bar{\rho}}$ is an embedding.*

Proof. Consider $\xi \in U^*S^2$ and $V \in T_\xi U^*S^2$ such that $d\phi_{\bar{\rho}}(\xi)(V) = 0$. This implies that the vector V lies in the kernel of the differential of the submersion $q_{\bar{\rho}}$ at ξ. By construction, it follows that the vector V is tangent to the $\bar{\rho}$-orbit of U^*S^2 at ξ. As π defines an embedding of each $\bar{\rho}$-orbit of U^*S^2 into S^2, we deduce from the relation $d\pi_\xi(V) = 0$ that $V = 0$. Thus, the map $\phi_{\bar{\rho}}$ is an immersion.

Let $\xi_1, \xi_2 \in U^*S^2$ such that $\phi_{\bar{\rho}}(\xi_1) = \phi_{\bar{\rho}}(\xi_2)$. As $\pi(\xi_1) = \pi(\xi_2)$, the covectors ξ_1 and ξ_2 are based at the same point x of S^2. Since the projections of the $\bar{\rho}$-orbits to S^2 are embeddings of S^1 into S^2, for every $\gamma \in \Gamma_{\bar{\rho}}$ passing through x, *i.e.*, $x \in \pi(q_{\bar{\rho}}^{-1}(\gamma))$, there is a unique $\xi \in U_x^*S^2$ such that $q_{\bar{\rho}}(\xi) = \gamma$. Thus, the relation $q_{\bar{\rho}}(\xi_1) = q_{\bar{\rho}}(\xi_2)$ implies that $\xi_1 = \xi_2$. As a result, the map $\phi_{\bar{\rho}}$ is injective, and so gives rise to an embedding. □

Define the volume form Ω on U^*S^2 as follows

$$\Omega = \Upsilon^*(\alpha \wedge d\alpha). \tag{6.1}$$

The actual expression of Ω only matters for Remark 6.7. Along with the S^1-action $\bar{\rho}$, this volume form gives rise to a two-form $\omega = \omega_{\bar{\rho}}$ on U^*S^2 by the following averaging construction:

$$\omega_{\bar{\rho}} = \frac{1}{2\pi} \int_{S^1} \bar{\rho}(\theta)^* [i_\nu(\Omega)] \, d\theta \tag{6.2}$$

where $\nu = \nu_{\bar{\rho}}$ is the vector field on U^*S^2 generating the S^1-action $\bar{\rho}$, that is,

$$\nu(\xi) = \frac{d}{d\theta}_{|\theta=0} \bar{\rho}(\theta)(\xi) \tag{6.3}$$

for every $\xi \in U^*S^2$. Thus,

$$\omega_\xi(u,v) = \frac{1}{2\pi} \int_{S^1} \Omega_{\bar{\rho}(\theta)(\xi)}(d\bar{\rho}(\theta)_{|\xi}(u), d\bar{\rho}(\theta)_{|\xi}(v), \tfrac{d}{d\theta}\bar{\rho}(\theta)(\xi)) \, d\theta \tag{6.4}$$

for every $\xi \in U^*S^2$ and $u, v \in T_\xi U^*S^2$. Here, despite the ambiguity in the notation, $d\bar{\rho}(\theta)_{|\xi}$ denotes the differential of the diffeomorphism $\bar{\rho}(\theta) : U^*S^2 \to U^*S^2$ at ξ. By construction, the two-form $\omega_{\bar{\rho}}$ is $\bar{\rho}$-invariant and projects to a two-form $\lambda_{\bar{\rho}}$ on the quotient surface $\Gamma_{\bar{\rho}} = U^*S^2/\bar{\rho}$ with

$$\omega_{\bar{\rho}} = q_{\bar{\rho}}^* \lambda_{\bar{\rho}}.$$

Up to the multiplicative factor $\frac{1}{2\pi}$, the form $\lambda_{\bar{\rho}}$ is the two-form induced by Ω by integration along the fibers of $q_{\bar{\rho}}$ (that is, the push-forward of Ω by the fibration $q_{\bar{\rho}}$). Note that both two-forms $\omega_{\bar{\rho}}$ and $\lambda_{\bar{\rho}}$ are determined by $\bar{\rho}$ (and so ρ).

We have the following straightforward result.

Lemma 6.3. *The two-form $\lambda_{\bar{\rho}}$ does not vanish (and so defines an area-form on $\Gamma_{\bar{\rho}}$). Furthermore,*

$$\int_{\Gamma_{\bar{\rho}}} |\lambda_{\bar{\rho}}| = 4\pi.$$

Proof. Consider two independent vectors $\bar{u}$ and $\bar{v}$ based at the same point tangent to $\Gamma_{\bar{\rho}}$. Let u and v be two lifts of $\bar{u}$ and $\bar{v}$, based at the same point $\xi \in U^*S^2$, under the submersion $q_{\bar{\rho}}$. That is, the vectors u and v tangent to U^*S^2 at ξ project to $\bar{u}$ and $\bar{v}$ under $dq_{\bar{\rho}}$. By construction, we have

$$\lambda_{\bar{\rho}}(\bar{u}, \bar{v}) = \omega_\xi(u, v).$$

Furthermore, for every $\theta \in S^1$, the vectors $d\bar{\rho}(\theta)_{|\xi}(u)$ and $d\bar{\rho}(\theta)_{|\xi}(v)$ also project to $\bar{u}$ and $\bar{v}$ under $dq_{\bar{\rho}}$. Now, since the vector $\frac{d}{d\theta}\bar{\rho}(\theta)(\xi)$ is tangent to the fibers of $q_{\bar{\rho}}$, it follows that the three vectors $d\bar{\rho}(\theta)_{|\xi}(u)$, $d\bar{\rho}(\theta)_{|\xi}(v)$ and $\frac{d}{d\theta}\bar{\rho}(\theta)(\xi)$ form a basis of $T_{\bar{\rho}(\theta)(\xi)}U^*S^2$. Thus, the value of the volume form $\Omega_{\bar{\rho}(\theta)(\xi)}$ at these vectors is nonzero (and so of constant sign) for every $\theta \in S^1$. From the expression (6.4), we conclude that both $\omega_\xi(u,v)$ and $\lambda_{\bar{\rho}}(\bar{u},\bar{v})$ are nonzero. Hence $\lambda_{\bar{\rho}}$ does not vanish.

By fiber integration, Fubini's theorem and change of variables, we have the following relation

$$\int_{U^*S^2} \Omega = 2\pi \int_{\Gamma_{\bar{\rho}}} \lambda_{\bar{\rho}}.$$

In particular, the integral of the (non-vanishing) area-form $|\lambda_{\bar{\rho}}|$ over $\Gamma_{\bar{\rho}}$ is equal to 4π, see (5.1). □

We can now apply the results of [AB] asserting that a (non-vanishing) area-form on $\Gamma_{\bar{\rho}}$ gives rise to a Finsler metric on S^2 via the Crofton formula. Indeed, the assumptions of [AB, Theorem 2.2] are satisfied from Lemma 6.1, Lemma 6.2 and Lemma 6.3. Thus, the existence and uniqueness of the Finsler metric follow from [AB, Theorem 2.2], while the expression of the metric is given by Lemma [AB, Lemma 2.3].

Theorem 6.4 ([AB]). *With the previous notations, there exists a unique Finsler metric F_ρ on S^2 satisfying the Crofton formula*

$$\text{length}_{F_\rho}(c) = \frac{1}{4} \int_{\gamma \in \Gamma_{\bar{\rho}}} \#(\gamma \cap c)\, |\lambda_{\bar{\rho}}| \tag{6.5}$$

for any smooth curve c on S^2.

Morevover, the Finsler metric F_ρ admits the following expression: for every $x \in S^2$, there exists a unique non-vanishing one-form β_x on $U^*_x S^2$ such that

$$F_\rho(x;v) = \int_{\xi \in U^*_x S^2} |\xi(v)|\, \beta_x \tag{6.6}$$

for every $v \in T_x S^2$, where the non-vanishing one-form β_x is defined for every $\xi \in U^*_x S^2$ by the relation

$$(\omega_{\bar{\rho}})_{(x,\xi)} = \pi^*\xi \wedge \beta_{(x,\xi)}. \tag{6.7}$$

Furthermore, the one-form β_x smoothly depends on x.

Note that the relation (6.7) allows us to define the one-form $\beta_{(x,\xi)}$ in a unique way only on $T_\xi U^*_x S^2$, not on $T_{(x,\xi)}U^*S^2$.

Before making use of Theorem 6.4, let us mention three important observations that will be useful in the proof of Theorem 6.8 below.

The first one deals with the dependance of F_ρ with respect to ρ.

Remark 6.5. By construction, both forms $\omega_{\bar{\rho}}$ and $\beta = \beta_{\bar{\rho}}$, see (6.2) and (6.7), continuously vary with the S^1-action ρ. It follows from the expression of F_ρ, see (6.6), that the Finsler metric F_ρ continuously varies with ρ too.

The second observation is about the geodesics of F_ρ.

Remark 6.6. When ρ is symmetric, see (4.6), it follows from the Crofton formula that the geodesics of F_ρ are exactly the curves $\pi(q_{\bar{\rho}}^{-1}(\gamma))$ given by $\gamma \in \Gamma_{\bar{\rho}}$, see [AB, Theorem 3.3]. Note that ρ is symmetric when it is given by the co-geodesic flow of the (reversible) Zoll Finsler metric F, and more generally, by its one-parameter family of deformations $(\rho_\tau)_{0 \leq \tau \leq 1}$ defined at the end of Section 4, assuming, in this case, that F is balanced.

The third observation deals with S^1-actions arising from the co-geodesic flow of the Zoll Finsler metric F on S^2.

Remark 6.7. In the special case when ρ is given by the co-geodesic flow of F, that is, $\rho = \rho_F$, the following properties hold. The vector field ν, see (6.3), agrees with the co-geodesic vector field X_F of F on U^*S^2 through the diffeomorphism Υ. Since $i_{X_F}(\alpha) = 1$, we deduce that $i_\nu(\Omega) = d\bar{\alpha}$ from the expression of the volume form Ω, see (6.1), where $\bar{\alpha} = \Upsilon^*\alpha$. Now, by the Liouville theorem and conjugation, the symplectic form $d\bar{\alpha}$ is $\bar{\rho}_F$-invariant, that is, $\bar{\rho}_F(\theta)^*(d\bar{\alpha}) = d\bar{\alpha}$ for every $\theta \in S^1$. This shows that $\omega = d\bar{\alpha}$. Hence, $\lambda_{\bar{\rho}} = \Upsilon_\Gamma^* \lambda_F$ by definition of λ_F, see (5.3), where $\Upsilon_\Gamma : \Gamma_{\bar{\rho}} \to \Gamma_F$ is the diffeomorphism induced by Υ. It follows from the Crofton formulas (5.4) and (6.5) that $F_\rho = F$.

We can now proceed to the proof of the following result.

Theorem 6.8. *The space of balanced Zoll Finsler metrics on the two-sphere whose geodesic length is equal to 2π strongly deformation retracts to the canonical round metric on the two-sphere.*

Furthermore, this strong deformation retraction is induced by the curvature flow on the canonical round two-sphere.

Proof. Let F be a balanced Zoll Finsler metric on S^2 all of whose geodesics are of length 2π. The unit tangent bundles US^2 and U_0S^2 are identified by radial projection, and are identified to the unit cotangent bundles U^*S^2 and $U_0^*S^2$ via the corresponding Legendre transforms. Consider the deformation $(\rho_\tau)_{0\leq\tau\leq 1}$ of S^1-actions on $U_0^*S^2$ defined at the end of Section 4, where ρ_0 is given by the co-geodesic flow of F and ρ_1 agrees with the co-geodesic flow of the canonical round metric g_0. Note that the S^1-action given by ρ_τ is symmetric, see (4.6). The space $U_0^*S^2$ is endowed with the tautological one-form α of the unit cotangent bundle U^*S^2 of F via the identification $U_0^*S^2 \simeq U^*S^2$. By construction, the ρ_τ-orbits are the images by the Legendre transform of the lifts to US^2 of simple closed curves in S^2. Therefore, the ρ_τ-orbits are transverse to the contact structure given by the kernel of the contact form on $U_0^*S^2$ induced by α and project to an embedding of S^1 into S^2 by the canonical projection $U_0^*S^2 \to S^2$.

Define a one-parameter family of Finsler metrics $F_\tau = F_{\rho_\tau}$, for $0 \leq \tau \leq 1$, from the volume form induced by the contact form on $U_0^*S^2$, as in Theorem 6.4. We will also write q_τ, Γ_τ and λ_τ for $q_{\bar{\rho}_\tau}$, $\Gamma_{\bar{\rho}_\tau}$ and $\lambda_{\bar{\rho}_\tau}$. From Remark 6.7, the metric deformation (F_τ) starts at F, that is, $F_0 = F$, since ρ_0 is conjugate to the co-geodesic flow of F via the contactomorphism $U_0^*S^2 \simeq U^*S^2$ (for our choice of contact form on $U_0^*S^2$). By Remark 6.6, the geodesics of F_τ are precisely the simple closed curves represented by Γ_τ, namely, the curves $\pi(q_\tau^{-1}(\gamma))$ where γ runs over Γ_τ. Thus, the geodesics of F_τ agree with the images of the geodesics of F under the curvature flow at some time depending on τ. In particular, the metrics F_τ are balanced Zoll Finsler metrics, see Proposition 3.1, and the metric F_1 has the same geodesics as the canonical round metric g_0.

At this point, we do not claim that F_1 agrees with g_0. Indeed, by construction, the metric F_1 is determined by its space of geodesics $\Gamma_1 = \Gamma_{g_0}$ *and* a smooth positive measure $|\lambda_1|$ on it, which, in this case, may differ from $|\lambda_{g_0}|$. The reason is that, though ρ_1 is the co-geodesic flow of the canonical round metric g_0, the volume form considered on $U_0^*S^2$ is not induced by the tautological one-form of $U_0^*S^2$, but of U^*S^2. Thus, we cannot apply Remark 6.7 to the action ρ_1. This leads us to extend the metric deformation (F_τ) in a natural way as follows. For every $1 \leq \tau \leq 2$, define a Finsler metric F_τ as in Theorem 6.4 with $\Gamma_\tau = \Gamma_1$, $q_\tau = q_1$ and

$$|\lambda_\tau| = (2 - \lambda)|\lambda_1| + (\tau - 1)|\lambda_{g_0}|.$$

As previously, the geodesics of these new metrics agree with those of the canonical round sphere, but now, F_2 is equal to g_0, since $|\lambda_2| = |\lambda_{g_0}|$.

We can also estimate the lengths of the geodesics of F_τ as follows. Let c_0 be a geodesic of F_τ. By Theorem 3.3, every pair of distinct closed geodesics of F_τ has exactly two intersection points. Hence, $\#(\gamma \cap c_0) = 2$ for almost every $\gamma \in \Gamma_\tau$. Since the integral of $|\lambda_\tau|$ equals 4π, it follows from the Crofton formula (6.5) that the length of c_0 is equal to 2π.

In conclusion, the metric deformation (F_τ) gives rise to a retraction from the space of balanced Zoll Finsler metrics on the two-sphere with geodesic length 2π to the canonical round metric g_0 on the two-sphere. □

Remark 6.9. Theorem 1.1 follows from Theorem 6.8 by taking the orientable double cover of the Zoll Finsler projective plane since, by construction, the strong deformation retraction on the two-sphere passes to the quotient by the antipodal map.

7. Appendix

In this appendix, we show how flexible Zoll Finsler metrics are. Given a closed Zoll Finsler n-manifold M, we construct an infinite-dimensional family of Zoll Finsler metrics on M with the same geodesic length. The results in this section are not new, but we provide details we were unable to find in the literature.

A classical way to perturb a Zoll Finsler metric F on a closed manifold M within the space of Zoll Finsler metrics with the same geodesic length is to deform the unit cotangent sphere bundle U^*M by a symplectomorphism of T^*M, symmetric in restriction to each fiber of the cotangent bundle and close to the identity in the smooth topology so as to preserve the quadratic convexity. This symplectomorphism can be chosen in such a way that the image of U^*M differs from the image of U^*M by the cotangent lift of any diffeomorphism of M. Indeed, the symplectomorphisms given by the cotangent lifts of diffeomorphisms of the base manifold M have a very special form: they are linear on each fiber of the cotangent bundle T^*M. For generic symplectomorphisms not of this form, the resulting Finsler metrics are not isometric to F, though their geodesic flows are symplectically conjugate to the one of F, in particular the metrics are Zoll with the same geodesic lengths as F (and have the same Holmes-Thompson volume).

Our presentation follows a more hand-on approach inspired by [I13] and [BI16] regarding boundary rigidity problems (and used in [C19] to study Finsler tori without conjugate points) and sheds a different light on this problem. It largely borrows from [C19]. The construction — proceeding by local perturbations of the initial metric — is loosely constrained and fairly easy to implement. Furthermore, all local Zoll perturbations are obtained by following this construction.

Let M be a closed n-manifold with a Finsler metric F. Fix an open ball D in M of radius less than $\frac{1}{4}\mathrm{inj}(M)$ centered at x_0 . For every $p \in \partial D$, the arclength parametrized geodesic γ_p with $\gamma_p(0) = x_0$ and $\gamma_p(r) = p$ defines a point $\gamma_p(-r)$ lying in ∂D denoted by $-p$. The set of points of D equidistant from p and $-p$ forms a hypersurface H_p passing through x_0 which divides D into two connected components: H_p^+ containing p and H_p^- containing $-p$.

Define a smooth function $f : \partial D \times D \to \mathbb{R}$ as

$$f(p,x) = \begin{cases} d(H_p, x) & \text{if } x \in H_p^+ \\ -d(H_p, x) & \text{otherwise} \end{cases}$$

For every $p \in \partial D$, denote $f_p = f(p,\cdot)$. The function f is an *enveloping function*, that is, it satisfies the following conditions:

1) for every $x \in D$, the map $p \mapsto df_p(x)$ is a diffeomorphism from ∂D to the boundary of a quadratically convex body of T_x^*M containing the origin;
2) for every $p \in \partial D$, we have $f_{-p} = -f_p$.

In our case, the boundary of the quadratically convex body in (1) is U_x^*M since $F^*(df_p(x)) = 1$ for every $p \in \partial D$ and $x \in D$. The condition (2) ensures that the convex bodies in (1) are symmetric with respect to the origin.

The distance function induced by F can be written as

$$d_F(x,y) = \sup_{p\in\partial D} f_p(x) - f_p(y) \tag{7.1}$$

for every $x, y \in D$. Similarly, the Finsler metric F can be expressed as

$$F(v) = \sup_{p\in\partial D} df_p(v)$$

for every $v \in TD$.

Consider a sufficiently small C^∞-perturbation $\tilde{f}$ of f such that $\tilde{f}$ is an enveloping function which agrees with f on $\partial D \times U$, where U is a tubular neighborhood of ∂D in D. Note that if the perturbation is small enough, the condition (1) is immediately satisfied by $\tilde{f}$.

Define a new Finsler metric $\tilde{F}$ on D as

$$\tilde{F}(v) = \sup_{p \in \partial D} d\tilde{f}_p(v)$$

for every $v \in TD$. The induced distance is given by

$$d_{\tilde{F}}(x, y) = \sup_{p \in \partial D} \tilde{f}_p(x) - \tilde{f}_p(y) \tag{7.2}$$

for every $x, y \in D$. Here, the reversibility of $\tilde{F}$ (and symmetry of $d_{\tilde{F}}$) follows from the condition (2). Also, the function $\tilde{f}_p$ satisfies $\tilde{F}^*(d\tilde{f}_p(x)) = 1$ for every $x \in D$. Moreover, for every $x \in D$, there exists a unique $\tilde{F}$-unit tangent vector $v \in T_x D$ such that $d\tilde{f}_p(v) = 1$. This tangent vector smoothly depends on x (and p) and defines an $\tilde{F}$-unit vector field $\tilde{\nabla} \tilde{f}_p$ on D, called the $\tilde{F}$-*gradient* of $\tilde{f}_p$. Since F and $\tilde{F}$ agree on U (as do f and $\tilde{f}$ on $\partial D \times U$), we can extend $\tilde{F}$ by letting $\tilde{F} = F$ outside D.

The geodesics of $\tilde{F}$ can be described as follows.

Proposition 7.1. *The geodesics of $\tilde{F}$ agree with the integral curves of $\tilde{\nabla} \tilde{f}_p$ on D, with $p \in \partial D$. Furthermore, these curves are $\tilde{F}$-minimizing on D.*

Proof. Consider an integral curve γ of $\tilde{\nabla} \tilde{f}_p$. By construction, the curve γ is parametrized by its $\tilde{F}$-arclength and $d\tilde{f}_p(\gamma'(t)) = 1$ for every $t \in [a, b]$. Thus,

$$b - a = \int_a^b d\tilde{f}_p(\gamma'(t))\, dt = \tilde{f}_p(\gamma(b)) - \tilde{f}_p(\gamma(a)) \leq b - a$$

since $\tilde{f}_p$ is $\tilde{F}$-nonexpanding. This implies

$$d_{\tilde{F}}(\gamma(b), \gamma(a)) = \tilde{f}_p(\gamma(b)) - \tilde{f}_p(\gamma(a)) = b - a$$

Hence, the arc γ is a minimizing $\tilde{F}$-geodesic on D.

Conversely, let γ_v be the $\tilde{F}$-geodesic induced by some $\tilde{F}$-unit tangent vector $v \in T_x D$ based at x. From the condition (1), there exists a (unique) $p \in \partial D$ such that v agrees with $\tilde{\nabla} \tilde{f}_p$ at x. The integral curve of $\tilde{\nabla} \tilde{f}_p$ passing through x is an $\tilde{F}$-geodesic with the same initial condition v as the $\tilde{F}$-geodesic γ_v. Therefore, the two $\tilde{F}$-geodesics agree on D. □

Let $\tilde{\gamma}$ be an $\tilde{F}$-geodesic arc of D. Since $\tilde{\gamma}$ is $\tilde{F}$-minimizing, it leaves D through two points x and y in ∂D.

Proposition 7.2. *Let γ be the F-geodesic arc of D with the same endpoints as $\tilde{\gamma}$. Then, the arcs γ and $\tilde{\gamma}$ satisfy $\ell_{\tilde{F}}(\tilde{\gamma}) = \ell_F(\gamma)$ and have the same tangent vectors at their endpoints x and y.*

Proof. Let us show this latter statement holds for the tangent vectors of γ and $\tilde{\gamma}$ at y (and so at x, by symmetry). Fix a point y_+ close to y lying slightly outside $\bar{D}$ on the F-geodesic extension of γ. Denote by α the F-minimizing arc (lying on the F-geodesic extension of γ) joining y to y_+, see Figure 2. The arc α is also minimizing for $\tilde{F}$ since $\tilde{F} = F$ in the neighborhood of y.

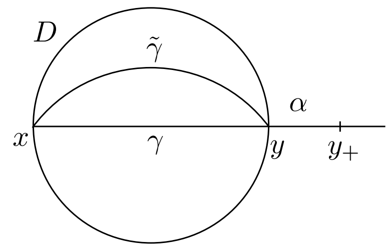

Figure 2: Geodesics of D for F and $\tilde{F}$.

Since f and $\tilde{f}$ agree on $\partial D \times U$, we deduce from the expression of d_F and $d_{\tilde{F}}$, see (7.1) and (7.2), that $d_F = d_{\tilde{F}}$ not only for pairs of points in a neighborhood of y but on $\partial D \times \partial D$. Thus, for every $z \in \partial D$ close enough to y, we have

$$d_F(x,z) + d_F(z,y_+) = d_{\tilde{F}}(x,z) + d_{\tilde{F}}(z,y_+).$$

The infimum of the left-hand side of this equation over such z is attained for $z = y$. By a first variation argument applied to the right-hand side of the equation, we deduce that $\tilde{\gamma} \cup \alpha$ is smooth. (Recall that $\tilde{\gamma}$ is an $\tilde{F}$-minimizing arc joining x to y, and that α is an $\tilde{F}$-minimizing arc joining y to y_+.) Thus, the unit tangent vectors of $\tilde{\gamma}$ at y is the same as the unit tangent vector of α (and so γ) at y.

Now, since γ and $\tilde{\gamma}$ are minimizing with respect to F and $\tilde{F}$, the relation $d_F = d_{\tilde{F}}$ on $\partial D \times \partial D$ also implies

$$\ell_{\tilde{F}}(\tilde{\gamma}) = d_{\tilde{F}}(x,y) = d_F(x,y) = \ell_F(\gamma).$$

□

Suppose now that F is a Zoll Finsler metric and that, in addition, the radius of D is small enough so that the geodesics of F pass at most once through D. We derive from Proposition 7.2 that the geodesics of $\tilde{F}$ are closed and agree with those of F outside D. Furthermore, they are simple and have the same length as those of F. In other words, $\tilde{F}$ is also a Zoll Finsler metric with the same geodesic length as F.

Moreover, it is possible to choose the perturbation $\tilde{f}$ so that the Finsler metrics are not pairwise isometric as follows. Fix a point x_0 in M. Consider an ε-deformation $||\cdot||$ of the Minkowski norm $F^*_{x_0}$ on $T^*_{x_0}M$ in the C^∞-topology for ε small enough. Recall that the map $p \mapsto df_p(x_0)$ is a diffeomorphism from ∂D to the boundary $\partial B^{F^*}_{x_0}$ of the F^*-unit tangent ball in $T^*_{x_0}M$. It is possible to deform f into an enveloping function $\tilde{f}$ which agrees with f in a neighborhood of $\partial D \times \partial D$ such that the map $p \mapsto d\tilde{f}_p(x_0)$ is a diffeomorphism from ∂D to the boundary of the unit ball $||\cdot||$ in $T^*_{x_0}M$. Moreover, the deformation $\tilde{f}$ can be chosen arbitrarily close to f for the C^∞-norm if ε is small enough (the main point is to be able to set $d\tilde{f}_p$ at x_0). This gives rise to a Zoll Finsler metric $\tilde{F}$ with the same geodesic length as F whose restriction of the Fenchel-dual to $T^*_{x_0}M$ agrees with $||\cdot||$, that is, $\tilde{F}^*_{x_0} = ||\cdot||$. Now, the norm $||\cdot||$ describes an infinite-dimensional subspace in the Banach-Mazur compactum $Q(n)$ of isometry classes of n-dimensional normed spaces. By choosing $||\cdot||$ not lying in the finite-dimensional subspace of $Q(n)$ given by the norms F^*_x on T^*_xM where x runs over M, we ensure that $\tilde{F}$ is not isometric to F. A similar dimension argument shows that the Zoll Finsler metrics $\tilde{F}$ obtained from the perturbed norms $||\cdot||$ form an infinite-dimensional space.

References

[ABHS17] A. Abbondandolo, B. Bramham, U. Hryniewicz, and P. Salomão, *A systolic inequality for geodesic flows on the two-sphere*, Math. Ann. **367** (2017), no. 1-2, 701–753.

[AB06] J. C. Álvarez Paiva and G. Berck, *What is wrong with the Hausdorff measure in Finsler spaces*, Adv. Math. **204** (2006), no. 2, 647–663.

[AB] J. C. Álvarez Paiva and G. Berck, *Finsler surfaces with prescribed geodesics*, preprint, `arXiv:1002.0243`.

[A88] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96.

[A90] S. Angenent, *Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature*, Ann. of Math. (2) **132** (1990), no. 3, 451–483.

[A91] S. Angenent, *Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions*, Ann. of Math. (2) **133** (1991), no. 1, 171–215.

[B03] M. Berger, A Panoramic View of Riemannian Geometry, Springer-Verlag, (2003).

[Be78] A. L. Besse, Manifolds All of Whose Geodesics Are Closed, Ergebnisse Math. Grenzg. **93**, Springer, (1978).

[BI16] D. Burago and S. Ivanov, *Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics*, Geom. Topol. **20** (2016), no. 1, 469–490.

[C19] D. Chen, *On total flexibility of local structures of Finsler tori without conjugate points*, J. Topol. Anal. **11** (2019), no. 2, 349–355.

[E72] D. B. A. Epstein, *Periodic flows on three-manifolds*, Ann. of Math. (2) **95** (1972), no. 1, 66–82.

[Ga90] M. Gage, *Curve shortening on surfaces*, Ann. Sci. École Norm. Sup. (4) **23** (1990), no. 2, 229–256.

[Gr89] M. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), no. 1, 71–111.

[G76] V. Guillemin, *The Radon transform on Zoll surfaces*, Adv. Math. **22** (1976), no. 1, 85–119.

[H] Y.-W. Hsu, *Curve shortening flow and smooth projective planes*, preprint, `arXiv:1308.3537`.

[I13] S. Ivanov, *Local monotonicity of Riemannian and Finsler volume with respect to boundary distances*, Geom. Dedicata **164** (2013), no. 1, 83–96.

[J17] D. Jane, *The Ricci flow does not preserve the set of Zoll metrics*, Differential Geom. Appl. **54** (2017), part B, 397–401.

[LM02] C. Lebrun and L. Mason, *Zoll manifolds and complex surfaces*, J. Differential Geom. **61** (2002), no. 3, 453–535.

[P52] P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. **2** (1952), 55–71.

[S01] Z. Shen, Lectures on Finsler Geometry, World Scientific, (2001).

[T96] A. C. Thompson, Minkowski Geometry, Encyclopedia of Math. and its Applications, Vol. 63, Cambridge Univ. Press, Cambridge, (1996).

[W75] A. Weinstein, *Fourier integral operators, quantization, and the spectra of Riemannian manifolds*, in: Géométrie Symplectique et Physique Mathématique, Colloq. Internat. CNRS, no. 237, pp. 289–298, Presses du CNRS, (1975).

[Z03] O. Zoll, *Ueber Flächen mit Scharen geschlossener geodätischer Linien*, Math. Ann. **57** (1903), no. 1, 108–133.

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Received March 28, 2016
Accepted November 29, 2017