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Presymplectic convexity and (ir)rational polytopes

Tudor Ratiu* and Nguyen Tien Zung

In this paper, we extend the Atiyah–Guillemin–Sternberg convexity theorem and Delzant's classification of symplectic toric manifolds to presymplectic manifolds. We also define and study the Morita equivalence of presymplectic toric manifolds and of their corresponding framed momentum polytopes, which may be rational or non-rational. Toric orbifolds [16], quasifolds [3] and noncommutative toric varieties [14] may be viewed as the quotient of our presymplectic toric manifolds by the kernel isotropy foliation of the presymplectic form.

1. Introduction

The celebrated convexity theorem of Atiyah [1] and Guillemin–Sternberg [10] states that if a connected compact symplectic manifold (M, ω) admits an effective Hamiltonian torus $\mathbb{T}^n$-action with corresponding momentum map

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$F : M \to \mathbb{R}^n$, then the image $F(M) \subset \mathbb{R}^n$ is a convex n-dimensional polytope, called the *momentum polytope*, which is *rational*, i.e., each facet is given by a linear equation with rational linear coefficients in $\mathbb{R}^n$.

A particularly important special case, related to algebraic toric geometry, is when the dimension of the symplectic manifold is exactly $2n$, where n is the dimension of the torus which acts on it. In this case, the momentum polytope is not only rational, but also *simple* (i.e., each s-dimensional face has exactly $n - s$ faces of dimension $s + 1$ adjacent to it) and *regular* (i.e., for any point x_0 on any s-dimensional face there is a complete integral affine coordinate system $(h_1, \ldots, h_n)$ such that the polytope is locally given near x_0 by the system of linear inequalities $\{h_1(x) \geq 0, \ldots, h_{n-s}(x) \geq 0\}$). Convex polytopes which satisfy the three conditions of rationality, simplicity and regularity are called ***Delzant polytopes***, because Delzant [7] found a natural 1-to-1 correspondence between such polytopes and connected compact symplectic toric manifolds (i.e., those symplectic manifolds which admit an effective Hamiltonian torus action of half the dimension).

In this paper, we give a natural extension of these theorems to *presymplectic* manifolds. Our main results can be roughly formulated as follows:

• (Theorem 2.4 and Theorem 2.7) *The image of the momentum map of a Hamiltonian torus action on a connected compact presymplectic manifold with a regular presymplectic form, under a natural flatness condition, is a convex polytope (of lower dimension in general) in a Euclidean space. Moreover, any such presymplectic manifold admits an equivariant symplectization, which is unique in a natural sense.*

• (Theorem 3.13, Theorem 3.14 and Theorem 3.15). *Connected compact presymplectic toric manifolds are classified, up to equivariant presymplectic diffeomorphisms, by their associated framed momentum polytopes. The classification, up to Morita equivalence, of connected presymplectic toric manifolds is given by the Morita equivalence classes of their framed momentum polytopes.*

Our motivation for this work comes from our desire to understand the role in symplectic geometry of convex simple polytopes which do not satisfy the rationality or the regularity conditions of Delzant polytopes. Many other authors have worked on this question. In particular, Lerman and Tolman [16] obtained the relation between non-regular simple rational polytopes and symplectic toric orbifolds.

Battaglia and Prato (see, e.g., [2–5] and references therein) and Katzarkov–Lupercio–Meersseman–Verjovsky (see, [14]) have worked on irrational analogues of symplectic toric manifolds. However, we wanted to have a simpler understanding of the geometric structure and so developed our own approach, which uses presymplectic realizations.

In the process of studying quotient spaces of presymplectic toric manifolds, we are naturally led to the notion of *Morita equivalence* of these manifolds, and of their corresponding *framed momentum polytopes*, borrowing the idea from the theory of Lie groupoids and stacks. Using our language of Morita-equivalent framed polytopes, we recover the results of Lerman and Tolman [16] on symplectic toric orbifolds, and also give a clear, easy to understand, definition of what it means for two toric quasifolds (in the sense of Prato [23]) to be isomorphic.

We also note that, in order to turn an irrational convex polytope into a momentum polytope of a (pre)symplectic toric object, one first needs to lift it *non-isomorphically* to a *rational-faced* polytope in a higher-dimensional space! This simple but important observation clarifies the role of irrational polytopes in toric (pre)symplectic geometry.

The paper is structured as follows. Section 2 is devoted to the presymplectic version of the Atiyah–Guillemin-Sternberg convexity theorem. Section 3 is about presymplectic toric manifolds, their framed momentum polytopes, and their Morita equivalence classes. Section 4, the last section of this paper, contains some final remarks about related works by other authors and related questions.

2. Presymplectic convexity theorem

2.1. The flatness condition

The goal of this section is to prove an analogue of the Atiyah-Guillemin-Sternberg convexity theorem [1, 10] for Hamiltonian torus actions on presymplectic manifolds.

Let (M,ω) be a connected compact presymplectic manifold of dimension $2n+d$, i.e., $\omega \in \Omega^2(M)$ is closed. Suppose that the dimension of the image of the linear maps $T_xM \ni v_x \mapsto \omega(x)(v_x,\cdot) \in T_x^*M$ is $2n$ for all $x \in M$, i.e., the presymplectic form ω has constant corank d. Assume that there is a presymplectic torus $\mathbb{T}^{q+d}$-action on M which is effective, i.e., the intersection of all isotropy groups of the torus action is the identity element. In addition, suppose that this action is Hamiltonian in the following presymplectic sense.

Let the vector fields $X_1,\ldots,X_{q+d}$ on M be a family of generators of the torus action, i.e., the flow of each X_i is periodic of period 1 and their flows together form the $\mathbb{T}^{q+d}$-action. We assume that for each $i=1,\ldots,q+d$, there is a function $F_i : M \to \mathbb{R}$, called the ***Hamiltonian function*** of X_i, such that

$$\omega(X_i, \cdot) = \mathrm{d}F_i,$$

similarly to the symplectic case. This condition implies automatically that ω is preserved by X_i: $\mathcal{L}_{X_i}\omega = \mathrm{d}(X_i \lrcorner \omega) + X_i \lrcorner (\mathrm{d}\omega) = \mathrm{d}(\mathrm{d}F) + 0 = 0$. Hence ω is invariant with respect to the torus $\mathbb{T}^{q+d}$-action, which implies that each function F_i is also $\mathbb{T}^{q+d}$-invariant. Indeed, for every i and j, the function $X_j(F_i) = \omega(X_i, X_j)$ is $\mathbb{T}^{q+d}$-invariant, but on each orbit of the $\mathbb{T}^{q+d}$-torus action there is a critical point of the restriction of F_i to that orbit, and $X_j(F_i) = 0$ at such a critical point; hence $X_j(F_i) = 0$ everywhere. The map

$$F = (F_1, \ldots, F_{q+d}) : M \to \mathbb{R}^{q+d}$$

is called the ***momentum*** map of the Hamiltonian $\mathbb{T}^{q+d}$-action. Since the momentum map F is invariant with respect to the $\mathbb{T}^{q+d}$-action, it factors through a map from the space of orbits $M/\mathbb{T}^{q+d}$ of the $\mathbb{T}^{q+d}$-action to $\mathbb{R}^{q+d}$. We will assume that *the kernel of ω lies in the tangent spaces to the orbits of the $\mathbb{T}^{q+d}$-action.* Under this assumption, the rank of F is at most q at every point of M, and so the image $F(M) \subset \mathbb{R}^{q+d}$ also has dimension at most q.

In the symplectic case, when $d = 0$ and hence ω is non-degenerate, the celebrated Atiyah–Guillemin–Sternberg theorem [1, 10] states that $F(M)$ is a convex polytope. We want to obtain a similar result for the presymplectic case, i.e., we want to see when the image $F(M)$ of the momentum map F of a presymplectic manifold M is still a convex polytope. If this is the case, then the image (which has dimension at most q by our assumptions) must lie in a q-dimensional affine subspace, i.e., the intersection of d hyperplanes in $\mathbb{R}^{q+d}$. We call this the ***flatness condition*** of the momentum map.

Definition 2.1. With the hypotheses and notations above, we say that the momentum map $F = (F_1, \ldots, F_{q+d}) : M \to \mathbb{R}^{q+d}$ is ***flat*** if it satisfies d linearly independent affine relations on M:

$$\sum_{j=1}^{q+d} a_{ij} F_j = b_i, \quad a_{ij}, b_i \in \mathbb{R}, \quad i = 1, \ldots, d. \tag{1}$$

In other words, the image $F(M)$ of the momentum map lies in the q-dimensional intersection

$$L = \bigcap_{i=1}^{d} L_i$$

of the hyperplanes

$$L_i = \left\{ x = (x_1, \ldots, x_{q+d}) \in \mathbb{R}^{q+d} \;\middle|\; \sum_{j=1}^{q+d} a_{ij} x_j = b_i \right\}, \quad i = 1, \ldots, d.$$

The above flatness condition is equivalent to the inclusions $Y_i \in \ker \omega$ for all $i = 1, \ldots, d$, where

$$Y_i = \sum_{j=1}^{q+d} a_{ij} X_j, \quad i = 1, \ldots, d, \tag{2}$$

with the same constant coefficients a_{ij} as in equation (1).

Example 2.2 (Flat slice). Let $F : (\hat{M}^{2(n+d)}, \omega) \to \mathbb{R}^{q+d}$ be the momentum map of a Hamiltonian effective torus $\mathbb{T}^{q+d}$-action on a connected compact symplectic manifold $(\hat{M}^{2(n+d)}, \omega)$, and let L be an arbitrary q-dimensional affine subspace of $\mathbb{R}^{q+d}$ which intersects the $(q+d)$-dimensional polytope $F(\hat{M}^{2(n+d)})$ *transversally* at $P = L \cap F(\hat{M}^{2(n+d)})$. Then $(M = F^{-1}(P), \omega)$ is a $(2n+d)$-dimensional presymplectic manifold with the inherited Hamiltonian torus $\mathbb{T}^{q+d}$-action from $(\hat{M}^{2(n+d)}, \omega)$, the presymplectic form ω on M has constant corank d, the inherited momentum map F is flat on M, and its image $F(M) = P$ is a q-dimensional convex polytope. We say that M is a ***flat presymplectic slice*** of $(\hat{M}^{2(n+d)}, \omega)$ by L.

If, instead of taking the transversal intersection of $F(\hat{M}^{2(n+d)})$ with an affine q-dimensional subspace $L \subset \mathbb{R}^{q+d}$, we take its transversal intersection

$$P' = S \cap F(\hat{M}^{2(n+d)})$$

with a curved q-dimensional submanifold $S \subset \mathbb{R}^{q+d}$, then $M' = F^{-1}(P')$ is still a presymplectic manifold with a Hamiltonian torus $\mathbb{T}^{q+d}$-action on it, the kernel of the presymplectic form still lies in the tangent space to the orbits of the $\mathbb{T}^{q+d}$-action at every point, but its image under the momentum map is now P', which is a non-convex set.

Remark 2.3. Usually, the target space of the momentum map of a Hamiltonian action of a Lie group G is the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g} = Lie(G)$ of G. In this paper, our Lie group is always a torus $\mathbb{T}$ of some dimension k. The target space of the momentum map is identified with a Euclidean space $\mathbb{R}^k \cong \mathfrak{t}^*$ by fixing a canonical basis of $\mathfrak{t} = Lie(\mathbb{T})$ and the corresponding dual basis of $\mathfrak{t}^*$.

2.2. The presymplectic convexity theorem

It turns out that the above flatness condition, which is of course a necessary condition for the convexity of $F(M)$ under our assumptions, is also the only additional condition that one needs in order to ensure that $F(M)$ is a q-dimensional convex polytope. Moreover, if we assume that $F(M)$ is flat q-dimensional, then the condition that the kernel of ω is tangent to the orbits of the torus action is automatically satisfied, at least at regular points of the torus action.

Theorem 2.4. *Let $F : M^{2n+d} \to \mathbb{R}^{q+d}$ be a flat momentum map of a Hamiltonian torus $\mathbb{T}^{q+d}$-action on a connected compact presymplectic manifold (M^{2n+d}, ω) whose presymplectic form ω has constant corank d. Then the image $F(M)$ is a convex q-dimensional polytope lying in a q-dimensional affine subspace L of $\mathbb{R}^{q+d}$. Moreover, the fibers of F are connected and $F : M \to F(M)$ is open, with $F(M) \subset \mathbb{R}^{q+d}$ endowed with the subspace topology.*

We reduce the proof of the above theorem to the symplectic case. In order to do so, we first study the kernel of the presymplectic form on M^{2n+d}. Then we show that (M^{2n+d}, ω), together with the Hamiltonian torus action, admits a natural symplectization (Theorem 2.7). This local symplectization theorem allows us to deduce the local normal form of a Hamiltonian torus action in the presymplectic case from the one in symplectic case, which, in turn, reduces the convexity problem in the presymplectic case to the well-known convexity result in the symplectic case.

2.3. On the kernel of the presymplectic form

We begin with the following statement linking the kernel of the presymplectic form with the tangent spaces to the $\mathbb{T}^{q+d}$-action.

Proposition 2.5. *Under the assumptions of Theorem 2.4, for every point $y \in M^{2n+d}$ we have*

$$\ker \omega(y) \subset T_y(\mathbb{T}^{q+d} \cdot y),$$

where $\mathbb{T}^{q+d} \cdot y$ denotes the orbit of the Hamiltonian $\mathbb{T}^{q+d}$-action through y and $T_y(\mathbb{T}^{q+d} \cdot y)$ is its tangent space at y. Moreover, the vector fields $Y_1, \ldots, Y_d$ given by equation (2) *are linearly independent and span* $\ker \omega$ *at every point of M^{2n+d}.*

Proof. First consider the generic case, when the $\mathbb{T}^{q+d}$-action is locally free at y. Then $T_y(\mathbb{T}^{q+d} \cdot y) = \mathrm{span}(X_1, \dots, X_{q+d})(y)$ has dimension $q+d$. Its image under the contraction map $X \mapsto \omega(X, \cdot)$ has dimension at most q, so the kernel of this linear map has dimension at least d. However, the kernel of this linear map lies in $\ker \omega$, which has dimension exactly d. Therefore, the kernel of the linear map $X \mapsto \omega(X, \cdot)$ on $T_y(\mathbb{T}^{q+d} \cdot y)$ coincides with $\ker \omega(y)$, which implies $\ker \omega(y) \subset T_y(\mathbb{T}^{q+d} \cdot y)$.

Consider now the case when y is a singular point for the torus $\mathbb{T}^{q+d}$-action, i.e., $\dim(\mathbb{T}^{q+d} \cdot y) < q+d$. We show that the inclusion $\ker \omega(y) \subset T_y(\mathbb{T}^{q+d} \cdot y)$ still holds.

By the slice theorem, there is a local submanifold $N(y)$ which intersects the orbit $\mathbb{T}^{q+d} \cdot y$ transversally at y and which is saturated by the orbits of the action of $\mathbb{T}_y$, where $\mathbb{T}_y$ denotes the connected component of the identity of isotropy subgroup of the $\mathbb{T}^{q+d}$-action at y. The singular foliation by the orbits of the torus action is locally a direct product of the orbits of $\mathbb{T}_y$ on $N(y)$ with a small neighborhood of y in the orbit $\mathbb{T}^{q+d} \cdot y$. Moreover, by a local linearization, the orbits of $\mathbb{T}_y$ on $N(y)$ can be assumed to lie on the concentric spheres centered at y.

If $\ker \omega(y) \not\subset T_y(\mathbb{T}^{q+d} \cdot y)$, there would exist a non-zero vector $Y \in \ker \omega(y) \cap T_y N(y)$. By continuity, for every point $y' \in N(y)$ near y which is regular with respect to the $\mathbb{T}^{q+d}$-action, there is also a vector $Y' \in \ker \omega(y')$ which is "almost equal to Y"; y' can be chosen so that Y' is transverse to the cylinder which is the direct product of the sphere centered at y in $N(y)$ with a small neighborhood of y in $\mathbb{T}^{q+d} \cdot y$ in the local linearized model for the torus $\mathbb{T}^{q+d}$-action. On the other hand, locally the orbit through y' lies on this cylinder, so $Y' \notin T_{y'}(\mathbb{T}^{q+d} \cdot y')$, which is a contradiction, because y' is a regular point and we must have $Y' \in \ker \omega(y') \subset T_{y'}(\mathbb{T}^{q+d} \cdot y')$.

Recall that the vector fields $Y_1, \dots, Y_d$ are tangent to $\ker \omega$, and are given by linearly independent linear combinations of $X_1, \dots, X_{q+d}$, so at a regular point y, where $X_1, \dots, X_{q+d}$ are linearly independent, we also have that $Y_1, \dots, Y_d$ are linearly independent and span $\ker \omega$, because $\dim \ker \omega(y) = d$.

Let us show that if y is a singular point, i.e., the connected component $\mathbb{T}_y$ of the isotropy group of the torus $\mathbb{T}^{q+d}$-action has positive dimension $s \geq 1$, then $Y_1, \dots, Y_d$ are still linearly independent at y. Without loss of generality, we may assume that the subgroup $\mathbb{T}_y \subset \mathbb{T}^{q+d}$ is generated by $X_1, \dots, X_s$. Assume that $Y_1, \dots, Y_d$ are linearly dependent at y. Then, without loss of generality, we may assume that $Y_1(y) = 0$. In the linear combination expression $Y_1 = \sum a_{1i} X_i$ we must have $a_{1i} = 0$ for all $i > s$ (because otherwise $Y_1(y)$ would be non-zero), so $Y_1 = \sum_{i=1}^{s} a_{1i} X_i$. Since the torus $\mathbb{T}^{q+d}$-action preserves the regular integrable distribution $\ker \omega$, we can linearize simultaneously this torus action and $\ker\omega$ near y, i.e., find a coordinate system $(x_1, \ldots, x_{q+d-s}, z_1, \ldots, z_{2n-q+s})$ of M centered at y in which $\ker\omega$ is a constant distribution, $X_{s+i} = \partial/\partial x_i$ for every $i = 1, \ldots, q+d-s$, and $X_1, \ldots, X_s$ and Y_1 are linear vector fields in $z_1, \ldots, z_{2n-q+s}$ with imaginary eigenvalues. In order for $\ker\omega$ to contain Y_1 in this linearized coordinate system, $\ker\omega$ must have non-trivial intersection with the distribution spanned by $\partial/\partial z_1, \ldots, \partial/\partial z_{n+s}$ (at point y, and hence at any point in a small neighborhood of y because both distributions are constant). But such a nontrivial intersection is not tangent to the vector space generated by $X_1, \ldots, X_s$ at a generic point, and hence $\ker\omega$ is not tangent to the vector space generated by $X_1, \ldots, X_s, X_{s+1}, \ldots, X_{q+d}$ at a generic point, which is a contradiction. Thus $Y_1, \ldots, Y_d$ must be linearly independent at y. □

Proposition 2.6. *Under the assumptions of Theorem 2.4, for every point $y \in M^{2n+d}$ we have the following equality:*

$$(q+d) - \dim(\mathbb{T}^{q+d} \cdot y) = q - \text{rank } (\mathrm{d}F_1, \ldots, \mathrm{d}F_{q+d})(y)$$

Proof. This equality is a direct consequence of the previous proposition: the map $X \mapsto \omega(X, \cdot)$ sends $T_y(\mathbb{T}^{q+d} \cdot y)$ to $\text{span}(\mathrm{d}F_1, \ldots, \mathrm{d}F_{q+d})(y)$ and its kernel is $\ker\omega(y) \subset T_y(\mathbb{T}^{q+d} \cdot y)$ which is d-dimensional, hence $\dim(\mathbb{T}^{q+d} \cdot y) = d + \text{rank } (\mathrm{d}F_1, \ldots, \mathrm{d}F_{q+d})(y)$. □

The number $(q+d) - \dim \mathbb{T}^{q+d} \cdot y = q - \text{rank } (\mathrm{d}F_1, \ldots, \mathrm{d}F_{q+d})(y)$ is called the ***corank*** of y with respect to the Hamiltonian torus action and is denoted by corank y. The point y is ***regular*** if and only if its corank is 0, or, equivalently, the orbit through y has dimension $q+d$, or, equivalently, the momentum map has rank q at y. The point y is ***maximally singular*** if and only if the momentum map has rank zero at y, in which case the orbit through y has dimension d and is a leaf of the isotropic foliation of ω.

2.4. Local symplectization

In order to prove Theorem 2.4, we will need the following symplectization result.

Theorem 2.7 (Local Symplectization). *Under the hypotheses of Theorem 2.4, there exists a Hamiltonian torus $\mathbb{T}^{q+d}$-action on $M \times D^d$ (where $D^d \subset \mathbb{R}^d$ denotes a small d-dimensional open disk) equipped with an appropriate symplectic form $\tilde{\omega}$ and momentum map $\tilde{F}$ such that:*

(i) *For each* $z \in D^d$*,* $M \times \{z\}$ *is presymplectic of constant corank* d *and is invariant with respect to the torus* $\mathbb{T}^{q+d}$*-action.*

(ii) *Denote by* $O \in D^d$ *the origin of the disk. Then* $M \times \{O\}$ *together with the pull back of* $\tilde{\omega}$*, the restriction of the* $\mathbb{T}^{q+d}$*-action and of* $\tilde{F}$*, coincides with the original* M *with its presymplectic form, Hamiltonian* $\mathbb{T}^{q+d}$*-action, and momentum map* F*.*

Moreover, this local symplectization is unique in the following natural sense. If there is an equivariant presymplectic embedding $\phi : M \to (\hat{M}^{2(n+d)}, \hat{\omega})$ *from* M *to a symplectic manifold* $(\hat{M}^{2(n+d)}, \hat{\omega})$ *equipped with a Hamiltonian* $\mathbb{T}^{q+d}$*-action, then* ϕ *can be extended to an equivariant symplectic diffeomorphism from a neighborhood of* $M \cong M \times \{O\}$ *in* $M \times D^d$ *(equipped with the above symplectic from and Hamiltonian* $\mathbb{T}^{q+d}$*-action) into* $(\hat{M}^{2(n+d)}, \hat{\omega})$*.*

Proof. Put an arbitrary $\mathbb{T}^{q+d}$-invariant Riemannian metric g on M. At each point $y \in M$ denote by $V_y = (\ker \omega(y))^\perp \subset T_y M$ the $2n$-dimensional subspace of the tangent space $T_y M$ which is g-orthogonal to $\ker \omega(y)$. Then the distribution $\mathcal{V} = \{V_y \mid y \in M\}$ is smooth and invariant with respect to the $\mathbb{T}^{q+d}$-action. For each $i = 1, \dots, d$, define the 1-form α_i on M by

(3) $$\alpha_i(Y_i) = 1, \quad \alpha_i(Y_j) = 0,\ \forall j \neq i, \quad \alpha_i(V) = 0,\ \forall\, V \text{ a section of } \mathcal{V}.$$

Then put

(4) $$\tilde{\omega} = \sum_{i=1}^{d} dh_i \wedge \alpha_i + \sum_{i=1}^{n} h_i d\alpha_i + proj^* \omega,$$

where $(h_1, \dots, h_d)$ is a coordinate system on D^d which vanishes at O, and $proj : M \times D^d \to M$ is the natural projection onto M.

Lift the $\mathbb{T}^{q+d}$-action from M to $M \times D^d$ by making it acting trivially on D^d.

It is clear that ω is closed and non-degenerate (if the radius of the disk D^d is small) and invariant with respect to the $\mathbb{T}^{q+d}$-action, which shows that this is action is symplectic. This symplectic action is actually Hamiltonian for cohomological reasons. Indeed, when restricted to $M \times \{O\} \cong M$, the first cohomology class of $\tilde{\omega}(X_i, \cdot) = \omega(X_i, \cdot) = \mathrm{d}F_i$ is trivial, so on $M \times D^d$ the cohomology class of $\tilde{\omega}(X_i, \cdot)$ is also trivial by homotopy, and hence $\tilde{\omega}(X_i, \cdot) = \mathrm{d}\tilde{F}_i$ for some $\tilde{F}_i$ which can be chosen to be equal to F_i on $M \cong M \times \{O\}$.

Note that, by construction, we have $X_{H_i} = Y_i$, where $H_i = \sum_{j=1}^{q} a_{ij} \tilde{F}_j$ is constant on M for each $i = 1, \dots, d$.

To show the uniqueness of the local symplectization, we invoke Gotay's coisotropic embedding theorem [9], or, more precisely, its equivariant version, which is proved using the equivariant Moser path method. □

We remark that, if we forget about the torus action, then the situation studied by Gotay is more general than ours, because, in his case, the isotropic tangent vector bundle can be non-parallelizable, while in our case this bundle is parallelizable (precisely because of the torus action).

2.5. Normal form near an orbit of the torus action

In this subsection we recall the normal form theorem, due to Marle [19] and Guillemin–Sternberg [11], for a Hamiltonian torus $\mathbb{T}^k$-action ($k \geq 1$) on a symplectic manifold in the neighborhood of an orbit of the action, and then adapt this theorem to our presymplectic case using the Symplectization Theorem 2.7; see [21, Chapter 7] for the details and proofs of the well-known results stated in this subsection.

Let us start with the following simplified (Hamiltonian instead of symplectic) version of the so-called Witt–Artin decomposition, which was first proved by Witt [24] for symmetric bilinear forms. Fix a point m in a symplectic manifold M with a Hamiltonian $\mathbb{T}^k$-action. Since the torus action is Hamiltonian, every orbit is isotropic. We split $\mathfrak{t}$, the Lie algebra of $\mathbb{T}^k$, into the direct sum of two summands,

$$\mathfrak{t} = \mathfrak{t}_m \oplus \mathfrak{m}, \tag{5}$$

where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{t}_m$ in $\mathfrak{t}$ with respect to some positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}$. The splitting in (5) induces a similar one on the dual

$$\mathfrak{t}^* = \mathfrak{t}_m^* \oplus \mathfrak{m}^*, \tag{6}$$

where $\mathfrak{t}_m^* = \{\langle \eta, \cdot \rangle \mid \eta \in \mathfrak{t}_m\}$ and $\mathfrak{m}^* = \{\langle \xi, \cdot \rangle \mid \xi \in \mathfrak{m}\}$.

We use the following notation. If (V, Ω) is a symplectic vector space and $S \subset V$ is an arbitrary subset, then $S^\Omega = \{v \in V \mid \Omega(v, s) = 0, \forall x \in S\}$ denotes the ***Ω-orthogonal complement*** of S in V. Note that S^Ω is a vector subspace of V.

If G is a Lie group acting on a manifold N whose Lie algebra is denoted by $\mathfrak{g}$, then the tangent space to the orbit $G \cdot n \subset N$ equals $\mathfrak{g} \cdot n = \{\xi_N(n) \mid \xi \in \mathfrak{g}\}$, where $\xi_N(n) = \frac{d}{dt}\big|_{t=0} \exp(t\xi) \cdot n$ is the value at n of the infinitesimal generator vector field ξ_N defined by $\xi \in \mathfrak{g}$.

Theorem 2.8 (Witt–Artin decomposition). *Let (M,ω) be a symplectic manifold together with a Hamiltonian $\mathbb{T}^k$-action. Then for any point $m \in M$ we have*

$$T_mM = \mathfrak{t}\cdot m \oplus V \oplus W, \tag{7}$$

where:

(i) *V is the orthogonal complement to $\mathfrak{t}\cdot m$ in $(\mathfrak{t}\cdot m)^{\omega(m)}$ with respect to a $\mathbb{T}^k_m$-invariant inner product $\ll\cdot,\cdot\gg$ in T_mM. The subspace V is a symplectic $\mathbb{T}^k_m$-invariant subspace of $(T_mM,\omega(m))$.*

(ii) *$\mathfrak{t}\cdot m := \{\xi_M(m) \mid \xi \in \mathfrak{t}\}$ is a Lagrangian subspace of $V^{\omega(m)}$.*

(iii) *W is a $\mathbb{T}^k_m$-invariant Lagrangian complement to $\mathfrak{t}\cdot m$ in $V^{\omega(m)}$.*

(iv) *The map $f : W \to \mathfrak{m}^*$ defined by*

$$\langle f(w),\eta\rangle := \omega(m)(\eta_M(m),w), \qquad \textit{for all} \qquad \eta\in\mathfrak{m}$$

is a $\mathbb{T}^k_m$-equivariant isomorphism.

The space V in Theorem 2.8 is called a ***symplectic normal space*** at m. The $\mathbb{T}_m$-action on $(V,\omega(m)|_V)$ is linear Hamiltonian and has a standard associated momentum map $J_V : V \to \mathfrak{t}^*_m$ given by

$$\langle J_V(v),\xi\rangle = \frac{1}{2}\omega(m)(\xi\cdot v, v), \tag{8}$$

where $\xi\cdot v = \xi_V(v)$.

Let (M,ω) be a symplectic manifold together with a Hamiltonian $\mathbb{T}^k$-action and $m\in M$. Let V be a symplectic normal space at m and $\mathfrak{m}\subset\mathfrak{t}$ the subspace introduced in the splitting (5). Define the smooth manifold

$$Y_r = \mathbb{T}^k \times_{\mathbb{T}^k_m} (\mathfrak{m}^*_r \times V_r) \tag{9}$$

as the quotient of the product $\mathbb{T}^k\times(\mathfrak{m}^*_r\times V_r)$ by the $\mathbb{T}^k_m$-action defined by $h\cdot(t,\alpha,v) = (th,\alpha,h^{-1}\cdot v)$ for any $h\in\mathbb{T}^k_m$, $t\in\mathbb{T}^k$, $\alpha\in\mathfrak{m}^*$, and $v\in V$. Let $\pi : \mathbb{T}^k\times(\mathfrak{m}^*_r\times V_r)\to\mathbb{T}^k\times_{\mathbb{T}^k_m}(\mathfrak{m}^*_r\times V_r)$ be the projection. The torus $\mathbb{T}^k$ acts on Y_r by the formula $g\cdot[h,\eta,v] := [gh,\eta,v]$, for any $g\in\mathbb{T}^k$ and any $[h,\eta,v]\in Y_r$.

There exist $\mathbb{T}^k_m$-invariant disks $\mathfrak{m}^*_r\subset\mathfrak{m}^*$ and $V_r\subset V$ of some small radius $r>0$ centered at the origin, such that Y_r is a symplectic manifold with the

$\mathbb{T}^k$-invariant symplectic two-form ω_{Y_r} defined by

$$\begin{aligned}(10)\qquad &\omega_{Y_r}([g,\rho,v])(T_{(g,\rho,v)}\pi(T_eL_g(\xi_1),\alpha_1,u_1),T_{(g,\rho,v)}\pi(T_eL_g(\xi_2),\alpha_2,u_2))\\ &= \langle \alpha_2 + T_vJ_V(u_2),\xi_1\rangle - \langle \alpha_1 + T_vJ_V(u_1),\xi_2\rangle + \omega(m)(u_1,u_2),\end{aligned}$$

where $[g,\rho,v]\in Y_r$, $\xi_1,\xi_2\in\mathfrak{t}$, $\alpha_1,\alpha_2\in\mathfrak{m}^*$, and $u_1,u_2\in V$.

The symplectic manifold (Y_r,ω_{Y_r}) constructed above is called a ***symplectic tube*** of (M,ω) at the point m with respect to the Hamiltonian torus action. The importance of this symplectic tube is in the fact that it models the symplectic manifold (M,ω) as a Hamiltonian $\mathbb{T}^k$-space in a neighborhood of the orbit $\mathbb{T}^k\cdot m$.

Theorem 2.9 (Symplectic Slice Theorem). *Let (M,ω) be a symplectic manifold together with a Hamiltonian $\mathbb{T}^k$-action. Let $m\in M$ and let (Y_r,ω_{Y_r}) be the $\mathbb{T}^k$-symplectic tube at that point constructed above. Then there is a $\mathbb{T}^k$-invariant neighborhood U of m in M and a $\mathbb{T}^k$-equivariant symplectomorphism $\phi: U\to Y_r$ satisfying $\phi(m)=[e,0,0]$.*

Theorem 2.10 (The Marle–Guillemin–Sternberg normal form). *Let (M,ω) be a connected symplectic manifold with a Hamiltonian torus $\mathbb{T}^k$-action and associated momentum map $F: M\to\mathbb{R}^k$. Let $m\in M$ and (Y_r,ω_{Y_r}) be the symplectic tube at m constructed above that models a $\mathbb{T}^k$-invariant open neighborhood U of the orbit $\mathbb{T}^k\cdot m$ via the $\mathbb{T}^k$-equivariant symplectomorphism $\phi:(U,\omega|_U)\to(Y_r,\omega_{Y_r})$. Then the momentum map $F_{Y_r}=F|_U\circ\phi^{-1}: Y_r\to\mathbb{R}^k$ of the Hamiltonian $\mathbb{T}^k$-action on (Y_r,ω_{Y_r}) has the expression*

$$(11)\qquad \begin{array}{rrcl} F_{Y_r}: & Y_r=\mathbb{T}^k\times_{\mathbb{T}^k_m}(\mathfrak{m}_r^*\times V_r) & \longrightarrow & \mathbb{R}^k\\ & [g,\rho,v] & \longmapsto & F(m)+\rho+J_V(v),\end{array}$$

where $J_V: V\to\mathfrak{t}_m^*$ is given by (8).

The above normal forms plus the symplectization theorem (Theorem 2.7) yields the following normal form around an orbit in the presymplectic case under the flatness condition for the momentum map.

Theorem 2.11 (Presymplectic normal form under the flatness condition). *Let (M^{2n+d},ω) be a connected presymplectic manifold, whose presymplectic form ω has constant corank d, with a Hamiltonian torus $\mathbb{T}^{q+d}$-action and associated momentum map $F: M\to\mathbb{R}^{q+d}$ which satisfies the flatness condition (see Definition 2.1). Let $m\in M$ and (Y_r,ω_{Y_r}) be the symplectic tube at m constructed above that models a $\mathbb{T}^{q+d}$-invariant open* *neighborhood U of the orbit $\mathbb{T}^{q+d}\cdot m$ in the symplectization $(\hat{M}^{2n+2d},\hat{\omega})$ of (M^{2n+d},ω) via the $\mathbb{T}^{q+d}$-equivariant symplectomorphism $\phi:(U,\omega|_U)\to(Y_r,\omega_{Y_r})$. Then the momentum map*

$$F_{Z_r}=F|_U\circ\phi^{-1}|_{Z_r}:Z_r\to\mathbb{R}^{q+d}$$

of the Hamiltonian $\mathbb{T}^{q+d}$-action on $Z_r=Y_r\cap\phi(M^{2n+d})$ has the expression

$$\begin{array}{llcc} F_{Z_r}: & Z_r=\mathbb{T}^{q+d}\times_{\mathbb{T}^{q+d}_m}B_r & \longrightarrow & \mathbb{R}^{q+d} \\ & [g,\rho,v] & \longmapsto & F(m)+\rho+J_V(v), \end{array} \tag{12}$$

where $J_V:V\to\mathfrak{t}^*_m$ is given by (8) and $B_r\subset\mathfrak{m}^*_r\times V_r$ consists of points (ρ,v) such that $\rho+J_V(v)\in\mathfrak{l}$, where $\mathfrak{l}$ is a q-dimensional linear subspace of $\mathfrak{t}^*\cong\mathbb{R}^{q+d}$ such that $\mathfrak{l}+\mathfrak{m}^*=\mathfrak{t}^*$.

Remark 2.12. The space $\mathfrak{l}$ in the above theorem is nothing else but the vector subspace which is parallel to the q-dimensional affine subspace which contains the image $F(M)$ of the momentum map by the flatness condition. The equality $\mathfrak{l}+\mathfrak{m}^*=\mathfrak{t}^*$ in the theorem assures that the set B_r in the theorem is a manifold, and hence the normal form model is regular. The intersection $\mathfrak{l}\cap\mathfrak{m}^*$ is not trivial in general: in fact, it corresponds to the face of $F(M)$ which contains the point $F(m)$.

2.6. Proof of convexity Theorem 2.4

Equipped with symplectization (Theorem 2.7) and presymplectic normal forms (Theorem 2.11) we can now obtain Theorem 2.4 by simply repeating the same steps in the proofs of the classical Atiyah–Guillemin–Sternberg theorem.

For example, we can use the approach based on the local-global convexity principle, as outlined in [27].

Denote by $\mathcal{B}$ the q-dimensional ***base space*** of a presymplectic toric manifold M with momentum F, which satisfies the conditions of Theorem 2.4. By definition, $\mathcal{B}$ is the set of all connected components of all fibers $F^{-1}(c)\subset M$, $c\in F(M)$, endowed with the quotient topology, i.e., a subset $U\subset\mathcal{B}$ is open if and only if $\pi^{-1}(U)$ is open in M, where $\pi:M\to\mathcal{B}$ is the map that sends $y\in M$ to the connected component of $F^{-1}(F(y))$ containing y.

The momentum map $F:M\to\mathbb{R}^{q+d}$ factorizes through $\mathcal{B}$, i.e., $F=\tilde{F}\circ\pi$, where π is the projection map from M to $\mathcal{B}$ and $\tilde{F}$ is a map from $\mathcal{B}$ to $\mathbb{R}^{q+d}$. F may be viewed as the restriction of the momentum map $\hat{F}:\hat{M}\to\mathbb{R}^{q+d}$ of the torus action on the symplectization $\hat{M}$ of M to M, and $\mathcal{B}$ may be viewed as a subspace of the base space $\hat{\mathcal{B}}$ of the torus action on $\hat{M}$. The base space $\hat{\mathcal{B}}$ admits an intrinsic integral affine structure (see [27] for a definition of this integral affine structure using period integrals over 1-cycles), whose local integral affine functions are those functions which generate local Hamiltonian $\mathbb{T}^1$-actions on $\hat{M}$ (i.e., local action functions).The Normal Form Theorem 2.11 implies, in particular, that $\mathcal{B}$ is a locally flat subspace of $\hat{\mathcal{B}}$ (this is exactly what the flatness condition is about). More precisely, locally $\hat{\mathcal{B}}$ can be identified with a corner of $\mathfrak{t}^* \cong \mathbb{R}^{q+d}$ while $\mathcal{B}$ can be identified with the intersection of that corner with a linear q-dimensional subspace $\mathfrak{l}$ of $\mathfrak{t}^*$. Thus, $\mathcal{B}$ inherits an intrinsic affine structure from $\hat{\mathcal{B}}$. (A corner means a subspace of the type $\{(x_1, \ldots, x_{q+d}) \in \mathbb{R}^{q+d} \mid x_1 \geq 0, \ldots, x_s \geq 0\}$ for some $s \geq 0$).

The Atiyah–Guillemin–Sternberg theorem (more precisely, its local version) states that $\hat{\mathcal{B}}$ is locally convex with locally polyhedral boundary with respect to its integral affine structure, and the projected momentum map $\hat{\tilde{F}} : \hat{\mathcal{B}} \to \mathbb{R}^{q+d}$ is a locally injective integral affine map. Since $\mathcal{B}$ is locally a flat slice of $\hat{\mathcal{B}}$, it inherits these properties from $\mathcal{B}$, i.e., $\mathcal{B}$ is locally convex with locally polyhedral boundary with respect to its affine structure, and the projected momentum map $\tilde{F} : \mathcal{B} \to \mathbb{R}^{q+d}$ is a locally injective affine map. Moreover, $\mathcal{B}$ is connected compact. The local-global convexity principle (see Lemma 3.7 of [27]) then says that $\tilde{F}$ is an injective affine map on $\mathcal{B}$, and its image $\tilde{F}(\mathcal{B}) = F(M)$ is a convex polytope of dimension q in $\mathbb{R}^{q+d}$. The injectivity of $\tilde{F}$ also implies that the preimages of F are connected. Similarly, the openness of F follows immediately from the openness of $\tilde{F}$.

Theorem 2.4 is proved. □

3. Presymplectic toric manifolds

3.1. Framed momentum polytopes of presymplectic toric manifolds

Definition 3.1. A compact ***presymplectic toric manifold*** is a compact connected manifold M^{2n+d} of dimension $2n + d$ $(n, d \geq 0)$ equipped with a presymplectic structure ω with constant corank d and a Hamiltonian torus $\mathbb{T}^{n+d}$-action $\rho : \mathbb{T}^{n+d} \times M^{2n+d} \to M^{2n+d}$ which is free almost everywhere and which satisfies the flatness condition given in Definition 2.1.

Remark 3.2. Karshon and Tolman in [13] also introduced a notion of presymplectic toric manifolds, but their notion is completely different and should not be confused with ours. In fact, the presymplectic structure in their manifolds are symplectic almost everywhere and is degenerate only at a small subset, while our presymplectic structure is a regular presymplectic structure of constant corank. Their manifolds do not have convexity properties for the momentum maps, while our presymplectic toric manifolds do have convex momentum polytopes.

It follows from Theorem 2.4 and Theorem 2.7 that if (M^{2n+d}, ω, ρ) is a compact presymplectic toric manifold with a corresponding momentum map $F = (F_1, \dots, F_{n+d})$, then its image $P = F(M^{2n+d})$ is an n-dimensional convex polytope lying in $\mathbb{R}^{n+d}$. Moreover, by Theorem 2.7, (M^{2n+d}, ω, ρ) admits a unique (up to local isomorphisms) symplectization in the form of an open symplectic manifold $(\hat{M}^{2n+2d}, \omega, \rho)$, together with a Hamiltonian torus $\mathbb{T}^{n+d}$-action, which contains (M^{2n+d}, ω, ρ), such that $\hat{M}^{2n+2d}$ is a small tubular neighborhood of M^{2n+d}, the inclusion map $i : (M^{2n+d}, \omega, \rho) \hookrightarrow (\hat{M}^{2n+2d}, \omega, \rho)$ is compatible with the (pre)symplectic forms, the torus actions, and the momentum maps, which are denoted by the same letters for both M^{2n+d} and $\hat{M}^{2n+2d}$.

In particular, for a sufficiently small local tubular symplectization $(\hat{M}^{2n+2d}, \omega, \rho)$ of (M^{2n+d}, ω, ρ), the image $F(\hat{M}^{2n+2d})$ of $(\hat{M}^{2n+2d}, \omega, \rho)$ under the momentum map F is a $(n+d)$-dimensional locally-polyhedral set $Q \subset \mathbb{R}^{n+d}$, and $P = L \cap Q$ where L is an n-dimensional affine subspace of $\mathbb{R}^{n+d}$; Q is like an open subset of a Delzant polytope, i.e., its faces satisfy the rationality, simplicity, and regularity conditions of a Delzant polytope. Indeed, denoting by $\mathcal{B}$ and $\hat{\mathcal{B}}$ the base spaces of M^{2n+d} and $\hat{M}^{2n+2d}$ with respect to the torus actions, then the momentum map from $\hat{M}^{2n+2d}$ to $\mathbb{R}^{q+d}$ projects to a map from $\hat{\mathcal{B}}$ to $\mathbb{R}^{q+d}$ which is integral affine locally injective and which is injective on $\mathcal{B}$. But since $\hat{\mathcal{B}}$ is a small neighborhood of $\mathcal{B}$, the local injectivity of the momentum map on $\hat{\mathcal{B}}$ plus its injectivity on $\mathcal{B}$ implies its injectivity on $\hat{\mathcal{B}}$. From this fact we get that Q, which is the image of $\hat{\mathcal{B}}$ in $\mathbb{R}^{n+d}$ under an injective integral affine map, is locally rational simple polyhedral, because $\hat{\mathcal{B}}$ has these local properties.

We are only interested in the germ of a neighborhood of P in Q. That germ is called the *framed momentum polytope*. To make things more precise, we introduce the following definitions.

Definition 3.3. (i) A n-dimensional convex polytope $P \subset \mathbb{R}^N$ $(N \geq n)$ is called ***rational-faced*** if for every facet Z of P there is a linear function with integral coefficients $H_Z = \sum_{j=1}^{N} c_j f_j$, where $c_j \in \mathbb{Z}$ and $(f_1, \dots, f_N)$ is an integral affine coordinate system on $\mathbb{R}^N$, such that H_Z is constant on Z but is not constant on P.

(ii) An injective affine map $\eta : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ $(N_1 \leq N_2)$ is called an ***integral affine embedding*** from $\mathbb{R}^{N_1}$ to $\mathbb{R}^{N_2}$ if, up to a translation, the image $\eta(\mathbb{Z}^{N_1})$ of the integral lattice of $\mathbb{R}^{N_1}$ is a sub-lattice of the integral lattice $\mathbb{Z}^{N_2}$ of $\mathbb{R}^{N_1}$ such that the quotient $\mathbb{Z}^{N_2}/\eta(\mathbb{Z}^{N_1})$ is without torsion, or equivalently, the pull-back of the space of integral affine functions on $\mathbb{R}^{N_2}$ to $\mathbb{R}^{N_1}$ via the map η is exactly equal to the space of integral affine functions on $\mathbb{R}^{N_1}$: $\eta^*(Aff_{\mathbb{Z}}\mathbb{R}^{N_2}) = Aff_{\mathbb{Z}}\mathbb{R}^{N_1}$. If $P \subset \mathbb{R}^{N_1}$ is a polytope and $\eta : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ is an integral affine embedding then the restriction of η to P is also called an ***integral affine embedding*** from P to $\mathbb{R}^{N_2}$.

(iii) Two convex polytopes $P_1 \subset \mathbb{R}^{N_1}$ and $P_2 \subset \mathbb{R}^{N_2}$ $(N_1 \leq N_2)$ are called (integral-affinely) ***isomorphic*** if there is an integral affine embedding $\eta : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ such that the restriction of η to P is a homeomorphism from P_1 to $P_2 = \eta(P_1)$. In other words, there is integral affine embedding from P_1 to $\mathbb{R}^{N_2}$ whose image is P_2.

Remark 3.4. It is easy to see that the above notion of integral-affinely isomorphic polytopes is really an equivalence relation. If two polytopes $P, P' \subset \mathbb{R}^N$ in the same Euclidean space are isomorphic then it means that there is an integral affine transform $\phi \in GL(N, \mathbb{Z}) \ltimes \mathbb{R}^N$ such that $\phi(P) = P'$.

Definition 3.5. (i) A ***regular rational-faced framing*** of a convex simple polytope P of dimension n in $\mathbb{R}^N$ $(N \geq n)$ is a pair (L, Q), where L is the n-dimensional affine subspace of $\mathbb{R}^N$ which contains P, and Q is a locally-polyhedral set in $\mathbb{R}^N$ whose faces satisfy the rationality, simplicity, and regularity conditions of a Delzant polytope, such that L intersects Q transversally and $L \cap Q = P$. The convex polytope P together with a regular rational-faced framing given by Q is called a regular rational-faced ***framed polytope***.

(ii) If $P = F(M^{2n+d}) \subset L \subset \mathbb{R}^{n+d}$ is the image under the flat momentum map F of a presymplectic toric manifold (M^{2n+d}, ω, ρ) and $Q = F(\hat{M}^{2n+2d})$ is the image under the momentum map F of a symplectization $(\hat{M}^{2n+2d}, \omega, \rho)$ of (M^{2n+d}, ω, ρ), then P is called the ***momentum polytope*** of (M^{2n+d}, ω, ρ), and P together with the framing given by Q is called the ***framed momentum polytope*** of (M^{2n+d}, ω, ρ).

(iii) Two n-dimensional regular rational-faced framed polytopes $P_1 = L_1 \cap Q_1$ and $P_2 = L_2 \cap Q_2$ in $\mathbb{R}^{n+d}$ are called (integral-affinely) ***isomorphic*** if there is an integral affine transform $\Phi \in GL(n + d, \mathbb{Z}) \ltimes \mathbb{R}^{n+d}$ such that $\Phi(L_1) = L_2$ and $\Phi(U(P_1)) = U(P_2)$ where $U(P_1)$ (resp., $U(P_2)$) is a small neighborhood of P_1 (resp., P_2) in Q_1 (resp., Q_2).

(iv) Two presymplectic toric manifolds are called ***isomorphic*** if there is a diffeomorphism from one to the other which preserves the presymplectic structure and the torus action, up to an automorphism of the torus.

Theorem 3.6. (i) *If (M^{2n+d}, ω, ρ) is a connected compact presymplectic toric manifold, then its momentum polytope is a convex rational-faced simple polytope and its framed momentum polytope is a regular rational-faced framed polytope.*

(ii) *Conversely, any convex regular rational-faced framed polytope is the framed momentum polytope of a compact connected presymplectic toric manifold.*

(iii) *Connected compact presymplectic toric manifolds are classified by their framed momentum polytopes: two compact presymplectic toric manifolds are isomorphic if and only if their corresponding framed momentum polytopes are isomorphic.*

Proof. Part (i) is just a special case of the results obtained in Section 2. The image $Q = F(M^{2n+d})$ of a symplectization $(\hat{M}^{2n+2d}, \omega, \rho)$ of our presymplectic manifold is locally-polyhedral and satisfies the rationality, simplicity and regularity properties at its face because the singularities of the Hamiltonian torus action, viewed as a toric integrable Hamiltonian system on it, are all non-degenerate elliptic. The momentum polytope P of (M^{2n+d}, ω, ρ) is a convex polytope because of the convexity Theorem 2.4, and this polytope is simple because it is a slice of Q, which is simple, by an affine submanifold which cuts Q transversally (the transversality condition is implied by the regularity condition of the presymplectic form ω on M^{2n+d}).

To prove Part (ii), we can use the method of integrable surgery for constructing integrable Hamiltonian systems [26]. Let us recall that, from the point of view of integrable Hamiltonian systems, a Hamiltonian torus action of maximal dimension on a symplectic manifold is also an integrable Hamiltonian system, whose singularities are all non-degenerate elliptic (see [25, 26]). Given a regular framing (P, Q) of P, there is a unique integrable Hamiltonian system with elliptic singularities which admits Q (together with its induced integral affine structure) as the base space, according to general results of [26] on the construction and classification of integrable Hamiltonian systems. (See Section 4 of [26] about classification and integrable surgery, and in particular Example 4.14 about the case of Delzant polytopes. Our situation here is absolutely similar, i.e., the monodromy sheaf is constant and there is no room for characteristic classes, hence we have both existence and uniqueness). See also Karshon and Lerman [12] where this result is also proved in more detail. Due to the type of the base space and singularities, this integrable Hamiltonian system is actually a Hamiltonian torus action of half the dimension of the symplectic manifold $(\hat{M}^{2n+2d}, \omega)$. By taking $M^{2n+d} = F^{-1}(P)$ with the pull-back of the symplectic form and the restriction of the torus action on it, where $F : \hat{M}^{2n+2d} \to \mathbb{R}^{n+d}$ is the momentum map of the Hamiltonian torus $\mathbb{T}^{n+d}$-action such that $F(\hat{M}^{2n+2d}) = Q$, we get the required presymplectic toric manifold whose framed momentum polytope is (P, Q). Part (iii) also follows from these same arguments, together with Theorem 2.7 about the existence and uniqueness of equivariant symplectization. □

Remark 3.7 (Slices of Delzant polytopes). When Q is a framing of P then we also say that P is a ***slice*** of Q. If Q is a Delzant polytope and $P = L \cap Q$ is a slice of Q by an affine subspace which intersects Q transversally, then, of course, by Delzant's theorem $Q = F(M)$ is the image of the momentum map F of a symplectic toric manifold M and $M_P = F^{-1}(P)$ is the presymplectic toric submanifold of M whose momentum polytope is P. In general, we do not need Q to be a regular simple polytope, we just need that locally Q looks like a regular simple polytope at its faces.

3.2. Lifting and framing of polytopes

In the definition of a rational-faced polytope P, the affine subspace $L \subset \mathbb{R}^N$ containing P may be irrational, in the sense that the linear equations defining it may have irrational linear coefficients, even though each facet of P must satisfy a rational linear equation. If L is rational, then we also say that P is ***rational***, and if L is irrational then we also say that P is ***irrational***.

More precisely, we can define the ***degree of irrationality*** to be the minimal number of linear equations which must have at least one irrational linear coefficient in the definition of the supporting affine subspace L of P. It is clear that P is rational if and only if its degree of irrationality is 0, and if P and P' are isomorphic then they have the same degree of irrationality.

For each convex rational-faced polytope $P \subset \mathbb{R}^N$, denote by $Aff_{\mathbb{Z}}(P)$ the Abelian group of all integral affine functions restricted to P, i.e., the quotient of the space of all integral affine functions on $\mathbb{R}^N$ by those which vanish on P. The rational-faced condition means that each facet of P is given by an equation of the form $F = 0$, where $F \in Aff_{\mathbb{Z}}(P)$. The quotient $Daff_{\mathbb{Z}}(P) = Aff_{\mathbb{Z}}(P)/\mathbb{R}$ of $Aff_{\mathbb{Z}}(P)$ by constant functions is a free finitely generated Abelian group, and is called the ***Abelian group of integral affine 1-forms*** on P.

Up to isomorphisms, each convex rational-faced polytope P is uniquely characterized by its group of integral affine functions $Aff_{\mathbb{Z}}(P)$, i.e., P_1 is isomorphic to P_2 (even if they live in different Euclidean spaces) if and only if there is a homeomorphism from P_1 to P_2 which induces a group isomorphism from $Aff_{\mathbb{Z}}(P_1)$ to $Aff_{\mathbb{Z}}(P_2)$. The number

$$I = \mathrm{rank}_{\mathbb{Z}} Daff_{\mathbb{Z}}(P) - \dim P$$

is nothing else but the *degree of irrationality* of P. In addition, P is a subset of a Euclidean space of dimension at least $n + I$, where n is the dimension of P and I is the degree of irrationality of P.

One can embed P isomorphically into $\mathbb{R}^{n+I}$ by a map $G = (G_1, \ldots, G_{n+I}) : P \to \mathbb{R}^{n+d}$, where $(G_1, \ldots, G_{n+I})$ modulo constants is a basis of $Daff_{\mathbb{Z}}(P)$. However, in order to find a *regular* simple rational-faced framing for P, we may need more than $n + I$ dimensions. Indeed, already if P is n-dimensional simple rational but not regular in $\mathbb{R}^n$, we need more dimensions in order to produce a regular framing of P.

By increasing the dimension of the Euclidean space, if necessary, it is always possible to find a regular rational-faced framing for a rational-faced simple convex polytope, as will be shown in Theorem 3.8.

By an *irrational polytope*, many authors, including Prato–Battaglia [3] and Katzarkov–Lupercio–Meersseman–Verjovsky [14], mean an n-dimensional polytope in $\mathbb{R}^n$ with an irrational face. However, the rational-faced property of a polytope is preserved under isomorphisms, so if a polytope P has an irrational face, there is no way to turn it into the momentum polytope of a presymplectic toric manifold by integral affine isomorphisms.

Nevertheless, one can always *lift* an arbitrary convex polytope $P \subset \mathbb{R}^n$ to a rational-faced polytope $P' \subset \mathbb{R}^N$ with a regular rational-faced framing $Q \subset \mathbb{R}^N$ for some $N > n$, such that the linear projection map $proj : \mathbb{R}^N \to \mathbb{R}^n$ on the first n components of $\mathbb{R}^N$ projects P' to P homeomorphically. If P is rational-faced then projection map $proj$ is an integral-affine isomorphism from P' to P, and if P is not rational-faced, then it is not. We call such a P' a ***rational-faced lifting*** of P.

We describe below an easy construction of rational-faced liftings together with a regular framing, which is already used by Prato in [23].

Let $P \subset \mathbb{R}^n$ be a arbitrary convex polytope of dimension n. Let $F_i(x) = \sum_{j=1}^n a_{ij} x_j + b_i$, where $a_{ij}, b_i \in \mathbb{R}$ and $(x_1, \ldots, x_n)$ is an integral affine coordinate system of $\mathbb{R}^n$, be linear functions on $\mathbb{R}^n$ which determine the facets of P ($i = 1, \dots, d$), and such that $F_i \geq 0$ on P. Then P can be written as

$$P = \{x \in \mathbb{R}^n \mid F_i(x) \geq 0,\ i = 1, \dots, d\}.$$

Denote the coordinates of points in $\mathbb{R}^{n+d}$ by $(x_1, \dots, x_n, y_1, \dots, y_d)$. For $M > 0$ sufficiently large, define the box

$$\begin{aligned} Q = \{(x_1, \dots, x_n, y_1, \dots, y_d) \in \mathbb{R}^{n+d} \mid -M \leq x_i \leq M,\ i = 1, \dots, n, \\ 0 \leq y_j \leq M,\ j = 1, \dots, d\}. \end{aligned}$$

Cut Q by the d hyperplanes L_i, $i = 1, \dots, d$, where

$$L_i = \left\{ (x_1, \dots, x_n, y_1, \dots, y_d) \in \mathbb{R}^{n+d} \;\middle|\; y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i \right\}$$

to obtain $P' = Q \cap L_1 \cap \cdots \cap L_d$. It is easy to see that if M is large enough, then $P' \subset \mathbb{R}^{n+d}$ projects bijectively to $P \subset \mathbb{R}^n$, and that Q is a Delzant polytope, hence a regular framing for P'.

When P is rational-faced, we can choose all the above coefficients a_{ij} to be integers. In this case, because the coefficient of y_i in the equation $y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$ is 1 for every $1 \leq i \leq d$, it is easy to verify that the projection from P' to P is an integral affine isomorphism.

If the dimension of $P \subset \mathbb{R}^n$ is smaller than n, the situation is the same: just add the same defining linear equations for P to the above linear equations $y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$ for P'; Q remains the same. From this construction, we obtain the following result.

Theorem 3.8. *For any convex simple polytope $P \subset \mathbb{R}^N$ there is another polytope $P' \subset \mathbb{R}^{N'}$ ($N' \geq N$) which projects to P under the natural projection from $\mathbb{R}^{N'}$ to $\mathbb{R}^N$, such that P' admits a regular rational-faced framing in $\mathbb{R}^{N'}$, and hence is the momentum polytope of a presymplectic toric manifold. Moreover, if P is rational-faced then P' can be chosen to be integral-affinely isomorphic to P.*

Remark 3.9. Even when P is not simple, we can use the same construction. The only difference is that L does not intersect Q transversally, if P is not simple.

Example 3.10. In general, the same convex rational-faced polytope admits infinitely many regular framings which are not isomorphic, and thus correspond to infinitely many non-isomorphic presymplectic toric manifolds. Take, for example, the interval $P = [O, A] \subset \mathbb{R}^2$, where $O = (0, 0)$ and $A = (a, 0)$. Then P is rational-faced and admits infinitely many non-isomorphic 2-dimensional regular rational-faced framings. In fact, for any positive integer p and any integer q such that $gcd(p, q) = 1$, the set

$$Q_{p,q} = \{(x, y) \in \mathbb{R}^2 \mid -\varepsilon < y < \varepsilon,\ x \geq 0,\ p(x - a) + qy \leq 0\}$$

(for $\varepsilon > 0$ small enough) is a regular rational-faced framing of P, and two framings $Q_{p,q}$ and $Q_{p',q'}$ are isomorphic if and only if $p = p'$ and $q = \pm q'$.

3.3. Morita equivalence

The notion of Morita equivalence that we want to introduce in this subsection is inspired by the notion of Morita equivalence for Lie groupoids (see, e.g., [18], [8, Section 7.2]). Intuitively speaking, two presymplectic toric manifolds (M_1, ω_1, ρ_1) and (M_2, ω_2, ρ_2) (ρ_1 and ρ_2 are the torus actions) are Morita equivalent if their quotient spaces with respect to the corresponding kernel isotropic foliations are isomorphic.

In the case of rational momentum polytopes, the quotient spaces of the presymplectic toric manifolds are symplectic toric orbifolds and we can really compare them directly. However, when the momentum polytopes are irrational, the quotient spaces are quasifolds which are not Hausdorff, and it is rather inconvenient to compare such bad quotient spaces directly. Instead, we will develop the Morita equivalence as an indirect way to verify when two presymplectic toric manifolds should be considered as having the same quotient space.

Definition 3.11. (i) Let $\phi : (M_1^{2n+d_1}, \omega_1, \rho_1) \to (M_2^{2n+d_2}, \omega_2, \rho_2)$ be a submersion with connected fibers between two presymplectic toric manifolds M_1 and M_2, with $d_1 \geq d_2$ (d_i is the corank of ω_i). Then ϕ is called a ***Morita equivalence submersion*** if $\omega_1 = \phi^*\omega_2$, and $\phi(\rho_1(t, x)) = \rho_2(\theta(t), \phi(x))$ for any $x \in M_1^{2n+d_1}$ and any $t \in \mathbb{T}^{n+d_1}$, where $\theta : \mathbb{T}^{n+d_1} \to \mathbb{T}^{n+d_2}$ is a surjective homomorphism whose kernel is connected.

(ii) Two presymplectic toric manifolds $(M_1^{2n+d_1}, \omega_1, \rho_1)$ and $(M_2^{2n+d_2}, \omega_2, \rho_2)$ are called ***Morita equivalent*** if there is a third presymplectic toric manifold $(M_3^{2n+d_3}, \omega_3, \rho_3)$ together with two Morita equivalence submersions $\phi_1 : M_3^{2n+d_3} \to M_1^{2n+d_1}$ and $\phi_2 : M_3^{2n+d_3} \to M_2^{2n+d_2}$.

It is not obvious from the definition that the above Morita equivalence notion is a true equivalence relation among presymplectic toric manifolds, but it really is. In order to see why, we can translate this equivalence relation to an equivalence relation among framed momentum polytopes of presymplectic toric manifolds.

Definition 3.12. (i) Let $Q_1 \subset \mathbb{R}^{N_1}$ and $Q_2 \subset \mathbb{R}^{N_2}$ be two regular rational-faced framings of two rational-faced simple polytopes $P_1 \subset \mathbb{R}^{N_1}$ and $P_2 \subset \mathbb{R}^{N_2}$, respectively. Assume that there is an integral affine embedding $\eta : \mathbb{R}^{N_2} \to \mathbb{R}^{N_1}$ from $\mathbb{R}^{N_2}$ to $\mathbb{R}^{N_1}$ ($N_1 \geq N_2$), such that $\eta(P_2) = P_1$ and $\eta(U(P_2)) = U(P_1)$, where $U(P_i)$ is a small neighborhood of P_i in Q_i, respectively ($i = 1, 2$). Then we say that the framed polytope (P_1, Q_1) is ***Morita-equivalent*** to the framed polytope (P_2, Q_2), and that η is a ***Morita equivalence embedding*** from (P_2, Q_2) to (P_1, Q_1).

(ii) Two framed polytopes are called ***Morita-equivalent*** if both of them admit Morita equivalence embeddings to a third framed polytope.

Theorem 3.13. *The Morita equivalence of regular rational-faced framed simple convex polytopes is a true equivalence relation.*

Proof. All framings in this proof are assumed to be regular simple. It is easy to see directly from Definition 3.12 that if η is a Morita equivalence embedding from a framed polytope (P_1, Q_1) to a framed polytope (P_2, Q_2), and ν is a Morita equivalence embedding from (P_2, Q_2) to a framed polytope (P_3, Q_3), then the composition $\nu \circ \eta$ is a Morita equivalence embedding from (P_1, Q_1) to (P_3, Q_3).

The main point in the proof of the above theorem is the verification of the following statement: *if* (P, Q) *admits two Morita equivalence embeddings to* (P_1, Q_1) *and* (P_2, Q_2) *then there exist Morita equivalence embeddings from* (P_1, Q_1) *and* (P_2, Q_2) *to another framed polytope* (P_3, Q_3).

We will construct (P_3, Q_3) as a "*crossed product of* (P_1, Q_1) *and* (P_2, Q_2) *over* (P, Q)". The construction goes as follows: for each $i = 1, 2$, decompose the ambient Euclidean space $V_i \cong \mathbb{R}^{N_i}$ of Q_i in an integral affine way as $V_i = K_i \oplus V$ (the integral affine structure on V_i is the direct sum of the integral affine structures on K_i and V), where $V \cong \mathbb{R}^N$ is identified with the ambient Euclidean space of Q via the Morita equivalence embedding from (P, Q) to (P_i, Q_i). P is identified with P_1 and P_2 via these embeddings. Put $V_3 = K_1 \oplus K_2 \oplus V \cong \mathbb{R}^{N_1+N_2-N}$, which contains both V_1 and V_2 via natural identifications, and define the framing Q_3 of $P_3 = P$ in V_3 as follows:

Each facet ζ of Q is contained in exactly one facet ζ_1 of Q_1 and exactly one facet ζ_2 of Q_2 in V, $\zeta_1 \cap \zeta_2 = \zeta$. The (smallest) affine subspace of V_3 which contains both ζ_1 and ζ_2 is a hyperplane (i.e., of codimension 1) in

V_3. Denote by $\tilde{\zeta}_3$ the half-space of V_3 bounded by this hyperplane which contains P. Take $\tilde{Q}_3$ to be the intersection of all these half-spaces (one for each facet of Q), and define Q_3 to be a small neighborhood of P in this intersection. This is the framing of P that we wanted to construct.

It is clear from the construction that Q_3 is simple, rational, and that both (P, Q_1) and (P, Q_2) are embedded in (P, Q_3) in an integral affine way. It remains to check that Q_3 is regular, but this fact is a consequence of our assumption that the three frames Q, Q_1, Q_2 are all regular.

Indeed, consider a face of Q_3 and prove that Q_3 is regular at that face. It's enough to show that Q_3 is regular at one point x of that face, and we can choose x to be in Q, because Q is a transversal slice of Q_3 and each face of Q_3 contains a face of Q.

Denote by $\zeta^1, \ldots, \zeta^m$ the facets of Q which contain x $(m \geq 1)$. The regularity of Q at x means that on the tangent space T_xQ, equipped with the integral lattice induced from $V \supset Q$, there is a basis $(\alpha_1, \ldots, \alpha_N)$ of this integral lattice such that $\alpha_i \in T_x(\zeta^1 \cap \cdots \cap \zeta^{i-1} \cap \zeta^{i+1} \cap \cdots \cap \zeta^m)$ for each $i = 1, \ldots, m$ and $(\zeta_{m+1}, \ldots, \zeta_N)$ is a basis for the integral lattice of $T_x(\zeta^1 \cap \cdots \cap \zeta^m)$.

Because Q_1 is also regular at x, we have a similar basis for the integral lattice of T_xQ_1. Actually, because of the embedding of Q in Q_1, we can choose the basis of the integral lattice of T_xQ_1 to be of the form $(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_{N_1-N})$, where $(\alpha_1, \ldots, \alpha_N)$ is the above basis for T_xQ and $\beta_1, \ldots, \beta_{N_1-N}$ are additional vectors in the tangent space to the intersection of the facets of Q_1 at x. For the same reasons, we have a similar basis $(\alpha_1, \ldots, \alpha_N, \gamma_1, \ldots, \gamma_{N_2-N})$ for the integral affine lattice of T_xQ_2. Then $(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_{N_1-N}, \gamma_1, \ldots, \gamma_{N_2-N})$ is a basis for the integral lattice of T_xQ_3, which implies that Q_3 is regular at x. $\square$

Theorem 3.14. *There is a Morita equivalence submersion* $\phi : (M_1^{2n+d_1}, \omega_1, \rho_1) \to (M_2^{2n+d_2}, \omega_2, \rho_2)$ *between two presymplectic toric manifolds if and only if there is a Morita equivalence embedding* η *from the corresponding second framed momentum polytope* (P_2, Q_2) *to the first framed momentum polytope* (P_1, Q_1).

Proof. Let $\phi : (M_1^{2n+d_1}, \omega_1, \rho_1) \to (M_2^{2n+d_2}, \omega_2, \rho_2)$ be a Morita equivalence submersion between two presymplectic toric manifolds. By definition, we have that $\omega_1 = \phi^*\omega_2$, hence the tangent spaces to the fibers of the projection map $\phi : M_1 \to M_2$ lie in the kernel of ω_1. These tangent spaces have dimension $s = d_1 - d_2$.

Denote by $\mathbb{T}^s = \theta^{-1}(0)$ the kernel of θ (it is a torus of dimension s), where $\theta : \mathbb{T}^{n+d_1} \to \mathbb{T}^{n+d_2}$ is the surjective homomorphism given in Definition 3.11 of Morita equivalence submersion. For any $x \in M_1$ and $t \in \mathbb{T}^s$ we have $\phi(\rho_1(t,x)) = \rho_2(\theta(t), \phi(x)) = \rho_2(0, \phi(x)) = \phi(x)$, which means that the orbit of the action of $\mathbb{T}^s$ through x in M_1 lies in the fiber of ϕ which contains x.

Consider a symplectization $(\hat{M}_1^{2n+2d_1}, \omega_1, \rho_1)$ of $(M_1^{2n+d_1}, \omega_1, \rho_1)$ and the sub-Hamiltonian $\mathbb{T}^s$-action of the Hamiltonian $\mathbb{T}^{n+d_1}$-action on it. Denote by $(H_1, \ldots, H_s) : \hat{M}_1^{2n+2d_1} \to \mathbb{R}^s$ the momentum map of this $\mathbb{T}^s$-action, and by $X_1, \ldots, X_s$ the corresponding infinitesimal generators, with X_i being the Hamiltonian vector field of H_i. Since X_i lies in the kernel of ω_1 on M_1, it follows that H_i is constant on M_1 for every $i = 1, \ldots, s$, and without losing generality we can assume that $H_i = 0$ on M_1.

Since X_i are in the kernel of ω_1 on M_1, the results of the previous section say that the vector fields $X_1, \ldots, X_s$ are independent everywhere on M_1, i.e., the action of $\mathbb{T}^s$ on M_1 is locally free everywhere. It follows that the orbits of $\mathbb{T}^s$ on M_1 have dimension s everywhere, and hence coincide with the fibers of the submersion ϕ. The action of $\mathbb{T}^s$ is free almost everywhere (being a sub-action of a $\mathbb{T}^{n+d_1}$-action which is free almost everywhere), and its orbits form a locally trivial fibration, so the action must actually be free everywhere (without any discrete isotropy at any point).

Consider the Marsden-Weinstein reduction of $(\hat{M}_1^{2n+2d_1}, \omega_1)$ with respect to the $\mathbb{T}^s$-action at the zero level $\{H_1 = \cdots = H_s = 0\}$. One verifies easily that the reduced symplectic manifold is a symplectization of $(M_2^{2n+d_2}, \omega_2, \rho_2)$ and the image of the momentum map of the Hamiltonian $\mathbb{T}^{n+d_2}$-action on it coincides with $Q' = Q_1 \cap \{H_1 = 0\} \cap \cdots \cap \{H_s = 0\}$, which is a regular framing of P in an $(n + d_2)$-dimensional space. Due to the uniqueness of symplectization of $(M_2^{2n+d_2}, \omega_2, \rho_2)$ up to isomorphisms, we have that the framed polytope (P_2, Q_2) is isomorphic to (P, Q'), which means that there is a Morita equivalence embedding from (P_2, Q_2) to (P_1, Q_1).

The converse statement can be proved in a similar way. □

Theorem 3.15. *The Morita equivalence of compact presymplectic toric manifolds is a true equivalence relation, and two compact presymplectic toric manifolds are Morita-equivalent if and only if their corresponding framed momentum polytopes are Morita-equivalent.*

Proof. It is a direct consequence of Theorem 3.13 and Theorem 3.14. □

Example 3.16. The two intervals $P_1 = [A, B]$ and $P_2 = [G, H]$ in Figure 1 are isomorphic. Their framings, also shown in Figure 1 are not isomorphic, but are Morita-equivalent. The framed P_2 corresponds to the presymplectic manifold S^3 with the Hopf circle fibration as the kernel isotropic foliation, while the framed P_1 corresponds to the presymplectic manifold $S^2 \times S^1$ whose kernel isotropic foliation is the projection to S^2. Both have S^2 as the quotient space. By taking a direct product of the framing of P_2 with an interval, one can easily realize a 3-dimensional framing of P_2 which is Morita-equivalent to both the framed P_1 and the framed P_2 via integral affine embeddings. On the other hand, Figure 2 shows an example of non-Morita-equivalent framings of an interval: one of them corresponds to a symplectic sphere while the other one corresponds to a symplectic orbifold (see the next subsection).

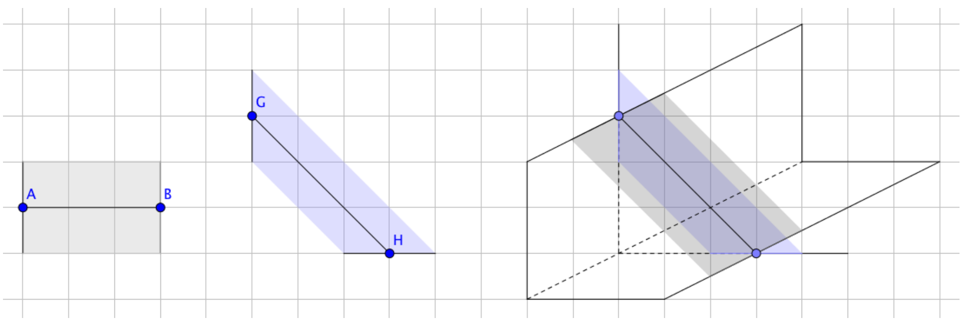

Figure 1: Non-isomorphic but Morita-equivalent framed intervals.

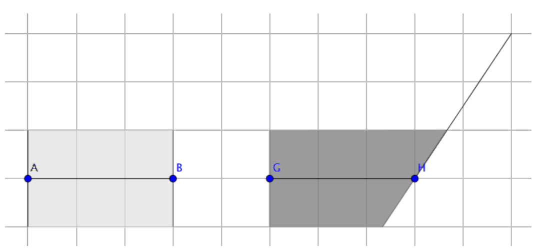

Figure 2: Non-Morita-equivalent framings of an interval.

3.4. Toric orbifolds and quasifolds

Theorem 3.15 reduces the problem of classification of presymplectic toric manifolds, up to Morita equivalence, to the combinatorial problem of classification of framed polytopes up to Morita equivalence. In particular, if two presymplectic toric manifolds are Morita equivalent, then their momentum polytopes must be integral-affinely isomorphic, though the same polytope (without framing) can correspond to infinitely many non-Morita-equivalent presymplectic toric manifolds.

It is clear that the following three conditions are equivalent:

(i) the quotient of a presymplectic toric manifold (M^{2n+k}, ω, ρ) by the kernel isotropic foliation is Hausdorff;

(ii) all the leaves of the foliation are closed;

(iii) the momentum polytope is rational.

In the rational case, the leaves of the kernel isotropic foliation are k-dimensional tori (so we get a higher-dimensional analogue of Seifert fibrations), and the quotient space $(M^{2n+k}/\text{kernel}, \omega/\text{kernel}, \rho/\text{kernel})$ is a $2n$-dimensional symplectic toric orbifold.

Compact symplectic toric orbifolds have been classified (up to equivariant symplectomorphisms) by Lerman and Tolman [16]: they put on each facet of the momentum polytope a positive integer m, which corresponds to the orbifold type $D^{2(n-1)} \times (D^2/\mathbb{Z}_m)$ of points whose image under the momentum map lies in the interior of the facet.

A rational convex polytope together with one positive integer for each facet is called a ***weighted rational convex polytope***. Lerman and Tolman [16] proved that connected compact symplectic toric orbifolds are classified by their weighted rational convex polytopes (up to natural isomorphisms), and any weighted rational convex polytope can be realized by a compact symplectic toric orbifold.

We can recover the above-mentioned result of Lerman and Tolman from our language of Morita-equivalent framed momentum polytopes as follows.

Let (P, Q) be a rational simple polytope with a regular framing, Q is of dimension $n+k$ and sits in $\mathbb{R}^{n+d}$, $P = L \cap Q$, where L is a rational n-dimensional affine subspace of $\mathbb{R}^{n+d}$ which intersects Q transversally. Let ζ_P be a facet of P and ζ_Q be the corresponding facet of Q, $\zeta_P \subset \zeta_Q$. Fix a point $x \in \zeta_P$ and a basis $\alpha_1, \ldots, \alpha_{n+d}$ of the integral lattice of T_xQ. This basis can be chosen so that $\alpha_1, \ldots, \alpha_{n-1} \in T_x\zeta_P$ and $\alpha_1, \ldots, \alpha_{n+d-1} \in T_x\zeta_Q$. The vector α_n does not belong to T_xP, in general, but there exists a linear combination $\beta = \sum_{i=1}^{n+d} c_i\alpha_i$ with integer coefficients c_i such that $\beta \in T_x\zeta_P$ and $c_{n+d} > 0$. The minimal positive number $c_{n+d} > 0$ for which such an integral linear combination $\beta = \sum_{i=1}^{n+d} c_i\alpha_i \in T_x\zeta_P$ exists will be called the ***weight*** of the facet ζ_P in the framing (P,Q). It is easy to see that this number does not depend on the choice of the basis $(\alpha_1,\ldots,\alpha_{n+d})$. So each facet has a weight which is a positive integer, which depends only on the framed polytope.

For any choice of weights for the facets of a given rational simple polytope, there always exists a regular framing with those weights. Indeed, in the construction of the cubic framing Q given in Subsection 3.2, each facet of Q, which corresponds to a facet ζ_i of P, is given by an equation of the type

$$y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i,$$

(where a_{ij} are integers because P is rational), and it is easy to check that the weight of this facet is nothing else but the greatest common divisor of the numbers $a_{i1},\ldots,a_{in}$. By multiplying all the coefficients $a_{i1},\ldots,a_{in}$ and b_i by p/q, where q is the greatest common divisor of $a_{i1},\ldots,a_{in}$, and p is any new weight that we want to have, we can change Q to a new regular frame (while leaving P unchanged), such that the weight of the facet ζ_i is changed from q to p.

If a regular framed polytope (P,Q_1) admits a Morita equivalence embedding into a regular framed polytope (P,Q_2), and x is a point in a facet ζ_P of P, then a basis $(\alpha_1,\ldots,\alpha_{n+d})$ of the integral lattice of T_xQ_1 with the above properties can be completed to a basis of the integral lattice of T_xQ_2 with similar properties. This implies that the weight of ζ_P given by the framing (P,Q_1) is the same as its weight given by the framing (P,Q_2). Hence facet weights are invariants with respect to Morita equivalence transformations of regular framed polytopes.

Let us now show the converse: if two regular framings Q_1 and Q_2 of a rational simple convex polytope P give rise to the same weight for each facet of P, then (P,Q_1) and (P,Q_2) are Morita equivalent. In order to show it, we need to construct a third regular framing (P,Q_3) with Morita equivalence embeddings from (P,Q_1) and (P,Q_2) to (P,Q_3). Q_3 can be constructed as the crossed product of Q_1 and Q_2 relative to P_3 in a way which is absolutely similar to the construction in the proof of Theorem 3.13. One then verifies directly that Q_3 is regular, also in a similar way to the proof of Theorem 3.13. So we obtain the following result, which incorporates the classification theorem of Lerman and Tolman [16].

Theorem 3.17. *Consider two connected compact presymplectic toric manifolds with rational momentum polytopes. The following conditions are equivalent:*

(i) *they are Morita equivalent;*

(ii) *their quotients by the kernel isotropic foliations are isomorphic as symplectic toric orbifolds;*

(iii) *their momentum polytopes are isomorphic and, moreover, have the same facet weights given by the respective regular framings.*

For irrational polytopes, we do not have orbifolds but quasifolds in the sense of Prato [23] and Battaglia–Prato [3] (after a lifting and framing). In this case, our Morita equivalence for framed polytopes and presymplectic toric manifolds can be understood as a natural isomorphism relation among symplectic toric quasifolds.

4. Some final remarks

Remark 4.1. In this paper we considered only simple polytopes, but in fact any non-simple convex polytope P also admits a lifting and rational framing (P', Q) by the same constructions. Q still satisfies the rationality, simplicity and regularity conditions at its faces, P' is still a slice of Q by an affine subspace L. The only difference is that if P' is not simple then L intersects Q *non-transversally.* We still have a symplectic $2(n+d)$-dimensional manifold (M^{2n+2d}, ω) with a Hamiltonian $\mathbb{T}^{n+d}$-action ρ on it with a momentum map F such that $F(M) = Q$, and can still take $M_P = F^{-1}(P')$ to be the $(2n+d)$-dimensional presymplectic toric variety corresponding to the framed polytope (P', Q). When P is not simple then this presymplectic toric variety is singular (not a manifold) but still has very reasonable topology and geometry. When P is rational non-simple then P' is isomorphic to P, we can talk about a framing (P, Q) of P, take the quotient of the singular presymplectic toric variety $M_P = F^{-1}(P')$ by the kernel isotropy foliation to get a *singular symplectic toric variety* corresponding to (P, Q) (with algebraic singularities). Some results about such singular symplectic toric varieties can be found, for example, in [6].

Remark 4.2. In [16], Lerman and Tolman extended the convexity theorem of Atiyah–Guillemin–Sternberg to the case of Hamiltonian torus actions on symplectic orbifolds. In this paper, we didn't study presymplectic orbifolds, but we are pretty sure that the presymplectic convexity theorem (Theorem 2.4) can be naturally extended to the case of presymplectic orbifolds, with essentially the same arguments for the proof.

Remark 4.3. One can extend in a natural way the theory of *symplectic cuts* [15] to the presymplectic setting, and to presymplectic toric manifolds in particular. The corresponding operations on the level of framed momentum polytopes will also be cuts by rational hyperplanes. Some results concerning cuts for irrational polytopes and associated quasifolds were obtained recently by Battaglia and Prato [4]. We recall again that their non-rational polytopes need to be lifted (non-isomorphically) before they can be framed and then cut.

Remark 4.4. In [2, 5], Battaglia and Zaffran also worked on foliation and quotient modelings of irrational analogs of toric varieties. Their approach is complex-analytic, based on ideas from complex geometric invariant theory and earlier results of Meersseman and Verjovsky [20] and others; it is quite different from our real presymplectic approach. In the case of rational polytopes, different approaches should give basically the same results.

Remark 4.5. In this paper we didn't talk about Kähler structures at all, but one can put compatible Kähler structures on symplectizations of presymplectic toric manifolds (these symplectizations are "semi-local" versions of symplectic toric manifolds), and use reduction (with respect to kernel torus actions) to get Kähler structures on quotient spaces, which are toric orbifolds or quasifolds or "non-commutative toric varieties" in the language of [14].

Remark 4.6. There is a theory of so-called *moment-angle manifolds*, developed by Panov and many other authors (see, e.g., [22]), which is closely related to our notion of presymplectic toric manifolds satisfying the flatness condition. Indeed, even though the authors studying moment-angle manifolds were mainly interested in other things like complex structures and cohomological rings and didn't care much about the presymplectic structures, the definition of moment-angle manifolds by itself shows that these manifolds are a special case of flat presymplectic toric manifolds (without the specified presymplectic structure). One may hope that the two theories can be combined together, and in particular the notion of Morita equivalence may prove useful for the study of moment-angle manifolds.

Remark 4.7. After our preprint was posted on arXiv, we were informed by Reyer Sjamaar that he and Y. Lin also studied convexity properties of presymplectic manifolds in a preprint [17] which appeared on arXiv a few weeks after ours. Their work is complementary to our work, though there are some overlaps. They studied more general compact group actions while we restricted our attention to torus actions; on the other hand, we studied Morita equivalence of framed polytopes and presymplectic toric manifolds, which they did not.

Remark 4.8. We have the following *global symplectization conjecture*, which is the global version of Theorem 2.7, and which looks very reasonable to us: *With the assumptions of Theorem 2.7 there exists a connected compact symplectic manifold $\hat{M}$ with an effective Hamiltonian $\mathbb{T}^{q+d}$-action such that M is a flat presymplectic cut of $\hat{M}$ with this action.* If one weakens this conjecture and requires $\hat{M}$ to be an orbifold instead of a manifold, then it becomes rather easy.

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School of Mathematical Sciences and Ministry of
Education Laboratory of Scientific Computing (MOE-LSC)
Shanghai Jiao Tong University
800 Dongchuan Road, Minhang District, Shanghai, 200240 China
Section de Mathématiques, Université de Genève
2-4 rue du Lièvre, Case postale 64, 1211 Genève 4
and Ecole Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland
E-mail address: `ratiu@sjtu.edu.cn, tudor.ratiu@epfl.ch`

School of Mathematics, Shanghai Jiao Tong University
800 Dongchuan Road, Minhang District, Shanghai, 200240 China
and Institut de Mathématiques de Toulouse, UMR5219
Université Paul Sabatier, 118 route de Narbonne
31062 Toulouse, France
E-mail address: `tienzung@math.univ-toulouse.fr`

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