JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 17, Number 6, 1773–1792, 2019

On L_2-cohomology of almost Hermitian manifolds

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We prove two results regarding the L_2 cohomology of almost-complex manifolds. First we show that there exist complete, d-bounded almost Kähler manifolds of any complex dimension $n \geq 2$ such that the space of harmonic 1-forms in L_2 has infinite dimension. By contrast a theorem of Gromov [6] states that a complete d-bounded Kähler manifold X has no nontrivial harmonic forms of degree different from $n = \dim_{\mathbb{C}} X$. Second let (X, J, g) be a complete almost Hermitian manifold of dimension four. We prove that the reduced L_2 2^{nd}-cohomology group decomposes as direct sum of the closure of the invariant and anti-invariant L_2-cohomology. This generalizes a decomposition theorem by Drăghici, Li and Zhang [4] for 4-dimensional closed almost complex manifolds to the L_2-setting.

1. Introduction

Cohomological properties of closed complex manifolds have recently been studied by many authors, focusing on their relations with other special structures (see e.g. [1, 4, 9] and the references therein). The aim of this paper is to study cohomological properties of non compact almost complex manifolds. In this context, L_2-cohomology provides a useful tool to study the relationship between such properties and the existence of further structures, e.g., Kähler, almost Kähler structures.

In [6] Gromov developed L_2-Hodge theory for complete Riemannian manifolds, respectively Kähler manifolds, proving an L_2-Hodge decomposition Theorem for L_2-forms.

The first author is partially supported by Simons Foundation grant # 317510. The second author is partially supported by the Project PRIN "Varietà reali e complesse: geometria, topologia e analisi armonica", Project PRIN 2017 "Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics" and by GN-SAGA of INdAM

As a consequence, for a complete and d-bounded Kähler manifold X, denoting by $\mathcal{H}_2^k$, respectively $\mathcal{H}_2^{p,q}$, the space of Δ-harmonic L_2-forms of degree k, respectively $\Delta_{\overline{\partial}}$-harmonic L_2-forms of bi-degree (p,q), he showed that $\mathcal{H}_2^k \simeq \oplus_{p+q=k}\mathcal{H}_2^{p,q}$; furthermore, denoting by $m = \dim_{\mathbb{C}} X$, that $\mathcal{H}_2^k = \{0\}$, for all $k \neq m$ and hence $\mathcal{H}_2^{p,q} = \{0\}$, for all (p,q) such that $p+q \neq m$. A key ingredient in the proof is the Hard Lefschetz Theorem.

In the present paper we show that such a conclusion no longer holds in the category of non compact almost Kähler manifolds. Indeed, by using methods of contact geometry, starting with a contact manifold (M,α) having an exact symplectic filling (see Definition 3.1), we construct a d-bounded complete almost Kähler manifold Y satisfying $L_2H^1(Y) \neq \{0\}$. More precisely, we prove the following main result (see Theorem 3.9).

Theorem. *There exist d-bounded, complete almost Kähler manifolds Y of every real dimension $2n \geq 4$ with $L_2H^1(Y)$ infinite dimensional.*

Next we focus on L_2-cohomology of almost complex 4-dimensional manifolds. In the closed case a theorem of Drăghici, Li and Zhang [4] states that the 2^{nd}-de Rahm cohomology group decomposes as the direct sum of J-invariant and J-anti-invariant cohomology subgroups, which can be viewed as a sort of L_2-Hodge decomposition theorem for almost complex manifolds. We generalize this to L_2 cohomology defined with respect to a complete Hermitian metric, see Theorem 4.8.

Theorem. *Let (X,J,g) be a complete almost Hermitian 4-dimensional manifold. Then,*

$$L_2H^2(X;\mathbb{R}) = \overline{L_2H^+(X)} \oplus \overline{L_2H^-(X)}.$$

The paper is organized as follows: in Section 2 we recall some generalities regarding L_2-cohomology. Section 3 is devoted to the proof of non vanishing of the first L_2-cohomology group. In Section 4 we prove the decomposition Theorem 4.8 and also give cohomological obstructions for an almost complex structure to admit a compatible complete symplectic form.

Finally, we would like mention an open question. An almost-complex manifold of dimension at least 6 may have a taming symplectic form but not a compatible symplectic form (see e.g., [8]). For closed 4-dimensional manifolds however, there are no local obstructions and Donaldson in [2] raised the following question:

Donaldson's Question([2])*If J is an almost complex structure on a compact 4-manifold which is tamed by a symplectic form, is there a symplectic form compatible with J?*

Moving to the complex case, it is still unknown whether a closed complex manifold X of dimension at least 6 with a taming symplectic form also has a compatible symplectic form, in other words, whether it is Kähler. Such a question has a positive answer by Li and Zhang for complex surfaces [9, Theorem 1.2]. Here is an analogue of the question for open manifolds.

Question. *Let (X, J) be a complex $2n$-dimensional manifold. Suppose there exists a d-bounded symplectic form ω taming J such that $g(\cdot,\cdot) = \frac{1}{2}(\omega(\cdot, J\cdot) - \omega(J\cdot,\cdot))$ is complete. Does (X, J) admit a complete d-bounded Kähler structure whose corresponding metric is uniformly comparable to g?*

Our construction in section 3 gives d-bounded complete almost complex manifolds Y which admit a compatible symplectic form and corresponding complete metric satisfying $L_2H^1(Y;\mathbb{R}) \neq \{0\}$. If our construction could be upgraded to give examples of (integrable) complex manifolds which still admit a taming symplectic form with corresponding complete metric and satisfy $L_2H^1(Y;\mathbb{R}) \neq \{0\}$ then by Gromov's theorem there could not be a compatible Kähler structure with comparable metric, thus implying a negative answer.

Acknowledgments. We thank Weiyi Zhang for useful comments. We are also pleased to thank the referee for fruitful suggestions and remarks leading to a better presentation of the results. The second author would like to thank the Math Departments of Stanford and Notre Dame Universities for their warm hospitality.

2. Preliminaries

We start by recalling some notions about L_2-cohomology. Let (X, g) be a Riemannian manifold and denote by $\Omega^k(X)$ the space of smooth k-forms on X. Then $\alpha \in \Omega^k(X)$ is said to be *bounded* if the L_∞-norm of α is finite, namely,

$$\|\alpha\|_{L_\infty(X)} = \sup_{x\in X} |\alpha(x)| < +\infty$$

where $|\alpha(x)|$ denotes the pointwise norm induced by the metric g on the space of forms. By definition, a (smooth) k-form α is said to be *d-bounded* if $\alpha = d\beta$, where β is a bounded $(k-1)$-form. Furthermore, a k-form α is said to be L_2, namely $\alpha \in L_2\Omega^k(X)$ if

$$\|\alpha\|_{L_2(X)} := \left(\int_X |\alpha(x)|^2\, dx\right)^{\frac{1}{2}} < +\infty,$$

that is the pointwise norm $|\alpha|^2$ is integrable. Denote by $(L_2A^\bullet(X), d)$ the sub-complex of $(\Omega^\bullet(X), d)$ formed by differential forms α such that both α and $d\alpha$ are in L_2. Then the *reduced* L_2*-cohomology group* of *degree* k of X is defined as

$$L_2H^k(X;\mathbb{R}) = L_2A^k(X) \cap \ker d \Big/ \overline{dL_2(A^{k-1}(X))}.$$

We recall the following (see [6, Lemma 1.1.A])

Lemma 2.1. *Let* (X, g) *be a complete Riemannian manifold of dimension* n *and let* $\alpha \in L_1\Omega^{n-1}(X)$*, that is*

$$\int_X |\alpha(x)|\, dx < +\infty.$$

Assume that also $d\alpha \in L_1\Omega^n(X)$*. Then*

$$\int_X d\alpha = 0.$$

Let $\Delta = d\delta + \delta d$ denote the Hodge Laplacian and set

$$\mathcal{H}_2^k = \{\alpha \in L_2\Omega^k(X) \ \mid \ \Delta\alpha = 0\}$$

namely, $\mathcal{H}_2^k$ is the space of harmonic L_2-forms on (X, g) of degree k. Then, under the assumption that (X, g) is complete, Gromov proved the following Hodge decomposition for L_2-forms (see [6]), namely,

$$L_2\Omega^k(X) = \mathcal{H}_2^k \oplus \overline{d(L_2\Omega^{k-1}(X))} \oplus \overline{\delta(L_2\Omega^{k+1}(X))}, \tag{1}$$

where $\overline{d(L_2\Omega^{k-1}(X))}$ means the closure in $L_2\Omega^k(X)$ of

$$L_2\Omega^k(X) \cap d(L_2\Omega^{k-1}(X))$$

and similarly for $\overline{\delta(L_2(\Omega^{k+1}(X))}$. Given any $\alpha, \beta \in \Omega^k(X)$, we set

$$\langle \alpha, \beta \rangle = \int_X g(\alpha, \beta) dx.$$

We have the following

Lemma 2.2. *Let (X, g) be a complete Riemannian manifold and let $\alpha \in L_2\Omega^k(X)$. Denote by*

$$\alpha = \alpha_H + \lambda + \mu$$

the Hodge decomposition of α, where

$$\alpha_H \in \mathcal{H}_2^k, \quad \lambda \in \overline{d(L_2\Omega^{k-1}(X))}, \quad \mu \in \overline{\delta(L_2\Omega^{k+1}(X))}.$$

Then

i) $d\lambda = 0$.

ii) *If $d\alpha = 0$, then $\mu = 0$.*

Proof. i) Let λ be a smooth k-form. First of all note that $d\lambda = 0$ if and only if for every compactly supported $(k+1)$-form φ we have $\langle \lambda, \delta\varphi \rangle = 0$.
Let $\{d\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of d-exact forms in $L_2\Omega^k(X)$ such that $\lambda_j \in L_2\Omega^{k-1}(X)$ for every $j \in \mathbb{N}$ and $d\lambda_j \to \lambda$ in L_2. Then

$$\langle \lambda, \delta\varphi \rangle = \lim_{j\to\infty} \langle d\lambda_j, \delta\varphi \rangle = 0,$$

since δ is the adjoint of d.
ii) Let $\{\delta\mu_j\}_{j\in\mathbb{N}}$ be a sequence of δ-exact forms in $L_2\Omega^k(X)$ such that $\mu_j \in L_2\Omega^{k+1}(X)$ for every $j \in \mathbb{N}$ and $\delta\mu_j \to \mu$ in L_2. Then, by Lemma 2.1, we obtain

$$\langle \alpha, \delta\mu_j \rangle = \langle d\alpha, \mu_j \rangle = 0.$$

Now

$$\begin{aligned} |\langle \alpha, \mu \rangle - \langle \alpha, \delta\mu_j \rangle| &= \left| \int_X g(\alpha, \mu - \delta\mu_j) dx \right| \le \int_X |\alpha|\, |\mu - \delta\mu_j| dx \\ &\le \|\alpha\|_{L_2(X)} \|\mu - \delta\mu_j\|_{L_2(X)}. \end{aligned}$$

Hence, the sequence $\{\langle \alpha, \delta\mu_j \rangle\}_{j\in\mathbb{N}}$ converges to $\langle \alpha, \mu \rangle$. and consequenltly, $\langle \alpha, \mu \rangle = 0$. Therefore, by the L_2-orthogonality of the L_2-Hodge decomposition, it follows that $\mu = 0$. □

3. L_2-cohomology and contact structures

Let now (X, J) be a complex manifold and g be a Hermitian metric. Then according to Gromov [6, 1.2.B], if X is a complete n-dimensional Kähler manifold whose Kähler form ω is d-bounded, then $\mathcal{H}_2^k = \{0\}$, unless $k = \frac{n}{2}$.

In this section we will see that the same conclusion does not hold in the category of almost Kähler manifolds.

To begin, let M be a $(2n-1)$-dimensional compact contact manifold, $n > 1$ and denote by α a contact form. Let $\xi = \ker\alpha$ be the contact distribution and R be the Reeb vector field.
On the product manifold $X = M \times (3, +\infty)$, with t the coordinate on $(3, \infty)$, let $\rho = \rho(t)$ be a positive smooth function, such that $\rho' > 0$ and let $\omega_\rho = d(\rho\alpha)$. Then ω_ρ is a symplectic form on X.

Definition 3.1. We say that a contact manifold M with contact form α has an *exact symplectic filling* if there exists a compact exact symplectic manifold $(W, \omega = d\lambda)$ with $\partial W = M$ and $\lambda|_M = \alpha$. Furthermore we require the Liouville field ζ defined by $\zeta \rfloor \omega = \lambda$ to be outward pointing along M.

We remark that if a particular contact form on M has an exact symplectic filling then so do all other contact forms which generate the same contact structure, that is, all α' such that $\ker\alpha' = \ker\alpha$.

A version of Darboux' Theorem implies that a tubular neighborhood of $M = \partial W$ in W can be identified symplectically with $(M \times (-\delta, 0], d(e^t\alpha))$, and we may choose a primitive on W equal to $e^t\alpha$ in this neighborhood.

Proposition 3.2. *Suppose that (M, α) has an exact symplectic filling and $\rho(3) > 1$. Then there exists an exact symplectic manifold $(Y, \omega = d\beta)$ such that the complement of a compact set may be identified with $X = M \times (3, +\infty)$ via a diffeomorphism pulling back $\rho\alpha$ to β.*

Proof. We set $Y = W \cup (M \times (-\delta, \infty))$ where we identify $M \times (-\delta, 0]$ with a tubular neighborhood of $M = \partial W$ as above. Then define $\beta|_W = \lambda$ and $\beta|_{M\times(0,\infty)} = \rho(t)\alpha$ where ρ is extended to $(-\delta, +\infty)$ such that $\rho = e^t$ for t close to 0 and $\rho' > 0$ for all $t > 0$. □

Remark 3.3. We note that if $\rho\alpha$ is bounded (with respect to a choice of metric) on $M \times [3, +\infty) \subset Y$ then it is globally bounded; any compatible almost complex structure on $M \times [3, +\infty)$ extends to a compatible almost complex structure on Y; any exact 1-form γ on $M \times [3, +\infty)$ extends to an exact 1-form on Y, and if γ lies in L_2 then so does its extension.

Suppose the contact form has a closed Reeb orbit and we can choose coordinates $(x_i, y_i, z) \in \mathbb{R}^{2(n-1)} \times \mathbb{R}/\mathbb{Z}$ in a tubular neighborhood of the orbit such that the contact form is given by $\alpha = dz + \frac{1}{2}(\sum_{i=1}^{n-1} x_i dy_i - y_i dx_i)$.

Hence the Reeb vector field $R = \frac{\partial}{\partial z}$. Set

$$r^2 = \sum_{i=1}^{n-1} x_i^2 + y_i^2, \quad U = \frac{\partial}{\partial r^2} = 2\sum_{i=1}^{n-1}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right),$$
$$V = \sum_{i=1}^{n-1}\left(-y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}\right) + \frac{1}{2}r^2\frac{\partial}{\partial z}$$

Observe that the vector fields $U, V \in \xi = \ker\alpha$. Then define $\xi' \subset \xi$ by

$$\xi' = \langle U, V\rangle^{\perp\, d\alpha},$$

the symplectic orthogonal complement of the span of U and V in ξ with respect to $d\alpha$. Accordingly, we have a direct sum decomposition

$$TX = \langle R\rangle \oplus \left\langle \frac{d}{dt}\right\rangle \oplus \langle U\rangle \oplus \langle V\rangle \oplus \xi'. \tag{2}$$

We will call an almost complex structure J on X *adapted* to the local coordinates, if there exist smooth functions $f, \varepsilon : (3, +\infty) \to (0, +\infty)$ such that

$$J\frac{d}{dt} = fR, \qquad JR = -\frac{1}{f}\frac{d}{dt}, \qquad J(\xi) = \xi$$

on TX. Further, on a subset $A < r^2 < B$, we have in addition

$$JU = \frac{1}{\varepsilon}V, \qquad JV = -\varepsilon U, \qquad J\xi' = \xi'.$$

We will denote by $g_{J,\rho}$ the Riemannian metric associated with (ω_ρ, J), that is $g_{J,\rho}(\cdot,\cdot) = \omega_\rho(\cdot, J\cdot)$. Note that the direct sum (2) is an orthogonal decomposition with respect to $g_{J,\rho}$.

Theorem 3.4. *Let $\rho(t) = \log t$, $\varepsilon(t) = \rho(t)t^{1-n}$ and $f(t) = \frac{1}{t\log^2 t}$. Then (X, ω_ρ, J) is an almost Kähler manifold, and if (M, α) has an exact symplectic filling then the structure extends to Y. Further:*

i) *ω_ρ is d-bounded.*

ii) *$(Y, \omega_\rho, J, g_{J,\rho})$ is complete.*

iii) *Let $b = b(r^2)$ be a smooth function satisfying $b' = 0$ if $r^2 \notin (A, B)$. Set $\gamma = db$. Then $\gamma \in L_2\Omega^1(X)$. Moreover, $\gamma \in \overline{dL_2(\mathcal{C}^\infty(X))}$ only if $\gamma = 0$. The 1-form γ extends to a 1-form on Y and the conclusions hold for the extension.*

For the proof of Theorem 3.4, we will need the following general

Lemma 3.5. *Let* (Z, g) *be a Riemannian manifold and let* $\gamma \in L_2\Omega^k(Z)$, $\gamma \neq 0$. *Let* $\{d\varphi_j\}_{j\in\mathbb{N}}$ *be a sequence in* $d(L_2\Omega^{k-1}(Z)) \cap L_2\Omega^k(Z)$ *such that* $d\varphi_j \to \gamma$ *in* L_2. *Then, for all* $j >> 1$,

$$\|\varphi_j\|_{L_2(Z)} \geq C(\gamma)\,,$$

for a suitable positive constant $C(\gamma)$.

Proof. Since $\gamma \neq 0$ there exists a bump function a such that $\langle\gamma, a\gamma\rangle > 0$. We have:

$$\begin{aligned}\langle\gamma, a\gamma\rangle - \langle d\varphi_j, a\gamma\rangle = \langle\gamma - d\varphi_j, a\gamma\rangle &= \int_Z g(\gamma - d\varphi_j, a\gamma)dx \\ &\leq \int_Z |\gamma - d\varphi_j||a\gamma|dx \leq \|\gamma - d\varphi_j\|_{L_2(Z)}\,\|a\gamma\|_{L_2(Z)}.\end{aligned}$$

Set

$$C_j = \|\gamma - d\varphi_j\|_{L_2(Z)}\,\|a\gamma\|_{L_2(Z)}.$$

Note that $C_j \to 0$ for $j \to +\infty$. We obtain

$$\langle\gamma, a\gamma\rangle - C_j \leq \langle d\varphi_j, a\gamma\rangle = \langle\varphi_j, \delta(a\gamma)\rangle \leq \|\varphi_j\|_{L_2(Z)}\,\|\delta(a\gamma)\|_{L_2(Z)}\,.$$

For j large the left hand side is positive, hence $\|\delta(a\gamma)\|_{L_2(Z)} > 0$ and therefore setting

$$C(\gamma) = \frac{\langle\gamma, a\gamma\rangle}{2\|\delta(a\gamma)\|_{L_2(Z)}}$$

we get

$$\|\varphi_j\|_{L_2(Z)} \geq C(\gamma) > 0.$$

□

We give now the proof of Theorem 3.4

Proof of Theorem 3.4. By Remark 3.3 it suffices to work on X. By construction, J is an almost complex structure on X which is compatible with ω_ρ. Therefore, $g_{J,\rho}(\cdot,\cdot) = \omega_\rho(\cdot, J\cdot)$ is a Riemannian metric on X and

$(X, \omega_\rho, J, g_{J,\rho})$ is an almost Kähler manifold. Then

$$\omega_\rho^n = 2\rho'\rho^{n-1} dt \wedge \alpha \wedge (d\alpha)^{n-1}$$

is a volume form on X and $\mathrm{Vol}_M = 2\alpha \wedge (d\alpha)^{n-1}$ is a volume form on the compact contact manifold M, so that

$$\omega_\rho^n = \rho'\rho^{n-1} dt \wedge \mathrm{Vol}_M\,.$$

i) By assumption, $\omega = d(\rho\alpha) = d\lambda$, where $\lambda = \rho\alpha$; by definition ω_ρ is d-bounded if $\lambda \in L_\infty(X)$. Recalling that $JR = -\frac{1}{f}\frac{d}{dt}$, we have,

$$|R|^2 = \omega_\rho(R, JR) = -\omega_\rho\left(R, \frac{1}{f}\frac{d}{dt}\right) = \frac{\rho'}{f}.$$

Since J preserves the contact distribution ξ, we see that α is dual to $\frac{f}{\rho'}R$ with respect to $g_{J,\rho}$. Therefore $|\lambda|^2 = \rho^2\frac{f}{\rho'}$. Hence $\lambda \in L_\infty(X)$ if and only if

$$f \le C\frac{\rho'}{\rho^2}, \tag{3}$$

where C is a positive constant. By our assumptions,

$$\rho = \log t, \quad f = \frac{1}{t\log^2 t},$$

so that (3) is satisfied.

ii) In order to check completeness of (Y, ω_ρ, J) it is enough to estimate $\int_3^{+\infty} |\frac{d}{dt}| dt$. We obtain:

$$\left|\frac{d}{dt}\right|^2 = \omega_\rho\left(\frac{d}{dt}, J\frac{d}{dt}\right) = \omega_\rho\left(\frac{d}{dt}, fR\right) = f\rho'.$$

Therefore,

$$\int_3^{+\infty}\left|\frac{d}{dt}\right| = \int_3^{+\infty}\sqrt{\rho' f}\,dt = +\infty\,,$$

that is $(Y, \omega_\rho, J_{\varepsilon,f}, g_{\varepsilon,f,\rho})$ is complete.

iii) First of all we check that $\gamma \in L_2\Omega^1(X)$. We have the pointwise estimate valid on the support of γ:

$$|\gamma|^2 = b'^2|dr^2|^2 = b'^2\frac{(dr^2(U))^2}{|U|^2},$$

since dr^2 vanishes on the orthogonal complement of $\langle U\rangle$ in the decomposition (2), indeed, $dr^2 = -2V\lfloor d\alpha$. Therefore,

$$|\gamma|^2 = b'^2\frac{(dr^2(U))^2}{|U|^2} \leq \|b'\|^2_{L_\infty}\frac{\varepsilon}{2\rho A},$$

since

$$|U|^2 = \omega(U, JU) = \frac{1}{\epsilon}\omega(U, V) = \frac{2\rho r^2}{\epsilon}.$$

Therefore, as $\varepsilon = \rho t^{1-n}$, for suitable constants c_1, c_2 we get:

$$\begin{aligned}\|\gamma\|^2_{L_2(X)} \leq c_1\int_X \frac{\varepsilon}{\rho}\omega^n_\rho &= c_1\int_X t^{1-n}\rho^{n-1}\rho' dt\wedge \mathrm{Vol}_M\\ &= c_2\int_3^{+\infty}\frac{\log^{n-1} t}{t^n}dt < +\infty.\end{aligned}$$

Let $\gamma \neq 0$. We show that $\gamma \notin \overline{dL_2(\mathcal{C}^\infty(X))}$.
By contradiction: assume that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ in $L_2(\mathcal{C}^\infty(X))$ such that $d\varphi_j \in L^2\Omega^1(X)$ for every $j \in \mathbb{N}$ and $d\varphi_j \to \gamma$ in $L_2\Omega^1(X)$. Set $\gamma_j = d\varphi_j$. We also write

$$f_j(t) := \int_{M\times\{t\}}(\varphi_j)^2\,\mathrm{Vol}_M\,.$$

Then

$$\begin{aligned}\|\varphi_j\|^2_{L_2(M\times[a,b])} &= \int_a^b\frac{(\log t)^{n-1}}{t}dt\int_{M\times\{t\}}(\varphi_j)^2\,\mathrm{Vol}_M\\ &= \int_a^b\frac{(\log t)^{n-1}}{t}f_j(t)dt.\end{aligned}$$

We will show that $f_j(t)$ is bounded away from 0 for large j, contradicting the assumption that $\varphi_j \in L_2(\mathcal{C}^\infty(X))$. First, for the pointwise norm of γ, since $\gamma(\frac{d}{dt}) = 0$, we have the estimate:

$$\begin{aligned} (4) \qquad \left|\gamma_j\left(\frac{d}{dt}\right)\right| &= \left|(\gamma_j - \gamma)\left(\frac{d}{dt}\right)\right| \\ &\leq |\gamma - \gamma_j| \left|\frac{d}{dt}\right| = |\gamma - \gamma_j|\sqrt{f\rho'} = \frac{1}{t\log t}|\gamma - \gamma_j|. \end{aligned}$$

Now

$$f_j'(t) = \int_{M\times\{t\}} 2\varphi_j d\varphi_j\left(\frac{d}{dt}\right)\mathrm{Vol}_M = \int_{M\times\{t\}} 2\varphi_j\gamma_j\left(\frac{d}{dt}\right)\mathrm{Vol}_M .$$

Therefore, by (4), setting

$$\psi_j(t) := 2\|\gamma - \gamma_j\|_{L_2(M\times\{t\})},$$

we obtain:

$$\begin{aligned} (5) \qquad |f_j'(t)| &\leq \int_{M\times\{t\}} 2|\varphi_j|\left|\gamma_j\left(\frac{d}{dt}\right)\right|\mathrm{Vol}_M \\ &\leq \frac{2}{t\log t}\|\varphi_j\|_{L_2(M\times\{t\})}\|\gamma - \gamma_j\|_{L_2(M\times\{t\})} \\ &= \frac{1}{t\log t}\sqrt{f_j(t)}\psi_j(t). \end{aligned}$$

From the last expression,

$$\begin{aligned} (6) \qquad \left[\sqrt{f_j(t)}\right]_a^b &\leq \frac{1}{2}\int_a^b \frac{1}{t\log t}\psi_j(t)dt \\ &\leq \frac{1}{2}\left(\int_a^b \frac{1}{t(\log t)^{n+1}}dt\right)^{\frac{1}{2}}\left(\int_a^b \frac{(\log t)^{n-1}}{t}\psi_j^2(t)dt\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{\left[-\frac{1}{n(\log t)^n}\right]_a^b}\left(\int_a^b \frac{(\log t)^{n-1}}{t}\psi_j^2(t)dt\right)^{\frac{1}{2}}. \end{aligned}$$

Now, by definition,

$$\begin{aligned} (7) \qquad \int_a^b \frac{(\log t)^{n-1}}{t}\psi_j^2(t)dt &= 4\int_a^b\left(\int_{M\times\{t\}}|\gamma - \gamma_j|^2\mathrm{Vol}_M\right)\frac{(\log t)^{n-1}}{t}dt \\ &\leq 4\|\gamma - \gamma_j\|^2_{L_2(X)}. \end{aligned}$$

In view of (6) and (7), we obtain

$$\sqrt{f_j(b)} - \sqrt{f_j(a)} \le 2\sqrt{\frac{1}{n(\log a)^n} - \frac{1}{n(\log b)^n}}\|\gamma - \gamma_j\|_{L_2(X)}. \tag{8}$$

By assumption, $d\varphi_j \to \gamma$ in $L_2(X)$, and consequently $d\varphi_j \to \gamma$ also in $L_2(M \times [3, 3+\delta])$. By Lemma 3.5, it follows that there exists a constant $C(\gamma, \delta)$ such that that

$$\|\varphi_j\|^2_{L_2(M\times[3,3+\delta])} \ge C(\gamma, \delta)\,,$$

for all $j >> 1$, where $C(\gamma, \delta) > 0$ is independent of j, that is

$$C(\gamma, \delta) \le \int_3^{3+\delta} \frac{(\log t)^{n-1}}{t} f_j(t)dt.$$

But

$$\int_3^{3+\delta} \frac{(\log t)^{n-1}}{t} f_j(t)dt \le \sup_{3\le t\le 3+\delta} |f_j(t)| \int_3^{3+\delta} \frac{(\log t)^{n-1}}{t} dt,$$

which implies

$$\sup_{3\le t\le 3+\delta} |f_j(t)| \ge \frac{nC(\gamma, \delta)}{(\log(3+\delta))^n - (\log 3)^n}.$$

Therefore, for all large j there exists a $t \in [3, 3+\epsilon]$ such that

$$f_j(t) \ge \frac{nC(\gamma, \delta)/2}{(\log(3+\delta))^n - (\log 3)^n}$$

and we note that the lower bound is independent of j. By (8), this implies that $f_j(t)$ is bounded below for large j and all t, since $\gamma_j \to \gamma$ in $L_2(X)$. This gives our contradiction as required. □

Corollary 3.6. *Let $(Y, \omega_\rho, J, g_{J,\rho})$ be an almost Kähler structure adapted to a contact form α as above. Then for γ, ε, f as in Theorem 3.4 we have*

$$L_2H^1(Y; \mathbb{R}) \ne \{0\}.$$

Proof. Let γ be a non-zero exact 1-form on M satisfying the hypothesis iii) of Theorem 3.4. Then the pull-back of γ to X extends to a 1-form on Y, still denoted by γ, such that $[\gamma] \ne 0$. □

Corollary 3.7. *Suppose $b_1(r^2), b_2(r^2), \ldots$ is an infinite family of linearly independent real valued smooth functions on M such that $b_1' = 0, b_2' = 0, \ldots$ for $r^2 \notin (A, B)$. Let γ_j be an extension to Y of the pull-back of $\gamma_j = db_j$ to X. Then $(Y, \omega_\rho, J, g_{J,\rho})$ is an almost Kähler structure and $L_2H^1(Y)$ is infinite dimensional.*

Proof. Any finite linear combination $\sum_{j=1}^{r} c_j\gamma_j$, for $c_i \in \mathbb{R}, j = 1, \ldots, r$ satisfies the assumptions of Corollary 3.6. Hence

$$\sum_{j=1}^{r} c_j[\gamma_j] \neq 0.$$

□

Corollary 3.8. *The almost complex structure J is not integrable.*

Proof. By [6, 1.2.B], complete d-bounded Kähler manifolds have $L_2H^1 = \{0\}$. □

We collect the previous results in the following

Theorem 3.9. *There exist d-bounded, complete almost Kähler manifolds Y of every dimension $2n \geq 4$ so that $L_2H^1(Y)$ is infinite dimensional.*

4. L_2-Decomposition for almost complex 4-manifolds

Let (X, J, g) be a 4-dimensional almost Hermitian manifold. Then J acts as an involution on the space of smooth 2-forms $\Omega^2(X)$: given $\alpha \in \Omega^2(X)$, for every pair of vector fields u, v on X

$$J\alpha(u, v) = \alpha(Ju, Jv)\,.$$

Therefore the bundle $\Lambda^2 X$ splits as the direct sum of ±1-eigenspaces $\Lambda_\pm$, i.e., $\Lambda^2 X = \Lambda_+ \oplus \Lambda_-$. We will refer to the sections of Λ_J^+, respectively Λ_J^- as to the *invariant* respectively *anti-invariant* forms, denoted by $\Omega^+(X)$, respectively $\Omega^-(X)$. Let us denote by $L_2\mathcal{Z}(X)$ the space of closed 2-forms which are in L_2 and set

$$L_2\mathcal{Z}^\pm = L_2\mathcal{Z}(X) \cap \Omega^\pm(X).$$

Define

$$L_2H^\pm(X) = \{\mathfrak{a} \in L_2H^2(X; \mathbb{R}) \mid \exists \alpha \in L_2\mathcal{Z}^\pm \text{ such that } \mathfrak{a} = [\alpha]\}.$$

We will assume that g is a complete J-Hermitian metric on X and we will denote by ω the corresponding fundamental form. Let $\Lambda_g^{\pm}$ be the ± 1-eigenbundle of the $*$ Hodge operator associated with g. Then, we have the following relations

$$\Lambda_J^+ = Span_{\mathbb{R}}\langle\omega\rangle \oplus \Lambda_g^-, \qquad \Lambda_g^+ = Span_{\mathbb{R}}\langle\omega\rangle \oplus \Lambda_J^-. \tag{9}$$

In general if α^- is anti-invariant then

$$* \alpha^- = \alpha^-. \tag{10}$$

Therefore,

Corollary 4.1. *Closed anti-invariant forms are harmonic, that is, we have an inclusion* $L_2\mathcal{Z}^- \hookrightarrow \mathcal{H}_2^2$*. All anti-invariant forms are self-dual, while anti self-dual forms are invariant.*

For closed almost complex 4-manifolds Drăghici, Li and Zhang showed in [4] that there is a direct sum decomposition

$$H^2_{dR}(X;\mathbb{R}) = H^+(X) \oplus H^-(X).$$

In this section we generalize such a decomposition to the L_2 setting. The arguments follow closely those in [4].
First of all, by the L_2-Hodge decomposition and Lemma 2.2 the vector space $L_2H^2(X;\mathbb{R})$ is isomorphic to the space $\mathcal{H}_2^2$ of L_2-harmonic forms on X, which is a topological subspace of the Hilbert space $L_2\Omega^2(X)$. The following lemma is well known.

Lemma 4.2. $\mathcal{H}_2^2$ *is a closed subspace of* $L_2\Omega^2(X)$*, and hence inherits the structure of a Hilbert space.*

Proof. We recall the proof for the sake of completeness. Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{H}_2^2$ such that $\alpha_j \to \alpha$, for $j \to +\infty$ in L_2. Then, for every smooth compactly supported 2-form φ on X we have:

$$\langle\alpha, \delta\varphi\rangle = \lim_{j\to+\infty}\langle\alpha_j, \delta\varphi\rangle = 0.$$

In the same way,

$$\langle\alpha, d\varphi\rangle = \lim_{j\to+\infty}\langle\alpha_j, d\varphi\rangle = 0,$$

that is α is harmonic in the sense of distributions. Therefore, by elliptic regularity, $\alpha \in \mathcal{H}_2^2$. □

Lemma 4.3. *Let $\alpha \in L_2\Omega^2(X)$ be self-dual and let $\alpha = \alpha_H + \lambda + \mu$ be its L_2-Hodge decomposition* (1). *Then,*

$$\lambda_g^{sd} = \mu_g^{sd}, \qquad \lambda_g^{asd} = -\mu_g^{asd}$$

where $\lambda = \lambda_g^{sd} + \lambda_g^{asd}$ and $\mu = \mu_g^{sd} + \mu_g^{asd}$ denote the $*$-decomposition. Furthermore, the forms

$$\alpha + 2\lambda_g^{asd} = \alpha_H + 2\lambda. \tag{11}$$

are closed.

Proof. By assumption $*\alpha = \alpha$. Hence, if

$$\alpha = \alpha_H + \lambda + \mu,$$

where $\lambda \in \overline{d(L_2\Omega^1(X))}$, $\mu \in \overline{\delta(L_2\Omega^3(X))}$ then,

$$*\alpha = *\alpha_H + *\lambda + *\mu = \alpha_H + \lambda + \mu$$

Now, if $\{\lambda_j\}_{j\in\mathbb{N}}$, $\{\mu_j\}_{j\in\mathbb{N}}$ are sequences in $d(L_2\Omega^1(X))$, respectively $\delta(L_2\Omega^3(X))$ such that

$$\lambda_j = d\lambda_j', \quad \mu_j = d\mu_j', \quad \lambda_j \in L_2\Omega^1(X), \mu_j \in L_2\Omega^3(X), \quad d\lambda_j' \to \lambda, d\mu_j' \to \mu$$

in the L_2-norm, then

$$\|\lambda_j - \lambda\|_{L_2(X)} = \|d\lambda_j' - \lambda\|_{L_2(X)} = \|*d\lambda_j' - *\lambda\|_{L_2(X)},$$

so that $*d\lambda_j' \to *\lambda$ in L_2 and, similarly, $*d\mu_j' \to *\mu$. Therefore, since

$$*d\lambda_j' \in \delta(L_2\Omega^3(X)), \qquad *\delta\mu_j' \in d(L_2\Omega^2(X)),$$

we obtain that

$$*\lambda \in L_2\Omega^2(X), \quad *\mu \in L_2\Omega^2(X), \quad *\lambda \in \overline{\delta(L_2\Omega^3(X)}, \quad *\mu \in \overline{d(L_2\Omega^2(X)}.$$

Therefore, by the uniqueness of the L_2-Hodge decomposition,

$$*\lambda = \mu, \qquad *\mu = \lambda.$$

Then (11) follows. The form $\alpha_H + 2\lambda$ is closed since α_H is harmonic and λ is closed by Lemma 2.2. □

Lemma 4.4. *The following holds*

$$\overline{L_2H^+(X)} \cap \overline{L_2H^-(X)} = \{0\}.$$

Proof. Let $\{\alpha_i\}_{i\in\mathbb{N}}$, $\{\beta_j\}_{i\in\mathbb{N}}$ be sequences of harmonic forms in $L_2\Omega^2(X)$ with $[\alpha_i] \in L_2H^+(X)$ and $[\beta_i] \in L_2H^-(X)$ such that $\alpha_i \to \alpha$, for $i \to +\infty$ in L_2 and $\beta_i \to \alpha$, for $i \to +\infty$ in L_2. Then, using Lemma 2.2 and Corollary 4.1 we can write

$$\alpha_i = \theta_i^+ + \lambda_i\,, \qquad \beta_i = \eta_i^-$$

where $\theta_i^+ \in L_2\mathcal{Z}^+$, $\eta_i^- \in L_2\mathcal{Z}^-$, $\lambda_i \in \overline{dL_2(\Omega^1(X))}$. Then, as anti-invariant forms are self-dual we can use Lemma 2.1 to obtain

$$0 = \int_X \theta_i^+ \wedge \eta_i^- = \int_X \theta_i^+ \wedge *\eta_i^- = \int_X \alpha_i \wedge *\beta_i = \langle \alpha_i, \beta_i\rangle.$$

Taking a limit this implies

$$\|\alpha\|^2_{L_2(X)} = 0,$$

that is, $\overline{L_2H^+(X)} \cap \overline{L_2H^-(X)} = \{0\}$. □

Lemma 4.5.

$$(L_2H^+(X) \oplus L_2H^-(X))^\perp = \{0\}.$$

The orthogonal complement is defined by recalling that $L_2H^\pm(X)$ can be thought of as subspaces of the Hilbert space $\mathcal{H}_2^2$.

Proof. By contradiction: assume that there exists $0 \neq [\alpha] \in L_2H^2(X;\mathbb{R})$ such that, for every $[\theta^+] + [\theta^-] \in L_2H^+(X) \oplus L_2H^-(X)$,

$$< \alpha, \theta^+ + \theta^- >= 0.$$

To compute the inner product we assume that α, θ^+ and θ^- are harmonic representatives. By taking θ^+ to be the anti self-dual part of α (which is invariant by Corollary 4.1) and $\theta^- = 0$ we see immediately that the anti self-dual part of α must vanish, that is, α is self-dual. Therefore, by (9) we have

$$\alpha = c\omega + \theta^-,$$

where c is a function on X such that $c \neq 0$ and $\theta^- \in \Omega^-(X)$.

Since $[\alpha] \notin L_2H^-(X)$ we may assume that there exists $x \in X$ such that $c(x) > 0$. Let a be a bump function and W be a compact neighborhood of x such that $a|_W = 1$ and

$$\operatorname{supp} a \subset \{x \in X \mid c(x) > 0\}.$$

Let $\Phi : X \to \mathbb{R}$ be defined as

$$\Phi(x) = g(\alpha, a\omega)(x).$$

Then

$$\Phi(x) = g(\alpha, a\omega)(x) = g(c\omega + \theta^-, a\omega)(x) = c(x)a(x) \geq 0.$$

Now we apply Lemma 4.3 to the self-dual form $a\Phi\omega$. Let λ be the exact part of the L_2 Hodge decomposition of $a\Phi\omega$. Then Lemma 4.3 gives

$$(a\Phi\omega)_H + 2\lambda = a\Phi\omega + 2\lambda_g^{asd} \in L_2H^+(X).$$

Therefore, using Lemma 2.1, 2.2 and noting that self-dual and anti self-dual forms are pointwise g-orthogonal, we obtain

$$\begin{aligned} 0 &= < (a\Phi\omega + 2\lambda_g^{asd})_H, \alpha > = \int_X (a\Phi\omega + 2\lambda_g^{asd}) \wedge *\alpha \\ &= \int_X g(\alpha, a\Phi\omega) + 2g(\alpha, \lambda_g^{asd})\mathrm{Vol}_X = \int_X g(\alpha, a\Phi\omega)\mathrm{Vol}_X = \int_X \Phi^2 \mathrm{Vol}_X. \end{aligned}$$

Hence $\Phi = 0$ and $c(x) = 0$. This gives a contradiction. □

Lemma 4.6. *We have*

$$L_2H^2(X;\mathbb{R}) = \overline{L_2H^+(X) \oplus L_2H^-(X)}.$$

Proof.

$$\overline{L_2H^+(X) \oplus L_2H^-(X)} = ((L_2H^+(X) \oplus L_2H^-(X))^\perp)^\perp = \{0\}^\perp = \mathcal{H}_2^2$$

using Lemma 4.5. □

Lemma 4.7. *The subspace* $\overline{L_2H^+(X)} \oplus \overline{L_2H^-(X)}$ *is closed in* $L_2H^2(X;\mathbb{R})$.

Proof. As $\overline{L_2H^+(X)}$ and $\overline{L_2H^-(X)}$ are orthogonal, we can check that a sequence $\{(\alpha_i, \beta_i)\}$ in t he direct sum is Cauchy if and only if both $\{\alpha_i\}$ and $\{\beta_i\}$ are Cauchy. □

Theorem 4.8. *Let (X, J, g) be a complete almost Hermitian 4-dimensional manifold. Then, we have the following decomposition*

$$L_2H^2(X;\mathbb{R}) = \overline{L_2H^+(X)} \oplus \overline{L_2H^-(X)}.$$

Proof. Indeed, by Lemma 4.7 the direct sum is closed and so by Lemma 4.6 contains $L_2H^2(X;\mathbb{R}) = \overline{L_2H^+(X) \oplus L_2H^-(X)}$. □

Let now J be an almost complex structure on a manifold X of any dimension. The following Proposition provides a cohomological obstruction on J in order that there exists a compatible symplectic form ω such that the associated Hermitian metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is complete.

Proposition 4.9. *Let (X, J, ω, g_J) be an almost Kähler manifold such that g_J is complete. Then*

$$L_2H^+(X) \cap L_2H^-(X) = \{0\}.$$

Proof. Let $[\alpha] \in L_2H^+(X) \cap L_2H^-(X)$. Then, there exist $\alpha^\pm \in L_2\mathcal{Z}^\pm$ such that

$$\alpha = \alpha^+ + \lambda\,, \qquad \alpha = \alpha^- + \mu,$$

where $\alpha^\pm \in L_2\mathcal{Z}^\pm$. Then

$$\alpha^+ = \alpha^- + \eta \tag{12}$$

where $\eta \in \overline{d(L_2A^1(X))}$. Let $\{\eta_j\}_{j\in\mathbb{N}}$ be a sequence in $L_2(X)$ such that $\eta_j = d\eta'_j$, $\eta'_j \in L_2(X)$ and $d\eta'_j \to \eta$ in $L_2(X)$. Then, by bi-degree reasons,

$$\int_X \alpha^+ \wedge \alpha^- \wedge \omega^{n-2} = 0. \tag{13}$$

We claim that

$$\int_X \eta \wedge \alpha^- \wedge \omega^{n-2} = 0. \tag{14}$$

Indeed,

$$\begin{aligned}
&|\langle d\eta'_j \wedge \alpha^-, *\omega^{n-2}\rangle - \langle \eta \wedge \alpha^-, *\omega^{n-2}\rangle| \\
&\leq \int_X |(d\eta'_j - \eta) \wedge \alpha^-||*\omega^{n-2}|\mathrm{Vol}_X \leq C\int_X |(d\eta'_j - \eta)||\alpha^-|\mathrm{Vol}_X \\
&\leq \|d\eta'_j - \eta\|_{L_2(X)}\|\alpha^-\|_{L_2(X)},
\end{aligned}$$

that is $\langle d\eta'_j \wedge \alpha^-, *\omega^{n-2}\rangle \to \langle \eta \wedge \alpha^-, *\omega^{n-2}\rangle$, for $j \to +\infty$. On the other hand, by Lemma 2.1,

$$0 = \lim_{j\to\infty} \langle d\eta'_j \wedge \alpha^-, *\omega^{n-2}\rangle = \int_X \eta \wedge \alpha^- \wedge \omega^{n-2},$$

that is, (14). Therefore, by (10), (12) and (13) we have

$$\begin{aligned} 0 &= \int_X \alpha^+ \wedge \alpha^- \wedge \omega^{n-2} \\ &= \int_X (\alpha^- + \eta) \wedge \alpha^- \wedge \omega^{n-2} = \int_X \alpha^- \wedge \alpha^- \wedge \omega^{n-2} \\ &= (n-2)! \int_X \alpha^- \wedge *\alpha^- = (n-2)!\|\alpha^-\|^2_{L_2(X)}, \end{aligned}$$

since given any J-anti-invariant form α^- on a $2n$-dimensional almost Hermitian manifold X, we have $*\alpha^- = \frac{1}{(n-2)!}\alpha^- \wedge \omega^{n-2}$. Hence $[\alpha] = 0$. □

Example 4.10. Let $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < r_1, |z_2| < r_2\}$ be a polydisc in $\mathbb{C}^2$ endowed with the complete and d-bounded Kähler metric

$$\omega = i\partial\overline{\partial} \sum_{j=1}^{2} \log(1 - |z_j|^2).$$

Then, the real J-anti-invariant forms

$$\frac{1}{2}(dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2), \qquad \frac{1}{2i}(dz_1 \wedge dz_2 - d\bar{z}_1 \wedge d\bar{z}_2)$$

and the real J-invariant forms

$$\frac{1}{2}(dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2), \qquad \frac{1}{2i}(dz_1 \wedge d\bar{z}_2 - d\bar{z}_1 \wedge dz_2)$$

are L_2-harmonic, so that $L_2H^{\pm}(\Delta^2) \neq \{0\}$.

Remark 4.11. Notice that for the de Rham cohomology, Drăghici, Li and Zhang in [5, Theorem 3.24] constructed non-compact complex surfaces for which $H^+(M) \cap H^-(M) \neq \{0\}$.

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Received August 29, 2017
Accepted October 17, 2018