

# Symplectic divisorial capping in dimension 4

TIAN-JUN LI AND CHEUK YU MAK

We investigate the notion of symplectic divisorial compactification for symplectic 4-manifolds with either convex or concave type boundary. This is motivated by the notion of compactifying divisors for open algebraic surfaces. Our main classification result is that if the symplectic form of a symplectic divisor is exact on the boundary of its plumbing, then the symplectic divisor admits either a concave or convex neighborhood after a symplectic deformation that keeps the divisor symplectic.

## 1. Introduction

In this paper, a *symplectic divisor* refers to a connected configuration of finitely many closed embedded symplectic surfaces  $D = C_1 \cup \cdots \cup C_k$  in a symplectic 4 dimensional manifold (possibly with boundary or non-compact)  $(W, \omega)$ .  $D$  is further required to have the following properties:  $D$  has empty intersection with  $\partial W$ , no three  $C_i$  intersect at a point, and any intersection between two surfaces is transversal and positive. The orientation of each  $C_i$  is chosen to be positive with respect to  $\omega$ . Since we are interested in the germ of a symplectic divisor,  $W$  is sometimes omitted in the writing and  $(D, \omega)$ , or simply  $D$ , is used to denote a symplectic divisor.

A closed regular neighborhood of  $D$  is called a plumbing of  $D$ . The plumbings are well defined up to orientation preserving diffeomorphism. We call the boundary of a plumbing of  $D$  the boundary of  $D$ . In the same vein, when  $\omega$  is exact on the boundary of a plumbing, we say that  $\omega$  is exact on the boundary of  $D$ .

A plumbing  $P(D)$  of  $D$  is called a *concave (resp. convex) neighborhood* if  $P(D)$  is a strong concave (resp. convex) filling of its boundary. A symplectic divisor  $D$  is called *concave (resp. convex)* if for any neighborhood  $N$  of  $D$ , there is a concave (resp. convex) neighborhood  $P(D) \subset N$  for the divisor. Through out this paper, all concave (resp. convex) fillings are symplectic

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Both authors are supported by NSF-grant DMS 1065927.

strong concave (resp. strong convex) fillings and we simply call it cappings or concave fillings (resp. fillings or convex fillings).

**Definition 1.1.** Suppose that  $D$  is a concave (resp. convex) divisor. If a symplectic gluing ([7]) can be performed for a concave (resp. convex) neighborhood of  $D$  and a symplectic manifold  $Y$  with convex (resp. concave) boundary to obtain a closed symplectic manifold, then we call  $D$  a **capping** (resp. **filling**) divisor. In both cases, we call  $D$  a **compactifying** divisor of  $Y$ .

### 1.1. Motivation

We provide some motivation from two typical families of examples in algebraic geometry together with some general symplectic compactification phenomena.

Suppose  $Y$  is a smooth affine algebraic variety over  $\mathbb{C}$ . Then  $Y$  can be compactified by a divisor  $D$  to a projective variety  $X$ . By Hironaka's resolution of singularities theorem, we could assume that  $X$  is smooth and  $D$  is a simple normal crossing divisor. In this case,  $Y$  is a Stein manifold and  $D$  has a concave neighborhood induced by a plurisubharmonic function on  $Y$  ([6]). Moreover,  $Y$  is symplectomorphic to the completion of a suitably chosen Stein domain  $\bar{Y} \subset Y$  (see e.g. [23]). Therefore, compactifying  $Y$  by  $D$  in the algebro geometric situation is analogous to gluing  $\bar{Y}$  with a concave neighborhood of  $D$  along their contact boundaries [7].

On the other hand, suppose we have a compact complex surface with an isolated normal singularity. We can resolve the isolated normal singularity and obtain a pair  $(W, D)$ , where  $W$  is a smooth compact complex surface and  $D$  is a simple normal crossing resolution divisor. In this case, we can define a Kähler form near  $D$  such that  $D$  has a convex neighborhood  $P(D)$ . If the Kähler form can be extended to  $W$ , then the Kähler compactification of  $W - D$  by  $D$  is analogous to gluing the symplectic manifold  $W - \text{Int}(P(D))$  with  $P(D)$  along their contact boundaries.

From the symplectic point of view, there are both flexibility and constraints for capping a symplectic 4 manifold  $Y$  with convex boundary. For flexibility, there are infinitely many ways to embed  $Y$  in closed symplectic 4-manifolds (Theorem 1.3 of [9]). This still holds even when  $Y$  has only weak convex boundary (see [5] and [8]). For constraints, it is well-known that (e.g. [15])  $Y$  does not have any exact capping. From these perspectives, divisor cappings might provide a suitable capping model to study (see also [12] and [11]).

On the other hand, divisor fillings have been studied by several authors. For instance, it is known that they are the maximal fillings for the canonical contact structures on Lens spaces (see [20] and [3]). They also naturally arise in the study of symplectic fillings of link of complex surface singularities (see eg. [2] and [26]).

In this setting, the following questions are natural: Suppose  $D$  is a symplectic divisor. (i) When is  $D$  a concave/convex divisor? and (ii) When is it also a compactifying divisor?

We offer a comprehensive study of the first question in this paper. The second question for a specific kind of  $D$ , namely, when the fundamental group of  $\partial P(D)$  is finite is addressed in [17].

## 1.2. A flowchart

Regarding the first question, observe that a divisor is a capping (resp. filling) divisor if it is concave (resp. convex), and embeddable in the following sense:

**Definition 1.2.** A symplectic divisor  $D \subset (W, \omega)$  is called **embeddable** if there is a neighborhood  $U \subset W$  of  $D$  which admits a symplectic embedding into a closed symplectic manifold  $W$ .

We recall some results from the literature. It is proved in [13] that when the graph of a symplectic divisor is negative definite, it can always be perturbed to be a convex divisor. Moreover, a convex divisor is always embeddable, by [9], hence a filling divisor. However, a concave divisor is not necessarily embeddable. An obstruction is provided by [21].

It is convenient to associate an augmented graph  $(\Gamma, a)$  to a symplectic divisor  $(D, \omega)$ , where  $\Gamma$  is the graph of  $D$  and  $a$  is the area vector for the embedded symplectic surfaces (see Section 2 for details). The intersection form of  $\Gamma$  is denoted by  $Q_\Gamma$ .

**Definition 1.3.** Suppose  $(\Gamma, a)$  is an augmented graph with  $k$  vertices. Then, we say that  $(\Gamma, a)$  satisfies the positive (resp. negative) **GS criterion** if there exists  $z \in \mathbb{R}_{>0}^k$  (resp  $\mathbb{R}_{\leq 0}^k$ ) such that  $Q_\Gamma z = a$ . A symplectic divisor is said to satisfy the positive (resp. negative) GS criterion if its associated augmented graph does.

The first observation is:

**Proposition 1.4 (see [24]).** *A symplectic divisor  $(D, \omega)$  is a capping divisor if  $(D, \omega)$  is embeddable and satisfies the positive GS criterion.*

When  $(D, \omega)$  satisfies the negative GS criterion, a construction of a convex neighborhood of  $(D, \omega)$  is given in Gay-Stipsicz [13] in the  $\omega$ -orthogonal case, and McLean [24] in the general case (and even for higher dimensions). When  $(D, \omega)$  satisfies the positive GS criterion, we realize that their construction can be applied to obtain Proposition 1.4. We remark that GS criteria can be verified easily.

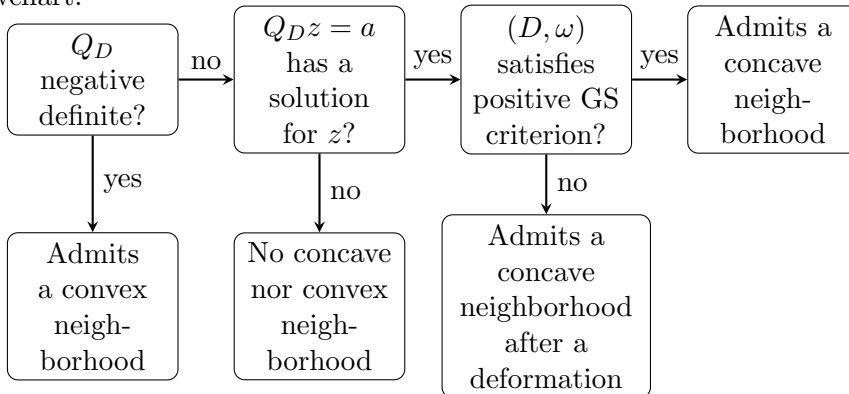
Surprisingly, there is an easily verified sufficient condition for us to get a concave divisor after symplectic deformation. Our main result is:

**Theorem 1.5.** *Let  $D \subset (W, \omega_0)$  be a symplectic divisor such that the intersection form of  $D$  is not negative definite and  $\omega_0$  restricted to the boundary of  $D$  is exact. Then for any neighborhood  $N$  of  $D$ , there is a family of symplectic forms  $\omega_t$  on  $W$  keeping  $D$  symplectic for all  $t \in [0, 1]$  such that  $(D, \omega_1)$  is a concave divisor and  $\omega_t = \omega_0$  outside  $N$  for all  $t$ . In particular, if  $D$  is also an embeddable divisor, then it is a capping divisor after a deformation.*

Summarizing Proposition 1.4, Theorem 1.5 and the known results for negative definite symplectic divisors, we have

**Corollary 1.6.** Let  $(D, \omega)$  be a symplectic divisor with  $\omega$  exact on the boundary of  $D$ . Then  $D$  is either a convex divisor or a concave divisor possibly after a symplectic deformation.

More complete information is illustrated by the following schematic flowchart.



We would like to mention here an application of Theorem 1.5. It provides a rather general construction of uniruled caps and Calabi-Yau caps in [18]. These caps capture most known contact 3-manifolds with bounded topological complexity of strong fillings and Stein fillings.

## Acknowledgements

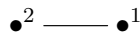
The authors would like to thank Mark McLean for many helpful discussions, in particular for explaining the canonical contact structure in [24]. They would also like to thank David Gay, Ko Honda, Burak Ozbagci, Andras Stipsicz, Weiwei Wu and Weiyi Zhang for their interest in this work. They are also grateful to Laura Starkston for stimulating discussions.

## 2. Construction

Essential topological information of a symplectic divisor can be encoded by its *graph*, with vertices representing the surfaces and each edge joining two vertices representing an intersection between the two surfaces corresponding to the two vertices. Moreover, each vertex is weighted by its genus (a non-negative integer) and its self-intersection number (an integer). If each vertex is also weighted by its symplectic area (a positive real number), then we call it an *augmented graph*. Sometimes, the genera (and the symplectic area) are not explicitly stated. For simplicity, we would like to assume the symplectic divisors are connected. For a graph (resp. an augmented graph)  $\Gamma$  (resp.  $(\Gamma, a)$ ), we use  $Q_\Gamma$  to denote the intersection matrix for  $\Gamma$  (resp. and  $a$  to denote the area weights for  $\Gamma$ ).

We remark that the germ of a symplectic divisor  $(D, \omega)$  with  $\omega$ -orthogonal intersections is uniquely determined by its augmented graph  $(\Gamma, a)$  (see [25] and Theorem 3.1 of [13]) and a symplectic divisor can always be made  $\omega$ -orthogonal after a perturbation (see [14]).

**Example 2.1.** The graph



where both vertices are of genus zero, represents a symplectic divisor consisting of two spheres with self-intersection 2 and 1, respectively, and intersecting positively transversally at a point.

### 2.1. GS criteria versus wrapping numbers

A compact symplectic manifold  $(P, \omega)$  is a strong concave (resp. convex) filling of its boundary if there exists a Liouville vector field  $X$  (ie.  $d\iota_X\omega = \omega$ ) defined near  $\partial P$  such that  $X$  points inward (resp. outward) along  $\partial P$ . In particular,  $\omega|_{\partial P}$  is exact and it gives  $\partial P$  the structure of a contact manifold with a contact form  $\alpha := \iota_X\omega|_{\partial P}$ .

Conversely, if we have a symplectic manifold  $(P, \omega)$  together with  $\alpha \in \Omega^1(\partial P)$  such that  $d\alpha = \omega|_{\partial P}$ . We call  $(P, \omega)$  a strong concave (resp. convex) filling of  $(\partial P, \alpha)$  if we can find a Liouville vector field  $X$  defined near  $\partial P$  such that  $\iota_X \omega|_{\partial P} = \alpha$  and  $X$  points inward (resp. outward) along  $\partial P$ .

Let  $(D, P(D), \omega)$  be a plumbing of a symplectic divisor.  $\omega|_{\partial P(D)}$  being exact is equivalent to  $[\omega] \in H^2(P(D), \mathbb{R})$  being able to be lifted to a relative cohomological class  $H^2(P(D), \partial P(D), \mathbb{R})$ . Using Lefschetz duality, this is in turn equivalent to  $[\omega]$  being able to be expressed as a linear combination  $\sum_{i=1}^k z_i [C_i]$ , where  $z_i \in \mathbb{R}$  and  $D = C_1 \cup \dots \cup C_k$ . As a result,  $\omega|_{\partial P(D)}$  being exact if and only if there exist a solution  $z$  for the equation  $Q_D z = a$ . In particular,  $(D, \omega)$  satisfies the positive (resp. negative) GS criteria if and only if there exists a primitive  $\alpha$  of  $\omega|_{\partial P(D)}$  such that  $PD([\omega, \alpha]) = \sum z_i [C_i]$  and all  $z_i$  are positive (resp. non-positive).

Moreover,  $Q_D$  being non-degenerate is equivalent to the choices of lift of  $[\omega]$  to a class in  $H^2(P(D), \partial P(D), \mathbb{R})$  being unique, which is in turn equivalent to the connecting homomorphism

$$H^1(\partial P(D); \mathbb{R}) \rightarrow H^2(P(D), \partial P(D); \mathbb{R})$$

being zero. When  $Q_D$  is degenerate, the equation  $Q_D z = a$  having no solution for  $z$  is equivalent to  $\omega|_{\partial P(D)}$  being not exact. Similarly, when  $Q_D z = a$  has a solution for  $z$ , then the solution is unique up to the kernel of  $Q_D$ , which corresponds to the unique lift of  $\omega$  up to the image of the connecting homomorphism  $H^1(\partial P(D); \mathbb{R}) \rightarrow H^2(P(D), \partial P(D); \mathbb{R})$ .

**Definition 2.2.** Suppose that  $d\alpha = \omega|_{\partial P(D)}$  and  $PD([\omega, \alpha]) = \sum z_i [C_i]$ , the *wrapping numbers* of  $\alpha$  around  $C_i$  are defined as  $\lambda_i := -z_i$ .

This is the terminology used in [24] and [23].

**Remark 2.3.** There is another equivalent interpretation of wrapping numbers. Let  $i : U \rightarrow P(D)$  be a symplectic embedding of a small disc  $U$  to  $P(D)$  meeting  $C_i$  positively transversally once at the origin of  $U$ , Then  $i|_{U-0}^* \alpha - \frac{r^2}{2} d\vartheta$  is closed and hence cohomologous to  $[\frac{\lambda_i}{2\pi} d\vartheta]$  for some  $\lambda_i$ , where  $(r, \vartheta)$  are the polar coordinates of  $U$ . The  $\lambda_i$  is the wrapping number of  $\alpha$  around  $C_i$ .

The following remark explains that the definitions for positive and negative GS criteria can be made more symmetric.

**Remark 2.4.** For an augmented graph  $(\Gamma, a)$ , the negative GS criterion is equivalent to the existence of  $z \in \mathbb{R}_{<0}^k$  such that  $Q_\Gamma z = a$  (instead of  $z \in \mathbb{R}_{\leq 0}^k$ ). This is because if  $z \in \mathbb{R}_{\leq 0}^k$  and one of the entries, say  $z_1$  equals 0, then we have  $a_1 = \sum_j (Q_\Gamma)_{1j} z_j \leq 0$  which contradicts to the assumption that  $a_1 > 0$ . Here, we use the fact that off-diagonal entries of  $Q_\Gamma$  are non-negative.

With the preceding remark understood, we can summarize our discussions on GS criteria and wrapping numbers as follows.

**Lemma 2.5.** *Let  $(D, \omega)$  be a symplectic divisor. Then the set of possible lifts of  $[\omega]$  to a class in  $H^2(P(D), \partial P(D); \mathbb{R})$  are in one-to-one correspondence to the solution  $z$  of  $Q_D z = a$ .*

*The positive (negative) GS criterion is satisfied if and only if the wrapping numbers are negative (positive).*

## 2.2. McLean's construction

For each  $i$ , let  $N_i$  be a neighborhood of  $C_i$  such that we have a smooth projection  $p_i : N_i \rightarrow C_i$  with a connection rotating the disc fibers. Hence, for each  $i$ , we have a well-defined radial coordinate  $r_i$  with respect to the fiber bundle  $p_i$  such that  $C_i$  corresponds to  $r_i = 0$ .

Let  $\mathcal{W}$  be the space of all smooth functions  $\bar{\rho} : [0, \delta) \rightarrow [0, 1]$  such that  $\bar{\rho}(x) = x^2$  near  $x = 0$ ,  $\bar{\rho}(x) = 1$  when  $x$  is close to  $\delta$  and  $\bar{\rho}'(x) \geq 0$  for all  $x$ .

A smooth function  $f : W - D \rightarrow \mathbb{R}$  is called **compatible** with  $D$  if  $f = \sum_{i=1}^k \log(\bar{\rho}(r_i)) + \bar{\tau}$  for some  $\bar{\tau} \in C^\infty(W)$  and  $\bar{\rho} \in \mathcal{W}$ . Here,  $r_i$  are the radial coordinates with respect to the fiber bundle  $p_i$ .

For a one-form  $\beta$ , we use  $X_\beta$  to denote the vector field  $\omega$ -dual of  $\beta$  (ie.  $\iota_{X_\beta} \omega = \beta$ ).

**Proposition 2.6 (cf. Proposition 5.8 of [24]).** *Let  $D \subset (W, \omega)$  be a symplectic divisor. Suppose  $\theta \in \Omega^1(W - D)$  is a primitive of  $\omega$  on  $W - D$  such that it has positive (resp. negative) wrapping numbers for all  $i = 1, \dots, k$ . Then there exists  $f : W - D \rightarrow \mathbb{R}$  compatible with  $D$  and  $g : W - D \rightarrow \mathbb{R}$  such that  $df(X_{\theta+dg}) > 0$  (resp.  $df(-X_{\theta+dg}) > 0$ ) near  $D$ , where  $X_{\theta+dg}$  is the dual of  $\theta + dg$  with respect to  $\omega$ .*

*In particular,  $D$  is a convex (resp. concave) divisor.*

This is essentially contained in Proposition 5.8 of [24]—the only new statement is the last sentence. And Proposition 5.8 in [24] is stated only for the

case in which wrapping numbers are all positive, however, the proof there goes through without additional difficulty for the case where all wrapping numbers are negative. We remark that the  $\omega$ -orthogonal intersection condition is not required in his construction.

Proposition 1.4 is a consequence of Proposition 2.6.

McLean also proves a uniqueness statement for his construction.

**Proposition 2.7.** *[cf. Proposition 5.10 and Corollary 5.11 of [24]] Let  $D \subset (W, \omega_j)$  for  $j = 0, 1$  be symplectic divisors. Suppose  $\theta_j \in \Omega^1(W - D)$  is a primitive of  $\omega_j$  on  $W - D$  such that it has positive (resp. negative) wrapping numbers for all  $i = 1, \dots, k$  and for both  $j = 0, 1$ . If  $f_j : W - D \rightarrow \mathbb{R}$  are compatible with  $D$  and there are  $g_j : W - D \rightarrow \mathbb{R}$  such that  $df(X_{\theta_j + dg_j}^j) > 0$  (resp.  $df(-X_{\theta_j + dg_j}^j) > 0$ ) near  $D$  for both  $j$ , then  $(f_0^{-1}(l), \theta_0 + dg_0|_{f_0^{-1}(l)})$  is contactomorphic to  $(f_1^{-1}(l), \theta_1 + dg_1|_{f_1^{-1}(l)})$  for sufficiently negative  $l$ .*

Here are a few remarks regarding the uniqueness of contact structures on  $\partial P(D)$ .

**Remark 2.8.** Proposition 5.10 and Corollary 5.11 of [24] requires that  $\omega_0$  and  $\omega_1$  are connected by a path of symplectic forms  $\omega_t$  making  $D$  symplectic for all  $t$ . However, in dimension four, we can take  $\omega_t = (1 - t)\omega_0 + t\omega_1$  which is symplectic in a small neighborhood of  $D$  and making  $D$  symplectic for all  $t$ . If both  $\omega_0$  and  $\omega_1$  have positive (resp. negative) wrapping numbers, then so is  $\omega_t$ . As a result, Proposition 2.7 implies that in dimension four, for any symplectic form  $\omega_0$  and  $\omega_1$  making  $D$  a symplectic divisor such that they have primitives  $\theta_0$  and  $\theta_1$  on  $W - D$  with positive (resp. negative) wrapping numbers, the contact structures constructed by McLean's construction with respect to  $\theta_0$  and  $\theta_1$  are contactomorphic.

**Remark 2.9.** When  $(D, \omega)$  is  $\omega$ -orthogonal, the Gay-Stipsicz [13] construction also produces a concave (resp. convex) neighborhood structure on  $(P(D), \omega)$  if  $(D, \omega)$  satisfies the positive (resp. negative) GS criteria. Since the Gay-Stipsicz construction does not involve the use of a compatible function, it is not clear a priori that the contact structure on  $\partial P(D)$  constructed by Gay-Stipsicz is contactomorphic to the one constructed by McLean. We remark that these two contact structures are indeed contactomorphic to each other and one can prove it by constructing an appropriate function on  $P(D) - D$  compatible to  $D$  in the Gay-Stipsicz setting. Since this is not the focus of our paper, we leave it to the interested readers.



**Remark 2.10 (cf. [24]).** Another application of Proposition 2.7 implies the following: If  $D$  is a simple normal crossing ample divisor in a smooth projective variety  $W$  and  $f : W - D \rightarrow \mathbb{R}_{\geq 0}$  is a plurisubharmonic function, then the induced contact structure on  $f^{-1}(R)$  is contactomorphic to the one by McLean's construction. Here,  $R$  is larger than all critical values of  $f$ .

### 3. Deformation

In this section we first apply the inflation operation to establish Theorem 1.5. Then, we explain in details the resulting flowchart. Finally, we give some examples of concave symplectic divisor.

#### 3.1. Theorem 1.5

The proof of Theorem 1.5 involves two inputs. The first input is a linear algebra lemma. The second input is inflation lemma, which allows us to deform the symplectic form to our desired one so as to apply Proposition 2.6.

**3.1.1. A key lemma.** The following linear algebra lemma is related to the positive GS criterion and is crucial to the proof of Theorem 1.5.

**Lemma 3.1.** *Let  $Q$  be a  $k$  by  $k$  symmetric matrix with off-diagonal entries being all non-negative. Assume that there exist  $a \in \mathbb{R}_{>0}^k$  such that there exist  $z \in \mathbb{R}^k$  with  $Qz = a$ . Suppose also that  $Q$  is not negative definite. Then, there exists  $z \in \mathbb{R}_{>0}^k$  such that  $Qz \in \mathbb{R}_{>0}^k$ .*

*Proof.* When  $k = 1$ , it is trivial. Suppose the statement is true for  $(k-1)$  by  $(k-1)$  matrix and now we consider a  $k$  by  $k$  matrix  $Q$ . Let  $q_{i,j}$  be the  $(i, j)^{th}$ -entry of  $Q$ . First observe that if  $q_{i,i} \geq 0$ , for all  $i = 1, \dots, k$ , then the statement is true for the reason below. In this case, all entries are non-negative. If, in addition that, each row has a positive entry, then  $z = (1, \dots, 1)$  works. If not, then there exist a row with all 0 and there is no  $a \in \mathbb{R}_{>0}^k$  such that there exist  $z \in \mathbb{R}^k$  with  $Qz = a$ .

Therefore, we might assume  $q_{k,k} < 0$ . Let  $l_j = -\frac{q_{k,j}}{q_{k,k}} \geq 0$ , for  $j < k$ , and let  $B$  be the lower triangular matrix given by

$$b_{i,j} = \begin{cases} \delta_{i,j} & \text{if } i \neq k \text{ or } (i, j) = (k, k) \\ l_j & \text{if } i = k \text{ and } (i, j) \neq (k, k) \end{cases}$$

Let  $M = B^T Q B$ . Then,

$$m_{i,j} = \begin{cases} q_{i,j} - \frac{q_{i,k} q_{k,j}}{q_{k,k}} & \text{if } (i,j) \neq (k,k) \\ q_{k,k} & \text{if } (i,j) = (k,k) \end{cases}$$

In particular,  $m_{i,k} = m_{k,j} = 0$ , for all  $i$  and  $j$  less than  $k$ . We can write  $M$  as a direct sum of a  $k-1$  by  $k-1$  matrix  $M'$  with the  $1$  by  $1$  matrix  $q_{k,k}$  in the obvious way. Notice that the off diagonal entries of  $M'$  are all non-negative.

Let  $a = (a_1, \dots, a_k)^T$  and  $z = (z_1, \dots, z_k)^T$  such that  $Qz = a$ . Let also  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_k)^T = B^{-1}z$  and  $\bar{a} = (\bar{a}_1, \dots, \bar{a}_k)^T = B^T a$ . Then,  $Qz = a$  is equivalent to  $M\bar{z} = \bar{a}$ . Here,  $\bar{z}_i = z_i$ , for  $i < k$ , and  $\bar{z}_k = z_k - \sum_{i=1}^{k-1} l_i z_i$ . On the other hand,  $\bar{a}_i = a_i + l_i a_k$ , for all  $i < k$ , and  $\bar{a}_k = a_k$ .

By assumption, there exist  $a \in \mathbb{R}_{>0}^k$  such that there exist  $z \in \mathbb{R}^k$  with  $Qz = a$ . So we have  $(\bar{a}_1, \dots, \bar{a}_{k-1})^T \in \mathbb{R}_{>0}^k$  and

$$M'(z_1, \dots, z_{k-1})^T = (\bar{a}_1, \dots, \bar{a}_{k-1})^T.$$

Apply induction hypothesis, we can find  $y \in \mathbb{R}_{>0}^{k-1}$  such that  $M'y \in \mathbb{R}_{>0}^{k-1}$ . Pick  $y_k > 0$  such that  $q_{k,k}(y_k - \sum_{i=1}^{k-1} l_i y_i) > 0$  but sufficient close to zero. Then, let  $\bar{z} = (y_1, \dots, y_{k-1}, y_k - \sum_{i=1}^{k-1} l_i y_i)^T$  and tracing it back. We have  $Q(y_1, \dots, y_k)^T \in \mathbb{R}_{>0}^k$ .  $\square$

Regarding the negative GS criterion, we remark that one can show the following. (It is mentioned in [10] with an additional assumption but the additional assumption can be removed.) Suppose  $Q$  is a symmetric matrix with off-diagonal entries being non-negative. Then, the following statements are equivalent.

- (a) For any  $a \in \mathbb{R}_{>0}^n$ , there exist  $z \in \mathbb{R}_{<0}^n$  satisfying  $Qz = a$ .
- (a2) For any  $a \in \mathbb{R}_{>0}^n$ , there exist  $z \in \mathbb{R}_{\leq 0}^n$  satisfying  $Qz = a$ .
- (b) There exist  $a \in \mathbb{R}_{>0}^n$  such that there exist  $z \in \mathbb{R}_{<0}^n$  satisfying  $Qz = a$ .
- (b2) There exist  $a \in \mathbb{R}_{>0}^n$  such that there exist  $z \in \mathbb{R}_{\leq 0}^n$  satisfying  $Qz = a$ .
- (c)  $Q$  is negative definite.

The implication from (a) to (b), (a2) to (b2), (a) to (a2), (b) to (b2) are trivial. (c) implying (a2) is Lemma 3.3 of [13] and a moment thought will justify (a2) implying (a), which is explained in Remark 2.4. (b) implying (c) is similar to the proof of Lemma 3.1. To be more precise, one again use induction on the size of  $Q$  and change the basis using  $B$ . Therefore, an augmented graph  $(\Gamma, a)$  satisfies the negative GS criterion if and only if  $Q_\Gamma$

is negative definite. In particular, when a graph  $\Gamma$  is negative definite, the negative GS criterion is always satisfied, independent of the area weights.

**3.1.2. Inflation.** Now, it comes the second input.

**Lemma 3.2 (Inflation, See [16] and [19]).** *Let  $C$  be a smooth symplectic surface inside  $(W, \omega)$ . If  $[C]^2 \geq 0$ , then there exists a family of symplectic form  $\omega_t$  on  $W$  such that  $[\omega_t] = [\omega] + tPD(C)$  for all  $t \geq 0$ . If  $[C]^2 < 0$ , then there exists a family of symplectic form  $\omega_t$  on  $W$  such that  $[\omega_t] = [\omega] + tPD(C)$  for all  $0 \leq t < -\frac{\omega[C]}{[C]^2}$ . Also,  $C$  is symplectic with respect to  $\omega_t$  for all  $t$  in the range above. Moreover, if there is another smooth symplectic surface  $C'$  intersecting  $C$  positively and  $\omega$ -orthogonally, then  $C'$  is also symplectic with respect to  $\omega_t$  for all  $t$  in the range above. Here,  $PD(C)$  denotes the Poincare dual of  $[C]$ .*

When  $[C]^2 < 0$ , one can see that  $([\omega] + tPD(C))[C] > 0$  if and only if  $t < -\frac{\omega[C]}{[C]^2}$ . Therefore, the upper bound of  $t$  in this case comes directly from  $\omega_t[C] > 0$ . We remark that one can actually do inflation for  $t > -\frac{\omega[C]}{[C]^2}$  but one cannot hope for  $C$  being symplectic anymore when  $t$  goes beyond  $-\frac{\omega[C]}{[C]^2}$ .

**3.1.3. Proof.**

*Proof of Theorem 1.5.* First of all, we can isotope  $D$  to  $D'$  such that every intersection of  $D'$  is  $\omega_0$ -orthogonal, using Theorem 2.3 of [14]. Since every intersection of  $D$  is transversal and no three of  $C_i$  intersect at a common point, such an isotopy can be extended to an ambient isotopy. Now, instead of isotoping  $D$ , we can deform  $\omega_0$  through the pull back of  $\omega_0$  along the isotopy. As a result, we can assume  $D$  is  $\omega_0$ -orthogonal.

Now, we want to construct a family of realizations  $D_t$  of  $\Gamma$ , by deforming the symplectic form, such that the augmented graph of  $D_1$  satisfies the positive GS criterion.

Let  $D = D_0 = C_1 \cup \dots \cup C_k$  and let also the area weights of  $D_0$  with respect to  $\omega_0$  be  $a$ . Since  $\omega$  is exact on  $\partial P(D)$ , there exists  $z$  such that  $Q_\Gamma z = a$ . Also, by assumption and Lemma 3.1, there exists  $\bar{z} \in \mathbb{R}_{>0}^k$  such that  $Q_\Gamma \bar{z} = \bar{a} \in \mathbb{R}_{>0}^k$ . Let  $z^t = z + t(\bar{z} - z)$  and  $a^t = a + t(\bar{a} - a) = Q_\Gamma z^t \in \mathbb{R}_{>0}^k$ . We want to construct a realization  $D_1$  of  $\Gamma$  with area weights  $a^1$ . If this can be done, then the augmented graph of  $D_1$  will satisfy the positive GS criterion.

Observe that, it suffices to find a family of symplectic forms  $\omega_t$  such that  $[\omega_t] = [\omega_0] + t \sum_i (\bar{z}_i - z_i) PD([C_i])$  and a corresponding family of  $\omega_t$ -symplectic divisor  $D_t = C_1 \cup \dots \cup C_k$ . The reason is that  $C_i$  has symplectic area equal the  $i^{\text{th}}$  entry of  $a^t$  under the symplectic form  $[\omega_t] = [\omega_0] +$

$t \sum_i (\bar{z}_i - z_i) PD([C_i])$ . However, we need to modify this natural choice of family a little bit. By possibly replacing  $\bar{a}$  with a large multiple of it, we can assume  $\bar{z}_i > z_i$  for all  $1 \leq i \leq k$ . We can choose a piecewise linear path  $p^t$  arbitrarily close to  $z^t$  such that each piece is parallel to a coordinate axis and moving in the positive axis direction. Since satisfying the positive GS criterion is an open condition, we can choose  $p^t$  such that  $Q_\Gamma p^t \in \mathbb{R}_{>0}^k$ . The fact that  $p^t$  is chosen such that  $Q_\Gamma p^t \in \mathbb{R}_{>0}^k$  allows us to do inflation along  $p^t$  to get out desired family of  $\omega_t$  and  $D_t$ , by Lemma 3.2. Therefore, we arrive at a symplectic form  $\omega_1$  such that the augmented graph of  $(D, \omega_1)$ , denoted by  $(\Gamma, a)$ , satisfies the positive GS criterion. We finish the proof by applying Proposition 2.6.  $\square$

**Remark 3.3.** The proof of Theorem 1.5 implies that for any  $a \in \mathbb{R}_{>0}^k \cap Q_D \mathbb{R}_{>0}^k$ , there is a symplectic deformation making the augmented graph of  $(D, \omega_1)$  to be  $(\Gamma, a)$ .

For negative definite symplectic divisor, we can use inflation to obtain the following amusing observation.

**Proposition 3.4.** *Let  $D \subset (W, \omega_0)$  be a negative definite symplectic divisor such that  $\omega_0|_{W-D}$  is exact. There exists a symplectic deformation  $\omega_t$  supported in an arbitrarily small neighborhood of  $D$  such that  $\omega_1 \in \Omega^2(W)$  is exact.*

*Proof.* First, we can perturb  $D$  to make it  $\omega$ -orthogonal. Since  $D$  is negative definite, we have  $[C_i]^2 < 0$  for all  $i$ . We can apply inflation to  $C_1$  for  $t = -\frac{\omega[C_1]}{[C_1]^2}$ . As explained in the paragraph after Lemma 3.2, this can be done but  $C_1$  will no longer be symplectic. In the meanwhile, all other  $C_i$  for  $i \neq 1$  is still symplectic after inflation along  $C_1$ . We can inductively apply inflation to all the  $C_i$ . It is then clear that the resulting symplectic structure is exact.  $\square$

*Proof of Corollary 1.6.* First suppose  $D$  is not negative definite. By Theorem 1.5,  $\omega$  being exact on the boundary implies  $D$  is a concave divisor after a symplectic deformation. If  $D$  is negative definite, then  $\omega$  is necessarily exact on the boundary with a unique lift of  $[\omega]$  to a relative second cohomology class. Moreover, the discussion after the proof of Lemma 3.1 implies that  $D$  satisfies negative GS criterion and hence  $D$  is a convex divisor.  $\square$

### 3.2. A flowchart

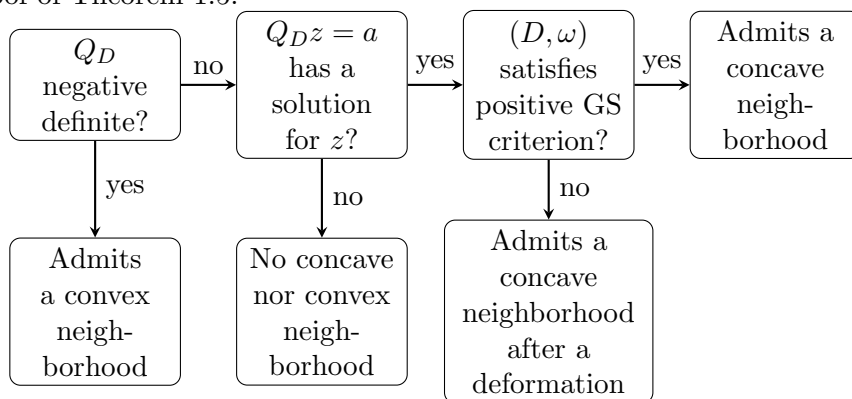
We offer a detailed explanation of the flowchart.

Given a divisor  $(D, \omega)$  (not necessarily  $\omega$ -orthogonal, see Proposition 2.6), we first consider whether  $Q_D$  is negative definite. If it is, then  $\omega$  is necessarily exact on the boundary and there is a unique solution  $z$  to the equation  $Q_D z = a$ , and all the entries of  $z$  are negative. Therefore,  $(D, \omega)$  satisfies the negative GS criterion and  $D$  is convex (Proposition 2.6).

If  $Q_D$  is not negative definite, we want to know whether  $\omega|_{\partial P(D)}$  is exact or not. This is equivalent to solving  $Q_D z = a$  for  $z$ . If there is no solution, then  $D$  cannot have a concave nor convex neighborhood.

If  $Q_D$  is not negative definite and  $\omega$  is exact on the boundary, the situation becomes a bit more complicated. There might be more than one solution for  $z$  (when  $Q_D$  is degenerate). If we are lucky that there is one solution  $z$  with all entries being positive, then  $D$  is concave (Proposition 2.6).

However, it is possible that all the solutions  $z$  have at least one entry being non-positive. Fortunately, we can choose an area vector  $\bar{a}$  such that there is a solution  $\bar{z}$  for  $Q_D \bar{z} = \bar{a}$  with all entries of  $\bar{z}$  being positive (Lemma 3.1). Geometrically, we can do inflation (Lemma 3.2) to deform the symplectic form in an arbitrarily small neighborhood of  $D$  such that  $(D, \bar{\omega})$  has area vector  $\bar{a}$ . Then,  $(D, \bar{\omega})$  is concave (Proposition 2.6). This is exactly the proof of Theorem 1.5.



From a more topological perspective, we could start with a graph  $\Gamma$  instead of  $(D, \omega)$ . There is a well-defined smooth manifold  $P(D)$  associated to  $\Gamma$  (but no symplectic structure). One can ask whether  $P(D)$  can be equipped with the structure of a concave/convex neighborhood such that  $D$  is symplectic. From this point of view, we have a trichotomy.

- 1)  $Q_\Gamma$  is negative definite

- 2)  $Im(Q_{\Gamma}) \cap \mathbb{R}_{>0}^k = \emptyset$   
 3) neither (1) or 2.

In case (1),  $P(D)$  admits a convex neighborhood structure. In case (2),  $P(D)$  does not admit a convex nor a concave neighborhood structure. In case (3),  $P(D)$  admits a concave neighborhood structure. Moreover, when  $P(D)$  admits a concave/convex neighborhood structure, the contact structure on  $\partial P(D)$  constructed by Gay-Stipsicz/McLean is independent of choices of  $\omega$  as long as  $\omega$  makes  $D$  symplectic, and the augmented graph of  $(D, \omega)$  satisfies positive/negative GS criteria (see Remarks 2.8, 2.9).

### 3.3. Examples of Concave Divisors

In this subsection, we are going to see five illuminating examples. The first one is the simplest kind of symplectic divisor. The second one illustrates that a concave divisor can admit a convex neighborhood. The third one is a frequently used example when studying Lefschetz fibration. The fourth one is a concave divisor with non-fillable contact structure on the boundary. The last one shows that the constructed contact structure on the boundary is not necessarily contactomorphic to the standard one that one might expect if the divisor is concave.

**Example 3.5.** A symplectic surface with self-intersection  $n$  admits a concave (resp convex) boundary when  $n > 0$  (resp  $n < 0$ ). When  $n = 0$ , a symplectic form cannot make both the surface symplectic and the restriction to boundary be exact so it has no convex or concave neighborhood. In fact, more is true, by a result of Eliashberg [4],  $\mathbb{S}^1 \times \mathbb{S}^2$  cannot be a convex boundary of any symplectic form on  $\mathbb{D}^2 \times \mathbb{S}^2$ . In contrast, although a symplectic torus with self-intersection zero has no concave nor convex neighborhood, a Lagrangian torus has self-intersection zero and has a convex neighborhood.

**Example 3.6.** ([22]) In [22], McDuff constructed a symplectic form on  $(S\Sigma_g \times [0, 1], \omega)$  such that it has disconnected convex boundary, where  $S\Sigma_g$  is a circle bundle of a genus  $g$  surface and  $g > 1$ . The contact structure near  $S\Sigma_g \times \{0\}$  is contactomorphic to the concave boundary near a self-intersection  $2g - 2$  symplectic genus  $g$  surface. The contact structure near  $S\Sigma_g \times \{1\}$  is contactomorphic to the convex boundary near a Lagrangian genus  $g$  surface. If one glues a symplectic closed disc bundle  $P(D)$  over a symplectic genus  $g$  surface  $D$  with  $(S\Sigma_g \times [0, 1], \omega)$  along  $S\Sigma_g \times \{0\}$ . One gets a plumbing of the surface with convex boundary. This suggests that

a symplectic genus  $g$  ( $g > 1$ ) surface can have both concave and convex neighborhood, depending on the symplectic form and the neighborhood.

**Example 3.7.** Suppose there is a symplectic Lefschetz fibration  $(X, \omega)$  over  $\mathbb{C}\mathbb{P}^1$  with generic fibre  $F$  and a symplectic section  $S$  of self-intersection  $-n$  ( $n \geq 0$ ). Let  $D = F \cup S$ , then the augmented graph of  $D$  always satisfies the positive GS criterion regardless the area weights of the surfaces. Then Proposition 2.6 shows that  $D$  is a concave divisor. In other words, the complement of a concave neighborhood of  $D$  is a convex filling of its boundary.

This fits well to the well-known fact that the complement of a regular neighborhood of  $D$  is a Stein domain. Moreover, this construction has been successfully used to find exotic Stein fillings [1].

**Lemma 3.8.** *Let  $(\Gamma, a)$  be an augmented graph satisfying the positive GS criterion and  $D$  be a realization. Suppose there are two genera zero vertices with self-intersection  $s_1, s_2$  such that either*

- (i) *they are adjacent to each other and  $s_1 > s_2 \geq 1$ , or*
- (ii) *they are not adjacent to each other with  $s_1 \geq 1$  and  $s_2 \geq 0$ .*

*Then,  $D$  is a concave divisor but not a capping divisor.*

*Proof.* Suppose on the contrary, the boundary has a convex fillings  $Y$ . Then, we can glue  $D$  with  $Y$  to obtain a closed symplectic 4 manifold  $W$ . By McDuff's theorem [21],  $W$  is rational or ruled and hence have  $b_2^+ = 1$ . For (i), the two spheres generates a positive two dimensional subspace of  $H_2(W)$  with respect to the intersection form. Thus, we get a contradiction. For (ii), it suffices to consider the case  $s_1 = 1$  and  $s_2 = 0$ . By the Theorem in [21], one can assume the sphere with self-intersection 1 represent the hyperplane class  $H$ , with respect to an orthonormal basis  $\{H, E_1, \dots, E_n\}$  for  $H_2(W)$ . The two spheres being disjoint implies the one with self-intersection 0 has homology class being a linear combination of exceptional classes. Since the sphere is symplectic, the linear combination is non-trivial. Thus, we get a contradiction.  $\square$

**Example 3.9.** Let  $\Gamma$  be the graph in Example 2.1.

$$Q_\Gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

Then the boundary fundamental group of  $\Gamma$  is the free group generated by  $e_1$  and  $e_2$  modulo the relations  $e_1 e_2^1 = e_2^1 e_1$ ,  $1 = e_1^2 e_2$  and  $1 = e_1 e_2$ . Therefore,

the boundary of the plumbing according to  $\Gamma$  has trivial fundamental group and hence diffeomorphic to a sphere. It is easily seen that the corresponding augmented graph  $(\Gamma, a)$  satisfies the positive GS criterion if and only if the area weights satisfy  $a_1 < a_2 < 2a_1$ , where  $a_i$  is the area weight of  $v_i$ . In other words, if  $a_1 < a_2 < 2a_1$ , by Proposition 2.6 and Lemma 3.8, we get an overtwisted contact structure on  $S^3$  ( $S^3$  has only one tight contact structure which is fillable).

## References

- [1] Anar Akhmedov, John B. Etnyre, Thomas E. Mark, and Ivan Smith, *A note on Stein fillings of contact manifolds*, Math. Res. Lett. **15** (2008), no. 6, 1127–1132.
- [2] Mohan Bhupal and Kaoru Ono, *Symplectic fillings of links of quotient surface singularities*, Nagoya Math. J. **207** (2012), 1–45.
- [3] Mohan Bhupal and Burak Ozbagci, *Canonical contact structures on some singularity links*, Bull. Lond. Math. Soc. **46** (2014), no. 3, 576–586.
- [4] Yakov Eliashberg, *Filling by holomorphic discs and its applications*, in: Geometry of Low-Dimensional Manifolds, 2 (Durham, 1989), Vol. 151 of London Math. Soc. Lecture Note Ser., pages 45–67. Cambridge Univ. Press, Cambridge, (1990).
- [5] Yakov Eliashberg, *A few remarks about symplectic filling*, Geom. Topol. **8** (2004), 277–293.
- [6] Yakov Eliashberg and Mikhael Gromov, *Convex symplectic manifolds*, in: Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989), Vol. 52 of Proc. Sympos. Pure Math., pages 135–162. Amer. Math. Soc., Providence, RI, (1991).
- [7] John B. Etnyre, *Symplectic convexity in low-dimensional topology*, Topology Appl. **88** (1998), no. 1-2, 3–25. Symplectic, Contact and Low-Dimensional Topology (Athens, GA, 1996).
- [8] John B. Etnyre, *On symplectic fillings*, Algebr. Geom. Topol. **4** (2004), 73–80.
- [9] John B. Etnyre and Ko Honda, *On symplectic cobordisms*. Math. Ann. **323** (2002), no. 1, 31–39.



- [10] David Gay and Thomas E. Mark, *Convex plumbings and Lefschetz fibrations*, J. Symplectic Geom. **11** (2013), no. 3, 363–375.
- [11] David T. Gay, *Correction to: “Open books and configurations of symplectic surfaces”*, Algebr. Geom. Topol. **3** (2003), 1275–1276 (electronic).
- [12] David T. Gay, *Open books and configurations of symplectic surfaces*, Algebr. Geom. Topol. **3** (2003), 569–586 (electronic).
- [13] David T. Gay and András I. Stipsicz, *Symplectic surgeries and normal surface singularities*, Algebr. Geom. Topol. **9** (2009), no. 4, 2203–2223.
- [14] Robert E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), no. 3, 527–595.
- [15] Michael Hutchings, *Lecture notes on embedded contact homology*, in: Contact and Symplectic Topology, Vol. 26 of Bolyai Soc. Math. Stud., pages 389–484. János Bolyai Math. Soc., Budapest, (2014).
- [16] François Lalonde and Dusa McDuff, *The classification of ruled symplectic 4-manifolds*, Math. Res. Lett. **3** (1996),no. 6, 769–778.
- [17] Tian-Jun Li and Cheuk Yu Mak, *Concave divisor with finite boundary fundamental group*.
- [18] Tian-Jun Li, Cheuk Yu Mak, and Kouichi Yasui, *Calabi-Yau caps, uniruled caps and symplectic fillings*, Proc. Lond. Math. Soc. (3) **114** (2017), no. 1, 159–187.
- [19] Tian-Jun Li and Michael Usher, *Symplectic forms and surfaces of negative square*, J. Symplectic Geom. **4** (2006), no. 1, 71–91.
- [20] Paolo Lisca, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. **360** (2008), no. 2, 765–799.
- [21] Dusa McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. **3** (1990), no. 3, 679–712.
- [22] Dusa McDuff, *Symplectic manifolds with contact type boundaries*, Invent. Math. **103** (1991), no. 3, 651–671.
- [23] Mark McLean, *The growth rate of symplectic homology and affine varieties*, Geom. Funct. Anal. **22** (2012), no. 2, 369–442.
- [24] Mark McLean, *Reeb orbits and the minimal discrepancy of an isolated singularity*, Invent. Math. **204** (2016), no. 2, 505–594.

- [25] Alan S. McRae, *Symplectic geometry for pairs of submanifolds*, Rocky Mountain J. Math. **35** (2005), no. 5, 1755–1764.
- [26] Hiroshi Ohta and Kaoru Ono, *Simple singularities and symplectic fillings*, J. Differential Geom. **69** (2005), no. 1, 1–42.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA  
MINNEAPOLIS, MN 55455, USA  
*E-mail address:* tjli@math.umn.edu

CENTRE FOR MATHEMATICAL SCIENCES  
UNIVERSITY OF CAMBRIDGE  
CB3 0WB, UK  
*E-mail address:* cym22@dpms.cam.ac.uk

RECEIVED MAY 3, 2018

ACCEPTED AUGUST 10, 2018