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QP-structures of degree 3 and CLWX 2-algebroids

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In this paper, we give the notion of a CLWX 2-algebroid and show that a QP-structure (symplectic NQ structure) of degree 3 gives rise to a CLWX 2-algebroid. This is the higher analogue of the result that a QP-structure of degree 2 gives rise to a Courant algebroid. A CLWX 2-algebroid can also be viewed as a categorified Courant algebroid. We show that one can obtain a Lie 3-algebra from a CLWX 2-algebroid. Furthermore, CLWX 2-algebroids are constructed from split Lie 2-algebroids and split Lie 2-bialgebroids.

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1. Introduction

This paper is motivated by the following questions:

- A QP-structure of degree 2 gives rise to a Courant algebroid. What is the geometric structure underlying a QP-structure of degree 3?
- What is a categorified Courant algebroid? Or, equivalently, what is the L_∞-analogue of a Courant algebroid?
- Split Lie 2-algebroids have become a useful tool to study problems related to NQ-manifolds. What is a split Lie 2-bialgebroid? What is the double of a split Lie 2-bialgebroid?

The CLWX 2-algebroid that we introduce in this paper provides answers of above questions.

A QP-manifold of degree n is a graded manifold equipped with a graded symplectic structure of degree n and a degree $n+1$ function satisfying the master equation. A QP-manifold is also called a symplectic NQ manifold in some literature, e.g. [Roy02]. QP-manifolds are very important in the topological field theory. Classical QP-manifolds of degree 1 are in one-to-one correspondence with Poisson manifolds. The 2-dimensional topological field theory constructed by AKSZ formulation [AKSZ] is the Poisson sigma model. Classical QP-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids [Roy02]. Courant algebroids can be used as target spaces for a general class of 3-dimensional topological field theory [Roy07B]. The notion of a Courant algebroid was introduced by Liu, Weinstein and Xu in [LWX97] in the study of the double of a Lie bialgebroid [MX]. An alternative definition was given in [Roy]. See the review article [KS] for more information. Roughly speaking, a Courant algebroid is a vector bundle, whose section space is a Leibniz algebra, together with an anchor map and a nondegenerate symmetric bilinear form, such that some compatibility conditions are satisfied. If a skew-symmetric bracket is used, in [RW98], the authors showed that the underlying algebraic structure of a Courant algebroid is a Lie 2-algebra, which is the categorification of a Lie algebra [BC, Roy07A].

In [IU], the authors studied QP-manifolds of degree 3 and derived a new 4-dimensional topological field theory by the AKSZ construction. The authors showed that a QP-manifold of degree 3 gives rise to a Lie algebroid up to homotopy (Ikeda-Uchino algebroid), and analyzed its algebraic and geometric structures.

In this paper, we restudy QP-manifolds of degree 3 and find that a QP-manifold of degree 3 can give rise to a more fruitful geometric structure, which we call a CLWX 2-algebroid. Roughly speaking, a CLWX 2-algebroid is a graded vector bundle $\mathcal{E} = E_0 \oplus E_{-1}$ over M, whose section space is a Leibniz 2-algebra, together with an anchor map $\rho : E_0 \longrightarrow TM$ and a nondegenerate graded symmetric bilinear form of degree 1, such that some compatibility conditions are satisfied. See Definition 3.1 for details. Since Leibniz 2-algebras are the categorification of Leibniz algebras, CLWX 2-algebroids can be viewed as the categorification of Courant algebroids. This viewpoint can also be justified by another fact: a Courant algebroid over a point is a quadratic Lie algebra while a CLWX 2-algebroid over a point is a quadratic Lie 2-algebra. Generalizing Li-Bland and Meinrenken's construction of a Courant algebroid from a coisotropic action of a quadratic Lie algebra on a manifold [LM], we construct a CLWX 2-algebroid, called the transformation CLWX 2-algebroid, using an action of a quadratic Lie 2-algebra on a manifold. We show that we can obtain a Lie 3-algebra (3-term L_∞-algebras) from a CLWX 2-algebroid if we use the skew-symmetric bracket. This is a higher analogue of Roytenberg and Weinstein's result given in [RW98].

Usually an NQ-manifold of degree n is considered as a Lie n-algebroid [Vor10]. In [SZ], the authors defined split Lie n-algebroids using graded vector bundles. The equivalence between the category of split Lie n-algebroids and the category of NQ-manifolds of degree n is given in [BP]. The language of split Lie n-algebroids has slowly become a useful tool to study problems related to NQ-manifolds [Jot, Jot18, Jot19]. There is a Courant algebroid structure on $A \oplus A^*$ associated to any Lie algebroid A. Similarly, we construct a CLWX 2-algebroid structure on $\mathcal{A} \oplus \mathcal{A}^*[1]$ associated to any split Lie 2-algebroid $(\mathcal{A} = A_0 \oplus A_{-1}, l_1, l_2, l_3, a)$. The notion of a Lie bialgebroid was introduced in [MX] as the infinitesimal object of a Poisson groupoid. Using the graded Poisson bracket on $T^*[3]E[1]$, where $E = A_0 \oplus A_{-1}^*$, we introduce the notion of a split Lie 2-bialgebroid. Furthermore, we show that there is a CLWX 2-algebroid structure on the double $\mathcal{A} \oplus \mathcal{A}^*[1]$ of a split Lie 2-bialgebroid $(\mathcal{A}, \mathcal{A}^*[1])$, which is a higher analogue of the fact that there is a Courant algebroid structure on the double $A \oplus A^*$ of a Lie bialgebroid (A, A^*). Recently, the notion of an L_∞-bialgebroid is introduced in [BV], which is a natural generalization of the Kravchenko's notion of an L_∞-bialgebra [Kra]. Even though the 2-term truncation of an L_∞-algebroid is a split Lie 2-algebroid, the 2-term truncation of an L_∞-bialgebroid is not a split Lie 2-bialgebroid.

The theory of Courant algebroids is very rich, and we can go on to study analogously for CLWX 2-algebroids. In [LSh], we introduce the notion of a weak Dirac struture of a CLWX 2-algebroid and establish its relation with Maurer-Cartan elements of certain homotopy Poisson algebra. In [She], transitive CLWX 2-algebroids are studied in detail, and it is shown that a quadratic Lie 2-algebroid admits a CLWX-extension if and only if its first Pontryagin class, which is represented by a closed 5-form, is trivial.

The paper is organized as follows. In Section 2, we recall QP-manifolds, Courant algebroids, Lie n-algebras, Leibniz 2-algebras and Lie 2-algebroids. In Section 3, we give the definition of a CLWX 2-algebroid and analyze its properties. We construct "transformation CLWX 2-algebroid" from a quadratic Lie 2-algebra action on a manifold. We show that a CLWX 2-algebroid gives rise to a Lie 3-algebra (Theorem 3.14). In Section 4, we construct a CLWX 2-algebroid from a split Lie 2-algebroid directly (Theorem 4.4). In Section 5, we show that the degree 3 QP-manifold $T^*[3]A[1]$ gives rise to a CLWX 2-algebroid through the derived bracket (Theorem 5.1). In Section 6, we give the definition of a split Lie 2-bialgebroid using the canonical graded Poisson bracket on $T^*[3]\mathcal{A}[1]$, where $\mathcal{A} = A_0 \oplus A_{-1}$ is a graded vector bundle. Then we show that the double $\mathcal{A} \oplus \mathcal{A}^*[1]$ of a split Lie 2-bialgebroid $(\mathcal{A}, \mathcal{A}^*[1])$ is a CLWX 2-algebroid (Theorem 6.2).

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2. Preliminaries

2.1. QP-manifolds and Courant algebroids

Recall that a graded manifold $\mathcal{M}$ is a sheaf of a graded commutative algebra over an ordinary smooth manifold M. The structure sheaf of $\mathcal{M}$ is locally isomorphic to a graded commutative algebra $C^\infty(U) \otimes S(V)$, where U is an ordinary local chart of M, $S(V)$ is the polynomial algebra over V and where $V := \sum_{i\geq 1} V_i$ is a graded vector space such that the dimension of V_i is finite for each i.

Definition 2.1. A graded manifold $\mathcal{M}$ equipped with a graded symplectic structure ω of degree n is called a ***P*-manifold** of degree n.

The structure sheaf $C^\infty(\mathcal{M})$ of a P-manifold becomes a graded Poisson algebra. The graded Poisson bracket is defined by

$$\{f, g\} = -\iota_{X_f}\iota_{X_g}\omega, \tag{1}$$

where $f, g \in C^\infty(\mathcal{M})$ and X_f is the Hamiltonian vector field of f, i.e. $\iota_{X_f}\omega = -df$. We recall the basic properties of the graded Poisson bracket,

$$\{f, g\} = -(-1)^{(|f|-n)(|g|-n)}\{g, f\}, \tag{2}$$

$$\{f, gh\} = \{f, g\}h + (-1)^{(|f|-n)|g|}g\{f, h\}, \tag{3}$$

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|-n)(|g|-n)}\{g, \{f, h\}\}, \tag{4}$$

where $|\, f \,|$ is the degree of f and n is the degree of the symplectic structure. The degree of the Poisson bracket is $-n$.

Definition 2.2. Let $(\mathcal{M}, \omega)$ be a P-manifold of degree n. A function $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ is called a ***Q*-structure**, if it is a solution of the classical master equation

$$\{\Theta, \Theta\} = 0. \tag{5}$$

The triple $(\mathcal{M}, \omega, \Theta)$ is called a ***QP*-manifold**.

It is well-known that QP-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids [Roy02, Theorem 4.5].

Definition 2.3. [LWX97] A **Courant algebroid** is a vector bundle E together with a bundle map $\rho : E \longrightarrow TM$, a nondegenerate symmetric bilinear form S, and an operation $\diamond : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E)$ such that for all $e_1, e_2, e_3 \in \Gamma(E)$, the following axioms hold:

(i) $(\Gamma(E), \diamond)$ is a Leibniz algebra;

(ii) $S(e_1 \diamond e_1, e_2) = \frac{1}{2}\rho(e_2)S(e_1, e_1)$;

(iii) $\rho(e_1)S(e_2, e_3) = S(e_1 \diamond e_2, e_3) + S(e_2, e_1 \diamond e_3)$.

Given a QP-manifold of degree 2, the Courant algebroid structure is obtained by the derived bracket using the Q-structure Θ [Roy02]. See [Get, Vor05] for more information about higher derived brackets.

For a vector bundle A, the graded manifold $T^*[2]A[1]$ is a P-manifold of degree 2. Let (x^i, ξ^a) be local coordinates on $A[1]$, we denote by $(x^i, \xi^a, \theta_a, p_i)$

the local coordinates on $T^*[2]A[1]$. About their degrees, we have

$$\text{degree}(x^i, \xi^a, \theta_a, p_i) = (0, 1, 1, 2).$$

The graded Poisson bracket satisfies

$$\{x^i, p_j\} = \delta^i_j = -\{p_j, x^i\}, \quad \{\xi^a, \theta_b\} = \delta^a_b = \{\theta_b, \xi^a\}.$$

A Lie algebroid structure on A is equivalent to a degree 3 function $\mu = \rho^i_b p_i \xi^b + \frac{1}{2}\mu^a_{bc}\xi^b\xi^c\theta_a$ such that $\{\mu, \mu\} = 0$. A **Lie bialgebroid** structure on A is given by a degree 3 function $\mu + \gamma$, which can be locally written as

$$\mu = \rho^i_b p_i \xi^b + \frac{1}{2}\mu^a_{bc}\xi^b\xi^c\theta_a, \quad \gamma = \varrho^{ib} p_i \theta_b + \frac{1}{2}\gamma^{bc}_a \xi^a \theta_b \theta_c,$$

and they satisfy

$$\{\mu + \gamma, \mu + \gamma\} = 0.$$

On $A \oplus A^*$, there is a natural Courant algebroid structure, in which the Q-structure Θ is exactly $\mu + \gamma$.

2.2. Lie n-algebras, Leibniz 2-algebras and Lie 2-algebroids

A Lie 2-algebra is a 2-vector space C equipped with a skew-symmetric bilinear functor, such that the Jacobi identity is controlled by a natural isomorphism, which satisfies the coherence law of its own. It is well-known that a Lie 2-algebra is equivalent to a 2-term L_∞-algebra [BC]. L_∞-algebras, also called strongly homotopy Lie algebras, were introduced in [Sta]. See [LM95, LS] for more details.

Definition 2.4. An **L_∞-algebra** is a graded vector space $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_{-i}$ equipped with a system $\{l_k|\ 1 \le k < \infty\}$ of linear maps $l_k : \wedge^k\mathfrak{g} \longrightarrow \mathfrak{g}$ with degree $\deg(l_k) = 2 - k$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign "Ksgn" is satisfied for all $n \ge 0$:

$$(6) \quad \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma) \times l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

Here the summation is taken over all $(i, n - i)$-unshuffles with $i \ge 1$.

People usually refer to an L_∞-algebra with $\mathfrak{g}_{-i} = 0$ for all $i \geq n$ and $i < 0$ as an n-term L_∞-algebra and we will call an n-term L_∞-algebra a Lie n-algebra.

As a model for "Leibniz algebras that satisfy Jacobi identity up to all higher homotopies", the notion of a strongly homotopy Leibniz algebra, or a Lod_∞-algebra was given in [Liv] by Livernet, which was further studied by Ammar, Poncin and Uchino in [AP, Uch]. In [SL], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and prove that the category of Leibniz 2-algebras and the category of 2-term Lod_∞-algebras are equivalent.

Definition 2.5. A **Leibniz 2-algebra** $\mathcal{V}$ consists of the following data:

- a complex of vector spaces $\mathcal{V} : V_{-1} \xrightarrow{\mathrm{d}} V_0$,
- bilinear maps $l_2 : V_{-i} \times V_{-j} \longrightarrow V_{-i-j}$, where $0 \leq i + j \leq 1$,
- a trilinear map $l_3 : V_0 \times V_0 \times V_0 \longrightarrow V_{-1}$,

such that for all $w, x, y, z \in V_0$ and $m, n \in V_{-1}$, the following equalities are satisfied:

(a) $\mathrm{d}l_2(x, m) = l_2(x, \mathrm{d}m)$,

(b) $\mathrm{d}l_2(m, x) = l_2(\mathrm{d}m, x)$,

(c) $l_2(\mathrm{d}m, n) = l_2(m, \mathrm{d}n)$,

(d) $\mathrm{d}l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(l_2(x, y), z) - l_2(y, l_2(x, z))$,

(e$_1$) $l_3(x, y, \mathrm{d}m) = l_2(x, l_2(y, m)) - l_2(l_2(x, y), m) - l_2(y, l_2(x, m))$,

(e$_2$) $l_3(x, \mathrm{d}m, y) = l_2(x, l_2(m, y)) - l_2(l_2(x, m), y) - l_2(m, l_2(x, y))$,

(e$_3$) $l_3(\mathrm{d}m, x, y) = l_2(m, l_2(x, y)) - l_2(l_2(m, x), y) - l_2(x, l_2(m, y))$,

(f) the Jacobiator identity:

$$\begin{aligned}
&l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) + l_2(l_3(w, x, y), z) \\
&- l_3(l_2(w, x), y, z) - l_3(x, l_2(w, y), z) - l_3(x, y, l_2(w, z)) \\
&+ l_3(w, l_2(x, y), z) + l_3(w, y, l_2(x, z)) - l_3(w, x, l_2(y, z)) = 0.
\end{aligned}$$

We usually denote a Leibniz 2-algebra by $(V_{-1}, V_0, \mathrm{d}, l_2, l_3)$, or simply by $\mathcal{V}$.

Definition 2.6. A **split Lie 2-algebroid** is a graded vector bundle $\mathcal{A} = A_0 \oplus A_{-1}$ over a manifold M equipped with a bundle map (called the anchor) $a : A_0 \longrightarrow TM$, and brackets $l_i : \Gamma(\wedge^i \mathcal{A}) \longrightarrow \Gamma(\mathcal{A})$ with degree $2 - i$ for $i = 1, 2, 3$, such that

(i) $(\Gamma(\mathcal{A}), l_1, l_2, l_3)$ is a Lie 2-algebra;

(ii) l_2 satisfies the Leibniz rule with respect to the anchor a:

$$l_2(X^0, fY) = fl_2(X^0, Y) + a(X^0)(f)Y,$$

for all $X^0 \in \Gamma(A_0),\ f \in C^\infty(M),\ Y \in \Gamma(\mathcal{A})$;

(iii) l_1 and l_3 are $C^\infty(M)$-linear.

Denote a Lie 2-algebroid by $(\mathcal{A}, l_1, l_2, l_3, a)$.

Remark 2.7. In our definition of a Lie n-algebroid, the section space is an L_∞-algebra. In [Bru], the author introduced a notion of an L_∞-algebroid, where the section space is a superized ($\mathbb{Z}_2$-graded) L_∞-algebra.

Lemma 2.8. *Let $(\mathcal{A}, l_1, l_2, l_3, a)$ be a Lie 2-algebroid. Then we have*

$$a \circ l_1 = 0, \tag{7}$$

$$a(l_2(X^0, Y^0)) = [a(X^0), a(Y^0)], \quad \forall X^0, Y^0 \in \Gamma(A_0). \tag{8}$$

Proof. On one hand, for all $X^0 \in \Gamma(A_0)$, $X^1 \in \Gamma(A_{-1})$ and $f \in C^\infty(M)$, we have

$$l_2(l_1(X^1), fX_0) = fl_2(l_1(X^1), X_0) + a(l_1(X^1))(f)X^0.$$

On the other hand, since $(\Gamma(\mathcal{A}), l_1, l_2, l_3)$ is a Lie 2-algebra, we have

$$l_2(l_1(X^1), fX_0) = l_1(l_2(X^1, fX_0)) = l_1(fl_2(X^1, X_0)) = fl_1(l_2(X^1, X_0)).$$

Therefore, we have $a(l_1(X^1))(f)X^0 = 0$, which implies that (7) holds.

For all $X^0, Y^0, Z^0 \in \Gamma(A_0)$ and $f \in C^\infty(M)$, by

$$\begin{aligned} &l_2(l_2(X^0, Y^0), fZ^0) + l_2(l_2(Y^0, fZ^0), X^0) + l_2(l_2(fZ^0, X^0), Y^0) \\ = &-l_3(X^0, Y^0, fZ^0) = -fl_3(X^0, Y^0, Z^0), \end{aligned}$$

we can deduce that (8) holds. □

3. CLWX 2-algebroids and Lie 3-algebras

3.1. CLWX 2-algebroids

In this subsection, we introduce the notion of a CLWX 2-algebroid (named after Courant-Liu-Weinstein-Xu) and analyze its properties.

Definition 3.1. A CLWX **2-algebroid** is a graded vector bundle $\mathcal{E} = E_{-1} \oplus E_0$ over M equipped with a non-degenerate graded symmetric bilinear form[1] S on $\mathcal{E}$, a bilinear operation $\diamond : \Gamma(E_{-i}) \times \Gamma(E_{-j}) \longrightarrow \Gamma(E_{-(i+j)})$, $0 \le i + j \le 1$, which is skewsymmetric on $\Gamma(E_0) \times \Gamma(E_0)$, an E_{-1}-valued 3-form Ω on E_0, two bundle maps $\partial : E_{-1} \longrightarrow E_0$ and $\rho : E_0 \longrightarrow TM$, such that E_{-1} and E_0 are isotropic and the following axioms are satisfied:

- (i) $(\Gamma(E_{-1}), \Gamma(E_0), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra;
- (ii) for all $e \in \Gamma(\mathcal{E})$, $e \diamond e = \frac{1}{2}\mathcal{D}S(e, e)$, where $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E_{-1})$ is defined by

$$S(\mathcal{D}f, e^0) = \rho(e^0)(f), \quad \forall e^0 \in \Gamma(E_0); \tag{9}$$

- (iii) for all $e^1_1, e^1_2 \in \Gamma(E_{-1})$, $S(\partial(e^1_1), e^1_2) = S(e^1_1, \partial(e^1_2))$;
- (iv) for all $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$, $\rho(e_1)S(e_2, e_3) = S(e_1 \diamond e_2, e_3) + S(e_2, e_1 \diamond e_3)$;
- (v) for all $e^0_1, e^0_2, e^0_3, e^0_4 \in \Gamma(E_0)$, $S(\Omega(e^0_1, e^0_2, e^0_3), e^0_4) = -S(e^0_3, \Omega(e^0_1, e^0_2, e^0_4))$.

Denote a CLWX 2-algebroid by $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$, or simply by $\mathcal{E}$. Since the section space of a CLWX 2-algebroid is a Leibniz 2-algebra, the section space of a Courant algebroid is a Leibniz algebra and Leibniz 2-algebras are the categorification of Leibniz algebras, we can view CLWX 2-algebroids as the categorification of Courant algebroids.

Remark 3.2. When M is a point, both E_0 and E_{-1} are vector spaces and the operators $\mathcal{D}$ and ρ vanish. In this case, the operation $\diamond$ is skew-symmetric. It follows that $(E_{-1}, E_0, \partial, \diamond, \Omega)$ is a Lie 2-algebra. Furthermore, S is a degree 1 pairing. Axioms (iii)-(iv) imply that S is invariant. Thus, what we obtain is a **metric (quadratic) Lie 2-algebra**. This is a higher analogue of the fact that a Courant algebroid over a point is a metric (quadratic) Lie algebra.

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[1] Here graded symmetry means $S(e^i, h^j) = (-1)^{ij} S(h^j, e^i)$ for all $e^i \in \Gamma(E_{-i})$, $h^j \in \Gamma(E_{-j})$.

See [BSZ] and [Kra] for more information about general notion of an L_∞-algebra with a degree k nondegenerate graded symmetric invariant bilinear form.

Remark 3.3. Note that via the nondegenerate bilinear form S, we obtain that $E_{-1} \cong E_0^*$. Comparing to the Lie algebroid up to homotopy introduced in [IU], the main difference is that our bilinear operation $\diamond$ is defined from $\Gamma(E_{-i}) \times \Gamma(E_{-j})$ to $\Gamma(E_{-(i+j)})$, $0 \leq i + j \leq 1$, while their bilinear operation $[\cdot, \cdot]$ is only defined from $\Gamma(E_0) \wedge \Gamma(E_0)$ to $\Gamma(E_0)$. Consequently, we have a Leibniz 2-algebra underlying a CLWX 2-algebroid, which is the higher analogue of the fact that there is a Leibniz algebra underlying a Courant algebroid. It turns out that the operation $\diamond : \Gamma(E_{-i}) \times \Gamma(E_{-j}) \longrightarrow \Gamma(E_{-(i+j)})$, $i + j = 1$, behaves more like the Courant-Dorfman bracket in a Courant algebroid. Thus, CLWX 2-algebroids are more fruitful structures than Lie algebroids up to homotopy.

Remark 3.4. The standard Courant algebroid $TM \oplus T^*M$ can be viewed as a CLWX 2-algebroid $(T^*[1]M, TM, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = 0)$, where S is the natural symmetric pairing between TM and T^*M, and $\diamond$ is the standard Dorfman bracket given by

$$(X + \alpha) \diamond (Y + \beta) = [X, Y] + L_X\beta - \iota_Y d\alpha, \tag{10}$$

for all $X,\ Y \in \mathfrak{X}(M),\ \alpha,\ \beta \in \Omega^1(M)$. Similarly, a Courant algebroid $A \oplus A^*$, in which A is a Lie algebroid and A^* is abelian, can also be viewed as a CLWX 2-algebroid. However, there is not a canonical way to obtain a CLWX 2-algebroid from an arbitrary Courant algebroid. See Remark 5.5 for an interpretation from the viewpoint of QP-manifolds.

Example 3.5. Let $H \in \Omega^4(M)$ be a closed 4-form, which can be viewed as a bundle map from $\wedge^3 TM \longrightarrow T^*M$. Then $(T^*[1]M, TM, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = H)$ is a CLWX 2-algebroid, where S and $\diamond$ are the same as the ones given in the above remark.

Lemma 3.6. *Let $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ be a CLWX 2-algebroid. For all $e_1, e_2 \in \Gamma(\mathcal{E})$, $e_1^0, e_2^0 \in \Gamma(E_0)$ and $f \in C^\infty(M)$, we have*

$$e_1 \diamond f e_2 = f(e_1 \diamond e_2) + \rho(e_1)(f)e_2, \tag{11}$$

$$(fe_1) \diamond e_2 = f(e_1 \diamond e_2) - \rho(e_2)(f)e_1 + S(e_1, e_2)\mathcal{D}f, \tag{12}$$

$$\rho(e_1^0 \diamond e_2^0) = [\rho(e_1^0), \rho(e_2^0)]. \tag{13}$$

Proof. By axiom (iv) in Definition 3.1 and the nondegeneracy of S, we have

$$\begin{aligned}S(e_1 \diamond fe_2, e_3) &= \rho(e_1)S(fe_2, e_3) - S(fe_2, e_1 \diamond e_3)\\ &= f\rho(e_1)S(e_2, e_3) + S(e_2, e_3)\rho(e_1)(f) - fS(e_2, e_1 \diamond e_3)\\ &= S(f(e_1 \diamond e_2), e_3) + S(\rho(e_1)(f)e_2, e_3),\end{aligned}$$

which implies that (11) holds.

By axiom (ii) in Definition 3.1, (12) follows immediately.

By (d) in Definition 2.5, for $f \in C^\infty(M)$, we have

$$\begin{aligned}f\partial\Omega(e_1^0, e_2^0, e_3^0) &= e_1^0 \diamond (e_2^0 \diamond fe_3^0) - (e_1^0 \diamond e_2^0) \diamond fe_3^0 - e_2^0 \diamond (e_1^0 \diamond fe_3^0)\\ &= f\big(e_1^0 \diamond (e_2^0 \diamond e_3^0) - (e_1^0 \diamond e_2^0) \diamond e_3^0 - e_2^0 \diamond (e_1^0 \diamond e_3^0)\big)\\ &\quad + \Big(\rho(e_1^0)\rho(e_2^0)(f) - \rho(e_2^0)\rho(e_1^0)(f) - \rho(e_1^0 \diamond e_2^0)(f)\Big)e_3^0\\ &= f\partial\Omega(e_1^0, e_2^0, e_3^0) + \Big([\rho(e_1^0), \rho(e_2^0)](f) - \rho(e_1^0 \diamond e_2^0)(f)\Big)e_3^0,\end{aligned}$$

which implies that (13) holds. □

Lemma 3.7. *Let $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ be a* CLWX *2-algebroid. For all $e^0 \in \Gamma(E_0)$ and $f \in C^\infty(M)$, we have*

$$\rho \circ \partial = 0, \tag{14}$$

$$\partial \circ \mathcal{D} = 0, \tag{15}$$

$$e^0 \diamond \mathcal{D}f = \mathcal{D}S(e^0, \mathcal{D}f), \tag{16}$$

$$\mathcal{D}f \diamond e^0 = 0. \tag{17}$$

Proof. By (c) in Definition 2.5 and (11), for all $e_1^1, e_2^1 \in \Gamma(E_{-1})$, we have

$$\begin{aligned}\rho(\partial(e_1^1))(f)e_2^1 &= (\partial(e_1^1)) \diamond (fe_2^1) - f\partial(e_1^1) \diamond e_2^1\\ &= e_1^1 \diamond \partial(fe_2^1) - f\partial(e_1^1) \diamond e_2^1 = 0,\end{aligned}$$

which imply that (14) holds.

By axiom (iii) in Definition 3.1 and (14), (15) follows immediately.

Finally, for all $h^0 \in \Gamma(E_0)$, by axiom (iv) in Definition 3.1 and (13), we have

$$\begin{aligned}\rho(e^0)\rho(h^0)(f) &= \rho(e^0)S(\mathcal{D}f, h^0) = S(e^0 \diamond \mathcal{D}f, h^0) + S(\mathcal{D}f, e^0 \diamond h^0)\\ &= S(e^0 \diamond \mathcal{D}f, h^0) + \rho(e^0 \diamond h^0)(f)\\ &= S(e^0 \diamond \mathcal{D}f, h^0) + \rho(e^0)\rho(h^0)(f) - \rho(h^0)\rho(e^0)(f).\end{aligned}$$

Hence,

$$S(e^0 \diamond \mathcal{D}f, h^0) = \rho(h^0)\rho(e^0)(f) = S(h^0, \mathcal{D}S(e^0, \mathcal{D}f)).$$

Since S is nondegenerate, we deduce that (16) holds.

By axiom (ii) in Definition 3.1, (17) follows immediately. □

3.2. Transformation CLWX 2-algebroids

One can obtain a transformation Courant algebroid from a coisotropic action of a quadratic Lie algebra on a manifold, see [LM] for more details. The notion of an L_∞-algebra action on a graded manifold was given by Mehta and Zambon in [MZ]. One can obtain a transformation L_∞-algebroid from an L_∞-algebra action. Here we give explicit formulas of a Lie 2-algebra action on a usual manifold and the corresponding transformation Lie 2-algebroid, by which we construct a CLWX 2-algebroid, called the transformation CLWX 2-algebroid.

Definition 3.8. An action of a Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3)$ on a manifold M is a linear map $\rho : \mathfrak{g}_0 \longrightarrow \mathfrak{X}(M)$ such that

$$\rho(l_2(x^0, y^0)) = [\rho(x^0), \rho(y^0)], \quad \forall x^0, y^0 \in \mathfrak{g}_0, \tag{18}$$

$$\rho \circ l_1 = 0. \tag{19}$$

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be an action of a Lie 2-algebra $\mathfrak{g}$ on a manifold M. Then ρ induces a bundle map from $M \times \mathfrak{g}_0$ to TM, which we use the same notation ρ. On the graded bundle $(M \times \mathfrak{g}_{-1}) \oplus (M \times \mathfrak{g}_0)$, define $\bar{l}_1 : M \times \mathfrak{g}_{-1} \longrightarrow M \times \mathfrak{g}_0$, $\bar{l}_2 : \Gamma(M \times \mathfrak{g}_{-i}) \times \Gamma(M \times \mathfrak{g}_{-j}) \longrightarrow \Gamma(M \times \mathfrak{g}_{-i-j}), 0 \le i + j \le 1$, and $\bar{l}_3 : \wedge^3(M \times \mathfrak{g}_0) \longrightarrow M \times \mathfrak{g}_{-1}$ by

$$\begin{cases} \bar{l}_1(X^1) = l_1(X^1), \\ \bar{l}_2(X^0, Y^0) = l_2(X^0, Y^0) + L_{\rho(X^0)}Y^0 - L_{\rho(Y^0)}X^0, \\ \bar{l}_2(X^0, Y^1) = -\bar{l}_2(Y^1, X^0) = l_2(X^0, Y^1) + L_{\rho(X^0)}Y^1, \\ \bar{l}_3(X^0, Y^0, Z^0) = l_3(X^0, Y^0, Z^0). \end{cases} \tag{20}$$

Then $(M \times \mathfrak{g}_{-1}, M \times \mathfrak{g}_0, \rho, \bar{l}_1, \bar{l}_2, \bar{l}_3)$ is a Lie 2-algebroid, called the transformation Lie 2-algebroid. See [MZ] for the general case of transformation L_∞-algebroids.

Now let $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3)$ be a quadratic Lie 2-algebra, i.e. there is a degree 1 nondegenerate graded symmetric invariant bilinear form S on $\mathfrak{g}$.

In this case $\mathfrak{g}_{-1}$ is isomorphic to $\mathfrak{g}_0^*$. More precisely, the invariant condition reads

$$\begin{aligned} S(l_1(x^1), y^1) &= S(x^1, l_1(y^1)), \\ S(l_2(x^0, y^0), z^1) &= -S(y^0, l_2(x^0, z^1)), \\ S(l_3(x^0, y^0, z^0), w^0) &= -S(z^0, l_3(x^0, y^0, w^0)), \end{aligned}$$

for all $x^0, y^0, z^0, w^0 \in \mathfrak{g}_0$ and $x^1, y^1, z^1 \in \mathfrak{g}_{-1}$. Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be an action of $\mathfrak{g}$ on M. With the same notations as above, on the graded bundle $(M \times \mathfrak{g}_{-1}) \oplus (M \times \mathfrak{g}_0)$, we define the operation $\diamond : \Gamma(M \times \mathfrak{g}_{-i}) \times \Gamma(M \times \mathfrak{g}_{-j}) \longrightarrow \Gamma(M \times \mathfrak{g}_{-i-j}), 0 \leq i + j \leq 1$, by

$$\begin{cases} X^0 \diamond Y^0 = \bar{l}_2(X^0, Y^0), \\ X^0 \diamond Y^1 = \bar{l}_2(X^0, Y^1) + \rho^* S(dX^0, Y^1), \\ Y^1 \diamond X^0 = \bar{l}_2(Y^1, X^0) + \rho^* S(dY^1, X^0), \end{cases} \tag{21}$$

for all $X^0, Y^0 \in \Gamma(M \times \mathfrak{g}_0)$ and $Y^1 \in \Gamma(M \times \mathfrak{g}_{-1})$.

Theorem 3.9. *Let $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3)$ be a quadratic Lie 2-algebra with a degree 1 nondegenerate graded symmetric invariant bilinear form S on $\mathfrak{g}$ and $\rho : \mathfrak{g}_0 \longrightarrow TM$ an action of $\mathfrak{g}$ on M such that*

$$l_1 \circ \rho^* = 0, \tag{22}$$

where $\rho^* : T^*M \longrightarrow M \times \mathfrak{g}_{-1}$ is defined by

$$S(\rho^*(\alpha), X^0) = \langle \alpha, \rho(X^0) \rangle, \quad \forall X^0 \in \Gamma(M \times \mathfrak{g}_0), \alpha \in \Omega^1(M).$$

Then $(M \times \mathfrak{g}_{-1}, M \times \mathfrak{g}_0, \partial = \bar{l}_1, \rho, S, \diamond, \Omega = \bar{l}_3)$ is a CLWX 2-algebroid, where $\diamond$ is given by (21).

We call this CLWX 2-algebroid the **transformation CLWX 2-algebroid.**

Proof. Obviously, for all $X^0 \in \Gamma(M \times \mathfrak{g}_0)$ and $Y^1 \in \Gamma(M \times \mathfrak{g}_1)$, we have

$$X^0 \diamond Y^1 + Y^1 \diamond X^0 = \rho^*(S(dX^0, Y^1) + S(X^0, dY^1)) = \rho^* dS(X^0, Y^1),$$

which implies that axiom (ii) in Definition 3.1 holds.

For all $X^0, Y^0 \in \Gamma(M \times \mathfrak{g}_0)$ and $Z^1 \in \Gamma(M \times \mathfrak{g}_1)$, since S is an invariant bilinear form on $\mathfrak{g}$, we have

$$\begin{aligned}
&S(X^0 \diamond Y^0, Z^1) + S(Y^0, X^0 \diamond Z^1)\\
&= S(l_2(X^0, Y^0) + L_{\rho(X^0)}Y^0 - L_{\rho(Y^0)}X^0, Z^1)\\
&\quad + S(Y^0, l_2(X^0, Z^1) + L_{\rho(X^0)}Z^1 + \rho^* S(dX^0, Z^1))\\
&= S(L_{\rho(X^0)}Y^0, Z^1) + S(Y^0, L_{\rho(X^0)}Z^1)\\
&= \rho(X^0)S(Y^0, Z^1),
\end{aligned}$$

which implies that axiom (iv) in Definition 3.1 holds.

Also by the fact that S is an invariant bilinear form on $\mathfrak{g}$, axioms (iii) and (v) in Definition 3.1 hold naturally.

Finally, we show that $(\Gamma(M \times \mathfrak{g}_{-1}), \Gamma(M \times \mathfrak{g}_0), \partial = \bar{l}_1, \diamond, \Omega = \bar{l}_3)$ is a Leibniz 2-algebra. By (22), we have

$$\begin{aligned}
\partial(X^0 \diamond X^1) &= \bar{l}_1(\bar{l}_2(X^0, X^1) + \rho^* S(dX^0, X^1))\\
&= \bar{l}_1(\bar{l}_2(X^0, X^1)) = \bar{l}_2(X^0, \bar{l}_1(X^1)) = X^0 \diamond \partial(X^1),
\end{aligned}$$

which implies that Condition (a) in Definition 2.5 holds. Similarly, we can deduce that Condition (b) holds. Since S is an invariant bilinear form on $\mathfrak{g}$, we have

$$\begin{aligned}
\partial(X^1) \diamond Y^1 &= \bar{l}_2(\bar{l}_1(X^1), Y^1) + \rho^* S(d\bar{l}_1(X^1), Y^1)\\
&= \bar{l}_2(X^1, \bar{l}_1(Y^1)) + \rho^* S(dX^1, \bar{l}_1(Y^1)) = X^1 \diamond \partial(Y^1),
\end{aligned}$$

which implies that Condition (c) in Definition 2.5 holds.

Since for all $X^0, Y^0 \in \Gamma(M \times \mathfrak{g}_0)$, we have $X^0 \diamond Y^0 = \bar{l}_2(X^0, Y^0)$. Thus, Condition (d) in Definition 2.5 holds naturally.

For all $X^0, Y^0 \in \Gamma(M \times \mathfrak{g}_0)$ and $Z^1 \in \Gamma(M \times \mathfrak{g}_{-1})$, by axiom (iv) in Definition 3.1 that we have proved above, we have

$$\begin{aligned}
&S\Big(X^0 \diamond (Y^0 \diamond Z^1) - (X^0 \diamond Y^0) \diamond Z^1 - Y^0 \diamond (X^0 \diamond Z^1)\\
&\quad - \Omega(X^0, Y^0, \partial(Z^1)), Z^0\Big)\\
&= S\Big(X^0 \diamond (\bar{l}_2(Y^0, Z^1) + \rho^* S(dY^0, Z^1)) - \bar{l}_2(X^0, Y^0) \diamond Z^1\\
&\quad - Y^0 \diamond (\bar{l}_2(X^0, Z^1) + \rho^* S(dX^0, Z^1)) - \bar{l}_3(X^0, Y^0, \bar{l}_1(Z^1)), Z^0\Big)
\end{aligned}$$

$$
\begin{aligned}
&= S\Big(\bar{l}_2(X^0,\bar{l}_2(Y^0,Z^1)) + \rho^*S(dX^0,\bar{l}_2(Y^0,Z^1)) + X^0 \diamond \rho^*S(dY^0,Z^1)\\
&\qquad - \bar{l}_2(\bar{l}_2(X^0,Y^0),Z^1) - \rho^*S(d\bar{l}_2(X^0,Y^0),Z^1) - \bar{l}_2(Y^0,\bar{l}_2(X^0,Z^1))\\
&\qquad - \rho^*S(dY^0,\bar{l}_2(X^0,Z^1)) - Y^0 \diamond \rho^*S(dX^0,Z^1) - \bar{l}_3(X^0,Y^0,\bar{l}_1(Z^1)),Z^0\Big)\\
&= S\Big(\rho^*S(dX^0,\bar{l}_2(Y^0,Z^1)) + X^0 \diamond \rho^*S(dY^0,Z^1) - \rho^*S(d\bar{l}_2(X^0,Y^0),Z^1)\\
&\qquad - \rho^*S(dY^0,\bar{l}_2(X^0,Z^1)) - Y^0 \diamond \rho^*S(dX^0,Z^1),Z^0\Big)\\
&= S(L_{\rho(Z^0)}X^0, Y^0 \diamond Z^1 - \rho^*S(dY^0,Z^1)) + \rho(X^0)S(L_{\rho(Z^0)}Y^0,Z^1)\\
&\quad - S(L_{[\rho(X^0),\rho(Z^0)]}Y^0,Z^1) - S(L_{\rho(Z^0)}\bar{l}_2(X^0,Y^0),Z^1)\\
&\quad - S(L_{\rho(Z^0)}Y^0, X^0 \diamond Z^1 - \rho^*S(dX^0,Z^1)) - \rho(Y^0)S(L_{\rho(Z^0)}X^0,Z^1)\\
&\quad + S(L_{[\rho(Y^0),\rho(Z^0)]}X^0,Z^1)\\
&= -S(\bar{l}_2(Y^0,L_{\rho(Z^0)}X^0),Z^1) - S(L_{\rho(L_{\rho(Z^0)}X^0)}Y^0,Z^1)\\
&\quad - S(L_{[\rho(X^0),\rho(Z^0)]}Y^0,Z^1) - S(L_{\rho(Z^0)}\bar{l}_2(X^0,Y^0),Z^1)\\
&\quad + S(\bar{l}_2(X^0,L_{\rho(Z^0)}Y^0),Z^1) + S(L_{\rho(L_{\rho(Z^0)}Y^0)}X^0,Z^1)\\
&\quad + S(L_{[\rho(Y^0),\rho(Z^0)]}X^0,Z^1)\\
&= -S\Big(L_{\rho(Z^0)}\bar{l}_2(X^0,Y^0) - \bar{l}_2(L_{\rho(Z^0)}X^0,Y^0) - \bar{l}_2(X^0,L_{\rho(Z^0)}Y^0)\\
&\qquad + L_{\rho(L_{\rho(Z^0)}X^0)}Y^0 - L_{\rho(L_{\rho(Z^0)}Y^0)}X^0 + L_{[\rho(X^0),\rho(Z^0)]}Y^0\\
&\qquad - L_{[\rho(Y^0),\rho(Z^0)]}X^0, Z^1\Big)\\
&= 0.
\end{aligned}
$$

The last equality is due to the following Lemma 3.10. Thus, Condition (e_1) in Definition 2.5 holds. Similarly we can show that Conditions (e_2), (e_3) and (f) in Definition 2.5 hold. Thus, $(\Gamma(M\times\mathfrak{g}_{-1}),\Gamma(M\times\mathfrak{g}_0),\partial=\bar{l}_1,\diamond,\Omega=\bar{l}_3)$ is a Leibniz 2-algebra. The proof is finished. ☐

Lemma 3.10. *For all $Z\in\mathfrak{X}(M)$ and $X,Y\in\Gamma(M\times\mathfrak{g}_0)$, we have*

$$
Z\bar{l}_2(X,Y) - \bar{l}_2(L_ZX,Y) - \bar{l}_2(X,L_ZY) + L_{\rho(L_ZX)}Y - L_{\rho(L_ZY)}X + L_{[\rho(X),Z]}Y - L_{[\rho(Y),Z]}X = 0. \tag{23}
$$

Proof. If $X,Y\in\mathfrak{g}$ are constant sections, it is obvious that the above equality holds. Generally, since $\Gamma(M\times\mathfrak{g}_0)=C^\infty(M)\otimes\mathfrak{g}_0$, we can assume that $X=fu, Y=gv$, where $u,v\in\mathfrak{g}_0$ are constant sections and $f,g\in C^\infty(M)$, then it is straightforward to deduce the above equality. ☐

3.3. Lie 3-algebras

In this subsection we prove that we can obtain a Lie 3-algebra from a CLWX 2-algebroid via skewsymmetrization.

We introduce a skew-symmetric bracket on $\Gamma(\mathcal{E})$,

$$\llbracket e_1, e_2 \rrbracket = \frac{1}{2}(e_1 \diamond e_2 - e_2 \diamond e_1), \quad \forall\, e_1, e_2 \in \Gamma(\mathcal{E}), \tag{24}$$

which is the skew-symmetrization of $\diamond$. By axiom (ii) in Definition 3.1, (24) can be written by

$$\llbracket e_1, e_2 \rrbracket = e_1 \diamond e_2 - \frac{1}{2}\mathcal{D}S(e_1, e_2). \tag{25}$$

Lemma 3.11. *Let* $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ *be a* CLWX 2*-algebroid. For all* $e^0 \in \Gamma(E_0), e^1, e^1_1, e^1_2 \in \Gamma(E_{-1})$ *and* $f \in C^\infty(M)$*, we have*

$$\partial \left\llbracket e^0, e^1 \right\rrbracket = \left\llbracket e^0, \partial(e^1) \right\rrbracket, \tag{26}$$

$$\left\llbracket \partial(e^1_1), e^1_2 \right\rrbracket = \left\llbracket e^1_1, \partial(e^1_2) \right\rrbracket, \tag{27}$$

$$\left\llbracket e^0, \mathcal{D}f \right\rrbracket = \frac{1}{2}\mathcal{D}S(e^0, \mathcal{D}f). \tag{28}$$

Proof. By (a) in Definition 2.5 and (15), we have

$$\partial \left\llbracket e^0, e^1 \right\rrbracket = \partial(e^0 \diamond e^1) - \frac{1}{2}\partial \circ \mathcal{D}S(e_0, e_1) = e^0 \diamond \partial(e^1),$$

which implies that (26) holds.

By (c) in Definition 2.5 and axiom (iii) in Definition 3.1, (27) follows immediately.

By (16) and (17), (28) is obvious. □

For simplicity, for all $e_i \in \Gamma(\mathcal{E}), i = 1, 2, 3$, we let

$$K(e_1, e_2, e_3) = e_1 \diamond (e_2 \diamond e_3) - (e_1 \diamond e_2) \diamond e_3 - e_2 \diamond (e_1 \diamond e_3), \tag{29}$$

$$J(e_1, e_2, e_3) = \llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \llbracket \llbracket e_2, e_3 \rrbracket, e_1 \rrbracket + \llbracket \llbracket e_3, e_1 \rrbracket, e_2 \rrbracket. \tag{30}$$

By (13) and (17), we can deduce that K is totally skew-symmetric.

Lemma 3.12. *Let* $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ *be a* CLWX *2-algebroid. For all* $e^0, e^0_1, e^0_2, e^0_3 \in \Gamma(E_0)$, $e^1, e^1_1, e^1_2 \in \Gamma(E_{-1})$, *we have*

$$J(e^0_1, e^0_2, e^0_3) = -\partial\Omega(e^0_1, e^0_2, e^0_3), \tag{31}$$

$$J(e^0_1, e^0_2, e^1) = \mathcal{D}T(e^0_1, e^0_2, e^1) - \Omega(e^0_1, e^0_2, \partial e^1), \tag{32}$$

$$T(\partial e^1_1, e^0, e^1_2) = -T(\partial e^1_2, e^0, e^1_1), \tag{33}$$

where the totally skew-symmetric $T : \Gamma(E_0) \times \Gamma(E_0) \times \Gamma(E_{-1}) \longrightarrow C^\infty(M)$ *is given by*

$$T(e^0_1, e^0_2, e^1) = \frac{1}{6}\big(S(e^0_1, [\![e^0_2, e^1]\!]) + S(e^1, [\![e^0_1, e^0_2]\!]) + S(e^0_2, [\![e^1, e^0_1]\!])\big). \tag{34}$$

Proof. It is obvious that $J(e^0_1, e^0_2, e^0_3) = -K(e^0_1, e^0_2, e^0_3)$, which implies that (31) holds.

By straightforward computations, we have

$$K(e^0_1, e^0_2, e^1) = -J(e^0_1, e^0_2, e^1) + R(e^0_1, e^0_2, e^1),$$

where

$$\begin{aligned} R(e^0_1, e^0_2, e^1) = \frac{1}{2}\big(&\mathcal{D}S(e^0_1, [\![e^0_2, e^1]\!]) - \mathcal{D}S(e^1, [\![e^0_1, e^0_2]\!]) - \mathcal{D}S(e^0_2, [\![e^0_1, e^1]\!]) \\ &+ \mathcal{D}S(e^0_1, \mathcal{D}S(e^1, e^0_2)) - \mathcal{D}S(e^0_2, \mathcal{D}S(e^1, e^0_1))\big). \end{aligned}$$

Similarly, we have

$$K(e^1, e^0_1, e^0_2) = -J(e^1, e^0_1, e^0_2) + R(e^1, e^0_1, e^0_2),$$

where

$$\begin{aligned} R(e^1, e^0_1, e^0_2) = \frac{1}{2}\Big(&\mathcal{D}S(e^0_2, [\![e^0_1, e^1]\!]) + \mathcal{D}S(e^1, [\![e^0_1, e^0_2]\!]) \\ &+ \mathcal{D}S(e^0_1, [\![e^0_2, e^1]\!]) - \mathcal{D}S(e^0_1, \mathcal{D}S(e^1, e^0_2))\Big), \end{aligned}$$

and

$$K(e^0_2, e^1, e^0_1) = -J(e^0_2, e^1, e^0_1) + R(e^0_2, e^1, e^0_1),$$

where

$$\begin{aligned} R(e^0_2, e^1, e^0_1) = \frac{1}{2}\Big(&-\mathcal{D}S(e^0_2, [\![e^0_1, e^1]\!]) - \mathcal{D}S(e^0_1, [\![e^0_2, e^1]\!]) \\ &+ \mathcal{D}S(e^1, [\![e^0_1, e^0_2]\!]) + \mathcal{D}S(e^0_2, \mathcal{D}S(e^1, e^0_1))\Big). \end{aligned}$$

Since both J and K are completely skew-symmetric, we have

$$3K(e^0_1, e^0_2, e^1) = -3J(e^0_1, e^0_2, e^1) + 3\mathcal{D}T(e^0_1, e^0_2, e^1).$$

Then by axiom (e_1) in Definition 2.5, we have

$$K(e^0_1, e^0_2, e^1) = \Omega(e^0_1, e^0_2, \partial e^1),$$

which implies that (32) holds.

Finally, by axiom (iii) in the Definition 3.1, (26) and (27), we have

$$\begin{aligned}
&T(\partial(e^1_1), e^0, e^1_2)\\
&= \frac{1}{6}\big(S(\partial(e^1_1), [\![e^0, e^1_2]\!]) + S(e^0, [\![e^1_2, \partial(e^1_1)]\!]) + S(e^1_2, [\![\partial(e^1_1), e^0]\!])\big)\\
&= \frac{1}{6}\big(S(e^1_1, [\![e^0, \partial(e^1_2)]\!]) + S(e^0, [\![\partial(e^1_2), e^1_1]\!]) + S(\partial(e^1_2), [\![e^1_1, e^0]\!])\big)\\
&= -T(\partial e^1_2, e^0, e^1_1).
\end{aligned}$$

The proof is finished. □

Lemma 3.13. *For all $e^1 \in \Gamma(E_{-1})$ and $e^0_1, e^0_2, e^0_3, e^0_4 \in \Gamma(E_0)$, we have*

$$\begin{aligned}
&\Omega([\![e^0_1, e^0_2]\!], e^0_3, e^0_4) - \Omega([\![e^0_1, e^0_3]\!], e^0_2, e^0_4) + \Omega([\![e^0_1, e^0_4]\!], e^0_2, e^0_3)\\
&+ \Omega([\![e^0_2, e^0_3]\!], e^0_1, e^0_4) - \Omega([\![e^0_2, e^0_4]\!], e^0_1, e^0_3) + \Omega([\![e^0_3, e^0_4]\!], e^0_1, e^0_2)\\
&- [\![\Omega(e^0_1, e^0_2, e^0_3), e^0_4]\!] - [\![\Omega(e^0_1, e^0_3, e^0_4), e^0_2]\!] + [\![\Omega(e^0_1, e^0_2, e^0_4), e^0_3]\!]\\
&+ [\![\Omega(e^0_2, e^0_3, e^0_4), e^0_1]\!] + \mathcal{D}S(\Omega(e^0_1, e^0_2, e^0_3), e^0_4) = 0,
\end{aligned}$$

and

$$2\mathbf{J} + \mathbf{K} = -S(\Omega(\partial e^1, e^0_2, e^0_3), e^0_4),$$

where

$$\begin{aligned}
\mathbf{J} &= S(J(e^1, e^0_2, e^0_3), e^0_4) - S(J(e^1, e^0_2, e^0_4), e^0_3) + S(J(e^1, e^0_3, e^0_4), e^0_2)\\
&\quad + 3S(\Omega(\partial e^1, e^0_2, e^0_3), e^0_4),\\
\mathbf{K} &= S([\![e^1, e^0_2]\!], [\![e^0_3, e^0_4]\!]) - S([\![e^1, e^0_3]\!], [\![e^0_2, e^0_4]\!]) + S([\![e^1, e^0_4]\!], [\![e^0_2, e^0_3]\!]).
\end{aligned}$$

Proof. By axiom (f) in Definition 2.5, axiom (v) in Definition 3.1 and (25), we have

$$\begin{aligned}
&\Omega(\llbracket e_1^0, e_2^0\rrbracket, e_3^0, e_4^0) - \Omega(\llbracket e_1^0, e_3^0\rrbracket, e_2^0, e_4^0) + \Omega(\llbracket e_1^0, e_4^0\rrbracket, e_2^0, e_3^0)\\
&+ \Omega(\llbracket e_2^0, e_3^0\rrbracket, e_1^0, e_4^0) - \Omega(\llbracket e_2^0, e_4^0\rrbracket, e_1^0, e_3^0) + \Omega(\llbracket e_3^0, e_4^0\rrbracket, e_1^0, e_2^0)\\
&- \llbracket\Omega(e_1^0, e_2^0, e_3^0), e_4^0\rrbracket - \llbracket\Omega(e_1^0, e_3^0, e_4^0), e_2^0\rrbracket + \llbracket\Omega(e_1^0, e_2^0, e_4^0), e_3^0\rrbracket\\
&+ \llbracket\Omega(e_2^0, e_3^0, e_4^0), e_1^0\rrbracket + \mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0)\\
=& \Omega(e_1^0 \diamond e_2^0, e_3^0, e_4^0) - \Omega(e_1^0 \diamond e_3^0, e_2^0, e_4^0) + \Omega(e_1^0 \diamond e_4^0, e_2^0, e_3^0) + \Omega(e_2^0 \diamond e_3^0, e_1^0, e_4^0)\\
&- \Omega(e_2^0 \diamond e_4^0, e_1^0, e_3^0) + \Omega(e_3^0 \diamond e_4^0, e_1^0, e_2^0) - \Omega(e_1^0, e_2^0, e_3^0) \diamond e_4^0\\
&+ \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0) + e_2^0 \diamond \Omega(e_1^0, e_3^0, e_4^0) - \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_3^0, e_4^0), e_2^0)\\
&- e_3^0 \diamond \Omega(e_1^0, e_2^0, e_4^0) + \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_2^0, e_4^0), e_3^0) - e_1^0 \diamond \Omega(e_2^0, e_3^0, e_4^0)\\
&+ \frac{1}{2}\mathcal{D}S(\Omega(e_2^0, e_3^0, e_4^0), e_1^0) + \mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0)\\
=& \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0) - \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_3^0, e_4^0), e_2^0) + \frac{1}{2}\mathcal{D}S(\Omega(e_1^0, e_2^0, e_4^0), e_3^0)\\
&+ \frac{1}{2}\mathcal{D}S(\Omega(e_2^0, e_3^0, e_4^0), e_1^0) + \mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0)\\
=& -\mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0) + \mathcal{D}S(\Omega(e_1^0, e_2^0, e_3^0), e_4^0) = 0.
\end{aligned}$$

The second equality can be proved by the same method in the proof of Lemma 2.5.2 in [Roy]. We omit the details. □

Let $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ be a CLWX 2-algebroid. Consider the graded vector space $\mathfrak{e} = \mathfrak{e}_{-2} \oplus \mathfrak{e}_{-1} \oplus \mathfrak{e}_0$, where $\mathfrak{e}_0 = \Gamma(E_0)$, $\mathfrak{e}_{-1} = \Gamma(E_{-1})$ and $\mathfrak{e}_{-2} = C^\infty(M)$.

Theorem 3.14. *A* CLWX 2*-algebroid* $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ *gives rise to a Lie* 3*-algebra* $(\mathfrak{e}, l_1, l_2, l_3, l_4)$*, where* l_i *are given by the following formulas:*

$$\begin{aligned}
l_1(f) &= \mathcal{D}(f), && \forall\ f \in C^\infty(M),\\
l_1(e^1) &= \partial(e^1), && \forall\ e^1 \in \Gamma(E_{-1}),\\
l_2(e_1^0 \wedge e_2^0) &= \llbracket e_1^0, e_2^0\rrbracket, && \forall\ e_1^0, e_2^0 \in \Gamma(E_0),\\
l_2(e^0 \wedge e^1) &= \llbracket e^0, e^1\rrbracket, && \forall\ e^0 \in \Gamma(E_0), e^1 \in \Gamma(E_{-1}),\\
l_2(e^0 \wedge f) &= \frac{1}{2}S(e^0, \mathcal{D}f), && \forall\ e^0 \in \Gamma(E_0), f \in C^\infty(M),\\
l_2(e_1^1 \vee e_2^1) &= 0, && \forall\ e_1^1, e_2^1 \in \Gamma(E_{-1}),
\end{aligned}$$

$$
\begin{aligned}
l_3(e^0_1\wedge e^0_2\wedge e^0_3) &= \Omega(e^0_1,e^0_2,e^0_3), && \forall\, e^0_1,e^0_2,e^0_3\in\Gamma(E_0),\\
l_3(e^0_1\wedge e^0_2\wedge e^1) &= -T(e^0_1,e^0_2,e^1), && \forall\, e^0_1,e^0_2\in\Gamma(E_0), e^1\in\Gamma(E_{-1}),\\
l_4(e^0_1\wedge e^0_2\wedge e^0_3\wedge e^0_4) &= \overline{\Omega}(e^0_1,e^0_2,e^0_3,e^0_4), && \forall\, e^0_1,e^0_2,e^0_3,e^0_4\in\Gamma(E_0),
\end{aligned}
$$

where $\overline{\Omega}:\wedge^4\Gamma(E_0)\longrightarrow C^\infty(M)$ *is given by*

$$\overline{\Omega}(e^0_1,e^0_2,e^0_3,e^0_4)=S(\Omega(e^0_1,e^0_2,e^0_3),e^0_4).$$

Proof. We need to show that (6) holds for $n=1,2,3,4,5$. For $n=1$, we need to show that $l_1^2=0$, which follows from $\partial\circ\mathcal{D}=0$.

For $n=2$, we need to verify that for all $x_i\in\mathfrak{e}$,

$$-l_2(l_1(x_1),x_2)+(-1)^{|x_1||x_2|}l_2(l_1(x_2),x_1)+l_1l_2(x_1,x_2)=0. \tag{35}$$

For $x_1=e^0\in\mathfrak{e}_0, x_2=f\in\mathfrak{e}_{-2}$, by (28), we have

$$
\begin{aligned}
l_2(\mathcal{D}f,e^0)+\mathcal{D}l_2(e^0,f) &= -\left[\!\left[e^0,\mathcal{D}f\right]\!\right]+\frac{1}{2}\mathcal{D}S(e^0,\mathcal{D}f)\\
&= -\frac{1}{2}\mathcal{D}S(e^0,\mathcal{D}f)+\frac{1}{2}\mathcal{D}S(e^0,\mathcal{D}f)\\
&= 0,
\end{aligned}
$$

which implies that (35) holds for $x_1\in\mathfrak{e}_0$ and $x_2\in\mathfrak{e}_{-2}$. The other cases can be proved similarly and we omit the details.

For $n=3$, we need to prove that for all $x_i\in\mathfrak{e}$,

$$
\begin{aligned}
& l_3(l_1(x_1),x_2,x_3)-(-1)^{|x_1||x_2|}l_3(l_1(x_2),x_1,x_3)\\
&+(-1)^{|x_3|(|x_1|+|x_2|)}l_3(l_1(x_3),x_1,x_2)+l_2(l_2(x_1,x_2),x_3)\\
&-(-1)^{|x_2||x_3|}l_2(l_2(x_1,x_3),x_2)+(-1)^{|x_1|(|x_2|+|x_3|)}l_2(l_2(x_2,x_3),x_1)\\
&+l_1l_3(x_1,x_2,x_3)\\
&=0.
\end{aligned} \tag{36}
$$

By (31), we can deduce that (36) holds for $x_1,x_2,x_3\in\mathfrak{e}_0$. By (32), we can deduce that (36) holds for two elements in $\mathfrak{e}_0$ and one element in $\mathfrak{e}_{-1}$. By (33), we can deduce that (36) holds for one element in $\mathfrak{e}_0$ and two elements in $\mathfrak{e}_{-1}$. The other cases can be proved similarly and we omit the details.

For $n = 4$, we need to verify the following equality:

$$\begin{aligned}
&- l_4(l_1(x_1), x_2, x_3, x_4) + (-1)^{|x_1||x_2|} l_4(l_1(x_2), x_1, x_3, x_4) \\
&- (-1)^{|x_3|(|x_1|+|x_2|)} l_4(l_1(x_3), x_1, x_2, x_4) \\
&+ (-1)^{|x_4|(|x_1|+|x_2|+|x_3|)} l_4(l_1(x_4), x_1, x_2, x_3) \\
&- (-1)^{|x_2||x_3|} l_3(l_2(x_1, x_3), x_2, x_4) + (-1)^{|x_4|(|x_3|+|x_2|)} l_3(l_2(x_1, x_4), x_2, x_3) \\
&+ (-1)^{|x_1|(|x_3|+|x_2|)} l_3(l_2(x_2, x_3), x_1, x_4) \\
&- (-1)^{|x_1|(|x_4|+|x_2|)+|x_3||x_4|} l_3(l_2(x_2, x_4), x_1, x_3) \\
&+ (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)} l_3(l_2(x_3, x_4), x_1, x_2) - l_2(l_3(x_1, x_2, x_3), x_4) \\
&- (-1)^{|x_2|(|x_3|+|x_4|)} l_2(l_3(x_1, x_3, x_4), x_2) + (-1)^{|x_3||x_4|} l_2(l_3(x_1, x_2, x_4), x_3) \\
&+ (-1)^{|x_1|(|x_2|+|x_3|+|x_4|)} l_2(l_3(x_2, x_3, x_4), x_1) + l_3(l_2(x_1, x_2), x_3, x_4) \\
&+ l_1 l_4(x_1, x_2, x_3, x_4) = 0.
\end{aligned}$$

For $x_1 = e^0_1, x_2 = e^0_2, x_3 = e^0_3, x_4 = e^0_4 \in \mathfrak{e}_0$, we need to prove that

$$\begin{aligned}
&\Omega(\llbracket e^0_1, e^0_2\rrbracket, e^0_3, e^0_4) - \Omega(\llbracket e^0_1, e^0_3\rrbracket, e^0_2, e^0_4) + \Omega(\llbracket e^0_1, e^0_4\rrbracket, e^0_2, e^0_3) \\
&+\Omega(\llbracket e^0_2, e^0_3\rrbracket, e^0_1, e^0_4) - \Omega(\llbracket e^0_2, e^0_4\rrbracket, e^0_1, e^0_3) + \Omega(\llbracket e^0_3, e^0_4\rrbracket, e^0_1, e^0_2) \\
&- \llbracket \Omega(e^0_1, e^0_2, e^0_3), e^0_4\rrbracket - \llbracket \Omega(e^0_1, e^0_3, e^0_4), e^0_2\rrbracket + \llbracket \Omega(e^0_1, e^0_2, e^0_4), e^0_3\rrbracket \\
&+ \llbracket \Omega(e^0_2, e^0_3, e^0_4), e^0_1\rrbracket + \mathcal{D}\overline{\Omega}(e^0_1, e^0_2, e^0_3, e^0_4) = 0,
\end{aligned}$$

which holds by Lemma 3.13.

For $x_1 = e^1 \in \mathfrak{e}_{-1}, x_2 = e^0_2, x_3 = e^0_3, x_4 = e^0_4 \in \mathfrak{e}_0$, we need to prove that

$$\begin{aligned}
&-\overline{\Omega}(\partial e^1, e^0_2, e^0_3, e^0_4) - T(\llbracket e^1, e^0_2\rrbracket, e^0_3, e^0_4) + T(\llbracket e^1, e^0_3\rrbracket, e^0_2, e^0_4) \\
&-T(\llbracket e^1, e^0_4\rrbracket, e^0_2, e^0_3) - T(\llbracket e^0_2, e^0_3\rrbracket, e^0_1, e^0_4) + T(\llbracket e^0_2, e^0_4\rrbracket, e^0_1, e^0_3) \\
&-T(\llbracket e^0_3, e^0_4\rrbracket, e^1, e^0_2) + \llbracket T(e^1, e^0_2, e^0_3), e^0_4\rrbracket + \llbracket T(e^1, e^0_3, e^0_4), e^0_2\rrbracket \\
&- \llbracket T(e^0_2, e^0_3, e^0_4), e^1\rrbracket - \llbracket T(e^1, e^0_2, e^0_4), e^0_3\rrbracket = 0.
\end{aligned}$$

On one hand, by direct calculation, we have

$$\begin{aligned}
&\llbracket T(e^1, e^0_2, e^0_3), e^0_4\rrbracket + \llbracket T(e^1, e^0_3, e^0_4), e^0_2\rrbracket \\
&\qquad - \llbracket T(e^0_2, e^0_3, e^0_4), e^0_1\rrbracket - \llbracket T(e^1, e^0_2, e^0_4), e^0_3\rrbracket = -\frac{1}{2}\mathbf{J}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&-\overline{\Omega}(\partial e^1,e^0_2,e^0_3,e^0_4)-T(\llbracket e^1,e^0_2\rrbracket,e^0_3,e^0_4)+T(\llbracket e^1,e^0_3\rrbracket,e^0_2,e^0_4)\\&-T(\llbracket e^1,e^0_4\rrbracket,e^0_2,e^0_3)-T(\llbracket e^0_2,e^0_3\rrbracket,e^1,e^0_4)+T(\llbracket e^0_2,e^0_4\rrbracket,e^1,e^0_3)\\&-T(\llbracket e^0_3,e^0_4\rrbracket,e^1,e^0_2)=-\frac{1}{6}(\mathbf{J}+2\mathbf{K})-\frac{1}{3}\overline{\Omega}(\partial e^1,e^0_2,e^0_3,e^0_4).\end{aligned}$$

Therefore, by Lemma 3.13, we prove the equality above.

Finally, we can show that (6) holds for $n=5$. We omit the details. The proof is finished. □

Remark 3.15. In [Roy07A], Roytenberg showed that one can obtain a semistrict Lie 2-algebra from a weak Lie 2-algebra via the skew-symmetrization. For a CLWX 2-algebroid $(E_{-1},E_0,\partial,\rho,S,\diamond,\Omega)$, the Leibniz 2-algebra $(\Gamma(E_{-1}),\Gamma(E_0),\partial,\diamond,\Omega)$ is not necessarily a weak Lie 2-algebra. Thus, we obtain a Lie 3-algebra rather than a Lie 2-algebra via the skew-symmetrization.

Remark 3.16. In this remark, we give a possible way to understand Theorem 3.14 conceptually. In [Roy07A], Roytenberg introduced the notion of a weak Lie 2-algebra and showed that via skew-symmetrization, one can obtain a Lie 2-algebra. Assume that this result could be generalized to the higher case: one can obtain a Lie n-algebra from a weak Lie n-algebra via skew-symmetrization. Then hopefully our Leibniz 2-algebra in a CLWX 2-algebroid can naturally be completed to a weak Lie 3-algebra and the Lie 3-algebra given in Theorem 3.14 is exactly its skew-symmetrization.

4. The CLWX 2-algebroid associated to a split Lie 2-algebroid

In this section, we first describe a split Lie 2-algebroid structure on a graded vector bundle $A_{-1}\oplus A_0$ using the graded Poisson bracket on $T^*[3](A_0\oplus A^*_{-1})[1]$. Then we construct a CLWX 2-algebroid $\mathcal{A}\oplus\mathcal{A}^*[1]$ from a split Lie 2-algebroid $\mathcal{A}$ with explicit formulas using the usual language of differential calculus. In Section 6, we will generalize this result to the case of split Lie 2-bialgebroids using the tool of derived brackets and graded geometry.

Let $\mathcal{A}=A_{-1}\oplus A_0$ be a graded bundle. The shifted cotangent bundle $T^*[3](A_0\oplus A^*_{-1})[1]$ is a P-manifold of degree 3 over M. Denote by $(x^i,\xi^j,\theta_k,p_i,\xi_j,\theta^k)$ a canonical Darboux coordinate on $\mathcal{M}$, where x^i is a coordinate on M, (ξ^j,θ_k) is the fiber coordinate on $A_0\oplus A^*_{-1}$, (p_i,ξ_j,θ^k) is the momentum coordinate on $\mathcal{M}$ for (x^i,ξ^j,θ_k). The degrees of variables

$(x^i, \xi^j, \theta_k, p_i, \xi_j, \theta^k)$ are respectively $(0,1,1,3,2,2)$. The degree of the symplectic structure $\omega = dx^i dp_i + d\xi^j d\xi_j + d\theta_k d\theta^k$ is 3 and the degree of the corresponding graded Poisson structure is -3.

Now we consider the following function[2] μ of degree 4 on $\mathcal{M}$:

$$\begin{aligned}\mu &= \mu_1{}^i_j(x)p_i\xi^j + \mu_2{}^i_j(x)\xi_i\theta^j \\ &\quad + \frac{1}{2}\mu_3{}^k_{ij}(x)\xi_k\xi^i\xi^j + \mu_4{}^k_{ij}\theta^j\xi^i\theta_k + \frac{1}{6}\mu_5{}^l_{ijk}(x)\theta_l\xi^i\xi^j\xi^k,\end{aligned} \tag{37}$$

where $\mu_1{}^i_j, \mu_2{}^i_j, \mu_3{}^k_{ij}, \mu_4{}^k_{ij}, \mu_5{}^l_{ijk}$ are functions on M. The function μ can be uniquely decomposed into[3]

$$\mu = \mu_2 + \mu_{134} + \mu_5,$$

where μ_2, μ_{134} and μ_5 are given by

$$\begin{aligned}\mu_2 &= \mu_2{}^i_j(x)\xi_i\theta^j, \\ \mu_{134} &= \mu_1{}^i_j(x)p_i\xi^j + \frac{1}{2}\mu_3{}^k_{ij}(x)\xi_k\xi^i\xi^j + \mu_4{}^k_{ij}\xi^i\theta^j\theta_k, \\ \mu_5 &= \frac{1}{6}\mu_5{}^l_{ijk}(x)\theta_l\xi^i\xi^j\xi^k.\end{aligned}$$

Define a bundle map $l_1 : A_{-1} \longrightarrow A_0$ by

$$l_1(X^1) = \left\{X^1, \mu_2\right\}. \tag{38}$$

Define $l_2 : \Gamma(A_{-i}) \times \Gamma(A_{-j}) \longrightarrow \Gamma(A_{-i-j}),\ 0 \le i+j \le 1$ by

$$\begin{cases} l_2(X^0, Y^0) = \left\{Y^0, \left\{X^0, \mu_{134}\right\}\right\}, \\ l_2(X^0, Y^1) = \left\{Y^1, \left\{X^0, \mu_{134}\right\}\right\}, \\ l_2(Y^1, X^0) = -\left\{X^0, \left\{Y^1, \mu_{134}\right\}\right\}. \end{cases} \tag{39}$$

Define a bundle map $l_3 : \wedge^3 A_0 \longrightarrow A_{-1}$ by

$$l_3(X^0, Y^0, Z^0) = \left\{Z^0, \left\{Y^0, \left\{X^0, \mu_5\right\}\right\}\right\}, \tag{40}$$

where $X^0, Y^0, Z^0 \in \Gamma(A_0)$ and $X^1, Y^1 \in \Gamma(A_{-1})$.

---

[2]We thank very much the referee for pointing out that such a function is linear on $\mathcal{A}^*$.

[3]It is routine to check that the decomposition does not depend on the choice of local coordinates. See also [IU] for more details.

Finally, define a bundle map $a : A_0 \longrightarrow TM$ by

$$a(X^0)(f) = \{f, \{X^0, \mu_{134}\}\}, \quad \forall\, X^0 \in \Gamma(A_0), f \in C^\infty(M). \tag{41}$$

Theorem 4.1. *Let $\mathcal{A} = A_{-1} \oplus A_0$ be a graded vector bundle and μ a degree 4 function given by (37). If $\{\mu, \mu\} = 0$, $(\mathcal{A}, l_1, l_2, l_3, a)$ is a split Lie 2-algebroid, where l_1, l_2, l_3 and a are given by (38)–(41) respectively.*

Conversely, if $(\mathcal{A}, l_1, l_2, l_3, a)$ is a split Lie 2-algebroid, we have $\{\mu, \mu\} = 0$, where μ is given by (37), in which $\mu_1{}^i_j, \mu_2{}^i_j, \mu_3{}^k_{ij}, \mu_4{}^k_{ij}, \mu_5{}^l_{ijk}$ are given by:

$$a(\xi_j) = \mu_1{}^i_j \frac{\partial}{\partial x^i}, \quad l_1(\theta_j) = \mu_2{}^i_j \xi_i,$$
$$l_2(\xi_i, \xi_j) = \mu_3{}^k_{ij} \xi_k, \quad l_2(\theta_j, \xi_i) = \mu_4{}^k_{ij} \theta_k, \quad l_3(\xi_i, \xi_j, \xi_k) = \mu_5{}^l_{ijk} \theta_l.$$

Proof. One can easily prove that $\{\mu, \mu\} = 0$ is equivalent to the following three identities:

$$\{\mu_{134}, \mu_2\} = 0,$$
$$\frac{1}{2}\{\mu_{134}, \mu_{134}\} + \{\mu_2, \mu_5\} = 0,$$
$$\{\mu_{134}, \mu_5\} = 0.$$

It is straightforward to deduce that Conditions (ii) and (iii) in Definition 2.6 holds.

In the following, we prove that $(\Gamma(\mathcal{A}), l_1, l_2, l_3)$ is a Lie 2-algebra. It is easy to see that l_2 and l_3 are totally skew-symmetric. For all $X^0 \in \Gamma(A_0), X^1 \in \Gamma(A_{-1})$, we have

$$\{X^1, \{X^0, \{\mu_2, \mu_{134}\}\}\} = -l_2(X^0, l_1(X^1)) + l_1 l_2(X^0, X^1) = 0,$$

which implies that $l_1 l_2(X^0, X^1) = l_2(X^0, l_1(X^1))$.

For all $X^1, Y^1 \in \Gamma(A_{-1})$, we have

$$\{Y^1, \{X^1, \{\mu_2, \mu_{134}\}\}\} = l_2(l_1(X^1), Y^1) - l_2(X^1, l_1(Y^1)) = 0,$$

which implies that $l_2(l_1(X^1), Y^1) = l_2(X^1, l_1(Y^1))$.

For all $X^0, Y^0, Z^0 \in \Gamma(A_0)$, by

$$\left\{Z^0, \left\{Y^0, \left\{X^0, \frac{1}{2}\{\mu_{134}, \mu_{134}\} + \{\mu_2, \mu_5\}\right\}\right\}\right\} = 0,$$

we get

$$\begin{aligned}&l_2(X^0, l_2(Y^0, Z^0)) + l_2(Z^0, l_2(X^0, Y^0)) + l_2(Y^0, l_2(Z^0, X^0))\\ =\ & l_1 l_3(X^0, Y^0, Z^0).\end{aligned}$$

For all $X^0, Y^0 \in \Gamma(A_0), Z^1 \in \Gamma(A_{-1})$, by

$$\left\{Z^1, \left\{Y^0, \left\{X^0, \frac{1}{2}\{\mu_{134}, \mu_{134}\} + \{\mu_2, \mu_5\}\right\}\right\}\right\} = 0,$$

we get

$$\begin{aligned}&l_2(X^0, l_2(Y^0, Z^1)) + l_2(Z^1, l_2(X^0, Y^0)) + l_2(Y^0, l_2(Z^1, X^0))\\ =\ & l_3(X^0, Y^0, l_1(Z^1)).\end{aligned}$$

For all $X^0, Y^0, Z^0, W^0 \in \Gamma(A_0)$, by

$$\left\{W^0, \left\{Z^0, \left\{Y^0, \left\{X^0, \frac{1}{2}\{\mu_{134}, \mu_{134}\} + \{\mu_2, \mu_5\}\right\}\right\}\right\}\right\} = 0,$$

we deduce that (6) holds for $n = 4$. Therefore, $(\Gamma(\mathcal{A}), l_1, l_2, l_3)$ is a Lie 2-algebra.

The proof of the converse part is similar as the above deduction. We omit the details. The proof is finished. □

Let $(\mathcal{A}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid with the structure function μ. Then we have a generalized Chevalley-Eilenberg complex

$$(\Gamma(Sym(\mathcal{A}[1])^*), \delta),$$

where δ is defined by

$$\delta(\cdot) = \{\mu, \cdot\}. \tag{42}$$

In particular, for all $f \in C^\infty(M), \alpha^0 \in \Gamma(A_0^*), \alpha^1 \in \Gamma(A_{-1}^*)$, we have

$$\begin{cases}\delta(f)(X^0) = a(X^0)(f),\\ \delta(\alpha^0)(X^0, Y^0) = a(X^0)\langle\alpha^0, Y^0\rangle - a(Y^0)\langle\alpha^0, X^0\rangle - \langle\alpha^0, l_2(X^0, Y^0)\rangle,\\ \delta(\alpha^1)(X^0, Y^1) = a(X^0)\langle\alpha^1, Y^1\rangle - \langle\alpha^1, l_2(X^0, Y^1)\rangle,\end{cases} \tag{43}$$

where $X^0, Y^0 \in \Gamma(A_0), Y^1 \in \Gamma(A_{-1})$.

Given a split Lie 2-algebroid $(\mathcal{A}, l_1, l_2, l_3, a)$, define $l_1^*: A_0^* \longrightarrow A_{-1}^*$ by

$$\langle l_1^*(\alpha^0), X^1\rangle = \langle \alpha^0, l_1(X^1)\rangle, \quad \forall \alpha^0 \in \Gamma(A_0^*), Y^1 \in \Gamma(A_{-1}). \tag{44}$$

For all $X^0 \in \Gamma(A_0)$, define $L^0_{X^0}: \Gamma(A^*_{-i}) \longrightarrow \Gamma(A^*_{-i})$, $i = 0, 1$, by

$$\begin{aligned}\langle L^0_{X^0}\alpha^0, Y^0\rangle &= \rho(X^0)\langle Y^0, \alpha^0\rangle - \langle \alpha^0, l_2(X^0, Y^0)\rangle,\\ \langle L^0_{X^0}\alpha^1, Y^1\rangle &= \rho(X^0)\langle Y^1, \alpha^1\rangle - \langle \alpha^1, l_2(X^0, Y^1)\rangle,\end{aligned}$$

where $\alpha^0 \in \Gamma(A_0^*),\ Y^0 \in \Gamma(A_0),\ \alpha^1 \in \Gamma(A^*_{-1}),\ Y^1 \in \Gamma(A_{-1})$.

For all $X^1 \in \Gamma(A_{-1})$, define $L^1_{X^1}: \Gamma(A^*_{-1}) \longrightarrow \Gamma(A^*_0)$ by

$$\langle L^1_{X^1}\alpha^1, Y^0\rangle = -\langle \alpha^1, l_2(X^1, Y^0)\rangle, \quad \forall \alpha^1 \in \Gamma(A^*_{-1}), Y^0 \in \Gamma(A_0). \tag{45}$$

For all $X^0, Y^0 \in \Gamma(A_0)$, define $L^3_{X^0,Y^0}: \Gamma(A^*_{-1}) \longrightarrow \Gamma(A^*_0)$ by

$$\langle L^3_{X^0,Y^0}\alpha^1, Z^0\rangle = -\langle \alpha^1, l_3(X^0, Y^0, Z^0)\rangle, \quad \forall \alpha^1 \in \Gamma(A^*_{-1}), Z^0 \in \Gamma(A_0). \tag{46}$$

The following lemmas list some properties of the above operators.

Lemma 4.2. *For all $X^0 \in \Gamma(A_0), X^1 \in \Gamma(A_{-1}), f \in C^\infty(M)$, $\alpha^0 \in \Gamma(A_0^*)$, $\alpha^1 \in \Gamma(A^*_{-1})$, we have*

$$\begin{aligned}L^0_{X^0} f\alpha^0 &= f(L^0_{X^0}\alpha^0) + a(X^0)(f)\alpha^0,\\ L^0_{fX^0}\alpha^0 &= f(L^0_{X^0}\alpha^0) + \langle X^0, \alpha^0\rangle\delta(f),\\ L^0_{X^0} f\alpha^1 &= f(L^0_{X^0}\alpha^1) + a(X^0)(f)\alpha^1,\\ L^0_{fX^0}\alpha^1 &= f(L^0_{X^0}\alpha^1),\\ L^1_{X^1} f\alpha^1 &= f(L^1_{X^1}\alpha^1),\\ L^1_{fX^1}\alpha^1 &= f(L^1_{X^1}\alpha^1) + \langle X^1, \alpha^1\rangle\delta(f),\\ L^0_{X^0}\alpha^0 &= \iota_{X^0}\delta\alpha^0 + \delta\iota_{X^0}\alpha^0,\\ L^1_{X^1}\alpha^1 &= \delta\iota_{X^1}\alpha^1 - \iota_{X^1}\delta\alpha^1.\end{aligned}$$

Proof. It is straightforward. □

Lemma 4.3. *For $X^0, Y^0 \in \Gamma(A_0), X^1 \in \Gamma(A_{-1}), \alpha^0 \in \Gamma(A_0^*), \alpha^1 \in \Gamma(A^*_{-1})$, we have*

$$L^0_{l_2(X^0,Y^0)}\alpha^0 - L^0_{X^0}L^0_{Y^0}\alpha^0 + L^0_{Y^0}L^0_{X^0}\alpha^0 = -L^3_{X^0,Y^0}l_1^*\alpha^0, \tag{47}$$

$$L^0_{l_2(X^0,Y^0)}\alpha^1 - L^0_{X^0}L^0_{Y^0}\alpha^1 + L^0_{Y^0}L^0_{X^0}\alpha^1 = -l_1^*L^3_{X^0,Y^0}\alpha^1, \tag{48}$$

$$L^1_{l_2(X^1,Y^0)}\alpha^1 - L^1_{X^1}L^0_{Y^0}\alpha^1 + L^0_{Y^0}L^1_{X^1}\alpha^1 = -L^3_{l_1(X^1),Y^0}\alpha^1. \tag{49}$$

Proof. For all $Z^0 \in \Gamma(A_0)$, we have

$$\begin{aligned}
&\langle L^0_{l_2(X^0,Y^0)}\alpha^0 - L^0_{X^0}L^0_{Y^0}\alpha^0 + L^0_{Y^0}L^0_{X^0}\alpha^0, Z^0\rangle \\
&= (a(l_2(X^0,Y^0)) - a(X^0)a(Y^0) + a(Y^0)a(X^0))\langle\alpha^0, Z^0\rangle \\
&\quad + \langle\alpha^0, -l_2(l_2(X^0,Y^0),Z^0) - l_2(Y^0, l_2(X^0,Z^0)) + l_2(X^0, l_2(Y^0,Z^0))\rangle \\
&= \langle\alpha^0, l_1 l_3(X^0,Y^0,Z^0)\rangle \\
&= \langle -L^3_{X^0,Y^0} l_1^*\alpha^0, Z^0\rangle,
\end{aligned}$$

which implies that the first equality holds. The others can be proved similarly. □

Let $(\mathcal{A}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid. Now let $E_0 = A_0 \oplus A^*_{-1}$, $E_{-1} = A_{-1} \oplus A^*_0$ and $\mathcal{E} = E_0 \oplus E_{-1}$. Let $\partial : E_{-1} \longrightarrow E_0$ and $\rho : E_0 \longrightarrow TM$ be bundle maps defined by

$$\partial(X^1 + \alpha^0) = l_1(X^1) + l_1^*(\alpha^0), \tag{50}$$

$$\rho(X^0 + \alpha^1) = a(X^0). \tag{51}$$

On $\Gamma(\mathcal{E})$, there is a natural symmetric bilinear form $(\cdot,\cdot)_+$ given by

$$\begin{aligned}
&(X^0 + \alpha^1 + X^1 + \alpha^0, Y^0 + \beta^1 + Y^1 + \beta^0)_+ \\
&= \langle X^0, \beta^0\rangle + \langle Y^0, \alpha^0\rangle + \langle X^1, \beta^1\rangle + \langle Y^1, \alpha^1\rangle,
\end{aligned} \tag{52}$$

where $X^0, Y^0 \in \Gamma(A_0), X^1, Y^1 \in \Gamma(A_{-1}), \alpha^0, \beta^0 \in \Gamma(A^*_0), \alpha^1, \beta^1 \in \Gamma(A^*_{-1})$.

On $\Gamma(\mathcal{E})$, we introduce the operation $\diamond$ by

$$\begin{cases}
(X^0 + \alpha^1) \diamond (Y^0 + \beta^1) = l_2(X^0, Y^0) + L^0_{X^0}\beta^1 - L^0_{Y^0}\alpha^1, \\
(X^0 + \alpha^1) \diamond (X^1 + \alpha^0) = l_2(X^0, X^1) + L^0_{X^0}\alpha^0 + \iota_{X^1}\delta(\alpha^1), \\
(X^1 + \alpha^0) \diamond (X^0 + \alpha^1) = l_2(X^1, X^0) + L^1_{X^1}\alpha^1 - \iota_{X^0}\delta(\alpha^0).
\end{cases} \tag{53}$$

An E_{-1}-valued 3-form Ω is defined by

$$\begin{aligned}
&\Omega(X^0 + \alpha^1, Y^0 + \beta^1, Z^0 + \zeta^1) \\
&= l_3(X^0, Y^0, Z^0) + L^3_{X^0,Y^0}\zeta^1 + L^3_{Z^0,X^0}\beta^1 + L^3_{Y^0,Z^0}\alpha^1,
\end{aligned} \tag{54}$$

where $X^0, Y^0, Z^0 \in \Gamma(A_0), \alpha^1, \beta^1, \zeta^1 \in \Gamma(A^*_{-1})$.

It is easy to see that the operator $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E_{-1})$ is given by

$$\mathcal{D}(f) = \delta(f), \quad \forall\ f \in C^\infty(M). \tag{55}$$

Theorem 4.4. *Let* $(\mathcal{A}, l_1, l_2, l_3, a)$ *be a split Lie* 2*-algebroid. Then*

$$(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$$

is a CLWX 2*-algebroid, where* ∂ *is given by* (50)*,* ρ *is given by* (51)*,* S *is given by* (52)*,* $\diamond$ *is given by* (53) *and* Ω *is given by* (54)*.*

Proof. It is easy to verify that $e \diamond e = \frac{1}{2}\mathcal{D}(e,e)_+$ for all $e \in \Gamma(\mathcal{E})$.

In the following, we verify that $(\Gamma(E_{-1}), \Gamma(E_0), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra. For all $e^0 = X^0 + \alpha^1 \in \Gamma(E_0), e^1 = X^1 + \alpha^0$, we have

$$\begin{aligned}\partial((X^0+\alpha^1)\diamond(X^1+\alpha^0)) &= l_1 l_2(X^0, X^1) + l_1^*\big(L^0_{X^0}\alpha^0 + \iota_{X^1}\delta(\alpha^1)\big),\\ (X^0+\alpha^1)\diamond\partial(X^1+\alpha^0) &= l_2(X^0, l_1(X^1)) + L^0_{X^0} l_1^*(\alpha^0) - L^0_{l_1(X^1)}\alpha^1.\end{aligned}$$

Since $(\Gamma(\mathcal{A}), l_1, l_2, l_3)$ is a Lie 2-algebra, we have

$$l_1 l_2(X^0, X^1) = l_2(X^0, l_1(X^1)), \quad l_2(l_1(X^1), Y^1) = l_2(X^1, l_1(Y^1)).$$

Then by the fact that $a \circ l_1 = 0$, we get

$$l_1^*\big(L^0_{X^0}\alpha^0 + \iota_{X^1}\delta(\alpha^1) = L^0_{X^0} l_1^*(\alpha^0) - L^0_{l_1(X^1)}\alpha^1.$$

Therefore we have

$$\partial(e^0 \diamond e^1) = e^0 \diamond \partial(e^1), \tag{56}$$

which implies that Condition (a) in Definition 2.5 holds.

Also by the fact $a \circ l_1 = 0$, we have

$$\begin{aligned}\partial(e^1 \diamond e^0) &= l_1^*(\delta(e^1, e^0)_+) - \partial(e^0 \diamond e^1)\\ &= -\partial(e^0 \diamond e^1) = -e^0 \diamond \partial(e^1) = \partial(e^1) \diamond e^0,\end{aligned}$$

which implies that Condition (b) in Definition 2.5 holds.

Similarly, for all $e^1_i \in \Gamma(E_{-1}), i = 1, 2$, we have

$$\partial(e^1_1) \diamond e^1_2 = e^1_1 \diamond \partial(e^1_2), \tag{57}$$

which implies that Condition (c) in Definition 2.5 holds.

For all $X_i^0 \in \Gamma(A_0), i = 1, 2, 3$, it is obvious that

$$K(X_1^0, X_2^0, X_3^0) = l_1 l_3(X_1^0, X_2^0, X_3^0) = \partial\Omega(X_1^0, X_2^0, X_3^0).$$

Furthermore, for all $X_i^0 \in \Gamma(A_0), i = 1, 2$ and $\alpha^1 \in \Gamma(A_{-1}^*)$, by Lemma 4.3, we have

$$\begin{aligned} K(X_1^0, X_2^0, \alpha^1) &= -(L^0_{l_2(X_1^0, X_2^0)}\alpha^1 - L^0_{X_1^0} L^0_{X_2^0}\alpha^1 + L^0_{X_2^0} L^0_{X_1^0}\alpha^1) \\ &= l_1^* L^3_{X_1^0, X_2^0}\alpha^1 = \partial\Omega(X_1^0, X_2^0, \alpha^1). \end{aligned}$$

Therefore, for all $e_i^0 \in \Gamma(E_0), i = 1, 2, 3$, we get

$$K(e_1^0, e_2^0, e_3^0) = \partial\Omega(e_1^0, e_2^0, e_3^0), \tag{58}$$

which implies that Condition (d) in Definition 2.5 holds.

Similarly, for all $e_i^0 \in \Gamma(E_0), i = 1, 2$ and $e^1 \in \Gamma(E_{-1})$, we have

$$\begin{aligned} K(e_1^0, e_2^0, e^1) &= \Omega(e_1^0, e_2^0, \partial e^1), \\ K(e_1^0, e^1, e_2^0) &= \Omega(e_1^0, \partial e^1, e_2^0), \\ K(e^1, e_1^0, e_2^0) &= \Omega(\partial e^1, e_1^0, e_2^0), \end{aligned}$$

which implies that Conditions (e_1)-(e_3) in Definition 2.5 holds.

By the coherence law that l_3 satisfies in the definition of a Lie 2-algebra, we can deduce that Condition (f) in Definition 2.5 also holds. We omit the details. Thus, $(\Gamma(E_{-1}), \Gamma(E_0), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra.

Finally, for all $e_1^1, e_2^1 \in \Gamma(E_{-1})$, $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$ and $e_1^0, e_2^0, e_3^0, e_4^0 \in \Gamma(E_0)$, it is straightforward to deduce that

$$\begin{aligned} (\partial(e_1^1), e_2^1)_+ &= (e_1^1, \partial(e_2^1))_+, \\ \rho(e_1)(e_2, e_3)_+ &= (e_1 \diamond e_2, e_3)_+ + (e_2, e_1 \diamond e_3)_+, \\ (\Omega(e_1^0, e_2^0, e_3^0), e_4^0)_+ &= -(e_3^0, \Omega(e_1^0, e_2^0, e_4^0))_+, \end{aligned}$$

which implies that axioms (iii), (iv) and (v) in Definition 3.1 hold. The proof is finished. □

Example 4.5. Let $(\mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3)$ be a Lie 2-algebra. Denote by $\mathfrak{d}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}^*$ and $\mathfrak{d}_{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^*$. Then the CLWX 2-algebroid given by Theorem 4.4 is over a point. By remark 3.2, we obtain a metric Lie 2-algebra structure on the graded vector space $\mathfrak{d}_0 \oplus \mathfrak{d}_{-1}$. The Lie 2-algebra $(\mathfrak{d}_{-1}, \mathfrak{d}_0, \partial, [\cdot,\cdot], \Omega)$

is given as follows:

$$\begin{aligned}
\partial &= l_1 + l_1^*,\\
[x^0+\alpha^1, y^0+\beta^1] &= l_2(x^0,y^0) + \mathrm{ad}^{0*}_{x^0}\beta^1 - \mathrm{ad}^{0*}_{y^0}\alpha^1,\\
[x^0+\alpha^1, y^1+\beta^0] &= l_2(x^0,y^1) + \mathrm{ad}^{0*}_{x^0}\beta^0 - \mathrm{ad}^{1*}_{y^1}\alpha^1,\\
\Omega(x^0+\alpha^1, y^0+\beta^1, z^0+\zeta^1) &= l_3(x^0,y^0,z^0) + \mathrm{ad}^{3*}_{x^0,y^0}\zeta^1 + \mathrm{ad}^{3*}_{y^0,z^0}\alpha^1\\
&\quad + \mathrm{ad}^{3*}_{z^0,x^0}\beta^1,
\end{aligned}$$

for all $x^0, y^0, z^0 \in \mathfrak{g}_0$, $x^1, y^1 \in \mathfrak{g}_{-1}$, $\alpha^1, \beta^1 \in \mathfrak{g}^*_{-1}$, $\alpha^0, \beta^0 \in \mathfrak{g}^*_0$, where $\mathrm{ad}^{0*}_{x^0} : \mathfrak{g}^*_{-i} \longrightarrow \mathfrak{g}^*_{-i}$, $\mathrm{ad}^{1*}_{x^1} : \mathfrak{g}^*_{-1} \longrightarrow \mathfrak{g}^*_0$ and $\mathrm{ad}^{3*}_{x^0,y^0} : \mathfrak{g}^*_{-1} \longrightarrow \mathfrak{g}^*_0$ are defined respectively by

$$\begin{aligned}
\langle \mathrm{ad}^{0*}_{x^0}\alpha^1, x^1\rangle &= -\langle \alpha^1, l_2(x^0,x^1)\rangle,\\
\langle \mathrm{ad}^{0*}_{x^0}\alpha^0, y^0\rangle &= -\langle \alpha^0, l_2(x^0,y^0)\rangle,\\
\langle \mathrm{ad}^{1*}_{x^1}\alpha^1, y^0\rangle &= -\langle \alpha^1, l_2(x^1,y^0)\rangle,\\
\langle \mathrm{ad}^{3*}_{x^0,y^0}\alpha^1, z^0\rangle &= -\langle \alpha^1, l_3(x^0,y^0,z^0)\rangle.
\end{aligned}$$

Thus, this Lie 2-algebra is exactly the semidirect product of the Lie 2-algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3)$ with its dual $\mathfrak{g}^*_0[1] \oplus \mathfrak{g}^*_{-1}[1]$ via the coadjoint representation.

5. QP-manifolds $T^*[3]A[1]$ and CLWX 2-algebroids

Let A be a vector bundle over M and A^* its dual bundle. The shifted bundle $A[1]$ is a graded manifold whose fiber space has degree -1. We consider the shifted cotangent bundle $\mathcal{M} := T^*[3]A[1]$. It is a P-manifold of degree 3 over M. In this section, we construct a CLWX 2-algebroid from the degree 3 QP-manifold $T^*[3]A[1]$.

Denote by $(q^i, \xi^\alpha, \xi_\alpha, p_i)$ a canonical Darboux coordinate on $T^*[3]A[1]$, where q^i is a coordinate on M, ξ^α is the fiber coordinate on $A[1]$, (p_i, ξ_α) is the momentum coordinate on $T^*[3]A[1]$ for (q^i, ξ^α). The degrees of variables $(q^i, \xi^\alpha, \xi_\alpha, p_i)$ are respectively $(0,1,2,3)$. The degree of the symplectic structure $\omega = dq^i dp_i + d\xi^\alpha d\xi_\alpha$ is 3 and the degree of the corresponding graded Poisson structure is -3. In the local coordinate, any Q-structure Θ is of the following form:

$$\Theta = f_{1a}^{\ i}(x) p_i \xi^a + f_2^{ab}(x)\xi_a\xi_b + \frac{1}{2} f_{3ab}^{\ \ c}(x)\xi^a\xi^b\xi_c + \frac{1}{6} f_{4abcd}(x)\xi^a\xi^b\xi^c\xi^d. \tag{59}$$

We write $\Theta = \theta_2 + \theta_{13} + \theta_4$, where the substructures are

$$\theta_2 = f_2{}^{ab}\xi_a\xi_b,$$
$$\theta_{13} = f_1{}^i_a(x)p_i\xi^a + \frac{1}{2}f_3{}^c_{ab}\xi^a\xi^b\xi_c,$$
$$\theta_4 = \frac{1}{6}f_{4abcd}\xi^a\xi^b\xi^c\xi^d.$$

The classical master equation $\{\Theta, \Theta\} = 0$ is equivalent to the following three identities:

$$\{\theta_{13}, \theta_2\} = 0, \tag{60}$$
$$\frac{1}{2}\{\theta_{13}, \theta_{13}\} + \{\theta_2, \theta_4\} = 0, \tag{61}$$
$$\{\theta_{13}, \theta_4\} = 0. \tag{62}$$

Define two bundle maps $\partial : A^* \longrightarrow A$ and $\rho : A \longrightarrow TM$ by the following identities respectively:

$$\partial\alpha = \{\alpha, \theta_2\}, \quad \forall\ \alpha \in \Gamma(A^*), \tag{63}$$
$$\rho(X)(f) = \{f, \{X, \theta_{13}\}\}, \quad \forall\ X \in \Gamma(A), f \in C^\infty(M). \tag{64}$$

A natural non-degenerate bilinear form S on $A^* \oplus A$ is given by

$$S(X + \alpha, Y + \beta) = \langle X, \beta\rangle + \langle Y, \alpha\rangle, \quad \forall\ X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*). \tag{65}$$

Define the operation $\diamond$ by

$$\begin{cases} X \diamond Y = \{Y, \{X, \theta_{13}\}\}, & \forall\ X, Y \in \Gamma(A), \\ X \diamond \alpha = \{\alpha, \{X, \theta_{13}\}\}, & \forall\ X \in \Gamma(A), \alpha \in \Gamma(A^*), \\ \alpha \diamond X = -\{X, \{\alpha, \theta_{13}\}\}, & \forall\ X \in \Gamma(A), \alpha \in \Gamma(A^*). \end{cases} \tag{66}$$

An A^*-valued 3-form Ω is defined by

$$\Omega(X, Y, Z) = \{Z, \{Y, \{X, \theta_4\}\}\}, \quad \forall\ X, Y, Z \in \Gamma(A). \tag{67}$$

Theorem 5.1. *Let $(T^*[3]A[1], \Theta)$ be a QP-manifold of degree 3. Then $(A^*[1], A, \partial, \rho, S, \diamond, \Omega)$ is a* CLWX *2-algebroid, where ∂ is given by* (63), *ρ is given by* (64), *S is given by* (65), *$\diamond$ is given by* (66) *and Ω is given by* (67).

The proof follows from the following Lemma 5.2–5.4 directly.

Lemma 5.2. *With the above notations,* $e \diamond e = \frac{1}{2}\mathcal{D}S(e,e)$*, where* S *is given by* (65) *and* $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(A^*)$ *is given by* $\langle \mathcal{D}(f), X\rangle = \rho(X)(f)$*, in which* ρ *is given by* (64)*.*

Proof. By (66) and (43), we can deduce that

$$X \diamond Y = -Y \diamond X, \tag{68}$$

$$X \diamond \alpha + \alpha \diamond X = \delta\langle X, \alpha\rangle, \tag{69}$$

which finishes the proof. □

Lemma 5.3. *With the above notations,* $(\Gamma(A^*), \Gamma(A), \partial, \diamond, \Omega)$ *is a Leibniz* 2*-algebra, where* ∂ *is given by* (63)*,* $\diamond$ *is given by* (66) *and* Ω *is given by* (67) *respectively.*

Proof. By (60), we have $\{\theta_2, \{X, \theta_{13}\}\} = 0$. Thus we have

$$\begin{aligned} \partial(X \diamond \alpha) &= \{\{\alpha, \{X, \theta_{13}\}\}, \theta_2\} \\ &= -\{\{\theta_2, \alpha\}, \{X, \theta_{13}\}\} - \{\alpha, \{\theta_2, \{X, \theta_{13}\}\}\} \\ &= \{\{\alpha, \theta_2\}, \{X, \theta_{13}\}\} = X \diamond \partial(\alpha). \end{aligned} \tag{70}$$

By (60), we get

$$\rho \circ \partial = 0. \tag{71}$$

Then by (69), we have

$$\partial(\alpha \diamond X) = \partial(\delta\langle X, \alpha\rangle - X \diamond \alpha) = \partial(\delta\langle X, \alpha\rangle) - X \diamond \partial(\alpha) = \partial(\alpha) \diamond X. \tag{72}$$

Similarly, we have

$$\partial(\alpha) \diamond \beta = \alpha \diamond \partial(\beta). \tag{73}$$

By (61) and the following two facts:

$$\begin{aligned} \{Z, \{Y, \{X, \{\theta_{13}, \theta_{13}\}\}\}\} &= -2\big(X \diamond (Y \diamond Z) - (X \diamond Y) \diamond Z - Y \diamond (X \diamond Z)\big), \\ \{Z, \{Y, \{X, \{\theta_2, \theta_4\}\}\}\} &= \partial\Omega(X, Y, Z), \end{aligned}$$

where $X, Y, Z \in \Gamma(A)$, we have

$$X \diamond (Y \diamond Z) - (X \diamond Y) \diamond Z - Y \diamond (X \diamond Z) = \partial\Omega(X, Y, Z). \tag{74}$$

Similarly, we can obtain

$$X \diamond (Y \diamond \alpha) - (X \diamond Y) \diamond \alpha - Y \diamond (X \diamond \alpha) = \Omega(X, Y, \partial(\alpha)), \tag{75}$$
$$X \diamond (\alpha \diamond Y) - (X \diamond \alpha) \diamond Y - \alpha \diamond (X \diamond Y) = \Omega(X, \partial(\alpha), Y), \tag{76}$$
$$\alpha \diamond (X \diamond Y) - (\alpha \diamond X) \diamond Y - X \diamond (\alpha \diamond Y) = \Omega(\partial(\alpha), X, Y). \tag{77}$$

Finally, expanding $\{W, \{Z, \{Y, \{X, \{\theta_{13}, \theta_4\}\}\}\}\} = 0$ by the graded Jacobi identity, we have

$$\begin{aligned} &W \diamond \Omega(X, Y, Z) - X \diamond \Omega(W, Y, Z) + Y \diamond \Omega(W, X, Z) + \Omega(W, X, Y) \diamond Z \\ &- \Omega(W \diamond X, Y, Z) - \Omega(X, W \diamond Y, Z) - \Omega(X, Y, W \diamond Z) \\ &+ \Omega(W, X \diamond Y, Z) + \Omega(W, Y, X \diamond Z) - \Omega(W, X, Y \diamond Z) = 0. \end{aligned} \tag{78}$$

By (70), (72), (73), (74)–(78), we deduce that $(\Gamma(A^*), \Gamma(A), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra. □

Lemma 5.4. *With the above notations, for all $\alpha, \beta \in \Gamma(A^*)$, $X, Y, Z, W \in \Gamma(A)$ and $e_1, e_2, e_3 \in \Gamma(A) \oplus \Gamma(A^*)$, we have*

$$\langle \partial\alpha, \beta \rangle = \langle \alpha, \partial\beta \rangle, \tag{79}$$
$$\rho(e_1) S(e_2, e_3) = S(e_1 \diamond e_2, e_3) + S(e_2, e_1 \diamond e_3), \tag{80}$$
$$S(\Omega(X, Y, Z), W) = -S(Z, \Omega(X, Y, W)). \tag{81}$$

Proof. By the Jacobi identity of the graded Poisson bracket $\{\cdot, \cdot\}$, we have

$$\begin{aligned} \langle \partial\alpha, \beta \rangle &= \{\partial\alpha, \beta\} = \{\{\alpha, \theta_2\}, \beta\} \\ &= \{\alpha, \{\theta_2, \beta\}\} - \{\theta_2, \{\alpha, \beta\}\} = -\{\alpha, \partial\beta\} = \{\partial\beta, \alpha\} = \langle \partial\beta, \alpha \rangle. \end{aligned}$$

For $X, Y \in \Gamma(A), \alpha \in \Gamma(A^*)$, we have

$$\{Y, \{\alpha, \{X, \theta_{13}\}\}\} = \{\{Y, \alpha\}, \{X, \theta_{13}\}\} + \{\alpha, \{Y, \{X, \theta_{13}\}\}\},$$

which implies that

$$\langle Y, X \diamond \alpha \rangle = \rho(X) \langle Y, \alpha \rangle - \langle \alpha, X \diamond Y \rangle.$$

That is $\rho(X) S(Y, \alpha) = S(X \diamond Y, \alpha) + S(Y, X \diamond \alpha)$. Therefore, (80) holds when $e_1, e_2 \in \Gamma(A)$ and $e_3 \in \Gamma(A^*)$. Similarly, we can show that (80) holds for all the other cases.

Finally, (81) follows from

$$\begin{aligned} S(\Omega(X,Y,Z),W) &= \{W,\{Z,\{Y,\{X,\theta_4\}\}\}\} \\ &= \{\{W,Z\},\{Y,\{X,\theta_4\}\}\} - \{Z,\{W,\{Y,\{X,\theta_4\}\}\}\} \\ &= -S(\Omega(X,Y,W),Z). \end{aligned}$$

The proof is finished. □

Remark 5.5. The P-manifold of degree 3, $T^*[3]A[1]$, can be viewed as a shifted manifold of $T^*[2]A[1]$, which is a P-manifold of degree 2. However, in general, a degree 3 function Θ on $T^*[2]A[1]$ is not a degree 4 function on $T^*[3]A[1]$. Thus, there is not a canonical way to obtain a QP-manifold of degree 3 from a given QP-manifold of degree 2. Therefore, we can not obtain a CLWX 2-algebroid from an arbitrary Courant algebroid.

Remark 5.6. Let us consider the degree 3 QP-manifold $T^*[3]T[1]M$ where the Q-structure is given by $p_i\xi^i$ in local coordinates. On one hand, according to Theorem 5.1, we obtain the CLWX 2-algebroid $(T^*[1]M, TM, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = 0)$ given in Remark 3.4. Then according to Theorem 3.14, we have a Lie 3-algebra structure on $C^\infty(M)[2] \oplus \Omega^1(M)[1] \oplus \mathfrak{X}(M)$. On the other hand, according to [Zam], there is also a Lie 3-algebra structure on $C^\infty(M)[2] \oplus \Omega^1(M)[1] \oplus (\mathfrak{X}(M) \oplus \Omega^2(M))$. However, we do not find any connection between the two Lie 3-algebras.

Furthermore, if we consider the Q-structure given by

$$p_i\xi^i + \frac{1}{6} f_{4abcd}\xi^a\xi^b\xi^c\xi^d,$$

we obtain the CLWX 2-algebroid $(T^*[1]M, TM, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = H)$ given in Example 3.5.

6. The CLWX 2-algebroid associated to a split Lie 2-bialgebroid

In this section, we introduce the notion of a split Lie 2-bialgebroid and show that there is a CLWX 2-algebroid structure on $\mathcal{A} \oplus \mathcal{A}^*[1]$ associated to any split Lie 2-bialgebroid $(\mathcal{A}, \mathcal{A}^*[1])$.

Now assume that there is a split Lie 2-algebroid structure on the dual bundle $\mathcal{A}^*[1] = A_0^*[1] \oplus A_{-1}^*[1]$. Since $T^*[3]((A_0 \oplus A_{-1}^*)^*[1])[1]$, $T^*[3](A_0 \oplus A_{-1}^*)[1]$ and $T^*[3](A_0 \oplus A_{-1}^*)^*[2]$ are naturally isomorphic, by Theorem 4.1, the dual split Lie 2-algebroid $(\mathcal{A}^*[1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ gives rise to a degree 4 function γ on $T^*[3](A_0 \oplus A^*_{-1})[1]$ satisfying $\{\gamma, \gamma\} = 0$. It is given in local coordinates $(x^i, \xi^j, \theta_k, p_i, \xi_j, \theta^k)$ by

$$\gamma = {\gamma_1}^{ij}(x) p_j \theta_i + {\gamma_2}_i^j(x) \xi_j \theta^i + \frac{1}{2} {\gamma_3}_k^{ij}(x) \theta^k \theta_i \theta_j + {\gamma_4}_k^{ij} \xi_i \theta_j \xi^k + \frac{1}{6} {\gamma_5}_l^{ijk}(x) \xi^l \theta_i \theta_j \theta_k. \tag{82}$$

We will also write $\gamma = \gamma_2 + \gamma_{134} + \gamma_5$.

Definition 6.1. Let $(\mathcal{A}, l_1, l_2, l_3, a)$ be a split Lie 2-algebroid with the structure function μ given by (37) and $(\mathcal{A}^*[1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ a split Lie 2-algebroid with the structure function γ given by (82). The pair $(\mathcal{A}, \mathcal{A}^*[1])$ is called a **split Lie 2-bialgebroid** if $\gamma_2 = \mu_2$ and

$$\{\mu + \gamma - \mu_2, \mu + \gamma - \mu_2\} = 0, \tag{83}$$

where $\{\cdot, \cdot\}$ is the graded Poisson bracket corresponding to the symplectic structure $\omega = dx^i dp_i + d\xi^j d\xi_j + d\theta_k d\theta^k$ on $T^*[3](A_0 \oplus A^*_{-1})[1]$.

Denote a split Lie 2-bialgebroid by $(\mathcal{A}, \mathcal{A}^*[1])$.

We denote by $\mathcal{L}^0, \mathcal{L}^1, \mathcal{L}^3, \delta_*$ the operations for the dual split Lie 2-algebroid $(\mathcal{A}^*[1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ corresponding to the operations L^0, L^1, L^3, δ for the split Lie 2-algebroid $(\mathcal{A}, l_1, l_2, l_3, a)$.

Now we assume that $(\mathcal{A}, l_1, l_2, l_3, a)$ and $(\mathcal{A}^*[1], \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{a})$ are split Lie 2-algebroids. Let $E_0 = A_0 \oplus A^*_{-1}$, $E_{-1} = A_{-1} \oplus A^*_0$ and $\mathcal{E} = E_0 \oplus E_{-1}$.

Let $\partial : E_{-1} \longrightarrow E_0$ and $\rho : E_0 \longrightarrow TM$ be bundle maps defined by

$$\partial(X^1 + \alpha^0) = l_1(X^1) + \mathfrak{l}_1(\alpha^0), \tag{84}$$

$$\rho(X^0 + \alpha^1) = a(X^0) + \mathfrak{a}(\alpha^1). \tag{85}$$

On $\Gamma(\mathcal{E})$, we introduce the operation $\diamond$ by

$$\begin{cases} (X^0 + \alpha^1) \diamond (Y^0 + \beta^1) = l_2(X^0, Y^0) + L^0_{X^0}\beta^1 - L^0_{Y^0}\alpha^1 \\ \qquad\qquad + \mathfrak{l}_2(\alpha^1, \beta^1) + \mathcal{L}^0_{\alpha^1} Y^0 - \mathcal{L}^0_{\beta^1} X^0, \\ (X^0 + \alpha^1) \diamond (X^1 + \alpha^0) = l_2(X^0, X^1) + L^0_{X^0}\alpha^0 + \iota_{X^1}\delta(\alpha^1) \\ \qquad\qquad + \mathfrak{l}_2(\alpha^1, \alpha^0) + \mathcal{L}^0_{\alpha^1} X^1 + \iota_{\alpha^0}\delta_*(X^0), \\ (X^1 + \alpha^0) \diamond (X^0 + \alpha^1) = l_2(X^1, X^0) + L^1_{X^1}\alpha^1 - \iota_{X^0}\delta(\alpha^0) \\ \qquad\qquad + \mathfrak{l}_2(\alpha^0, \alpha^1) + \mathcal{L}^1_{\alpha^0} X^0 - \iota_{\alpha^1}\delta_*(X^1). \end{cases} \tag{86}$$

An E_{-1}-valued 3-form Ω is defined by

$$\begin{aligned}(87)\qquad &\Omega(X^0+\alpha^1, Y^0+\beta^1, Z^0+\zeta^1)\\ &= l_3(X^0,Y^0,Z^0) + L^3_{X^0,Y^0}\zeta^1 + L^3_{Y^0,Z^0}\alpha^1 + L^3_{Z^0,X^0}\beta^1\\ &\quad + \mathfrak{l}_3(\alpha^1,\beta^1,\zeta^1) + \mathcal{L}^3_{\alpha^1,\beta^1}Z^0 + \mathcal{L}^3_{\beta^1,\zeta^1}X^0 + \mathcal{L}^3_{\zeta^1,\alpha^1}Y^0,\end{aligned}$$

where $X^0, Y^0, Z^0 \in \Gamma(A_0), \alpha^1, \beta^1, \zeta^1 \in \Gamma(A^*_{-1})$.

Theorem 6.2. *Let $(\mathcal{A}, \mathcal{A}^*[1])$ be a split Lie 2-bialgebroid. Then*

$$(E_{-1}, E_0, \partial, \rho, (\cdot,\cdot)_+, \diamond, \Omega)$$

is a CLWX *2-algebroid, where $E_0 = A_0 \oplus A^*_{-1}$, $E_{-1} = A_{-1} \oplus A^*_0$, ∂ is given by* (84)*, ρ is given by* (85)*, $(\cdot,\cdot)_+$ is given by* (52)*, $\diamond$ is given by* (86) *and Ω is given by* (87)*.*

Proof. Since $\mu+\gamma-\mu_2$ is a degree 4 function on $T^*[3]E_0[1]$ satisfying

$$\{\mu+\gamma-\mu_2, \mu+\gamma-\mu_2\} = 0,$$

by Theorem 5.1, there is a CLWX 2-algebroid defined by $\mu+\gamma-\mu_2$ through derived brackets. It is straightforward to deduce that (84)–(87) are exactly the one obtained through derived brackets. The proof is finished. $\square$

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