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Cahen–Gutt moment map, closed Fedosov star product and structure of the automorphism group

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We show that if a compact Kähler manifold M with non-negative Ricci curvature admits closed Fedosov star product then the reduced Lie algebra of holomorphic vector fields on M is reductive. This comes in pair with the obstruction previously found by La Fuente-Gravy [20]. More generally we consider the squared norm of Cahen–Gutt moment map as in the same spirit of Calabi functional for the scalar curvature in cscK problem, and prove a Cahen–Gutt version of Calabi's theorem on the structure of the Lie algebra of holomorphic vector fields for extremal Kähler manifolds.

1. Introduction.

A deformation quantization is a formal associative deformation of a Poisson algebra $(C^\infty(M), \cdot, \{\cdot,\cdot\})$ into the space $C^\infty(M)[[\nu]]$ of formal power series in ν with a composition law $*$ called the star product with the following property. The constant function 1 is a unit, and if we write for $f,\ g \in C^\infty(M)$

$$f * g = \sum_{r=0}^{\infty} C_r(f,g)\nu^r, \tag{1}$$

then $*$ is required to satisfy

$$C_0(f,g) = f \cdot g, \qquad C_1(f,g) - C_1(g,f) = \{f,g\}, \tag{2}$$

and C_r's are required to be bidifferential operators. For symplectic manifolds, the existence of star products was shown by Dewilde and Lecompte [6], Fedosov [8] and Omori, Maeda and Yoshioka [24]. For general Poisson manifolds, the existence of star products was shown by Kontsevich [18]. A star product on a compact symplectic manifold (M,ω) of dimension $2m$ is called closed (in the sense of Connes-Flato-Sternheimer [5]) if

$$\int_M F * H \, \omega^m = \int_M H * F \, \omega^m \tag{3}$$

for all $F,\ H \in C^\infty(M)[[\nu]]$.

Let (M, ω) be a compact connected symplectic manifold of dimension $2m$. In [3], Cahen and Gutt defined a moment map μ on the space of symplectic connections for the action of the group of Hamiltonian diffeomorphisms. In [19], La Fuente-Gravy showed that if the Fedosov star product $*_\nabla$ is closed for a symplectic connection ∇ then $\mu(\nabla)$ is constant. Assuming M is Kähler and fixing a Kähler class, La Fuente-Gravy further defined in [20] a Lie algebra character $\mathrm{Fut} : \mathfrak{g} \to \mathbf{R}$ where $\mathfrak{g}$ is the reduced Lie algebra of holomorphic vector fields on M, and showed that if there exists a Kähler metric in the fixed Kähler class such that $\mu(\nabla)$ is constant for the Levi-Civita connection ∇ then the character Fut vanishes. In particular, non-vanishing of the character Fut obstructs the existence of a Kähler metric in the fixed Kähler class such that the Fedosov star product $*_\nabla$ is closed. (Recall that, by definition, the reduced Lie algebra $\mathfrak{g}$ of holomorphic vector fields on a compact Kähler manifold consists of holomorphic vector fields of the form $\mathrm{grad}' f$ for some complex valued smooth function f (c.f. [17]). In fact, $\mathfrak{g}$ does not depend on the choice of the Kähler metric.) La Fuente-Gravy showed that, when the fixed Kähler class is integral and equal to $c_1(L)$ for some ample line bundle L, the character he defined is one of the obstructions for the polarized manifold (M, L) to be asymptotically Chow semistable obtained in [12], see also [15] for more applications.

In the problem of finding constant scalar curvature Kähler (cscK) metrics, Fujiki [10] and Donaldson [7] set up a moment map τ on the space of complex structures compatible with a fixed symplectic form ω where $\tau(J)$ at a complex structure J is the scalar curvature of the Kähler manifold (M, ω, J). In this cscK problem, we also have the Lie algebra character which obstructs the existence of cscK metrics ([11]). On the other hand, we also have another obstruction which claims that the Lie algebra of all holomorphic vector fields of a cscK manifold has to be reductive ([23], [22]). This is further extended by Calabi [4] to a structure theorem of the Lie algebra for compact extremal Kähler manifolds. As can be seen in other similar problems (see e.g. [14], [16], [21], [1]), the two obstructions of the Lie algebra character and the reductiveness come always in pair. The purpose of this paper is to show that this is the case, namely the following is the main result of this paper.

Theorem 1.1. *Let M be a compact Kähler manifold. If there exists a Kähler metric with non-negative Ricci curvature such that $\mu(\nabla)$ is constant for the Cahen–Gutt moment map μ and the Levi-Civita connection ∇ then the reduced Lie algebra $\mathfrak{g}$ of holomorphic vector fields is reductive. In particular, if $\mathfrak{g}$ is not reductive then there is no Kähler metric with non-negative Ricci curvature such that the Fedosov star product $*_\nabla$ for the Levi-Civita connection ∇ is closed.*

To show this we define Cahen–Gutt version of extremal Kähler metrics and prove a similar structure theorem as the Calabi extremal Kähler metrics. The strategy of the proof of the structure theorem for Cahen–Gutt extremal Kähler manifold is to use the formal finite dimensional argument for the Hessian formula of the squared norm of the moment map given by Wang [25]. The merit of Wang's argument is that once the suitable modification of the Lichnerowicz operator is made we can apply his formal argument without using the explicit expression of the modified Lichnerowicz operator. This strategy has been used previously for perturbed extremal Kähler metrics in [14] and for conformally Kähler, Einstein-Maxwell metrics in [16].

This paper is organized as follows. In section 2 we recall Cahen–Gutt moment map. We show in Kähler situation an explicit expression of the Lie derivative of the connection by a Hamiltonian vector field (Lemma 2.2). From this lemma we see that the Lie derivative of the connection by a Hamiltonian vector field vanishes if and only if the Hamiltonian vector field is a holomorphic Killing vector field. Using the Cahen–Gutt moment map formula we reprove the result of La Fuente-Gravy that the Lie algebra character is independent of the choice of Kähler metric in the fixed Kähler class. But our set-up is ω-fixed and J-varying, and thus what we prove is independence of the choice of J. In section 3 we give an alternate proof of Lemma 2.2. The computations in this section are used in section 4. In section 4 we apply Wang's formal argument to prove the structure theorem for Cahen–Gutt extremal Kähler manifolds. As we will only use the fact that closedness of the Fedosov star product $*_\nabla$ implies that $\mu(\nabla)$ is constant, we will not reproduce the detailed account on closedness of Fedosov star product. We expect an interested reader will refer to La Fuente-Gravy's articles [19], [20] for it.

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2. Cahen–Gutt moment map.

Let (M, ω) be a symplectic manifold of dimension $2m$. A symplectic connection ∇ is a torsion free affine connection such that $\nabla\omega = 0$. There always exists a symplectic connection on any symplectic manifold, see e.g. [2], section 2.1. Unlike the Levi-Civita connection on a Riemannian manifold, a symplectic connection is not unique on a symplectic manifold. Given two symplectic connections ∇ and ∇', we write the difference by S:

$$\nabla_X Y - \nabla'_X Y = S(X, Y).$$

Then $\omega(S(X,Y),Z)$ is totally symmetric in X, Y and Z. Conversely, if ∇ is a symplectic connection and $\omega(S(X,Y),Z)$ is totally symmetric, then $\nabla' := \nabla + S$ is a symplectic connection, see [2], section 2.1. In the geometry of symplectic connections, $\omega = \omega_{ij} dx^i \wedge dx^j$ and $(\omega^{ij}) = (\omega_{ij})^{-1}$ are used to raise and lower the indices, and we write

$$\underline{S} = \underline{S}_{ijk}\, dx^i \otimes dx^j \otimes dx^k \tag{4}$$

for

$$S = S_{ij}{}^k\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{5}$$

with $\underline{S}_{ijk} = S_{ij}{}^\ell \omega_{\ell k}$. With this notation, $\underline{S}_{ijk}$ is symmetric in i, j and k. Thus, on a symplectic manifold (M, ω), the space of symplectic connections, denoted by $\mathcal{E}(M, \omega)$, is an affine space modeled on the set of all smooth sections $\Gamma(S^3(T^*M))$ of symmetric covariant 3-tensors. Thus we may identify $\mathcal{E}(M, \omega)$ as

$$\mathcal{E}(M, \omega) \cong \nabla + \Gamma(S^3(T^*M)).$$

From now on we assume M is a closed manifold. On $\mathcal{E}(M, \omega)$ there is a natural symplectic structure $\Omega^{\mathcal{E}}$ defined at ∇ given by

$$\Omega^{\mathcal{E}}_\nabla(\underline{A}, \underline{B}) = \int_M \omega^{i_1 j_1} \omega^{i_2 j_2} \omega^{i_3 j_3} \underline{A}_{i_1 i_2 i_3}\, \underline{B}_{j_1 j_2 j_3}\ \omega_m \tag{6}$$

for $\underline{A}$, $\underline{B} \in T_\nabla \mathcal{E}(M, \omega) \cong \Gamma(S^3(T^*M))$ where $\omega_m := \frac{\omega^m}{m!}$. Since $\Omega^{\mathcal{E}}_\nabla$ is independent of ∇ we may omit ∇ and write $\Omega^{\mathcal{E}}$. There is a natural action of the group of symplectomorphisms (i.e. symplectic diffeomorphisms) of (M, ω)

on $\mathcal{E}(M,\omega)$, which is given for a symplectomorphism φ by

$$(\varphi(\nabla))_X Y = \varphi_*(\nabla_{\varphi_*^{-1}X}\varphi_*^{-1}Y)$$

for any $\nabla \in \mathcal{E}(M,\omega)$ and any smooth vector fields X and Y on M. This action preserves the symplectic structure $\Omega^{\mathcal{E}}$ on $\mathcal{E}(M,\omega)$. In particular, the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms acts on $\mathcal{E}(M,\omega)$ as symplectomorphisms. Let X_f be a Hamiltonian vector field on M for a smooth function f on M, that is

$$i(X_f)\omega = df. \tag{7}$$

Then the induced infinitesimal action of $-X_f$ on $\mathcal{E}(M,\omega)$ is computed as

$$\begin{aligned}(L_{X_f}\nabla)_Y Z &= [X_f,\nabla_Y Z] - \nabla_{[X_f,Y]}Z - \nabla_Y[X_f,Z]\\ &= R^{\nabla}(X_f,Y)Z + (\nabla\nabla X_f)(Y,Z)\end{aligned} \tag{8}$$

where R^{∇} is the curvature tensor of ∇, i.e. $R^{\nabla}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and $(\nabla\nabla X)(Y,Z) = \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z}X$. For $\nabla \in \mathcal{E}(M,\omega)$ we put

$$\begin{aligned}\mu(\nabla) = {}&\nabla_p\nabla_q \mathrm{Ric}(\nabla)^{pq}\\ &- \frac{1}{2}\mathrm{Ric}(\nabla)_{pq}\mathrm{Ric}(\nabla)^{pq} + \frac{1}{4}\mathrm{R}(\nabla,\omega)_{pqrs}\mathrm{R}(\nabla,\omega)^{pqrs}\end{aligned} \tag{9}$$

where

$$\mathrm{R}(\nabla,\omega)(X,Y,Z,W) = \omega(R(X,Y)Z,W) \tag{10}$$

and

$$\mathrm{Ric}(X,Y) = -\mathrm{tr}(Z \mapsto R(X,Z)Y).$$

Theorem 2.1 (Cahen–Gutt [3]). *The functional μ on $\mathcal{E}(M,\omega)$ gives a moment map for the action of* $\mathrm{Ham}(M,\omega)$.

This follows from the formula

$$\left.\frac{d}{dt}\right|_{t=0}\int_M \mu(\nabla + tA)\, f\,\omega_m = \Omega^{\mathcal{E}}(\underline{L_{X_f}\nabla},\underline{A}). \tag{11}$$

Note from (8) that

$$\underline{L_X\nabla} = (X^s R(\nabla,\omega)_{squt} + \nabla_q\nabla_u X^s\,\omega_{st})\,dx^q\otimes dx^u\otimes dx^t. \tag{12}$$

Now we assume that M is a compact Kähler manifold and that ω is a fixed symplectic form. We set

$$\begin{aligned}\mathcal{J}(M,\omega) = \{J \text{ integrable complex structure } | \\ (M,\omega,J) \text{ is a Kähler manifold.}\}\end{aligned}$$

La Fuente-Gravy [19], [20] considered the *Levi-Civita map* $lv : \mathcal{J}(M,\omega) \to \mathcal{E}(M,\omega)$ sending J to the Levi-Civita connection ∇^J of the Kähler manifold (M,ω,J). The following is a key lemma to this paper.

Lemma 2.2. *If we choose local holomorphic coordinates $z^1,\dots,z^m$ then for any smooth function f we have*

$$\begin{aligned}\underline{L_{X_f}\nabla^J} = {} & f_{ijk}dz^i\otimes dz^j\otimes dz^k + f_{\bar i\bar j\bar k}dz^{\bar i}\otimes dz^{\bar j}\otimes dz^{\bar k} \\ & + f_{ij\bar k}dz^i\otimes dz^j\otimes dz^{\bar k} + f_{\bar i\bar j k}dz^{\bar i}\otimes dz^{\bar j}\otimes dz^k \\ & + f_{ik\bar j}dz^i\otimes dz^{\bar j}\otimes dz^k + f_{\bar i\bar k j}dz^{\bar i}\otimes dz^j\otimes dz^{\bar k} \\ & + f_{jk\bar i}dz^{\bar i}\otimes dz^j\otimes dz^k + f_{\bar j\bar k i}dz^i\otimes dz^{\bar j}\otimes dz^{\bar k}\end{aligned} \tag{13}$$

where the lower indices of f stand for the covariant derivatives, e.g. $f_{ij\bar k} = \nabla_{\bar k}\nabla_j\nabla_i f$.

Proof. Write $\partial_i = \partial/\partial z^i$, $\overline{\partial}_i = \partial/\partial\overline{z^i}$ for short. We use the standard tensor calculus notations in Kähler geometry. Thus, R_{ABCD} denotes the Kählerian curvature tensor with $A,\ B,\ C,D$ running from $1,\dots,m,\bar 1,\dots,\bar m$ and we use the metric tensor or its inverse to lower and raise indices. First of all, from (7) we have

$$X_f = -J\mathrm{grad} f = -\sqrt{-1}f^\ell\partial_\ell + \sqrt{-1}f^{\bar\ell}\overline{\partial}_\ell. \tag{14}$$

Secondly, from (10) we have

$$R(\nabla,\omega)_{ABi\bar j} = -\sqrt{-1}R_{ABi\bar j}. \tag{15}$$

From (12), (14) and (15) we obtain

$$\underline{L_X\nabla}(\partial_i,\partial_j,\partial_k) = f_{ijk} \tag{16}$$

since $R_{ijCD} = 0$ on Kähler manifolds, and

$$\underline{L_X\nabla}(\partial_i, \partial_j, \partial_{\overline{k}}) = -R_{\overline{\ell}i\overline{k}j} f^{\overline{\ell}} + \nabla_i \nabla_j f_{\overline{k}} \tag{17}$$
$$= -R_{\overline{k}i\overline{\ell}j} f^{\overline{\ell}} + \nabla_i \nabla_{\overline{k}} f_j$$
$$= \nabla_{\overline{k}} \nabla_i f_j = f_{ji\overline{k}} = f_{ij\overline{k}}.$$

Since we know $\underline{L_X\nabla}$ is totally symmetric, taking the complex conjugates of (16) and (17) we obtain (13).

Alternatively, one may compute

$$R(-\sqrt{-1}f^\ell\partial_\ell + \sqrt{-1}f^{\overline{\ell}}\overline{\partial}_\ell, \partial_i + \overline{\partial}_i)(\partial_j + \overline{\partial}_j)$$
$$+ \nabla_{\partial_i+\overline{\partial}_i} \nabla_{\partial_j+\overline{\partial}_j}(-\sqrt{-1}f^\ell\partial_\ell + \sqrt{-1}f^{\overline{\ell}}\overline{\partial}_\ell),$$

from which one obtains (13), and the result shows that $\underline{L_X\nabla}$ is totally symmetric. □

An alternate proof of Lemma 2.2 is given in the next section, see (34), where we use an explicit description of the differential $lv_* : T_J\mathcal{J}(M,\omega) \to T_{\nabla^J}\mathcal{E}(M,\omega)$, see Lemma 3.1.

If $lv^*\Omega^{\mathcal{E}}$ is non-degenerate, $lv^*\mu$ gives the moment map for the action of $\mathrm{Ham}(M,\omega)$ with respect to the symplectic structure $lv^*\Omega^{\mathcal{E}}$. However the nondegeneracy of $lv^*\Omega^{\mathcal{E}}$ is not obvious, and in [19], Proposition 17, a sufficient condition for the non-degeneracy of $lv^*\Omega^{\mathcal{E}}$ is given (see also Lemma 4.9 and Remark 4.10 of the present paper). Disregarding this difficulty, notice that this symplectic structure is different from the one used in Donaldson [7] and Fujiki [10]. As we noted in the introduction, by choosing different symplectic structures on $\mathcal{J}(M,\omega)$, we obtain similar results for other non-linear geometric problems just as the cscK problem, see [13], [14], [16]. Each case of them should be studied in terms K-stability.

Proposition 2.3. *For a real smooth function f, $L_{X_f}\nabla^J = 0$ if and only if $L_{X_f}J = 0$. In this case, X_f is a holomorphic Killing vector field.*

Proof. We write ∇ instead of ∇^J for notational simplicity. By Lemma 2.2, $L_{X_f}\nabla = 0$ implies $\nabla'\nabla''\nabla''f = 0$. By integration by parts, this shows

$$\int_M |\nabla''\nabla''f|^2\omega_m = 0.$$

This implies X_f is holomorphic, that is, $L_{X_f}J = 0$ since for a smooth vector field $X = X' + X''$ we have

$$L_X J = 2\sqrt{-1}\nabla'' X' - 2\sqrt{-1}\nabla' X'', \tag{18}$$

as shown in Lemma 2.3 in [13]. Lemma 2.2 also shows, conversely if $L_{X_f}J = 0$ then $L_{X_f}\nabla = 0$. In this case, since X_f preserves ω it is a Killing vector field. This completes the proof of Proposition 2.3. □

Lemma 2.2 gives an alternate proof of the following result of La Fuente-Gravy. We consider $\mathfrak{g}_{\mathbf{R}}$ consisting of $\mathrm{grad}' f \in \mathfrak{g}$ of some real smooth function.

Corollary 2.4 (La Fuente-Gravy [20]). *Let (M, ω) be a compact Kähler manifold, and $\mathfrak{g}_{\mathbf{R}}$ be the real reduced Lie algebra of holomorphic vector fields. We normalize the Hamiltonian functions f so that $\int_M f\ \omega_m = 0$. Then*

$$\mathrm{Fut}(\mathrm{grad}' f) := \int_M \mu(\nabla^J)\, f\ \omega_m$$

is independent of the choice of $J \in \mathcal{J}(M, \omega)$.

Proof. By the moment map formula (11), the derivative of $\mathrm{Fut}(\mathrm{grad}' f)$ vanishes when $L_{X_f}\nabla = 0$. But by Proposition 2.3, $L_{X_f}\nabla = 0$ is equivalent to $L_{X_f}J = 0$. This completes the proof of Corollary 2.4 □

La Fuente-Gravy [20] shows that this is an obstruction to the existence of an integrable complex structure J such that the Levi-Civita connection ∇^J gives rise to closed Fedosov star product. This follows from his observation that the closedness of Fedosov star product implies $\mu(\nabla^J)$ is constant. He also observed the invariant $\mathrm{Fut}(\mathrm{grad}' f)$ in Corollary 2.4 is one of the invariants considered in [12]. The latter family of invariants includes the standard obstruction to the existence of Kähler-Einstein metrics [11]. In the last section of this paper we will obtain another obstruction for the Levi-Civita connection ∇^J to give rise to closed Fedosov star product.

3. An alternate proof of Lemma 2.2.

In this section we give an alternate proof of Lemma 2.2. The results in this section are used in the next section. Let (M, ω) be a compact symplectic manifold of dimension $2m$ with a fixed symplectic form ω. Let $\mathcal{J}(M, \omega)$ be the set of all ω-compatible integrable complex structures where J is said to be ω-compatible if $\omega(JX, JY) = \omega(X, Y)$ for all vector fields X and Y and

$\omega(X, JX) > 0$ for all non-zero X. Thus, for each $J \in \mathcal{J}(M, \omega)$, the triple (M, ω, J) determines a Kähler structure.

Consider J as acting on the cotangent bundle and decompose the complexified cotangent bundle into holomorphic and anti-holomorphic parts, i.e. $\pm\sqrt{-1}$-eigenspaces of J:

$$T^*M \otimes \mathbf{C} = T_J^{*\prime}M \oplus T_J^{*\prime\prime}M, \qquad T_J^{*\prime\prime}M = \overline{T_J^{*\prime}M}. \tag{19}$$

Take arbitrary $J' \in \mathcal{J}(M, \omega)$, then we also have the decomposition with respect to J'

$$T^*M \otimes \mathbf{C} = T_{J'}^{*\prime}M \oplus T_{J'}^{*\prime\prime}M, \qquad T_{J'}^{*\prime\prime}M = \overline{T_{J'}^{*\prime}M}. \tag{20}$$

If J' is sufficiently close to J then $T_{J'}^{*\prime}M$ can be expressed as a graph over $T_J^{*\prime}M$ in the form

$$T_{J'}^{*\prime}M = \{\ \alpha + \mu(\alpha) \mid \alpha \in T_J^{*\prime}M\ \} \tag{21}$$

for some endomorphism μ of $T_J^{*\prime}M$ into $T_J^{*\prime\prime}M$. We use the identification of $T_J^{*\prime\prime}M$ with $T_J^{\prime}M$ by the Kähler metric defined by the pair (ω, J), and then μ is regarded as

$$\begin{aligned} \mu &\in \Gamma(\mathrm{End}(T_J^{*\prime}M, T_J^{*\prime\prime}M)) \\ &\cong \Gamma(T_J^{\prime}M \otimes T_J^{*\prime\prime}M) \cong \Gamma(T_J^{\prime}M \otimes T_J^{\prime}M). \end{aligned} \tag{22}$$

In the tensor calculus notations this is expressed as

$$\mu^i{}_{\overline{k}} \mapsto g^{j\overline{k}}\mu^i{}_{\overline{k}} =: \mu^{ij}$$

where we chose a local holomorphic coordinate system $(z^1, \dots, z^m)$ with respect to J and wrote ω as $\omega = \sqrt{-1}\ g_{i\overline{j}}dz^i \wedge d\overline{z^j}$. The following is known:

$$\mu^{ij} = \mu^{ji}, \tag{23}$$

see the proof of Lemma 2.1 in [13].

If J_t is a smooth curve in $\mathcal{J}(M, \omega)$ with $J_0 = J$ and $\mu(t)$ is the curve in $\Gamma(\mathrm{End}(T_J^{*\prime}M, T_J^{*\prime\prime}M))$ satisfying

$$J_t(\alpha + \mu(t)\alpha) = \sqrt{-1}(\alpha + \mu(t)\alpha)$$

with $\dot{\mu}(0) = \mu$, then we have

$$\dot{J}|_{t=0} = 2\sqrt{-1}\mu - 2\sqrt{-1}\overline{\mu}, \tag{24}$$

see [13], pp.353–354. Let g_t be the Kähler metric of (ω, J_t), i.e. $g_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$. We write the Christoffel symbol in terms of real coordinates

$$\Gamma^i_{t,kj} = \frac{1}{2} g_t^{i\ell}\left(\frac{\partial g_{t,\ell j}}{\partial x^k} + \frac{\partial g_{t,\ell k}}{\partial x^j} - \frac{\partial g_{t,jk}}{\partial x^\ell}\right),$$

take the derivative at $t = 0$ and express it in terms of normal coordinates to obtain the derivative of the covariant derivative

$$\dot\nabla^i_{kj} dx^k = \dot\Gamma^i_{kj} dx^k = \frac{1}{2}(\dot g^i{}_{j,k} + \dot g^i{}_{k,j} - \dot g_{jk,}{}^i) dx^k \tag{25}$$

where i and j are respectively row and column indices. Note $g_{t\,ij} = \omega_{ik} J^k_{t\ j}$ and that ω is fixed. Thus we have $\dot g_{ij} = \omega_{ik} \dot J^k{}_j$.

In local complex coordinates $z^1, \dots, z^m, \overline{z}^1, \dots, \overline{z}^m$ giving complex struture J we have $\omega = \sqrt{-1} g_{i\bar j} dz^i \wedge dz^{\bar j}$, $\omega_{\bar j i} = -\omega_{i\bar j} = -\sqrt{-1} g_{i\bar j}$. It follows from (24) that

$$\dot g_{\bar i\bar j} = \omega_{\bar i k} 2\sqrt{-1} \mu^k{}_{\bar j} = 2\mu_{\bar i\bar j}, \tag{26}$$

$$\dot g_{ij} = \omega_{i\bar k}(-2\sqrt{-1})\overline{\mu}^{\bar k}{}_j = 2\overline{\mu}_{ij}, \tag{27}$$

$$\dot g_{\bar i j} = 0, \quad \dot g_{i\bar j} = 0. \tag{28}$$

The derivative $\dot\nabla$ of the connection is the section of $\mathrm{End}(TM) \otimes T^*M \cong T^*M \otimes T^*M \otimes TM$. Using ω a section A of $\mathrm{End}(TM) \otimes T^*M \cong T^*M \otimes T^*M \otimes TM$ can be identified with a section $\underline{A}$ of $\otimes^3 T^*M$. For example, this means

$$\underline{A}_{\bar i jk} = A^p{}_{jk}\omega_{p\bar i} = \sqrt{-1} g_{p\bar i} A^p{}_{jk}, \quad \underline{A}_{ijk} = A^{\overline{p}}{}_{jk}\omega_{\overline{p} i} = -\sqrt{-1} g_{i\overline{p}} A^{\overline{p}}{}_{jk}.$$

Lemma 3.1. *With the identification above, we have*

$$\begin{aligned} -\sqrt{-1}\underline{\dot\nabla} = {} & \overline{\mu}_{ij,k} dz^i \otimes dz^j \otimes dz^k - \mu_{\overline{i}\overline{j},\overline{k}} dz^{\overline{i}} \otimes dz^{\overline{j}} \otimes dz^{\overline{k}} \\ & + \overline{\mu}_{ij,\overline{k}} dz^i \otimes dz^j \otimes dz^{\overline{k}} - \mu_{\overline{i}\overline{j},k} dz^{\overline{i}} \otimes dz^{\overline{j}} \otimes dz^k \\ & + \overline{\mu}_{ik,\overline{j}} dz^i \otimes dz^{\overline{j}} \otimes dz^k - \mu_{\overline{i}\overline{k},j} dz^{\overline{i}} \otimes dz^j \otimes dz^{\overline{k}} \\ & + \overline{\mu}_{jk,\overline{i}} dz^{\overline{i}} \otimes dz^j \otimes dz^k - \mu_{\overline{j}\overline{k},i} dz^i \otimes dz^{\overline{j}} \otimes dz^{\overline{k}} \end{aligned} \tag{29}$$

where the comma , denotes the covariant derivative. In particular, $\underline{\dot\nabla}$ is totally symmetric in terms of the three components.

Proof. We lower the upper indices of (25) using ω, and then use (26), (27) and (28) to obtain

$$
\begin{aligned}
(30)\qquad -\sqrt{-1}\dot{\underline{\nabla}} &= -\sqrt{-1}\dot{\underline{\Gamma}} \\
&= \overline{\mu}_{ij,k}dz^i\otimes dz^j\otimes dz^k + \overline{\mu}_{ij,\overline{k}}dz^i\otimes dz^j\otimes dz^{\overline{k}} \\
&\quad - \mu_{\overline{i}\overline{j},k}dz^{\overline{i}}\otimes dz^{\overline{j}}\otimes dz^k - \mu_{\overline{i}\overline{j},\overline{k}}dz^{\overline{i}}\otimes dz^{\overline{j}}\otimes dz^{\overline{k}} \\
&\quad + \overline{\mu}_{ik,j}dz^i\otimes dz^j\otimes dz^k + \overline{\mu}_{ik,\overline{j}}dz^i\otimes dz^{\overline{j}}\otimes dz^k \\
&\quad - \mu_{\overline{i}\overline{k},j}dz^{\overline{i}}\otimes dz^j\otimes dz^{\overline{k}} - \mu_{\overline{i}\overline{k},\overline{j}}dz^{\overline{i}}\otimes dz^{\overline{j}}\otimes dz^{\overline{k}} \\
&\quad - \overline{\mu}_{jk,i}dz^i\otimes dz^j\otimes dz^k + \overline{\mu}_{jk,\overline{i}}dz^{\overline{i}}\otimes dz^j\otimes dz^k \\
&\quad - \mu_{\overline{j}\overline{k},i}dz^i\otimes dz^{\overline{j}}\otimes dz^{\overline{k}} + \mu_{\overline{j}\overline{k},\overline{i}}dz^{\overline{i}}\otimes dz^{\overline{j}}\otimes dz^{\overline{k}}.
\end{aligned}
$$

By Proposition 11 in [19], we see that $J((\nabla_X\mu)Y) - (\nabla_{JX}\mu)Y$ is symmetric in X and Y. Taking $X = \overline{\partial}_j$ and $Y = \overline{\partial}_k$ we see that $2\sqrt{-1}\nabla_{\overline{k}}\mu^i{}_{\overline{j}}$ is symmetric in j and k. By lowering the upper index i, we see $\nabla_{\overline{k}}\mu_{\overline{i}\overline{j}}$ is symmetric in j and k. By (23), $\mu_{\overline{i}\overline{j}}$ is symmetric in i and j. Thus $\nabla_{\overline{k}}\mu_{\overline{i}\overline{j}}$ is totally symmetric in i, j and k. Similarly $\nabla_k\overline{\mu}_{ij}$ is totally symmetric in i, j and k. It follows that the right hand side of (30) is totally symmetric in i, j and k. Further, using this symmetry, four terms in (30) cancel out, and we obtain (29). Instead of Proposition 11 in [19], one can use the torsion-freeness of the Levi-Civita connection for the Kähler metrics g_t, see [2]. □

We apply Lemma 3.1 when the family J_t is induced by the flow generated by a Hamiltonian vector field X_f of a smooth function f. Then by Lemma 2.3 in [13],

$$(31)\qquad L_{X_f}J = 2\sqrt{-1}\nabla''X'_f - 2\sqrt{-1}\nabla'X''_f.$$

Thus we have

$$\mu = \nabla''X'_f.$$

But X_f is given by (14), and thus

$$(32)\qquad \mu = -\sqrt{-1}\nabla''\mathrm{grad}'f,$$

and

$$(33)\qquad L_{X_f}J = 2\nabla''\mathrm{grad}'f + 2\nabla'\mathrm{grad}''f.$$

Thus (29) becomes

$$
\begin{aligned}
(34)\qquad -\underline{\dot{\nabla}} = {} & f_{ijk}dz^i \otimes dz^j \otimes dz^k + f_{\overline{ijk}}dz^{\bar{i}} \otimes dz^{\bar{j}} \otimes dz^{\overline{k}} \\
& + f_{ij\overline{k}}dz^i \otimes dz^j \otimes dz^{\overline{k}} + f_{\overline{ij}k}dz^{\bar{i}} \otimes dz^{\bar{j}} \otimes dz^k \\
& + f_{ik\bar{j}}dz^i \otimes dz^{\bar{j}} \otimes dz^k + f_{\overline{ik}j}dz^{\bar{i}} \otimes dz^j \otimes dz^{\overline{k}} \\
& + f_{jk\bar{i}}dz^{\bar{i}} \otimes dz^j \otimes dz^k + f_{\overline{jk}i}dz^i \otimes dz^{\bar{j}} \otimes dz^{\overline{k}}.
\end{aligned}
$$

Since the flow generated by X_f induces the infinitesimal action $-L_{X_f}\nabla$ on $\mathcal{E}(M,\omega)$, the result of (34) coincides with Lemma 2.2.

4. Cahen–Gutt extremal Kähler metrics

We consider the following functional on $\mathcal{J}(M,\omega)$ which is similar to the Calabi functional [4]. For $J \in \mathcal{J}(M,\omega)$ we consider the squared L^2-norm Φ of the moment map: $\Phi(J) := \int_M \mu(\nabla^J)^2\omega_m$.

Theorem 4.1. *A complex structure $J \in \mathcal{J}(M,\omega)$ is a critical point of Φ if* $\mathrm{grad}'\mu(\nabla^J)$ *is a holomorphic vector field.*

Proof. Let J_t be a smooth curve in $\mathcal{J}(M,\omega)$ with $J_0 = J$. Then by Theorem 2.1, we have

$$
\begin{aligned}
(35)\qquad \left.\frac{d}{dt}\right|_{t=0} \int_M \mu(\nabla^{J_t})^2\ \omega_m &= 2\left.\frac{d}{dt}\right|_{t=0} \int_M \mu(\nabla^{J_t})\mu(\nabla^J)\ \omega_m \\
&= 2\Omega^{\mathcal{E}}\left(L_{X_\mu}\nabla^J, \left.\frac{d}{dt}\right|_{t=0} \nabla^{J_t}\right)
\end{aligned}
$$

where we have put $\mu = \mu(\nabla^J)$. By Proposition 2.3, $L_{X_\mu}\nabla^J = 0$ if and only if $L_{X_\mu}J = 0$, that is, if $\mathrm{grad}'\mu(\nabla^J)$ is a holomorphic vector field. □

In this paper we call the Kähler metric $g = \omega(\cdot, J\cdot)$ a Cahen–Gutt extremal Kähler metric if $\mathrm{grad}'\mu(\nabla^J)$ is a holomorphic vector field. Note that this is different from the critical metrics of Fox [9].

Now we follow the arguments of [14] and [16] to obtain a Hessian formula of Φ, but we essentially follow the finite dimensional formal arguments of Wang [25]. Here the Hessian is considered on the subspace of the tangent space of $\mathcal{J}(M,\omega)$ at J consisting of the tangent vectors of the form

$$\dot{J} = 4\Re\nabla^i\nabla_{\bar{j}}\sqrt{-1}f\,\frac{\partial}{\partial z^i}\otimes\overline{dz^j} \\ = L_{JX_f}J \tag{36}$$

for a real smooth function $f \in C^\infty(M)$.

Let (M,ω,J) be a compact Kähler manifold, and ∇ the Levi-Civita connection. We define Lichnerowicz operator $\mathcal{L}$ of order 6 by

$$(h,\mathcal{L}f)_{L^2} = 3(\nabla'\nabla''\nabla''h, \nabla'\nabla''\nabla''f)_{L^2} - (\nabla''\nabla''\nabla''h, \nabla''\nabla''\nabla''f)_{L^2} \\ = 3\int_M \overline{\nabla^{\bar{i}}\nabla^j\nabla^k h}\,\nabla_i\nabla_{\bar{j}}\nabla_{\bar{k}}f\,\omega_m - \int_M \overline{\nabla^i\nabla^j\nabla^k h}\,\nabla_{\bar{i}}\nabla_{\bar{j}}\nabla_{\bar{k}}f\;\omega_m \tag{37}$$

for any complex valued smooth functions f and h. This is a self-adjoint elliptic differential operator of order 6. We further define the sixth order self-adjoint elliptic differential operator $\overline{\mathcal{L}} : C^\infty_{\mathbf{C}}(M) \to C^\infty_{\mathbf{C}}(M)$ by

$$\overline{\mathcal{L}}u = \overline{\mathcal{L}\overline{u}}. \tag{38}$$

Lemma 4.2. *If $\dot{J} = 4\Re\nabla^i\nabla_{\bar{j}}\sqrt{-1}f\,\frac{\partial}{\partial z^i}\otimes\overline{dz^j}$ for a real valued smooth function $f \in C^\infty(M)$, we have*

$$\left.\frac{d}{dt}\right|_{t=0}\mu(\nabla^{J(t)}) = \mathcal{L}f + \overline{\mathcal{L}}f.$$

Proof. Note that

$$\dot{J} = 2\sqrt{-1}\nabla_{\bar{j}}(-\sqrt{-1}(\sqrt{-1}f)^i) - 2\sqrt{-1}\nabla_j(-\sqrt{-1}(\sqrt{-1}f)^{\bar{i}}) \tag{39}$$

For any real smooth function h we obtain from Theorem 2.1, (39) and Lemma 3.1 that

$$\left.\frac{d}{dt}\right|_{t=0}\int_M h\,\mu(\nabla^{J(t)})\,\omega_m = \Omega^{\mathcal{E}}(\underline{L_{X_h}\nabla^J}, \underline{A})$$

where

$$\underline{A} = \sqrt{-1}f_{ijk}dz^i\otimes dz^j\otimes dz^k - \sqrt{-1}f_{\bar{i}\bar{j}\bar{k}}dz^{\bar{i}}\otimes dz^{\bar{j}}\otimes dz^{\bar{k}} \\ + \sqrt{-1}f_{ij\bar{k}}dz^i\otimes dz^j\otimes dz^{\bar{k}} - \sqrt{-1}f_{\bar{i}\bar{j}k}dz^{\bar{i}}\otimes dz^{\bar{j}}\otimes dz^k \\ + \sqrt{-1}f_{ik\bar{j}}dz^i\otimes dz^{\bar{j}}\otimes dz^k - \sqrt{-1}f_{\bar{i}\bar{k}j}dz^{\bar{i}}\otimes dz^j\otimes dz^{\bar{k}} \\ + \sqrt{-1}f_{jk\bar{i}}dz^{\bar{i}}\otimes dz^j\otimes dz^k - \sqrt{-1}f_{\bar{j}\bar{k}i}dz^i\otimes dz^{\bar{j}}\otimes dz^{\bar{k}}.$$

Thus, we obtain

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0}\int_M h\,\mu(\nabla^{J(t)})\,\omega_m \\
&= 2\Re(-(\nabla''\nabla''\nabla''h, \nabla''\nabla''\nabla''f)_{L^2} + 3(\nabla'\nabla''\nabla''h, \nabla'\nabla''\nabla''f)_{L^2}) \\
&= (h, \mathcal{L}f) + \overline{(h, \mathcal{L}f)} \\
&= (h, \mathcal{L}f) + (h, \overline{\mathcal{L}f}) = (h, \mathcal{L}f + \overline{\mathcal{L}}f).
\end{aligned}$$

This completes the proof. □

Recall that the Poisson bracket $\{h, f\}$ of two real smooth functions h and f is defined by

$$\{h, f\} := X_h f$$

where X_h is the Hamiltonian vector field of h, that is $i(X)\omega = dh$. For a Kähler form ω, the Poisson bracket is expressed in terms of local holomorphic coordinates as

$$\begin{aligned}
\{h, f\} &= \omega(X_h, J\mathrm{grad} f) \\
&= dh(J\mathrm{grad} f) \\
&= \sqrt{-1} f^\alpha h_\alpha - \sqrt{-1} f_\alpha h^\alpha
\end{aligned}$$

Lemma 4.3. *For real valued smooth functions f and h in $C^\infty(M)$ we have*

$$\Omega^{\mathcal{E}}(L_{X_h}\nabla^J, L_{X_f}\nabla^J) = -\int_M \{h, f\}\,\mu(\nabla^J)\omega_m.$$

Proof. Let σ_t be the flow generated by the Hamiltonian vector field of $h \in C^\infty(M)$. Since $\mu(\nabla^J)$ gives a $\mathrm{Ham}(M, \omega)$-equivariant moment map by Proposition 2.1 we have

$$\int_M f\,\mu(\sigma_t(\nabla^J))\,\omega_m = \int_M f \circ \sigma_t^{-1}\,\mu(\nabla^J)\,\omega_m. \tag{40}$$

Taking the time differential of σ_t we obtain the lemma by (11). □

Lemma 4.4. *For any smooth complex valued smooth function $f \in C^\infty(M)_{\mathbf{C}}$ we have*

$$(\overline{\mathcal{L}} - \mathcal{L})f = \sqrt{-1}\{f, \mu(\nabla^J)\} = f^\alpha \mu(\nabla^J)_\alpha - \mu(\nabla^J)^\alpha f_\alpha$$

where $f^\alpha = g^{\alpha\overline{\beta}}\partial f/\partial\overline{z^\beta}$ for local holomorphic coordinates $z^1, \ldots, z^m$.

Proof. It is sufficient to prove when f is real valued. For any real valued smooth function $h \in C^\infty(M)$ it follows from Lemma 4.3 that

$$\begin{aligned}(h, \overline{\mathcal{L}} f - \mathcal{L} f)_{L^2} &= \overline{3(\nabla'\nabla''\nabla'' h, \nabla'\nabla''\nabla'' f)_{L^2} - (\nabla''\nabla''\nabla'' h, \nabla''\nabla''\nabla'' f)_{L^2}} \\ &\quad - (3(\nabla'\nabla''\nabla'' h, \nabla'\nabla''\nabla'' f)_{L^2} \\ &\quad - (\nabla''\nabla''\nabla'' h, \nabla''\nabla''\nabla'' f)_{L^2}) \\ &= -\sqrt{-1}\Omega^{\mathcal{E}}(L_{X_h}\nabla^J, L_{X_f}\nabla^J) \\ &= -\sqrt{-1}(\{f, h\}, \mu(\nabla^J))_{L^2} \\ &= \sqrt{-1}(h, \{f, \mu(\nabla^J)\})_{L^2} \\ &= (h, \mu(\nabla^J)_\alpha f^\alpha - \mu(\nabla^J)^\alpha f_\alpha)_{L^2}.\end{aligned}$$

This completes the proof of Lemma 4.4. □

Lemma 4.5. *If* $f \in C^\infty(M)$ *and* $\dot{J} = 4\Re \nabla^i \nabla_{\bar{j}} \sqrt{-1} f \frac{\partial}{\partial z^i} \otimes d\overline{z^j}$, *then*

$$\frac{d}{dt}\bigg|_{t=0} \int_M \mu(\nabla^{J_t})^2 \omega_m = 4(f, \mathcal{L}\mu(\nabla^J))_{L^2} \tag{41}$$
$$= 4(f, \overline{\mathcal{L}}\mu(\nabla^J))_{L^2}.$$

Proof. In this proof we write ∇ instead of ∇^J for notational simplicity. We apply Theorem 2.1 and Lemma 4.2 to show that the left hand side of (41) is equal to

$$\begin{aligned}&2\Omega^{\mathcal{E}}(\underline{L_{X_{\mu(\nabla)}}\nabla, \dot{\nabla}}) \\ &= 4\Re(-(\nabla''\nabla''\nabla''\mu(\nabla), \nabla''\nabla''\nabla'' f)_{L^2} + 3(\nabla'\nabla''\nabla''\mu(\nabla), \nabla'\nabla''\nabla'' f)_{L^2}) \\ &= 2(\mu(\nabla), \mathcal{L} f)_{L^2} + 2\overline{(\mu(\nabla), \mathcal{L} f)_{L^2}} \\ &= 2(f, \mathcal{L}\mu(\nabla) + \overline{\mathcal{L}}\mu(\nabla))_{L^2}.\end{aligned}$$

But Lemma 4.4 implies

$$\overline{\mathcal{L}}\mu(\nabla) = \mathcal{L}\mu(\nabla).$$

Hence the left hand side of (41) is equal to

$$4(f, \mathcal{L}\mu(\nabla))_{L^2} = 4(f, \overline{\mathcal{L}}\mu(\nabla))_{L^2}.$$

□

Lemma 4.6. *Suppose that (ω, J) gives a Cahen–Gutt extremal Kähler metric so that $J\mathrm{grad}\,\mu(\nabla^J)$ is a holomorphic vector field. If*

$$\dot{J} = 4\Re\nabla^i\nabla_{\overline{j}}\sqrt{-1}f\,\frac{\partial}{\partial z^i}\otimes d\overline{z^j}$$

for some real smooth function $f \in C^\infty(M)$ then we have

$$\begin{aligned}\left(\frac{d}{dt}\Big|_{t=0}\mathcal{L}\right)\mu(\nabla^J) &= -\mathcal{L}(\mu(\nabla^J)^\alpha f_\alpha - f^\alpha\mu(\nabla^J)_\alpha)\\ &= \mathcal{L}(\overline{\mathcal{L}} - \mathcal{L})f\end{aligned}$$

Proof. First note that if

$$i(X_f)\omega = df$$

then

$$L_{JX_f}\omega = 2i\partial\overline{\partial}f. \tag{42}$$

Let $\{\varphi_s\}$ be the flow generated by $-JX_f$. Let S be a smooth function on M such that $\mathrm{grad}\,S$ is a holomorphic vector field. We shall compute $\frac{d}{ds}\big|_{s=0}\mathcal{L}(\varphi_s J,\omega)S$, and apply to $S = \mu(\nabla^J)$, and obtain the conclusion of Lemma 4.6. Let $\{S_s\}$ be a family of smooth functions such that $S_0 = S$, that

$$\mathrm{grad}'_s\,S_s = \mathrm{grad}'\,S,$$

where grad_s denotes the gradient with respect to $\varphi^*_{-s}\omega$, and that

$$\int_M S_s(\varphi^*_{-s}\omega)_m = \int_M S\omega_m.$$

This implies

$$\mathcal{L}(\varphi_s J,\omega)\varphi^*_s S_s = \varphi^*_s(\mathcal{L}(J,\varphi^*_{-s}\omega)S_s) = 0. \tag{43}$$

On the other hand, in general, if $\varphi^*_{-s}\omega = \omega + i\partial\overline{\partial}h$ then $S_s = S + S^\alpha h_\alpha$. Therefore, since $L_{JX_f}\omega = 2i\partial\overline{\partial}f$ by (42) we have

$$S_s = S + 2sS^\alpha f_\alpha + O(s^2). \tag{44}$$

Thus taking the derivative of (43), we obtain

$$\left(\frac{d}{ds}|_{s=0}\mathcal{L}\right)S + \mathcal{L}(-(JX_f)S + 2S^\alpha f_\alpha) = 0. \tag{45}$$

By an elementary computation we see

$$\begin{aligned}(JX_f)S &= g(JX_f, \operatorname{grad} S) = \omega(X_f, \operatorname{grad} S) = df(\operatorname{grad} S)\\ &= (\partial f + \overline{\partial} f)(\operatorname{grad}' S + \operatorname{grad}'' S) = f_\alpha S^\alpha + f^\alpha S_\alpha.\end{aligned}$$

Thus, from (45) and the above computation, we obtain

$$\begin{aligned}\left(\frac{d}{ds}|_{s=0}\mathcal{L}\right) S &= \mathcal{L}((f_\alpha S^\alpha + f^\alpha S_\alpha) - 2S^\alpha f_\alpha) = 0\\ &= \mathcal{L}(f^\alpha S_\alpha - f_\alpha S^\alpha)\\ &= \mathcal{L}(\overline{\mathcal{L}} - \mathcal{L})f.\end{aligned}$$

This completes the proof of Lemma 4.6. □

To express the Hessian formula of Φ, we consider its restriction to the subspace consisting of tangent vectors of the form $\dot{J} = 4\Re\nabla^i\nabla_{\bar{j}}\sqrt{-1}f\,\frac{\partial}{\partial z^i} \otimes \overline{dz^j} = L_{JX_f}J$ for a real smooth function $f \in C^\infty(M)$.

Theorem 4.7. *Suppose that J gives a Cahen–Gutt extremal Kähler metric so that J is a critical point of Φ. Let f and h be real smooth functions in $C^\infty(M)$. Then the Hessian* $\operatorname{Hess}(\Phi)_J$ *at J is given by*

$$\operatorname{Hess}(\Phi)_J(L_{JX_f}J, L_{JX_h}J) = 8(f, \mathcal{L}\overline{\mathcal{L}}h)_{L^2} = 8(f, \overline{\mathcal{L}}\mathcal{L}h)_{L^2}.$$

In particular, at any point J giving a Cahen–Gutt extremal Kähler metric, we have $\mathcal{L}\overline{\mathcal{L}} = \overline{\mathcal{L}}\mathcal{L}$ on the space $C^\infty_{\mathbf{C}}(M)$ of smooth complex valued functions since $\mathcal{L}\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}\mathcal{L}$ are both **C***-linear.*

Proof. Suppose $\dot{J} = 4\Re\nabla^i\nabla_{\bar{j}}\sqrt{-1}h\,\frac{\partial}{\partial z^i} \otimes \overline{dz^j}$. Then by Lemma 4.5, Lemma 4.6 and Lemma 4.2 we obtain

$$\begin{aligned}\operatorname{Hess}(\Phi)_J(L_{JX_f}J, L_{JX_h}J) &= \left.\frac{d}{dt}\right|_{t=0} 4(f, \mathcal{L}\mu(\nabla^J))_{L^2}\\ &= 4\left(f, \left(\left.\frac{d}{dt}\right|_{t=0}\mathcal{L}\right)\mu(\nabla^J) + \mathcal{L}\left.\frac{d}{dt}\right|_{t=0}\mu(\nabla^{J_t})\right)_{L^2}\\ &= 4(f, \mathcal{L}(\overline{\mathcal{L}} - \mathcal{L})h + \mathcal{L}(\mathcal{L} + \overline{\mathcal{L}})h)_{L^2}\\ &= 8(f, \mathcal{L}\overline{\mathcal{L}}h)_{L^2}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\mathrm{Hess}(\Phi)_J(L_{JX_f}J, L_{JX_h}J) &= \frac{d}{dt}\Big|_{t=0} 4(f, \overline{\mathcal{L}}\mu(\nabla^J))_{L^2} \\ &= 4\left(f, \left(\frac{d}{dt}\Big|_{t=0} \overline{\mathcal{L}}\right)\mu(\nabla) + \overline{\mathcal{L}}\,\frac{d}{dt}\Big|_{t=0}\mu(\nabla^{J_t})\right)_{L^2} \\ &= 4(f, \overline{\mathcal{L}}(\mathcal{L}-\overline{\mathcal{L}})h + \overline{\mathcal{L}}(\mathcal{L}+\overline{\mathcal{L}})h)_{L^2} \\ &= 8(f, \overline{\mathcal{L}}\mathcal{L}h)_{L^2}.\end{aligned}$$

This completes the proof of Theorem 4.7. □

Let $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda\neq 0}\mathfrak{g}_\lambda$ be the eigenspace decomposition of the reduced Lie algebra $\mathfrak{g}$ of holomorphic vector fields with respect to the action of $\mathrm{ad}(\mathrm{grad}'\mu(\nabla))$, and $\mathfrak{e} = \mathfrak{e}_0 + \sum_{\lambda\neq 0}\mathfrak{e}_\lambda$ be the corresponding decomposition of the space of complex valued potentials functions. Thus $\dim\mathfrak{e} = \dim\mathfrak{g} + 1$ because of the constant functions.

Theorem 4.8. *Let $g = \omega J$ be a Cahen–Gutt extremal Kähler metric on a compact Kähler manifold. Then we have the following:*

(a) *The space $\mathfrak{e}$ of potential functions of the reduced Lie algebra $\mathfrak{g}$ is included in* $\ker\mathcal{L}$.

(b) *$\overline{\mathcal{L}}$ maps $\mathfrak{e}$ into itself and coincides with the Poisson bracket with $\mu(\nabla)$. In particular, the eigenspace decomposition of $\overline{\mathcal{L}} : \mathfrak{e} \to \mathfrak{e}$ coincides with the decomposition $\mathfrak{e} = \mathfrak{e}_0 + \sum_{\lambda\neq 0}\mathfrak{e}_\lambda$.*

(c) *$\mathcal{L}$ and $\overline{\mathcal{L}}$ coincide when restricted to $\mathfrak{e}_0$, and are real operators on $\mathfrak{e}_0$.*

Proof. By the definition (37) of $\mathcal{L}$ it is clear that $\mathfrak{e}$ is included in $\ker\mathcal{L}$. Thus (a) follows. By Lemma 4.4, we have for $f \in \mathfrak{e}$

$$\begin{aligned}\overline{\mathcal{L}}f &= (\overline{\mathcal{L}} - \mathcal{L})f \\ &= \mu(\nabla)^\alpha f_\alpha - f^\alpha \mu(\nabla)_\alpha,\end{aligned} \tag{46}$$

and the right hand side is the Poisson bracket $\{\mu(\nabla), f\}$ and belongs to $\mathfrak{e}$. Further we have

$$\mathrm{grad}'\{\mu(\nabla), f\} = [\mathrm{grad}'\mu(\nabla), \mathrm{grad}'f].$$

Thus the eigenspace decompositions of $\overline{\mathcal{L}}$ coincides with $\mathfrak{e} = \mathfrak{e}_0 + \sum_{\lambda\neq 0}\mathfrak{e}_\lambda$. This proves (b). The equality (46) shows $\mathcal{L} = \overline{\mathcal{L}}$ on $\mathfrak{e}_0$. This proves (c). □

Lemma 4.9. *Let M be a compact Kähler manifold. If $g = \omega J$ is a Cahen–Gutt extremal Kähler metric with non-negative Ricci curvature then $\mathfrak{e} = \ker \mathcal{L}$.*

Proof. By (a) of Theorem 4.8 we have only to show $\ker \mathcal{L} \subset \mathfrak{e}$. Since

$$-\nabla^i \nabla_i \nabla_{\overline{j}} \nabla_{\overline{k}} f + \nabla^{\overline{i}} \nabla_{\overline{i}} \nabla_{\overline{j}} \nabla_{\overline{k}} f = 2 R^{\overline{\ell}}{}_{\overline{j}} \nabla_{\overline{k}} \nabla_{\overline{\ell}} f$$

for any complex valued smooth function f we see using (37) that if the Ricci curvature is non-negative

$$(f, \mathcal{L}f)_{L^2} \geq 2(\nabla'\nabla''\nabla'' f, \nabla'\nabla''\nabla'' f)_{L^2}. \tag{47}$$

Thus $\mathcal{L}f = 0$ implies $\nabla'\nabla''\nabla'' f = 0$. By integration by parts, this shows

$$\int_M |\nabla''\nabla'' f|^2 \omega_m = 0.$$

This implies $\mathrm{grad}' f$ is holomorphic. This shows $\ker \mathcal{L} \subset \mathfrak{e}$. □

Remark 4.10. *The condition of non-negative Ricci curvature coincides with the non-degenracy condition of $lv^*\Omega^{\mathcal{E}}$ due to La Fuente-Gravy, Proposition 17 in [19].*

Now we show a Cahen–Gutt version of Calabi's theorem [4] for extremal Kähler metrics. Before stating it we remark that it is well-known that for a Killing vector field X on a compact Kähler manifold M, the complex vector field $X - iJX$ is a holomorphic vector field. We identify the real Lie algebras $\mathfrak{i}(M)$ of all Killing vector fields on M with the real Lie subalgebra of the complex Lie algebra of all holomorphic vector fields by the identification $X \mapsto X - iJX$.

Theorem 4.11. *Let M be a compact Kähler manifold. If $g = \omega J$ is a Cahen–Gutt extremal Kähler metric with non-negative Ricci curvature then the reduced Lie algebra $\mathfrak{g}$ of holomorphic vector fields has the following structure:*

(a) $\mathrm{grad}'\mu(\nabla) = g^{i\overline{j}} \frac{\partial \mu(\nabla)}{\partial \overline{z^j}} \frac{\partial}{\partial z^i}$ *is in the center of* $\mathfrak{g}_0$.

(b) $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda > 0} \mathfrak{g}_\lambda$ *where $\mathfrak{g}_\lambda$ is the λ-eigenspace of* $\mathrm{ad}(\mathrm{grad}'\mu(\nabla))$. *Moreover, we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.*

(c) $\mathfrak{g}_0$ *is isomorphic to* $\mathfrak{i}(M) \otimes \mathbf{C}$*, and is the maximal reductive subalgebra of* $\mathfrak{g}$ *where* $\mathfrak{i}(M)$ *denotes the real Lie algebra of all Killing vector fields. In particular* $\mathfrak{g}_0$ *is reductive. Further, the identity component of the isometry group is a maximal compact subgroup of the identity component of the group of all biholomorphisms of* M*.*

Proof. Since $\mathfrak{g}_0$ is the 0-eigenspace with respect to the action of $\mathrm{ad}(\mathrm{grad}'\mu(\nabla))$, $\mathrm{grad}'\mu(\nabla)$ is in the center of $\mathfrak{g}_0$. This proves (a). By (47) $\mathcal{L}$ is a non-negative operator the non-zero eigenvalues of $\mathcal{L}$ are positive. Taking the complex-conjugate the same is true for $\overline{\mathcal{L}}$. Since by Theorem 4.8, (b), the non-zero eigenvalues of $\mathcal{L}$ coincide with those of $\mathrm{ad}(\mathrm{grad}'\mu(\nabla))$ we have $\lambda > 0$ for all non-zero λ's. This proves (b). From Theorem 4.8, (c), we see $f \in \mathfrak{e}_0$ satisfies both $\mathcal{L}f = 0$ and $\mathcal{L}\overline{f} = 0$. Since $\overline{\mathcal{L}}f = 0$ is equivalent to $\mathcal{L}\overline{f} = 0$, this implies $\mathcal{L}\Re f = 0$ and $\mathcal{L}\Im f = 0$. Hence by Lemma 4.9, both the real and imaginary part of f are potential functions of holomorphic vector fields. In general if $\mathrm{grad}'f$ is a holomorphic vector field for a real smooth function f then $J\mathrm{grad}\, f$ is a Killing vector field. It is also well-known that for a Killing vector field X on a compact Kähler manifold, $X - iJX$ is a holomorphic vector field. Hence we obtain $\mathfrak{g}_0 = \mathfrak{i}(M) \otimes \mathbf{C}$. In particular $\mathfrak{g}_0$ is reductive. To show that this is a maximal reductive Lie algebra, suppose we have a reductive Lie subalgebra $\mathfrak{l}$ containing $\mathfrak{g}_0$. If $\mu(\nabla)$ is constant we have $\mathfrak{g} = \mathfrak{g}_0$ and thus $\mathfrak{g}_0$ is maximal. Thus we may assume $\mu(\nabla)$ is not constant. Let X be an element of $\mathfrak{l}$ in the form $X = X_0 + \sum_\lambda X_\lambda$ where $X_0 \in \mathfrak{g}_0$ and $X_\lambda \in \mathfrak{g}_\lambda$. Since

$$Ad(\exp(t\,\mathrm{grad}'\mu(\nabla))X = X_0 + \sum_{\lambda>0} e^{\lambda t} X_\lambda \in \mathfrak{l}$$

for any $t \in \mathbf{R}$, by considering this for many values of t we have $X_\lambda \in \mathfrak{l}$ for each λ. If $X_\lambda \neq 0$ for some $\lambda > 0$ then $\mathrm{grad}'\mu(\nabla) + X_\lambda$ generates a solvable Lie subalgebra. This contradicts the reductiveness of $\mathfrak{l}$. Thus $X_\lambda = 0$ for all $\lambda > 0$. Thus $\mathfrak{g}_0$ is a maximal reductive Lie subalgebra. To show the last statement let K be a compact connected Lie subgroup of the group of all biholomorphisms of M including the identity component of the group of all isometries of M, and $\mathfrak{k}$ be its real Lie algebra. Since $\mathfrak{k} \otimes \mathbf{C}$ is reductive and $\mathfrak{g}_0$ is a maximal reductive Lie subalgebra we must have $\mathfrak{k} \otimes \mathbf{C} = \mathfrak{g}_0 = \mathfrak{i}(M) \otimes \mathbf{C}$. Thus any element of $\mathfrak{k}$ can be written in the form $J\mathrm{grad}\, f + \mathrm{grad}\, h$ for some real potential functions f and h where $J\mathrm{grad}\, f$ and $J\mathrm{grad}\, h$ are Killing vector fields (by using the identification remarked before the statement of Theorem 4.11). Since $\mathfrak{i}(M)$ is a Lie subalgebra of $\mathfrak{k}$ we have $\mathrm{grad}\, h \in \mathfrak{k}$. If $\mathrm{grad}\, h$ is non-zero it generates a non-compact group since h has at least two critical points and K can not be compact. Thus $\operatorname{grad} h$ has to be zero and $\mathfrak{k} = \mathfrak{i}(M)$. This proves (c). $\square$

Proof of Theorem 1.1. As shown by La Fuente-Gravy [19], if $lv(J)$ gives rise to closed Fedosov star product then $\mu(\nabla)$ is constant and thus $\mathfrak{g} = \mathfrak{g}_0$. It follows from Theorem 4.11, (c), that $\mathfrak{g}$ is reductive. This completes the proof. $\square$

Example 4.12. *Let M be a one point blow-up of the complex projective plane $\mathbf{CP}^2$. Since M is simply connected the reduced Lie algebra of holomorphic vector fields coincides with the Lie algebra of all holomorphic vector fields. The corresponding Lie group is the group of all biholomorphic automorphisms. Any automorphism of M leaves the exceptional divisor invariant, and thus descends to an automorpism fixing the point where the blow-up is performed. Thus the reduced Lie algebra is of the form*

$$\left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \Big/ \text{center}$$

which is not reductive. Thus any Kähler metric with non-negative Ricci curvature on M does not give closed Fedosov star product. This example is also the simplest example of a compact Kähler manifold with no cscK metric.

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