

Quantization of Hamiltonian coactions via twist

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In this paper we introduce a notion of quantum Hamiltonian (co)action of Hopf algebras endowed with Drinfel'd twist structure (resp., 2-cocycles). First, we define a classical Hamiltonian action in the setting of Poisson Lie groups compatible with the 2-cocycle structure and we discuss a concrete example. This allows us to construct, out of the classical momentum map, a quantum momentum map in the setting of Hopf coactions and to quantize it by using Drinfel'd approach.

Introduction

Deformation quantization has been introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [3] and since then many developments occurred. A (formal) star product on a Poisson manifold M is defined as a formal associative deformation of the algebra of smooth functions $\mathcal{C}^\infty(M)$ on M . Existence and classification of star products on Poisson manifolds has been proved via formality theory in [17]. In the same spirit, Drinfel'd introduced the notion of quantum groups as deformations of Hopf algebras, whose semiclassical limit are the so-called Poisson Lie groups which are Lie groups with multiplicative Poisson structures (see e.g. the textbooks [8, 23] for a detailed discussion).

In this paper we focus on particular classes of star products which are induced by a (formal) Drinfel'd twist by means of universal deformation formulas (UDF) as discussed e.g. in [9, 10]. Roughly speaking, a Drinfel'd twist of an enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an element $\mathcal{F} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ compatible with the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$. Given a Hopf algebra action of $\mathcal{U}(\mathfrak{g})$ on an associative algebra one can deform the $\mathcal{U}(\mathfrak{g})$ -module algebra and the deformed product turns out to be a star product. It is important to stress that the $\mathcal{U}(\mathfrak{g})$ -module algebra is automatically endowed with a Poisson bracket defined as the semiclassical limit of such star product. In recent works the UDF has been further studied, e.g. [5, 7, 11, 13, 15]. Also, a twist

defines a 2-cocycle on the Hopf algebra $F(G)$ of regular functions on G and it can be seen that the star products induced via UDF coincide with star products induced by the 2-cocycle on $F(G)$ -comodule algebras. Finally, a non-formal version of Drinfel'd twist and its corresponding UDF has been discussed in [6].

Given a Lie algebra action $\varphi: \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ on a smooth manifold M , we can always obtain a Hopf algebra action $\mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$. Thus, Drinfel'd approach can be interpreted by saying that symmetries encoded by Lie algebra actions induce quantization. Also, this approach provides a notion of quantized action. In this paper we prove that this approach is compatible with Hamiltonian actions. In other words, given a classical Hamiltonian action our goal is to quantize it by using Drinfel'd approach and get a notion of quantum momentum map. The problem of quantizing the momentum map has been the main topic of many works, e.g. [14] and [22]. In general, the interest for the quantization of the momentum map is motivated by the fact that conserved quantities described via the momentum map lead to phase space reduction which constructs from the high-dimensional original phase space one of a smaller dimension. Thus, it is highly desirable to find an analogue in the quantum setting. A study of the compatibility of the notion of quantum action provided by Drinfeld and the notion of Hamiltonian action was so far absent. In this paper we prove that the two notions are actually compatible and we construct a quantum momentum map via twist.

The content of this work is as follows.

In Section 1 we discuss the well-known notions of Drinfel'd twist and its corresponding 2-cocycle and the construction of the universal deformation formula. Twist and 2-cocycle induce a quantum group structure which is briefly recalled.

Section 2 contains a definition of Hamiltonian actions in the setting of Poisson Lie groups which generalizes the one contained in [19, 21]. More precisely, we need to introduce a notion of classical Hamiltonian action which is compatible with twist, which is necessary in order to quantize Hamiltonian actions by using Drinfel'd approach.

It is known that the semiclassical limit of a twist gives rise to an element $r \in \mathfrak{g} \wedge \mathfrak{g}$, called r -matrix, satisfying the condition $\llbracket r, r \rrbracket = 0$ (for a detailed treatment of the relation between r -matrices and twist see [13]). It can be proved that r -matrices always induces a Lie bialgebra structure on \mathfrak{g} . Thus, the corresponding Lie group G automatically becomes a Poisson Lie group, since the Poisson tensor obtained by integrating the Lie bialgebra structure on \mathfrak{g} is multiplicative. The concept of momentum map for Poisson Lie groups

acting on Poisson manifolds has been first introduced by Lu in [19, 21], in the case in which the Poisson structures of G , its dual G^* and M are fixed. In contrast to the ordinary momentum map it takes values in G^* and the equivariance is defined in relation to the so-called dressing action of G on G^* . Here we introduce a slight generalization and then focus on the case in which, in the same spirit as Drinfel'd, the Poisson structure on G^* is induced by r via the dressing action and on M via the action φ .

In Section 3 we construct a momentum map in the setting of Hopf algebra actions and coactions and study its quantization. More precisely, given a classical Hamiltonian action $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ with momentum map $J : M \rightarrow G^*$ we construct a corresponding Hopf algebra action and we prove that J^* defines a momentum map for this action. This allows us to define the notion of Hamiltonian Hopf algebra action. Motivated by the significance of coactions in the theory of quantum groups in the C^* -algebraic framework, we give a dual version of the above result and prove that given φ the corresponding Hopf algebra coaction $\delta_\Phi : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \otimes F(G)$ is also Hamiltonian. Finally, using the UDF we obtain the quantized algebras $\mathcal{C}_\hbar^\infty(M)$ and we prove that the quantum group coaction $\delta_\Phi : \mathcal{C}_\hbar^\infty(M) \rightarrow \mathcal{C}_\hbar^\infty(M) \otimes F_\hbar(G)$ is again Hamiltonian.

1. Preliminaries

Let \mathfrak{g} be a (finite-dimensional) Lie algebra and consider the algebra $\mathcal{U}(\mathfrak{g})[[\hbar]]$ of formal power series with coefficients in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. It can be endowed with a (topologically free) Hopf algebra structure, denoted by $(\mathcal{U}(\mathfrak{g})[[\hbar]], \Delta, \epsilon, S)$. Let us recall the definition of a Drinfel'd twist and its semiclassical limit, see [9, 10].

Definition 1.1 (Twist). An element $\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ is said to be a twist on $\mathcal{U}(\mathfrak{g})[[\hbar]]$ if the following three conditions are satisfied:

- i) $\mathcal{F} = 1 \otimes 1 + \sum_{k=1}^\infty \hbar^k \mathcal{F}_k$.
- ii) $(\mathcal{F} \otimes 1)(\Delta \otimes 1)(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta)(\mathcal{F})$.
- iii) $(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1$.

We sometimes use the notation $\mathcal{F} = \mathcal{F}^\alpha \otimes \mathcal{F}_\alpha$. The semiclassical limit of a twist gives rise to a well-known structure on the Lie algebra \mathfrak{g} called r -matrix, as proved in [10] or [16, Thm. 1.14]. In fact, we have the following claim.

Proposition 1.2. *Given a twist \mathcal{F} on $\mathcal{U}(\mathfrak{g})[[\hbar]]$, the antisymmetric part of its first order is a classical r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$.*

Given a twist we can obtain a deformed Hopf algebra structure on $\mathcal{U}(\mathfrak{g})[[\hbar]]$.

Proposition 1.3. *Let \mathcal{F} be a twist on $\mathcal{U}(\mathfrak{g})[[\hbar]]$. Then the algebra $\mathcal{U}(\mathfrak{g})[[\hbar]]$ endowed with coproduct given by*

$$(1.1) \quad \Delta_{\mathcal{F}} := \mathcal{F} \Delta \mathcal{F}^{-1},$$

undeformed counit and antipode $S_{\mathcal{F}} := u_{\mathcal{F}} S(X) u_{\mathcal{F}}^{-1}$, where $u_{\mathcal{F}} := \mathcal{F}^{\alpha} S(\mathcal{F}_{\alpha})$ is again a Hopf algebra denoted by $\mathcal{U}_{\mathcal{F}}(\mathfrak{g})$.

As a consequence, the twist automatically defines a Lie bialgebra structure. Given a twist on the universal enveloping algebra, we can always define a star product on any $\mathcal{U}(\mathfrak{g})$ -module algebra. In particular, let us consider the algebra $\mathcal{C}^{\infty}(M)$ of smooth functions on a manifold M with pointwise multiplication m_M and a Hopf algebra action

$$(1.2) \quad \Phi: \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^{\infty}(M) \longrightarrow \mathcal{C}^{\infty}(M): (X, f) \mapsto \Phi(X, f)$$

This action can be immediately extended to formal power series, allowing the following result.

Lemma 1.4 (Universal deformation formula). *The product defined by*

$$(1.3) \quad f \star_{\mathcal{F}} g = m_M(\Phi(\mathcal{F}^{-1}, (f \otimes g)))$$

for $f, g \in \mathcal{C}^{\infty}(M)[[\hbar]]$ is an associative star product quantizing the Poisson structure induced by the semiclassical limit r of \mathcal{F} via the action.

We denote the deformed algebra by $\mathcal{C}_{\mathcal{F}}^{\infty}(M)$. Moreover, it is important to remark that the deformed algebra $\mathcal{C}_{\mathcal{F}}^{\infty}(M)$ is now a left $\mathcal{U}_{\mathcal{F}}(\mathfrak{g})$ -module algebra

We can give a dual version of the above discussion by using the notions of 2-cocycles and coactions. A more detailed discussion about 2-cocycles and their duality with twist can be found in [2]. Consider the Hopf algebra $F(G)$ of regular functions on G , where G is the Lie group corresponding to the finite-dimensional Lie algebra \mathfrak{g} . It is known that $F(G)$ and $\mathcal{U}(\mathfrak{g})$ are dually paired Hopf algebras with pairing denoted by $\langle \cdot, \cdot \rangle$. Thus,

given a twist \mathcal{F} there corresponds an element $\gamma : (F(G) \otimes F(G))[[\hbar]] \rightarrow \mathbb{K}$ on $F(G)[[\hbar]]$ defined by

$$(1.4) \quad \gamma(f \otimes g) := \langle \mathcal{F}^\alpha, f \rangle \langle \mathcal{F}_\alpha, g \rangle,$$

for all $f, g \in F(G)$. Roughly, from the second condition mentioned in Definition 1.1 it is easy to see that γ satisfies the 2-cocycle condition

$$(1.5) \quad \gamma(f_{(1)} \otimes g_{(1)})\gamma(f_{(2)}g_{(2)} \otimes h) = \gamma(g_{(1)} \otimes h_{(1)})\gamma(f \otimes g_{(2)}h_{(2)}),$$

for any $f, g, h \in F(G)$. Here we used the Sweedler notation. Thus, dualizing the deformed Hopf algebra $\mathcal{U}_{\mathcal{F}}(\mathfrak{g})$ obtained in Proposition 1.3 we immediately obtain a twisted Hopf algebra denoted by $\mathcal{C}_\gamma^\infty(G)$ with a new associative product m_G^γ defined by

$$(1.6) \quad \langle \Delta_{\mathcal{F}}(X), f \otimes g \rangle = \langle X, m_G^\gamma(f \otimes g) \rangle$$

As it will be used in the following we resume this structure in the following defining.

Definition 1.5 (Quantum group). The quantum group corresponding to G is defined to be the Hopf algebra $\mathcal{C}_\gamma^\infty(G)$ given by $(F(G)[[\hbar]], m_G^\gamma, \Delta, \epsilon, S)$ where the deformed product is given by (1.6) and coproduct and counit are undeformed.

The deformed Hopf algebras $\mathcal{C}_\gamma^\infty(G)$ and $\mathcal{U}_{\mathcal{F}}(\mathfrak{g})$ are again dually paired via the same pairing. Finally, if $\mathcal{C}^\infty(M)$ is a left $\mathcal{U}(\mathfrak{g})$ -module algebra via (1.2), it is automatically a right- $F(G)$ -comodule algebra (the coaction $\delta : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \otimes F(G)$ can be easily obtained by dualizing Φ , see [23, Prop. 1.6.11]). In the same spirit of Lemma 1.4, the algebra structure of $\mathcal{C}^\infty(M)$ can be equivalently deformed by considering a 2-cocycle on $F(G)$ and pushing its deformation on $\mathcal{C}^\infty(M)$ via the coaction δ .

2. Hamiltonian actions

In this section we introduce the notion of Hamiltonian action in the setting of Poisson Lie groups. This notion has been first defined in [19, 21] in the case of a Poisson Lie group acting on a Poisson manifold with both Poisson structures fixed. In our work we are mainly interested in the case in which the Poisson structure on the manifold is the one induced by the action. This requires a slight generalization of the notion of Hamiltonian action.

2.1. Dressing generators

In the same spirit of [19, 21], the notion of Hamiltonian action relies on the definition of momentum map, which provides us of a comparison tool between the dressing orbits and the orbit of the considered action. For this reason, we first focus on the dressing action and in particular on the possible descriptions of the corresponding fundamental vector fields.

Let us consider a Lie bialgebra \mathfrak{g} with dual and double denoted by \mathfrak{g}^* and \mathfrak{d} , respectively. The Lie groups G and G^* associated to \mathfrak{g} and \mathfrak{g}^* , respectively, turn into Poisson Lie groups. Furthermore, the Lie group D corresponding to the double Lie algebra \mathfrak{d} is called double of the Poisson Lie group G .

Consider $g \in G$, $u \in G^*$ and let $ug \in D$ be their product. Since $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, elements in D close to the unit can be decomposed in a unique way as a product of an element in G and an element in G^* . Then, there exist elements ${}^u g \in G$ and $u^g \in G^*$ such that

$$(2.1) \quad ug = {}^u g u^g.$$

Hence, the action of $g \in G$ on $u \in G^*$ is given by

$$(2.2) \quad (u, g) \mapsto (ug)_G^*$$

where $(ug)_G^*$ denotes the G^* -factor of $ug \in D$. This defines a left action of G on G^* , called *dressing action*. This action plays an important role in the context of Poisson actions since its orbits coincides with the symplectic leaves of G^* and its linearization is the coadjoint action. Let us denote by ℓ_X the corresponding fundamental vector field for $X \in \mathfrak{g}$. In the following we introduce the notion of dressing generators, which are one-forms that give us the fundamental vector fields ℓ_X if contracted with the Poisson bitensor. As it will be seen in the next sections these forms are in general not globally defined, so we use the notation $\Omega_{\text{loc}}^1(G^*)$ to denote local forms on G^* .

Definition 2.1 (Dressing generator). The map $\alpha : \mathfrak{g} \rightarrow \Omega_{\text{loc}}^1(G^*) : X \mapsto \alpha_X$ is said to be dressing generator with respect to the Poisson structure π on G^* if the fundamental vector field ℓ_X of the dressing action can be written as

$$(2.3) \quad \ell_X = \pi^\sharp(\alpha_X)$$

and satisfies

$$(2.4) \quad \alpha_{[X,Y]} = [\alpha_X, \alpha_Y]_{\pi_\ell},$$

$$(2.5) \quad d\alpha_X = \alpha \wedge \alpha \circ \delta(X).$$

Here δ denotes the Lie bialgebra structure on \mathfrak{g} .

Remark 2.2. The first example of dressing generators with respect to the standard dual Poisson structure π_* is given by the left-invariant one-forms corresponding to the element X , as proved in [18, Appendix 2, page 66]. As already mentioned, the dressing generators with respect to a generic Poisson structure on G^* are in general not globally defined (a concrete example is computed in the next section). However, the contraction with the Poisson tensor still gives rise to a smooth vector field.

Here we are interested to the case in which \mathfrak{g} is endowed with an r -matrix and we consider the Poisson structure π_ℓ induced by the infinitesimal dressing action $\ell: \mathfrak{g} \rightarrow \Gamma^\infty(TG^*)$ via

$$(2.6) \quad \pi_\ell = r_{ij} \ell_{X_i} \wedge \ell_{X_j}.$$

This is a natural candidate since the contraction of π_ℓ with one-forms satisfying (2.4)–(2.5) gives rise automatically to an infinitesimal Poisson action, as proved in the following Lemma.

Lemma 2.3. *Given a map $\alpha: \mathfrak{g} \rightarrow \Omega_{\text{loc}}^1(G^*)$ satisfying (2.4)–(2.5) then we have:*

- i) *The map $\mathfrak{g} \ni X \mapsto \pi_\ell^\sharp(\alpha_X) \in \Gamma^\infty(TG^*)$ is a Lie algebra morphism*
- ii) *The map $\mathfrak{g} \ni X \mapsto \pi_\ell^\sharp(\alpha_X) \in \Gamma^\infty(TG^*)$ is an (infinitesimal) Poisson action.*

Proof. Let us compute:

$$\begin{aligned} \pi_\ell^\sharp(\alpha_{[X,Y]}) &\stackrel{(2.4)}{=} \pi_\ell^\sharp([\alpha_X, \alpha_Y]_{\pi_\ell}) \\ &\stackrel{(*)}{=} [\pi_\ell^\sharp(\alpha_X), \pi_\ell^\sharp(\alpha_Y)]. \end{aligned}$$

In (*) we used the fact that π_ℓ^\sharp is a Lie algebra morphism with respect to the Lie bracket of one-forms $[a, b]_{\pi_\ell} = \mathcal{L}_{\pi_\ell^\sharp(a)}b - \mathcal{L}_{\pi_\ell^\sharp(b)}a - d\pi_\ell(a, b)$. Furthermore, we have:

$$\begin{aligned} \wedge^2 \pi_\ell^\sharp(\alpha \wedge \alpha \circ \delta(X)) &\stackrel{(2.5)}{=} \wedge^2 \pi_\ell^\sharp(d\alpha_X) \\ &\stackrel{(*)}{=} d_{\pi_\ell} \pi_\ell^\sharp(\alpha_X). \end{aligned}$$

In (*) we used $d_\pi(\wedge^p \pi^\sharp)(\xi) = (\wedge^{p+1} \pi^\sharp)(d\xi)$. □

Example 2.4 (Dressing generators on $ax + b$). Let us denote by \mathfrak{s} the Lie algebra with basis H, E and commutation relation

$$(2.7) \quad [H, E] = 2E,$$

also known as $ax + b$. The corresponding group is denoted by S and we consider the dressing action $S \times S^* \rightarrow S$. Then we have that the dressing generators with respect to π_ℓ are given by the local forms

$$(2.8) \quad \alpha_H = \frac{1}{y} dx \quad \text{and} \quad \alpha_E = \frac{1}{2y} dy.$$

The complete discussion of this example can be found in the Appendix A.

2.2. Hamiltonian actions

Using the notion of dressing generator we give a new definition of Hamiltonian action in this context.

Definition 2.5 (Momentum map). Let $\Phi : G \times M \rightarrow M$ be an action of (G, π_G) on (M, π) and α_X the dressing generator with respect to a Poisson structure π_{G^*} on G^* .

i) A momentum map for Φ is a map $J : M \rightarrow G^*$ such that

$$(2.9) \quad \varphi(X) = \pi^\sharp(J^*(\alpha_X)),$$

where $\varphi(X)$ is the fundamental vector field of Φ . In other words, J is defined by the commutativity of the following diagram:

$$(2.10) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & \Gamma^\infty(TM) \\ \alpha \downarrow & & \uparrow \pi_M^\sharp \\ \Gamma^\infty(T^*G^*) & \xrightarrow{J^*} & \Gamma^\infty(T^*M) \end{array}$$

ii) A map $J : M \rightarrow G^*$ is said to be ℓ -equivariant if it intertwines the fundamental vector field $\varphi(X)$ and the dressing action ℓ_X for any X .

Lemma 2.6. *The momentum map J defined above is ℓ -equivariant if and only if it is Poisson.*

Proof. Let us consider generic Poisson structures π on M and π_{G^*} on G^* . Thus, J is a Poisson map if and only if

$$J_*(\pi^\sharp(J^*(\alpha))) = \pi_{G^*}^\sharp(\alpha).$$

Let α be the dressing generator corresponding to π_{G^*} . Thus $\pi_{G^*}^\sharp(\alpha_X) = \ell_X$ and $\pi^\sharp(J^*(\alpha_X)) = \varphi(X)$ and the equation above coincides with the ℓ -equivariance. □

Now the notion of Hamiltonian follows naturally:

Definition 2.7 (Hamiltonian action). An action Φ of (G, π_G) on (M, π_M) is said to be Hamiltonian if it is Poisson and is generated by a ℓ -equivariant momentum map $J : M \rightarrow G^*$.

Since in the following we mainly use the infinitesimal action φ , we say that it is Hamiltonian whenever the corresponding Φ is Hamiltonian.

Remark 2.8. i) If we choose the standard dual Poisson structure on G^* , the dressing generators are the left-invariant one-forms and the above definition boils down to the definition of momentum map and Hamiltonian action given by Lu in [19, 21].

ii) Let \mathfrak{g} be a triangular Lie algebra with r -matrix r , acting on a manifold M by $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$. We denote by π_r the Poisson structure

induced by r via

$$(2.11) \quad \pi_r = r^{ij} \varphi(X_i) \wedge \varphi(X_j).$$

In this case the action φ and its global corresponding are automatically Poisson. (The proof is the same as the one given in Lemma 2.3).

Example 2.9 (Dressing action). The easiest example is given by the dressing action. Here the momentum map is just the identity.

Example 2.10 (Coadjoint action). Let us consider the Poisson structure π_r induced by the coadjoint action. Notice that π_r does not coincide with the linear one. As proved in [1, Section 3.3] one can define a map $j : \mathfrak{g}^* \rightarrow \mathfrak{d}$ by $j(\xi) = \xi - r(\xi, \cdot)$. Thus, the modified exponential is given by

$$\text{Exp} : \mathfrak{g}^* \rightarrow G^* : \text{Exp}(\xi) := \text{pr}_{G^*}(\exp(j(\xi))).$$

In contrast to the usual exponential map it intertwines the coadjoint action with the dressing action, hence it takes symplectic leaves to symplectic leaves. In other words, we have

$$\ell_X = \text{Exp}_* \varphi(X).$$

If G is compact with the Lu-Weinstein Poisson structure [20], Exp is a global diffeomorphism (see [1, Remarks 3.5]). An easy computation shows that Exp is a momentum map for the coadjoint action.

Remark 2.11. From the above example we can construct other Hamiltonian actions. Given a standard momentum map $\mu : M \rightarrow \mathfrak{g}^*$ which is ad^* -equivariant we can always construct a momentum map $J : M \rightarrow G^*$ by composing μ and Exp . For instance, observing that $r^\sharp : \mathfrak{g}^* \rightarrow \mathfrak{g}$ intertwines adjoint and coadjoint actions we can conclude that the adjoint action is Hamiltonian with momentum map given by the composition of r^\sharp with Exp .

Remark 2.12. The reduction can be obtained with various techniques (see e.g. [12]). We here remark that the preimage $C = J^{-1}(\{0\})$ of a ℓ -invariant momentum map is a coisotropic submanifold and \mathcal{J}_C the corresponding vanishing ideal. Thus the reduced algebra can be easily obtained by the quotient $\mathcal{B}_C/\mathcal{J}_C$ where $\mathcal{B}_C = \{f \in \mathcal{C}^\infty(M) \mid \{f, \mathcal{J}_C\} \subseteq \mathcal{J}_C\}$.

3. Hamiltonian Hopf algebra (co)actions

In this section we aim to give a definition of Hamiltonian (co)action in the setting of Hopf algebra (co)actions and a possible quantization procedure. In the same spirit of Definition 2.7, given an Hopf algebra action Φ , a momentum map has to be an intertwiner between dressing action and Φ . In order to introduce this notion we first prove that given a classical Hamiltonian action we can always associate a Hopf algebra action and construct, out of the classical momentum map, the desired intertwiner.

First, we observe that any Lie algebra action gives rise to a Hopf algebra action.

Lemma 3.1. *Consider the infinitesimal action $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$. This is equivalent to a Hopf algebra action $\Phi : \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ by setting*

$$(3.1) \quad \Phi(X, f) := \mathcal{L}_{\varphi_X} f,$$

where \mathcal{L} denotes the Lie derivative. Equivalently, it defines a Hopf algebra coaction $\delta_\Phi : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(G)$

Proof. The Lie algebra elements act as derivations of $\mathcal{C}^\infty(M)$, thus Φ defines a Lie algebra action $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$. Since the elements of \mathfrak{g} generate $\mathcal{U}(\mathfrak{g})$, the action Φ is given by differential operators with order determined by the natural filtration of the universal enveloping algebra. Conversely, every Lie algebra action φ of \mathfrak{g} on M determines via the fundamental vector fields $\varphi_X \in \Gamma^\infty(TM)$ a representation of \mathfrak{g} on $\mathcal{C}^\infty(M)$ by derivations which therefore extends to a Hopf algebra action Φ as above. The action Φ and the coaction δ_Φ are always equivalent. \square

In particular, given the infinitesimal dressing action $\ell : \mathfrak{g} \rightarrow \Gamma^\infty(TG^*)$ we obtain the Hopf algebra action $\Lambda : \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(G^*)$ by setting:

$$(3.2) \quad \Lambda(X, f) := \mathcal{L}_{\ell_X} f.$$

We denote by δ_Λ the corresponding Hopf algebra coaction. As a next step we lift the notion of dressing generator to the setting of Hopf algebra actions. We observe that, given the Lie algebra representation $\alpha : \mathfrak{g} \rightarrow \Omega_{\text{loc}}^1(G^*)$, we can define another Hopf algebra action by using the Lie derivative in the direction of a one-form \mathcal{L}_α which has been defined by Bhaskara and Viswanath

[4]. In particular, for $f \in \mathcal{C}^\infty(G^*)$

$$(3.3) \quad \mathcal{L}_\alpha f = \mathcal{L}_{\pi^\#(\alpha)} f.$$

More precisely, we have:

Lemma 3.2. *Given a dressing generator $\alpha : \mathfrak{g} \rightarrow \Omega_{\text{loc}}^1(G^*)$, the corresponding map given by $\mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(G^*) : (X, f) \mapsto \mathcal{L}_{\alpha_X} f$ is a Hopf algebra action. Furthermore we have*

$$(3.4) \quad \Lambda(X, f) = \mathcal{L}_{\alpha_X} f,$$

where $\Lambda(X, f)$ is given by (3.2).

Proof. First, as in Lemma 3.1 the map $\mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(G^*) : (X, f) \mapsto \mathcal{L}_{\alpha_X}$ immediately satisfies the condition to be a Hopf algebra action. Also, from the definition of dressing generator we have $\ell_X = \pi_\ell^\#(\alpha_X)$. Thus

$$\begin{aligned} \Lambda(X, f) &= \mathcal{L}_{\ell_X} f \\ &= \mathcal{L}_{\pi_\ell^\#(\alpha_X)} f \\ &= \mathcal{L}_{\alpha_X} f. \end{aligned}$$

□

Now, let us consider a Hamiltonian action $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ with momentum map $J : M \rightarrow G^*$. Notice that its pullback of functions $J^* : \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(M)$ is an algebra morphism. With an abuse of notation, we also refer to J^* as the pullback of forms. Since the latter is always defined, we can extend J to a map J^* acting on \mathcal{L}_α by

$$(3.5) \quad J^* \mathcal{L}_\alpha := \mathcal{L}_{J^* \alpha} \circ J^*.$$

Theorem 3.3. *Let $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ be an Hamiltonian action with momentum map $J : M \rightarrow G^*$ and consider the corresponding Hopf algebra action $\Phi : \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ given by $\Phi(X) = \mathcal{L}_{\varphi(X)}$. Then we have:*

- i) *The pullback $J^* : \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(M)$ of J intertwines Φ and the Hopf algebra action Λ corresponding to the dressing action via (3.2).*
- ii) *The pullback $J^* : \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(M)$ of J intertwines the corresponding Hopf algebra coaction δ_Φ and the Hopf algebra coaction δ_Λ corresponding to the dressing action.*

Proof. The two claims above can be rephrased by saying that J^* defines a $\mathcal{U}(\mathfrak{g})$ -module algebra morphism and $F(G)$ -comodule algebra morphism.

- i) We already observed that $J^* : \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(M)$ is an algebra morphism. Thus, we only need to prove that it is a module morphism, i.e. the commutativity of the following diagram:

$$(3.6) \quad \begin{array}{ccc} \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(G^*) & \xrightarrow{\Lambda} & \mathcal{C}^\infty(G^*) \\ \text{id} \times J^* \downarrow & & \downarrow J^* \\ \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) & \xrightarrow{\Phi} & \mathcal{C}^\infty(M) \end{array}$$

In other words, we need to prove

$$(3.7) \quad \Phi(X, J^* f) = J^*(\Lambda(X, f)).$$

Using (3.5) we can easily compute:

$$\begin{aligned} J^*(\Lambda(X, f)) &= J^*(\mathcal{L}_{\alpha_X} f) \\ &= \mathcal{L}_{J^* \alpha_X} J^* f \\ &= \mathcal{L}_{\pi^\sharp(J^*(\alpha_X))} J^* f \\ &= \mathcal{L}_{\varphi(X)} J^* f \\ &= \Phi(X, J^* f). \end{aligned}$$

Here we used the fact that, from Definition 2.7, we have $\varphi(X) = \pi^\sharp(J^*(\alpha_X))$.

- ii) Given the Hopf algebra action Φ we can always find the corresponding Hopf algebra coaction δ_Φ , as discussed in Section 1. Thus we can immediately state the dual version of the above claim. In fact, dualizing the commutative diagram (3.6) we immediately get the following commutative diagram

$$(3.8) \quad \begin{array}{ccc} \mathcal{C}^\infty(G^*) & \xrightarrow{\delta_\Lambda} & \mathcal{C}^\infty(G^*) \otimes \mathcal{C}^\infty(G) \\ J^* \downarrow & & \downarrow J^* \otimes \text{id} \\ \mathcal{C}^\infty(M) & \xrightarrow{\delta_\Phi} & \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(G) \end{array}$$

which gives the comodule morphism condition $\delta_\Phi \circ J^* = (J^* \otimes \text{id}) \circ \delta_\Lambda$. Since J^* is an algebra morphism the claim is proved. \square

Finally, the above discussion motivates the following definition. Let $\mathcal{C}^\infty(M)$ be a $\mathcal{U}(\mathfrak{g})$ -module algebra where the module structure is given by a generic Hopf algebra action $\Phi : \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$. Equivalently, $\mathcal{C}^\infty(M)$ is endowed with a $F(G)$ -comodule algebra structure. Furthermore, given the dressing action ℓ we showed that $\mathcal{C}^\infty(G^*)$ automatically turns into a $\mathcal{U}(\mathfrak{g})$ -module algebra where the Hopf algebra action Λ is given by (3.2) (and equivalently into a $F(G)$ -comodule algebra).

Definition 3.4 (Hamiltonian (co)action). i) A Hopf algebra action $\Phi : \mathcal{U}(\mathfrak{g}) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ is said to be Hamiltonian if there exist a $\mathcal{U}(\mathfrak{g})$ -module algebra morphism, called momentum map, $\mathbf{J} : \mathcal{C}^\infty(G^*) \rightarrow \mathcal{C}^\infty(M)$. In other words, Φ is Hamiltonian if it allows a map \mathbf{J} satisfying the following condition:

$$(3.9) \quad \Phi(X, \mathbf{J}f) = \mathbf{J}(\Lambda(X, f)).$$

ii) A Hopf algebra coaction $\delta_\Phi : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \otimes F(G)$ is said to be Hamiltonian if there exist $F(G)$ -module algebra morphism \mathbf{J} , called momentum map, which intertwines it with the Hopf algebra coaction δ_Λ corresponding to the dressing action.

3.1. Quantum Hamiltonian coactions via 2-cocycles

In this section we prove that, using Drinfeld approach, we obtain a quantization of the Hamiltonian coactions as in Definition 3.4. Since actions and coactions are completely equivalent we here prefer to focus only on the coaction case.

Let us consider a twist \mathcal{F} on $\mathcal{U}(\mathfrak{g})$ with corresponding 2-cocycle γ on $F(G)$. As seen in Definition 1.5, the 2-cocycle γ induces a deformed product \star_γ and we denote by $F_\hbar(G)$ the corresponding quantum group. Furthermore, we obtain a deformed product on the comodule algebras $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(G^*)$. More precisely, the action $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ induces a star product \star_φ on M whose semiclassical limit is the Poisson structure π_r induced by r via φ . Similarly, the dressing action $\ell : \mathfrak{g} \rightarrow \Gamma^\infty(TG^*)$ induces a star product \star_ℓ on G^* . Let us denote by $\mathcal{C}_\hbar^\infty(G^*)$ the deformed algebra given by the pair $(\mathcal{C}^\infty(G^*)[[\hbar]], \star_\ell)$ and by $\mathcal{C}_\hbar^\infty(M)$ the pair $(\mathcal{C}^\infty(M)[[\hbar]], \star_\varphi)$. Notice that $\mathcal{C}_\hbar^\infty(G^*)$ and $\mathcal{C}_\hbar^\infty(M)$ are now $F_\hbar(G)$ -comodule algebras. In other

words, the coactions

$$(3.10) \quad \begin{aligned} \delta_\Phi &: \mathcal{C}_\hbar^\infty(M) \rightarrow \mathcal{C}_\hbar^\infty(M) \otimes F_\hbar(G) \\ \text{and } \delta_\Lambda &: \mathcal{C}_\hbar^\infty(G^*) \rightarrow \mathcal{C}_\hbar^\infty(G^*) \otimes F_\hbar(G) \end{aligned}$$

are morphisms of algebras. Thus we can state our main result.

Theorem 3.5. *Let $\varphi : \mathfrak{g} \rightarrow \Gamma^\infty(TM)$ be an Hamiltonian action with momentum map $J : M \rightarrow G^*$. Then the corresponding quantum group coaction $\delta_\Phi : \mathcal{C}_\hbar^\infty(M) \rightarrow \mathcal{C}_\hbar^\infty(M) \otimes F_\hbar(G)$ is Hamiltonian in the sense of Definition 3.4.*

Proof. Since in the Drinfeld approach the coactions do not change but they only intertwine different algebraic structures, the classical momentum map is still a comodule morphism as in Theorem 3.3. More explicitly, the diagram

$$(3.11) \quad \begin{array}{ccc} \mathcal{C}_\hbar^\infty(G^*) & \xrightarrow{\delta_\Lambda} & \mathcal{C}_\hbar^\infty(G^*) \otimes \mathcal{C}_\hbar^\infty(G) \\ J^* \downarrow & & \downarrow J^* \otimes \text{id} \\ \mathcal{C}_\hbar^\infty(M) & \xrightarrow{\delta_\Phi} & \mathcal{C}_\hbar^\infty(M) \otimes \mathcal{C}_\hbar^\infty(G) \end{array}$$

commutes. Thus, we only need to prove that $J^* : \mathcal{C}_\hbar^\infty(G^*) \rightarrow \mathcal{C}_\hbar^\infty(M)$ is a morphism of algebras. This can be immediately checked by using the UDF (1.3) and Theorem 3.3. We can extend the action (3.2) by

$$(3.12) \quad \Lambda(\mathcal{F}, f \otimes g) = \Lambda(\mathcal{F}_\alpha, f) \otimes \Lambda(\mathcal{F}^\alpha, g).$$

As a consequence, we have:

$$\begin{aligned} J^*(f \star_\ell g) &= J^*(m(\Lambda(\mathcal{F}, f \otimes g))) \\ &= J^*(m(\Lambda(\mathcal{F}_\alpha, f), \Lambda(\mathcal{F}^\alpha, g))) \\ &= (J^* \Lambda(\mathcal{F}_\alpha, f))(J^* \Lambda(\mathcal{F}^\alpha, g)) \\ &\stackrel{(3.7)}{=} \Phi(\mathcal{F}_\alpha, J^* f) \Phi(\mathcal{F}^\alpha, J^* g) \\ &= m(\Phi(\mathcal{F}^{-1}, J^* f \otimes J^* g)) \\ &= J^* f \star_\varphi J^* g. \end{aligned}$$

□

Appendix A. Dressing generators on $ax + b$

In this appendix we discuss a concrete example of dressing generators. Let \mathfrak{s} be the Lie algebra with basis H, E and commutation relation

$$(A.1) \quad [H, E] = 2E,$$

also known as the Lie algebra $ax + b$. Consider the triangular r -matrix $r = H \wedge E$. This induces the Lie bialgebra structure on \mathfrak{g}^* :

$$\begin{aligned} \delta(H) &= [r, H \otimes 1 + 1 \otimes H] \\ &= H \otimes [E, H] - [E, H] \otimes H \\ &= -2H \wedge E, \\ \delta(E) &= [r, E \otimes 1 + 1 \otimes E] \\ &= 0. \end{aligned}$$

As a consequence, the dual basis H^*, E^* satisfies the following commutation relation:

$$(A.2) \quad [H^*, E^*] = -2H^*.$$

Note that the element r corresponds to the Poisson structure associated to the bilinear symplectic structure ω on \mathfrak{s} defined by $\omega(H, E) := 1$. Within this set up the Lie algebra structure (A.2) on \mathfrak{s}^* is simply obtained by transporting the Lie bracket on \mathfrak{s} to \mathfrak{s}^* under the linear musical isomorphism $\flat : \mathfrak{s} \rightarrow \mathfrak{s}^* : X \mapsto \flat X := \iota_X \omega$ i.e.

$$(A.3) \quad [\flat X, \flat Y]_{\mathfrak{s}^*} := \flat[X, Y].$$

In our case we have:

$$(A.4) \quad \flat H = E^* \quad \flat E = -H^*$$

The double $\mathfrak{g} := D(\mathfrak{s})$ is given by the vector space $\mathfrak{s} \oplus \mathfrak{s}^*$ equipped with the following Lie brackets (using the notation induced by musical isomorphism)

$$(A.5) \quad \begin{aligned} [H, E] &= 2E, & [\flat H, \flat E] &= 2\flat E, & [\flat H, H] &= 2(\flat H - H), \\ [H, \flat E] &= 2E, & [E, \flat E] &= 0, & [E, \flat H] &= -2\flat E. \end{aligned}$$

We observe that the the first derivative $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is spanned by E , bE and $F := {}^bH - H$ and admits the table:

$$[E, F] = 2E - 2 {}^bE =: Z, \quad [E, Z] = [F, Z] = 0.$$

Thus, \mathfrak{g}' is isomorphic to the Heisenberg algebra $\mathfrak{h}_1 := V \oplus \mathbb{R}Z$ associated to the symplectic plane (V, Ω) spanned by E and F and structured by

$$[v + zZ, v' + z'Z] = \Omega(v, v')Z \quad \text{with } v, v' \in V \quad \text{and} \quad \Omega(E, F) := 1.$$

In this setting, the double $D(\mathfrak{s})$ can be viewed as the semidirect product of the Lie algebra \mathfrak{h}_1 with the abelian Lie algebra $\mathbb{R}H$:

$$D(\mathfrak{s}) \simeq \mathbb{R} \ltimes_{\rho} \mathfrak{h}_1$$

whose Lie algebra homomorphism

$$\rho : \mathbb{R}H \rightarrow \text{Der}(\mathfrak{h}_1)$$

is defined in the basis E, F, Z by

$$\rho(H) := \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma A.1. *Let $\mathfrak{s} = ax + b$. Then we have:*

- i) *The connected simply connected Lie group $G := D(\mathfrak{s})$, with Lie algebra given by the vector space $\mathfrak{g} := D(\mathfrak{s}) := \mathfrak{s} \oplus \mathfrak{s}^*$ with Lie algebra structure given by (A.5), is diffeomorphic to the product manifold:*

$$(A.6) \quad G = \mathbb{R} \times V \times \mathbb{R}.$$

- ii) *Within this model, the group law is given by*

$$(A.7) \quad (a, v, z) \cdot (a', v', z') = \left(a + a', v + e^{2aB}v', z + z' + \frac{1}{2}\Omega(v, e^{2aB}v') \right)$$

where

$$(A.8) \quad B := \frac{1}{2}\rho(H)|_V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{in basis } \{E, F\}.$$

iii) *Realizing the Lie algebra \mathfrak{g} as*

$$(A.9) \quad \mathfrak{g} = \mathbb{R}H \oplus V \oplus \mathbb{R}Z = \{(a_0, v_0, z_0)\},$$

the exponential mapping is given by

$$(A.10) \quad \exp(a_0, v_0, z_0) = \left(a_0, \frac{1}{2a_0}(e^{2a_0B} - I)Bv_0, \right. \\ \left. z_0 + \frac{1}{4a_0}\Omega(Bv_0, v_0) + \frac{1}{8a_0^2}\Omega(v_0, e^{2a_0B}v_0) \right).$$

Proof. The connected simply connected Lie group H_1 corresponding to \mathfrak{h}_1 can be modelled on $V \times \mathbb{R}Z$ with group law given by

$$(A.11) \quad (v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\Omega(v, v')).$$

Within this setting, we observe that the symplectic group $\mathrm{Sp}(V, \Omega)$ (which in our two-dimensional case just coincides with the group $\mathrm{SL}_2(\mathbb{R})$) acts by centre-fixing group-automorphisms on H_1 under:

$$(A.12) \quad R : \mathrm{Sp}(V, \Omega) \times H_1 \rightarrow H_1 : (\mathbf{a}, (v, z)) \mapsto R_{\mathbf{a}}(v, z) := (\mathbf{a}(v), z).$$

Every sub-group A of $\mathrm{Sp}(V, \Omega)$ therefore determines the semi-direct product group

$$(A.13) \quad G := A \ltimes_{\mathbb{R}} H_1$$

modelled on the Cartesian product $G = A \times H_1$ with group law defined by $(\mathbf{a}, \mathbf{a}' \in A)$:

$$(A.14) \quad (\mathbf{a}, v, z) \cdot (\mathbf{a}', v', z') := (\mathbf{a} \cdot \mathbf{a}', (v, z) \cdot R_{\mathbf{a}}(v', z')) \\ = \left(\mathbf{a} \cdot \mathbf{a}', v + \mathbf{a}(v'), z + z' + \frac{1}{2}\Omega(v, \mathbf{a}(v')) \right).$$

In the case

$$(A.15) \quad A := \left\{ \exp(2aB) = \begin{pmatrix} e^{2a} & 0 \\ 0 & e^{-2a} \end{pmatrix} \right\}_{a \in \mathbb{R}}$$

the semi-direct product is therefore the Lie group

$$(A.16) \quad G = \mathbb{R} \times H_1$$

with group law given by (A.7). One then readily verifies that the given expression in (A.10) satisfies the condition $\exp t(a_0, v_0, z_0) \cdot \exp s(a_0, v_0, z_0) = \exp(t+s)(a_0, v_0, z_0)$ for all $s, t \in \mathbb{R}$. The fact that $B^2 = \mathbb{I}$ then implies $\frac{d}{dt} \Big|_{t=0} \exp t(a_0, v_0, z_0) = (a_0, v_0, z_0)$. The computation of the Lie algebra of G is then performed using the expression of the above exponential mapping (A.10). It identifies with the one of \mathfrak{g} . \square

We now pass to realize \mathfrak{s} and \mathfrak{s}^* in the double G . For this we start from expressing the generators at the Lie algebra level:

$$(A.17) \quad H^* = -\left(\frac{1}{2}Z + E\right) \quad \text{and} \quad E^* = H + F.$$

The coordinates on \mathfrak{s}^* are given by $(\nu, \kappa)_* := \exp \nu E^* \exp \kappa H^*$ where

$$(A.18) \quad \begin{aligned} \exp \kappa H^* &= \exp \kappa \left(-E - \frac{1}{2}Z\right) = \left(0, -\kappa E, -\frac{\kappa}{2}\right) \\ \text{and} \quad \exp \nu E^* &= \left(\nu, \frac{1}{2}(e^{-2\nu} - 1)F, 0\right). \end{aligned}$$

Using the group law (A.7) we get

$$(A.19) \quad (\nu, \kappa)_* = \left(\nu, -\kappa e^{2\nu} E, \frac{1}{2}(e^{-2\nu} - 1)F, -\frac{\kappa}{4}(1 + e^{2\nu})\right).$$

Similarly, we have

$$(A.20) \quad (a, n) := \exp(aH) \exp(nE) = (a, e^{2a}nE, 0).$$

Lemma A.2. *Let us consider the dressing action $S \times S^* \rightarrow S$. Then we have*

- i) *The dressing generators with respect to the standard dual Poisson structure π_* are given by the left-invariant forms*

$$(A.21) \quad \alpha_H = -\frac{1}{y+1} dx \quad \text{and} \quad \alpha_E = \frac{1}{2(y+1)} dy$$

- ii) *The dressing generators with respect to π_ℓ are given by the local forms*

$$(A.22) \quad \alpha_H = \frac{1}{y} dx \quad \text{and} \quad \alpha_E = \frac{1}{2y} dy$$

Proof. The first step consists in computing the fundamental vector field of the dressing action by using the realization obtained above of \mathfrak{s} and \mathfrak{s}^* in

terms of the double. More explicitly, using the coordinates (A.19)–(A.20) and the group law (A.7) we have that

$$(A.23) \quad (a, n)(\nu, \kappa)_* = \left(a + \nu, e^{2a}(n - \kappa e^{2\nu})E, \right. \\ \left. \frac{e^{-2a}}{2}(e^{-2\nu} - 1)F, -\frac{1}{4}(\kappa + e^{2\nu}\kappa - ne^{-2\nu} + n) \right).$$

Similarly, we have

$$(A.24) \quad (\underline{\nu}, \underline{\kappa})_*(\underline{a}, \underline{n}) = \left(\underline{\nu} + \underline{a}, e^{2\underline{\nu}}(e^{2\underline{a}}\underline{n} - \underline{\kappa})E, \frac{1}{2}(e^{-2\underline{\nu}} - 1)F, \right. \\ \left. -\frac{1}{4}(\underline{\kappa}(1 + e^{2\underline{\nu}}) + (1 - e^{2\underline{\nu}})e^{2\underline{a}}\underline{n}) \right).$$

The dressing action $S^* \times S \rightarrow S^*$ therefore amounts to solve the equation $(a, n)(\nu, \kappa)_* = (\underline{\nu}, \underline{\kappa})_*(\underline{a}, \underline{n})$ for $(\underline{\nu}, \underline{\kappa})_*$ as a function of a, n, κ, ν . From an easy computation it follows that the solution is given by

$$(A.25) \quad \begin{cases} \underline{\kappa} = \kappa - n\eta(\nu) \\ \eta(\underline{\nu}) = e^{-2a}\eta(\nu) \end{cases}$$

where η is the diffeomorphism defined by $\eta: \mathbb{R} \rightarrow]-1, \infty[: x \mapsto \eta(x) := e^{-2x} - 1$. Considering the coordinate system $S^* \hookrightarrow \mathbb{R}^2 : \xi := (\nu, \kappa)_* \mapsto (x, y) := (\kappa, \eta(\nu))$, the local right dressing action then reads:

$$(A.26) \quad (x, y) \cdot (a, n) := (x - ny, e^{-2a}y).$$

Indeed, the multiplication map

$$(A.27) \quad S^* \times S \rightarrow G : (\xi, x) \mapsto \xi \cdot s$$

is an open embedding. Hence locally one may set:

$$(A.28) \quad s \cdot \xi = \xi^s \cdot s^\xi \quad \text{with} \quad s^\xi \in S \quad \text{and} \quad \xi^s \in S^*.$$

One then notes that for all $s_1, s_2 \in S$ and $\xi \in S^*$:

$$(A.29) \quad \xi^{s_1 s_2} (s_1 s_2)^\xi = s_1 s_2 \xi = s_1 \xi^{s_2} s_2^\xi = (\xi^{s_2})^{s_1} s_1^{\xi^{s_2}} s_2^\xi$$

which implies

$$(A.30) \quad \xi^{s_1 s_2} = (\xi^{s_2})^{s_1}.$$

Hence the map $S^* \times S \rightarrow S^* : (\xi, s) \mapsto \xi^s$ which given elements $s \in S$ and $\xi \in S^*$ expresses the S^* -component (local) of the product $s \cdot \xi$ in terms of the decomposition (A.27) is a right action of S on S^* . The latter globalizes under the usual matrix left-action of the affine group on the plane as

$$(A.31) \quad S \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (s = (a, n), v = (x, y)) \\ \mapsto s.v := v.s^{-1} := \begin{pmatrix} e^{2a} & 0 \\ ne^{2a} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now we express the group multiplication in S^* within the above coordinate system:

$$(A.32) \quad (x, y).(x', y') := \Phi(\Phi^{-1}(x, y).\Phi^{-1}(x', y')) \\ = ((y' + 1)x + x', (y' + 1)y + y').$$

The unit consists in the vector origin $(0, 0)$ and the inverse (which is only local at the level of the entire ambient space \mathbb{R}^2) is given by $(x, y)^{-1} = \frac{1}{y+1}(-x, -y)$. It is useful to rewrite the dressing action using musical notation; in this case we consider the coordinate system

$$(A.33) \quad S^* \hookrightarrow \mathbb{R}^2 : \xi := (\nu, \kappa)_* \mapsto (x, y) := (\kappa, \eta(\nu)),$$

where $\eta(x) = 1 - e^{-2x}$ and the local right dressing action $\xi.(a, n) := (\underline{\kappa}, \underline{\nu})$ then reads:

$$(A.34) \quad (x, y) \cdot (a, n) = (x + ny, e^{-2a}y).$$

This implies that the dressing action is infinitesimally generated by the following fields:

$$(A.35) \quad {}^b\widehat{H}_{(x,y)} := \left. \frac{d}{dt} \right|_{t=0} (x, y)(t, 0) = -2y\partial_y \\ \text{and } {}^b\widehat{E}_{(x,y)} := \left. \frac{d}{dt} \right|_{t=0} (x, y)(0, t) = y\partial_x.$$

The next step consists in computing explicitly the dressing generators. Note that there are 3 Poisson structures involved here on the image U of $S^* \hookrightarrow \mathbb{R}^2$, the dual Poisson Lie group structure

$$(A.36) \quad \pi_* = 2y(y + 1)\partial_x \wedge \partial_y,$$

the Poisson structure π_ℓ induced by the action

$$(A.37) \quad \pi_\ell = 2y^2 \partial_x \wedge \partial_y$$

and the linear one $\pi_{\mathfrak{g}^*}$. It is easy to see that

$$(A.38) \quad \pi_* = \pi_\ell + \pi_{\mathfrak{g}^*}.$$

Finally, imposing the condition (2.3) we obtain that the dressing generators with respect to π_* and π_ℓ we get the expressions (A.21) and (A.22), resp. \square

References

- [1] A. Alekseev and E. Meinrenken, *Linearization of Poisson Lie group structures*, J. Symplectic Geom. **14** (2016), 227–267.
- [2] P. Aschieri, P. Bieliavsky, C. Pagani, and A. Schenkel, *Noncommutative principal bundles through twist deformation*, Comm. Math. Phys. **352** (2017), 287–344.
- [3] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Ann. Phys. **111** (1978), 61–151.
- [4] K. Bhaskara and K. Viswanath, *Calculus on Poisson manifolds*, Bull. London Math. Soc. **20** (1988), 68–72.
- [5] P. Bieliavsky, Ph. Bonneau, and Y. Maeda, *Universal deformations formulae, symplectic Lie groups and symmetric spaces*, Pacific J. Math. **230** (2007), 41–57.
- [6] P. Bieliavsky and V. Gayral, *Deformation quantization for actions of Kählerian Lie groups*, Mem. Amer. Math. Soc. **236** (2015).
- [7] P. Bieliavsky, C. Esposito, S. Waldmann, and T. Weber, *Obstructions for twist star products*, Lett. Math. Phys. **108** (2018), 1341–1350.
- [8] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, (1994).
- [9] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras, and the geometric meaning of the classical Yang-Baxter equations*, Soviet Math. Dokl. **27** (1983), 285–287.
- [10] V. G. Drinfeld, *Quantum groups*, J. Sov. Math. **41** (1988), 898–918.

- [11] F. D'Andrea and T. Weber, *Twist star products and Morita equivalence*, C.R. Acad. Sci. Paris, Ser. I **355** (2017), 1178–1184.
- [12] C. Esposito, *Poisson reduction*, Trends in Mathematics, Geom. Methods Phys. (2014), 131–142.
- [13] C. Esposito, J. Schnitzer, and S. Waldmann, *A universal construction of universal deformation formulas, Drinfel'd twists and their positivity*, Pacific J. Math. **291** (2017), 319–358.
- [14] B. Fedosov, *Non Abelian reduction in deformation quantization*, Lett. Math. Phys. **43** (1998), 137–154.
- [15] M. Gerstenhaber, *New universal deformation formulas for deformation quantization*, preprint, [arXiv:1802.04919](https://arxiv.org/abs/1802.04919).
- [16] A. Giaquinto and J. J. Zhang, *Bialgebra actions, twists, and universal deformation formulas*, J. Pure Appl. Algebra **128** (1998), no. 2, 133–152.
- [17] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.
- [18] Y. Kosmann-Schwarzbach, *Lie bialgebras, Poisson Lie groups and dressing transformations*, in: Integrability of Nonlinear Systems, Vol. 638 in Lecture Notes in Physics, 107–173. Springer-Verlag, second edition, (2004).
- [19] J.-H. Lu, *Multiplicative and Affine Poisson structure on Lie groups*, PhD thesis, University of California (Berkeley), (1990).
- [20] J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526.
- [21] J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, in: Symplectic Geometry, Groupoids, and Integrable Systems, Math. Sci. Res. Inst. Publ., Berkeley, California, (1991).
- [22] J.-H. Lu, *Moment maps at the quantum level*, Comm. Math. Phys. **157** (1993), no. 2, 389–404.
- [23] S. Majid, *Foundations of Quantum Group Theory*. Cambridge University Press, (1995).

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