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The fundamental groups of contact toric manifolds

HUI LI

Let M be a connected compact contact toric manifold. Most of such manifolds are of Reeb type. We show that if M is of Reeb type, then $\pi_1(M)$ is finite cyclic, and we describe how to obtain the order of $\pi_1(M)$ from the moment map image.

Let M be a contact manifold, and α be a contact 1-form on M. Let T^k be a connected compact k-dimensional torus. If T^k acts on M preserving the contact form α, then it preserves the contact structure $\xi = ker(\alpha)$.

A contact manifold M of dimension $2n+1$ with an effective T^{n+1}-action preserving the contact structure is called a **contact toric manifold**. If the Reeb vector field of a contact form on M is generated by a one parameter subgroup action of T^{n+1}, then the contact T^{n+1}-manifold M is called a **contact toric manifold of Reeb type**.

Recall that a $2n$-dimensional symplectic manifold equipped with an effective Hamiltonian T^n-action is called a symplectic toric manifold. Contact toric manifolds are the odd dimensional analog of symplectic toric manifolds. Compact symplectic toric manifolds and compact contact toric manifolds are both classified ([3] and [4]). Most of the compact contact toric manifolds are of Reeb type.

Compact symplectic toric manifolds are simply connected ([1] p. 235, [6, 7]). In contrast, the fundamental groups of connected compact contact toric manifolds are finite abelian if they are of Reeb type ([5]), and are infinite abelian if they are not of Reeb type. (The latter fact can be derived by listing the non-Reeb type manifolds using the classification in [4].)

In this paper, we prove a result on the fundamental groups of compact contact toric manifolds of Reeb type. To describe the result, we define some terms and state a known result as follows. Let (M, α) be a connected compact contact toric manifold of dimension $2n+1$. Let $\mathfrak{t}$ be the Lie algebra of the torus T^{n+1}, and $\mathfrak{t}^*$ be the dual Lie algebra. The **contact moment map**

$\Phi\colon M \to \mathfrak{t}^*$ is defined to be

$$\langle \Phi(x), X\rangle = \alpha_x\big(X_M(x)\big),\ \forall\, x \in M, \text{ and } \forall\ X \in \mathfrak{t},$$

where X_M is the vector field on M generated by the X-action. The **moment cone** of Φ is defined as

$$C(\Phi) = \big\{t\Phi(x) \,|\, t \geq 0,\ x \in M\big\}.$$

It is known ([2, 4, 5] etc.) that, if M is of Reeb type, then $C(\Phi)$ is a *strictly convex rational good polyhedral cone* (of dimension $n+1$). *Strictly convex* means that $C(\Phi)$ contains no linear subspaces of $\mathfrak{t}^*$ of positive dimension, *polyhedral* means that $C(\Phi)$ is a cone over a polytope, *rational* means that the normal vectors of the facets of the cone lie in the integral lattice of $\mathfrak{t}$, and *good* means that for any codimension l face $\mathcal{F}_l$ of $C(\Phi)$, the normal vectors of the facets which intersect at $\mathcal{F}_l$ form a $\mathbb{Z}$-basis of the lattice of an l-dimensional linear subspace of $\mathfrak{t}$.

Theorem. *Let (M, α) be a connected compact contact toric manifold of Reeb type with dimension $2n+1$. Let*

$$I = \big\{v_1, v_2, \dots, v_d\big\}$$

be the set of primitive inward normal vectors of the facets of the moment cone, ordered in the way that the first n vectors are the normal vectors of the facets which intersect at a (any) 1-dimensional face of the moment cone. Then

$$\pi_1(M) = \mathbb{Z}_k,$$

where

$$k = \gcd\big(\det[v_1, v_2, \dots, v_n, v_{n+1}], \\ \det[v_1, v_2, \dots, v_n, v_{n+2}], \dots, \det[v_1, v_2, \dots, v_n, v_d]\big).$$

The 3-dimensional lens spaces are compact contact toric manifolds of Reeb type. Hence any finite cyclic group can be the fundamental group of a contact toric manifold of Reeb type.

Proof of Theorem. Let $\mathbb{Z}_T \subset \mathfrak{t}$ be the integral lattice of the torus T^{n+1}, and $\mathcal{L}$ the sublattice of $\mathbb{Z}_T$ generated by the elements in I. By Lerman's Theorem [5],

$$\pi_1(M) = \mathbb{Z}_T/\mathcal{L}.$$

We identify $\mathbb{Z}_T = \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$. Since the moment cone $C(\Phi)$ is a good cone, $v_1, \dots, v_n$ is a $\mathbb{Z}$-basis of an n-dimensional subspace of $\mathbb{Z}^{n+1}$. So there exists another vector $u \in \mathbb{Z}^{n+1}$ such that $\{v_1, \ldots, v_n, u\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$. Let $\mathcal{L}'$ be the sublattice generated by the elements in $\{v_1, \ldots, v_n\}$. Then

$$\mathbb{Z}_T/\mathcal{L}' = \mathbb{Z}^{n+1}/\mathbb{Z}^n = \mathbb{Z} = \mathbb{Z}\langle u\rangle.$$

Since $\{v_1, \ldots, v_n, u\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$, for $\forall\ n+1 \leq j \leq d$, we have

$$v_j = l_j u \mod \mathcal{L}', \text{ where } l_j \in \mathbb{Z}.$$

Let $k = \gcd(l_j)_{j=n+1}^d$. Since the elements in I span an $n+1$-dimensional vector space, at least one $l_j \neq 0$. So $k \neq 0$. Then

$$\mathbb{Z}_T/\mathcal{L} = \mathbb{Z}\langle u\rangle / k\mathbb{Z}\langle u\rangle = \mathbb{Z}_k$$

is finite cyclic. Moreover, notice that

$$l_j = \pm \det[v_1, \ldots, v_n, v_j], \ \forall\ n+1 \leq j \leq d,$$

where $[v_1, \ldots, v_n, v_j]$ denotes the matrix with column vectors $v_1, \ldots, v_n$ and v_j. □

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School of mathematical Sciences, Soochow University
Suzhou, Jiangsu 215006, China
E-mail address: hui.li@suda.edu.cn

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