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Positive loops of loose Legendrian embeddings and applications

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In this paper, we prove that there exist contractible positive loops of Legendrian embeddings based at any loose Legendrian submanifold. As an application, we define a partial order on $\widetilde{Cont}_0(M,\xi)$, called strong orderability, and prove that overtwisted contact manifolds are not strongly orderable.

Introduction

In this paper, we focus on the study of positive contact and Legendrian isotopies in a co-oriented contact manifold (M,ξ).

A contact manifold (M^{2n+1},ξ) is a $2n+1$ dimensional smooth manifold M with a non-integrable hyperplane field ξ which is called a contact structure. When ξ is co-oriented, it is given by the kernel of a *contact* 1-form α. For example, in $\mathbb{R}^4$ with the usual coordinates (x_1,x_1,y_1,y_2), the sphere $\mathbb{S}^3$ carries a contact form $\alpha_{std}=(y_1dx_1-x_1dy_1+y_2dx_2-x_2dy_2)|_{\mathbb{S}^3}$. We denote ξ_{std} the contact structure defined by α_{std}. It induces a contact structure on the quotient $\mathbb{R}P^3$ which is also denoted by ξ_{std}.

One class of submanifolds of (M^{2n+1},ξ) with an interesting behavior is that of Legendrian submanifolds. A n-dimensional submanifold $L\subset M^{2n+1}$ is called a Legendrian submanifold if $\alpha|_L=0$. A contactomorphism of (M,ξ) is a diffeomorphism which preserves ξ and a contact isotopy $(\varphi_t)_{t\in[0,1]}$ is a path of contactomorphisms with $\varphi_0=id$. We say a contact isotopy $(\varphi_t)_{t\in[0,1]}$ is positive if $\alpha(\partial_t\phi_t)>0$. That is to say, the infinitesimal generator of the isotopy is positively transverse to ξ everywhere. An isotopy $(\varphi_t)_{t\in[0,1]}$ based at a Legendrian submanifold L is said to be a Legendrian isotopy if $\varphi_t(L)$ is a Legendrian submanifold for all t. Similarly, we say φ_t is positive if $\alpha(\partial_t\varphi_t)>0$. This notion of positivity does only depend on the image $L_t=\varphi_t(L)$ of the isotopy. For us, a Legendrian isotopy will be such a family of unparametrized Legendrian submanifolds.

With the concept of positive contact isotopy, Eliashberg and Polterovich defined a partial order on the universal cover $\widetilde{Cont}_0(M,\xi)$ of the identity component of the contactomorphisms group of (M, ξ). A class of contact isotopy $[(\psi_t)_{t\in[0,1]}]$ is greater than another class $[(\varphi_t)_{t\in[0,1]}]$ if there exists a positive contact isotopy from φ_1 to ψ_1 which is homotopic to the concatenation of the opposite of $(\varphi_t)_{t\in[0,1]}$ and $(\psi_t)_{t\in[0,1]}$.

Proposition 0.1. *[EP99] If (M, ξ) is a contact manifold, the following conditions are equivalent:*

(i). *(M, ξ) is non-orderable;*

(ii). *There exists a contractible positive loop of contactomorphisms for (M, ξ).*

This order is closely related to *squeezing* properties in contact geometry [EP99] as well as to the existence of bi-invariant metrics on $\widetilde{Cont}_0(M, \xi)$ or on the space of Legendrian submanifolds [CS12].

From the beginning of the 80's, it is known that the world of contact structures splits in two classes with opposite behaviors. Following Eliashberg, we say that a contact structure ξ on M^3 is overtwisted if there exists an overtwisted disk $D_{OT} \subset M$, i.e. an embedded disk which is tangent to ξ along its boundary. The overtwisted contact structures are flexible and classified by an adequate h-principle [Eli89]. We denote α_{OT} a contact form for an overtwisted contact structure ξ defined on a neighborhood of an overtwisted disk. More recently, the work of Niederkrüger [Nie06] and Borman-Eliashberg-Murphy [BEM15] have described a similar dichotomy in the higher dimensional case. Following a suggestion of Niederkrüger, we say a contact structure ξ is overtwisted if (M^{2n+1}, α) contains $D^3 \times D^{2n-2}(r)$ with $\alpha|_{D^3\times D^{2n-2}(r)} = \alpha_{OT} - (ydx - xdy)$ for some constant $r > 0$ large enough depending on the dimension of M [CMP15]. As in dimension three, Borman, Eliashberg and Murphy [BEM15] have shown that overtwisted contact structures are purely topological objects and are flexible.

On the contrary, we say ξ is a tight contact structure if it is not overtwisted. For example, the contact manifolds $(\mathbb{S}^3, \xi_{std})$ and $(\mathbb{R}P^3, \xi_{std})$ are tight according to the fundamental result of Bennequin [Ben83]. Similar results hold in higher dimension, where holomorphic methods give that a Liouville fillable contact structure is tight, see [Nie06].

The orderability property is not shared by all contact manifolds (see the work of Albers, Frauenfelder, Fuchs and Merry [AF12, AM13, AFM15] for more examples).

Theorem 0.2. (i). *$(\mathbb{S}^3, \xi_{std})$ is non-orderable while $(\mathbb{R}P^3, \xi_{std})$ is orderable [EKP06];*

(ii). *There are some overtwisted contact manifolds which are non-orderable [CPS14].*

It is interesting to see that tight contact manifolds can be orderable or not despite their rigid nature. At the same time we guess overtwisted contact manifolds are non-orderable.

Question 0.3. *Are all overtwisted contact manifolds non-orderable?*

In order to answer the above question, we transfer the study of positive contact isotopies to that of positive Legendrian isotopies by the trick of contact product. Indeed, a positive contact isotopy of (M, ξ) can be lifted to a negative Legendrian isotopy of the diagonal $\Delta_{M\times M} \times \{0\}$ in the contact product $(M \times M \times \mathbb{R}, \alpha_1 - e^s\alpha_2)$. Here α_1 and α_2 denote the pull-backs of α by the first and second projection from $M \times M \times \mathbb{R}$ to M. The advantage is that the study of positive Legendrian isotopies should be easier than that of contact isotopies.

In that context, there is a natural question regarding positive Legendrian isotopies:

Question 0.4. *Let (M, ξ) be a contact manifold and let L_0 and L_1 be Legendrian submanifolds in (M, ξ) which are Legendrian isotopic. Does there exist a positive Legendrian isotopy connecting them?*

Example 0.5. *Let $(\mathbb{S}^2, g)$ be the 2-sphere with the round metric g, and let $ST^*\mathbb{S}^2$ be the space of contact elements on $\mathbb{S}^2$. Denoting S, N the poles, then the geodesic flow of g induces a positive Legendrian isotopy L_t connecting the Legendrian fibers $ST^*_N\mathbb{S}^2$ and $ST^*_S\mathbb{S}^2$.*

Generally, the answer to Question 0.4 is negative.

Theorem 0.6. *Let M^n, $n > 1$ be a manifold with open universal cover. Then*

(i). *the fibers of ST^*M are not in a positive loop of Legendrian embeddings [CFP10, CN10];*

(ii). *the zero-section of $(T^*M \times \mathbb{R}, dz - ydx)$ is not in a positive loop of Legendrian embeddings [CFP10].*

We let F be the front projection

$$(J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}, dz - ydx) \to M \times \mathbb{R} : (x,y,z) \mapsto (x,z).$$

For a Legendrian submanifold $L \subset (T^*M \times \mathbb{R}, dz - ydx)$, the subset $L_F := F(L) \subset M \times \mathbb{R}$ is the front of L. We usually identify L_F with L, since the y coordinates are given by the slopes of the front. In the case where L and M are of dimension 1, we can replace a smooth segment of L_F by a zigzag with two cusps. The zig-zag either has a z-shape, as in Figure 1, or an s-shape (the symmetric of Figure 1 by the vertical axis). The Legendrian submanifold obtained by this operation is denoted by $S(L)$ and is called a stabilization of L. When we want to make it clearer, we will discrimiante between the z-shape/positive stabilization denoted $S_+(L)$ and the s-shape/negative stabilization $S_-(L)$.

We have:

Proposition 0.7. *[CFP10] Let L be the zero-section of $T^*\mathbb{S}^1 \times \mathbb{R}$ and $S(L)$ a stabilization of L. Then there exists a loop of positive Legendrian embeddings based at $S(L)$.*

For a contact manifold (M,ξ) of dimension strictly higher than three, Murphy [Mur12] introduced the class of loose Legendrian submanifolds. This is a higher dimensional generalization of the stabilized $S(L)$ in dimension three. Loose Legendrian submanifolds satisfy a h-principle discovered by Murphy which make them flexible. The main result of this article extends this flexible behavior.

Theorem 0.8. *Let (M,ξ) be a contact manifold of dimension ≥ 5 and $L \subset (M,\xi)$ be a Legendrian submanifold. If L is loose then there exists a contractible positive loop of Legendrian embeddings based at L.*

Without the looseness assumption, F. Laudenbach has proven that there always exist positive loops of Legendrian immersions [Lau07].

As an application of Theorem 0.8, we obtain a holomorphic curve free proof of the existence of tight (i.e. non overtwisted in the Borman-Eliashberg-Murphy sense [BEM15]) contact structures in every dimensions. The "hard part" of the argument uses Theorem 0.6 whose proof relies on the existence of a generating function for a specific class of Legendrians (in that case the Legendrian fibers of the Legendrian fibration in $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$).

Corollary 0.9. *[MNPS13] The contact manifold $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$ is tight.*

This corollary is proved in Subsection 3.1.

In the last section, we define a new partial order on certain groups $\widetilde{Cont}_0(M,\xi)$, called strong orderability, based on the transfer of an isotopy of contactomorphisms to a Legendrian isotopy of their graphs in the contact product. We then drop the graph condition to stick to Legendrian isotopies and get a (possibly) different notion than that of Eliashberg-Polterovich's [EP99].

Proposition 0.10. *Let (M,ξ) be a contact manifold. Then (M,ξ) is strongly orderable if and only if there does not exist a contractible positive loop of Legendrian embeddings based at the diagonal of the contact product of (M,ξ).*

As an example we prove that the contact manifold $(\mathbb{S}^1, d\theta)$ is strongly orderable.

In that context, we explain the following result which was first suggested by Klaus Niederkrüger and also observed by Casals and Presas.

Proposition 0.11. *Let (M^{2n+1},α) be a compact overtwisted contact manifold. Then the contact product $(M\times M\times\mathbb{R}, \alpha_1 - e^s\alpha_2)$ is also overtwisted and the diagonal $\Delta\subset(M\times M\times\mathbb{R}, \alpha_1 - e^s\alpha_2)$ is loose.*

Therefore, according to Proposition 0.11, we have the following result:

Theorem 0.12. *Overtwisted contact manifolds are not strongly orderable.*

Organisation of the paper: In section 1, we recall some basic definitions including Murphy's loose Legendrian embeddings. In section 2, we give the proof of Theorem 0.8. Finally, we prove all the other results mentioned above in the last section.

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1. Basic definitions in Contact Geometry

Let $L : Y \hookrightarrow (J^1(Y), \alpha)$ be a smooth Legendrian embedding. We denote its front map by $L_F : Y \to Y \times \mathbb{R}$.

Given a Legendrian submanifold L', there is a neighborhood $U(L')$ of L' contactomorphic to a neighborhood of the zero section in $J^1(L', \alpha)$, according to the Weinstein neighborhood theorem. If L is a Legendrian submanifold close to L' then we can talk about the front L_F of L in this Weinstein neighborhood.

If $\phi_t : Y \to Y \times \mathbb{R}$ is a homotopy of fronts (with $\phi_t(Y)$ transverse to the $\mathbb{R}$-factor), we denote $\widetilde{\phi_t}$ its Legendrian lift and write v_{ϕ_t} and $v_{\widetilde{\phi_t}}$ for the corresponding time dependent generating vector fields.

1.1. Positive Legendrian isotopies

Definition 1.1. [CFP10, CN10](**Positive Legendrian isotopy**) Let $(M, \xi = ker\alpha)$ be a contact manifold, $L \subset M$ a Legendrian submanifold, $\varphi : L \times [0, 1] \to M$ a Legendrian isotopy and let $X_t = \frac{d\varphi}{dt}$ where $t \in [0, 1]$. We say φ is **positive** if X_t is transverse to ξ positively, i.e.

$$\alpha(X_t) > 0.$$

Moreover, φ is said to be a **positive loop** if in addition $\varphi_0(L) = \varphi_1(L)$.

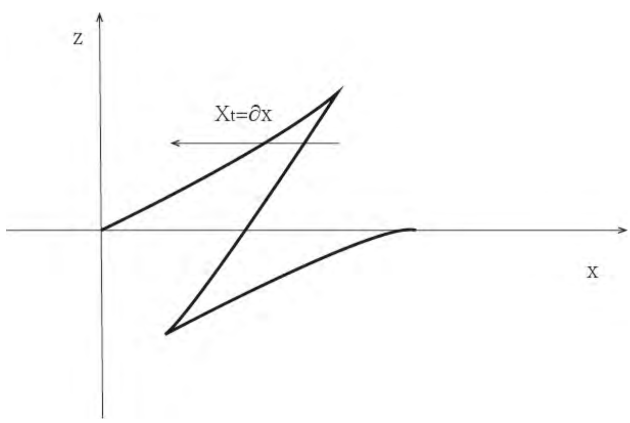

Figure 1: A positive stabilized front.

The following remark is the starting point of our study.

Remark 1.2. [CFP10] Let $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ be a Legendrian embedding whose front has positive slopes everywhere. Then there exists a positive Legendrian loop based at L.

Proof. Regard $\mathbb{S}^1$ as $\mathbb{R}/\mathbb{Z}$ with coordinate x. Denote $Z = L_F(\mathbb{S}^1)$. On Z, the slopes $\partial z/\partial x > 0$ are positive (see Figure 1). Consider the vector field $X_t := -\partial_x$ on $J^1(\mathbb{S}^1)$ and its flow φ_t.

Because $\alpha(X_t) > 0$ on $\varphi_t(Z)$ for every $t \in [0, 1]$, then φ_t is a positive Legendrian isotopy. Since $\varphi_1 = Id$, then we have a positive loop. □

Remark 1.3. If the front of L has negative slopes everywhere, we can choose $v = \partial_x$ so that its flow is a positive loop.

1.2. Loose Legendrian embeddings

In this section, we recall Murphy's notion of loose Legendrian embeddings, wrinkled Legendrian embeddings and the idea for resolving wrinkles [Mur12]. For simplicity, we give the following equivalent definition of a loose Legendrian.

Definition 1.4. Let $L : Y^n \hookrightarrow (J^1(Y^n), \xi_{std})$ be a Legendrian embedding. Let Λ be a one-dimensional zigzag and N be a closed $n-1$ dimensional manifold. We say L is *loose* if its front contains $\Lambda \times N$. In particular, it is obtained from a Legendrian L' by replacing a neighborhood of $N \subset L'$ by N times a zigzag. We denote $L = S^N_\pm(L')$, where $\pm$ stands for the z- or s-shape of the zig-zag.

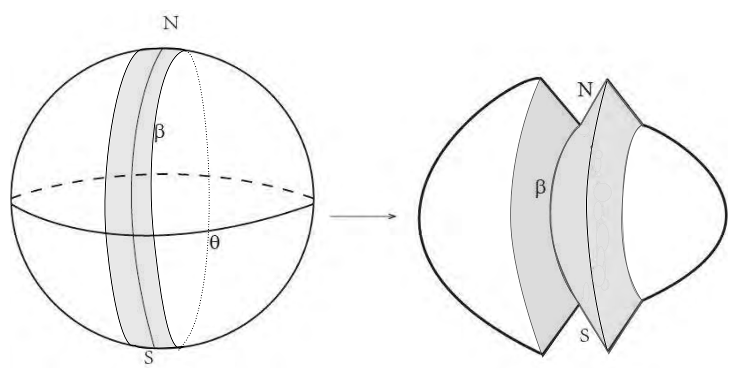

Figure 2: A loose embedding of $\mathbb{S}^2$.

Definition 1.5. [EM11] and [Mur12, Definition 4.1](**Wrinkled embeddings**) See Figure 3. Let $W : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a smooth, proper map, which is a topological embedding. Suppose W is a smooth embedding away from a finite collection of spheres $\{S_j^{n-1}\}$. Suppose, in some coordinates near these spheres, that W can be parametrized by

$$W(u, v) = (v, u^3 - 3u(1 - |v|^2), \frac{1}{5}u^5 - \frac{2}{3}u^3(1 - |v|^2) + u(1 - |v|^2)^2,$$

where our domain coordinates lies in a small neighborhood of the sphere $\{|v|^2 + u^2 = 1\} \subset \mathbb{R}^n$. Then W is called a *wrinkled embedding*, and the spheres S_j^{n-1} are called the *wrinkles*.

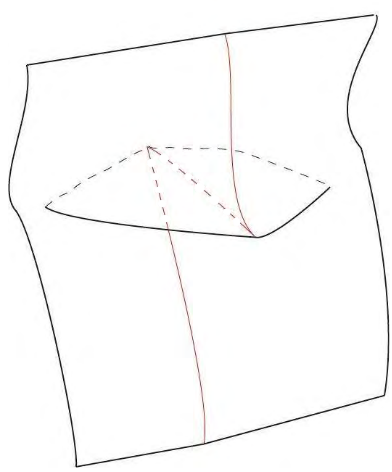

Figure 3: Wrinkled embedding.

Definition 1.6. [Mur12, Definition 5.1](**Wrinkled Legendrians**) Let Y^n be a closed and connected manifold and (M^{2n+1}, ξ) be a contact manifold. A wrinkled Legendrian is a smooth map $L : Y \to M$, which is a topological embedding, satisfying the following properties: The image of dL is contained in ξ everywhere and dL is full rank outside a subset of codimension 2. This singular set is required to be diffeomorphic to a disjoint union of $(n-2)$-spheres $\{S_j^{n-2}\}$, whose images are called *Legendrian wrinkles*. We assume the image of each S_j^{n-2} is contained in a Darboux chart U_j, so that the front projection of $L(Y) \cap U_j$ is a wrinkled embedding, smooth outside of a compact set.

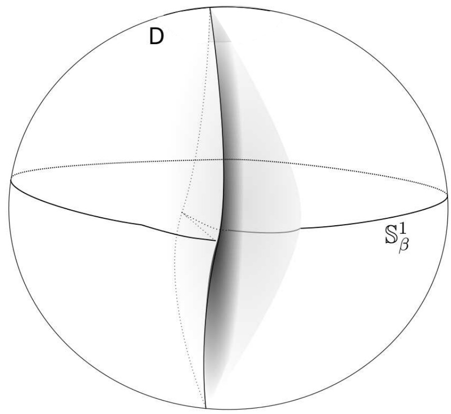

Figure 4: A wrinkled sphere.

Definition 1.7. (**Adding a wrinkle along a disk**) Let L be a Legendrian of dimension n and $D \subset L$ an embedded $(n-1)$-disk together with a collar neighborhood $D \times [-1,1] \subset L$, $D = D \times \{0\}$. We consider the Legendrian L' obtained by replacing $D \times [-1,1]$ by the image of W in Definition 1.6. In particular, we get a wrinkled Legendrian with singular $(n-2)$-sphere $\partial D \times \{0\}$. We say L' is obtained form L by adding a wrinkle along D. Viewed in the front projection, the front of L' is obtained from the front of L by replacing parametrically every smooth interval $x \times [-1,1]$ for every x in the interior of D by a zig-zag shaped interval. When x approaches $y \in \partial D$, the zig-zag degenerates to an unfurled swallowtail singularity at $y \times \{0\}$.

Definition 1.8. [Mur12, Definition 6.1](**Twist marking**) Let $L : Y \to (M, \xi)$ be a wrinkled Legendrian embedding, and $\{S_j^{n-2}\}$ be the set of singular spheres corresponding to the equator spheres $\{u = 0\}$ of the spheres S_j^{n-1} in the previous definition 1.6. Let $N \subset Y$ be a submanifold with $\partial N = \cup_j S_j^{n-2}$ and whose interior is disjoint from $\cup_j S_j^{n-1}$. Denote $\Phi := L|_N$. Then (Φ, N) is called a *twist marking*.

Remark 1.9. We will put the C^∞-topology on the space of wrinkled Legendrian embeddings. Thus we can talk about a smooth family of wrinkled embeddings (L_t, Φ_t, N_t).

Given a Legendrian L, we denote L^w a wrinkled Legendrian obtained by adding some wrinkles to L along an embedded collection of codimension 1 disks. Given a twist marking N on L^w and $\eta > 0$, we denote by $W^{-1}_{\eta,N}(L^w)$, or $W^{-1}_{\eta}(L^w)$ when N is understood, the operation of resolving the wrinkles along N with an η-small operation (see [Mur12, Proposition 6.3]). To measure proximity, we can first perform an immersed resolution $W^{-1}_0(L^w)$, where along N we incorporate a completely flat zig-zag, covering a segment times N. Then $W^{-1}_{\eta}(L^w)$ is η-C^∞-close to $W^{-1}_0(L^w)$.

This operation can be done parametrically, as summarized in the following theorem of [Mur12].

Theorem 1.10. *[Mur12, Proposition 6.3] Let L^w_t be a smooth family of wrinkled Legendrian embeddings, let (Φ_t, N_t) be the twist markings. Then there is a smooth family of Legendrian embeddings L_t, such that L_t is identical to L^w_t outside of any small neighborhood of N_t for all t. Also, the resolution L_t can be taken to be as close as we want from L^w_t.*

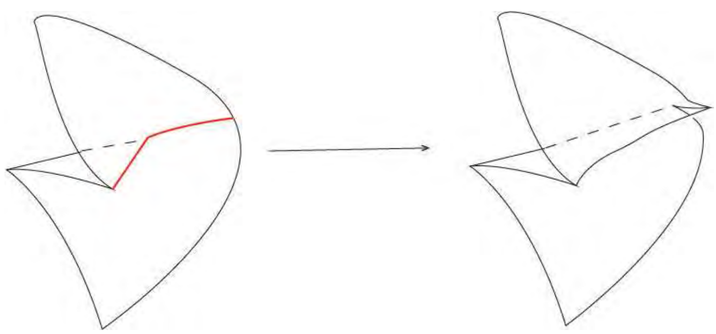

Figure 5: Resolve a wrinkle.

In the front picture of L, the operation of adding a wrinkle along D and resolving it along a twist marking N amounts to replace a neighborhood $N' \times [-1, 1]$ of $N' = N \cup D$ in L with $N' \times \{\text{zig-zag}\}$.

Theorem 1.11. *[Mur12, essentially Proposition 2.6] Let L^n be a loose Legendrian and $N \subset L$ be a closed codimension 1 submanifold of Euler characteristic 0. If we stabilize L positively and negatively along N, we obtain a Legendrian $S^N_-(S^N_+(L))$ which is Legendrian isotopic to L.*

Sketch of proof. First of all, the stabilized Legendrian $S^N_-(S^N_+(L))$ is loose. By Murphy's h-principle, we have to show that $S^N_-(S^N_+(L))$ is formally Legendrian isotopic to L. This is a consequence of the fact that $S^N_-(S^N_+(L))$ is obtained from a Murphy N-stabilization of L by an isotopy, which doesn't change the formal Legendrian isotopy class when $\chi(N) = 0$ (Proposition 2.6 in [Mur12]). □

In our case, we note that the stabilization operation passing from L to $S_+^N(L)$ might change the formal isotopy class of the Legendrian L, even if N has Euler characteristic zero. However, we can go back to the original formal class by stabilizing again to $S_-^N(S_+^N(L))$. This fact will be used later in the proof of our main theorem to correct formal classes.

2. Contractible positive Legendrian loops

In this section we prove our main theorem 0.8 in a geometric way.

Proof. We start with a loose Legendrian L and work in a compact region of its standard neighborhood $J^1(L)$.

A. Construction of a positive loop

We first describe an elementary operation that will be applied repeatedly. Recall L_F is the front projection of L. Since L is the zero-section in $J^1(L)$, one can canonically identify L_F with L.

We consider a n-disk $D_0^n \subset L_F$, written as $D_0^{n-2} \times D^2(3)$ together with coordinates (u, ρ, θ), where $(\rho \leq 3, \theta)$ are polar coordinates on D^2. We let L^w be the wrinkled Legendrian obtained by adding one wrinkled disk D_0^w along the $(n-1)$-disk $D_0 = \{1 \leq \rho \leq 2, \theta = 0\}$ to L_F, so that $D_0^w \subset \{1 \leq \rho \leq 2\}$ (see Figure 6 for a picture in the front projection). We moreover slightly modify L^w along $D_0 \times S^1 = \{1 \leq \rho \leq 2\}$, where S^1 corresponds to the θ-direction, by spreading the slope of the wrinkle in the θ-direction so that:

- every circle $L^w \cap \{\rho = \rho_0 \in (1, 2), u = u_0\}$, contained in the $\{(u_0, \rho_0, \theta, z)\}$ cylinder, has a positive slope, i.e. is positively transverse to ∂_θ as in Figure 1, where ∂_θ is interchanged with ∂_x;
- L^w is equal to L_F away from $\{1 \leq \rho \leq 2\}$.

The situation is pictured in Figure 6. We take a twist marking $N \subset D_0^n$ for D_0^w so that if N' is the closed submanifold $N \cup D_0 \subset L_F$ then the Euler characteristic $\chi(N')$ of N' is zero. Note that the twist marking N can be chosen in $\{1/2 \leq \rho \leq 5/2\}$.

For example, if the Legendrian L_F is 2-dimensional, a portion of the front of the wrinkled L^w is realistically represented in Figure 6. The twist marking N will then be an arc joining the two critical points and the resolution will

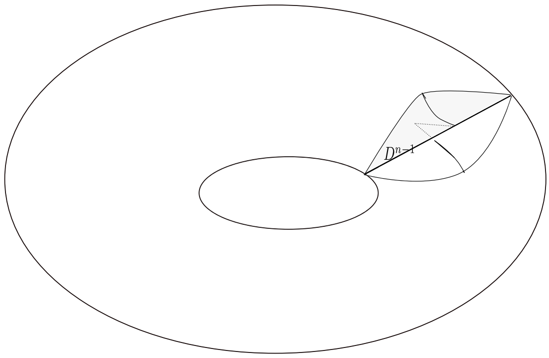

Figure 6: Rotation of the wrinkle.

give a circle of zig-zags (see Figure 7). When we cut the front over the red circle (see Figure 8) we get a circle with four cusps and whose slope is positively transverse to the (horizontal) ∂_θ-direction: this comes from the fact that the s-shape zig-zag to the right, coming from the resolution of the wrinkle, is taken small in front of the z-shape zig-zag and in a region where the slope is already positive (positivity given by spreading the positivity of the z-shape all around the θ-circle).

Step 1. **We rotate the wrinkle positively in the θ-direction**.

Given some constant $K_0 > 0$, we rotate positively the wrinkle in the θ-direction with (large) speed $K_0 \in \mathbb{N}$: We take a z-invariant path of diffeomorphisms ϕ_t^0 in the front such that $\phi_t^0(u, \rho, \theta, z) = (u, \rho, \theta - 2K_0\pi t, z)$ and ϕ_t^0 globally preserves $L^w \setminus \{1 \leq \rho \leq 2\}$. By construction $L^w_{t,K_0} = \phi_t^0(L^w)$ is a loop of wrinkled fronts, even if ϕ_t^0 is only a path. Its lift $\widetilde{L^w_{t,K_0}}$ is a non-negative loop of wrinkled Legendrians based in L^w. Non-negativity comes from the fact that the infinitesimal generator of the isotopy ϕ_t^0 is either tangent (outside $\{1 \leq \rho \leq 2\}$) or positively transverse (inside $\{1 \leq \rho \leq 2\}$) to the front of L^w_{t,K_0}.

Step 2. **We resolve the wrinkle**. We now parametrically resolve the wrinkle $\phi_t^0(D_0^w) \subset L^w_{t,K_0}$ along the marking $\phi_t^0(N) \subset L^w_{t,K_0}$ to get a loop $W_{\eta_0}^{-1}(L^w_{t,K_0})$ of Legendrian fronts (notice that $\phi_1^0(N) = N$). Doing so, we might introduce some negative displacement near $\phi_t^0(N)$, but, taking the size of the resolution η_0 small enough in front of K_0 and the slope of the

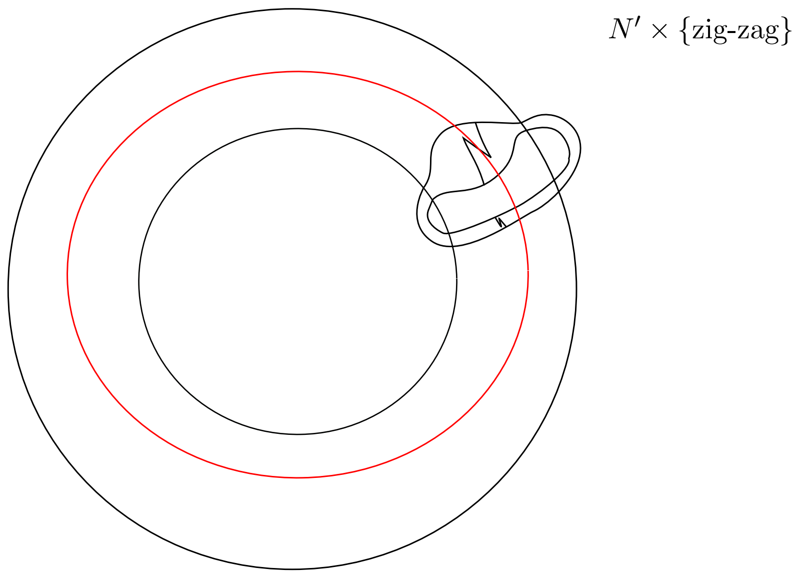

Figure 7: Resolution of the wrinkle along the twist marking.

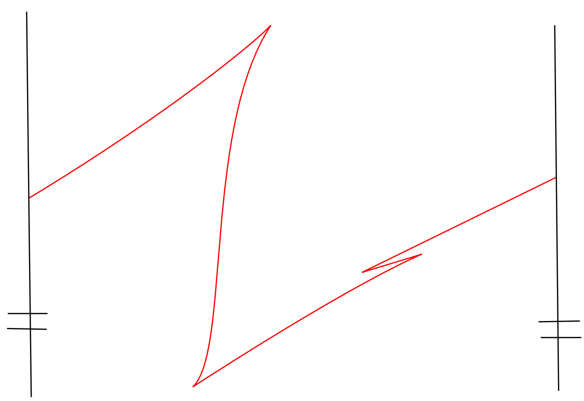

Figure 8: A section of the front over the red middle circle of radius ρ_0 of Figure 7, in the $(\rho_0, \theta, z)_{\{(\theta,z)\}\in\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}}$ cylinder.

circles $\{\rho = \rho_0 \in [5/4, 7/4], u = u_0\}$, we can make sure that the isotopy is still positive in the region $\{5/4 \leq \rho \leq 7/4\}$.

Here, we notice that the loop of fronts $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ is obtained from L_F by replacing a neighborhood of $\phi_t^0(N')$ by the product of N' with a z-shape. In particular, the loop of fronts $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ admits a parametrization by a loop of homeomorphisms (which are diffeomorphisms except at the cusps of the fronts) $\psi^0_{t,K_0} : L \to W^{-1}_{\eta_0}(L^w_{t,K_0})$ which is constant away from D^n_0. When we lift the loop of fronts $W^{-1}_{\eta_0}(L^w_{t,K_0})$ to a loop of Legendrians L^0_{t,K_0} in $J^1(L)$, we can lift the loop of parametrizations ψ^0_{t,K_0} to a loop of smooth parametrizations Ψ^0_{t,K_0}, which is constant away from D^n_0. The loop L^0_{t,K_0} is positive along $\Psi^0_{t,K_0}(\{5/4 \le \rho \le 7/4\})$.

Step 3. **We adjust the formal class**. In steps 1 and 2, we have constructed a loop for a stabilization $S^{N'}_+(L)$ of L, which might not be formally Legendrian isotopic to L. Following Proposition 1.11, we correct this by stabilizing again parametrically along a parallel copy of $\psi^0_{t,K_0}(N')$ to obtain a loop of Legendrians based at $S^{N'}_-(S^{N'}_+(L))$. This second stabilization can be made small at will (in the sense that the s-shape is squashed) so that the positivity property of step 2 is still unchanged and we still have a loop of parametrizations, that we persist to write Ψ^0_{t,K_0}.

This preparatory work being done, the proof starts from a covering of L by open sets $A_i \subset D^n_i$, $i = 0, \dots, k$, of the form $S^1 \times D^{n-1} = \{5/4 \le \rho \le 7/4\} \subset D^n_i$ as before.

We then construct our loop by induction: by step 1,2,3, we construct a loop of Legendrians $L^0_{t,K_0} = \Psi^0_{t,K_0}(L)$ which is positive along $\Psi^0_{t,K_0}(A_0)$, by rotating a wrinkle with speed K_0. We now take a loop of standard Weinstein neighborhoods $N(L^0_{t,K_0})$ of L^0_{t,K_0} parametrized by $t \in S^1$, in which L^0_{t,K_0} is the zero section diffeomorphic to L. This is given by a family of embeddings $J^1(L) \to J^1(L)$ sending the zero section to $L^0_{t,K_0} = \Psi^0_{t,K_0}(L)$. These embeddings can be chosen to extend Ψ^0_{t,K_0}, so we still denote them $\Psi^0_{t,K_0} : J^1(L) \to N(L^0_{t,K_0})$.

We then apply steps 1,2,3 to A_1, D^n_1 in $J^1(L)$, which is thought of as the source of Ψ^0_{t,K_0}, by rotating a wrinkle with (relative) speed K_1 and resolving it with size η_1. This means we are performing this step in the moving neighborhood $\Psi^0_{t,K_0}(J^1(L))$ of L^0_{t,K_0}. We get a loop of Legendrians $L^1_{t,K_1} = \Psi^1_{t,K_1}(L)$ in the moving neighborhood $J^1(L)$. Viewed in the original jet-space, we are considering the loop $\Psi^0_{t,K_0}(L^1_{t,K_1}) = \Psi^0_{t,K_0}(\Psi^1_{t,K_1}(L))$.

We see that if we take K_1 large enough, in particular with respect to η_0 and K_0, then the loop $\Psi^0_{t,K_0}(\Psi^1_{t,K_1}(L))$ becomes positive along $\Psi^0_{t,K_0}(\Psi^1_{t,K_1}(A_1))$ — where it was before possibly negative. We also have to take η_1 small enough so that the isotopy remains positive along $\Psi^0_{t,K_0}(\Psi^1_{t,K_1}(A_0))$ after resolving the wrinkle with size η_1.

Precise computations are described by the following composition of speeds:

Since

$$v_{\widetilde{\Psi}^0_{t,K_0}\circ\widetilde{\Psi}^1_{t,K_1}}(x) = v_{\widetilde{\Psi}^0_{t,K_0}}(\widetilde{\Psi}^1_{t,K_1}(x)) + D\widetilde{\Psi}^0_{t,K_0}(v_{\widetilde{\Psi}^1_{t,K_1}}(x)),$$

we have

$$\alpha(v_{\widetilde{\Psi}^0_{t,K_0}\circ\widetilde{\Psi}^1_{t,K_1}}(x)) = \alpha(v_{\widetilde{\Psi}^0_{t,K_0}}(\widetilde{\Psi}^1_{t,K_1}(x))) + (\widetilde{\Psi}^0_{t,K_0})_*\alpha(v_{\widetilde{\Psi}^1_{t,K_1}}(x)).$$

Now, we have that $\alpha(v_{\widetilde{\Psi}^0_{t,K_0}}) > -k_0$ independent of K_1. Moreover, since the isotopy of Legendrians is compactly supported, there exists some $c_0 > 0$ independent of K_1 such that $\widetilde{\Psi}^{0*}_{t,K_0}\alpha = f\alpha$, where $f > c_0 > 0$ in a neighborhood of the original L which contains all the deformations.

We can thus see that in the neighborhood of $\Psi^0_{t,K_0}(\Psi^1_{t,K_1}(A_1))$ where the slope of the front is larger than some $c_1 > 0$, $\alpha(v_{\widetilde{\Psi}^1_{t,K_1}\circ\widetilde{\Psi}^0_{t,K_0}}) > -k_0 + c_0c_1K_1$. Thus, for K_1 large enough $\widetilde{\Psi}^0_{t,K_0}\circ\widetilde{\Psi}^1_{t,K_1}$ is positive in the neighborhood of $\Psi^0_{t,K_0}(\Psi^1_{t,K_1}(A_1))$. Near A_0 where the loop was already positive, we do not alter positivity if the size η_1 of the resolution is small enough.

Once this is understood, it is clear that we can repeat the process until we get a loop which is positive everywhere. At each step the rotation speed has to be higher and higher with respect to previous operations.

To conclude, we observe that we have been producing a loop based at a loose Legendrian which is formally isotopic to L, and thus by Murphy's Theorem 1.2 in [Mur12] Legendrian isotopic to L.

B. Contractibility

We show that the positive loop that we have been constructing is contractible amongst Legendrian loops.

The construction was inductive on the set of annuli (A_i) and thus it is enough to check that the first loop $W^{-1}_{\eta_0}(L^w_{t,K_0})$ is homotopic to a constant loop.

We first treat the case when the dimension of the Legendrian L is greater of equal to 3. Define $\phi^0_{s,t}$ such that $\phi^0_{s,t}(u,\rho,\theta,z) = (u,\rho - s,\theta - 2K_0\pi t, z)$. We can see that $\phi^0_{s,t}(L^w)$ is a homotopy from $\phi^0_{t,K_0}(L^w)$ to $\phi^0_{1,t}(L^w)$ which is a loop of rotation of a wrinkled disk D^w around some point, says x_0. Up to homotopy, the wrinkled disk $\phi^0_{1,t}(D^w)$ is completely determined by its normal vector in L at x_0, and thus by a map $S^1 \to S^{n-1}$. Since $n \geq 3$, this map is homotopic to a point and thus we can deform our loop of wrinkled Legendrians to a constant loop. Moreover, this homotopy can be extended to a homotopy of twist markings from the original loop of twists markings to a constant loop. Resolving parametrically the markings, we get a homotopy from $W_{\eta_0}^{-1}(L_{t,K_0}^w)$ to a constant loop. The extra stabilization of step 3 to fix the formal isotopy class enters the same scheme and can be also homotoped to a constant operation. This concludes the proof.

The case when the dimension of L is 2 follows the same scheme, except that we homotope the loop of resolved wrinkles $\Psi_{t,K_0}^0(U)$, where U is a circle times a z-shape segment, to a constant annuli around the circle $\{\rho = 1\}$. □

3. Applications

In this chapter, we give some applications of our main theorem. First, we reprove tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$. Second, we define a partial order on the universal cover $\widetilde{Cont}_0(M, \xi)$ of the identity component of the group of contactomorphisms of a contact manifold (M, ξ) and prove that overtwisted contact structures are not orderable.

3.1. Tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$

In this section we prove Corollary 0.9. A similar proof for $\mathbb{S}^1 \times \mathbb{R}^2$ was given in [CFP10].

Proof. Assume $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is overtwisted, and $D_{OT} \subset (\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is an overtwisted disk. Denote $\pi : \mathbb{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$ the projection. There exists some point $x \in \mathbb{R}^n$ such that the fiber $\pi^{-1}(x) \cap D_{OT} = \emptyset$. According to [CMP15], the fiber $\pi^{-1}(x)$ is loose. Thus, there exists a positive loop based at it by Theorem 0.8. That contradicts Theorem 0.6. Therefore, the manifold $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is tight. □

3.2. Positive loops and orderings

Definition 3.1. Given a contact manifold (M, α), the manifold $(\Gamma_M, \tilde{\alpha}) = (M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is called a contact product. Here $\alpha_i = \pi_i^* \alpha$ where π_i project Γ_M to the i-th factor. The Legendrian submanifold of $(\Gamma_M, \tilde{\alpha})$ $\Delta = \{(x, x, 0)\}$ is called the diagonal.

The contact product Γ_M is a special case of a contact fibration. We recall the definition from [Pre07].

Definition 3.2. Let $(E, \xi = ker\alpha)$ be a contact manifold, and $E \longrightarrow B$ be a fibration with fiber F. Then $(E, \xi = ker\alpha) \longrightarrow B$ is called a contact fibration if $(F, \alpha|_F)$ is a contact manifold. Let $(E, \xi = ker\alpha) \longrightarrow B$ be a contact fibration we say that the horizontal distribution $H = (TF \cap \xi)^{\perp d\alpha}$ is the contact connection associated to the fibration.

Remark 3.3. The horizontal distribution depends on the contact form α.

The connection defined above has the following properties:

Proposition 3.4. *[Pre07] For a path $\gamma : [0,1] \to B$, the monodromy $m_\gamma : F(\gamma(0)) \to F(\gamma(1))$ induced by γ is a contactomorphism.*

Corollary 3.5. *Let $\phi \in Diff_0(B)$. Then it lifts to a contactomorphism $\widetilde{\phi}$.*

Note that Γ_M is a contact fibration with $F = (M, \alpha)$ and $B = M \times \mathbb{R}$.

We now explain the following result which was first suggested by Klaus Niederkrüger and also observed by Casals and Presas.

Proposition 3.6. *Let (M^{2n+1}, α) be a compact overtwisted contact manifold and let $(\Gamma_M, \tilde{\alpha})$ be the associated contact product. Then $(\Gamma_M, \tilde{\alpha})$ is also overtwisted and the diagonal $\Delta \subset \Gamma_M$ is loose.*

Proof. We apply the overtwisted criterion from [CMP15]. If $\lambda = ydx - xdy$, it is enough to construct a higher dimensional overtwisted ball $D = (B^{2n+1}_{OT} \times D^{2n+2}(r), \alpha_{OT} - \lambda) \subset (\Gamma_M, \tilde{\alpha})$ for some r large enough, such that D does not intersect Δ.

Let $\mathbb{S}^{2n+1} = \{(x,y) \mid x^2 + y^2 = 1\}$ with its standard contact form α_{std}, and let $\varphi_0 : \mathbb{S}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+2}, (x,y,s) \mapsto (e^s x, e^s y)$. Note that $\varphi_0^* \lambda = \alpha_{std}$. We take a Darboux ball $B \subset (M, \alpha)$ and we regard it as a subset of $(\mathbb{S}^{2n+1}, \alpha_{std})$. Then we can construct a contact embedding $\varphi : (M \times B \times \mathbb{R}, \tilde{\alpha}) \hookrightarrow (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$ by the following series of contact embeddings

$$\begin{aligned} (M \times B \times \mathbb{R}, \tilde{\alpha}) &\overset{i}{\hookrightarrow} (M \times \mathbb{S}^{2n+1} \times \mathbb{R}, \alpha_1 - e^s \alpha_{std}) \\ &\xrightarrow{id \times \varphi_0} (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda). \end{aligned}$$

Let $B^{2n+1}_{OT} \subset M$ be a overtwisted ball, then $D_0 = (B^{2n+1}_{OT} \times \mathbb{D}^{2n+2}(r), \alpha_1 - \lambda)$ is the overtwisted ball in $(M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$. We can move D_0 away from $\varphi(\Delta)$ by Corollary 3.5. More precisely, we take the vector field $V = 2r\partial_x + 2r\partial_y$ on $\mathbb{R}^{2n+2}$, then lift it to a contact vector field $V' = V + 2r(y - x)R_\alpha$ on $M \times \mathbb{R}^{2n+2}$ where R_α is the Reeb vector field of (M, α). Let ϕ_t be the contact isotopy of V'. Denote $C = \{rx \mid x \in B, r > 0\}$ the cone defined by B. Then $D_1 = \phi_1(D_0) \subset M \times (C \setminus \{0\}) = \varphi(M \times B)$ does not intersect $\varphi(\Delta)$. Therefore $D = \varphi^{-1}(D_1)$ is one of the overtwisted balls we want. □

Corollary 3.7. *Let (M,α) be a compact overtwisted contact manifold and $(\Gamma_M,\tilde{\alpha})$ the contact product. Then there exists a positive loop of Legendrian embeddings based at Δ.*

Let $Leg(M,\Gamma_M)$ be the set of Legendrian embeddings $M \hookrightarrow (\Gamma_M,\widetilde{\alpha})$. Given $\phi \in Cont_0(M,\xi = ker\alpha)$ with $\phi^*\alpha = e^{g(x)}\alpha$, it induces a contactomorphism

$$\bar{\phi}(x,y,s) := (x,\phi(x),s-g(y))$$

on $(\Gamma_M,\widetilde{\alpha})$. We denote $gr(\phi) = \bar{\phi}|_\Delta$ which is in $Leg(M,\Gamma_M)$. In fact, given a positive contact isotopy ϕ_t, we can see that $gr(\phi_t)$ is a negative Legendrian isotopy. Therefore, we would like to transfer the study of positive contact isotopies to that of negative Legendrian isotopies.

Definition 3.8. Let $f = [f_t]$ and $g = [g_t]$ be two elements in $\widetilde{Cont}_0(M,\xi)$. We say $f \succeq g$ if there exists a non-positive path $L_t \in Leg(M,\Gamma_M)$ from $gr(g_1)$ to $gr(f_1)$ and $gr(g_t)*L_t$ is homotopic to $gr(f_t)$. The space $\widetilde{Cont}_0(M,\xi)$ and (M,ξ) are said to be **strongly orderable** if $\succeq$ defines a partial order[1] on it. Otherwise, they are said to be non strongly orderable.

Remark 3.9. Let C be the set generated by all the homotopy classes of non-positive paths in $Leg(M,\Gamma_M)$. Then $f \succeq g$ equals to $gr(g^{-1}f) \in C$. Given $[L_t] \in C$ and $\phi \in Cont_0(M,\xi)$, then we have $[\bar{\phi}L_t] \in C$. Therefore, the order $\succeq$ is left invariant, that is to say, given f and g in $\widetilde{Cont}_0(M,\xi)$, if $f \succeq g$, then $hf \succeq hg$ for all $h \in \widetilde{Cont}_0(M,\xi)$. Because if L_t is a non-positive path from g_1 to f_1, then $\bar{h}_1 L_t$ is a non-positive path from $h_1 g_1$ to $h_1 f_1$.

Proposition 3.10. *Let (M,ξ) be a contact manifold. Then (M,ξ) is strongly orderable if and only if there does not exist a contractible negative loop of Legendrian embeddings based at Δ.*

Proof. Let $f = [f_t]$, $g = [g_t]$ and $h = [h_t]$ be elements in $\widetilde{Cont}_0(M,\xi)$. The relation $\succeq$ is reflective, since we have $f \succeq f$ by the definition of $\succeq$. If there are two non-positive paths L_t^1 from $gr(g_1)$ to $gr(f_1)$ and L_t^2 from $gr(h_1)$ to $gr(g_1)$, then $L_t^2 * L_t^1$ is a non-positive path from $gr(h_1)$ to $gr(f_1)$. Thus, the relation is transitive. Now we check the antisymmetry of $\succeq$. According to [CN13, Propostion 4.5], the existence of contractible non-positive non-trivial loop of Legendrian embeddings is equivalent to the existence of contractible negative loop of Legendrian embeddings. Thus, for any $f \neq 1$, on one hand,

[1] in the sense of a partial order on sets

if there does not exist any negative loop based at Δ, we can not find a non-negative path L_t^1 and a non-positive path L_t^1 in the homotopy class of $gr(f_t)$ at the same times. Otherwise, $L_t^1 * L_t^2$ would be a contractible non-negative loop. On the other hand, if there exists a non-positive loop f_t based at Δ, then $f_{1/2} \succeq 1$ and $1 \succeq f_{1/2}$. That means (M, ξ) is not strongly orderable. □

Our definition is stronger than that of [EP99], since we do not require the path of Legendrian embeddings $\widetilde{\phi}_t$ to be graphical for all t.

Corollary 3.11. *Let (M, ξ) be a contact manifold. If (M, ξ) is strongly orderable, then it is orderable.*

A contact manifold which is not strongly orderable is said to be weakly non-orderable. Immediately, according to Proposition 3.6 and Corollary 3.7, we deduce theorem 0.12 saying that overtwisted contact manifolds are weakly non-orderable.

We have the following example of strong orderability.

Theorem 3.12. *$(\mathbb{S}^1, \xi_{std})$ is strongly orderable.*

Proof. Denote $d\theta$ the standard contact form for $\mathbb{S}^1$. We have a contactomorphism $\varphi : (\Gamma_{\mathbb{S}^1}, d\theta_1 - e^s d\theta_2) \to (\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx), (\theta_1, \theta_2, s) \mapsto (z = \theta_1 - \theta_2, x = \theta_2, y = e^s - 1)$ such that $\varphi(\Delta)$ is the zero-section. Assume there exists a contractible positive loop based at the zero-section of $(\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx)$, then it lifts to a positive loop based at the zero-section of $(\mathbb{R}^1 \times T^*\mathbb{S}^1, dz - ydx)$. However, such loops do not exist according to [CFP10] (notice this is not a trivial result). Thus $(\mathbb{S}^1, \xi_{std})$ is strongly orderable. □

Question 3.13. *Is $(\mathbb{R}P^3, \xi_{std})$ strongly orderable?*

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