A new approach to the symplectic isotopy problem

Laura Starkston

The symplectic isotopy conjecture states that every smooth symplectic surface in $\mathbb{CP}^2$ is symplectically isotopic to a complex algebraic curve. Progress began with Gromov's pseudoholomorphic curves [Gro85], and progressed further culminating in Siebert and Tian's proof of the conjecture up to degree 17 [ST05], but further progress has stalled. In this article we provide a new direction of attack on this problem. Using a solution to a nodal symplectic isotopy problem we guide model symplectic isotopies of smooth surfaces. This results in an equivalence between the smooth symplectic isotopy problem and an existence problem of certain embedded Lagrangian disks. This redirects study of this problem from the realm of pseudoholomorphic curves of high genus to the realm of Lagrangians and Floer theory. Because the main theorem is an equivalence going both directions, it could theoretically be used to either prove or disprove the symplectic isotopy conjecture.

1. Introduction

One of the first examples of a symplectic manifold is the complex projective plane. Despite $\mathbb{CP}^2$'s simplicity and homogeneous symmetries, we remain unable to answer a fundamental question about its symplectic geometry: classify its smooth symplectic submanifolds up to symplectic isotopy. The adjunction formula ensures that the genus of a smooth symplectic surface in $\mathbb{CP}^2$ is determined by its degree (homology class). The question of whether there is a unique smooth symplectic representative of each homology class up to symplectic isotopy is known as *the symplectic isotopy problem*. Since every homology class is represented by a smooth complex algebraic curve, this is equivalent to asking whether every embedded symplectic surface in $\mathbb{CP}^2$ is isotopic through symplectic embeddings to a complex algebraic curve. This was solved in the affirmative by Gromov [Gro85] for curves of degree one and two, extended to degree three by Sikorav [Sik03], and to degree

six by Shevchishin [She]. The furthest progress to date of Siebert and Tian
establishes the result up to degree 17 [ST05].

The general approach which has been taken to prove the low degree
symplectic isotopy classification follows Gromov's original outline. A symplectically embedded surface in $\mathbb{CP}^2$ is J_0-holomorphic with respect to some
almost complex structure J_0 compatible with the standard symplectic structure ω_{std} on $\mathbb{CP}^2$. The space of almost complex structures compatible with
a given symplectic structure is contractible, so there exists a path $\{J_t\}$ of
almost complex structures connecting J_0 to the standard integrable complex
structure on $\mathbb{CP}^2$. Now, under sufficiently generic conditions, given a certain
number of points (the number depends on the degree d), there is a unique
J-holomorphic curve of degree d through those points. Therefore by fixing
the appropriate number of points, and varying J in the family J_t, one can
find a family of J_t-holomorphic curves interpolating between the original
symplectic curve and a curve which is complex in the standard sense. The
problem is that various types of degenerations can occur in the curves in this
family which prevent it from providing an isotopy. The analytic difficulties
involved in avoiding such degenerations have prevented further progress in
this direction after Siebert and Tian's 2005 paper.

In this article, we provide a new approach to solving this problem. We
will prove that the symplectic isotopy problem can be solved if one can
find a certain collection of disjoint Lagrangian disks with boundary on the
surface. While pseudoholomorphic curves provide powerful rigid techniques,
one has much greater flexibility to construct isotopies of symplectic surfaces if they are not required to be pseudoholomorphic with respect to a
specific almost complex structure. Our reduction utilizes this flexibility in
constructing a family of surfaces, as well as Gromov's pseudholomorphic
curve proof of the symplectic isotopy problem for degree one curves. Using the Lagrangian disks, we construct a certain nodal symplectic curve (a
generic line arrangement). The rigid techniques yield a symplectic isotopy
of the generic symplectic line arrangement to a complex line arrangement.
We use symplectic models to carry a smooth symplectic surface along the
guiding nodal symplectic surfaces. We also verify that the existence of the
hypothesized disjoint Lagrangian disks is a necessary as well as sufficient
condition for the existence of a symplectic isotopy to a complex curve.

More precisely, for a genus $g = (d-1)(d-2)/2$ surface Σ we are interested in finding a collection of disjointly embedded Lagrangian disks
$D_1, \ldots, D_k$ in $\mathbb{CP}^2$ ($k = d(d-1)/2$) with boundaries on Σ satisfying the
following property.

Definition 1.1. A collection of disjoint simple closed curves $C_1, \ldots, C_k$ on a surface Σ is a *d-splitting system* if $\Sigma \setminus (C_1 \cup \cdots \cup C_k)$ is a collection of d genus zero components each with $d-1$ boundary components, where each C_i separates a distinct pair of components.

It follows immediately from the definition that $k = \frac{d(d-1)}{2}$. An Euler characteristic calculation implies that a d-splitting system can only exist on a genus $g = \frac{(d-1)(d-2)}{2}$ surface. Additionally, the set of d-splitting systems on a genus $\frac{(d-1)(d-2)}{2}$ surface is unique up to diffeomorphism of Σ.

The main theorem is as follows.

Theorem 1.2. *Suppose $\Sigma \subset \mathbb{CP}^2$ is a symplectically embedded surface representing the homology class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. The following two statements are equivalent:*

1) *There exist $k := \frac{d(d-1)}{2}$ disjointly embedded Lagrangian disks $D_1, \ldots, D_k$ whose boundaries form a d-splitting system in Σ and whose interiors are disjoint from Σ.*

2) *There exists an isotopy of Σ through symplectically embedded surfaces to a smooth complex algebraic curve.*

Note that every 1-parameter family of symplectic surfaces can be realized as the image of the initial symplectic surface under an ambient symplectic isotopy by the following classical theorem.

Proposition 1.3 ([CdS06, Theorem 1.12] or [ST05, Proposition 0.3]). *Let (M, ω) be a symplectic manifold and $\Sigma_t \subset M$ a 1-parameter family of symplectic submanifolds. Then there exists a family of (Hamiltonian) symplectomorphisms $\Psi_t : (M, \omega) \to (M, \omega)$ such that $\Psi_0 = id$ and $\Sigma_t = \Psi_t(\Sigma_0)$ for every t.*

To use Theorem 1.2 to find a solution to the symplectic isotopy problem requires finding the hypothesized disjoint Lagrangian disks with boundaries on Σ. The search for disjoint Lagrangian submanifolds can utilize a rather different set of tools in symplectic geometry than previous attempts analyzing pseudoholomorphic curves of high genus. While this *embedded* Lagrangian disk problem presents its own challenges, we hope that this new approach will renew efforts to work on this long standing open problem. Additionally, because our criterion is both necessary and sufficient, it opens the possibility of discovering counterexamples if there are Floer theoretic obstructions to finding such Lagrangian disks.

Remark 1.4. The isotopy problem for algebraic curves of degree d in $\mathbb{CP}^2$ is easily solved, since the moduli space of degree d curves is a connected complex moduli space, and the singular curves form a subset of positive complex codimension (thus real codimension at least 2). This is because complex curves of degree d are parameterized by the coefficients of the degree d monomials. The conditions that the polynomial cutting out the curve is not smooth are algebraic equations in these complex coefficients and thus the space of non-smooth curves has real co-dimension at least two. On the other hand, in the smooth category, it is not true that every smooth surface is isotopic to a complex curve, even after requiring the surface to have the same homotopy invariants as a complex curve. Finashin [Fin02] constructed examples of infinitely many smooth surfaces in $\mathbb{CP}^2$ whose genus and homology class match those of a complex algebraic curve and whose complements have the same fundamental groups as those for algebraic curves, but which are not smoothly isotopic (or even equivalent by a diffeomorphism of $\mathbb{CP}^2$) to a complex algebraic curve. These surfaces are constructed by a certain kind of surface knotting (annular rim surgery) and are distinguished by the Seiberg-Witten invariants of their branched double covers.

1.1. Nodal case

The symplectic isotopy problem is interesting to study for curves with singularities as well. We require the singularities to be modeled on complex singularities, the simplest of which are nodal points, where two branches of the surface intersect transversally and positively. It is expected that symplectic surfaces with only nodal singularities are symplectically isotopic to complex curves. For curves of certain low degrees and small numbers of cusps and nodes the result has been proven (see [OO99] for the cuspidal cubic and [She04] for irreducible low genus nodal curves). The author proved for all degrees that a symplectic isotopy to a complex curve exists in certain completely reducible cases (line arrangements) in [Sta16] (and we utilize this result in this article).

The nodal version of Theorem 1.2 follows from the same proof, though it requires a stronger (but true) statement about deformations of nodal algebraic curves in $\mathbb{CP}^2$. Namely, that within the class of complex algebraic curves one can smooth each node independently while keeping the other nodes.

Theorem 1.5. *Suppose $\Sigma \subset \mathbb{CP}^2$ is a symplectic surface which is smooth except for n nodal singularities such that Σ represents the homology class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. The following two statements are equivalent:*

1) *There exist $k - n = \frac{d(d-1)}{2} - n$ disjointly embedded Lagrangian disks $D_1, \ldots, D_{k-n}$ with boundaries a collection of disjoint simple closed curves in $\Sigma \setminus U$ forming a d-splitting system for some U a union of small neighborhoods of the nodal points in Σ.*

2) *There exists an isotopy of Σ through symplectic surfaces with n nodes to a nodal complex algebraic curve.*

1.2. Formulation via Lagrangian disks in symplectic fillings

A symplectic degree $d > 0$ surface in $\mathbb{CP}^2$ has a neighborhood with concave contact type boundary. The boundary is a circle bundle over Σ with Euler class d^2, and the induced contact structure is invariant with respect to the circle action obtained by rotating the fibers. This contact manifold is called the prequantization bundle of Σ, $P_{d^2}(\Sigma)$. We note that the contact structure on the prequantization bundle is compatible with an open book decomposition whose pages are surfaces of genus $g = (d-1)(d-2)/2$ with d^2 boundary components, and whose monodromy is a product of right handed Dehn twists about one curve parallel to each boundary component [GM13]. (We will focus on the smooth case, but the complement of a nodal curve also has a concave neighborhood, where the induced contact structure is supported by the same open book as for the smooth case except that the monodromy has additional left handed Dehn twists along the vanishing cycles of the nodes.)

The complement of the concave neighborhood is a convex filling of this prequantization bundle. Note that the symplectic filling in the complement of a symplectic surface is symplectomorphic to the complement of a complex curve in $\mathbb{CP}^2$ if and only if the symplectic surface is isotopic to a complex curve. This follows from the fact that the symplectic mapping class group of $\mathbb{CP}^2$ deformation retracts to $PU(2, \mathbb{C})$ [Gro85]. While its precise symplectic topology depends on the symplectic isotopy class of Σ, the homology of $\mathbb{CP}^2 \setminus \nu(\Sigma)$ can be determined by the long exact sequence in cohomology of the pair $(\mathbb{CP}^2, \overline{\nu(\Sigma)})$.

$$H_1(\mathbb{CP}^2 \setminus \nu(\Sigma)) \cong \mathbb{Z}/d\mathbb{Z}, \quad H_2(\mathbb{CP}^2 \setminus \nu(\Sigma)) \cong \mathbb{Z}^{2g}, \quad H_3(\mathbb{CP}^2 \setminus \nu(\Sigma)) = 0$$

Example 1.6. In the case $d = 2$, $P(\Sigma) = (L(4,1), \xi_{can})$ is the lens space $L(4,1)$ with the canonical contact structure obtained as a quotient of (S^3, ξ_{std}) by the $\mathbb{Z}/4$ action. This contact manifold has exactly two strong symplectic fillings [McD90], only one of which is a rational homology ball as is required of $\mathbb{CP}^2 \setminus \Sigma$ by the above computations. This filling is symplecto-morphic to $T^*\mathbb{RP}^2$. The single Lagrangian disk required by Theorem 1.2 is provided by the cotangent fiber, or equivalently the co-core of the 2-handle in the Stein handle decomposition as in Figure 1. The boundary of this curve represents $2 \in \mathbb{Z}/4\mathbb{Z} = H_1(L(4,1))$.

Figure 1. Stein handlebody for $T^*\mathbb{RP}^2 = \mathbb{CP}^2 \setminus Q$ the complement of a degree 2 curve.

In general, we do not have classifications of the symplectic fillings of $P_{d^2}(\Sigma)$ (otherwise we would have already solved the symplectic isotopy problem). Using Theorem 1.2, we do not need to know exactly what the symplectic filling is, but rather that any filling of the fixed contact boundary with the same homology as the complement of an algebraic curve also contains that same collection of Lagrangian disks. One might hope that knowing the contact boundary and the homology of the filling would suffice to give Floer theoretic information about whether such Lagrangian disks exist. The author would be enthusiastic to build off of any Floer theoretic input to explore its geometric consequences.

2. Symplectic models

To prove Theorem 1.2 we will need to patch in certain model deformations. In this section we describe these models.

2.1. From the perspective of a Lagrangian disk

Our first model is for a coordinate neighborhood of a Lagrangian disk with boundary on a symplectic annulus (thought of as the piece of our symplectic surface Σ in the neighborhood of the boundary of the Lagrangian disk).

In $T^*\mathbb{R}^2$ with coordinates (q_1, p_1, q_2, p_2) and symplectic form $\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$, let D^{ml} be the unit disk in the 0-section (the (q_1, q_2)-plane). Let Σ^{ml} in a neighborhood of D^{ml} be the image of $(\theta, s) \in S^1 \times [-c, c]$ under the embedding

$$A(\theta, s) = (\cos(\theta), -s\sin(\theta), \sin(\theta), s\cos(\theta)).$$

Σ^{ml} is symplectic because

$$\omega\left(\frac{\partial A}{\partial \theta}, \frac{\partial A}{\partial s}\right) = \sin^2(\theta) + \cos^2(\theta) = 1 > 0.$$

D^{ml} and Σ^{ml} intersect along ∂D^{ml}. Since D^{ml} is Lagrangian and Σ^{ml} is symplectic, they intersect cleanly. We define the following small push-offs of D^{ml}:

$$D^{ml}_- = \{(q_1, -\varepsilon q_2, q_2, \varepsilon q_1) \mid q_1^2 + q_2^2 \leq 1\}$$
$$D^{ml}_+ = \{(q_1, \varepsilon q_2, q_2, -\varepsilon q_1) \mid q_1^2 + q_2^2 \leq 1\}$$

The signs indicate the comparison of the symplectic orientation with the orientation induced by the projection to the zero section oriented $(\partial_{q_1}, \partial_{q_2})$. More specifically,

$$TD^{ml}_- = \langle \partial_{q_1} + \varepsilon\partial_{p_2}, \partial_{q_2} - \varepsilon\partial_{p_1}\rangle \qquad \omega(\partial_{q_1} + \varepsilon\partial_{p_2}, \partial_{q_2} - \varepsilon\partial_{p_1}) = -2\varepsilon < 0$$
$$TD^{ml}_+ = \langle \partial_{q_1} - \varepsilon\partial_{p_2}, \partial_{q_2} + \varepsilon\partial_{p_1}\rangle \qquad \omega(\partial_{q_1} - \varepsilon\partial_{p_2}, \partial_{q_2} + \varepsilon\partial_{p_1}) = 2\varepsilon > 0$$

Observe that we have clean intersections

$$\Sigma^{ml} \cap D^{ml}_- = \{(\cos\theta, -\varepsilon\sin\theta, \sin\theta, \varepsilon\cos\theta)\} \cong S^1$$
$$\Sigma^{ml} \cap D^{ml}_+ = \{(\cos\theta, \varepsilon\sin\theta, \sin\theta, -\varepsilon\cos\theta)\} \cong S^1$$

Also, the intersection $D^{ml}_- \cap D^{ml}_+ = (0, 0, 0, 0)$ is ω-orthogonal: namely, by checking on the bases above, we see that $\omega(v, w) = 0$ for any $v \in T_0 D^{ml}_+$ and $w \in T_0 D^{ml}_-$.

One can explicitly check that concatenating bases for TD^{ml}_- and TD^{ml}_+ which are positively oriented with respect to the symplectic form gives a positively oriented basis for $T^*\mathbb{R}^2$ (the change of coordinates from $(\partial_{q_1}, \partial_{p_1}, \partial_{q_2}, \partial_{p_2})$ to $(\partial_{q_1} - \varepsilon\partial_{p_2}, \partial_{q_2} + \varepsilon\partial_{p_1}, \partial_{q_2} - \varepsilon\partial_{p_1}, \partial_{q_1} + \varepsilon\partial_{p_2})$ has positive determinant). That the intersection number is $+1$ between these push-offs is also more generally implied by Proposition 2.1 which gives a more topological reason.

The standard coordinates on $\mathbb{R}^2$ induce a trivialization of the cotangent fibration normal to the disk. Restricting this to the boundary of the disk, we get a standard trivialization (framing) of $\nu(D)|_{\partial D} \to \partial D$ which extends over D. We let this framing represent the 0 framing.

Proposition 2.1. *Let D be a Lagrangian disk with boundary in a symplectic surface Σ. Then the framing induced by Σ by taking the normal direction in Σ to ∂D is $+1$.*

Proof. Identify a neighborhood of D with a neighborhood of a disk in $T^*\mathbb{R}^2$. Because Σ contains ∂D and is symplectic, its tangent space is spanned by $T\partial D$ and another vector field V along ∂D which is not symplectically orthogonal to $T\partial D$. Since ∂D is one dimensional, $(T\partial D)^{\perp_\omega}$ is an orientable 3-dimensional sub-bundle of $T\mathbb{CP}^2|_{\partial D}$, thus it is separating. The vector field V must always lie on one side of this separating subspace. $(T\partial D)^{\perp_\omega}$ contains the tangent space to the 0-section $T\mathbb{R}^2$, as well as the co-normal directions to ∂D. Therefore the component of V in the normal direction to $T\mathbb{R}^2$ must be transverse to the co-normal line field. Since the co-normal line field to the boundary of a disk induces a framing $+1$ relative to the trivialization, (since the Gauss map of ∂D has degree 1), the normal component of V must also induce a $+1$ framing of ∂D. $\square$

A useful property of the above model is that each of the surfaces of interest intersects the (q_1, p_2) plane in a 1-dimensional subspace as shown in Figure 2, and the surfaces themselves are given by the orbit of these 1-dimensional subspaces under the following circle action:

$$\theta \cdot (q_1, p_2) = (q_1 \cos\theta, -p_2 \sin\theta, q_1 \sin\theta, p_2 \cos\theta).$$

We can construct more surfaces in this model which intersect Σ^{ml}, D^{ml}, D^{ml}_+, D^{ml}_- in a controlled way, by drawing a 1-dimensional curve in the (q_1, p_2) plane and taking the orbit under the same circle action. This idea is used for Polterovich's Lagrangian surgery [Pol91], but with a different circle action (since we want symplectic instead of Lagrangian surfaces.) If
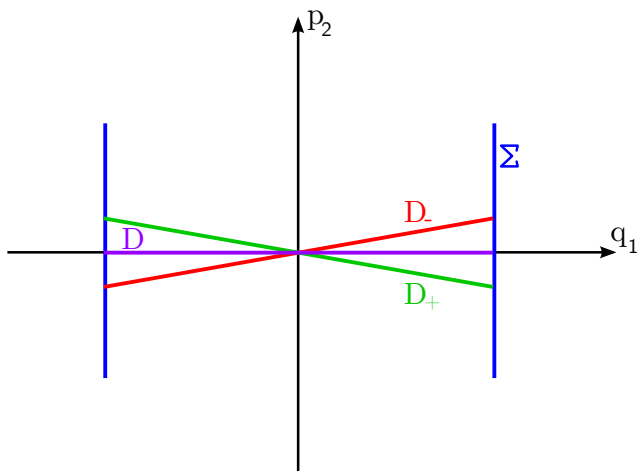
Figure 2. The (q_1, p_2)-plane slice of Σ^{ml}, D^{ml}, D^{ml}_+, and D^{ml}_-.
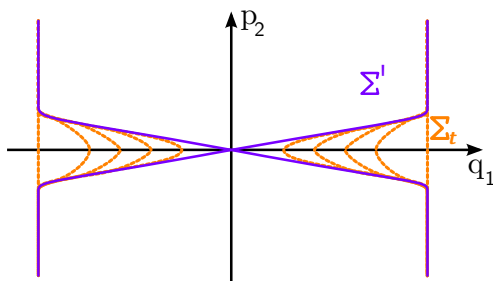

Figure 3. Curves in orange yield symplectic surfaces deforming Σ^{ml} to become close to the nodal surface Σ'.

we parametrize the curve as $q_1 = a(s), p_2 = b(s)$ then the surface obtained by applying the circle action is parametrized by

$$C(\theta, s) = (a(s) \cos \theta, -b(s) \sin \theta, a(s) \sin \theta, b(s) \cos \theta)$$

The symplectic form evaluates on a basis of the tangent space to this surface as

$$\omega \left(\frac{\partial C}{\partial \theta}, \frac{\partial C}{\partial s} \right) = a'(s)b(s) + b'(s)a(s).$$

Note that Σ^{ml} is the surface corresponding to, $a(s) = 1$, $b(s) = s$. In general, the above computation shows that we get a symplectic surface whenever $a'(s)b(s) + b'(s)a(s)$ is positive. Observe that this quantity is positive for any curve such that

$$\begin{cases} a(s) > 0 \\ b'(s) > 0 \\ a'(s) \leq 0 & \text{whenever } b(s) \leq 0 \\ a'(s) \geq 0 & \text{whenever } b(s) \geq 0 \end{cases}$$

For example rotating by the circle action the curves appearing in orange in Figure 3 gives a collection of symplectic surfaces deforming Σ^{ml} approaching $D^{ml}_- \cup D^{ml}_+$.

We also get an immersed surface Σ' which agrees with $D^{ml}_- \cup D^{ml}_+$ away from their boundaries and agrees with Σ^{ml} outside a neighborhood of the annulus between ∂D^{ml}_- and ∂D^{ml}_+. To do this simply smooth the corner between the intersections of Σ^{ml} with $D^{ml}_\pm$ in the (q_1, p_2) plane in Figure 2 and take the image under the circle action.

2.2. From the perspective of a symplectic node

Our second model rotates the coordinates to keep track of submanifolds relative to a fixed symplectically immersed surface in a neighborhood of an ω-orthogonal node modeled by

$$\{(x_1, y_1, 0, 0) \mid x_1^2 + y_1^2 \leq \varepsilon_1^2\} \cup \{(0, 0, x_2, y_2) \mid x_2^2 + y_2^2 \leq \varepsilon_2^2\}$$

in $(\mathbb{R}^4_{(x_1,y_1,x_2,y_2)}, \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$.

We can symplectically smooth the node in this model, by choosing a curve $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ in the real (x_1, x_2)-plane which follows the positive x_1 axis for s near 0 and follows the positive x_2 axis for s near 1 as in Figure 4. We may arrange that $\gamma_1(s), \gamma_2(s), \gamma_2'(s) \geq 0$ and $\gamma_1'(s) \leq 0$. Also we can ensure that $\gamma_2'(s) = 0$ and $\gamma_1(s) = 0$ only for s near 0 and $\gamma_1'(s) = 0$ and $\gamma_2(s) = 0$ only for s near 1. Consider the annulus

$$\mathcal{A}(\theta, s) = \{(\gamma_1(s)e^{i\theta}, \gamma_2(s)e^{-i\theta}) \mid s \in I, \theta \in S^1\}.$$

For s near 0, $\mathcal{A}(\theta, s)$ is an annular ring in the disk in the $z_1 = (x_1, y_1)$-plane, and for s near 1, $\mathcal{A}(\theta, s)$ is an annular ring in the disk in the $z_2 = (x_2, y_2)$-plane. Therefore $\mathcal{A}$ can be smoothly patched into the model node
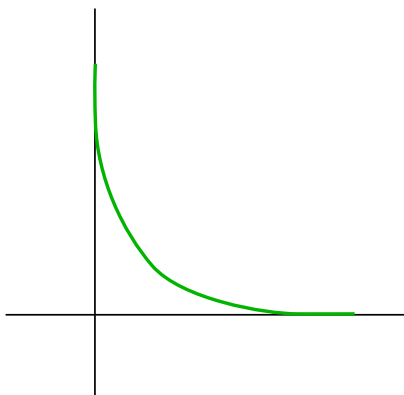
Figure 4. Curve $\gamma(s)$ in the real (x_1, x_2) plane.

after deleting the parts of the disks inside the rings where $\mathcal{A}$ intersects. We can verify that $\mathcal{A}$ patches in symplectically:

$$\omega \left(\frac{\partial A}{\partial \theta}, \frac{\partial A}{\partial s} \right) = \gamma_2(s)\gamma_2'(s) - \gamma_1(s)\gamma_1'(s) > 0$$

where the positivity comes from the assumptions on γ above.

2.3. Rotating nodal smoothings through ω-angles

Typical nodal singularities of symplectic curves need not be ω-orthogonal (though they can be symplectically isotoped to be ω-orthogonal see Section 3.1). Even nodal singularities of genuine complex curves need not be ω-orthogonal. We will consider a neighborhood of a complex linear nodal point where the ω-angles between the two branches is a varying parameter. Near the nodal curve, we provide a model symplectic smoothing which specializes to that of Section 2.2 in the ω-orthogonal case.

Consider a pair of distinct complex lines intersecting at the origin in $(\mathbb{R}^4_{(x_1,y_1,x_2,y_2)}, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ where we set $z_j = x_j + iy_j$. We can symplectically rotate one of the lines to $\{z_2 = 0\}$ so the pair is given by $\{(z_1 - (a + bi)z_2)z_2 = 0\}$. A complex deformation of this nodal singularity to a

smoothing is given by $\{(z_1 - (a + bi)z_2)z_2 = \varepsilon\}$. We can parametrize this deformation by choosing $\gamma_1(s)$ and $\gamma_2(s)$ satisfying $\gamma_1(s)\gamma_2(s) = \varepsilon$ and parametrizing by

$$z_2 = \gamma_2(s)e^{-i\theta} \qquad z_1 = (a + bi)\gamma_2(s)e^{-i\theta} + \gamma_1(s)e^{i\theta}$$
$$\mathcal{A}_{a+bi}(\theta, s) = \big((\gamma_1(s) + a\gamma_2(s))\cos\theta + b\gamma_2(s)\sin\theta,$$
$$(\gamma_1(s) - a\gamma_2(s))\sin\theta + b\gamma_2(s)\cos\theta, \gamma_2(s)\cos\theta, -\gamma_2(s)\sin\theta\big).$$

We can slightly modify the curve to $\widetilde{\gamma}(s) = (\widetilde{\gamma}_1(s), \widetilde{\gamma}_2(s))$ to agree with $\{(t, 0)\} \cup \{(0, u)\}$ outside a neighborhood of the origin. This has the effect of making the image of the corresponding parametrization $\widetilde{\mathcal{A}}_{a+bi}$ agree with the lines $(z_1 - (a + bi)z_2)z_2 = 0$ outside a neighborhood of the origin. If we choose ε sufficiently small, we can make this modification C^1 small so that the image of the resulting smoothing $\mathcal{A}_{a+bi}$ is C^1 close to the complex curve $\{(z_1 - (a + bi)z_2)z_2 = \varepsilon\}$ and thus is symplectic and agrees with the complex lines $\{(z_1 - (a + bi)z_2)z_2 = 0\}$ outside a chosen neighborhood of the origin.

3. Nodal curves

If Σ' is the union of symplectically embedded spheres in $\mathbb{CP}^2$ with each component homologous to $\mathbb{CP}^1$ such that each pair of components intersects exactly once (thus positively), we say that Σ' is a *symplectic line arrangement*. If each distinct pair of components intersects at a distinct point, we say that the symplectic line arrangement is generic. The following is a special case of [Sta16, Proposition 4.1].

Proposition 3.1 ([Sta16]). *Every generic symplectic line arrangement is isotopic through generic symplectic line arrangements to a complex line arrangement.*

The rough outline of the proof is to use Gromov's result that for each almost complex structure J compatible with the standard symplectic structure on $\mathbb{CP}^2$, any two distinct points contain a unique J-holomorphic "line". Furthermore, any two compatible almost complex structures are connected by a path of compatible almost complex structures. Therefore, starting with an almost complex structure J for which Σ' is J-holomorphic, there is a path J_t from J to the standard integrable complex structure J_{std}. Choosing two points on each component of Σ' and varying through the family J_t gives a family of J_t holomorphic lines interpolating between Σ' and a complex line arrangement. The only problem that could occur in this family is

that the lines may pass through non-generic intersections (intersections be-
tween more than two components may coincide at certain times). We show
in [Sta16] that a perturbation of the choice of points ensures that we avoid
this non-generic behavior.

Complex projective curves in $\mathbb{C}P^2$ of degree d are parametrized by the
coefficients of a homogeneous degree d polynomial. Some of these curves have
singularities, but the space of coefficients corresponding to singular curves
has positive complex co-dimension. Curves with only nodal singularities cor-
respond to a complex co-dimension one normal crossing divisor. Therefore
each nodal singularity has a one dimensional complex family of smoothings.
Moreover these nodal singularities can be smoothed independently (this nor-
mal crossing property is special to algebraic curves in $\mathbb{C}P^2$). See [Ful83] for
a reference on these facts about the space of nodal algebraic plane curves.
In a neighborhood of a nodal point on a curve, there are complex algebraic
coordinates (Z_1, Z_2) such that the algebraic curves obtained by varying the
normal parameter t_i which smooths the node of interest is given in these
coordinates by $C_{t_i} = \{Z_1 Z_2 = t_i\}$, so the two sheets of the nodal curve in
these coordinates are $\{Z_1 = 0\}$ and $\{Z_2 = 0\}$. Note that these complex an-
alytic coordinates need not be compatible with Darboux coordinates in the
standard way, in particular the two branches need not be ω-orthogonal.

3.1. Symplectically orthogonal nodes

Generic transverse intersections of symplectic surfaces can be either pos-
itive or negative (whether the symplectic orientations of the two compo-
nents add up to the 4-dimensional orientation). In this article, we restrict
only to positive nodes but even these have geometric variations. The most
canonical model for a positive transverse intersection of symplectic surfaces
is when the two branches meet ω-orthogonally as in the model of Sec-
tion 2.2. More generally, we can consider two transverse symplectic planes
in $(\mathbb{R}^4_{(x_1, y_1, x_2, y_2)}, \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. By a symplectic change of co-
ordinates, we may assume the first plane is spanned by $(\partial_{x_1}, \partial_{y_1})$, and then
the positivity and transversality assumptions imply that the second plane
has an oriented basis of the form

$$(u := \partial_{x_2} + a\partial_{x_1} + b\partial_{y_1}, v := \partial_{y_2} + c\partial_{x_1} + d\partial_{y_1}).$$

This second plane is symplectic if and only if $1 + ad - bc > 0$. It is ω-
orthogonal the first plane if and only if $a = b = c = d = 0$. It is complex

with respect to the standard complex structure $J_{std} = i$ if and only if $d = a$ and $c = -b$.

The need to find a symplectic isotopy taking positively intersecting symplectic submanifolds to ω-orthogonally intersecting symplectic submanifolds arose in Gompf's symplectic sum paper [Gom95, Lemma 2.3] and was proven there in greater generality than the 4-dimensional case we consider here. Symington made more explicit the 4-dimensional case of Gompf's construction in her thesis [Sym96, Lemma 3.2.3], and we present a modification of that here, in order to ensure the following additional property we will need later on. We state the lemma 1-parametrically and include the parameter in the construction in the proof since this is what we require later on.

Lemma 3.2. *Suppose a smoothly varying 1-parameter family of symplectic surfaces Σ_t in $\mathbb{CP}^2$ has a positive transverse node at p_t. Then there is a symplectic isotopy $\Sigma_{t,s}$, with $\Sigma_t = \Sigma_{t,0}$, supported in an arbitrarily small neighborhood of p_t, such that the symplectic node of $\Sigma_{t,1}$ at p_t is ω-orthogonal. Moreover, if Σ_1 agrees with two intersecting complex lines in $\mathbb{CP}^2$ in a neighborhood of p_1, then $\Sigma_{1,s}$ can be chosen to agree with a family of intersecting complex lines in $\mathbb{CP}^2$ in a neighborhood of p_1 for all $s \in [0,1]$.*

The second statement is not true of Symington's construction, and is more easily verified in the following simplified version of Gompf's more general construction.

Proof. Choose a smoothly varying 1-parameter family of Darboux coordinates $(x_1^t, y_1^t, x_2^t, y_2^t)$ giving a symplectomorphism of a neighborhood of p_t to $U \subset (\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ satisfying the following properties:

1) $p_t = (0,0,0,0)$.

2) One branch of Σ_t agrees with $\{x_2^t = y_2^t = 0\}$.

3) In the case that Σ_1 agrees with complex lines near p_1, require the Darboux coordinates $(x_1^1, y_1^1, x_2^1, y_2^1)$ to identify complex lines in $\mathbb{CP}^2$ through p_1 with complex lines in $\mathbb{C}^2$ through the origin.

We can find coordinates $(x_1^1, y_1^1, x_2^1, y_2^1)$ satisfying these three properties for $t = 1$ when Σ_1 agrees with complex lines by first choosing affine Kähler coordinates centered at p_1 such that one line is identified with $z_2 = 0$ (using the fact that $U(2)$ is transitive on lines) and then applying Lemma 3.3 to get Darboux coordinates sending the complex lines to complex lines. Then we can extend these Darboux coordinates smoothly to a family of Darboux

charts, satisfying the first two properties for all t using the relative Moser theorem.

Then the tangent space to the other branch is spanned by

$$(u^t := \partial_{x_2^t} + a^t \partial_{x_1^t} + b^t \partial_{y_1^t}, v^t := \partial_{y_2^t} + c^t \partial_{x_1^t} + d^t \partial_{y_1^t})$$

Using the fact that the non-degeneracy condition is open, we can perform a C^1 small isotopy of Σ^t so that the second branch agrees with the linear subspace spanned by (u^t, v^t) in a δ-neighborhood of p_t. When Σ^1 is complex linear near p_1 in these affine coordinates, the isotopy is trivial.

Next, we will rotate the linear space $\langle u^t, v^t \rangle$ to $\langle \partial_{x_2^t}, \partial_{y_2^t} \rangle$ with an appropriate cut-off function in the radial direction. By the symplectic condition $a^t d^t - b^t c^t > -1$ for all $t \in [0, 1]$, so there exists $\eta > 0$ such that $1 + (1 + \eta)(a^t d^t - b^t c^t) > 0$ for all $t \in [0, 1]$. As in [Gom95], choose a smooth cut-off function $\mu : [0, \infty) \to [0, 1]$ such that $\mu(r) = 1$ for $0 \le r \le \varepsilon \ll \delta$, $\mu(r) = 0$ for $r \ge \delta$, $\mu'(r) \le 0$ for all r and satisfying the additional property that $\mu(r) + r\mu'(r) \ge -\eta$.

Now for $s \in [0, 1]$ let

$$u_{s,r}^t = \partial_{x_2^t} + (1 - s\mu(r))a^t \partial_{x_1^t} + (1 - s\mu(r))b^t \partial_{y_1^t}$$
$$v_{s,r}^t = \partial_{x_2^t} + (1 - s\mu(r))c^t \partial_{x_1^t} + (1 - s\mu(r))d^t \partial_{y_1^t}$$

and let $\phi_s^t(r, \theta) = r \cos \theta \cdot u_{s,r}^t + r \sin \theta \cdot v_{s,r}^t$. Then the image of ϕ_s^t is symplectic because

$$\omega \left(\frac{\partial \phi_s^t}{\partial r}, \frac{\partial \phi_s^t}{\partial \theta} \right) = r \left[(1 - s\mu(r))(1 - s(\mu(r) + r\mu'(r)))(a^t d^t - b^t c^t) + 1 \right] > 0$$

where positivity follows from the assumption $\mu(r) + r\mu'(r) \ge -\eta$ so

$$(1 - s\mu(r))(1 - s(\mu(r) + r\mu'(r)))(a^t d^t - b^t c^t) + 1 > 0.$$

For $0 \le r \le \varepsilon$, $\mu(r)$ is identically 1 so $\phi_s^t(\{r \le \varepsilon\})$ is a disk in the linear plane spanned by $u_s^t = \partial_{x_2^t} + (1 - s)a^t \partial_{x_1^t} + (1 - s)b^t \partial_{y_1^t}$ and $v_s^t = \partial_{y_2^t} + (1 - s)c^t \partial_{x_1^t} + (1 - s)d^t \partial_{y_1^t}$. Since $u_1^t = \partial_{x_2^t}$ and $v_1^t = \partial_{y_2^t}$, the node of $\Sigma_{t,1}$ is ω-orthogonal.

Observe that if Σ^1 agrees with a pair of complex lines near p_1 in Kahler and thus Darboux coordinates then $a^1 = d^1$ and $b^1 = -c^1$, so $a_s^1 := (1 - s)a^1 = (1 - s)d^1 := d_s^1$ and $b_s^1 := (1 - s)b^1 = -(1 - s)c^1 = -c_s^1$, so $\Sigma_{1,s}$ also agrees with a pair of complex lines through p_1 (in both Darboux and Kahler coordinates) for all s in a sufficiently small neighborhood of p_1.   □

Lemma 3.3. *Let $\omega_K = -dd^{\mathbb{C}}(\log(1 + |z|^2))$ be the Kahler form on an affine chart of $\mathbb{CP}^2$, and let ω_D be the standard (Darboux) symplectic form on $\mathbb{C}^2$. There is a diffeomorphism $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Psi^* \omega_K = \omega_D$ which preserves all complex lines through the origin.*

Proof. The Kahler form on $\mathbb{CP}^2$ in an affine chart with coordinates $z = (z_1, z_2) = (x_1, y_1, x_2, y_2)$ is given by $\omega_K = -dd^{\mathbb{C}}(\log(1+|z|^2)) = 2i\partial\bar{\partial}(\log(1+|z|^2))$. The standard symplectic form on $\mathbb{C}^2$ in Darboux coordinates $z = (z_1, z_2) = (x_1, y_1, x_2, y_2)$ is $\omega_D = -dd^{\mathbb{C}}(|z|^2)$.

Because $[\omega_K] = [\omega_D] = 0$, Moser's trick allows one to find a diffeomorphism such that $\Psi^* \omega_K = \omega_D$. Because we are interested in how this symplectomorphism acts on complex lines through the origin, we will look more closely at how Ψ is generated in this case through Moser's argument.

Let $\omega_t = t\omega_K + (1-t)\omega_D = -dd^{\mathbb{C}}(t\log(1+|z|^2) + (1-t)|z|^2)$. Then

$$\frac{d\omega_t}{dt} = \omega_K - \omega_D = -dd^{\mathbb{C}}(\log(1+|z|^2) - |z|^2).$$

Let $\mu = -d^{\mathbb{C}}(\log(1+|z|^2) - |z|^2)$ so $d\mu = \frac{d\omega_t}{dt}$. The Moser trick is to find a vector field V_t such that $\iota_{V_t}\omega_t = -\mu$, then integrating the flow of V_t gives the symplectomorphism Ψ_t such that $\Psi_t^*\omega_t = \omega_0 = \omega_D$. We will calculate V_t in this case. Let $f(z) = |z|^2$, and let $\lambda = -d^{\mathbb{C}}(f)$. Note that $\omega_D = d\lambda$.

$$\omega_t = -dd^{\mathbb{C}}(t\log(1+|z|^2) - |z|^2)$$
$$= d\left(\left(\frac{t}{1+|z|^2} + (1-t)\right)\lambda\right)$$
$$= \left(\frac{t}{1+|z|^2} + (1-t)\right)d\lambda - \left(\frac{t}{(1+|z|^2)^2}\right)df \wedge \lambda$$

and

$$-\mu = d^{\mathbb{C}}(\log(1+|z|^2) - |z|^2) = -\left(\frac{1}{1+|z|^2} - 1\right)\lambda$$

Let $V_0 = x_1\partial_{x_1} + y_1\partial_{y_1} + x_2\partial_{x_2} + y_2\partial_{y_2}$ be the radial vector field. Observe that $df(V_0) = |z|^2$, $\lambda(V_0) = 0$, and $\iota_{V_0}d\lambda = \lambda$. Let $V_t = f_tV_0$ where f_t is a function. Then

$$\iota_{V_t}\omega_t = f_t\iota_{V_0}\omega_t = f_t\left[\left(\frac{t}{1+|z|^2} + (1-t)\right) - \left(\frac{t}{(1+|z|^2)^2}\right)|z|^2\right]\lambda$$

Therefore $\iota_{V_t}\omega_t = -\mu$ when

$$f_t(z) = \frac{-\left(\frac{1}{1+|z|^2} - 1\right)}{\left(\frac{t}{1+|z|^2} + (1-t)\right) - \left(\frac{t}{(1+|z|^2)^2}\right)|z|^2} = \frac{|z|^2(1+|z|^2)}{t + (1-t)(1+|z|^2)^2}$$

In particular, V_t is a scalar multiple of the radial vector field V_0. Therefore the diffeomorphism generated by the flow of V_t preserves each real line through the origin. Therefore the Darboux chart ϕ_1 preserves all linear subspaces through the origin, and in particular preserves the complex lines through the origin. $\qquad\square$

4. Building a symplectic isotopy from Lagrangian disks

Now we construct a symplectic isotopy of Σ to a complex algebraic curve, from the assumption we have Lagrangian disks as in condition (1), giving one direction of Theorem 1.2.

Proof. A smooth Lagrangian submanifold has a standard neighborhood modeled on its co-tangent bundle. For a Lagrangian with boundary, we obtain the same result by extending the Lagrangian submanifold to an open Lagrangian containing the closed Lagrangian with boundary. In particular, a neighborhood of a Lagrangian disk has coordinates identifying it with a subset of $T^*\mathbb{R}^2$, where the Lagrangian disk is identified with a disk in the 0-section. For each of the k disjoint Lagrangian disks in $\mathbb{C}P^2$ whose boundaries form a d-splitting system of Σ, we can identify the neighborhood and the submanifolds D and Σ with the model of Section 2.1 after a symplectic isotopy of Σ near ∂D as follows.

First, symplectically identify a neighborhood of D with a neighborhood of the unit disk in $\mathbb{R}^2$ inside $T^*\mathbb{R}^2$. We continue to write Σ to denote its image in the subset of $T^*\mathbb{R}^2$ under this identification. As a sub-bundle of $T(T^*\mathbb{R}^2)|_{\partial D}$, $T\Sigma|_{\partial D}$ and $T\Sigma^{ml}|_{\partial D}$ are both transverse to $(T\partial D)^{\perp_\omega}$ (which is a necessary and sufficient condition for a 2-dimensional sub-bundle containing $T\partial D$ to be a symplectic sub-bundle). Since $(T\partial D)^{\perp_\omega}$ is 3-dimensional, $T\Sigma|_{\partial D}$ and $T\Sigma^{ml}|_{\partial D}$ must be isotopic through symplectic sub-bundles of $T(T^*\mathbb{R}^2)|_{\partial D}$. Therefore Σ and Σ^{ml} induce the same framing on ∂D and there is a smooth isotopy Σ^t such that $\Sigma = \Sigma^0$, $\Sigma^t = \Sigma$ outside a small neighborhood of ∂D, Σ^1 agrees with Σ^{ml} in a smaller neighborhood of ∂D, and $T\Sigma^t|_{\partial D}$ is a symplectic sub-bundle. The non-degeneracy condition is open, and the isotopy is symplectic along ∂D. Therefore, by choosing an

isotopy supported in a sufficiently small neighborhood of ∂D, we can ensure that Σ^t is symplectic for all t.

Using the model of Section 2.1, we find an immersed symplectic surface Σ' with a positive ω-orthogonal self-intersection at a point on the Lagrangian D, such that Σ' agrees with Σ outside a neighborhood of D, and agrees with the model symplectic push-offs $D_\pm$ of D near the self-intersection point. Moreover, we can use the model to find a symplectic isotopy of Σ to a smooth symplectic surface which agrees with Σ' outside of an arbitrarily small neighborhood of the node.

Apply this construction along each of the k Lagrangian disks in disjoint neighborhoods. Rename the resulting nodal symplectic surface Σ'. Then we have shown that there is a symplectic isotopy Σ_t of Σ to a symplectic surface $\Sigma_{1/2}$ such that such that if U is the neighborhood of the k Lagrangian disks, and U' is an arbitrarily small neighborhood of the nodal points of Σ', then

- Σ_t agrees with Σ outside U for $t \in [0, 1/2]$,

- $\Sigma_{1/2}$ agrees with the nodal symplectic curve Σ' outside U',

- $\Sigma_{1/2}$ is an arbitrarily small symplectic smoothing of Σ' in U' as in the model of Section 2.1.

Because of the d-splitting condition on the boundaries of the Lagrangian disks, we can verify the nodal surface Σ' is a collection of d embedded symplectic spheres such that each pair intersects once positively as follows. The construction exhibits that Σ and Σ' are homologous (take the image under the S^1 action of the 2-dimensional region in the (q_1, p_2) plane bounded by the curves). Therefore, the total homology class of the components of Σ' is $d[\mathbb{C}P^1]$. Because the components of Σ' intersect once positively, they each represent non-trivial homology classes. Because they are symplectic spheres they must have degree 1 or 2. Since there are d components and the sum of the homology classes is $d[\mathbb{C}P^1]$, it follows that each sphere represents the class $[\mathbb{C}P^1]$ so Σ' is a symplectic line arrangement. Set $\Sigma'_{1/2} := \Sigma'$. Now use Proposition 3.1 to find a symplectic isotopy Σ'_t for $t \in [1/2, 3/4]$ to a complex line arrangement $\Sigma'_{3/4}$ (with respect to the standard complex structure on $\mathbb{C}P^2$).

For the family of generic symplectic line arrangements Σ'_t, $t \in [\frac{1}{2}, \frac{3}{4}]$, apply Lemma 3.2 to get a symplectic isotopy $\Sigma'_{t,s}$ supported in neighborhoods of the nodal points such that $\Sigma'_{t,0} = \Sigma'_t$ and the nodes of $\Sigma'_{t,1}$ are ω-orthogonal. Let $\widetilde{\Sigma}_t := \Sigma'_{t,1}$ denote the resulting family of ω-orthogonal

symplectic line arrangements. Since the nodes of $\Sigma'_{1/2}$ are ω-orthogonal by the model in Section 2.1, $\Sigma'_{1/2} = \widetilde{\Sigma}_{1/2}$.

Then by the relative Moser theorem, there exist sufficiently small neighborhoods of the ω-orthogonal nodal points of $\widetilde{\Sigma}_t$, where we can find a smooth family of Darboux coordinates centered at the nodal points such that $\widetilde{\Sigma}_t$ is given in this coordinate chart by $\{(x_1, y_1, 0, 0)\} \cup \{(0, 0, x_2, y_2)\}$ as in the model of Section 2.2. Moreover, we can choose this Darboux chart so that a smoothing of the model of Section 2.1 coincides with the smoothing of the model in Section 2.2.

More specifically, the symplectomorphism defined by

$$x_1 = \varepsilon q_1 + p_2, \quad y_1 = p_1 - \varepsilon q_2, \quad x_2 = p_2 - \varepsilon q_1, \quad y_2 = -p_1 - \varepsilon q_2$$

takes the disk D_-^{ml} to $\{(2\varepsilon q_1, -2\varepsilon q_2, 0, 0) \mid q_1^2 + q_2^2 = 1\}$ and D_+^{ml} to $\{(0, 0, -2\varepsilon q_1, -2\varepsilon q_2) \mid q_1^2 + q_2^2 = 1\}$. It takes the smoothing annulus $(a(s)\cos\theta, b(s)\sin\theta, -a(s)\sin\theta, b(s)\cos\theta)$ to

$$\begin{aligned}
&\big((\varepsilon a(s) + b(s))\cos\theta, (\varepsilon a(s) + b(s))\sin\theta, \\
&\quad (-\varepsilon a(s) + b(s))\cos(-\theta), (-\varepsilon a(s) + b(s))\sin(-\theta)\big) \\
&= (\gamma_1(s)e^{i\theta}, \gamma_2(s)e^{-i\theta})
\end{aligned}$$

when $\gamma_1(s) = \varepsilon a(s) + b(s)$ and $\gamma_2 = -\varepsilon a(s) + b(s)$.

Then implanting the parametric family of Darboux charts into the neighborhoods of the nodes over the isotopy $\widetilde{\Sigma}_t$, yields a symplectic isotopy from $\Sigma_{1/2}$ through smoothings Σ_t of the nodes of $\widetilde{\Sigma}_t$ for $t \in [1/2, 3/4]$.

Since $\Sigma'_{3/4}$ is a genuine complex line arrangement, it has a complex deformation smoothing its nodes to a smooth complex curve Σ_1. Finally, we connect $\Sigma_{3/4}$ to Σ_1 through a symplectic isotopy as follows. $\Sigma_{3/4}$ agrees with $\widetilde{\Sigma}_{3/4} = \Sigma'_{3/4,1}$ outside a small neighborhood V of the nodes. We can assume by Lemma 3.2 that inside each component of V, $\Sigma'_{3/4,s}$ agrees with some pair of complex lines for each s. In V, we can interpolate $\Sigma_{3/4}$ to a symplectic smoothing of $\Sigma'_{3/4,0}$ through symplectic smoothings of $\Sigma'_{3/4,s}$ using the model in Section 2.3 in the affine Darboux chart utilized for $\Sigma'_{3/4,s}$ in Lemma 3.2. We can extend this isotopy outside V to agree with $\Sigma'_{3/4,s}$. Because we use the affine chart where $\Sigma'_{3/4,0}$ is visibly linear, the smoothing of $\Sigma'_{3/4,0}$ by Section 2.3 is C^1 close to a complex deformation agreeing to first order with $(z_1 - (a + bi)z_2)z_2 = \varepsilon$. Therefore there is a symplectic isotopy from this smoothing of $\Sigma'_{3/4,0}$ to the complex curve Σ_1.

We summarize the collection of isotopies constructed in the following diagram where the smooth symplectic isotopy runs around the upper right boundary.
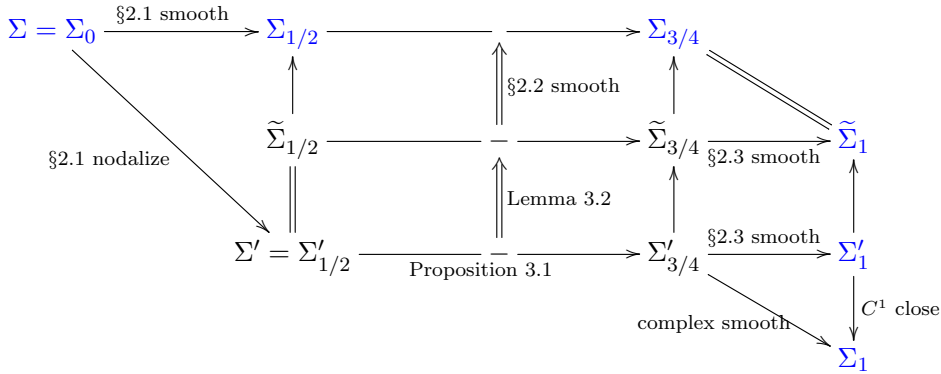
The nodal case follows precisely the same steps, we just require fewer Lagrangian disks to degenerate along to obtain a symplectic line arrangement. Since the symplectic degenerations are performed locally near the Lagrangian disks which are disjoint from Σ except along their boundaries, we simply keep some of the nodes of the symplectic line arrangement Σ'_t instead of resolving them. When we deform the complex line arrangement to a curve with some but not all nodes, we require the stronger result about deformations of complex planar nodal curves which allows us to deform the algebraic curve by independent complex parameters t_i where the neighborhoods of the ith node has complex analytic coordinates where the curves are given by $\{Z_1 Z_2 = t_i\}$.

5. The converse

Finally, we prove the reverse direction of Theorem 1.2.

Proof. Suppose there is a family of smooth symplectic surfaces Σ_t for $t \in [0,1]$ where Σ_1 is a complex curve of degree d. The space of all degree d algebraic curves is connected. The curves with nodal singularities can be deformed to smooth curves as discussed in Section 3. Therefore there is a path Σ_t for $t \in [1,2]$ of smooth complex algebraic curves from Σ_1 to a complex line arrangement Σ'_2.

We have a smooth symplectic isotopy from Σ_0 to $\Sigma_{2-\varepsilon}$ for $\varepsilon > 0$. We will show that for sufficiently small ε, $\Sigma_{2-\varepsilon}$ is symplectically isotopic to a

certain nearby smooth symplectic surface Σ_3 such that there are embedded Lagrangian disks with boundary on Σ_3 in a neighborhood of each node of Σ_2'. Because Σ_2' is a line arrangement, the boundaries of these disks will form a d-splitting system of Σ_3. To isotope $\Sigma_{2-\varepsilon}$ to such a Σ_3, first make a C^1 small isotopy of $\Sigma_{2-\varepsilon}$ to a smooth symplectic surface that agrees with Σ_2 outside a small neighborhood of the nodes and agrees with the model of Section 2.3 in the neighborhood of the nodes. Then follow an ω-orthogonalizing isotopy given by Lemma 3.2 using a parametrized family of smoothing models of Section 2.3. At the end of this isotopy the nodal surface Σ_3' is ω-orthogonal, and the smooth symplectic surface Σ_3 agrees with the model smoothing of Section 2.2 obtained by rotating a curve $\gamma(s) = (\gamma_1(s), \gamma_2(s)) \in \mathbb{R}^2_{(x_1,x_2)}$ by the given circle action. We can arrange that γ passes through $(\varepsilon', \varepsilon')$ for some $\varepsilon' > 0$. Then a Lagrangian disk in this model with boundary on the symplectic surface Σ_3 is given by $\{(r\cos\theta, r\sin\theta, r\cos\theta, -r\sin\theta) \mid r \leq \varepsilon'\}$. Because these disks are in arbitrarily small neighborhoods of the nodes, they are disjointly embedded and intersect Σ_3 only along their boundaries.

Finally, the smooth symplectic isotopy from Σ_0 to Σ_3 is realized by an ambient symplectic isotopy by Proposition 1.3 giving a family of symplectomorphisms Ψ_t for $t \in [0,3]$ such that $\Psi_0 = id$ and $\Sigma_t = \Psi_t(\Sigma_0)$. Since Lagrangians are sent to Lagrangians under symplectomorphisms, the converse direction in Theorem 1.2 then follows by pulling back the Lagrangian disks with boundary on Σ_3 by the symplectomorphism Ψ_3. $\square$

References

[CdS06] Ana Cannas da Silva, Symplectic Geometry, in: Handbook of Differential Geometry. Vol. II, pages 79–188. Elsevier/North-Holland, Amsterdam, (2006).

[Fin02] Sergey Finashin, *Knotting of algebraic curves in $\mathbb{C}P^2$*, Topology **41** (2002), no. 1, 47–55.

[Ful83] William Fulton, *On nodal curves*, in: Algebraic Geometry — Open Problems (Ravello, 1982), volume 997 of Lecture Notes in Math., pages 146–155. Springer, Berlin, (1983).

[GM13] David Gay and Thomas E. Mark, *Convex plumbings and Lefschetz fibrations*, J. Symplectic Geom. **11** (2013), no. 3, 363–375.

[Gom95] Robert E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), no. 3, 527–595.

[Gro85] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[McD90] Dusa McDuff, *Rational and ruled symplectic 4-manifolds*, in: Geometry of Low-Dimensional Manifolds, 2 (Durham, 1989), volume 151 of London Math. Soc. Lecture Note Ser., pages 7–14. Cambridge Univ. Press, Cambridge, (1990).

[OO99] Hiroshi Ohta and Kaoru Ono, *Simple singularities and topology of symplectically filling 4-manifold*, Comment. Math. Helv. **74** (1999), no. 4, 575–590.

[Pol91] L. Polterovich, *The surgery of Lagrange submanifolds*, Geom. Funct. Anal. **1** (1991), no. 2, 198–210.

[She] Vsevolod Shevchishin, *Pseudoholomorphic curves and the symplectic isotopy problem*, arXiv:math/0010262 [math.sg].

[She04] Vsevolod V. Shevchishin, *On the local Severi problem*, Int. Math. Res. Not. (2004), no. 5, 211–237.

[Sik03] Jean-Claude Sikorav, *The gluing construction for normally generic J-holomorphic curves*, in: Symplectic and Contact Topology: Interactions and Perspectives (Toronto, ON/Montreal, QC, 2001), volume 35 of Fields Inst. Commun., pages 175–199. Amer. Math. Soc., Providence, RI, (2003).

[ST05] Bernd Siebert and Gang Tian, *On the holomorphicity of genus two Lefschetz fibrations*, Ann. of Math. (2) **161** (2005), no. 2, 959–1020.

[Sta16] Laura Starkston, *Comparing star surgery to rational blow-down*, J. Gökova Geom. Topol. GGT **10** (2016), 60–79.

[Sym96] Margaret Fife Symington, *New constructions of symplectic four-manifolds*, Thesis (Ph.D.)–Stanford University, ProQuest LLC, Ann Arbor, MI, (1996).

Mathematics Department, University of California, Davis
1 Shields Avenue, Davis, CA 95616, USA
E-mail address: lstarkston@math.ucdavis.edu