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The geometric quantizations and the measured Gromov-Hausdorff convergences

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On a compact symplectic manifold (X, ω) with a prequantum line bundle (L, ∇, h), we consider the one-parameter family of ω-compatible complex structures which converges to the real polarization coming from the Lagrangian torus fibration. There are several researches which show that the holomorphic sections of the line bundle localize at Bohr-Sommerfeld fibers. In this article we consider the one-parameter family of the Riemannian metrics on the frame bundle of L determined by the complex structures and ∇, h, and we can see that their diameters diverge. If we fix a base point in some fibers of the Lagrangian fibration we can show that they measured Gromov-Hausdorff converge to some pointed metric measure spaces with the isometric S^1-actions, which may depend on the choice of the base point. We observe that the properties of the S^1-actions on the limit spaces actually depend on whether the base point is in the Bohr-Sommerfeld fibers or not.

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1. Introduction

In this article we introduce a new approach to the geometric quantization from the viewpoint of the convergence of the Riemannian manifolds with respect to the measured Gromov-Hausdorff topology. On a compact symplectic manifold (X, ω) of dimension $2n$, a prequantum line bundle (L, ∇, h) is a triple of a complex line bundle L, hermitian metric h and connection ∇ preserving h with curvature form $F^\nabla = -\sqrt{-1}\omega$. By considering the following geometric structures compatible with ω, we can equip L with the finite dimensional vector subspace consisting of the special sections of L. The first one is an ω-compatible complex structure J, then denote by

$$H^0(X_J, L) = \{s \in C^\infty(L);\ \nabla_{\overline{\partial}_J} s = 0\}$$

the space of all of the holomorphic sections of L.

The second one is a Lagrangian fibration $\mu\colon X \to Y$, where Y is a smooth manifold, all of the points in Y are regular values of μ, and all fibers are compact and connected. Then put

$$V_\mu := \bigoplus_{y \in Y} H^0\left(\mu^{-1}(y), L|_{\mu^{-1}(y)}, \nabla|_{\mu^{-1}(y)}\right),$$
$$H^0\left(\mu^{-1}(y), L|_{\mu^{-1}(y)}, \nabla|_{\mu^{-1}(y)}\right) := \left\{s \in C^\infty(L|_{\mu^{-1}(y)});\ \nabla|_{\mu^{-1}(y)} s \equiv 0\right\}.$$

$\mu^{-1}(y)$ is called a *Bohr-Sommerfeld fiber* if $L|_{\mu^{-1}(y)}$ has nontrivial parallel sections. Tyurin showed in [19, Proposition 3.2] that if X is compact then there are at most finitely many Bohr-Sommerfeld fibers, accordingly, $\dim V_\mu$ is finite.

In many examples of symplectic manifolds with some complex structures and Lagrangian fibrations, it is observed that

$$\dim V_\mu = \dim H^0(X_J, L)$$

when the Kodaira vanishing holds, which can be interpreted as the localization of the Riemann-Roch index to the Bohr-Sommerfeld fibers, and discussed by Andersen [1], by Fujita, Furuta and Yoshida [8], and by Kubota [14].

Moreover, on smooth toric varieties, Baier, Florentino, Mourão and Nunes have constructed a one parameter family of the pairs of the complex structures and the basis of the spaces of holomorphic sections of L, then showed that the holomorphic sections localize on the Bohr-Sommerfeld fibers in [3]. The similar phenomena were observed in the case of the abelian varieties by Baier, Mourão and Nunes [4] and the flag varieties by Hamilton and Konno [11]. In these examples, the family of complex structures and holomorphic sections are described concretely.

In the context of the geometric quantization, the ω-compatible complex structures and Lagrangian fibrations are treated uniformly by using the notion of polarizations. The one-parameter families of complex structures given in the above papers are taken such that the Kähler polarizations corresponding to them converge to the real polarization corresponding to the Lagrangian fibration.

Recently, Yoshida showed the localization of holomorphic sections of prequantum line bundle to the Bohr-Sommerfeld fiber if X admits a Lagrangian fibration with a complete base in [21], where the family of complex structures are taken such that it converges to the real polarization corresponding to the Lagrangian fibration.

In this article, we also study the behavior of holomorphic sections of L where the family of complex structures converges to the real polarization from the view of the point of the measured Gromov-Hausdorff convergence. Fix an ω-compatible complex structure J. Then $H^0(X_J, L)$ can be identified with the eigenspace of a Laplace operator as follows. Put $S := \{u \in L;\, h(u,u) = 1\} \subset L$ be the orthogonal frame bundle of (L, h), then there is the standard identification

$$C^\infty(X, L) \cong (C^\infty(S) \otimes \mathbb{C})^\rho,$$

where $\rho\colon S^1 \to GL(\mathbb{C})$ is the 1-dimensional unitary representation of S^1 defined by $\rho(e^{\sqrt{-1}t}) := e^{\sqrt{-1}t}$, and the S^1-action on $C^\infty(S) \otimes \mathbb{C}$ is defined by

$$\{e^{\sqrt{-1}t} \cdot (f \otimes \xi)\}(u) := e^{-\sqrt{-1}t} f(ue^{\sqrt{-1}t}) \otimes \xi$$

for any $f \in C^\infty(S)$, $\xi \in \mathbb{C}$ and $u \in S$. The connection ∇ gives the connection 1-form on S and the decomposition of TS into the horizontal and vertical subspaces. Then we have the Riemannian metric $\hat{g}$ on S which respects the connection form and the Kähler metric $g_J := \omega(\cdot, J\cdot)$. The precise definition of $\hat{g}$ is given by Section 3. Denote by $\Delta^{\hat{g}}$ the Laplace operator of $\hat{g}$. Since S^1 acts on $(S, \hat{g})$ isometrically, the $\mathbb{C}$-linear extension of $\Delta_{\hat{g}}$ gives the operator acting on $(C^\infty(S) \otimes \mathbb{C})^\rho$. Then we can see that $H^0(X_J, L)$ is identified with the eigenspace of

$$\Delta_{\hat{g}}\colon (C^\infty(S) \otimes \mathbb{C})^\rho \to (C^\infty(S) \otimes \mathbb{C})^\rho$$

associate with the eigenvalue $n + 1$.

Now, we suppose that a one-parameter family of the ω-compatible complex structures $\{J_s\}_{s>0}$ on X is given, then we consider the one-parameter family of the operators

$$\Delta_{\hat{g}_s}\colon (C^\infty(S) \otimes \mathbb{C})^\rho \to (C^\infty(S) \otimes \mathbb{C})^\rho.$$

There are several research of the spectral convergence of the metric Laplacian on Riemannian manifolds or the connection Laplacians on vector bundles under the convergence of the spaces in the sense of the measured Gromov-Hausdorff topology [5][9][13][15][16][17]. Therefore, there should be the significant relation between the convergence of principal bundle S with the connection metric $\hat{g}_s$ and the convergence of holomorphic sections with respect to J_s. This article focus on the convergence of $(S, \hat{g}_s, p)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology and we study the metric measure spaces appearing as the limit.

Now we explain the main result of this article. Let (X, ω) be a symplectic manifold of dimension $2n$, which is not necessarily to be compact, (L, ∇, h) be a prequantum line bundle and $\{J_s\}_{0<s\leq 1}$ be a smooth family of ω-compatible complex structures. Assume that there is a smooth map $\mu\colon X \to Y$, where Y is an n-dimensional smooth manifold, and for any regular values y of μ, $\mu^{-1}(y)$ is a compact connected Lagrangian submanifold. Fix a regular value y. We assume that $\{J_s\}_{0<s\leq 1}$ converges to the real polarizations induced by μ near $\mu^{-1}(y)$ as $s \to 0$, there is a constant $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_{J_s}} \geq \kappa g_{J_s}$ for all s. We also suppose additional assumptions

which are precisely described in ♠ of Section 7.2. Let $g_{m,\infty}$ and μ_∞ be a Riemannian metric and a measure on $\mathbb{R}^n \times S^1$ defined by

$$g_{m,\infty} := \frac{1}{m^2(1+\|y\|^2)}(dt)^2 + \sum_{i=1}^{n}(dy_i)^2,$$
$$d\mu_\infty := dy_1 \cdots dy_n dt,$$

where m is a positive integer, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $e^{\sqrt{-1}t} \in S^1$. We define the isometric S^1-action on $(\mathbb{R}^n \times S^1, g_{m,\infty}, \mu_\infty)$ by $(y, e^{\sqrt{-1}t}) \cdot e^{\sqrt{-1}\tau} := (y, e^{\sqrt{-1}(t+m\tau)})$ for $e^{\sqrt{-1}\tau} \in S^1$. The followings are the main results of this article.

Theorem 1.1. *Let m be a positive integer and $u \in S|_{\mu^{-1}(y)}$. Assume that $\mu^{-1}(y)$ is a Bohr-Sommerfeld fiber of $L^{\otimes m}$ and not a Bohr-Sommerfeld fiber of $L^{\otimes m'}$ for any $0 < m' < m$. Then for some positive constant $K > 0$, the family of pointed metric measure spaces with the isometric S^1-action*

$$\left\{\left(S, \hat{g}_s, \frac{\mu_{\hat{g}_s}}{K\sqrt{s}^n}, u\right)\right\}_s$$

converges to $\left(\mathbb{R}^n \times S^1, g_{m,\infty}, \mu_\infty, (0,1)\right)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology.

Theorem 1.2. *Let $u \in S|_{\mu^{-1}(y)}$ and assume that $\mu^{-1}(y)$ is not a Bohr-Sommerfeld fiber of $L^{\otimes m}$ for any positive integer m. Then $\{(S, \hat{g}_s, \frac{\mu_{\hat{g}_s}}{K\sqrt{s}^n}, u)\}_s$ converges to $(\mathbb{R}^n, {}^t dy \cdot dy, dy_1 \cdots dy_n, 0)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology. Here, the S^1-action on $\mathbb{R}^n$ is trivial.*

Now let S_∞ be the metric measure space appears as the limit in Theorem 1.1 or Theorem 1.2 and denote by Δ_∞ its Laplacian. Denote by $W(n+1)$ the eigenspace of

$$\Delta_\infty \colon (C^\infty(S_\infty) \otimes \mathbb{C})^\rho \to (C^\infty(S_\infty) \otimes \mathbb{C})^\rho$$

associate with the eigenvalue $n+1$.

Theorem 1.3. *If S_∞ be the metric measure space appears as the limit in Theorem 1.1, then* $\dim W(n+1) = 1$ *if* $m = 1$ *and* $\dim W(n+1) = 0$ *if* $m > 1$. *If S_∞ be the metric measure space appears as the limit in Theorem 1.2, then* $\dim W(n+1) = 0$.

This article is organized as follows. First of all, we explain how to identify the holomorphic sections of L on (X, J) with the eigenfunctions on the frame bundle S equipped with the connection metric in Section 2 and 3. In Section 4, we review the definition of Bohr-Sommerfeld fibers for the pairs of symplectic manifolds and prequantum line bundles. In Section 5, we review the notion of Polarizations, which enables us to treat the ω-compatible complex structures and the Lagrangian fibrations. In Section 6 we explain the notion of the pointed S^1-equivariant measured Gromov-Hausdorff convergence. This notion is the special case of the convergence introduced by Fukaya and Yamaguchi [10]. These sections are the preparations for the main argument. In Section 7, we show the pointed S^1-equivariant measured Gromov-Hausdorff convergence near the Bohr-Sommerfeld fibers. First of all we obtain the local description of the connection metric $\hat{g}_s$ on S, then discuss the condition equivalent to the existence of the lower bound of the Ricci curvatures. Then we show the convergence of $\hat{g}_s$ to $g_{m,\infty}$ as $s \to 0$. In Section 9 we consider the limit of $\hat{g}_s$ near the non Bohr-Sommerfeld fibers, then show that the S^1-action on the limit space is trivial. In Section 8, we study the spectral structure of the Laplacian of the metric measure spaces appearing as the limit of $\hat{g}_s$. In Section 10, we raise some examples to which these approaches can be applied.

2. Holomorphic line bundles

Let (X, J, ω) be a compact Kähler manifold. We write $X = X_J$ when we regard X as a complex manifold. Let $\pi_E \colon E \to X_J$ be a holomorphic line bundle over X_J. Suppose h is a hermitian metric on E and $\nabla \colon \Gamma(E) \to \Omega^1(E)$ is the Chern connection. Under the decomposition $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$, we have the decomposition $\nabla = \nabla^{1,0} + \nabla^{0,1}$. Let $\nabla^*, (\nabla^{1,0})^*, (\nabla^{0,1})^*$ are the formal adjoint of $\nabla, \nabla^{1,0}, \nabla^{0,1}$, respectively.

For a holomorphic coordinate $(U, z^1, \dots, z^n)$ on X_J, put $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then we may write

$$\begin{aligned}
\nabla^* &= (\nabla^{1,0})^* + (\nabla^{0,1})^*, \\
(\nabla^{1,0})^* &= -g^{i\bar{j}} \iota_{\partial_i} \nabla_{\bar{\partial}_j}, \\
(\nabla^{0,1})^* &= -g^{i\bar{j}} \iota_{\bar{\partial}_j} \nabla_{\partial_i},
\end{aligned}$$

where $\partial_i := \frac{\partial}{\partial z^i}$. Let $F \in \Omega^{1,1}(X_J)$ be the curvature form. Since we have

$$(\nabla^{1,0})^* \nabla^{1,0} s = (\nabla^{0,1})^* \nabla^{0,1} s + g^{i\bar{j}} F(\partial_i, \bar{\partial}_j) s,$$

we obtain

$$\begin{aligned} \nabla^*\nabla &= 2\Delta_{\bar\partial} + \Lambda_\omega F, \\ \Delta_{\bar\partial} &:= (\nabla^{0,1})^*\nabla^{0,1}, \\ \Lambda_\omega F &:= g^{i\bar j} F(\partial_i, \bar\partial_j) \in C^\infty(X). \end{aligned}$$

Let $L \to X_J$ be a holomorphic line bundle with hermitian metric h and hermitian connection ∇ such that the curvature form is equal to $-\sqrt{-1}\omega$, and put $E = L^k$. Then for the connection on E determined by ∇ we have $F = -k\sqrt{-1}\omega$, then

$$\Lambda_\omega F = nk.$$

Now, put

$$H^0(X_J, L^k) := \left\{ s \in C^\infty(L^k);\ \nabla^{0,1}s = 0 \right\}.$$

Since X is compact, we can see

$$H^0(X_J, L^k) = \left\{ s \in C^\infty(L^k);\ \nabla^*\nabla s = nks \right\}.$$

3. Holomorphic sections on line bundles and eigenfunctions on frame bundle

Let (X, ω) be a connected symplectic manifold of dimension $2n$ and $(\pi\colon L \to X, \nabla, h)$ be a prequantum line bundle over (X, ω), that is, a complex line bundle with a hermitian metric h a connection ∇ preserving h whose curvature form is equal to $-\sqrt{-1}\omega$.

The complex structure J on X is *ω-compatible* if $\omega(J\cdot, J\cdot) = \omega$ holds and $g_J := \omega(\cdot, J\cdot)$ is positive definite. If J is ω-compatible, then ω is a Kähler form on X_J.

Since ω is of type $(1,1)$, ∇ determines a holomorphic structure on L, consequently ∇ is the Chern connection determined by h and J.

By the previous section we have $\nabla^*\nabla = 2\Delta_{\bar\partial} + n$. Put

$$S := S(L, h) := \{u \in L;\ |u|_h = 1\},$$

which is a principal S^1-bundle over X equipped with the S^1-connection $\sqrt{-1}\Gamma \in \Omega^1(S, \sqrt{-1}\mathbb{R})$ corresponding to ∇. The S^1-connection induces the following decomposition

$$\begin{aligned} T_uS &:= H_u \oplus V_u, \\ H_u &:= \mathrm{Ker}\,(\Gamma_u \colon T_uS \to \mathbb{R}), \\ V_u &:= \{\xi_u^\sharp \in T_uS;\, \xi \in \sqrt{-1}\mathbb{R}\}, \end{aligned}$$

where $\xi_u^\sharp := \frac{d}{dt}e^{t\xi}|_{t=0}$. Then the connection metric $\hat{g} = \hat{g}(L, J, h, \sigma, \nabla)$ on S is defined by

$$\hat{g}(L, J, h, \sigma, \nabla) := \sigma \cdot \Gamma^2 + (d\pi|_H)^* g_J$$

for $\sigma > 0$.

Remark 3.1. By regarding $-\Gamma$ as a contact structure and $-\sqrt{-1}^\sharp$ as the Reeb vector field, $(S, \hat{g}(L, J, h, 2, \nabla))$ becomes a Sasakian manifold.

Now we can recover L by S as the associate bundle as follows. Let $\rho_k \colon S^1 \to GL_1(\mathbb{C})$ be defined by $\rho_k(\lambda) = \lambda^k$ for $k \in \mathbb{Z}$, then we have the identification $L^k \cong S \times_{\rho_k} \mathbb{C}$. Then there are natural isomorphisms

$$C^\infty(X, L^k) \cong (C^\infty(S) \otimes \mathbb{C})^{\rho_k},$$

where the action of S^1 on $C^\infty(S) \otimes \mathbb{C}$ is defined by $(\lambda \cdot f)(u) := \lambda^k f(u\lambda)$. By applying the argument in the previous section for $E = L^k$ we have $\nabla^*\nabla = 2\Delta_{\bar{\partial}} + kn$. Note that we may regard $\nabla^*\nabla$ and $\Delta_{\bar{\partial}}$ as operators acting on $(C^\infty(S) \otimes \mathbb{C})^{\rho_k}$, then by [13, Section 3] we have $\nabla^*\nabla = \Delta_{\hat{g}} - \frac{k^2}{\sigma}$, therefore we obtain

$$2\Delta_{\bar{\partial}} = \Delta_{\hat{g}} - \left(\frac{k^2}{\sigma} + kn\right) \colon (C^\infty(S) \otimes \mathbb{C})^{\rho_k} \to (C^\infty(S) \otimes \mathbb{C})^{\rho_k}.$$

On some open set $U \subset X$, suppose that $L|_U$ is trivial as C^∞ complex bundles, then there exists a global smooth section $E \in C^\infty(U, L)$ such that $h(E, E) \equiv 1$. Let $\gamma \in \Omega^1(U, \mathbb{R})$ be defined by $\nabla E = \sqrt{-1}\gamma \otimes E$. Under the diffeomorphism $U \times S^1 \to S(L|_U, h)$ defined by $(z, e^{\sqrt{-1}t}) \mapsto e^{\sqrt{-1}t}E_z$, one can obtain the following identification as Riemannian manifolds with isometric S^1-action;

$$(S|_U, \hat{g}) \cong (U \times S^1, g_J|_U + \sigma(dt + \gamma)^2). \tag{1}$$

4. Bohr-Sommerfeld fibers

Let $(\pi\colon L \to X, \nabla)$ be a prequantum line bundle over a symplectic manifold (X, ω). A Lagrangian fibration over (X, ω) is a smooth map $\mu\colon X \to B$, where B is a smooth manifold of dimension $\frac{\dim X}{2}$, such that $X_b := \mu^{-1}(b)$ is a Lagrangian submanifold for every $b \in B \setminus B_{\mathrm{sing}}$ and $B \setminus B_{\mathrm{sing}}$ is open dense in B. We suppose that B and all of the fibers X_b are path-connected. Then every X_b is diffeomorphic to a compact torus by Liouville-Arnold theorem.

For a subset $Y \subset X$, the holonomy $\mathrm{Hol}(L|_Y, \nabla)$ is defined by

$$\mathrm{Hol}(L|_Y, \nabla) := \{e^{\sqrt{-1}t} \in S^1;\ \tilde{c}(1) = \tilde{c}(0)e^{\sqrt{-1}t},\ c \in \mathcal{P}(a)\},$$

where $\mathcal{P}(a)$ consists of piecewise smooth curve $c\colon [0,1] \to X$ with $c(0) = c(1) = a \in Y$, $\mathrm{Im}(c) \subset Y$ and $\tilde{c}$ is the horizontal lift of c. Note that $\mathrm{Hol}(L|_Y, \nabla)$ does not depend on $a \in Y$ if Y is path-connected.

Definition 4.1.

(i) X_b is a *Bohr-Sommerfeld fiber of* $\mu\colon X \to B$ if $\mathrm{Hol}(L|_{X_b}, \nabla)$ is trivial.

(ii) X_b is an *m-BS fiber of* $\mu\colon X \to B$ if $\mathrm{Hol}(L|_{X_b}, \nabla)$ is a subgroup of $\mathbb{Z}/m\mathbb{Z}$. X_b is a *strict m-BS fiber of* $\mu\colon X \to B$ if $\mathrm{Hol}(L|_{X_b}, \nabla) \cong \mathbb{Z}/m\mathbb{Z}$.

Remark 4.2. X_b is a m-BS fiber of $\mu\colon X \to B$ iff $\mathrm{Hol}(L^m|_{X_b}, \nabla)$ is trivial.

Remark 4.3. In this article we suppose that

$$B_m := \{b \in B;\ X_b \text{ is an } m\text{-BS fiber}\}$$

are discrete in B for all $m > 0$. For example, if B_{sing} is empty, then Tyurin has shown in [19] that B_m is always discrete. If we put

$$B'_m := \{b \in B;\ X_b \text{ is a strict } m\text{-BS fiber}\},$$

then $B_m = \bigsqcup_{l|m} B'_l$ holds.

5. Polarizations

In this section we review the notion of polarizations in the sense of [20] to treat complex structures and Lagrangian fibrations uniformly.

Let $V_{\mathbb{R}}$ be a real vector space of dimension $2n$ with symplectic form $\alpha \in \bigwedge^2 V^*$ and put $V = V_{\mathbb{R}} \otimes \mathbb{C}$. Then α extends $\mathbb{C}$-linearly to a complex symplectic form on V. *A Lagrangian subspace* W *of* V is a complex vector subspace of V such that $\dim_{\mathbb{C}} W = n$ and $\alpha(u, v) = 0$ for all $u, v \in W$. Put

$$\mathrm{Lag}(V, \alpha) := \{W \subset V;\ W \text{ is a Lagrangian subspace}\},$$

which is a submanifold of Grassmannian $\mathrm{Gr}(n, V)$.

For a symplectic manifold (X, ω), put

$$\mathrm{Lag}_\omega := \bigsqcup_{x \in X} \mathrm{Lag}(T_x X \otimes \mathbb{C}, \omega_x).$$

This is a fiber bundle over X, and a section $\mathcal{P}$ of Lag_ω is a subbundle of $TX \otimes \mathbb{C}$. $\mathcal{P}$ is said to be *integrable* if

$$[\Gamma(\mathcal{P}|_U), \Gamma(\mathcal{P}|_U)] \subset \Gamma(\mathcal{P}|_U)$$

holds for any open set $U \subset X$, and we call such $\mathcal{P}$ a *polarization of* X. In this article we consider the following two types of polarizations.

Kähler polarizations. Let J be an ω-compatible complex structure. The subbundle

$$\mathcal{P}_J := T_J^{0,1} X \subset TX \otimes \mathbb{C}$$

is called a Kähler polarization.

Real polarizations. Let Y be a smooth manifold of dimension n, $\mu\colon X \to Y$ be a smooth map such that all $b \in \mu(X)$ are regular values and $\mu^{-1}(b)$ are Lagrangian submanifolds. Then

$$\mathcal{P}_\mu := \mathrm{Ker}(d\mu) \otimes \mathbb{C} \subset TX \otimes \mathbb{C}$$

is called a real polarization.

Define $l\colon \mathrm{Lag}(V, \alpha) \to \{0, 1, \ldots, n\}$ by $l(W) := \dim_{\mathbb{C}}(W \cap \overline{W})$. Then for any Kähler polarization $\mathcal{P}_J$ we have $l((\mathcal{P}_J)_x) = 0$, and for any real polarization $\mathcal{P}_\mu$ we have $l((\mathcal{P}_\mu)_x) = n$.

Conversely, for a polarization $\mathcal{P}$ such that $l(\mathcal{P}_x) = 0$ for all $x \in X$, there is a unique complex structure J such that $\omega(J\cdot, J\cdot) = \omega$ and $\mathcal{P} = T_J^{0,1} X$. For a polarization $\mathcal{P}$ such that $l(\mathcal{P}_x) = n$ for all $x \in X$, we obtain the Lagrangian foliation.

Next we observe the local structure of $\mathrm{Lag}(V,\alpha)$. For $W \in \mathrm{Lag}(V,\alpha)$, we can take a basis $\{w_1,\dots,w_n\} \subset W$ and vectors $u^1,\dots,u^n \in V$ such that $\{w_1,\dots,w_n,u^1,\dots,u^n\}$ is a basis of V and

$$\alpha(w_i,w_j)=\alpha(u^i,u^j)=0, \quad \alpha(u^i,w_j)=\delta^i_j$$

hold. Put $W' := \mathrm{span}_{\mathbb{C}}\{u^1,\dots,u^n\}$ and take $A \in \mathrm{Hom}(W,W')$. Then the subspace

$$W_A := \{w + Aw \in V;\, w \in W\}$$

is Lagrangian iff the matrix (A_{ij}) defined by $Aw_i = A_{ij}u^j$ is symmetric. Consequently, we have the identification

$$T_W\mathrm{Lag}(V,\alpha) = \left\{A \in \mathrm{Hom}(W,W');\, A_{ij}=A_{ji}\right\}. \tag{2}$$

Now, we fix W such that $l(W)=n$. Then $w_1,\dots,w_n,u^1,\dots,u^n$ can be taken to be real vectors, hence

$$l(W_A) = \dim \mathrm{Ker}(A-\overline{A}) = n - \mathrm{rank}(A-\overline{A})$$

holds. Moreover W_A comes from an almost complex structure which makes α positive hermitian iff $\mathrm{Im}A \in M_n(\mathbb{R})$ is the positive definite symmetric matrix. We define

$$T_W\mathrm{Lag}(V,\alpha)_+ := \left\{A \in \mathrm{Hom}(W,W');\, A_{ij}=A_{ji},\, \mathrm{Im}A>0\right\}$$

under the identification (2). If W_t is a smooth curve in $\mathrm{Lag}(V,\alpha)$ such that $l(W_0)=n$ and $\frac{d}{dt}W_t|_{t=0} \in T_{W_0}\mathrm{Lag}(V,\alpha)_+$, then there is $\delta>0$ such that $l(W_t)=0$ and $\alpha(w,\bar{w})>0$ for any $w \in W_t \setminus \{0\}$ and $0<t\le\delta$. Conversely, even if W_t satisfies $l(W_0)=n$ and

$$l(W_t)=0, \quad \alpha(w,\bar{w})>0 \text{ for any } w \in W_t \setminus \{0\}$$

for all $t>0$, $\frac{d}{dt}W_t|_{t=0}$ is not necessary to be in $T_{W_0}\mathrm{Lag}(V,\alpha)_+$ since the closure of positive definite symmetric matrices contains semi-positive definite symmetric matrices.

6. Topology

In this section we explain the notion of the S^1-equivariant measured Gromov-Hausdorff topology. The following notion is the special case of [10, Definition 4.1].

Definition 6.1. Let G be a compact topological group.

(i) Let (P', d') and (P, d) be metric spaces with isometric G-action. A map $\phi : P' \to P$ is an *G-equivariant ε-approximation* if ϕ is G-equivariant and ε-approximation. Here, ε-approximation means that $|d'(x', y') - d(\phi(x'), \phi(y'))| < \varepsilon$ holds for all $x', y' \in P'$ and $P \subset B(\phi(P'), \varepsilon)$. Moreover if ϕ is a Borel map then it is called a *Borel G-equivariant ε-approximation*.

(ii) Let $\{(P_i, d_i, \nu_i, p_i)\}_i$ be a sequence of pointed metric measure spaces with isometric G-action. $(P_\infty, d_\infty, \nu_\infty, p_\infty)$ is said to be *the pointed G-equivariant measured Gromov-Hausdorff limit of* $\{(P_i, d_i, \nu_i, p_i)\}_i$ if G acts on P_∞ isometrically and for any $R > 0$ there are positive numbers $\{\varepsilon_i\}_i$, $\{R_i\}_i$ with

$$\lim_{i\to\infty} \varepsilon_i = 0, \quad \lim_{i\to\infty} R_i = R,$$

and Borel G-equivariant ε_i-approximation

$$\phi_i \colon (\pi_i^{-1}(B(x_i, R_i)), p_i) \to (\pi_\infty^{-1}(B(x_\infty, R)), p_\infty)$$

for every i such that $\phi_{i*}(\nu_i|_{\pi_i^{-1}(B(x_i,R_i))}) \to \nu_\infty|_{\pi_\infty^{-1}(B(x_\infty,R))}$ vaguely (see Remark 6.2). Here, $\pi \colon P_i \to P_i/G$ is the quotient map and $x_i = \pi_i(p_i)$.

Remark 6.2.

(i) Let X, Y be topological spaces, $\phi \colon X \to Y$ be a Borel map and ν be a Borel measure on X. Then $\phi_*\nu$ is the pushforward measure, that is, the measure on Y defined by $\phi_*\nu(B) := \nu(\phi^{-1}(B))$ for any Borel subset $B \subset Y$.

(ii) Let X be a topological space and $(\nu_i)_{i=1}^\infty$ be a sequence of Borel measures on X. Then $(\nu_i)_i$ converges to a measure ν on X *vaguely* if

$$\int_X f d\nu_i \to \int_X f d\nu$$

as $i \to \infty$ for any continuous function $f \in C(X)$ with compact support.

7. Convergence

Throughout of this section let (X^{2n},ω) be a symplectic manifold, Y^n a smooth manifold and

$$\mu\colon X\to Y$$

be a smooth surjective map such that $\mu^{-1}(y)$ are smooth compact connected Lagrangian submanifolds for all regular value $y\in Y$. Assume that $y_0\in Y$ is a regular value of μ. Then by [2][7][18], there are open neighborhoods $U\subset X$ of $X_0:=\mu^{-1}(y_0)$, $B'\subset Y$ of y_0, $B\subset\mathbb{R}^n$ of the origin 0, diffeomorphisms $\tilde f\colon B\times T^n\overset{\cong}{\to}U$ and $f\colon B'\overset{\cong}{\to}B$ such that $\tilde f^*\omega=\sum_{i=1}^n dx_i\wedge d\theta^i$, and $f(y_0)=0$, where $x=(x_1,\dots,x_n)=f\circ\mu\circ\tilde f$ and $\theta=(\theta^1,\dots,\theta^n)\in T^n=\mathbb{R}^n/\mathbb{Z}^n$. Therefore, we may suppose

$$\begin{aligned}
U&=B\times T^n,\quad \mu=(x_1,\dots,x_n),\quad \omega=dx_i\wedge d\theta^i,\\
B&=\left\{x=(x_1,\dots,x_n)\in\mathbb{R}^n;\ \|x\|=\sqrt{x_1^2+\cdots+x_n^2}<R\right\},\\
X_0&=\{0\}\times T^n
\end{aligned}$$

for some $0<R\le 1$.

Let (L,∇) be the prequantum line bundle on (X,ω) and h be a hermitian metric such that $\nabla h=0$. Since $[\omega|_U]=0\in H^2(U)$, then the 1st Chern class of $(L,\nabla)|_U$ vanishes, hence $L|_U$ is trivial as C^∞ complex line bundle by [6, Section 5].

From now on we consider some covering spaces of U given by the followings. Let $\Phi\colon\mathbb{Z}^n\to\mathbb{Z}/m\mathbb{Z}$ be a homomorphism of $\mathbb{Z}$-modules. Then $\operatorname{Ker}\Phi$ is of rank n, hence $\mathbb{R}^n/\operatorname{Ker}\Phi$ is diffeomorphic to the n-dimensional torus. Now we have the natural projection

$$\begin{array}{ccc}
\mathbb{R}^n/\operatorname{Ker}\Phi & \to & T^n\\
\Psi & & \Psi\\
\theta\bmod\operatorname{Ker}\Phi & \mapsto & \theta\bmod\mathbb{Z}^n
\end{array}$$

which give a covering space and a covering map

$$U_\Phi:=B\times(\mathbb{R}^n/\operatorname{Ker}\Phi),\quad p_\Phi\colon U_\Phi\to U.$$

From now on we denote by θ the element of $\mathbb{R}^n/\mathrm{Ker}\,\Phi$ or T^n for the simplicity, if there is no fear of confusion. If we take $\mathbf{w} \in \mathbb{Z}^n$ then

$$\begin{array}{rccc} \beta(\Phi(\mathbf{w})): & U_\Phi & \to & U_\Phi \\ & \cup & & \cup \\ & (x,\theta) & \mapsto & (x,\theta+\mathbf{w}) \end{array}$$

gives the action of $\mathrm{Im}\,\Phi$ on U_Φ, which is the deck transformations of p_Φ.

Proposition 7.1. *Let X_0 be a strict m-BS fiber. Then there are surjective homomorphism $\Phi\colon \mathbb{Z}^n \to \mathbb{Z}/m\mathbb{Z}$ and $E \in C^\infty(p_\Phi^* L)$ such that $h(E,E) \equiv 1$ and $\nabla E = -\sqrt{-1}x_i d\theta^i \otimes E$. Moreover, the deck transformations of p_Φ satisfies $\beta(k)^* E = e^{\frac{2k\sqrt{-1}\pi}{m}} E$ for $k \in \mathbb{Z}/m\mathbb{Z}$.*

Proof. Since X_0 is the m-BS fiber, one can obtain the flat section $\hat{E}$ of $(L^m|_U)|_{x=0})$ such that $h^{\otimes m}(\hat{E},\hat{E}) \equiv 1$. Then $\hat{E}$ can be extended to the nowhere vanishing section of $C^\infty(L^m|_U)$ with $h^{\otimes m}(\hat{E},\hat{E}) \equiv 1$. Define $\gamma \in \Omega^1(U)$ by $\nabla \hat{E} = \sqrt{-1}\gamma \otimes \hat{E}$. By computing the curvature form of ∇ one obtain $d\gamma = -m\omega|_U = -m dx_i \wedge d\theta^i$ which implies that $\gamma + m x_i d\theta^i$ is a closed 1-form on U. Denote by α the cohomology class represented by $\gamma + m x_i d\theta^i$ and let $\iota\colon \{0\} \times T^n \to U$ be the natural embedding. Since $\hat{E}|_{x=0}$ is flat, then one can see that $\iota^*\gamma = 0$ and $\iota^*\alpha = 0$. Since $\iota^*\colon H^1(B \times T^n) \to H^1(\{0\} \times T^n)$ is isomorphic, one can see that $\alpha = 0$, therefore there exists $\tau \in C^\infty(U,\mathbb{R})$ such that $\gamma + m x_i d\theta^i = d\tau$.

Then one have

$$\nabla(e^{-\sqrt{-1}\tau}\hat{E}) = \sqrt{-1}(-d\tau + \gamma) \otimes e^{-\sqrt{-1}\tau}\hat{E} = -m\sqrt{-1}x_i d\theta^i \otimes e^{-\sqrt{-1}\tau}\hat{E},$$

accordingly, by replacing $e^{-\sqrt{-1}\tau}\hat{E}$ by $\hat{E}$, we may suppose

$$\nabla \hat{E} = -m\sqrt{-1}x_i d\theta^i \otimes \hat{E}.$$

Let $\tilde{p}\colon \tilde{U} = B \times \mathbb{R}^n \to B \times T^n$ be the universal cover of U. Then there is a nowhere vanishing section $E \in C^\infty(\tilde{p}^* L)$ such that $E^{\otimes m} = \tilde{p}^*\hat{E}$, consequently we obtain the homomorphism $\Phi\colon \pi_1(U) = \mathbb{Z}^n \to \mathbb{Z}/m\mathbb{Z}$ defiend by

$$E_{(x,\theta+\mathbf{k})} = e^{2\pi\sqrt{-1}\Phi(\mathbf{k})} E_{(x,\theta)}$$

for $\mathbf{k} \in \mathbb{Z}^n$. Therefore, E descends to the section of $p_\Phi^* L$, then

$$\nabla E = -\sqrt{-1}x_i d\theta^i \otimes E$$

holds. Since X_0 is the strict m-BS fiber, Φ is surjective and p_Φ is an m-fold covering. □

7.1. Local description of the complex structures and the metrics

We assume that an ω-compatible complex structure J on X is given such that $\mathcal{P}_J|_U$ is close to $\mathcal{P}_\mu|_U$ with respect to C^0-topology, as sections of $\mathrm{Lag}_\omega|_U \to U$. Define $\mathcal{P}'_\mu$ by

$$(\mathcal{P}'_\mu)_p := \mathrm{span}_{\mathbb{C}} \left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \right\} \subset T_pU \otimes \mathbb{C},$$

then we have the direct decomposition $TU \otimes \mathbb{C} = \mathcal{P}_\mu \oplus \mathcal{P}'_\mu$. Since $\mathcal{P}_J|_U$ is close to $\mathcal{P}_\mu|_U$, the identification (2) gives

$$A = (A_{ij}(x,\theta))_{i,j} \in C^\infty(U) \otimes M_n(\mathbb{C})$$

such that

$$A_{ij} = A_{ji}, \quad \mathrm{Im}A > 0$$

and

$$\frac{\partial}{\partial \theta^i} + \bar{A}_{ij}(x,\theta)\frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n$$

is a frame of $\mathcal{P}_J|_U = T_J^{0,1}U$. Moreover the integrability of J gives

$$\frac{\partial A_{jk}}{\partial \theta^i} - \frac{\partial A_{ik}}{\partial \theta^j} + A_{il}\frac{\partial A_{jk}}{\partial x_l} - A_{jl}\frac{\partial A_{ik}}{\partial x_l} = 0. \tag{3}$$

Conversely, if a complex matrix valued function A satisfies above properties then we can recover $J|_U$. Therefore, the ω-compatible J complex structure close to $\mathcal{P}_\mu$ is identified with the matrix valued function A on U.

If we put $A_{ij} = P_{ij} + \sqrt{-1}Q_{ij}$, where $P_{ij}, Q_{ij} \in \mathbb{R}$, and denote by (Q^{ij}) the inverse of (Q_{ij}), then one can see

$$J\left(\frac{\partial}{\partial\theta^i}\right) = -P_{ij}Q^{jk}\frac{\partial}{\partial\theta^k} - (Q_{ik} + P_{ij}Q^{jl}P_{lk})\frac{\partial}{\partial x_k}, \tag{4}$$

$$J\left(\frac{\partial}{\partial x_i}\right) = Q^{ik}\frac{\partial}{\partial\theta^k} + Q^{ij}P_{jk}\frac{\partial}{\partial x_k}, \tag{5}$$

$$Jd\theta^k = -P_{ij}Q^{jk}d\theta^i + Q^{ik}dx_i, \tag{6}$$

$$Jdx_k = -(Q_{ik} + P_{ij}Q^{jl}P_{lk})d\theta^i + Q^{ij}P_{jk}dx_i, \tag{7}$$

therefore we obtain

$$g_J|_U = g_A := (Q_{ij} + P_{ik}Q^{kl}P_{lj})d\theta^i d\theta^j - 2P_{ik}Q^{jk}d\theta^i dx_j + Q^{ij}dx_i dx_j.$$

Denote by d_g the Riemannian distance of a Riemannian metric g. Then $g_J|_U = g_A$, $d_{g_J}|_U \le d_{g_A}$ always holds, however, the opposite inequality does not hold in general since the shortest path connecting two points in U need not be included in U. Here we consider the lower estimate of d_{g_J} and the upper estimate of d_{g_A}.

For a real symmetric positive definite matrix valued function $S(x,\theta) = (S_{ij}(x,\theta))_{i,j}$ depending on $(x,\theta) \in U$ continuously, let $\lambda_1(x,\theta), \dots, \lambda_n(x,\theta)$ be the eigenvalues of $S(x,\theta)$. Define

$$U_r := \{(x,\theta) \in \mathbb{R}^n \times T^n;\ \|x\| < r\} \subset U \quad (r \le R),$$
$$\sup S := \sup_{i,(x,\theta)\in U_{\frac{R}{2}}} \lambda_i(x,\theta), \quad \inf S := \inf_{i,(x,\theta)\in U_{\frac{R}{2}}} \lambda_i(x,\theta).$$

Since $\overline{U_{\frac{R}{2}}}$ is compact, $0 < \inf S \le \sup S < \infty$ holds.

Proposition 7.2. *Put*

$$\Theta := Q + PQ^{-1}P$$

for $A = P + \sqrt{-1}Q$. The following inequalities

$$\sqrt{\inf(\Theta^{-1})}\,\|x - x'\| \le d_{g_J}(u,u'),$$
$$d_{g_A}(u,u') \le \sqrt{\sup(\Theta^{-1})}\|x - x'\| + \frac{\sqrt{n\sup\Theta}}{2}$$

hold for any $u = (x,\theta), u' = (x',\theta') \in U_{\frac{R}{2}}$.

Proof. First of all we show the first equality. If we write

$$d\theta = \begin{pmatrix} d\theta^1 \\ \vdots \\ d\theta^n \end{pmatrix}, \quad dx = \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then we may write

$$\begin{aligned} g_A &= {}^t d\theta \cdot \Theta \cdot d\theta - {}^t dx \cdot Q^{-1}P \cdot d\theta - {}^t d\theta \cdot PQ^{-1} \cdot dx + {}^t dx \cdot Q^{-1} \cdot dx \\ &= {}^t \left(\sqrt{\Theta} d\theta - \sqrt{\Theta^{-1}} PQ^{-1} dx\right) \cdot \left(\sqrt{\Theta} d\theta - \sqrt{\Theta^{-1}} PQ^{-1} dx\right) \\ &\quad + {}^t dx \cdot \left(Q^{-1} - Q^{-1}P\Theta^{-1}PQ^{-1}\right) \cdot dx. \end{aligned}$$

Since we have

$$\begin{aligned} &\Theta \left(Q^{-1} - Q^{-1}P\Theta^{-1}PQ^{-1}\right) \\ &= 1 + PQ^{-1}PQ^{-1} - P\Theta^{-1}PQ^{-1} - PQ^{-1}PQ^{-1}P\Theta^{-1}PQ^{-1} \\ &= 1 + PQ^{-1}PQ^{-1} - PQ^{-1}\left(Q + PQ^{-1}P\right)\Theta^{-1}PQ^{-1} \\ &= 1 + PQ^{-1}PQ^{-1} - PQ^{-1}PQ^{-1} = 1, \end{aligned}$$

we can see

$$\Theta^{-1} = Q^{-1} - Q^{-1}P\Theta^{-1}PQ^{-1}.$$

Therefore,

$$\begin{aligned} g_A = {}^t &\left(\sqrt{\Theta} d\theta - \sqrt{\Theta^{-1}} PQ^{-1} dx\right) \cdot \left(\sqrt{\Theta} d\theta - \sqrt{\Theta^{-1}} PQ^{-1} dx\right) \\ &+ {}^t dx \cdot \Theta^{-1} \cdot dx \end{aligned} \tag{8}$$

holds. Now let $c_1 \colon [0,1] \to X$ be a path connecting $u, u' \in U_{\frac{R}{2}}$, and put $u = (x, \theta)$ and $u' = (x', \theta')$ with $\|x\|, \|x'\| < \frac{R}{2}$. Note that the image of c_1 is not always contained in $U_{\frac{R}{2}}$. If $\mathrm{Im}(c_1) \subset U_{\frac{R}{2}}$ does not hold, then let

$$\tau_0 := \inf\{\tau \in [0,1];\ c_1(\tau) \notin U_{\frac{R}{2}}\}.$$

If $\mathrm{Im}(c_1) \subset U_{\frac{R}{2}}$ holds, then put $\tau_0 := 1$. Put $c_1(\tau) = (x(\tau), \theta(\tau))$. Then by (8) we can see

$$\begin{aligned}\mathcal{L}(c_1) &\geq \int_0^{\tau_0} \sqrt{{}^t x'(\tau) \cdot \Theta^{-1} \cdot x'(\tau)}\, d\tau \\ &\geq \sqrt{\inf(\Theta^{-1})} \int_0^{\tau_0} |x'(\tau)| d\tau \geq \sqrt{\inf(\Theta^{-1})}\, \|x - x'\|.\end{aligned}$$

Next we show the second inequality. To show it, we compute the length of two types of paths in $U_{\frac{R}{2}}$.

For $\theta \in \mathbb{R}^n$ put $c_2(\tau) := (x, \tau\theta)$, then (8) gives

$$\begin{aligned}\mathcal{L}(c_2) = \int_0^1 |c_2'(\tau)|_{g_A} d\tau &= \int_0^1 \sqrt{\Theta_{ij}\theta^i\theta^j}\, d\tau \\ &\leq \sqrt{\sup \Theta}\|\theta\|.\end{aligned}$$

If $c_3(\tau) := (\tau x + (1-\tau)x', \theta)$, where $\|x\| \leq \frac{R}{2}$, then

$$\begin{aligned}\mathcal{L}(c_3) = \int_0^1 |c_3'(\tau)|_{\hat{g}_A} d\tau &= \int_0^1 \sqrt{\Theta^{ij}(x_i - x_i')(x_j - x_j')}\, d\tau \\ &\leq \sqrt{\sup(\Theta^{-1})}\|x - x'\|.\end{aligned}$$

Connecting these two types of paths one can see

$$\begin{aligned}d_A(u, u') &\leq \sqrt{\sup(\Theta^{-1})}\|x\| + \sqrt{\sup \Theta} \cdot \mathrm{diam}(T^n) \\ &= \sqrt{\sup(\Theta^{-1})}\|x\| + \frac{\sqrt{n \sup \Theta}}{2}.\end{aligned}$$

□

Now, we describe Riemannian metric $\hat{g}(L|_U, J, h, \sigma, \nabla)$ using the identification (1) in the case of X_0 is a strict m-BS fiber. First of all we consider the connection metric with respect to the pullback of g_J and $L|_U$ by the covering map $p_\Phi \colon U_\Phi \to U$, which is obtained in Proposition 7.1. We also denote by $p_\Phi \colon {p_\Phi}^* L \to L|_U$ the lift of the covering map, then the following commutative diagram is obtained;

$$\begin{array}{ccc} p_\Phi^* L & \to & L|_U \\ \downarrow & \circlearrowleft & \downarrow \\ U_\Phi & \to & U \end{array}$$

Let $p_\Phi^* J$ be the complex structure on U_Φ inherited from U by the covering map. Then one can see

$$S({p_\Phi}^*L, {p_\Phi}^*h) = p_\Phi^{-1}(S(L,h))$$

and

$$\hat{g}({p_\Phi}^*L, {p_\Phi}^*J, {p_\Phi}^*h, \sigma, {p_\Phi}^*\nabla) = {p_\Phi}^*\hat{g}(L|_U, J, h, \sigma, \nabla).$$

Since ${p_\Phi}^*L$ is trivial as C^∞ complex line bundle, there is the identification

$$\begin{array}{ccc} U_\Phi \times S^1 & \to & S({p_\Phi}^*L, {p_\Phi}^*h) \\ \Cup & & \Cup \\ (x, \theta, e^{\sqrt{-1}t}) & \mapsto & e^{\sqrt{-1}t}\cdot E_{(x,\theta)} \end{array}$$

by (1), where $E \in C^\infty({p_\Phi}^*L)$ is taken as in Proposition 7.1. Under the identification we have

$$\begin{aligned} &\hat{g}({p_\Phi}^*L, {p_\Phi}^*J, {p_\Phi}^*h, \sigma, {p_\Phi}^*\nabla) \\ &= \sigma(dt - x_i d\theta^i)^2 + (Q_{ij} + P_{ik}Q^{kl}P_{lj})d\theta^i d\theta^j \\ &\quad - 2P_{ik}Q^{jk}d\theta^i dx_j + Q^{ij}dx_i dx_j. \end{aligned}$$

By Proposition 7.1, the deck transformation of

$$p_\Phi \colon (S({p_\Phi}^*L, {p_\Phi}^*h)) \to S(L|_U, h)$$

is identified with

$$(9) \qquad k\cdot(x,\theta,e^{\sqrt{-1}t}) := (x, \theta + k\mathbf{w}_0, e^{\sqrt{-1}(t-\frac{2k\pi}{m})}) \quad (k \in \mathbb{Z}/m\mathbb{Z}),$$

where $\mathbf{w}_0 \in \mathbb{Z}^n$ is taken such that $\Phi(\mathbf{w}_0) = 1 \in \mathbb{Z}/m\mathbb{Z}$. Thus we obtain the next proposition.

Proposition 7.3. *Define the Riemannian metric $\hat{g}_A$ on $U_\Phi \times S^1$ by*

$$\begin{aligned} \hat{g}_A = \sigma(dt - x_i d\theta^i)^2 &+ (Q_{ij} + P_{ik}Q^{kl}P_{lj})d\theta^i d\theta^j \\ &- 2P_{ik}Q^{jk}d\theta^i dx_j + Q^{ij}dx_i dx_j, \end{aligned}$$

which is invariant under the $\mathbb{Z}/m\mathbb{Z}$ action defined by (9)*. If X_0 is a strict m-BS fiber, then*

$${p_\Phi}^*\,\hat{g}(L|_U, J|_U, h, \sigma, \nabla) = \hat{g}_A$$

holds.

7.2. Boundedness of the Ricci curvatures

First of all we compute the Ricci curvature of $g_J|_U$. Since ω is the Kähler form on (U, J), it suffices to compute the Ricci form of ω. First of all we can see that

$$\partial\theta^i\left(\frac{\partial}{\partial\theta^j} + A_{jk}\frac{\partial}{\partial x_k}\right) = d\theta^i\left(\frac{\partial}{\partial\theta^j} + A_{jk}\frac{\partial}{\partial x_k}\right) = \delta^i_j,$$

hence $\partial\theta^1, \dots, \partial\theta^n$ forms the dual frame of $\Omega^{1,0}$.

Proposition 7.4. *The Kähler form $\omega|_U$ and the Ricci form $\rho_\omega|_U$ are given by*

$$\begin{aligned}\omega|_U &= 2\sqrt{-1}Q_{ij}\partial\theta^i \wedge \overline{\partial}\theta^j,\\ \rho_\omega|_U &= \sqrt{-1}\partial\overline{\partial}\log\det(Q_{ij}) - \sqrt{-1}\partial\alpha + \sqrt{-1}\,\overline{\partial\alpha},\end{aligned}$$

where

$$\alpha := \frac{\partial\bar{A}_{ij}}{\partial x_i}\overline{\partial}\theta^j \in \Omega^{0,1}(U).$$

Proof. Since $dx_i - A_{ij}d\theta^j$ is of type $(0,1)$, one can see $\partial x_i = A_{ij}\partial\theta^j$. Then we have

$$\omega|_U = dx_i \wedge d\theta^i = \partial x_i \wedge \overline{\partial}\theta^i + \overline{\partial}x_i \wedge \partial\theta^i = 2\sqrt{-1}Q_{ij}\partial\theta^i \wedge \overline{\partial}\theta^j.$$

Take $f \in C^\infty(U', \mathbb{C}^\times)$ such that $\Omega := f\partial\theta^1 \wedge \cdots \wedge \partial\theta^n$ is a nowhere vanishing holomorphic section of the canonical bundle $K_X|_{U'}$ on some open set $U' \subset U$. If we put $\beta = f^{-1}\overline{\partial}f$, then the Ricci form $\rho_\omega|_{U'}$ is given by

$$\begin{aligned}-\sqrt{-1}\partial\overline{\partial}\log\frac{\omega|_{U'}^n}{\Omega\wedge\overline{\Omega}} &= -\sqrt{-1}\partial\overline{\partial}\log\det(Q_{ij}) + \sqrt{-1}\partial\overline{\partial}\log|f|^2\\ &= -\sqrt{-1}\partial\overline{\partial}\log\det(Q_{ij}) + \sqrt{-1}\partial\beta - \sqrt{-1}\,\overline{\partial\beta}.\end{aligned}$$

Since we have

$$0 = f^{-1}\overline{\partial}\Omega = \beta \wedge \partial\theta^1 \wedge \cdots \wedge \partial\theta^n + \overline{\partial}(\partial\theta^1 \wedge \cdots \wedge \partial\theta^n), \tag{10}$$

it suffices to compute $\overline{\partial}\partial\theta^i$ to describe β. Now, we have

$$\begin{aligned}
&\overline{\partial}\partial\theta^i\left(\frac{\partial}{\partial\theta^k}+A_{kj}\frac{\partial}{\partial x_j},\frac{\partial}{\partial\theta^l}+\bar{A}_{lh}\frac{\partial}{\partial x_h}\right)\\
&=d\partial\theta^i\left(\frac{\partial}{\partial\theta^k}+A_{kj}\frac{\partial}{\partial x_j},\frac{\partial}{\partial\theta^l}+\bar{A}_{lh}\frac{\partial}{\partial x_h}\right)\\
&=-\partial\theta^i\left(\left[\frac{\partial}{\partial\theta^k}+A_{kj}\frac{\partial}{\partial x_j},\frac{\partial}{\partial\theta^l}+\bar{A}_{lh}\frac{\partial}{\partial x_h}\right]\right)\\
&=-\left(\frac{\partial\bar{A}_{lh}}{\partial\theta^k}+A_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}\right)\partial\theta^i\left(\frac{\partial}{\partial x_h}\right)+\left(\frac{\partial A_{kj}}{\partial\theta^l}+\bar{A}_{lh}\frac{\partial A_{kj}}{\partial x_h}\right)\partial\theta^i\left(\frac{\partial}{\partial x_j}\right).
\end{aligned}$$

Since

$$\frac{\partial}{\partial x_h}=\frac{Q^{hl}}{2\sqrt{-1}}\left(\frac{\partial}{\partial\theta^l}+A_{lk}\frac{\partial}{\partial x_k}-\frac{\partial}{\partial\theta^l}-\bar{A}_{lk}\frac{\partial}{\partial x_k}\right)$$

holds, we have $\partial\theta^i\left(\frac{\partial}{\partial x_h}\right)=\frac{Q^{hi}}{2\sqrt{-1}}$, which gives

$$\overline{\partial}\partial\theta^i=-\frac{Q^{hi}}{2\sqrt{-1}}\left(\frac{\partial\bar{A}_{lh}}{\partial\theta^k}+A_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}-\frac{\partial A_{kh}}{\partial\theta^l}-\bar{A}_{lj}\frac{\partial A_{kh}}{\partial x_j}\right)\partial\theta^k\wedge\overline{\partial}\theta^l.$$

Moreover, the integrability of J implies

$$\begin{aligned}
\frac{\partial\bar{A}_{lh}}{\partial\theta^k}+A_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}&=\frac{\partial\bar{A}_{lh}}{\partial\theta^k}+\bar{A}_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}+2\sqrt{-1}Q_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}\\
&=\frac{\partial\bar{A}_{kh}}{\partial\theta^l}+\bar{A}_{lj}\frac{\partial\bar{A}_{kh}}{\partial x_j}+2\sqrt{-1}Q_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j},
\end{aligned}$$

accordingly one can see that

$$\overline{\partial}\partial\theta^i=Q^{hi}\left(\frac{\partial Q_{kh}}{\partial\theta^l}+\bar{A}_{lj}\frac{\partial Q_{kh}}{\partial x_j}-Q_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}\right)\partial\theta^k\wedge\overline{\partial}\theta^l. \tag{11}$$

By combining (10), we have

$$\beta=\left(Q^{ih}\frac{\partial Q_{ih}}{\partial\theta^l}+\bar{A}_{lj}Q^{ih}\frac{\partial Q_{ih}}{\partial x_j}-\frac{\partial\bar{A}_{lj}}{\partial x_j}\right)\overline{\partial}\theta^l.$$

Now the Jacobi's formula yields

$$\overline{\partial}(\log\det(Q_{ij})) = Q^{ih}\overline{\partial}Q_{ih} = Q^{ih}\left(\frac{\partial Q_{ih}}{\partial\theta^l}\overline{\partial}\theta^l + \frac{\partial Q_{ih}}{\partial x^j}\overline{\partial}x^j\right)$$
$$= Q^{ih}\left(\frac{\partial Q_{ih}}{\partial\theta^l} + \bar{A}_{jl}\frac{\partial Q_{ih}}{\partial x^j}\right)\overline{\partial}\theta^l,$$

therefore, we obtain

$$\beta = \overline{\partial}(\log\det(Q_{ij})) - \frac{\partial\bar{A}_{lj}}{\partial x_j}\overline{\partial}\theta^l,$$

which gives the assertion. □

Proposition 7.5. *Let α be as in Proposition* 7.4. *Then we have*

$$\partial\alpha = \left(\frac{\partial^2\bar{A}_{il}}{\partial\theta^k\partial x_i} + A_{kh}\frac{\partial^2\bar{A}_{il}}{\partial x_h\partial x_i}\right)\partial\theta^k\wedge\overline{\partial}\theta^l$$
$$- Q^{mh}\frac{\partial\bar{A}_{im}}{\partial x_i}\left(\frac{\partial Q_{kh}}{\partial\theta^l} + \bar{A}_{lj}\frac{\partial Q_{kh}}{\partial x_j} - Q_{kj}\frac{\partial\bar{A}_{lh}}{\partial x_j}\right)\partial\theta^k\wedge\overline{\partial}\theta^l.$$

Proof. Since

$$\partial\alpha = \partial\left(\frac{\partial\bar{A}_{il}}{\partial x_i}\overline{\partial}\theta^l\right)$$
$$= \left(\frac{\partial^2\bar{A}_{il}}{\partial\theta^k\partial x_i} + A_{kh}\frac{\partial^2\bar{A}_{il}}{\partial x_h\partial x_i}\right)\partial\theta^k\wedge\overline{\partial}\theta^l + \frac{\partial\bar{A}_{il}}{\partial x_i}\partial\overline{\partial}\theta^l,$$

the assertion follows from (11). □

From now on we consider the one parameter family of ω-compatible complex structures $\{J_s\}_{0<s<\delta}$ on (X,ω). Then we denote by $A(s,\cdot)$ the matrix valued function corresponding to $J_s|_U$. For simplicity, we often write $A = A(s,\cdot)$ if there is no fear of confusion. We assume the following condition ♠ for $\{J_s\}$. Let $\mathrm{pr}\colon X\times[0,\delta)\to X$ be the projection and $\mathrm{pr}^*\mathrm{Lag}_\omega$ be the pullback bundle.

- ♠ There is a smooth section $\mathcal{P}$ of $\mathrm{pr}^*\mathrm{Lag}_\omega|_{U\times[0,\delta)}\to U\times[0,\delta)$ such that $\mathcal{P}(\cdot,s) = \mathcal{P}_{J_s}|_U$ for $s>0$, $\mathcal{P}(\cdot,0) = \mathcal{P}_\mu|_U$ and

$$\frac{d}{ds}\mathcal{P}(x,s)\Big|_{s=0} \in T_{\mathcal{P}_\mu(x)}\mathrm{Lag}(T_xX\otimes\mathbb{C},\omega_x)_+.$$

By assuming ♠, there are a constant $K > 0$ and $A^0 \in C^\infty(U) \otimes M_n(\mathbb{C})$ such that $\sup_{i,j} \|A_{ij}(s,\cdot) - sA^0_{ij}\|_{C^2(U)} \leq Ks^2$, $\mathrm{Im}(A^0)$ is a positive definite symmetric matrix and $\sup_{i,j} \|A^0_{ij}\|_{C^2(U)} < \infty$.

For a function $f_0(s,x,\theta)$ and $f_1(s,x,\theta)$ we write

$$f_0(s,x,\theta) = f_1(s,x,\theta) + \mathcal{O}_{C^l}(s^k)$$

if there exists a constant $K > 0$ such that $\|f_0(s,x,\theta) - f_1(s,x,\theta)\|_{C^l(U)} \leq Ks^k$. For instance, if $\{J_s\}_s$ satisfies ♠, then we may write

$$A_{ij} = sA^0_{ij} + \mathcal{O}_{C^2}(s^2).$$

Proposition 7.6. *Assume that $\{J_s\}_s$ satisfies ♠. Put*

$$A^0_{ij} = P^0_{ij} + \sqrt{-1}Q^0_{ij}$$

for $P^0_{ij}, Q^0_{ij} \in C^\infty(U,\mathbb{R})$.

(i) $\frac{\partial A^0_{ij}}{\partial \theta^k} = \frac{\partial A^0_{ik}}{\partial \theta^j}$ *hold for any i,j,k.*

(ii) *Let $\mathrm{Ric}_{g_{J_s}}$ be the Ricci curvature of g_{J_s}. There exists a constant $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_{J_s}} \geq \kappa g_{J_s}$ hold for all $0 < s < \delta$, if and only if $Q^0_{ij}(x,\theta)$ are independent of $\theta \in T^n$.*

Proof. We have $\frac{\partial A_{ij}}{\partial \theta^k} = s\frac{\partial A^0_{ij}}{\partial \theta^k} + \mathcal{O}_{C^1}(s^2)$ and $\frac{\partial A_{ij}}{\partial x_k} = s\frac{\partial A^0_{ij}}{\partial x_k} + \mathcal{O}_{C^1}(s^2)$, then by (3) and taking $s \to 0$ we obtain (i).

Next we show (ii). It suffices to discuss the existence of κ such that $\rho_\omega \geq \kappa\omega$ holds. To show it, we write $\rho_\omega = \sqrt{-1}\rho_{kl}\partial\theta^k \wedge \overline{\partial}\theta^l$ for $\rho_{kl} \in \mathbb{R}$, then we expand ρ_{kl} about $s = 0$.

We have

$$\begin{aligned} \det Q_{ij} &= s^n \left(\det Q^0_{ij} + \mathcal{O}_{C^2}(s)\right), \\ \log\det Q_{ij} &= \log(s^n) + \log\det Q^0_{ij} + \mathcal{O}_{C^2}(s), \end{aligned}$$

where $A^0_{ij} = P^0_{ij} + \sqrt{-1}Q^0_{ij}$, and

$$Q^{ij} = s^{-1}Q^{0,ij} + \mathcal{O}_{C^2}(1),$$

where $(Q^{0,ij})_{i,j}$ is the inverse of $(Q^0_{ij})_{i,j}$. Since $\frac{\partial}{\partial\theta^i} + A_{ij}\frac{\partial}{\partial x_j}$ forms the dual basis of $\partial\theta^i$, we have

$$\partial\overline{\partial}\log\det Q_{ij} = \left(\frac{\partial^2(\log\det Q^0_{ij})}{\partial\theta^k\partial\theta^l} + \mathcal{O}_{C^0}(s)\right)\partial\theta^k\wedge\overline{\partial}\theta^l,$$
$$\partial\alpha - \overline{\partial\alpha} = (\mathcal{O}_{C^0}(s))\,\partial\theta^k\wedge\overline{\partial}\theta^l.$$

Set $H = \log\det Q^0_{ij}$. We obtain

$$\rho_\omega = \sqrt{-1}\left(\frac{\partial^2 H}{\partial\theta^k\partial\theta^l} + \mathcal{O}_{C^0}(s)\right)\partial\theta^k\wedge\overline{\partial}\theta^l.$$

Put $Q = (Q_{ij})_{ij}$, $Q^0 = (Q^0_{ij})_{ij}$ and $\mathrm{Hess}_\theta H = (\frac{\partial^2 H}{\partial\theta^i\partial\theta^j})_{ij}$, and let $\sqrt{Q}$ be the symmetric matrix such that $\sqrt{Q}^2 = Q$. Since $\omega = 2\sqrt{-1}Q_{kl}\partial\theta^k\wedge\overline{\partial}\theta^l$, then $\rho_\omega \geq \kappa\omega$ holds for some $\kappa\in\mathbb{R}$ if and only if the eigenvalues of

$$\sqrt{Q^{-1}}(\mathrm{Hess}_\theta H + \mathcal{O}_{C^0}(s))\sqrt{Q^{-1}}$$

are bounded from the below by a constant. Since

$$\sqrt{Q^{-1}} = \sqrt{s^{-1}}\sqrt{(Q^0)^{-1} + \mathcal{O}_{C^2}(s)} = \sqrt{s^{-1}}\left(\sqrt{(Q^0)^{-1}} + \mathcal{O}_{C^2}(s)\right),$$

we obtain

$$\begin{aligned}&\sqrt{Q^{-1}}(\mathrm{Hess}_\theta H + \mathcal{O}_{C^0}(s))\sqrt{Q^{-1}}\\ &= s^{-1}\left(\sqrt{(Q^0)^{-1}} + \mathcal{O}_{C^2}(s)\right)(\mathrm{Hess}_\theta H + \mathcal{O}_{C^0}(s))\left(\sqrt{(Q^0)^{-1}} + \mathcal{O}_{C^2}(s)\right)\\ &= s^{-1}\sqrt{(Q^0)^{-1}}\mathrm{Hess}_\theta H\sqrt{(Q^0)^{-1}} + \mathcal{O}_{C^0}(1).\end{aligned}$$

Therefore, the existence of the lower bound of the Ricci curvatures of $\{g_{J_s}\}$ is equivalent to

$$\sqrt{(Q^0)^{-1}}\mathrm{Hess}_\theta H\sqrt{(Q^0)^{-1}} \geq 0,$$

moreover, it is equivalent to $\mathrm{Hess}_\theta H \geq 0$. Consequently, H should be constant by the maximum principle.

By the imaginary part of (i), we can see that $Q^0_{ij}d\theta^j$ is a closed 1-form on $\{x\}\times T^n$, hence there exists a constant $\bar{Q}_{ij}$ depends only on x such that $[\bar{Q}_{ij}d\theta^j] = [Q^0_{ij}d\theta^j] \in H^1(\{x\}\times T^n)$. Consequently, there are $F_i(x,\cdot)\in$

$C^\infty(\{x\}\times T^n)$ such that $Q^0_{ij} = \bar{Q}_{ij} + \frac{\partial F_i}{\partial \theta^j}$ holds. Integrating this equality over $\{x\}\times T^n$, we have

$$\int_{\{x\}\times T^n} Q^0_{ij}(x,\theta) d\theta^1 \cdots d\theta^n = \bar{Q}_{ij}(x),$$

which implies that $(\bar{Q}_{ij})_{i,j}$ is a positive definite symmetric matrix. Since $\frac{\partial F_i}{\partial \theta^j} = \frac{\partial F_j}{\partial \theta^i}$ holds, one can see that $F_i d\theta^i$ is a closed 1-form on $\{x\}\times T^n$, then by repeating the above argument, there are $F(x,\cdot) \in C^\infty(\{x\}\times T^n)$ and $\bar{Q}_i(x) \in \mathbb{R}$ such that $F_i = \bar{Q}_i + \frac{\partial F}{\partial \theta^i}$, hence we may write

$$Q^0_{ij} = \bar{Q}_{ij} + \frac{\partial^2 F}{\partial \theta^i \partial \theta^j}.$$

Since $\bar{Q}_{ij}$ can be obtained by integrating Q^0_{ij} along some cycles of $H_1(\{x\}\times T^n, \mathbb{Z})$, $(\bar{Q}_{ij})_{i,j}$ is also a positive definite symmetric matrix. Now we take another torus $T^n_{\mathrm{copy}} = \mathbb{R}^n/\mathbb{Z}^n$ and the coordinate $\tau^1,\ldots,\tau^n$ coming from $\mathbb{R}^n$. Next we regard $M_x := \{x\}\times T^n \times T^n_{\mathrm{copy}}$ as a complex manifold whose holomorphic coordinate is given by

$$z^1 := \theta^1 + \sqrt{-1}\tau^1, \ldots, z^n := \theta^n + \sqrt{-1}\tau^n.$$

Define the Kähler form $\hat{\omega}_x$ on M_x by $\hat{\omega}_x := \sqrt{-1}\bar{Q}_{ij}(x) dz^i \wedge d\bar{z}^j$. Since $\bar{Q}_{ij}$ is constant on M_x, it is a Ricci-flat Kähler metric. Moreover

$$\hat{\omega}_x + 4\sqrt{-1}\partial\bar{\partial}F = \sqrt{-1}Q^0_{ij}(x,\theta) dz^i \wedge d\bar{z}^j$$

is also a Ricci-flat Kähler metric since $\det Q^0$ is constant. By the uniqueness of the Ricci-flat Kähler metric in the fixed Kähler class, we obtain $Q^0_{ij} = \bar{Q}_{ij}$. □

7.3. Convergence

Set

$$\begin{aligned} U_{\Phi,r} &:= B_r \times (\mathbb{R}^n/\mathrm{Ker}\,\Phi) = p_\Phi^{-1}(U_r), \\ S_r &:= S(L|_{U_r}, h), \\ S_{\Phi,r} &:= U_{\Phi,r} \times S^1 = p_\Phi^{-1}(S_r) \end{aligned}$$

for $0 < r \le R$.

For the brevity, put

$$\begin{aligned}
\tilde{d}_A &:= \text{the Riemannian distance of } \hat{g}_A \text{ on } S_{\Phi,R},\\
\hat{g}_J &:= \hat{g}(L, J, h, \sigma, \nabla),\\
d_J &:= \text{the Riemannian distance of } \hat{g}_J \text{ on } S(L,h),\\
d_A &:= \text{the Riemannian distance of } \hat{g}_J|_{S(L|_U,h)} \text{ on } S(L|_U,h),
\end{aligned}$$

then

$$\begin{aligned}
d_A(p_\Phi(u), p_\Phi(v)) &= \inf_{k=0,1,\dots,m-1} \tilde{d}_A(k\cdot u, v),\\
d_J(p_\Phi(u), p_\Phi(v)) &\le d_A(p_\Phi(u), p_\Phi(v))
\end{aligned}$$

hold for all $u, v \in S_{\Phi,R}$.

Denote by $B_{g_J}(p,r)$ the geodesic ball in (X, g_J) of radius r centered at p, and denote by $B_{g_A}(p,r)$ the geodesic ball in (U, g_A). Put

$$\mathbf{0} := (0,0) \in U,$$

and

$$\begin{aligned}
B_{d_A}(r) &:= \{p \in S(L|_U, h);\ d_A(p_\Phi(u_0), p) < r\},\\
B_{d_J}(r) &:= \{u \in X;\ d_J(p_\Phi(u_0), u) < r\}.
\end{aligned}$$

The the connection metric $\hat{g}_A$ given in Proposition 7.3 is written as

$$\hat{g}_A = \sigma(dt - x_i d\theta^i)^2 + \Theta_{ij} d\theta^i d\theta^j - 2P_{ik}Q^{jk} d\theta^i dx_j + Q^{ij} dx_i dx_j.$$

Proposition 7.7.

(i) $B_{g_J}\left(\mathbf{0}, \sqrt{\inf(\Theta^{-1})}R'\right) \subset U_{R'}$ *holds for any* $0 < R' \le \frac{R}{2}$.

(ii) *Take* $R_0 > 0$ *such that*

$$2\left(1 + \frac{2\sqrt{\sup(\Theta^{-1})}}{\sqrt{\inf(\Theta^{-1})}}\right) R_0 + \frac{\sqrt{n\sup\Theta} + 2\sqrt{\sigma}\pi}{\sqrt{\inf(\Theta^{-1})}} \le R.$$

Then $d_J(p,p') = d_A(p,p')$ *holds for any* $p, p' \in S_{R_0}$.

(iii) *Assume that* $\{J_s\}_s$ *satisfies* ♠. *Then there are constants* $s_0 > 0$, $0 < R_0 < \frac{R}{2}$ *and* $C > 0$ *such that*

$$B_{g_{J_s}}\left(\mathbf{0}, \frac{CR'}{\sqrt{s}}\right) \subset U_{R'}, \quad d_{J_s}|_{S_{R_0}} = d_{A(s,\cdot)}|_{S_{R_0}}$$

hold for any $0 < s \leq s_0$ *and* $0 < R' \leq \frac{R}{2}$.

Proof. (i) Let $p \in B_{g_J}\left(\mathbf{0}, \sqrt{\inf(\Theta^{-1})}R'\right)$ and suppose $p \notin U_{R'}$. Then there is a piecewise smooth path $c_1 \colon [0,1] \to X$ such that $c_1(0) = \mathbf{0}$, $c_1(1) = p$ and the length $\mathcal{L}(c_1)$ is less than $\sqrt{\inf(\Theta^{-1})}R'$. Let

$$\tau_1 := \inf\{\tau \in [0,1];\, c_1(\tau) \notin U_{R'}\} \leq 1.$$

Then by the first inequality of Proposition 7.2,

$$\mathcal{L}(c_1) \geq \mathcal{L}(c_1|_{[0,\tau_1]}) \geq d_{g_J}(\mathbf{0}, c_1(\tau_1)) \geq \sqrt{\inf(\Theta^{-1})}R'$$

holds, hence we have the contradiction.

(ii) Take $R_0 > 0$ which satisfies the assumption. Let $p, p' \in S_{R_0}$ and suppose $d_J(p,p') < d_A(p,p')$. Then there is a piecewise smooth path $c_2 \colon [0,1] \to S(L,h)$ connecting p and p' such that $\mathrm{Im}(c_2)$ is not contained in $S_{\frac{R}{2}}$ and $\mathcal{L}(c_2)$ is less than $d_A(p,p')$. Put

$$\tau_2 := \inf\{\tau \in [0,1];\, c_2(\tau) \notin S_{\frac{R}{2}}\},$$

then

$$\mathcal{L}(c_2) \geq \mathcal{L}(c_2|_{[0,\tau_2]}) \geq d_J(c_2(0), c_2(\tau_2))$$

holds. Since $\pi \colon (S(L,h), \hat{g}_J) \to (X, g_J)$ is a Riemannian submersion,

$$d_J(c_2(0), c_2(\tau_2)) \geq d_{g_J}(\pi(c_2(0)), \pi(c_2(\tau_2)))$$

holds, then we can see

$$\mathcal{L}(c_2) \geq d_{g_J}(\pi(c_2(0)), \pi(c_2(\tau_2))) \geq \sqrt{\inf(\Theta^{-1})}\left(\frac{R}{2} - R_0\right),$$

by the first inequality of Proposition 7.2. The second inequality of Proposition 7.2 gives

$$\sqrt{\inf(\Theta^{-1})}\left(\frac{R}{2}-R_0\right) < d_A(p,p') \\ \leq 2\sqrt{\sup(\Theta^{-1})}R_0 + \frac{\sqrt{n\sup\Theta}}{2} + \sqrt{\sigma}\pi,$$

therefore we obtain

$$\frac{R}{2} < \left(1+\frac{2\sqrt{\sup(\Theta^{-1})}}{\sqrt{\inf(\Theta^{-1})}}\right)R_0 + \frac{1}{\sqrt{\inf(\Theta^{-1})}}\left(\frac{\sqrt{n\sup\Theta}}{2}+\sqrt{\sigma}\pi\right),$$

which contradicts the assumption.

(iii) Since we have

$$\sqrt{\inf(\Theta^{-1})} = \frac{1}{\sqrt{s}}\left(\sqrt{\inf((\Theta^0)^{-1})}+\mathcal{O}(s)\right),$$
$$\sqrt{\sup(\Theta^{-1})} = \frac{1}{\sqrt{s}}\left(\sqrt{\sup((\Theta^0)^{-1})}+\mathcal{O}(s)\right),$$
$$\sqrt{\sup(\Theta)} = \sqrt{s}\left(\sqrt{\sup(\Theta^0)}+\mathcal{O}(s)\right)$$

by the Hoffman-Wielandt's inequality [12], there exists $s_0 > 0$ such that

$$\frac{2\sqrt{\sup(\Theta^{-1})}}{\sqrt{\inf(\Theta^{-1})}} \leq \frac{3\sqrt{\sup((\Theta^0)^{-1})}}{\sqrt{\inf((\Theta^0)^{-1})}},$$
$$\frac{\sqrt{n\sup\Theta}+2\sqrt{\sigma}\pi}{\sqrt{\inf(\Theta^{-1})}} \leq \frac{R}{10}$$

for all $s \leq s_0$. If we take $0 < R_0 < \frac{R}{2}$ such that

$$2\left(1+\frac{3\sqrt{\sup((\Theta^0)^{-1})}}{\sqrt{\inf((\Theta^0)^{-1})}}\right)R_0 \leq \frac{9R}{10},$$

then the assumption of (ii) is satisfied for $s \leq s_0$, hence we have $d_{J_s}|_{S_{R_0}} = d_{A(s,\cdot)}|_{S_{R_0}}$. Moreover, if we put

$$C := \inf_{0<s\leq s_0}\sqrt{s\inf(\Theta^{-1})} > 0,$$

then we can see

$$\sqrt{\inf(\Theta^{-1})}R' \geq \frac{CR'}{\sqrt{s}}$$

for $R' \leq \frac{R}{2}$, hence we have $B_{g_{J_s}}\left(\mathbf{0}, \frac{CR'}{\sqrt{s}}\right) \subset U_{R'}$ by (i). $\square$

Next we consider ω-compatible complex structures J, J', and compare the Riemannian distances of g_J and $g_{J'}$. We will show that if g_J and $g_{J'}$ are close to each other in some sense then their Riemannian distances are also close to each other.

Now, we define the distance $d_{\mathrm{Sym}^+(\mathbb{R}^N)}$ on

$$\mathrm{Sym}^+(\mathbb{R}^N) := \{g \in M_N(\mathbb{R});\, g_{ij} = g_{ji},\, g > 0\}$$

as follows. For $g \in \mathrm{Sym}^+(\mathbb{R}^N)$, take $v_1, \ldots, v_N \in \mathbb{R}^N$ such that $g(v_i, v_j) = \delta_{ij}$. For $g' \in \mathrm{Sym}^+(\mathbb{R}^N)$ let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ be eigenvalues of $(g'(v_i, v_j))_{i,j}$. Then define

$$d_{\mathrm{Sym}^+(\mathbb{R}^N)}(g, g') := \max_i |\log \lambda_i|.$$

Moreover, if g, g' are Riemannian metrics on M, then define

$$d_{\mathrm{Sym}^+(M)}(g, g') := \sup_{x \in M} d_{\mathrm{Sym}^+(T_xM)}(g_x, g'_x).$$

Lemma 7.8. *Let M be a smooth manifold of dimension N, g, g' be Riemannian metrics on M and d, d' be the Riemannian distances of g, g', respectively. If we assume $d_{\mathrm{Sym}^+(M)}(g, g') \leq 2\log 2$, then*

$$|d(p_0, p_1) - d'(p_0, p_1)| \leq d_{\mathrm{Sym}^+(M)}(g, g') d'(p_0, p_1)$$

holds. Moreover, for any $f \in C_0(M)$

$$\left| \int_M f d\mu_g - \int_M f d\mu_{g'} \right| \leq N \sup |f| \cdot \mu_{g'}(\mathrm{supp}(f)) \cdot d_{\mathrm{Sym}^+(M)}(g, g')$$

holds if $d_{\mathrm{Sym}^+(M)}(g, g') \leq \frac{\log 2}{N}$.

Proof. Let $c\colon [a,b]\to M$ be a piecewise smooth path, and denote by $\mathcal{L}_g(c)$ be the length of c with respect to g. Since we have

$$\begin{aligned} g(c'(t),c'(t)) &\le \exp\left(d_{\mathrm{Sym}^+(T_{c(t)}M)}(g_{c(t)},g'_{c(t)})\right) g'(c'(t),c'(t)) \\ &\le \exp\left(d_{\mathrm{Sym}^+(M)}(g,g')\right) g'(c'(t),c'(t)) \end{aligned}$$

then we can see

$$\mathcal{L}_g(c)\le \exp\left(\frac{d_{\mathrm{Sym}^+(M)}(g,g')}{2}\right)\mathcal{L}_{g'}(c)$$

and

$$d(p_0,p_1)\le \exp\left(\frac{d_{\mathrm{Sym}^+(M)}(g,g')}{2}\right) d'(p_0,p_1).$$

By the symmetry we also have

$$\exp\left(-\frac{d_{\mathrm{Sym}^+(M)}(g,g')}{2}\right) d'(p_0,p_1)\le d(p_0,p_1).$$

Therefore, we obtain

$$d'(p_0,p_1)-d(p_0,p_1)\le \left(1-\exp\left(-\frac{d_{\mathrm{Sym}^+(M)}(g,g')}{2}\right)\right) d'(p_0,p_1)$$

and

$$d(p_0,p_1)-d'(p_0,p_1)\le \left(\exp\left(\frac{d_{\mathrm{Sym}^+(M)}(g,g')}{2}\right)-1\right) d'(p_0,p_1).$$

Since $1-e^{-\frac{t}{2}}\le t$ and $e^{\frac{t}{2}}-1\le t$ holds for any $0\le t\le 2\log 2$, we have the first inequality.

Next we take $f\in C_0(M)$ and denote by $d\mu_g$ the Riemannian measure of g. Then we have

$$\left|\int_M f d\mu_g-\int_M f d\mu_{g'}\right|\le \int_M |f|\left|\frac{\det g}{\det g'}-1\right| d\mu_{g'}.$$

Since $|\log\frac{\det g}{\det g'}|\le N d_{\mathrm{Sym}^+(M)}(g,g')$ holds and $|e^t-1|\le 2|t|$ holds for $|t|\le \log 2$, we can see

$$\left|\int_M f d\mu_g-\int_M f d\mu_{g'}\right|\le N\sup|f|\cdot\mu_{g'}(\mathrm{supp}(f))\cdot d_{\mathrm{Sym}^+(M)}(g,g')$$

if $d_{\mathrm{Sym}^+(M)}(g,g') \leq \frac{\log 2}{N}$. □

Lemma 7.9. *Let $g, g' \in \mathrm{Sym}^+(\mathbb{R}^N)$ and $\{v_1,\cdots,v_N\}$ be a basis of $\mathbb{R}^N$. Put $\mathbf{g} = (g(v_i,v_j))_{i,j}$ and $\mathbf{g}' = (g'(v_i,v_j))_{i,j}$. Denote by $\alpha_1,\dots,\alpha_N$ be the eigenvalues of $\mathbf{g}'\mathbf{g}^{-1}$. Then $\alpha_i \in \mathbb{R}$ and $d_{\mathrm{Sym}^+(\mathbb{R}^N)}(g,g') = \max_i |\log \alpha_i|$.*

Proof. Let $\sqrt{\mathbf{g}}$ be the square root of $\mathbf{g}$. If we put $e_i = \sum_j \sqrt{\mathbf{g}}^{-1}{}_{ij} v_j$, then $e_1,\cdots,e_N$ is an orthonormal basis of $(\mathbb{R}^N, g)$, therefore we have

$$d_{\mathrm{Sym}^+(\mathbb{R}^N)}(g,g') = \max_i |\log \lambda_i|,$$

where λ_i are the eigenvalues of

$$(g'(e_i,e_j))_{ij} = \sqrt{\mathbf{g}}^{-1}\mathbf{g}'\sqrt{\mathbf{g}}^{-1}.$$

Since we have

$$\sqrt{\mathbf{g}}^{-1}\cdot\left(\mathbf{g}'\mathbf{g}^{-1}\right)\cdot\sqrt{\mathbf{g}} = \sqrt{\mathbf{g}}^{-1}\mathbf{g}'\sqrt{\mathbf{g}}^{-1},$$

$\{\alpha_1,\dots,\alpha_N\} = \{\lambda_1,\dots,\lambda_N\}$ holds. □

Suppose that X_0 is a strict m-BS fiber and fix a small $s>0$ and a frame

$$dt - x_i d\theta^i,\ \sqrt{s}d\theta^1,\dots,\sqrt{s}d\theta^n,\ \frac{1}{\sqrt{s}}dx_1,\dots,\frac{1}{\sqrt{s}}dx_n$$

of $T^*(U_\Phi \times S^1)$. Then the matrix representation of $\hat{g}_A$ is given by

$$\mathbf{g}_A := \begin{pmatrix} \sigma & 0 & 0 \\ 0 & s^{-1}\Theta & -PQ^{-1} \\ 0 & -Q^{-1}P & sQ^{-1} \end{pmatrix},$$

and its inverse is

$$\mathbf{g}_A^{-1} = \begin{pmatrix} \sigma^{-1} & 0 & 0 \\ 0 & sQ^{-1} & Q^{-1}P \\ 0 & PQ^{-1} & s^{-1}\Theta \end{pmatrix}.$$

Suppose that $\{A(s,\cdot)\}_s$ corresponds to $\{J_s\}$ which satisfies ♠. Fix $r \geq 1$. Then there is a constant $K>0$ depending only on $\{A(s,\cdot)\}_s$ such that

$$|A(s,x,\theta) - sA^0(x,\theta)| \leq Ks^2$$
$$|A^0(x,\theta) - A^0(0,\theta)| \leq K\|x\|$$

for any $(x,\theta) \in U_\Phi$. If $(x,\theta) \in U_{\Phi,\sqrt{s}r}$ for $r \geq 1$ and $s > 0$ with $\sqrt{s}r \leq R$, then we have $s \leq \sqrt{s}\frac{R}{r} \leq \sqrt{s}r$ since $R \leq 1$, hence we obtain

$$\left|s^{-1}A(s,x,\theta) - A^0(0,\theta)\right| \leq K\sqrt{s}r.$$

Here we write

$$f_0(s,x,\theta) \cong_{\sqrt{s}r} f_1(s,x,\theta)$$

if there is a constant $K > 0$ such that $|f_0(s,x,\theta) - f_1(s,x,\theta)| \leq K\sqrt{s}r$ holds for any $(x,\theta) \in U_{\Phi,\sqrt{s}r}$.

Now $A'(s,x,\theta) := sA^0(0,\theta)$ gives another family of complex structures $\{J'_s\}_s$ which satisfies ♠, by (i) of Proposition 7.6. Since we have

$$\begin{aligned} s^{-1}\Theta &\cong_{\sqrt{s}r} \Theta^0(0,\theta), \\ PQ^{-1} &\cong_{\sqrt{s}r} P^0(0,\theta)Q^0(0,\theta)^{-1}, \\ Q^{-1}P &\cong_{\sqrt{s}r} Q^0(0,\theta)^{-1}P^0(0,\theta), \\ sQ^{-1} &\cong_{\sqrt{s}r} Q^0(0,\theta)^{-1}, \end{aligned}$$

where $\Theta^0(0,\theta) = Q^0(0,\theta) + P^0(0,\theta)Q^0(0,\theta)^{-1}P^0(0,\theta)$, then we obtain

$$\mathbf{g}_{A'}^{-1}\mathbf{g}_A \cong_{\sqrt{s}r} I_{2n+1}.$$

By Lemma 7.9, the eigenvalues of $\mathbf{g}_{A'}^{-1}\mathbf{g}_A$ are real. If $1 + \lambda \in \mathbb{R}$ is one of the eigenvalues, then

$$f(\lambda) := \det\left((1+\lambda)I_{2n+1} - \mathbf{g}_{A'}^{-1}\mathbf{g}_A\right) = 0$$

holds. Since we have

$$f(\lambda) = \det\left\{\lambda I_{2n+1} + (I_{2n+1} - \mathbf{g}_{A'}^{-1}\mathbf{g}_A)\right\},$$

there exists a constant $K > 0$ depending only on $\{A(s,\cdot)\}_s$, and there exist $c_0, c_1, \ldots, c_{2n} \in \mathbb{R}$ such that $\max_i |c_i| \leq K$ and

$$f(\lambda) = \lambda^{2n+1} + \sum_{i=0}^{2n} c_i(\sqrt{s}r)^{2n+1-i}\lambda^i.$$

Lemma 7.10. *For any $n \in \mathbb{Z}_{\geq 0}$, $K > 0$ and $r \geq 1$ there is a sufficiently large $N > 0$ depending only on n and K such that for any $c_0, c_1, \dots, c_{2n} \in [-K, K]$ and $\varepsilon > 0$, the solution λ of the equation*

$$f(\lambda) = \lambda^{2n+1} + \sum_{i=0}^{2n} c_i \varepsilon^{2n+1-i} \lambda^i = 0$$

always satisfies $|\lambda| \leq N\varepsilon$.

Proof. Put $\lambda = \varepsilon t$. Then $f(\lambda) = \varepsilon^{2n+1} \left(t^{2n+1} + \sum_{i=0}^{2n} c_i t^i \right)$. If $f(\lambda) = 0$ then we have $t^n = -\sum_{i=0}^{2n} c_i t^i$. Suppose $|t| \geq 1$. Then

$$|t|^{2n+1} \leq \sum_{i=0}^{2n} |c_i||t|^i \leq \sum_{i=0}^{2n} K|t|^{2n} = (2n+1)K|t|^{2n}$$

holds, hence $|t| \leq (2n+1)K$ is obtained. Consequently we can see $|\lambda| \leq \max\{1, (2n+1)K\}\varepsilon$. □

By Lemma 7.10 we can see $|\log(1+\lambda)| \leq N\sqrt{s}r$ for the eigenvalue $1+\lambda$ of $\mathbf{g}_{A'}^{-1}\mathbf{g}_A$, where N is the constant depending only on K. Therefore, we obtain the following proposition by Lemma 7.9.

Proposition 7.11. *Let A, A' be as above and let $r \geq 1$, $s > 0$ with $\sqrt{s}r \leq R$. Then there exists a constant $C > 0$ depending only on A such that*

$$d_{\mathrm{Sym}^+(U_{\Phi,\sqrt{s}r} \times S^1)}(\hat{g}_A, \hat{g}_{A'}) \leq C\sqrt{s}r.$$

From now on we assume $R > 0$ satisfies

$$CR \leq 2\log 2,$$

where C is the constant in Proposition 7.11. Then Lemma 7.8 holds for

$$M = S_{\Phi,R}, \quad g = \hat{g}_A, \quad g' = \hat{g}_{A'}.$$

and for

$$M = S_{\Phi,R}, \quad g = \hat{g}_{A'}, \quad g' = \hat{g}_A.$$

Proposition 7.12. *Let $\{J_s\}_s$ satisfy ♠ and $A'(s,x,\theta) := sA^0(0,\theta)$.*

(i) *There are positive constants* C_0', C_1' *depending only on* $A^0(0,\cdot)$ *and* σ *such that*

$$B_{g_{A'}}(\mathbf{0}, C_0'r) \subset U_{\sqrt{s}r} \subset B_{g_{A'}}(\mathbf{0}, C_1'r)$$

for any $r \geq 1$ *and* $s > 0$ *with* $\sqrt{s}r \leq R$.

(ii) *Suppose* X_0 *is a strict m-BS fiber. Then there are constants* $C > 0$ *and* $0 < R_0 < \frac{R}{2}$ *depending only on* A *and* σ *such that*

$$|d_{J_s}(p,q) - d_{A'}(p,q)| < C\sqrt{s}r^2$$

holds for any $r \geq 1$, $s > 0$ *with* $\sqrt{s}r \leq R_0$ *and* $p, q \in S_{\sqrt{s}r}$.

(iii) *There are positive constants* C_0, C_1 *and* $0 < R_0 < \frac{R}{2}$ *depending only on* A *and* σ *such that*

$$B_{g_{J_s}}(\mathbf{0}, C_0 r) \subset U_{\sqrt{s}r} \subset B_{g_{J_s}}(\mathbf{0}, C_1 r)$$

for any $r \geq 1$, $s > 0$ *with* $\sqrt{s}r \leq R_0$.

Proof. (i) Apply Proposition 7.2 for A'. Then there are positive constants C_2, C_3, C_4 depending only on $A^0(0,\cdot)$ and σ such that

$$C_2\sqrt{s}^{-1}\|x\| \leq d_{g_{A'}}(\mathbf{0}, u) \leq C_3\sqrt{s}^{-1}\|x\| + C_4$$

for any $u = (x, \theta)$ and $s > 0$. If $\|x\| < \sqrt{s}r$ then

$$d_{g_{A'}}(\mathbf{0}, u) < C_3 r + C_4 \leq (C_3 + C_4)r$$

holds since $r \geq 1$, which implies $U_{\sqrt{s}r} \subset B_{g_{A'}}(\mathbf{0}, (C_3 + C_4)r)$. On the other hand if $d_{g_{A'}}(\mathbf{0}, u) < C_2 r$ holds then

$$C_2\sqrt{s}^{-1}\|x\| \leq d_{g_{A'}}(\mathbf{0}, u) < C_2 r$$

gives $\|x\| < \sqrt{s}r$, hence $B_{g_{A'}}(\mathbf{0}, C_2 r) \subset U_{\sqrt{s}r}$ holds.

(ii) By applying Proposition 7.11, there is a constant $C_5 > 0$ such that

$$d_{\mathrm{Sym}^+(S_{\Phi,\sqrt{s}r})}(\hat{g}_A, \hat{g}_{A'}) \leq C_5\sqrt{s}r$$

holds if $\sqrt{s}r \leq R$. Now take $R_0 \leq \min\{\frac{2\log 2}{C_5}, R\}$ and assume $\sqrt{s}r \leq R_0$, then we may apply Lemma 7.8 and we have

$$|\tilde{d}_A(u,v) - \tilde{d}_{A'}(u,v)| \leq d_{\mathrm{Sym}^+(M)}(g, g')\tilde{d}_{A'}(u,v) \leq C_5\sqrt{s}r\tilde{d}_{A'}(u,v)$$

for all $u, v \in S_{\Phi,\sqrt{s}r}$. By the same argument in the proof of Proposition 7.2, we have the upper estimate

$$\begin{aligned}\tilde{d}_{A'}(u,v) \le \sqrt{\sup(Q^0)^{-1}}\frac{\|x-x'\|}{\sqrt{s}} \\ + \sqrt{s}\sqrt{\sigma r^2 + \cdot \sup \Theta^0} \cdot \mathrm{diam}(\mathbb{R}^n/\mathrm{Ker}\Phi) + \sqrt{\sigma}\pi,\end{aligned}$$

where $u = (x, \theta, e^{\sqrt{-1}t}), v = (x', \theta', e^{\sqrt{-1}t'})$. Since $\|x - x'\| \le 2\sqrt{s}r$, $\sqrt{s}r \le R \le 1$ and $r \ge 1$, we have $\frac{\|x-x'\|}{\sqrt{s}} \le 2r$ and $s \le \frac{R^2}{r^2} \le 1$, then there is a constant $C_6 > 0$ depending only on A^0, σ, Φ such that $\tilde{d}_{A'}(u,v) \le C_6 r$, which gives

$$|\tilde{d}_A(u,v) - \tilde{d}_{A'}(u,v)| < 2C_5C_6\sqrt{s}r^2.$$

Therefore, we can see

$$\begin{aligned}d_A(p_\Phi(u), p_\Phi(v)) &= \inf_{k=0,1,\ldots,m-1} \tilde{d}_A(k \cdot u, v) \\ &< \inf_{k=0,1,\ldots,m-1} \left\{\tilde{d}_{A'}(k \cdot u, v) + 2C_5C_6\sqrt{s}r^2\right\} \\ &= d_{A'}(k \cdot u, v) + 2C_5C_6\sqrt{s}r^2\end{aligned}$$

and similarly $d_{A'}(p_\Phi(u), p_\Phi(v)) < d_A(k \cdot u, v) + 2C_5C_6\sqrt{s}r^2$ is obtained. By (iii) of Proposition 7.7, we can take $0 < R'_0 < \frac{R}{2}$ and $s_0 > 0$ such that $d_{J_s}|_{S_{R'_0}} = d_A|_{S_{R'_0}}$ holds for any $0 < s \le s_0$. If we put $C = 2C_5C_6$ and $R_0 = \min\{\frac{2\log 2}{C_5}, R'_0, \sqrt{s_0}\}$, then $\sqrt{s}r \le R_0$ implies $s \le s_0$, hence we have (ii).

(iii) Take C, s_0, R_0 as in (iii) of Proposition 7.7 and replace R_0 by the smaller one such that $R_0 \le \sqrt{s_0}$. Then we have $B_{g_{J_s}}(\mathbf{0}, Cr) \subset U_{\sqrt{s}r}$ if $\sqrt{s}r \le R_0$. Next we assume $u \in U_{\sqrt{s}r}$. By (i), we have $u \in B_{g_{A'}}(\mathbf{0}, C'_1 r)$. Since $\pi\colon (S_R, \hat{g}_{J_s}) \to (U_R, g_{J_s})$ and $\pi\colon (S_R, \hat{g}_{A'}) \to (U_R, g_{A'})$ are Riemannian submersions, therefore (ii) gives

$$\begin{aligned}d_{g_{J_s}}(\pi(u), \pi(u')) &= \inf_{e^{\sqrt{-1}t} \in S^1} d_{J_s}(ue^{\sqrt{-1}t}, u') \\ &\le \inf_{e^{\sqrt{-1}t} \in S^1} d_{A'}(ue^{\sqrt{-1}t}, u') + C\sqrt{s}r^2 \\ &= d_{g_{A'}}(\pi(u), \pi(u')) + C\sqrt{s}r^2.\end{aligned}$$

Consequently we obtain

$$d_{g_{J_s}}(\mathbf{0}, u) \le d_{g_{A'}}(\mathbf{0}, u) + C\sqrt{s}r^2 < C'_1 r + CR_0 r,$$

which implies $U_{\sqrt{s}r} \subset B_{g_{J_s}}(\mathbf{0}, C_1 r)$ by putting $C_1 = C_1' + C$. □

Proposition 7.13. *Let $\{J_s\}_s$ satisfy ♠ and $A'(s,x,\theta) := sA^0(0,\theta)$. There exist constants $R_0, C > 0$ such that*

$$\mathrm{id}\colon \left(\pi^{-1}(B_{g_{J_s}}(\mathbf{0}, r - C\sqrt{s}r^2)), d_{J_s}\right) \to \left(\pi^{-1}(B_{g_{A'(s,\cdot)}}(\mathbf{0}, r)), d_{A'(s,\cdot)}\right)$$

is a Borel $C\sqrt{s}r^2$-S^1-equivariant Hausdorff approximation for any $r \geq 1$ and $s \leq \frac{R_0^2}{Cr^2}$. Moreover, if $f\colon S_R \to \mathbb{R}$ is a Borel function such that $\mathrm{supp}(f) \subset \overline{S_{R'}}$ *for some $R' \leq R$ and* $\sup|f| < \infty$, *then*

$$\left| \int_{S_R} f d\mu_{\hat{g}_{A'}} - \int_{S_R} f d\mu_{\hat{g}_{J_s}} \right| \leq C \sup|f| (R')^{n+1}$$

holds.

Proof. Fix $r \geq 1$. Take $R_0, C_0, C_1, C_0', C_1', C$ such that Proposition 7.12 holds. We may suppose $C > 1$ and $C_0 = C_0' = C^{-1}$, $C_1 = C_1' = C$. Then $B_{g_{J_s}}(\mathbf{0}, r) \subset U_{C\sqrt{s}r}$ and

$$|d_{J_s}(p,q) - d_{A'}(p,q)| < C^2\sqrt{s}r^2$$

hold for any $p, q \in S_{C\sqrt{s}r}$ and $0 < s \leq \frac{R_0^2}{C^2r^2}$. If $u \in B_{g_{J_s}}(\mathbf{0}, r - C^2\sqrt{s}r^2)$, then

$$d_{g_{A'}}(\mathbf{0}, u) < d_{g_{J_s}}(\mathbf{0}, u) + C^2\sqrt{s}r^2 < r,$$

which implies $B_{g_{J_s}}(\mathbf{0}, r - C^2\sqrt{s}r^2) \subset B_{g_{A'}}(\mathbf{0}, r)$.

Now, $B_{g_{A'}}(\mathbf{0}, r) \subset U_{C\sqrt{s}r}$ holds. We also have

$$B_{g_{A'}}(\mathbf{0}, r) \subset B_{g_{J_s}}(\mathbf{0}, r + C^2\sqrt{s}r^2).$$

Since $d_{g_{J_s}}$ is an intrinsic metric, we have

$$B_{g_{J_s}}(\mathbf{0}, r + C^2\sqrt{s}r^2) = B_{g_{J_s}}\left(B_{g_{J_s}}(\mathbf{0}, r - C^2\sqrt{s}r^2), 2C^2\sqrt{s}r^2\right).$$

hence we can see that

$$\mathrm{id}\colon \left(\pi^{-1}(B_{g_{J_s}}(\mathbf{0}, r - C^2\sqrt{s}r^2)), d_{J_s}\right) \to \left(\pi^{-1}(B_{g_{A'}}(\mathbf{0}, r)), d_{A'(s,\cdot)}\right)$$

is a Borel ε_i-S^1-equivariant Hausdorff approximation.

Let $f\colon S_R \to \mathbb{R}$ be a Borel function such that $\mathrm{supp}(f) \subset \overline{S_{R'}}$ for some $R' \le R$ and $\sup |f| < \infty$. Then one can see

$$\left| \int_{S_R} f d\mu_{\hat{g}_{A'}} - \int_{S_R} f d\mu_{\hat{g}_{J_s}} \right| \le 2n \sup |f| \cdot \mu_{\hat{g}_{A'}}(S_{R'}) \cdot CR'$$

by Lemma 7.8 and Proposition 7.11. Since

$$d\mu_{g_{A'}} = \det(\mathbf{g}_{A'}) dt d\theta^1 \cdots d\theta^n dx_1 \cdots dx_n$$

and

$$\det(\mathbf{g}_{A'}) = \sigma \det \begin{pmatrix} \Theta^0 & -P^0(Q^0)^{-1} \\ -(Q^0)^{-1}P^0 & (Q^0)^{-1} \end{pmatrix},$$

one can see that $\mu_{g_{A'}}(S_{R'}) = \sigma C'(R')^n$, which gives the assertion. □

Let $\{J_s\}_s$ satisfy ♠ and $A'(s,x,\theta) := sA^0(0,\theta)$. By Proposition 7.6, $P^0_{ij}(0,\theta)d\theta^j$ is a closed 1-form on T^n. Then there are constants $\bar{P}_{ij} \in \mathbb{R}$ such that

$$[P^0_{ij}(0,\cdot)d\theta^j] = [\bar{P}_{ij} d\theta^j] \in H^1(T^n, \mathbb{R}),$$

hence there are $\mathcal{H}_i \in C^\infty(T^n)$ such that

$$P^0_{ij}(0,\cdot)d\theta^j = \bar{P}_{ij} d\theta^j + d\mathcal{H}_i.$$

Since $P^0_{ij} = P^0_{ji}$ and

$$\bar{P}_{ij} = \int_{T^n} P^0_{ij}(0,\theta) d\theta^1 \cdots d\theta^n,$$

we have $\bar{P}_{ij} = \bar{P}_{ji}$ and $\frac{\partial \mathcal{H}_i}{\partial \theta^j} = \frac{\partial \mathcal{H}_j}{\partial \theta^i}$. Consequently, $\mathcal{H}_i d\theta^i$ is closed, therefore there are $\bar{P}_i \in \mathbb{R}$ and $\mathcal{H} \in C^\infty(T^n)$ such that $\mathcal{H}_i = \bar{P}_i + \frac{\partial \mathcal{H}}{\partial \theta^i}$, which gives

$$P^0_{ij}(0,\cdot) = \bar{P}_{ij} + \frac{\partial^2 \mathcal{H}}{\partial \theta^i \partial \theta^j}.$$

If $\mathrm{Ric}_{g_{J_s}}$ has the lower bound, then by Proposition 7.6 we have $Q^0_{ij}(0,\cdot) = \bar{Q}_{ij} \in \mathbb{R}$ and

$$\begin{aligned} g_{A'} &= s(\bar{Q}_{ij} + P^0_{ik}\bar{Q}^{kl}P^0_{lj})d\theta^i d\theta^j - 2P^0_{ik}\bar{Q}^{jk}d\theta^i dx_j + s^{-1}\bar{Q}^{ij}dx_i dx_j \\ &= s\bar{Q}_{ij}d\theta^i d\theta^j + \frac{\bar{Q}^{ij}}{s}\left(dx_i - sP^0_{ik}d\theta^k\right)\left(dx_j - sP^0_{jl}d\theta^l\right) \\ &= s\bar{Q}_{ij}d\theta^i d\theta^j \\ &\quad + \frac{\bar{Q}^{ij}}{s}\left\{d\left(x_i - s\frac{\partial \mathcal{H}}{\partial \theta^i}\right) - s\bar{P}_{ik}d\theta^k\right\}\left\{d\left(x_j - s\frac{\partial \mathcal{H}}{\partial \theta^j}\right) - s\bar{P}_{jl}d\theta^l\right\}. \end{aligned}$$

Now, define $F_s\colon \mathbb{R}^n \times T^n \to \mathbb{R}^n \times T^n$ by

$$F_s(x,\theta) := \left(x_1 + s\frac{\partial \mathcal{H}}{\partial \theta^1}, \ldots, x_n + s\frac{\partial \mathcal{H}}{\partial \theta^n}, \theta\right).$$

Then F_{-s} is the inverse of F_s and

$$F_s^* g_{A'} = s\bar{Q}_{ij}d\theta^i d\theta^j + \frac{\bar{Q}^{ij}}{s}\left(dx_i - s\bar{P}_{ik}d\theta^k\right)\left(dx_j - s\bar{P}_{jl}d\theta^l\right)$$

holds. Moreover, we can lift F_s to

$$\hat{F}_s\colon \mathbb{R}^n \times (\mathbb{R}^n/\mathrm{Ker}\Phi) \times S^1 \to \mathbb{R}^n \times (\mathbb{R}^n/\mathrm{Ker}\Phi) \times S^1$$

by

$$\hat{F}_s(x,\theta,e^{\sqrt{-1}t}) := \left(x_1 + s\frac{\partial \mathcal{H}}{\partial \theta^1}, \ldots, x_n + s\frac{\partial \mathcal{H}}{\partial \theta^n}, \theta, e^{\sqrt{-1}(t+s\mathcal{H}(\theta))}\right).$$

One can easy to check that $\hat{F}_s$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant and S^1-equivariant map, and

$$\begin{aligned} \hat{F}_s^* \hat{g}_{A'} &= \sigma(dt - x_i d\theta^i)^2 + s\bar{Q}_{ij}d\theta^i d\theta^j \\ &\quad + \frac{\bar{Q}^{ij}}{s}\left(dx_i - s\bar{P}_{ik}d\theta^k\right)\left(dx_j - s\bar{P}_{jl}d\theta^l\right). \end{aligned}$$

Put $\bar{P} = (\bar{P}_{ij})_{i,j}$, $\bar{Q} = (\bar{Q}_{ij})_{i,j}$, $\bar{\Theta} = \bar{Q} + \bar{P}\bar{Q}^{-1}\bar{P}$ and $y = \sqrt{s\bar{\Theta}}^{-1}x$, $\tau = \sqrt{s\bar{\Theta}}\theta$. Then we may write

$$\begin{aligned}
\hat{F}_s^*\hat{g}_{A'} &= \sigma(dt - {}^t x \cdot d\theta)^2 + {}^t d\theta \cdot s\bar{Q} \cdot d\theta \\
&\quad + {}^t \left(dx - s\bar{P}d\theta\right) \cdot \frac{\bar{Q}^{-1}}{s} \cdot \left(dx - s\bar{P}d\theta\right) \\
&= \sigma(dt)^2 - 2\sigma({}^t y \cdot d\tau)dt + {}^t d\tau \cdot \left(1 + \sigma y \cdot {}^t y\right) \cdot d\tau \\
&\quad + {}^t dy \cdot \sqrt{\bar{\Theta}}\bar{Q}^{-1}\sqrt{\bar{\Theta}} \cdot dy - 2 \cdot {}^t dy \cdot \sqrt{\bar{\Theta}}\bar{Q}^{-1}\bar{P}\sqrt{\bar{\Theta}}^{-1} \cdot d\tau.
\end{aligned}$$

Since $K_y := 1 + \sigma y \cdot {}^t y$ is positive definite, it has the inverse and the square root. Accordingly, we have

$$\begin{aligned}
\hat{F}_s^*\hat{g}_{A'} &= \sigma(dt)^2 - 2\sigma({}^t y \cdot d\tau)dt + {}^t d\tau \cdot K_y \cdot d\tau \\
&\quad + {}^t dy \cdot \sqrt{\bar{\Theta}}\bar{Q}^{-1}\sqrt{\bar{\Theta}} \cdot dy - 2 \cdot {}^t dy \cdot \sqrt{\bar{\Theta}}\bar{Q}^{-1}\bar{P}\sqrt{\bar{\Theta}}^{-1} \cdot d\tau \\
&= {}^t \left(\sqrt{K_y}d\tau - \sigma\sqrt{K_y}^{-1}ydt - \sqrt{K_y}^{-1}\sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}}dy\right) \\
&\quad \cdot \left(\sqrt{K_y}d\tau - \sigma\sqrt{K_y}^{-1}ydt - \sqrt{K_y}^{-1}\sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}}dy\right) \\
&\quad + \left(\sigma - (\sigma^2){}^t y K_y^{-1} y\right)(dt)^2 \\
&\quad - 2\sigma{}^t y \cdot K_y^{-1}\sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}}dydt \\
&\quad + {}^t dy \cdot \sqrt{\bar{\Theta}}\left(\bar{Q}^{-1} - \bar{Q}^{-1}\bar{P}\sqrt{\bar{\Theta}}^{-1}K_y^{-1}\sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\right)\sqrt{\bar{\Theta}} \cdot dy.
\end{aligned}$$

Here, we have

$$\begin{aligned}
{}^t y \cdot K_y^{-1} &= \frac{{}^t y}{1 + \sigma\|y\|^2}, \\
{}^t y \cdot K_y^{-1} \cdot y &= \frac{\|y\|^2}{1 + \sigma\|y\|^2},
\end{aligned}$$

and the similar computation as in the proof of Proposition 7.2 gives

$$\bar{\Theta}^{-1} = \bar{Q}^{-1} - \bar{Q}^{-1}\bar{P}\bar{\Theta}^{-1}\bar{P}\bar{Q}^{-1}.$$

Put

$$\begin{aligned}
\mathcal{T} &:= d\tau - \sigma K_y^{-1} \cdot ydt - K_y^{-1}\sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}}dy, \\
\bar{S} &:= \sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}}.
\end{aligned}$$

Then we may write

$$\hat{F}_s^*\hat{g}_{A'} = {}^t\mathcal{T}\cdot K_y\cdot\mathcal{T} + \frac{\sigma}{1+\sigma\|y\|^2}(dt)^2 - \frac{2\sigma}{1+\sigma\|y\|^2}{}^t y\cdot\bar{S}dydt$$
$$+ {}^t dy\cdot\left(1 + {}^t\bar{S}\left(1-K_y^{-1}\right)\bar{S}\right)\cdot dy.$$

Since we have

$$1-K_y^{-1} = (K_y-1)K_y^{-1} = \sigma y\cdot{}^t y\cdot K_y^{-1} = \frac{\sigma y\cdot{}^t y}{1+\sigma\|y\|^2},$$

we can see that

$$\begin{aligned}\hat{F}_s^*\hat{g}_{A'} &= {}^t\mathcal{T}\cdot K_y\cdot\mathcal{T} + \frac{\sigma}{1+\sigma\|y\|^2}(dt)^2 - \frac{2\sigma}{1+\sigma\|y\|^2}{}^t y\cdot\bar{S}dydt\\ &\quad + {}^t dy\cdot\left(1 + {}^t\bar{S}\left(\frac{\sigma y\cdot{}^t y}{1+\sigma\|y\|^2}\right)\bar{S}\right)\cdot dy\\ &= {}^t\mathcal{T}\cdot K_y\cdot\mathcal{T} + \frac{\sigma}{1+\sigma\|y\|^2}\left(dt - {}^t y\cdot\bar{S}dy\right)^2 + {}^t dy\cdot dy\end{aligned}$$

Define $\phi_{m,s}\colon S_{\Phi,R}\to\mathbb{R}^n\times S^1$ by

$$\phi_{m,s}(x,\theta,e^{\sqrt{-1}t}) := (\sqrt{s\bar{\Theta}}^{-1}x, e^{\sqrt{-1}t}).$$

and define $\mathbb{Z}/m\mathbb{Z}$-action on $\mathbb{R}^n\times S^1$ by $k\cdot(y,e^{\sqrt{-1}t}) := (y, e^{\sqrt{-1}(t-\frac{2k\pi}{m})})$. Then $\phi_{m,s}$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant map and

$$\phi_{m,s}\colon (\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1, \hat{F}_s^*\hat{g}_{A'}) \to (\mathbb{R}^n\times S^1, g_\infty)$$

is a Riemannian submersion, where

$$g_\infty = \frac{\sigma}{1+\sigma\|y\|^2}\left(dt - {}^t y\cdot\bar{S}dy\right)^2 + {}^t dy\cdot dy.$$

Denote by μ_∞ the measure on $\mathbb{R}^n\times S^1$ defined by $d\mu_\infty = dy_1\cdots dy_n dt$.

Proposition 7.14. *Let $f\in C_0(\mathbb{R}^n\times S^1)$. Then there is a constant $K>0$ depending only on $\Phi,\sigma,\bar{\Theta}$ such that*

$$\int_{\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1} f\circ\phi_{m,s}d\mu_{\hat{F}_s^*\hat{g}_{A'}} = K\sqrt{s}^n\int_{\mathbb{R}^n\times S^1} fd\mu_\infty.$$

Proof. Since

$$\begin{aligned} d\mu_{\hat{F}_s^*\hat{g}_{A'}} &= \left(\frac{\sigma}{1+\sigma\|y\|^2}\det(K_y)\right)^{\frac{1}{2}} dt d\tau^1\cdots d\tau^n dy_1\cdots dy_n \\ &= \left(\frac{\sigma}{1+\sigma\|y\|^2}(1+\sigma\|y\|^2)\right)^{\frac{1}{2}} dt d\tau^1\cdots d\tau^n dy_1\cdots dy_n \\ &= \sqrt{\sigma} dt d\tau^1\cdots d\tau^n dy_1\cdots dy_n, \end{aligned}$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1} f\circ\phi_{m,s} d\mu_{\hat{F}_s^*\hat{g}_{A'}} \\ &= \int_{\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1} f\circ\phi_{m,s}\sqrt{\sigma} dt d\tau^1\cdots d\tau^n dy_1\cdots dy_n \\ &= \sqrt{\sigma}\sqrt{s}^n\sqrt{\det\bar{\Theta}}\int_{\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1} f\circ\phi_{m,s} dt d\theta^1\cdots d\theta^n dy_1\cdots dy_n \\ &= \sqrt{\sigma}\mathrm{Vol}\left(\mathbb{R}^n/\mathrm{Ker}\,\Phi\right)\sqrt{\det\bar{\Theta}}\sqrt{s}^n\int_{\mathbb{R}^n\times S^1} f dt dy_1\cdots dy_n. \end{aligned}$$

□

Now, we put

$$\mathbf{S}_\Phi := \frac{\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1}{\mathbb{Z}/m\mathbb{Z}},$$

then $\hat{g}_{A'}$ and $\hat{F}_s^*\hat{g}_{A'}$ induces the Riemannian metrics on $\mathbf{S}_\Phi$ such that p_Φ is local isometry. We also denote by $\hat{g}_{A'}$ and $\hat{F}_s^*\hat{g}_{A'}$, respectively if there is no fear of confusion.

Since $\phi_{m,s}$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant, we have the following commutative diagram;

$$\begin{array}{ccc} (\mathbb{R}^n\times(\mathbb{R}^n/\mathrm{Ker}\,\Phi)\times S^1, \hat{F}_s^*\hat{g}_{A'}) & \overset{\phi_{m,s}}{\to} & (\mathbb{R}^n\times S^1, g_\infty) \\ p_\Phi\downarrow & & p_m\downarrow \\ (\mathbf{S}_\Phi, \hat{F}_s^*\hat{g}_{A'}) & \overset{\phi_s}{\to} & (\mathbb{R}^n\times S^1, g_{m,\infty}) \end{array}$$

where p_m is the quotient map defined by $p_m(y, e^{\sqrt{-1}t}) := (y, e^{\sqrt{-1}mt})$ and $g_{m,\infty}$ is defined by

$$g_{m,\infty} = \frac{\sigma}{1+\sigma\|y\|^2}\left(\frac{dt}{m} - {}^t y\cdot\bar{S}dy\right)^2 + {}^t dy\cdot dy \tag{12}$$

such that ${p_m}^* g_{m,\infty} = g_\infty$ and ϕ_s is the Riemannian submersion.

Proposition 7.15. *Let $\{J_s\}_s$ satisfy ♠ and $A'(s,x,\theta) := sA^0(0,\theta)$ and put $p_0 = p_\Phi(0,0,1) \in \mathbf{S}_\Phi$. Assume that there are constants $s_0 > 0$ and $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_{J_s}} \geq \kappa g_{J_s}$ for any $0 < s \leq s_0$. Then the family of pointed metric measure spaces with the isometric S^1-action*

$$\left\{\left(\mathbf{S}_\Phi, d_{A'}, \frac{\mu_{\hat{g}_{A'}}}{K\sqrt{s}^n}, p_0\right)\right\}_s$$

converges to $\left(\mathbb{R}^n \times S^1, d_{g_{m,\infty}}, \mu_\infty, (0,1)\right)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology.

Proof. Since $\hat{F}_s$ is an S^1-equivariant isometry, it suffices to show that

$$\left\{\left(\mathbf{S}_\Phi, d_{\hat{F}_s^*\hat{g}_{A'}}, \frac{\mu_{\hat{F}_s^*\hat{g}_{A'}}}{K\sqrt{s}^n}, p_0\right)\right\}_s$$

converges to $\left(\mathbb{R}^n \times S^1, d_{g_{m,\infty}}, \mu_\infty, (0,1)\right)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology. Since

$$\hat{F}_s^*\hat{g}_{A'} = {}^t\mathcal{T} K_y \mathcal{T} + g_\infty,$$

one can see that ϕ_s is a Riemannian submersion and the diameters of the fibers $\phi_s^{-1}(y,t)$ are at most $C\sqrt{s(1+\sigma\|y\|^2)}$, where $C > 0$ is a constant depending only on $\bar{P}, \bar{Q}$ and Φ, hence the pointed Gromov-Hausdorff convergence follows. Moreover, Proposition 7.14 implies that $(\phi_{m,s})_*\mu_{\hat{F}_s^*\hat{g}_{A'}} = K\sqrt{s}^n\mu_\infty$, especially we also have the vague convergence of the measures. □

Theorem 7.16. *Let $\{J_s\}_s$ satisfies ♠ and suppose that there there are constants $s_0 > 0$ and $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_{J_s}} \geq \kappa g_{J_s}$ for any $0 < s \leq s_0$. Put $p_0 = p_\Phi(0,0,1) \in S(L|_U, h)$. Then the family of pointed metric measure spaces with the isometric S^1-action*

$$\left\{\left(S(L,h), d_{J_s}, \frac{\mu_{\hat{g}_{J_s}}}{K\sqrt{s}^n}, p_0\right)\right\}_s$$

converges to $\left(\mathbb{R}^n \times S^1, d_{g_{m,\infty}}, \mu_\infty, (0,1)\right)$ as $s \to 0$ in the sense of the pointed S^1-equivariant measured Gromov-Hausdorff topology.

Proof. Put $A'(s,x,\theta) := sA^0(0,\theta)$. By Proposition 7.13, there exist constants $R_0, C > 0$ such that

$$\mathrm{id}\colon \left(\pi^{-1}(B_{g_{J_s}}(\mathbf{0}, r - C\sqrt{s}r^2)), d_{J_s}\right) \to \left(\pi^{-1}(B_{g_{A'(s,\cdot)}}(\mathbf{0}, r)), d_{A'(s,\cdot)}\right)$$

is a Borel $C\sqrt{s}r^2$-S^1-equivariant Hausdorff approximation for any $r \geq 1$ and $s \leq \frac{R_0^2}{Cr^2}$. Since $C\sqrt{s}r^2 \to 0$ as $s \to 0$ for any fixed r, therefore,

$$\{(S(L,h), d_{J_s}, p_0)\}_s \xrightarrow{S^1\text{-GH}} \left(\mathbb{R}^n \times S^1, d_{g_{m,\infty}}, (0,1)\right)$$

as $s \to 0$ by Proposition 7.15.

Next we show the vague convergence of the measures. Now the approximation from $(S(L,h), d_{J_s}, p_0)$ to $\left(\mathbb{R}^n \times S^1, d_{g_{m,\infty}}, (0,1)\right)$ is induced by the $\mathbb{Z}/m\mathbb{Z}$-equivariant maps $\psi_s := \phi_{m,s} \circ \hat{F}_{-s}$. Take $f \in C_0(\mathbb{R}^n \times S^1)$. Then Proposition 7.14 gives

$$\begin{aligned}\int_{\mathbb{R}^n \times S^1} f d\mu_\infty &= \frac{1}{K\sqrt{s}^n} \int_{\mathbb{R}^n \times (\mathbb{R}^n/\operatorname{Ker}\Phi) \times S^1} f \circ \phi_{m,s} d\mu_{\hat{F}_s^* \hat{g}_{A'}} \\ &= \frac{1}{K\sqrt{s}^n} \int_{\mathbb{R}^n \times (\mathbb{R}^n/\operatorname{Ker}\Phi) \times S^1} f \circ \psi_s d\mu_{\hat{g}_{A'}}.\end{aligned}$$

Note that $\sup|f \circ \psi_s| \leq \sup|f| < \infty$. By the definition of $\phi_{m,s}$, there is $r > 0$ independent of s such that $\operatorname{supp}(f \circ \psi_s) \subset S_{\sqrt{s}r}$ holds for any $0 < s \leq s_0$. Then Proposition 7.13 gives some constants $C_2 > 0$ such that

$$\begin{aligned}&\frac{1}{K\sqrt{s}^n}\left|\int_{\mathbb{R}^n \times (\mathbb{R}^n/\operatorname{Ker}\Phi) \times S^1} f \circ \psi_s d\mu_{\hat{g}_{J_s}} - \int_{\mathbb{R}^n \times (\mathbb{R}^n/\operatorname{Ker}\Phi) \times S^1} f \circ \psi_s d\mu_{\hat{g}_{A'}}\right| \\ &\leq \frac{C_2 \sup|f|(\sqrt{s}r)^{n+1}}{K\sqrt{s}^n} \to 0\end{aligned}$$

as $s \to 0$. □

8. The spectral structures on the limit spaces

In this section we consider the metric measure space $(\mathbb{R}^n \times S^1, g_{m,\infty}, \mu_\infty)$ defined by (12). Now, note that

$$\begin{aligned}\bar{S} = \sqrt{\bar{\Theta}}^{-1}\bar{P}\bar{Q}^{-1}\sqrt{\bar{\Theta}} &= \sqrt{\bar{\Theta}}^{-1}\left(\bar{P}\bar{Q}^{-1}\bar{\Theta}\right)\sqrt{\bar{\Theta}}^{-1} \\ &= \sqrt{\bar{\Theta}}^{-1}\left(\bar{P} + \bar{P}\bar{Q}^{-1}\bar{P}\bar{Q}^{-1}\bar{P}\right)\sqrt{\bar{\Theta}}^{-1},\end{aligned}$$

which implies $\bar{S}$ is symmetric. Consequently, we can see

$$\frac{dt}{m} - {}^t y \cdot \bar{S} \cdot dy = d\left(\frac{t}{m} - \frac{{}^t y \cdot \bar{S} \cdot y}{2}\right).$$

Here, by taking the pullback of $g_{m,\infty}$ by the diffeomorphism

$$\begin{array}{ccc} \mathbb{R}^n \times S^1 & \to & \mathbb{R}^n \times S^1 \\ \Cup & & \Cup \\ \left(y, e^{\sqrt{-1}t}\right) & \mapsto & \left(y, e^{\sqrt{-1}\left(t + m \cdot \frac{{}^t y \cdot \bar{S} \cdot y}{2}\right)}\right), \end{array}$$

we may suppose

$$\begin{aligned} g_{m,\infty} &= \frac{\sigma}{m^2(1+\sigma\|y\|^2)}(dt)^2 + {}^t dy \cdot dy \\ d\mu_\infty &= dy_1 \cdots dy_n dt \end{aligned}$$

and the isometric S^1-action on $(\mathbb{R}^n \times S^1, g_{m,\infty})$ is given by

$$e^{\sqrt{-1}\tau} \cdot \left(y, e^{\sqrt{-1}t}\right) = \cdot \left(y, e^{\sqrt{-1}(t+m\tau)}\right).$$

Then the Laplace operator $\Delta_{m,\infty}$ on $(\mathbb{R}^n \times S^1, g_{m,\infty}, \mu_\infty)$ is defined such that

$$\int_{\mathbb{R}^n \times S^1} (\Delta_{m,\infty} f_1) f_2 d\mu_\infty = \int_{\mathbb{R}^n \times S^1} \langle df_1, df_2 \rangle_{g_{m,\infty}} d\mu_\infty$$

holds for any $f_1, f_2 \in C^\infty(\mathbb{R}^n \times S^1)$, therefore we have

$$\Delta_{m,\infty} f = \Delta_{\mathbb{R}^n} f - \frac{m^2(1+\sigma\|y\|^2)}{\sigma}\frac{\partial^2 f}{\partial t^2},$$

where $\Delta_{\mathbb{R}^n} = -\sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$.

Let ρ_k be the representation of S^1 defined in Section 3, then we have

$$\left(L^2(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_{ml}} = \left\{\varphi(y) e^{-\sqrt{-1}lt};\ \varphi \in L^2(\mathbb{R}^n)\right\}$$

and $\left(L^2(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_k} = \{0\}$ if $k \notin m\mathbb{Z}$. Now we consider the operator

$$\Delta_{m,\infty} - \left(\frac{k^2}{\sigma} + kn\right) : \left(C^\infty(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_k} \to \left(C^\infty(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_k}$$

for $k = ml$, which corresponds to the limit of

$$2\Delta_{\overline{\partial}_{J_s}} : C^\infty(X, L^k) \to C^\infty(X, L^k)$$

as $s \to 0$. Let $(\mathbb{R}^n, {}^t dy \cdot dy, e^{-k\|y\|^2} d\mathcal{L}_{\mathbb{R}^n})$ be the Gaussian space, where $\mathcal{L}_{\mathbb{R}^n}$ is the Lebesgue measure on $\mathbb{R}^n$ and denote by $\Delta_{\mathbb{R}^n,k}$ the Laplacian of this metric measure space. Note that we have

$$\Delta_{\mathbb{R}^n,k}\varphi = \Delta_{\mathbb{R}^n}\varphi + 2k\sum_{i=1}^n y_i \frac{\partial \varphi}{\partial y_i}.$$

Then we can see that the following linear isomorphism

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) \otimes \mathbb{C} & \to & \left(C^\infty(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_k} \\ \Cup & & \Cup \\ \varphi & \mapsto & \varphi \cdot e^{-\frac{k\|y\|^2 + \sqrt{-1}lt}{2}} \end{array}$$

induces the isomorphism

$$L^2(\mathbb{R}^n, e^{-k\|y\|^2} d\mathcal{L}_{\mathbb{R}^n}) \otimes \mathbb{C} \cong \left(L^2(\mathbb{R}^n \times S^1, d\mu_\infty) \otimes \mathbb{C}\right)^{\rho_k}$$

and the identification of the operators

$$\Delta_{\mathbb{R}^n,k} \cong \Delta_{m,\infty} - \left(\frac{k^2}{\sigma} + kn\right).$$

Next we construct the eigenfunctions of $\Delta_{\mathbb{R}^n,k}$ by the hermitian polynomials. For $\xi \in \mathbb{R}$ the hermitian polynomials are defined by

$$H_{k,N}(\xi) := e^{k\xi^2} \frac{d^N}{d\xi^N} e^{-k\xi^2},$$

which is a polynomial in ξ of degree N, then it is known that $H_{k,N}$ solves

$$-\frac{d^2}{d\xi^2} H_{k,N} + 2k\xi \frac{d}{d\xi} H_{k,N} = 2kN H_{k,N}$$

and $\{H_{k,N}\}_{N=0}^{\infty}$ is a complete orthonormal system of $L^2(\mathbb{R}, e^{-k\xi^2}d\mathcal{L}_{\mathbb{R}})$. Let $N = (N_1, \ldots, N_n) \in \mathbb{Z}_{\geq 0}^n$ and put

$$\left(\frac{\partial}{\partial y}\right)^N := \frac{\partial^{N_1}}{\partial y_1^{N_1}} \cdots \frac{\partial^{N_n}}{\partial y_n^{N_n}},$$
$$|N| := \sum_{i=1}^n N_i.$$

Then

$$\varphi(y) = \prod_{i=1}^n H_{k,N_i}(y_i) = e^{k\|y\|^2} \left(\frac{\partial}{\partial y}\right)^N (e^{-k\|y\|^2})$$

solves

$$\Delta_{\mathbb{R}^n,k}\varphi = 2k|N|\varphi$$

and $\{\prod_{i=1}^n H_{k,N_i}(y_i);\ (N_1, \ldots, N_n) \in \mathbb{Z}_{\geq 0}\}$ is a complete orthonormal system of $L^2(\mathbb{R}^n, e^{-k\|y\|^2}d\mathcal{L}_{\mathbb{R}^n})$. Thus we have the following theorem.

Theorem 8.1. *Let $l \in \mathbb{Z}_{>0}$, $k = ml$ and*

$$W(k,\lambda) := \left\{ f \in \left(C^\infty(\mathbb{R}^n \times S^1) \otimes \mathbb{C}\right)^{\rho_k};\ \left(\Delta_{m,\infty} - \frac{k^2}{\sigma} - kn\right) f = 2\lambda f \right\}.$$

Then there is an orthogonal decomposition

$$(L^2(\mathbb{R}^n \times S^1) \otimes \mathbb{C})^{\rho_k} = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} W(k, kd)},$$

where

$$W(k,kd) = \mathrm{span}_{\mathbb{C}} \left\{ e^{\frac{k\|y\|^2}{2} - \sqrt{-1}lt} \left(\frac{\partial}{\partial y}\right)^N (e^{-k\|y\|^2});\ N \in \mathbb{Z}_{\geq 0},\ |N| = d \right\}.$$

As a consequence of Theorem 8.1, we obtain the former part of Theorem 1.3.

9. The fibers which are not m-BS fibers for any positive m

In this section we suppose (X^{2n}, ω) is a symplectic manifold with a prequantum line bundle (L, ∇, h), and assume that there is a continuous map

$\mu\colon X \to Y$ to a topological space Y. Moreover we fix $b_0 \in Y$ such that $\mu^{-1}(b_0)$ is not an m-BS fiber for any $m \in \mathbb{Z}$.

Let $\{J_s\}_{0<s\le s_0}$ be a one parameter family of ω-compatible complex structures, and denote by $\mathcal{L}_g(c)$ the length of a path c with respect to the Riemannian metric g. We fix $p_0 \in \mu^{-1}(b_0)$ and assume the followings.

- $\star 1$ For any $r > 0$ and open neighborhood $B \subset Y$ of b_0 there is $s_{r,B} > 0$ such that

$$\mu(B_{g_{J_s}}(p_0, r)) \subset B$$

holds for any $s \le s_{r,B}$.

- $\star 2$ For any piecewise smooth closed path $c_{b_0}\colon [0,1] \to X$ such that $c_{b_0}([0,1]) \subset \mu^{-1}(b_0)$ there exist an open neighborhood B of b_0 and a continuous map $c\colon B \times [0,1] \to X$ such that $\mu \circ c(b,t) = b$, $c(b,0) = c(b,1)$, $c(b_0,\cdot) = c_{b_0}$ and $c(b,\cdot)$ are piecewise smooth.
- $\star 3$ For any open neighborhood B of b_0 and a continuous map $c\colon B \times [0,1] \to X$ such that $\mu \circ c(b,t) = b$ and $c(b,\cdot)$ are piecewise smooth,

$$\lim_{s\to 0} \sup_{b\in B} \mathcal{L}_{g_{J_s}}(c(b,\cdot)) = 0$$

holds.

Let $\pi\colon S(L,h) \to X$ be the natural projection. By the connection ∇ we have the unique horizontal lift $\tilde{c}\colon [0,1] \to S(L,h)$ with $\tilde{c}(0) = u_0$ for any pair of a piecewise smooth path $c\colon [0,1] \to X$ and $u_0 \in \pi^{-1}(c(0))$.

Proposition 9.1. *Assume that $\mu^{-1}(b_0)$ is not an m-BS fiber for any $m \in \mathbb{Z}$. For any $p_0 \in \mu^{-1}(b_0)$, $e^{\sqrt{-1}t} \in S^1$ and $\delta > 0$, there is a piecewise smooth path $c\colon [0,1] \to \mu^{-1}(b_0)$ with $c(0) = c(1) = p_0$ such that its horizontal lift $\tilde{c}$ satisfies $\tilde{c}(1) = \tilde{c}(0)e^{\sqrt{-1}t'}$ and $|t'-t| < \delta$. In particular, if we assume $\star 3$, then $\lim_{s\to 0}\operatorname{diam}_{\hat{g}_{J_s}}(\pi^{-1}(p_0)) = 0$ holds.*

Proof. Since $\mu^{-1}(b_0)$ is not an m-BS fiber for any $m \in \mathbb{Z}$, the holonomy group of $\nabla|_{\mu^{-1}(b_0)}$ may not contained in any proper closed subgroup of S^1, hence we obtain the path c which satisfies the assertion. By $\star 3$,

$$\lim_{s\to 0} \mathcal{L}_{\hat{g}_{J_s}}(\tilde{c}) = \lim_{s\to 0} \mathcal{L}_{g_{J_s}}(c) = 0$$

holds, hence $d_{J_s}(\tilde{c}(0),\tilde{c}(1))\to 0$ as $s\to 0$. Therefore, for any $u_0\in\pi^{-1}(p_0)$, $e^{\sqrt{-1}t}\in S^1$ and δ we have

$$\lim_{s\to 0} d_{J_s}(u_0,u_0e^{\sqrt{-1}t})\le \lim_{s\to 0} d_{J_s}(u_0,u_0e^{\sqrt{-1}t'})+\sigma|t-t'|<\sigma\delta,$$

which implies $\lim_{s\to 0} d_{J_s}(u_0,u_0e^{\sqrt{-1}t})=0$, hence we have

$$\lim_{s\to 0}\mathrm{diam}_{\hat{g}_{J_s}}(\pi^{-1}(p_0))=0.$$

□

Let $B\subset Y$ be open and $\tilde{c}\colon B\times[0,1]\to S(L,h)$ be a map such that $\tilde{c}_y:=\tilde{c}(y,\cdot)$ is one of the horizontal lift of $c_y:=c(y,\cdot)$ with respect to ∇. Let $t_y\in\mathbb{R}$ be defined by $\tilde{c}(y,1)=\tilde{c}(y,0)e^{\sqrt{-1}t_y}$, which is determined independent of the choice of the initial point of $\tilde{c}(y,\cdot)$. Then the map $y\mapsto e^{\sqrt{-1}t_y}$ is continuous.

For a sufficiently large integer $N>0$, put $t=\frac{2\pi}{N}$ and $\delta=t=\frac{\pi}{N}$ and take c and t' as in Proposition 9.1, then we extend c to $c\colon B\times[0,1]\to X$ by $\star 2$, where B is an open neighborhood of b_0. Then by the continuity of $e^{\sqrt{-1}t_y}$, there is an open neighborhood $B_N\subset Y$ of b_0 such that $\frac{\pi}{N}<t_y<\frac{3\pi}{N}$ holds for any $y\in B_N$. If we consider the path obtained by connecting k copies of c_y, we can see that

$$d_{J_s}(\tilde{c}_y(0),\tilde{c}_y(0)e^{\sqrt{-1}kt_y})\le k\mathcal{L}_{g_{J_s}}(c_y).$$

If we consider the path along the fiber of $S(L,h)\to X$, we have

$$d_{J_s}(\tilde{c}_y(0)e^{\sqrt{-1}a},\tilde{c}_y(0)e^{\sqrt{-1}b})\le\sigma|a-b|.$$

Combining these estimates, we can see

$$d_{J_s}(\tilde{c}_y(0),\tilde{c}_y(0)e^{\sqrt{-1}\theta})\le N\mathcal{L}_{g_{J_s}}(c_y)+\frac{3\pi\sigma}{N}$$

for any $\theta\in\mathbb{R}$, which gives

$$\mathrm{diam}_{\hat{g}_{J_s}}(\pi^{-1}(\tilde{c}_y(0)))\le N\mathcal{L}_{g_{J_s}}(c_y)+\frac{3\pi\sigma}{N}.$$

Now we can take $s_N>0$ by $\star 3$ such that $\mathcal{L}_{g_{J_s}}(c_y)\le\frac{1}{N^2}$ for any $0<s\le s_N$ and $y\in B_N$. We can also take $0<s_{N,r}\le s_N$ by $\star 1$ such that $\mu(B_{g_{J_s}}(p_0,r))\subset$

B_N holds for all $0 < s \leq s_{N,r}$. Then we have

$$\mathrm{diam}_{\hat{g}_{J_s}}(\pi^{-1}(\tilde{c}_y(0))) \leq \frac{1+3\pi\sigma}{N}$$

for all $y \in \mu(B_{g_{J_s}}(p_0, r))$ and $0 < s \leq s_{N,r}$. Thus we obtain the following proposition.

Proposition 9.2. *Assume* $\star$1-3, $\mu^{-1}(b_0)$ *is not an* m*-BS fiber for any* m *and let* $u_0 \in \pi^{-1}(p_0)$*. Then for any* $r > 0$ *and* $\varepsilon > 0$ *there is* $0 < s_{r,\varepsilon} \leq s_0$ *such that*

$$\mathrm{diam}_{\hat{g}_{J_s}}(\pi^{-1}(x)) \leq \varepsilon$$

for all $x \in B_{g_{J_s}}(p_0, r)$ *and* $0 < s \leq s_{r,\varepsilon}$*.*

Before we prove Theorem 1.2, we describe the relation between the convergence of principal G-bundles and the convergence of the base spaces. Let G be a compact Lie group, (P, d, ν) be a metric measure space with an isometric G-action. Put $X := P/G$ and define the distance $\bar{d}$ on X by

$$\bar{d}(\bar{x}, \bar{y}) := \inf_{\gamma \in G} d(x, y\gamma),$$

where $\bar{x} \in X$ is the equivalence class represented by $x \in P$.

Proposition 9.3. *Let* $\{(P_i, d_i, \nu_i, p_i)\}_{i\in\mathbb{N}}$ *be a sequence of pointed metric measure spaces with isometric* G *actions and denote by* $\pi_i \colon P_i \to X_i = P_i/G$ *be the quotient maps. Suppose that for any* $r, \varepsilon > 0$ *there is* $i_{r,\varepsilon} \in \mathbb{N}$ *such that*

$$\sup_{x \in B(p_i, r)} \mathrm{diam}_{d_i} \pi^{-1}(x) < \varepsilon$$

holds for any $i \geq i_{r,\varepsilon}$*. If* $\{(X_i, \bar{d}_i, \bar{\nu}_i, \bar{p}_i)\}_i$ *converges to* $(X, \bar{d}, \bar{\nu}, \bar{p})$ *with respect to the pointed measured Gromov-Hausdorff topology, then* $\{(P_i, d_i, \nu_i, p_i)\}_s$ *converges to* $(X, \bar{d}, \bar{\nu}, \bar{p})$ *in the sense of the pointed* G*-equivariant measured Gromov-Hausdorff topology. Here, the* G*-action on* X *is the trivial action.*

Proof. Let $\bar{\phi}_i \colon (B_{X_i}(\bar{p}_i, r), \bar{p}_i) \to (X, \bar{p})$ be ε-approximations given by the pointed Gromov-Hausdorff convergence of $(X_i, \bar{p}_i)$. Then one can see that

$$\phi := \bar{\phi}_i \circ \pi_i \colon (\pi_i^{-1}(B(\bar{p}_i, r)), p_i) \to (X, \bar{p})$$

are G-equivariant 2ε-approximations. Using these maps one can show the assertion. □

Proof of Theorem 1.2. Assume ♠ and that there is $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_{g_{J_s}} \geq \kappa g_{J_s}$. Let $u \in S|_{\mu^{-1}(y)}$ and assume that $\mu^{-1}(y)$ is not a Bohr-Sommerfeld fiber of L^m for any $m > 0$. On the neighborhood U of $\mu^{-1}(y)$, we may write

$$g_{J_s}|_U = g_A$$

for some $A = A(s, x, \theta)$. Here we consider the pointed measured Gromov-Hausdorff limit of $(X, g_{J_s}, \frac{\mu_{g_{J_s}}}{K\sqrt{s}^n}, p)$ as $s \to 0$ for some $p \in \mu^{-1}(y)$ and $K > 0$. In the same way as Subsection 7.3, it suffices to consider the limit of $g_{A'}$, where $A'(s, x, \theta) = sA^0(0, \theta)$ and $\bar{Q} = \mathrm{Im}(A^0)(0, \theta)$ is independent of θ. Notice that we already had $P^0_{ij}(0, \theta) = \bar{P}_{ij} + \frac{\partial^2 \mathcal{H}}{\partial\theta^i \partial\theta^j}$ in Subsection 7.3 and

$$\begin{aligned} F_s^* g_{A'} = &{}^t\left(\sqrt{s\bar{\Theta}} d\theta - \sqrt{s\bar{\Theta}}^{-1} \bar{P}\bar{Q}^{-1} dx\right) \cdot \left(\sqrt{s\bar{\Theta}} d\theta - \sqrt{s\bar{\Theta}}^{-1} \bar{P}\bar{Q}^{-1} dx\right) \\ &+ s^{-1} \cdot {}^t dx \cdot \bar{\Theta}^{-1} \cdot dx \end{aligned}$$

holds by (8), where

$$\begin{aligned} F_s(x, \theta) &= \left(x_1 + s\frac{\partial \mathcal{H}}{\partial \theta^1}, \ldots, x_n + s\frac{\partial \mathcal{H}}{\partial \theta^n}, \theta\right), \\ \bar{\Theta} &= \bar{Q} + \bar{P}\bar{Q}^{-1}\bar{P}. \end{aligned}$$

Then by the transformation $y = \sqrt{s\bar{\Theta}}^{-1} x$ and $\tau = \sqrt{s\bar{\Theta}}\theta$, we have

$$F_s^* g_{A'} = {}^t\left(d\tau - \bar{P}\bar{Q}^{-1}dy\right) \cdot \left(d\tau - \bar{P}\bar{Q}^{-1}dy\right) + {}^t dy \cdot dy.$$

The above expression implies that $(y, \tau) \mapsto y$ is the Riemannian submersion to the Euclidean space. Since the diameters of the fibers of the submersion converge to 0 as $s \to 0$, we have proved that $(X, F_s^* g_{A'}, p_0)$ pointed Gromov-Hausdorff converges to $(\mathbb{R}^n, {}^t dy \cdot dy, 0)$. The convergence of the measure is shown by the similar argument with the proof of Proposition 7.14 and Theorem 7.16. By Proposition 9.3 we obtain the assertion. □

As a consequence of 1.2 we obtain the latter half of Theorem 1.3, since the S^1-action on $\mathbb{R}^n$ in Theorem 1.2 is trivial and $(C^\infty(\mathbb{R}^n) \otimes \mathbb{C})^{\rho_k} = \{0\}$ for any $k > 0$.

10. Examples

In this section we give some examples to which Theorems 1.1 and 1.2 can be applied.

Abelian varieties. Let $X = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\omega = 2\pi \sum_{i=1} dx_i \wedge d\theta^i$, where $x, \theta \in \mathbb{R}^n/\mathbb{Z}^n$. Then there is a prequantum line bundle L on (X, ω) (See [4]). Define the nonsingular Lagrangian fibration $\mu\colon X \to T^n$ by $\mu(x, \theta) = x$ and define ω-compatible complex structures $\{J_s\}_s$ such that

$$\left\{ \frac{\partial}{\partial \theta^i} + s\Omega_{ij} \frac{\partial}{\partial x_j};\ i = 1, \dots, n \right\}$$

is a frame of $T^{1,0}_{J_s} X$, where $\Omega = (\Omega_{ij})_{i,j} \in M_n(\mathbb{C})$ belongs to the Siegel upper half-space

$$\{\Omega \in M_n(\mathbb{C});\ \Omega_{ij} = \Omega_{ji},\ \mathrm{Im}(\Omega) > 0\}.$$

Then the family $\{J_s\}_s$ satisfies ♠ in Subsection 7.2 and $\star$1-$\star$3 in Section 9. Therefore, for any point $y \in T^n$, Theorems 1.1 and 1.2 hold for this family.

10.0.1. Toric symplectic manifolds. In [3], the asymptotic behavior of the vector spaces $H^0(X_{J_s}, L)$ as $s \to 0$ is considered where (X, ω) is a compact toric symplectic manifold, $\{J_s\}$ is the family of ω-compatible complex structures, given by the symplectic potentials, tending to the large complex structure limit. In this case the Lagrangian fibration $\mu\colon X \to P$ is given as the moment map and the image P is the delzant polytope in $\mathbb{R}^n$. Let $\breve{P} \subset P$ be the interior of P. If $y \in P \setminus \breve{P}$, then y is a critical value of μ and the inverse image $\mu^{-1}(y)$ is a torus whose dimension is less than n. Then we cannot apply Theorems 1.1 or 1.2 to these points. However, if J_s are the family given in [3] and y belongs to $\breve{P}$, then Theorems 1.1 and 1.2 hold. We should notice that the parameter s in [3] corresponds to $1/s$ in this paper.

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