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A compactness result for $\mathcal{H}-$holomorphic curves in symplectizations

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$\mathcal{H}-$holomorphic curves are solutions of a specific modification of the pseudoholomorphic curve equation in symplectizations involving a harmonic $1-$form as perturbation term. In this paper we compactify the moduli space of $\mathcal{H}-$holomorphic curves with a priori bounds on the harmonic $1-$forms.

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1. Introduction

The study of holomorphic curves in symplectizations of contact manifolds is closedly related to the Weinstein conjecture, which states that every closed contact manifold carries at least one periodic Reeb orbit. In this regard, Hofer [2] proved this conjecture for contact $3-$manifolds M with $\pi_2(M) \neq 0$, for overtwisted contact $3-$manifolds and for S^3 endowed with the standard contact structure. He obtained this result by studying compactness properties of holomorphic disks in the symplectization of M. In the overtwisted case, for example, he constructs a $1-$parameter family of pseudoholomorphic discs with boundary contained in an overtwisted disc. Using a bubbling-off analysis, which may be seen as the study of compactness properties of a $1-$parameter family of pseudoholomorphic discs, Hofer obtains a non-constant pseudoholomorphic plane of finite energy. He proves then the existence of a periodic Reeb orbit, by showing that a non-constant finite energy plane is necessarily asymptotic to a periodic Reeb orbit.

Subsequently, Hofer, Wysocki and Zehnder extended these results by constructing a holomorphic open book decomposition of S^3 adapted to a given contact form. Generally, an open book decomposition of a compact $3-$manifold is a decomposition of the complement of a collection K of links (called *binding*) into a S^1-family of $2-$dimensional manidolds (called *pages*) with boundary contained in K. In case of a contact $3-$manifold (M, α), Giroux proved in [16], that open book decompositions can always be chosen adapted to the contact structure, that is, the binding K consists of periodic Reeb orbits and the pages are transverse to Reeb flow for a supporting contact form (which has the same contact structure as α and is called *Giroux contact form*). Hofer, Wysocki and Zehnder constructed such decompositions for S^3 where the pages are images of pseudoholomorphic curves in the symplectization of M, which are asymptotic to the Reeb orbits in the binding of the open book. In what follows we will refer to such an open book decomposition as *holomorphic open book decomposition.* Whenever one has a holomorphic open book decomposition, one finds in particular a (nonempty) collection of Reeb orbits which together bound a surface (the pages). The strong Weinstein conjecture, first mentioned in [4] states, that for any contact $3-$manifold there is such a nonempty collection of Reeb orbits which together bound a surface. In [4] the authors initiated a program of proving the strong Weinstein conjecture in dimension three, with a method starting from a $\mathcal{H}-$holomorphic open book decomposition. We will describe their method later on. The reason for considering $\mathcal{H}-$holomorphic curves and not usual pseudoholomorphic curves for the pages is that in the case the pages have positive genus then the usual Cauchy-Riemann operator has negative index and thus it is not possible to find nearby solutions. To remedy this Hofer in [3] suggested to perturb the pseudoholomorphic curve equation by a harmonic $1-$form to obtain positive Fredholm index. Abbas in [14] developed a method of turning the pages of such an open book decomposition into $\mathcal{H}-$holomorphic curves. In what follows we will refer to such an open book decomposition as *$\mathcal{H}-$holomorphic open book decomposition.* Having such a holomorphic open book decomposition the strategy of the authors in [4] is to reduce the proof of this stronger version of the Weinstein conjecture to a compactness problem of the moduli space of solutions of the $\mathcal{H}-$holomorphic curve equation. However, Abbas et al. were only able to prove the strong Weinsten conjecture when the pages of the open book decomposition have genus 0 (called *planar*). For arbitrary genus this is still open. In the following we will describe their method. Starting with a closed contact 3$-$dimensional manifold (M, α), a cobordism between α and the Giroux contact form α' is introduced. For the Giroux contact form α', the holomorphic open book decomposition can be constructed following the guidelines of [14]. By the local behavior near punctures, we know that the $\mathcal{H}-$holomorphic curves are asymptotic to Reeb orbits; thus the generalized Weinstein conjecture for α' readily follows. In the next step, the cobordism and the classical SFT compactness result, see [1] and [8], is used to deform the $\mathcal{H}-$holomorphic curves for α' into $\mathcal{H}-$holomorphic curves with respect to the initial contact form α. In this deformation process the classical SFT compactness is invoked. If a compactness result for $\mathcal{H}-$holomorphic curves is established, the program can be adapted almost one to one to prove the strong Weinstein conjecture for genus different that 0. In this paper we describe a compactification of the moduli space of finite energy $\mathcal{H}-$holomorphic curves. However, we are only able to do this under certain conditions. In the case of vanishing harmonic perturbation 1$-$form, there exists a classical SFT compactness result which was established in [8] and [1].

In the general case of a closed contact 3$-$manifold, the Weinstein conjecture was proved by Taubes [17], using Seiberg-Witten Theory, a theory which we are not going to describe further in this paper. However, it is worthwhile to persue a proof of the Weinstein conjecture along the lines suggested by the program of Abbas et al. in [4].

Let M be a closed, connected, 3$-$dimensional manifold and α a 1$-$form on M such that (M, α) is a contact manifold. Further, let X_α be the Reeb vector field with respect to the contact form α on M, defined by $\iota_{X_\alpha}\alpha \equiv 1$ and $\iota_{X_\alpha} d\alpha \equiv 0$. Denote by $\xi = \ker(\alpha)$ the contact structure and $\pi_\alpha : TM \to \xi$ the canonical projection along the Reeb vector field X_α. Denote by ϕ^α_ρ the flow of X_α, and note that ϕ^α_ρ preserves the contact structure and the Reeb vector field X_α. Consider a $d\alpha-$compatible almost complex structure $J_\xi : \xi \to \xi$, and let J be the extension of J_ξ to a $\mathbb{R}-$invariant almost complex structure on $\mathbb{R} \times M$ by mapping $1 \in T\mathbb{R}$ to X_α and X_α to $-1 \in T\mathbb{R}$. M is equipped with the metric

$$g = \alpha \otimes \alpha + d\alpha(\cdot, J_\xi \cdot)$$

while we equip the $\mathbb{R} \times M$ with the metric

$$\overline{g} = dr \otimes dr + g \tag{1.1}$$

where r is the coordinate on $\mathbb{R}$. Throughout this paper we assume that all periodic orbits are non-degenerate. This means that for every periodic orbit x of period T, the linear map $d\phi_T^\alpha(x(0)) : \xi_{x(0)} \to \xi_{x(T)}$ does not contain 1 in its spectrum. Let (S, j) be a closed Riemann surface and $\mathcal{P} \subset S$ a finite subset whose elements are called "punctures". A proper and non-constant map $u = (a, f) : S \backslash \mathcal{P} \to \mathbb{R} \times M$ is called $\mathcal{H}-$holomorphic if

$$\begin{aligned} \pi_\alpha df \circ j &= J_\xi(u) \circ \pi_\alpha df \\ f^*\alpha \circ j &= da + \gamma \\ E(u; S\backslash\mathcal{P}) &< \infty \end{aligned} \tag{1.2}$$

for a harmonic $1-$form $\gamma \in \mathcal{H}_j^1(S)$, i.e. $d\gamma = d(\gamma \circ j) = 0$. The energy

$$E(u; S\backslash\mathcal{P}) = \sup_{\varphi \in \mathcal{A}} \int_{S\backslash\mathcal{P}} \varphi'(a) da \circ j \wedge da + \int_{S\backslash\mathcal{P}} f^* d\alpha \tag{1.3}$$

sums the contribution of the $\alpha-$energy (first term) and the $d\alpha-$energy (second term) of u on $S\backslash\mathcal{P}$. The set $\mathcal{A}$ consists of all smooth maps $\varphi : \mathbb{R} \to [0, 1]$ with $\varphi'(r) \geq 0$ for all $r \in \mathbb{R}$. Note that if the perturbation $1-$form γ vanishes, the energy of u is equal to the Hofer energy defined in [1]. This modification of the pseudoholomorphic curve equation, which was first suggested by Hofer [3], was used by Abbas et al. [4] to prove the generalized Weinstein conjecture in dimension three. However, due to a lack of a compactness result of the moduli space of $\mathcal{H}-$holomorphic curves, the generalized Weinstein conjecture was proved only in the planar case, i.e. when the leaves of the holomorphic open book decomposition [14] have zero genus. In this paper we describe a compactification of the moduli space of finite energy $\mathcal{H}-$holomorphic curves by imposing some additional conditions. These are outlined below.

The L^2-norm of the harmonic perturbation $1-$form γ is defined by

$$\|\gamma\|^2_{L^2(S)} = \int_S \gamma \circ j \wedge \gamma.$$

For an isotopy class $[c]$ which is represented by a smooth loop c the period and co-period of γ over $[c]$ are

$$P_\gamma([c]) = \int_c \gamma \tag{1.4}$$

and

$$S_\gamma([c]) = \int_c \gamma \circ j, \tag{1.5}$$

respectively. Let $R_{[c]}$ the conformal modulus of $[c]$, as defined in [8]. The *conformal period* and *co-period* of γ over $[c]$ are $\tau_{[c],\gamma} = R_{[c]} P_\gamma([c])$ and $\sigma_{[c],\gamma} = R_{[c]} S_\gamma([c])$, respectively. The significance of these two quantities will become apparent later on.

Before stating the main result we will briefly discuss the results of [15]. In the paper [15] the authors establish a notion of convergence for $\mathcal{H}-$holomorphic cylinders. These results will be extensively used in the present paper. In [15] the authors considered the case where the Riemann surface S is a cylinder $[-R, R] \times S^1$ equipped with the standard complex structure i. The period and co-period of γ over the cylinder is then

$$P(\gamma) = \int_{\{0\}\times S^1} \gamma, \quad S(\gamma) = \int_{\{0\}\times S^1} \gamma \circ i. \tag{1.6}$$

Furthermore, the conformal period and co-period are $\tau = P(\gamma)R$ and $\sigma = S(\gamma)R$. The goal of the analysis in [15] is to establish the asymptotic behavior of finite energy $\mathcal{H}-$holomorphic cylinders with a uniformly small $d\alpha-$energy and harmonic perturbation $1-$forms having uniformly bounded L^2-norms and uniformly bounded conformal periods and co-periods. In the following we will sketch the main results from [15]. For a detailed description the reader might consult the section 3.2.2.

For a $\mathcal{H}$-holomorphic cylinder $u = (a, f) : [-R, R] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ having bounded energy, L^2-norm of γ, small $d\alpha$-energy and bouned conformal periods and conformal co-periods by universal constants the center action, as in [9], may be defined as the unique element $A(u) \in \mathcal{P} = \{0\} \cup \{T > 0 \mid \text{there exists a } T-\text{priodic orbit of } X_\alpha\}$ which is sufficiently close to

$$\left| \int_{S^1} u(0)^*\alpha \right|.$$

For more details the reader might consult [15]. By Theorem 11 from [15] it can be shown that either $A(u) = 0$ or $A(u) \geq \hbar$.

For a sequence $u_n = (a_n, f_n)$ of $\mathcal{H}$-holomorphic cylinders u_n satisfying the above assumptions we distinguish between two cases: the first case is when there exists a subsequence of u_n with vanishing center action and the second case is when there is no subsequence of u_n with this property. In the case of vanishing center action it can be shown that there exists a subsequence such that the M-component f_n converges in the C^0-sense to $\phi^{\alpha}_{-2\tau s}(w)$ for some $w \in M$, where τ is the limit of the conformal periods $\tau_n = P_n R_n$ and $s \in [-1/2, 1/2]$. In case of positive center action the authors prove that the M-components f_n converge in the C^0-sense to a map $x(Tt - 2\tau s)$, where (x, T) is a periodic orbit of the Reeb vector field, τ is again the limit of conformal the conformal periods and $(s, t) \in [-1/2, 1/2] \times S^1$. In the zero center action case the limit object is a fine length Reeb trajectory, while in the positive center action case the limit object represents a finite twist of a periodic Reeb orbit.

The compactness result will be established for finite energy $\mathcal{H}-$holomorphic curves with harmonic perturbation $1-$forms having uniformly bounded L^2-norms and uniformly bounded conformal periods and co-periods. Specifically, we will consider a sequence of $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : (S_n \backslash \mathcal{P}_n, j_n) \to \mathbb{R} \times M$ with harmonic perturbations γ_n, satisfying the following conditions:

A1: (S_n, j_n) are compact Riemann surfaces of the same genus and $\mathcal{P}_n \subset S_n$ is a finite set of punctures whose cardinality is independent of n.

A2: The energy of u_n, as well as the L^2-norm of γ_n are uniformly bounded by the constants $E_0 > 0$ and $C_0 > 0$, respectively.

In [12] Bergmann introduced a model for the compactification of the moduli space of $\mathcal{H}-$holomorphic curves satisfying the conditions A1 and A2. However, he neglects the phenomena occuring when conformal periods and conformal coperiods of the sequence of the harmonic perturbation $1-$forms γ_n are unbounded. In this case, the convergence behaviour can be quite complicated; for instance in [15] the authors provide an example of a sequence of $\mathcal{H}-$holomorphic cylinders with bounded energy and L^2-norm, which asymptotically wrap infinitely often around a periodic Reeb orbit. It further might even happen, that a variant of such a sequence yields a sequence of curves which asymptotically follow some non-periodic Reeb trajectory for arbitrary long time. To avoid these complications we make additional assumptions on the conformal period and co-period. The task is to establish a notion of convergence of such curves as well as the description of the limit object similar to [1] and [8]. Essentially, we will prove the convergence of a sequence of $\mathcal{H}-$holomorphic maps to a stratified broken $\mathcal{H}-$holomorphic building. The concept of a stratified broken $\mathcal{H}-$holomorphic building of a certain level is similar to that given in [1]. Each level consists of a nodal $\mathcal{H}-$holomorphic curve having the same asymptotic properties at the positive and negative punctures. However, as compared to [1], there are two differences:

1) The nodes on each level are not just points; in our setting they are replaced by a finite length Reeb trajectory.
2) The breaking orbits between the levels have a twist in the sense made precise in Definition 14 from Section 2.3.

Remark 1. For a sequence of punctured Riemann surfaces $(S_n, j_n, \mathcal{P}_n)$, the Deligne-Mumford convergence result implies that there exists a punctured nodal Riemann surface $(S, j, \mathcal{P}, \mathcal{D})$ and a sequence of diffeomorphisms $\varphi_n : S^{D,r} \to S_n$, such that $\varphi_n^* j_n$ converges outside certain circles in C^∞_{loc} to j. Here, $S^{D,r}$ is the surface obtained by blowing up the points from $\mathcal{D}$ and identifying them via the decoration r (see Section 2). Denote by Γ_i^{nod}, for $i = 1, \dots, |\mathcal{D}|/2$, the equivalence classes of the boundary circles of $S^{\mathcal{D}}$in $S^{\mathcal{D},r}$. Let $\Gamma_{n,i}^{\mathrm{nod}} = (\varphi_n)_* \Gamma_i^{\mathrm{nod}}$ for all $n \in \mathbb{N}$ and $i = 1, \dots, |\mathcal{D}|/2$.

The main result of our analysis is the following

Theorem 2. *Let $(S_n, j_n, u_n, \mathcal{P}_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves in $\mathbb{R} \times M$ satisfying assumptions A1 and A2. Then there exists a subsequence that converges to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 16 from Section 2.3.3. Moreover, if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $1 \le i \le |\mathcal{D}|/2$ we have*

$$|\tau_{[\Gamma_{n,i}^{nod}],\gamma_n}|, |\sigma_{[\Gamma_{n,i}^{nod}],\gamma_n}| < C$$

then $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ is a stratified $\mathcal{H}-$holomorphic building of height N and after going over to a subsequence the $\mathcal{H}-$holomorphic curves $(S_n, j_n, u_n, \mathcal{P}_n, \gamma_n)$ converges to $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 18 from Section 2.3.3.

For applications we would like to get rid of the a priori bounds on the L^2-norm of γ_n. Such bounds are automatically satisfied if all the leafs of the foliation, given by $\ker(f^* \alpha \circ j)$, are compact.

1.1. Outline of the paper

The paper is organized as follows. In Chapter 2 we review the basic concepts related to the compactness of $\mathcal{H}-$holomorphic curves. More precisely, in Section 2.1 we recall the Deligne-Mumford convergence theorem for stable Riemann surfaces by following the analysis given in [1] and [6]. In Section 2.2 we provide the precise definition of $\mathcal{H}-$holomorphic curves. By Proposition 9, we recall a result established by Hofer et al. [10] stating that the behaviour of $\mathcal{H}-$holomorphic curves in a neighbourhood of the punctures is similar to that of usual pseudoholomorphic curves. This result will enable us to split the set of punctures into positive and negative punctures as in [1]. In Section 2.3 we introduce the notion of convergence and describe the limit object. In particular, the description of the limit object is given in Section 2.3.1, Definition 14, while the notion of convergence is defined in Section 2.3.3, Definition 16 and Definition 18. The limit object known as a stratified $\mathcal{H}-$holomorphic building of a certain hight is similar to the broken pseudoholomorphic building introduced in [1], [6] and [8]; the difference is that we allow two points, lying in the same level, to be connected by a finite length trajectory of the Reeb vector field. Essentially, the $\mathcal{H}-$holomorphic curves converge in C^{∞}_{loc} away from the punctures and certain loops that degenerate to nodes, while the projections of the $\mathcal{H}-$holomorphic curves to M converge in C^0.

The proof of the main compactness results on the thick part with certain points removed, as well as on the thin part and in a neighbourhood of the removed points, are carried out in Sections 3.1 and 3.2 of Chapter 3, respectively. For the thick part, we use the Deligne-Mumford convergence and the thick-thin decomposition to show that the domains converge in the Deligne-Mumford sense to a punctured nodal Riemann surface. By using bubbling-off analysis and the results of Appendix C (to generate a sequence of holomorphic coordinates that behaves well under Deligne-Mumford limit process) we prove, after introducing additional punctures, that the $\mathcal{H}-$holomorphic curves have uniformly bounded gradients in the complement of the special circles and certain marked points. By using the elliptic regularity theorem for pseudoholomorphic curves and Arzela-Ascoli theorem we show that the $\mathcal{H}-$holomorphic curves togeher with the harmonic perturbations converge in C^{∞}_{loc} on the thick part with certain points removed to a $\mathcal{H}-$holomorphic curve with harmonic perturbation. This result is similar to the bubbling-off analysis used in [1]. However, in contrast to Lemma 10.7 of [1], we do not change the hyperbolic structure each time after adding the additional marked point generated by the bubbling-off analysis.

The thin part is decomposed into cusps corresponding to neighbourhoods of punctures and hyperbolic cylinders corresponding to nodes in the limit. As the perturbation harmonic $1-$forms are exact in a neighbourhood of the punctures or the points that were removed in the first part, by means of a change of the $\mathbb{R}-$coordinate, the $\mathcal{H}-$holomorphic curves are turned into usual pseudoholomorphic curves on which the classical theory [1], [8] is applicable. The case of hyperbolic cylinders is more interesting because the difference from the classical SFT compactness result is evident. Due to a lack of the monotonicity lemma, we cannot expect the $\mathcal{H}-$holomorphic curves to have uniformly bounded gradients, and so, to apply the classical SFT convergence theory. To deal with this problem we decompose the hyperbolic cylinder into a finite uniform number of smaller cylinders; some of them having conformal modulus tending to infinity but $d\alpha-$energies strictly smaller than $\hbar$, and the rest of them having bounded modulus but $d\alpha-$energies possibly larger than $\hbar$. Here, $\hbar > 0$ is defined by

$$\hbar := \min\{|P_1 - P_2| \mid P_1, P_2 \in \mathcal{P}_\alpha, P_1 \neq P_2, P_1, P_2 \leq E_0\} \tag{1.7}$$

where $\mathcal{P}_\alpha$ is the action spectrum of α as defined in [9] and $E_0 > 0$ is the uniform bound on the energy. We refer to these cylinders as cylinders of types ∞ and b_1, respectively, and note that they appear alternately. Convergence results are derived for each cylinder type, and then glued together to obtain a convergence result on the whole hyperbolic cylinder. As cylinders of type ∞ have small $d\alpha-$energies, we assume by the classical bubbling-off analysis, that the $\mathcal{H}-$holomorphic curves have uniformly bounded gradients. To turn these maps into pseudoholomorphic curves, we perform a transformation by pushing them along the Reeb flow up to some specific time characterized by a uniformly bounded conformal period. These transformed curves are now pseudoholomorphic with respect to a domain-dependent almost complex structure on M, which due to the uniform boundedness of the conformal period varies in a compact set. For this part of our analysis, we use the results established in [15]. In the case of cylinders of type b_1 we proceed as follows. Relying on a bubbling-off argument, as we did in the case of the thick part, we assume that the gradient blows up only in a finite uniform number of points and remains uniformly bounded on a compact complement of them. In this compact region we use Arzelá-Ascoli theorem to show that the $\mathcal{H}-$holomorphic curves together with the harmonic perturbations converge in C^∞ to some $\mathcal{H}-$holomorphic curve. What is then left is the convergence in a neighbourhood of the finitely many punctures, where the gradient blows up. Here, a neighbourhood of a puncture is a disk on which the harmonic perturbation can be made exact and can be encoded in the $\mathbb{R}-$coordinate of the $\mathcal{H}-$holomorphic curve. By this procedure we transform the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve defined on a disk D. By the $C^{\infty}-$convergence on any compact complement of the punctures, established before, we assume that the transformed curves converge on an arbitrary neighbourhood of ∂D. Then we use the results of [8], especially Gromov compactness with free boundary, to obtain a convergence results for cylinders of type b_1. This part of our analysis uses extensively the results established in Appendix A and some of [15].

In Chapter 4 we discuss the condition imposed on the conformal period and co-period, namely their uniformly boundedness. The conformal period and co-period can be seen as a link between the conformal data and the topology on the Riemann surface, as well as the harmonic perturbation $1-$form. Without these conditions, the transformation performed in [15] cannot be established. The reason is that the domain dependent almost complex structure, which was constructed in order to change the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve, does not vary in a compact space, and so, the results established in [9] cannot be applied. By means of a counterexample stated in Proposition 39 we show that the condition on the uniform bound of the conformal period is not always satisfied. It should be pointed out that Bergmann [12] claimed to have established a compactification of the space of $\mathcal{H}-$holomorphic curves by performing the same transformation as we did in [15], i.e. by pushing the $M-$component of the $\mathcal{H}-$holomorphic curve by the Reeb flow up to some specific time determined by the conformal period, and then by assuming that the conformal period can be universally bounded by a quantity which depends only on the periods of the harmonic perturbation $1-$form (note that if the L^2-norm of a sequence of harmonic $1-$forms is uniformly bounded then their periods are also uniformly bounded). In this context, Proposition 39 contradicts his argument.

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2. Definitions and main results

In this chapter we present the basic concepts related to the compactness of $\mathcal{H}-$holomorphic curves. In particular, we provide the Deligne-Mumford compactness in order to describe the convergence of a sequence of Riemann surfaces, introduce the concept of a stratified $\mathcal{H}-$holomorphic buildings of height N, which serves as limit object, and discuss the convergence of such maps.

2.1. Deligne-Mumford convergence

In this section we review the Deligne-Mumford convergence following the analysis given in [1] and [6].

Consider the surface $(S, j, \mathcal{M} \amalg \mathcal{D})$, where (S, j) is a closed Riemann surface, and $\mathcal{M}$ and $\mathcal{D}$ are finite disjoint subsets of S. Assume that the cardinality of $\mathcal{D}$ is even. The points from $\mathcal{M}$ are called *marked points,* while the points from $\mathcal{D}$ are called *nodal points.* The points from $\mathcal{D}$ are organized in pairs, $\mathcal{D} = \{d_1', d_1'', d_2', d_2'', \ldots, d_k', d_k''\}$. A nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is said to be *stable* if the stability condition $2g + \mu \geq 3$ is satisfied for each component of the surface S, where $\mu = |\mathcal{M} \cup \mathcal{D}|$. With a nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ we can associate the following singular surface with double points,

$$\hat{S}_{\mathcal{D}} = S / \{d_i' \sim d_i'' \mid i = 1, \ldots, k\}.$$

The identified points $d_i' \sim d_i''$ are called *nodes*. The nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is said to be *connected* if the singular surface $\hat{S}_{\mathcal{D}}$ is connected. For each marked point $p \in \mathcal{M} \amalg \mathcal{D}$ of a stable nodal Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D})$, we define the surface S^p with boundary as the oriented blow-up of S at the point p. Thus S^p is the circle compactification of $S \backslash \{p\}$; it is a compact surface bounded by the circle $\Gamma_p = (T_p S \backslash \{0\}) / \mathbb{R}_+$. The canonical projection $\pi : S^p \to S$ sends the circle Γ_p to the point p and the maps $S^p \backslash \Gamma_p$ diffeomorphically to $S \backslash \{p\}$. Similarly, given a finite set $\mathcal{M}' = \{p_1, \ldots, p_k\} \subset \mathcal{M} \amalg \mathcal{D}$ of punctures, we consider a blow-up surface $S^{\mathcal{M}'}$ with k boundary components $\Gamma_1, \ldots, \Gamma_k$. It comes with the projection $\pi : S^{\mathcal{M}'} \to S$, which collapses the boundary circles $\Gamma_1, \ldots, \Gamma_k$ to points $p_1, \ldots, p_k$ and the maps $S^{\mathcal{M}'} \backslash \coprod_{i=1}^{k} \Gamma_i$ diffeomorphically to $\dot{S} = S \backslash \mathcal{M}'$.

If the nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is connected its arithmetic genus g is defined as

$$g = \frac{1}{2}|\mathcal{D}| - b_0 + \sum_{i=1}^{b_0} g_i + 1,$$

where $|\mathcal{D}| = 2k$ is the cardinality of $\mathcal{D}$, b_0 is the number of connected components of the surface S, and $\sum_{i=1}^{b_0} g_i$ is the sum of the genera of the connected components of S. The *signature* of a nodal curve $(S, j, \mathcal{M} \amalg \mathcal{D})$ is the pair (g, μ), where g is the arithmetic genus and $\mu = |\mathcal{M}|$. A stable nodal Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ is called *decorated* if for each node there is an orientation reversing orthogonal map

$$r_i : \overline{\Gamma}_i = (T_{\overline{d}_i} S \backslash \{0\})/\mathbb{R}_+ \to \underline{\Gamma}_i = (T_{\underline{d}_i} S \backslash \{0\})/\mathbb{R}_+. \tag{2.1}$$

For the orthogonal orientation reversing map r_i, we must have that

$$r_i(e^{2\pi i\vartheta} p) = e^{-2\pi i\vartheta} r(p) \text{ for all } p \in \overline{\Gamma}_i.$$

In the following we argue as in [1]. Consider the oriented blow-up $S^{\mathcal{D}}$ at the points of $\mathcal{D}$ as described above. The circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ defined by (2.1) are boundary circles for the points $d_i', d_i'' \in \mathcal{D}$. The canonical projection $\pi : S^{\mathcal{D}} \to S$, collapsing the circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ to the points d_i' and d_i'', respectively, induces a conformal structure on $S^{\mathcal{D}} \backslash \coprod_{i=1}^k \overline{\Gamma}_i \amalg \underline{\Gamma}_i$. The smooth structure of $S^{\mathcal{D}} \backslash \coprod_{i=1}^k \overline{\Gamma}_i \amalg \underline{\Gamma}_i$ extends to $S^{\mathcal{D}}$, while the extended conformal structure degenerates along the boundary circles $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$. Let $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ be a decorated surface, where $r = (r_1, \ldots, r_k)$. By means of the mappings r_i, $i = 1, \ldots, k$, $\overline{\Gamma}_i$ and $\underline{\Gamma}_i$ can be glued together to yield a closed surface $S^{\mathcal{D},r}$. The genus of the surface $S^{\mathcal{D},r}$ is equal to the arithmetic genus of $(S, j, \mathcal{M} \amalg \mathcal{D})$. There exists a canonical projection $p : S^{\mathcal{D},r} \to \hat{S}_{\mathcal{D}}$ which projects the circle $\Gamma_i = \{\overline{\Gamma}_i, \underline{\Gamma}_i\}$ to the node $d_i = \{d_i', d_i''\}$. The projection p induces on the surface $S^{\mathcal{D},r}$ a conformal structure in the complement of the special circles Γ_i; the conformal structure is still denoted by j. The continous extension of j to $S^{\mathcal{D},r}$ degenerates along the special circles Γ_i.

According to the uniformization theorem, for a stable surface $(S, j, \mathcal{M} \amalg \mathcal{D})$ there exists a unique complete hyperbolic metric of constant curvature -1 of finite volume, in the given conformal class j on $\dot{S} = S \backslash (\mathcal{M} \amalg \mathcal{D})$. This metric is denoted by $h^{j,\mathcal{M}\amalg\mathcal{D}}$. Each point in $\mathcal{M} \amalg \mathcal{D}$ corresponds to a cusp of the hyperbolic metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$. Assume that for a given stable Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D})$, the punctured surface $\dot{S} = S \backslash (\mathcal{M} \amalg \mathcal{D})$ is endowed with the uniformizing hyperbolic metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$.

Fix $\delta > 0$, and denote by

$$\mathrm{Thick}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) = \left\{ x \in \dot{S} \mid \rho(x) \geq \delta \right\}$$

and

$$\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) = \overline{\left\{x \in \dot{S} \mid \rho(x) < \delta\right\}},$$

the $\delta-$thick and $\delta-$thin parts, respectively, where $\rho(x)$ is the injectivity radius of the metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$ at the point $x \in \dot{S}$. A fundamental result of hyperbolic geometry states that there exists a universal constant $\delta_0 = \sinh^{-1}(1)$ such that for any $\delta < \delta_0$, each component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ is conformally equivalent either to a finite cylinder $[-R, R] \times S^1$ if the component C is not adjacent to a puncture, or to the punctured disk $D\backslash\{0\} \cong [0, \infty) \times S^1$ if it is adjacent to a puncture (see, for example, [5] and [6]). Each compact component C of the thin part contains a unique closed geodesic of length $2\rho(C)$ denoted by Γ_C, where $\rho(C) = \inf_{x\in C} \rho(x)$. When considering the $\delta-$thick$-$thin decompositions we always assume that δ is chosen smaller than δ_0.

The uniformization metric $h^{j,\mathcal{M}\amalg\mathcal{D}}$ can be lifted to a metric $\overline{h}^{j,\mathcal{M}\amalg\mathcal{D}}$ on $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r}\backslash\mathcal{M}$. The lifted metric degenerates along each circle Γ_i in the sense that the length of Γ_i is 0, and the distance of Γ_i to any other point in $\dot{S}^{\mathcal{D},r}$ is infinite. However, we can still speak about geodesics on $\dot{S}^{\mathcal{D},r}$ which are orthogonal to Γ_i, i.e., two geodesics rays, whose asymptotic directions at the cusps d_i' and d_i'' are related via the map r_i, and which correspond to a compact geodesic interval in $S^{\mathcal{D},r}$ intersecting orthogonally the circle Γ_i. It is convenient to regard $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ and $\mathrm{Thick}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ as subsets of $\dot{S}^{\mathcal{D},r}$. This interpretation provides a compactification of the noncompact components of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}})$ not adjacent to points from $\mathcal{M}$. Any compact component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) \subset \dot{S}^{\mathcal{D},r}$ is a compact annulus; it contains either a closed geodesic Γ_C, or one of the special circles, still denoted by Γ_C, which projects to a node (as described above).

Consider a sequence of decorated stable nodal marked Riemann surfaces $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ indexed by $n \in \mathbb{N}$.

Definition 3. The sequence $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ is said to converge to a decorated stable nodal surface $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ if for sufficiently large n, there exists a sequence of diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n^{\mathcal{D}_n,r_n}$ with $\varphi_n(\mathcal{M}) = \mathcal{M}_n$ such that the following are satisfied.

1) For any $n \geq 1$, the images $\varphi_n(\Gamma_i)$ of the special circles $\Gamma_i \subset S^{\mathcal{D},r}$ for $i = 1, \dots, k$, are special circles or closed geodesics of the metrics $h^{j_n,\mathcal{M}_n\amalg\mathcal{D}_n}$ on $\dot{S}^{\mathcal{D}_n,r_n}$. All special circles on $S^{\mathcal{D}_n,r_n}$ are among these images.

2) $h_n \to \overline{h}$ in $C^\infty_{\rm loc}(\dot{S}^{\mathcal{D},r} \backslash \coprod_{i=1}^k \Gamma_i)$, where $h_n := \varphi_n^* h^{j_n, \mathcal{M}_n \amalg \mathcal{D}_n}$ and $\overline{h} := \overline{h}^{j, \mathcal{M} \amalg \mathcal{D}}$.

3) Given a component C of $\mathrm{Thin}_\delta(S, h^{j,\mathcal{M}\amalg\mathcal{D}}) \subset \dot{S}^{\mathcal{D},r}$ containing a special circle Γ_i, and given a point $c_i \in \Gamma_i$, let δ_i^n be the geodesic arc corresponding to the induceed metric $h_n = \varphi_n^* h^{j_n, \mathcal{M}_n \amalg \mathcal{D}_n}$ for any $n \geq 1$, intersecting Γ_i orthogonally at the point c_i, and having the ends in the $\delta-$thick part of the metric h_n. Then, in the limit $n \to \infty$, $(C \cap \delta_i^n)$ converge in C^0 to a continous geodesic for a metric $\overline{h}$ passing through the point c_i.

Remark 4. In view of the uniformization theorem, one can see that the second property of Definition 3 is equivalent to the condition $\varphi_n^* j_n \to j$ in $C^\infty_{\rm loc}(\dot{S}^{\mathcal{D},r} \backslash \coprod_{i=1}^k \Gamma_i)$ which in turn, by the removable singularity theorem, is equivalent to the convergence in $C^\infty_{\rm loc}(S^{\mathcal{D},r} \backslash \coprod_{i=1}^k \Gamma_i)$.

We are now in the position to state the Deligne-Mumford convergence.

Theorem 5. *(Deligne-Mumford) Any sequence of nodal stable Riemann surfaces $(S_n, j_n, \mathcal{M}_n \amalg \mathcal{D}_n, r_n)$ of signature (g, μ) has a subsequence which converges in the sense of Definition 3 to a decorated nodal stable Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ of signature (g, μ).*

Corollary 6. *Any sequence of stable Riemann surfaces $(S_n, j_n, \mathcal{M}_n)$ of signature (g, μ) has a subsequence which converges to a decorated nodal stable Riemann surface $(S, j, \mathcal{M} \amalg \mathcal{D}, r)$ of signature (g, μ).*

2.2. Asymptotic behaviour of $\mathcal{H}-$holomorphic curves

To describe the behaviour of a $\mathcal{H}-$holomorphic curve near the puncture from $\mathcal{P}$ we need some auxiliary tools. One of these is the lemma about the removal of singularity. Consider a $\mathcal{H}-$holomorphic curve $(S, j, \mathcal{P}, u, \gamma)$, and assume that the set of punctures $\mathcal{P} \subset S$ is not empty. For $p \in \mathcal{P}$, consider a neighbourhood $U(p) = U \subset S$, which is biholomorphic to the standard open unit disk $D \subset \mathbb{C}$, such that, under this biholomorphism, the point p is mapped to 0.

First we mention a removable singularity result for a harmonic $1-$form γ defined on the punctured unit disk $D \backslash \{0\}$.

Lemma 7. *If γ is a harmonic $1-$form defined on the punctured disk $D\backslash\{0\}$, and having a bounded L^2-norm with respect to the standard complex structure i on D, i.e. $\|\gamma\|^2_{L^2(D\backslash\{0\})} < \infty$ then γ can be extended accross the puncture.*

Proof. With $z = s + it = (s,t)$ being the coordinates on D, we express γ as $\gamma = \zeta(s,t)ds + \chi(s,t)dt$, where $\zeta, \chi : D\backslash\{0\} \to \mathbb{R}$ are harmonic functions. As γ is harmonic with respect to the standard complex structure i, $F := \zeta + i\chi : D\backslash\{0\} \to \mathbb{C}$ is a mermorphic function with a bounded L^2-norm, i.e.,

$$\int_{D\backslash\{0\}} |F(s,t)|^2 dsdt = \int_{D\backslash\{0\}} \left(|\zeta(s,t)|^2 + |\chi(s,t)|^2\right) dsdt < \infty.$$

Consider the Laurent series of F,

$$F(z) = \sum_{n=-\infty}^{\infty} F_n z^n,$$

where $F_n \in \mathbb{C}$. Since the Laurent series converges in C^0_{loc} to F and $e^{2\pi in\theta}$ is an orthonormal system in $L^2(S^1)$, we infer that for every fixed $0 < \rho < 1$,

$$\int_0^1 |F(\rho e^{2\pi i\theta})|^2 d\theta = \sum_{n=-\infty}^{\infty} |F_n|^2 \rho^{2n}.$$

Consequently, due to Fubini's theorem,

$$\begin{aligned}\int_{D\backslash\{0\}} |F(z)|^2 dsdt &= 2\pi \int_{(0,1]\times S^1} \rho |F(\rho e^{2\pi i\theta})|^2 d\theta d\rho \\ &= 2\pi \int_0^1 \sum_{n=-\infty}^{\infty} |F_n|^2 \rho^{2n+1} d\rho.\end{aligned}$$

As the terms in the sum are all non-negative, it follows that

$$\int_{D\backslash\{0\}} |F(z)|^2 dsdt \geq 2\pi |F_n|^2 \int_0^1 \rho^{2n+1} d\rho$$

for all $n \in \mathbb{Z}$. However, for $n < 0$ and because of

$$\int_0^1 \rho^{2n+1} d\rho = \infty,$$

this yields a contradiction to the finiteness of the L^2-norm of F. Hence $F_{-n} = 0$ for all $n \geq 1$, and so, F can be extended to a holomorphic function on D. Therefore γ can be extended accross the puncture. □

A removable singularity result for $\mathcal{H}-$holomorphic curves is the following

Lemma 8. *Let $(D, i, \{0\}, u, \gamma)$ be a $\mathcal{H}-$holomorphic curve defined on $D\backslash\{0\}$ such that the image of u lies in a compact subset of $\mathbb{R} \times M$. Then u extends continously to a $\mathcal{H}-$holomorphic map on the whole disk D.*

Proof. Since D is contractible and $d\gamma = d(\gamma \circ i) = 0$, the harmonic perturbation γ can be written as $\gamma = d\Gamma$, where $\Gamma : D \to \mathbb{R}$ is a harmonic function. Hence $\overline{u} = (\overline{a}, \overline{f}) := (a + \Gamma, f)$ is a pseudoholomorphic curve (unperturbed), which still has the property that its image lies in a (maybe larger) compact subset of $\mathbb{R} \times M$. Application of the usual removable singularity theorem (see Lemma 5.5 of [1]) finishes the proof of the lemma. □

In a neighbourhood of a puncture, the map a is either bounded or unbounded. In the first case, Lemma 8 can be used to extend the $\mathcal{H}-$holomorphic curve across the puncture. In the second case, in which $a : D\backslash\{0\} \to \mathbb{R}$ is unbounded we have the following

Proposition 9. *Let $(D, i, \{0\}, u, \gamma)$ be a $\mathcal{H}-$holomorphic curve defined on $D\backslash\{0\}$ such that the image of u is unbounded in $\mathbb{R} \times M$. Then u is asymptotic to a trivial cylinder over a periodic orbit of X_α, i.e. after identifying $D\backslash\{0\}$ with the half open cylinder $[0, \infty) \times S^1$ there exists a periodic orbit x of period $|T|$ of X_α, where $T \neq 0$ such that*

$$\lim_{s\to\infty} f(s,t) = x(Tt) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a(s,t)}{s} = T \textit{ in } C^\infty(S^1)$$

where (s,t) denote the coordinates on $[0, \infty) \times S^1$.

Proof. As we restrict the curve to the disk, the harmonic perturbation γ is exact, i.e. there exists a harmonic function Γ defined on the open unit disk such that $\gamma = d\Gamma$. The new curve $\overline{u} = (\overline{a}, \overline{f}) = (a + \Gamma, f)$ is pseudoholomorphic. Let $\psi : \mathbb{R}_+ \times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto e^{-2\pi(s+it)}$ be a biholomorphism which maps $D\backslash\{0\}$ to the cylinder $\mathbb{R}_+ \times S^1$. We consider the pseudoholomorphic curve $\overline{u}$ as being defined on the cylinder $\mathbb{R}_+ \times S^1$ with finite energy and having an unbounded image in $\mathbb{R} \times M$. Since we assumed that X_α is of Morse type, we obtain by Proposition 5.6 of [1], that there exists $T \neq 0$ and a periodic orbit x of X_α of period $|T|$ such that

$$\lim_{s\to\infty} \overline{f}(s,t) = x(Tt) \quad \text{and} \quad \lim_{s\to\infty} \frac{\overline{a}(s,t)}{s} = T \text{ in } C^\infty(S^1).$$

By the boundedness of the harmonic function Γ, we have

$$\lim_{s\to\infty} \frac{a(s,t)}{s} = T \text{ in } C^\infty(S^1),$$

and the proof is finished. □

The puncture $p \in \mathcal{P}$ is called *positive* or *negative* depending on the sign of the coordinate function a when approaching the puncture. Keep in mind that the holomorphic coordinates near the puncture affects only the choice of the origin on the orbit x of X_α; the parametrization of the asymptotic orbits induceed by the holomorphic polar coordinates remains otherwise the same. Hence, the orientation induced on x by the holomorphic coordinates coincides with the orientation defined by the vector field X_α if and only if the puncture is positive.

Let $S^{\mathcal{P}}$ be the oriented blow-up of S at the punctures $\mathcal{P} = \{p_1, \ldots, p_k\}$ as defined in the previous section or in Section 4.3 of [1]. $S^{\mathcal{P}}$ is a compact surface with boundary circles $\Gamma_1, \ldots, \Gamma_k$. Noting that each of these circles is endowed with a canonical S^1-action and letting $\varphi_i : S^1 \to \Gamma_i$ be (up to a choice of the base point) the canonical parametrization of the boundary circle Γ_i, for $i = 1, \ldots, k$, we reformulate Proposition 9 as follows.

Proposition 10. *Let $(S, j, \mathcal{P}, u, \gamma)$ be a $\mathcal{H}-$holomorphic map without removable singularities. The map $f : \dot{S} \to M$ extends to a continous map $\overline{f} : S^{\mathcal{P}} \to M$ such that*

$$\overline{f}(\varphi_i(e^{2\pi i t})) = x_i(Tt), \tag{2.2}$$

where $x_i : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is a periodic orbit of the Reeb vector field X_α of period T, parametrized by the vector field X_α. The sign of T coincides with the sign of the puncture $p_i \in \mathcal{P}$.

2.3. Notion of convergence

2.3.1. Stratified $\mathcal{H}-$holomorphic buildings. In this section we introduce the notion of a stratified $\mathcal{H}-$holomorphic building. These are the objects which are needed for the compactification of the moduli space of

$\mathcal{H}-$holomorphic curves. First we define a $\mathcal{H}-$holomorphic building of height 1, and then we introduce the general notion of a $\mathcal{H}-$holomorphic building of height greater than 1. Let (S, j) be a Riemann surface, and $\underline{\mathcal{P}} \subset S$ and $\overline{\mathcal{P}} \subset S$ two disjoint unordered finite subsets called the sets of *negative* and *positive punctures,* respectively. Let $\underline{\mathcal{P}} = \{\underline{p}_1, \ldots, \underline{p}_l\}$, $\overline{\mathcal{P}} = \{\overline{p}_1, \ldots, \overline{p}_f\}$ and $\mathcal{P} = \underline{\mathcal{P}} \amalg \overline{\mathcal{P}}$. The set of nodes, defined by $\mathcal{D} = \{d'_1, d''_1, \ldots, d'_k, d''_k\} \subset S$, is a finite subset of S, where the significance of the pair $\{d'_i, d''_i\}$ will be clarified later on. Denote by $S^{\mathcal{P}}$ the blow-up of the surface $\dot{S} = S \backslash \mathcal{P}$ at the punctures $\mathcal{P}$. The surface $S^{\mathcal{P}}$ has $|\mathcal{P}|$ boundary components, which due to the splitting of $\mathcal{P}$, are denoted by $\underline{\Gamma} = \{\underline{\Gamma}_1, \ldots, \underline{\Gamma}_l\}$ and $\overline{\Gamma} = \{\overline{\Gamma}_1, \ldots, \overline{\Gamma}_f\}$.

Definition 11. $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$, where $\tau = \{\tau_i\}_{i=1,\ldots,|\mathcal{D}|/2}$, $\sigma = \{\sigma_i\}_{i=1,\ldots,|\mathcal{D}|/2}$ and $\tau_i, \sigma_i \in \mathbb{R}$ for all $i = 1, \ldots, |\mathcal{D}|/2$ is called a *stratified $\mathcal{H}-$holomorphic building of height* 1 if the following conditions are satisfied.

1) $(S, j, u, \mathcal{P}, \gamma)$ is a $\mathcal{H}-$holomorphic curve as defined in (1.2).
2) For each $\{d'_i, d''_i\} \in \mathcal{D}$, $\tau_i, \sigma_i \in \mathbb{R}$ the points $u(d'_i)$ and $u(d''_i)$ are connected by the map $[-1/2, 1/2] \to \mathbb{R} \times M$, $s \mapsto (-2\sigma_i s + b, \phi^{\alpha}_{-2\tau_i s}(w_f))$ for some $b \in \mathbb{R}$ and $w_f \in M$ such that $u(d'_i) = (\sigma_i + b, \phi^{\alpha}_{\tau_i}(w_f))$ and $u(d''_i) = (-\sigma_i + b, \phi^{\alpha}_{-\tau_i}(w_f))$.

Remark 12. The $M-$component $f : \dot{S} \to M$ of a stratified $\mathcal{H}-$holomorphic building $u = (a, f) : \dot{S} \to \mathbb{R} \times M$ of height 1 can be continously extended to $S^{\mathcal{P}}$. For the extension $\overline{f} : S^{\mathcal{P}} \to M$, it is apparent that $\overline{f}|_{\Gamma}$, where $\Gamma = \underline{\Gamma} \amalg \overline{\Gamma}$, defines parametrizations of Reeb orbits.

Remark 13. The energy of a stratified $\mathcal{H}-$holomorphic building of height 1 is the sum of the $\alpha-$ and $d\alpha-$energies of the $\mathcal{H}-$holomorphic curve, as defined in (1.3).

In a second step we define a stratified $\mathcal{H}-$holomorphic building of height N. Let $(S_1, j_1), \ldots, (S_N, j_N)$ be closed (possibly disconected) Riemann surfaces, and for any $i \in \{1, \ldots, N\}$, let $\underline{\mathcal{P}}_i \subset S_i$ and $\overline{\mathcal{P}}_i \subset S_i$ be the sets of *negative* and *positive punctures on level i,* respectively. There is a one-to-one correspondence between the elements $\overline{\mathcal{P}}_{i-1}$ and $\underline{\mathcal{P}}_i$ given by a bijective map $\varphi_i : \overline{\mathcal{P}}_{i-1} \to \underline{\mathcal{P}}_i$. The pair $\{\overline{p}_{i-1,j}, \underline{p}_{ij}\}$, where $\underline{p}_{ij} = \varphi_i(\overline{p}_{i-1,j})$, is called the *breaking point* between the levels S_{i-1} and S_i.

Let $\mathcal{P} = \coprod_{i=1}^{N} \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$ be the set of *punctures,* $\mathcal{P}_i = \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$ the set of *punctures at level i*, $\mathcal{D}_i = \{d'_{i1}, d''_{i1}, \ldots, d'_{ik_i}, d''_{ik_i}\}$ the set of *nodes at level i,* and $\mathcal{D} = \coprod_{i=1}^{N} \mathcal{D}_i$ the set of all nodes.

If $S_i^{\mathcal{P}_i}$ is the blow-up of S_i at the punctures $\mathcal{P}_i = \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i$, then accounting of the splitting of the punctures $\mathcal{P}_i$, we denote the boundary components of $S_i^{\mathcal{P}_i}$ by $\underline{\Gamma}_i$ and $\overline{\Gamma}_i$; they correspond to the negative and positive punctures $\underline{\mathcal{P}}_i$ and $\overline{\mathcal{P}}_i$, respectively. There is a one-to-one correspondence between the elements of $\overline{\Gamma}_{i-1}$ and $\underline{\Gamma}_i$ given by an orientation reversing diffeomorphism $\Phi_i : \overline{\Gamma}_{i-1} \to \underline{\Gamma}_i$. The pair $\{\overline{\Gamma}_{i-1,j}, \underline{\Gamma}_{ij}\}$, where $\underline{\Gamma}_{ij} = \Phi_i(\overline{\Gamma}_{i-1,j})$, is called a *breaking orbit* for all $i = 2, \ldots, N$. This gives an identification of the boundary components $\overline{\Gamma}_{i-1}$ from $S_{i-1}^{\mathcal{P}_{i-1}}$ and the boundary components $\underline{\Gamma}_i$ from $S_i^{\mathcal{P}_i}$. Furthermore, let

$$S^{P,\Phi} := S_1^{P_1} \cup_{\Phi_2} S_2^{P_2} \cup_{\Phi_3} \cdots \cup_{\Phi_N} S_N^{P_N} := \left(\coprod_{i=1}^N S_i^{P_i} \right) /\sim$$

where $\sim$ is defined by identifying the circles $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$ via the diffeomorphism Φ_i for all $i = 2, \ldots, N$ and $j = 1, \ldots, |\underline{\mathcal{P}}_i|$. Obviously, $S^{\mathcal{P},\Phi}$ is a compact surface with $|\underline{\mathcal{P}}_1| + |\overline{\mathcal{P}}_N|$ boundary components. The equivalence class of $\overline{\Gamma}_{i-1,j}$ in $S^{\mathcal{P},\Phi}$, denoted by Γ_{ij} for all $i = 2, \ldots, N$ and $j = 1, \ldots, |\underline{\mathcal{P}}_i|$, is called a *special circle*; the collection of all special circles is denoted by Γ.

A tuple $(S, j, \mathcal{P}, \mathcal{D})$ with the properties described above will be called a *broken building of height* N. We are now well prepared to introduce a stratified $\mathcal{H}-$holomorphic building of height N.

Definition 14. A tuple $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$, where

$$\begin{aligned} \tau &= \{\hat{\tau}_{ij_i} \mid i = 1, \ldots, N \text{ and } j_i = 1, \ldots, |\mathcal{D}_i|/2\} \\ &\quad \cup \{\tau_{ij_i} \mid i = 1, \ldots, N-1 \text{ and } j_i = 1, \ldots, |\overline{\Gamma}_i|\}, \\ \sigma &= \{\hat{\sigma}_{ij_i} \mid i = 1, \ldots, N \text{ and } j_i = 1, \ldots, |\mathcal{D}_i|/2\} \end{aligned}$$

and $(S, j, \mathcal{P}, \mathcal{D})$ is a broken building of height N, is called a *stratified $\mathcal{H}-$holomorphic building of height* N if the following are satisfied:

1) For any $i=1, \ldots, N$, $(S_i, j_i, u_i, \underline{\mathcal{P}}_i \amalg \overline{\mathcal{P}}_i, \mathcal{D}_i, \gamma_i, \{\hat{\tau}_{ij_i} \mid j_i=1, \ldots, |\mathcal{D}_i|/2\}, \{\hat{\sigma}_{ij_i} \mid j_i = 1, \ldots, |\mathcal{D}_i|/2\})$ is a stratified $\mathcal{H}-$holomorphic building of height 1, where $u_i = u|_{S_i \setminus \mathcal{P}_i}$, and j_i is the complex structure on S_i.
2) For all breaking points $\{\overline{p}_{i-1,j}, \underline{p}_{ij}\}$ and $\tau_{ij} \in \tau$, there exist $T_{ij} > 0$ such that the $\mathcal{H}-$holomorphic building of height 1, $u_{i-1} : \dot{S}_{i-1} \to \mathbb{R} \times M$ is asymptotic at $\overline{p}_{i-1,j}$ to a trivial cylinder over the Reeb orbit x_{ij} of period $T_{ij} > 0$, and $u_i : \dot{S}_i \to \mathbb{R} \times M$ is asymptotic at $\underline{p}_{ij}$ to the trivial cylinder over the Reeb orbit $x_{ij}(\cdot + \tau_{ij})$ of period $-T_{ij} < 0$.

Remark 15. The energy of a stratified $\mathcal{H}-$holomorphic buidling of height N is defined by

$$E(u) = \max_{1 \leq i \leq N} E_\alpha(u_i) + \sum_{i=1}^{N} E_{d\alpha}(u_i).$$

2.3.2. Collar blow-up. In this section we introduce a version of blow-up similar to that from [1]. Let $(S, j, \mathcal{P}, \mathcal{D})$ be a broken building of height N, and consider the setting used in the previous section. In addition, let $S_i^{\mathcal{P}_i \cup \mathcal{D}_i}$ be the blow-up of S_i at the punctures $\mathcal{P}_i$ and nodes $\mathcal{D}_i$. To each pair of nodes $\{d'_{ij}, d''_{ij}\}$, the corresponding boundary of $S_i^{\mathcal{P}_i \cup \mathcal{D}_i}$ is denoted by $\{\Gamma'_{ij}, \Gamma''_{ij}\}$, and for each such pair of boundary circles, let $r_{ij} : \Gamma'_{ij} \to \Gamma''_{ij}$ be orientation reversing diffeomorphisms. The diffeomorphisms r_{ij} are used to glue the boundary circles Γ'_{ij} and Γ''_{ij} together. Consider the surface $\hat{S} := S^{\mathcal{P} \cup \mathcal{D}, \Phi \cup r}$ which is obtained from S by blowing-up the punctures $\mathcal{P}$ and the nodes $\mathcal{D}$, and by using the orientation reversing diffomorphisms Φ and r_{ij}. $\hat{S}$ is a compact surface with boundary components given by the sets $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$. The equivalence class of Γ'_{ij} in $\hat{S}$ is denoted by Γ_{ij}^{nod} and is called *nodal special circles*; the set of all nodal special circles is denoted by Γ^{nod}. The *collar blow-up* $\overline{S}$ is a modification of the usual blow-up $\hat{S}$ defined in [1]. Essentially, we insert the cylinders $[-1/2, 1/2] \times S^1$ between the special circles $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$, and between Γ'_{ij} and Γ''_{ij}. To obtain a surface with boundary components $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$ that has the same topology as $\hat{S}$ we modify the orientation reversing the diffeomorphismsm Φ_{ij} and r_{ij} as follows:

- **B1:** The orientation reversing diffeomorphisms Φ_{ij} correspond to two orientation reversing diffeomorphisms $\overline{\Phi}_{ij} : \overline{\Gamma}_{i-1,j} \to \{-1/2\} \times S^1$ and $\underline{\Phi}_{ij} : \{1/2\} \times S^1 \to \underline{\Gamma}_{ij}$ for all $i = 2, \ldots, N$ and $j = 1, \ldots, |\underline{\mathcal{P}}_i|$.
- **B2:** Instead of gluing $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms Φ_{ij}, we glue $\overline{\Gamma}_{i-1,j}$, the cylinder $[-1/2, 1/2] \times S^1$, and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms $\overline{\Phi}_{ij}$ and $\underline{\Phi}_{ij}$ (see Figure 2.1).
- **B3:** For the nodal special circles Γ'_{ij} and Γ''_{ij}, we proceed analogously, and denote by $r'_{ij} : \Gamma'_{ij} \to \{-1/2\} \times S^1$ and $r''_{ij} : \{1/2\} \times S^1 \to \Gamma''_{ij}$ the orientation reversing diffeomorphisms that glue Γ'_{ij}, the cylinder $[-1/2, 1/2] \times S^1$ and Γ''_{ij} together.

Let $\overline{S}$ be the surface obtained by applying the above construction to all special and nodal special circles. The equivalence class of the cylinder $[-1/2, 1/2] \times S^1$ in $\overline{S}$ corresponding to the special circle Γ_{ij} is denoted by

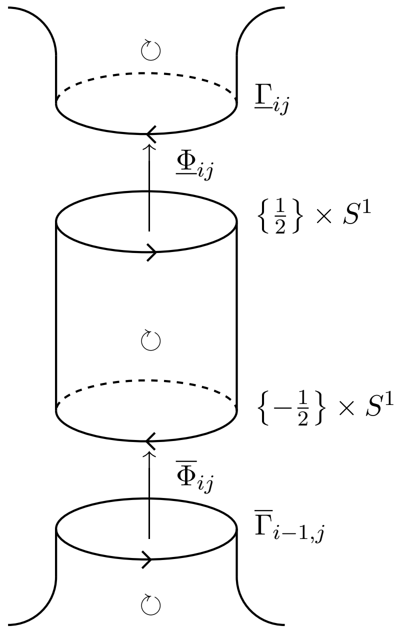

Figure 2.1: The gluing of $\overline{\Gamma}_{i-1,j}$, the cylinder $[-1/2, 1/2] \times S^1$ and $\underline{\Gamma}_{ij}$ via the orientation reversing diffeomorphisms $\overline{\Phi}_{ij} : \overline{\Gamma}_{i-1,j} \to \{-1/2\} \times S^1$ and $\underline{\Phi}_{ij} : \{1/2\} \times S^1 \to \underline{\Gamma}_{i,j}$.

A_{ij}, and is called a *special cylinder*. The equivalence class of the cylinder $[-1/2, 1/2] \times S^1$ in $\overline{S}$ corresponding to the nodal special circle Γ_{ij}^{nod} is denoted by A_{ij}^{nod}, and is called a *nodal special cylinder*. The boundary circles of A_{ij} are still denoted by $\overline{\Gamma}_{i-1,j}$ and $\underline{\Gamma}_{ij}$, while the boundary circles of A_{ij}^{nod} are also still denoted by Γ'_{ij} and Γ''_{ij}. Finally, the collections of all special and nodal special cylinders are denoted by A and A^{nod}, respectively. Take notice that there exists a natural projection between the collar blow-up $\overline{S}$ and the blow-up surface $\hat{S}$, which is defined similarly to [1], i.e. it maps $\overline{S} \backslash (A \amalg A^{\text{nod}})$ diffeomorphically to $\hat{S} \backslash (\Gamma \amalg \Gamma^{\text{nod}})$ and the annulli A and A^{nod} are mapped to Γ and Γ^{nod}. This induces a conformal structure on $\overline{S} \backslash (A \amalg A^{\text{nod}})$. Let $\tilde{S}$ be the closed surface obtained from $\overline{S}$ by identifying the boundary components $\underline{\Gamma}_1$ and $\overline{\Gamma}_N$ to points.

Having now a stratified $\mathcal{H}-$holomorphic building $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ of height N, we define the continous extension $\overline{f}$ of f on the surface $\overline{S}$ and the continous extension $\overline{a}$ of a on $\overline{S} \backslash A$. The extension $\overline{f}$ may be defined on the clinders A_{ij} and A_{ij}^{nod}, while the extension $\overline{a}$ is defined only on A_{ij}^{nod}. Set

$$\overline{f}(s,t) = \phi^{\alpha}_{-2s\hat{\tau}_{ij}}(w_f), \quad \text{for all } (s,t) \in A_{ij}^{\text{nod}} = [-1/2, 1/2] \times S^1,$$
$$\overline{f}(s,t) = \phi^{\alpha}_{-\left(s+\frac{1}{2}\right)\tau_{ij}}(x_{ij}(T_{ij}t)), \quad \text{for all } (s,t) \in A_{ij} = [-1/2, 1/2] \times S^1$$

and

$$\overline{a}(s,t) = 2\hat{\sigma}_{ij}s + b, \quad \text{for all } (s,t) \in A_{ij}^{\text{nod}} = [-1/2, 1/2] \times S^1$$

for some $b \in \mathbb{R}$ and $w_f \in M$. Here x_{ij} is the Reeb orbit of period $T_{ij} > 0$.

2.3.3. Convergence. In this section we define the notion of convergence using the notation from the previous two sections.

Definition 16. A sequence of $\mathcal{H}-$holomorphic curves $(S_n, j_n, u_n, \mathcal{P}'_n = \underline{\mathcal{P}}'_n \amalg \overline{\mathcal{P}}'_n, \gamma_n)$ converges in the C^{∞}_{loc} sense to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$, if the tuple $(S, j, \mathcal{P}, \mathcal{D})$ is a broken building of height N and there exists a sequence of diffeomorphisms $\varphi_n : \tilde{S} \to S_n$, where $\tilde{S}$ is the modified collar blow-up defined in Section 2.3.2, such that $\varphi_n^{-1}(\overline{\mathcal{P}}'_n) = \overline{\mathcal{P}}_1$ and $\varphi_n^{-1}(\underline{\mathcal{P}}'_n) = \underline{\mathcal{P}}_N$ and such that the following conditions are satisfied:

1) The sequence of complex structures $(\varphi_n)_* j_n$ converges in C^{∞}_{loc} on $\tilde{S} \backslash (A \amalg A^{\text{nod}})$ to j.
2) The special circles of $(S_n, j_n, \mathcal{P}_n)$ are mapped by φ_n^{-1} bijectively onto $\{0\} \times S^1$ of A_{ij} or A_{ij}^{nod}. For every special cylinder A_{ij} there exists an annulus $\overline{A}_{ij} \cong [-1,1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, (\varphi_n)_* j_n)$ and $(A_{ij}, (\varphi_n)_* j_n)$ are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n, h_n, R_n/h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s,t) \mapsto (\kappa(s), t)$.
3) The $\mathcal{H}-$holomorphic curves $u_n \circ \varphi_n : \dot{\tilde{S}} := \tilde{S} \backslash (\overline{\mathcal{P}}_1 \amalg \underline{\mathcal{P}}_N) \to \mathbb{R} \times M$ together with the harmonic perturbation $(\varphi_n)^* \gamma_n$ which are defined on $\tilde{S}$ converge in C^{∞}_{loc} on $\dot{\tilde{S}} \backslash \left(A \amalg A^{\text{nod}}\right)$ to the $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. Note that $\dot{\tilde{S}} \backslash (A \amalg A^{\text{nod}})$ may be conformally identified with $S \backslash (\mathcal{P} \amalg \mathcal{D})$.

Next we describe the C^0-convergence. Let $(S_n, j_n, u_n, \mathcal{P}'_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves. For any special circle Γ_{ij}, let $\tau_{ij}^n \in \mathbb{R}$ and $\sigma_{ij}^n \in \mathbb{R}$ be the conformal period of $\varphi_n^* \gamma_n$ on Γ_{ij} with respect to the complex structure $\varphi_n^* j_n$, and the conformal co-period of $\varphi_n^* \gamma_n$ on Γ_{ij} with respect to the complex structure $\varphi_n^* j_n$, respectively. For any nodal special circle Γ_{ij}^{nod} consider the numbers $\hat{\tau}_{ij}^n \in \mathbb{R}$ and $\hat{\sigma}_{ij}^n \in \mathbb{R}$, where $\hat{\tau}_{ij}^n$ is the conformal period of $\varphi_n^* \gamma_n$ on Γ_{ij}^{nod} with respect to the complex structure $\varphi_n^* j_n$, and $\hat{\sigma}_{ij}^n$ is the conformal co-period of $\varphi_n^* \gamma_n$ on Γ_{ij}^{nod} with respect to the complex structure $\varphi_n^* j_n$, respectively.

Remark 17. For a sequence $(S_n, j_n, u_n, \mathcal{P}_n', \gamma_n)$ of $\mathcal{H}-$holomorphic curves that converges to a $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 16, the quantities τ_{ij}^n, σ_{ij}^n, $\hat{\tau}_{ij}^n$ and $\hat{\sigma}_{ij}^n$ can be unbounded (see, e.g, Section 4). If τ_{ij}^n, σ_{ij}^n, $\hat{\tau}_{ij}^n$ and $\hat{\sigma}_{ij}^n$ are bounded, then after going over to a further subsequence, and assuming that there exist the real numbers $\tau_{ij}, \sigma_{ij}, \hat{\tau}_{ij}, \hat{\sigma}_{ij} \in \mathbb{R}$ such that

$$\tau_{ij}^n \to \tau_{ij}, \tag{2.3}$$

$$\sigma_{ij}^n \to \sigma_{ij}, \tag{2.4}$$

$$\hat{\tau}_{ij}^n \to \hat{\tau}_{ij}, \tag{2.5}$$

$$\hat{\sigma}_{ij}^n \to \hat{\sigma}_{ij} \tag{2.6}$$

as $n \to \infty$, we are able to derive a C^0- convergence result.

The convergence of a sequence of $\mathcal{H}-$holomorphic curves to a stratified $\mathcal{H}-$holomorphic building of height N should be understood in the following sense:

Definition 18. A sequence of $\mathcal{H}-$holomorphic curves $(S_n, j_n, \mathcal{P}_n', u_n, \gamma_n)$ converges in the C^0 sense to a statified $\mathcal{H}-$holomorphic building $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ of height N if the following conditions are satisfied.

1) The parameters τ_{ij}^n, σ_{ij}^n, $\hat{\tau}_{ij}^n$ and $\hat{\sigma}_{ij}^n$ converge as in (2.3)-(2.6).
2) The sequence $(S_n, j_n, \mathcal{P}_n', u_n, \gamma_n)$ converges to the underlying $\mathcal{H}-$holomorphic curve $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ in the sense of Definition 16 with respect to a sequence of diffeomorphisms $\varphi_n : \tilde{S} \to S_n$.
3) $(S, j, u, \mathcal{P}, \mathcal{D}, \gamma, \tau, \sigma)$ is a stratified $\mathcal{H}-$holomorphic building of height N corresponding to the constants τ_{ij}, $\hat{\tau}_{ij}$ and $\hat{\sigma}_{ij}$, as in Definition 14.
4) The maps $u_n \circ \varphi_n$ converges in C^0_{loc} on $\overline{S} \backslash A$ to the blow-up map $\overline{u}$ defined on $\dot{\tilde{S}} \backslash A$.
5) The maps $f_n \circ \varphi_n$ converges in C^0 on $\overline{S}$ to the blow-up map $\overline{f}$ defined on $\overline{S}$.

3. Proof of the compactness theorem

Let $(S_n, j_n, u_n, \mathcal{P}'_n, \gamma_n)$ be a sequence of $\mathcal{H}-$holomorphic curves satisfying Assumptions A1 and A2. After introducing an additional finite set of points $\mathcal{M}_n$ disjoint from the set of punctures $\mathcal{P}'_n$ we assume that the domains $(S_n, j_n, \mathcal{P}'_n \amalg \mathcal{M}_n)$ of the sequence of $\mathcal{H}-$holomorphic curves are stable. This condition enables us to use the Deligne-Mumford convergence (see Section 2.1) which makes it possible to formulate a convergence result for the domains $(S_n, j_n, \mathcal{P}'_n \amalg \mathcal{M}_n)$. Note that $\mathcal{M}_n$ can be choosen in such a way that their cardinality is independent of the index n. As an additional structure, let h^{j_n} be the hyperbolic metric on $\dot{S}_n := S_n \backslash (\mathcal{P}'_n \amalg \mathcal{M}_n)$. By the Deligne-Mumford convergence result (Corollary 6) there exists a stable nodal decorated surface $(S, j, \mathcal{P} \amalg \mathcal{M}, \mathcal{D}, r)$ and a sequence of diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n$, where $S^{\mathcal{D},r}$ is the closed surface obtained by blowing up the nodes and gluing pairs of nodal points according to the decoration r as described in Section 2.1, such that the following holds: Let h be the hyperbolic metric on $S \backslash (\mathcal{P} \amalg \mathcal{M} \amalg \mathcal{D})$. The diffeomorphisms φ_n map marked points into marked points and punctures into punctures, i.e. $\varphi_n(\mathcal{M}) = \mathcal{M}_n$ and $\varphi_n(\mathcal{P}) = \mathcal{P}'_n$. Via φ_n we pull-back the complex structures j_n and the hyperbolic metrics h^{j_n}, i.e. we define $j^{(n)} := \varphi_n^* j_n$ on $S^{\mathcal{D},r}$ and $h_n := \varphi_n^* h^{j_n}$ on $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r} \backslash (\mathcal{M} \amalg \mathcal{P})$. By the Deligne-Mumford convergence, $h_n \to h$ in $C^\infty_{\text{loc}}(\dot{S}^{\mathcal{D},r} \backslash \coprod_j \Gamma_j)$ as $n \to \infty$, where Γ_j are the special circles in $S^{\mathcal{D},r}$ (see Section 2.1 for the definition of special circles). This yields $j^{(n)} \to j$ in $C^\infty_{\text{loc}}(S^{\mathcal{D},r} \backslash \coprod_j \Gamma_j)$ as $n \to \infty$.

Consider now the maps $\tilde{u}_n = (\tilde{a}_n, \tilde{f}_n) := u_n \circ \varphi_n : S^{\mathcal{D},r} \backslash \mathcal{P} \to \mathbb{R} \times M$ and $\tilde{\gamma}_n := \varphi_n^* \gamma_n \in \mathcal{H}^1_{j^{(n)}}(S^{\mathcal{D},r})$. Then $\tilde{u}_n$ is a $\mathcal{H}-$holomorphic curve with harmonic perturbation $\tilde{\gamma}_n$; it satisfies the equation

$$\begin{aligned} \pi_\alpha d\tilde{f}_n \circ j^{(n)} &= J(\tilde{f}_n) \circ \pi_\alpha d\tilde{f}_n \\ (\tilde{f}_n^* \alpha) \circ j^{(n)} &= d\tilde{a}_n + \tilde{\gamma}_n \end{aligned} \quad \text{on } S^{\mathcal{D},r} \backslash \mathcal{P}$$

and has uniformly bounded energies, i.e. for $E_0 > 0$ and all $n \in \mathbb{N}$ we have $E(\tilde{u}_n; S^{\mathcal{D},r} \backslash \mathcal{P}) \leq E_0$. The L^2-norm of $\tilde{\gamma}_n$ over $S^{\mathcal{D},r}$ is equal to the L^2-norm of γ_n over S_n and it is apparent that the L^2-norm of $\tilde{\gamma}_n$ is uniformly bounded by the constant $C_0 > 0$. Hence A1 and A2 are satisfied for $\tilde{u}_n$.

In the following, we first establish a convergence result on the thick part, i.e. on $S^{\mathcal{D},r}$ away from special circles, punctures and certain additional marked points, and then treat the components from the thin part. With the convergence on the thick components, the first statement of Theorem 2 is proved.

3.1. The thick part

For the sequence $\tilde{u}_n : S^{\mathcal{D},r}\backslash\mathcal{P} \to \mathbb{R}\times M$ as defined above, we prove the $C^\infty_{\text{loc}}-$convergence in the complement of the special circles and of a finite collection of points in $\dot{S}^{\mathcal{D},r} := S^{\mathcal{D},r}\backslash(\mathcal{P}\amalg\mathcal{M})$. Set $\dot{\tilde{S}}^{\mathcal{D},r} := \dot{S}^{\mathcal{D},r}\backslash\coprod_j\Gamma_j$. To simplify the notation we continue to denote the maps $\tilde{u}_n$ by u_n and $\tilde{\gamma}_n$ by γ_n. The main result of this section is the following

Theorem 19. *There exists a subsequence of u_n, still denoted by u_n, a finite subset $\mathcal{Z}\subset\dot{\tilde{S}}^{\mathcal{D},r}$, and a $\mathcal{H}-$holomorphic curve $u : \dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z}\to\mathbb{R}\times M$ with harmonic perturbation γ defined on $\dot{\tilde{S}}^{\mathcal{D},r}$ with respect to the complex structure j such that $u_n\to u$ and $\gamma_n\to\gamma$ in $C^\infty_{loc}(\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z})$.*

Before proving Theorem 19 we establish some preliminary results.

Assume that there exists a point $z^1\in\mathcal{K}\subset\dot{\tilde{S}}^{\mathcal{D},r}$, where $\mathcal{K}$ is compact, and a sequence $z_n\in\mathcal{K}$ such that

$$z_n\to z^1 \quad\text{and}\quad \|du_n(z_n)\|\to\infty$$

as $n\to\infty$. The next lemma describing the convergence of conformal structures on Riemann surfaces is similar to Lemma 10.7 of [1].

Lemma 20. *There exist the open neighbourhoods $U_n(z^1) = U_n$ and $U(z^1) = U$ of z^1, and the diffeomorphisms*

$$\psi_n : D\to U_n,\ \ \psi : D\to U$$

such that

1) *ψ_n are $i-j^{(n)}-$biholomorphisms and ψ is a $i-j-$biholomorphism;*
2) *$\psi_n\to\psi$ in $C^\infty_{loc}(D)$ as $n\to\infty$ with respect to the Euclidean metric on D and the hyperbolic metric h on their images;*
3) *$\psi_n(0) = z^1$ for every n and $\psi(0) = z^1$;*
4) *$z_n\in U_n$ for every sufficiently large n;*
5) *$z^{(n)} := \psi_n^{-1}(z_n)\to 0$ as $n\to\infty$.*

Proof. Lemma 47 applied to the compact Riemann surface with boundary $\mathcal{K}$ and the interior point z^1, yields the diffeomorphisms $\psi_n : D\to U_n$ and $\psi : D\to U$ for which the first three assertions hold true. The fourth and fifth assertions are obvious since z_n converge to z^1. $\square$

Remark 21. The coordinate maps ψ_n and ψ have uniformly bounded gradients with respect to the Euclidian metric on D and the hyperbolic metric h on their images. This follows from the second assertion of Lemma 20.

Let $\hbar > 0$ be defined by (1.7). The next lemma essentially states that the $d\alpha-$energy concentrates around the point z^1 and is at least $\hbar/2 > 0$. The proof relies on bubbling-off analysis and proceeds as in Section 5.6 of [1].

Lemma 22. *For every open neighbourhood* $U(z^1) = U \subset \dot{\tilde{S}}^{\mathcal{D},r}$ *we have*

$$0 < \hbar \leq \lim_{n\to\infty} E_{d\alpha}(u_n; U) \leq E_0.$$

In particular, for each open neighbourhood U *of* z^1 *there exists an integer* $N_1 \in \mathbb{N}$ *such that for all* $n \geq N_1$ *we have*

$$E_{d\alpha}(u_n; U) \geq \frac{\hbar}{2}.$$

Proof. Consider the maps $\hat{u}_n := u_n \circ \psi_n : D \to \mathbb{R} \times M$, where ψ_n are the biholomorphisms given by Lemma 20. They satisfy the $\mathcal{H}-$holomorphic equations

$$\begin{aligned} \pi_\alpha d\hat{f}_n \circ i &= J(\hat{f}_n) \circ \pi_\alpha d\hat{f}_n \\ (\hat{f}_n^* \alpha) \circ i &= d\hat{a}_n + \hat{\gamma}_n \end{aligned} \quad \text{on } D,$$

where $\hat{\gamma}_n := \psi_n^* \gamma_n$ is a harmonic $1-$form on D with respect to i. The energy of $\hat{u}_n$ on D is uniformly bounded as $E(\hat{u}_n; D) \leq E_0$, while the L_2-norm of the $i-$harmonic $1-$form $\hat{\gamma}_n$ is uniformly bounded on D by the constant C_0. Furthermore, for $z^{(n)} := \psi_n^{-1}(z_n)$, $\left\| d\hat{u}_n(z^{(n)}) \right\| \to \infty$ as $n \to \infty$. This can be seen as follows. If $v_n \in T_{z^{(n)}}D$ with $\|v_n\|_{\text{eucl.}} = 1$ is such that

$$\left\| du_n(z_n) \frac{d\psi_n(z^{(n)})v_n}{\left\| d\psi_n(z^{(n)})v_n \right\|_{h_n}} \right\|_{\overline{g}} = \|du_n(z_n)\|,$$

then,

$$\begin{aligned} \left\| d\hat{u}_n(z^{(n)})v_n \right\|_{\tilde{g}} &= \left\| du_n(z_n) \frac{d\psi_n(z^{(n)})v_n}{\left\| d\psi_n(z^{(n)})v_n \right\|_{h_n}} \right\|_{\overline{g}} \left\| d\psi_n(z^{(n)})v_n \right\|_{h_n} \\ &= \|du_n(z_n)\| \left\| d\psi_n(z^{(n)})v_n \right\|_{h_n} \\ &\geq \|du_n(z_n)\| \frac{1}{2} \left\| d\psi_n(z^{(n)}) \right\| \\ &\geq \|du_n(z_n)\| \frac{1}{4} \|d\psi(0)\| \to \infty \end{aligned}$$

as $n \to \infty$. The first inequality follows from the $i - j^{(n)}-$holomorphicity of ψ_n. Applying Hofer's topological lemma (Lemma 2.39 of [6]), we obtain another sequence $\{\tilde{z}^{(n)}\}_{n\in\mathbb{N}} \subset D$ with $\tilde{z}^{(n)} \to 0$, $R_n := \left\|d\hat{u}_n(\tilde{z}^{(n)})\right\| \to \infty$, $\epsilon_n \searrow 0$, $\epsilon_n R_n \to \infty$ as $n \to \infty$ and $\|d\hat{u}_n(z)\| \leq 2R_n$ for all $z \in D_{\epsilon_n}(\tilde{z}^{(n)})$. Doing this rescaling we define the maps

$$v_n(z) = (b_n(z), g_n(z)) := \left(\hat{a}_n\left(\tilde{z}^{(n)} + \frac{z}{R_n}\right) - \hat{a}_n(\tilde{z}^{(n)}), \hat{f}_n\left(\tilde{z}^{(n)} + \frac{z}{R_n}\right)\right)$$

for all $z \in D_{\epsilon_n R_n}(0)$. The maps $v_n = (b_n, g_n) : D_{\epsilon_n R_n}(0) \to \mathbb{R} \times M$ satisfy $\|dv_n(0)\| = 1$ and $\|dv_n(z)\| \leq 2$ for all $z \in D_{\epsilon_n R_n}(0)$, and we have

$$E_\alpha(v_n; D_{\epsilon_n R_n}(0)) = E_\alpha(\hat{u}_n; D_{\epsilon_n}(\tilde{z}^{(n)})) \leq E_\alpha(\hat{u}_n; D)$$

and

$$E_{d\alpha}(v_n; D_{\epsilon_n R_n}(0)) = E_{d\alpha}(\hat{u}_n; D_{\epsilon_n}(\tilde{z}^{(n)})) \leq E_{d\alpha}(\hat{u}_n; D)$$

giving $E(v_n; D_{\epsilon_n R_n}(0)) \leq E_0$. Moreover, v_n solves the $\mathcal{H}-$holomorphic equations

$$\begin{aligned} \pi_\alpha dg_n \circ i &= J \circ \pi_\alpha dg_n, \\ (g_n^*\alpha) \circ i &= db_n + \underline{\gamma}_n, \end{aligned}$$

where $\underline{\gamma}_n := \hat{\gamma}_n / R_n$. Because v_n has a bounded gradient, there exists a smooth map $v : \mathbb{C} \to \mathbb{R} \times M$ with a bounded energy (by E_0) such that $v_n \to v$ in $C^\infty_{\text{loc}}(\mathbb{C})$ as $n \to \infty$. Nevertheless, because $\hat{\gamma}_n$ is bounded in L^2-norm, $\underline{\gamma}_n \to 0$ as $n \to 0$. Thus $v = (b, g) : \mathbb{C} \to \mathbb{R} \times M$ is a pseudoholomorphic plane, i.e. it solves the pseudoholomorphic curve equation

$$\begin{aligned} \pi_\alpha dg \circ i &= J \circ \pi_\alpha dg, \\ (g^*\alpha) \circ i &= db. \end{aligned}$$

We prove now that the $\alpha-$ and $d\alpha-$energies of v are bounded. Let $R > 0$ be arbitrary and for some $\tau_0 \in \mathcal{A}$ consider

$$\begin{aligned} \int_{D_R(0)} \tau_0'(b) db \circ i \wedge db &= \lim_{n\to\infty} \int_{D_R(0)} \tau_0'(b_n) db_n \circ i \wedge db_n \\ &= \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau_0'(\hat{a}_n - \hat{a}_n(\tilde{z}^{(n)})) d\hat{a}_n \circ i \wedge d\hat{a}_n \\ &= \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau_n'(\hat{a}_n) d\hat{a}_n \circ i \wedge d\hat{a}_n \end{aligned}$$

$$\leq \lim_{n\to\infty} \sup_{\tau\in\mathcal{A}} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \tau'(\hat{a}_n) d\hat{a}_n \circ i \wedge d\hat{a}_n$$
$$= \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})),$$

where $\tau_n = \tau_0(\cdot - \hat{a}_n(\tilde{z}^{(n)}))$ is a sequence of functions that belong to $\mathcal{A}$. Taking the supremum of the left-hand side over $\tau \in \mathcal{A}$, we get

$$E_\alpha(v; D_R(0)) \leq \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})),$$

while picking some arbitrary $\epsilon > 0$, we obtain

$$E_\alpha(v; D_R(0)) \leq \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_{R/R_n}(\tilde{z}^{(n)})) \leq \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_\epsilon(0)).$$

For the $d\alpha-$energy, we proceed analogously: for $R > 0$ we have

$$E_{d\alpha}(v; D_R(0)) = \lim_{n\to\infty} \int_{D_R(0)} g_n^* d\alpha = \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \hat{f}_n^* d\alpha,$$

while picking some arbitrary $\epsilon > 0$, we find

$$E_{d\alpha}(v; D_R(0)) = \lim_{n\to\infty} \int_{D_{R/R_n}(\tilde{z}^{(n)})} \hat{f}_n^* d\alpha$$
$$\leq \lim_{n\to\infty} \int_{D_\epsilon(0)} \hat{f}_n^* d\alpha \leq \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)).$$

Because the $\alpha-$ and $d\alpha-$energies are non-negative,

$$\begin{aligned} E(v; D_R(0)) &= E_\alpha(v; D_R(0)) + E_{d\alpha}(v; D_R(0)) \\ &\leq \lim_{n\to\infty} E_\alpha(\hat{u}_n; D_\epsilon(0)) + \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)) \\ &= \lim_{n\to\infty} E(\hat{u}_n; D_\epsilon(0)) \\ &\leq E_0, \end{aligned}$$

and since $R > 0$ was arbitrary, we obtain $E_{d\alpha}(v; \mathbb{C}) \leq E_0$. As v is a usual pseudoholomorphic curve, it follows that $E(v; \mathbb{C}) = E_{\mathrm{H}}(v; \mathbb{C})$, where E_{H} is the Hofer energy defined in [1]; thus $E_{\mathrm{H}}(v; \mathbb{C}) \leq E_0$. Moreover, as v is non-constant we have by Remark 2.38 of [6], that for any $\epsilon > 0$,

$$0 < \hbar \leq E_{d\alpha}(v; \mathbb{C}) \leq \lim_{n\to\infty} E_{d\alpha}(\hat{u}_n; D_\epsilon(0)) \leq \lim_{n\to\infty} E_{d\alpha}(u_n; \psi_n(D_\epsilon(0))).$$

Choosing $\epsilon > 0$ such that $\psi_n(D_\epsilon(0)) \subset U$ for all n, we end up with

$$0 < \hbar \leq \lim_{n\to\infty} E_{d\alpha}(u_n; U) \leq E_0,$$

and the proof is finished. □

The next proposition is proved by contradiction by means of Lemma 22.

Proposition 23. *There exists a subsequence of u_n, still denoted by u_n, and a finite subset $\mathcal{Z} \subset \dot{\tilde{S}}^{\mathcal{D},r}$ such that for every compact subset $\mathcal{K} \subset \dot{\tilde{S}}^{\mathcal{D},r} \backslash \mathcal{Z}$, there exists a constant $C_{\mathcal{K}} > 0$ such that*

$$\|du_n(z)\| := \sup_{v\in T_z S^{\mathcal{D},r}, \|v\|_{h_n}=1} \|du_n(z)v\|_{\overline{g}} \leq C_{\mathcal{K}}$$

for all $z \in \mathcal{K}$.

Proof. For the sequence u_n and any finite subset $\mathcal{Z} \subset \dot{\tilde{S}}^{\mathcal{D},r}$, we define

$$\mathcal{Z}_{\{u_n\},\mathcal{Z}} := \Big\{ z \in \dot{\tilde{S}}^{\mathcal{D},r} \backslash \mathcal{Z} \mid \text{there exists a subsequence } u_{n_k} \text{ of } u_n \text{ and a sequence } z_k \in \dot{\tilde{S}}^{\mathcal{D},r} \backslash \mathcal{Z} \text{ such that } z_k \to z \text{ and } \|du_{n_k}(z_k)\| \to \infty \text{ as } k \to \infty \Big\}.$$

If $\mathcal{Z}_{\{u_n\},\emptyset}$ is empty then the assertion is fullfilled for the sequence u_n and the finite set $\mathcal{Z} = \emptyset$. Otherwise, we choose $z^1 \in \mathcal{Z}_{\{u_n\},\emptyset}$. In this case, there exists a sequence $z_n^1 \in \dot{\tilde{S}}^{\mathcal{D},r}$ and a subsequence u_n^1 of u_n such that $z_n^1 \to z^1$ and $\left\|du_n^1(z_n^1)\right\| \to \infty$. Consider now the set $\mathcal{Z}_{\{u_n^1\},\{z^1\}}$. If $\mathcal{Z}_{\{u_n^1\},\{z^1\}}$ is empty then the assertion is fullfilled for the subsequence u_n^1 and the finite set $\mathcal{Z} = \{z^1\}$. Otherwise, we choose an element $z^2 \in \mathcal{Z}_{\{u_n^2\},\{z^1\}}$. In this case, by definition, there exists a sequence $z_n^2 \in \dot{\tilde{S}}^{\mathcal{D},r} \backslash \{z^1\}$ and a subsequence u_n^2 of u_n^1 such that $z_n^2 \to z^2$ and $\left\|du_n^2(z_n^2)\right\| \to \infty$. Let us show that the set of points $\mathcal{Z} = \{z^1, z^2, \dots\}$ constructed in this way is finite, or more precisely, that $|\mathcal{Z}| \leq 2E_0/\hbar$. Assume $|\mathcal{Z}| > 2E_0/\hbar$ and pick an integer $k > 2E_0/\hbar$ and pairwise different points $z^1, \dots, z^k \in \mathcal{Z}$. Let $U_1, \dots, U_k \subset \dot{\tilde{S}}^{\mathcal{D},r}$ be some open pairwise disjoint neighbourhoods of $z^1, \dots, z^k$. Applying Lemma 22 inductively, we deduce that there exists a positive integer N such that for every $n \geq N$,

$E_{d\alpha}(u_n;U_i)\geq \hbar/2$ for all $i=1,\dots,k$. Since the U_i are disjoint, we obtain

$$k\frac{\hbar}{2}\leq \sum_{i=1}^{k} E_{d\alpha}(u_n;U_i)\leq E_{d\alpha}(u_n;\dot{\tilde{S}}^{\mathcal{D},r})\leq E_0.$$

Thus $k\leq 2E_0/\hbar$ which is a contradiction to our assumption. □

By means of Proposition 23 we can prove the convergence of the $\mathcal{H}-$holomorphic maps in a punctured thick part of the Riemann surface.

Proof. (of Theorem 19) For some sufficiently small $k\in\mathbb{N}$ we consider the subsets $\Omega_k:=\text{Thick}_{1/k}(\dot{\tilde{S}}^{\mathcal{D},r},h)\backslash\bigcup_{i=1}^{N} D^h_{1/k}(z^i)$, where $\mathcal{Z}=\{z^1,\dots,z^N\}$ is the subset in Proposition 23 and $D^h_{1/k}(z_i)$ is the open disk around z_i of radius $1/k$ with respect to the metric h. In order to keep the notation simple, the subsequence obtained by applying Proposition 23 is still denoted by u_n. Obviously, Ω_k build an exhaustion by compact sets of $\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z}$. These sets are compact surfaces with boundary. By Proposition 23, the maps u_n have uniformly bounded gradients on Ω_1. Thus after a suitable translation of the maps u_n in the $\mathbb{R}-$coordinate, there exists a subsequence u^1_n of u_n that converges in $C^\infty(\Omega_1)$ to a map $u:\Omega_1\to\mathbb{R}\times M$. Iteratively, at step $k+1$ there exists a subsequence u^{k+1}_n of u^k_n that converges in $C^\infty(\Omega_{k+1})$ to a map $u:\Omega_{k+1}\to\mathbb{R}\times M$ which is an extension from Ω_k to Ω_{k+1}. This procedure allows us to define a map $u:\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z}\to\mathbb{R}\times M$. After passing to some diagonal subsequence u^n_n, the maps u^n_n converge in $C^\infty_{\text{loc}}(\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z})$ to the map $u:\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z}\to\mathbb{R}\times M$. Since the L^2-norms of γ_n are uniformly bounded on $S^{\mathcal{D},r}$, they converge in $C^\infty_{\text{loc}}(\dot{\tilde{S}}^{\mathcal{D},r})$ to some harmonic $1-$form γ with a bounded L^2-norm on $\dot{\tilde{S}}^{\mathcal{D},r}$. Hence the map u is a $\mathcal{H}-$holomorphic curve on $\dot{\tilde{S}}^{\mathcal{D},r}\backslash\mathcal{Z}$ with harmonic perturbation γ. □

3.2. Convergence on the thin part and around the points from $\mathcal{Z}$

In this section we investigate the convergence of the $\mathcal{H}-$holomorphic curves u_n on the components of the thin part and in the neighbourhood of the points from $\mathcal{Z}$ that were constructed in Theorem 19. For a sufficiently small $\delta>0$, the set $\text{Thin}_\delta(\dot{S}^{\mathcal{D},r},h_n)$ can be decomposed in two types of connected components: (I) the non-compact components that are called cusps, which are neighbourhoods of punctures with respect to the hyperbolic metric, and (II) the compact components called hyperbolic cylinders. Each of these cylinders can be biholomorphically identified with the standard cylinder $[-R,R]\times S^1$

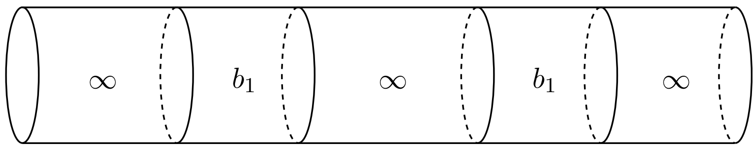

Figure 3.1: The component of the thin part, which is biholomorphic to a cylinder, is divided in cylinders of types b_1 and ∞ in an alternating order.

for a suitable $R > 0$. In the Deligne-Mumford limiting process R may tend to ∞ and nodes appear. For more details we refer to Chapter 1 of [6]. This section is organized as follows. First, we analyze the convergence of u_n on components that can be indentified with hyperbolic cylinders, and describe the limit object. Second, we treat the convergence of u_n on components that can be identified with cusps, and as before, describe the limit object. The convergence results estabished here can be used to describe the convergence of u_n in a neighbourhood of the points from $\mathcal{Z}$. Third, we use the description of the convergence of the $\mathcal{H}-$holomorpic curves u_n on the thick part (established in Section 3.1), the thin part, and in the neighbourhood of the points from $\mathcal{Z}$ (established in this section) to define a new surface by gluing the two parts together. On this surface we decribe the convergence of u_n completely.

3.2.1. Cylinders. We analyze the convergence of u_n on compact components of the thin part which are biholomorphic to hyperbolic cylinders. When restricted to these cylinders, the curves u_n can have a $d\alpha-$energy larger than the constant $\hbar > 0$ defined in (1.7). Since we do not have a version of the monotonicity lemma in the $\mathcal{H}-$holomorphic case, the classical results on the asymptotic of holomorphic cylinders from [1] and [9] are not directly applicable. To deal with this problem we shift the maps by the Reeb flow to make them pseudoholomorphic. Actually we proceed as follows. We decompose the hyperbolic cylinder into a finite uniform number of smaller cylinders; some of them having conformal modulus tending to infinity but a $d\alpha-$energy strictly smaller than $\hbar$, and the rest of them having bounded modulus but a $d\alpha-$energy possibly larger than $\hbar$. We refer to these cylinders as cylinders of types ∞ and b_1, respectively. We consider an alternating appearance of these cylinders, as it can be seen in Figure 3.1.

The convergence and the description of the limit object are first treated for cylinders of type ∞, and then for cylinders of type b_1.

As cylinders of type ∞ have a small $d\alpha-$energy, we can assume, by the classical bubbling-off analysis, that the maps u_n have uniformly bounded gradients. To make the curves u_n pseudoholomorphic, we perform a transformation by pushing them along the Reeb flow up to some specific time. This procedure is made precise in [15]. As the gradients of these transformed curves still remain uniformly bounded, we can adapt the results of [9] to formulate a convergence result for the transformed curves (see [15]). Undoing the transformation we obtain a convergence result for the $\mathcal{H}-$holomorphic curves.

In the case of cylinders of type b_1 we proceed as follows. Relying on a bubbling-off argument, as we did in the case of the thick part (see Section 3.1), we assume that the gradients blow up only in a finite uniform number of points and remain uniformly bounded in a compact complement of them. In this compact region, the Arzelá-Ascoli theorem shows that the curves u_n together with the harmonic perturbations γ_n converge in C^∞ to some $\mathcal{H}-$holomorphic curve. What is then left is the convergence in a neighbourhood of the finitely many punctures where the gradients blow up. Here, a neighbourhood of a puncture is a disk on which the harmonic perturbation can be made exact and can be encoded in the $\mathbb{R}-$coordinate of the curve u_n. By this procedure we transform the $\mathcal{H}-$holomorphic curve into a usual pseudoholomorphic curve defined on a disk D. By the $C^\infty-$convergence of u_n on any compact complement of the punctures, we assume that the transformed curves converge on an arbitrary neighbourhood of ∂D. This approach, which is described in detail in Section 3.2.3, uses a convergence result established in Appendix A. As for cylinders of type ∞, we undo the transformation and derive a convergence result for the $\mathcal{H}-$homolorphic curves on cylinders of type b_1. Finally, gluing all cylinders together, we are led to a convergence result for the entire component which is biholomorphic to a hyperbolic cylinder from the thin part.

Let C_n be a component of $\text{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ which is conformally equivalent to the cylinder $[-\sigma_n^\delta, \sigma_n^\delta] \times S^1$ via a map $\rho_n : [-\sigma_n^\delta, \sigma_n^\delta] \times S^1 \to C_n$. Observe that from the definition of Deligne-Mumford convergence, $\sigma_n^\delta \to \infty$ as $n \to \infty$. In the following, we drop the fixed, sufficiently small constant $\delta > 0$, and assume that the curves u_n are defined on $[-\sigma_n, \sigma_n] \times S^1$. Let $u_n = (a_n, f_n) : [-\sigma_n, \sigma_n] \times S^1 \to \mathbb{R} \times M$ be a sequence of $\mathcal{H}-$holomorphic curves with harmonic perturbations γ_n, i.e.,

$$\begin{aligned}\pi_\alpha df_n \circ i &= J(f_n) \circ \pi_\alpha df_n,\\ (f_n^*\alpha) \circ i &= da_n + \gamma_n\end{aligned}$$

on $[-\sigma_n, \sigma_n] \times S^1$, and let us assume that the energy of u_n, as well as the L^2-norm of γ_n on the cylinders are uniformly bounded, i.e. for the constants

$E_0, C_0 > 0$ we have $E(u_n; [-\sigma_n, \sigma_n] \times S^1) \leq E_0$ and $\|\gamma_n\|^2_{L^2([-\sigma_n,\sigma_n]\times S^1)} \leq C_0$ for all $n \in \mathbb{N}$.

Before describing the decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1 we give a proposition which states that the C^1-norm of the harmonic perturbation γ_n is uniformly bounded. This result will play an essential role in Section 3.2.3. We set $\gamma_n = \zeta_n ds + \chi_n dt$, where ζ_n and χ_n are harmonic functions defined on $[-\sigma_n, \sigma_n] \times S^1$ with coordinates (s,t) such that $\zeta_n + i\chi_n$ is holomorphic. By the uniform L^2-bound of γ_n, we have

$$\|\gamma_n\|^2_{L^2([-\sigma_n,\sigma_n]\times S^1)} = \int_{[-\sigma_n,\sigma_n]\times S^1} \left(\zeta_n^2 + \chi_n^2\right) dsdt \leq C_0$$

for all $n \in \mathbb{N}$. As a result, the L^2-norm of the holomorphic function $\zeta_n + i\chi_n$ is uniformly bounded. Denote this function by $G_n = \zeta_n + i\chi_n$.

Proposition 24. *For any $\delta > 0$ there exists a constant $C_\delta > 0$ such that*

$$\|G_n\|_{C^1([-\sigma_n+\delta,\sigma_n-\delta]\times S^1)} \leq C_\delta$$

for all $n \in \mathbb{N}$.

Proof. First, we prove that the sequence G_n is uniformly bounded in C^0-norm. As $G_n : [-\sigma_n, \sigma_n] \times S^1 \to \mathbb{C}$ is holomorphic, $\zeta_n = \Re(G_n)$ and $\chi_n = \Im(G_n)$ are harmonic functions defined on $[-\sigma_n, \sigma_n] \times S^1$. For a sufficiently small $\delta > 0$ we establish C^0-bounds for ζ_n on the subcylinders $[-\sigma_n + (\delta/2), \sigma_n - (\delta/2)] \times S^1$. By the mean value theorem for harmonic function, we have

$$\zeta_n(p) = \frac{16}{\pi\delta^2} \int_{D_{\frac{\delta}{4}}(p)} \zeta_n(s,t) dsdt$$

for all $p \in [-\sigma_n + (\delta/2), \sigma_n - (\delta/2)] \times S^1$, where $D_{\delta/4}(p) \subset [-\sigma_n, \sigma_n] \times S^1$. Then Hölder's inequality yields

$$|\zeta_n(p)| \leq \frac{4}{\sqrt{\pi}\delta} \left(\int_{D_{\frac{\delta}{4}}(p)} |\zeta_n(s,t)|^2 dsdt \right)^{\frac{1}{2}} \leq \frac{4}{\sqrt{\pi}\delta} \sqrt{C_0}$$

for all $n \in \mathbb{N}$. As a result, we obtain

$$\|\zeta_n\|_{C^0\left([-\sigma_n+\frac{\delta}{2},\sigma_n-\frac{\delta}{2}]\times S^1\right)} \leq \frac{4}{\sqrt{\pi}\delta} \sqrt{C_0},$$

and note that the same result holds for χ_n.

By means of bubbling-off analysis we prove now that the gradient of G_n is uniformly bounded. Assume

$$\sup_{p\in[-\sigma_n+\delta,\sigma_n-\delta]\times S^1} |\nabla G_n(p)| \to \infty$$

as $n \to \infty$. Let $p_n \in [-\sigma_n + \delta, \sigma_n - \delta] \times S^1$ be such that

$$|\nabla G_n(p_n)| = \sup_{p\in[-\sigma_n+\delta,\sigma_n-\delta]\times S^1} |\nabla G_n(p)|\,;$$

then $R_n := |\nabla G_n(p_n)| \to \infty$ as $n \to \infty$. Set $\epsilon_n := R_n^{-\frac{1}{2}} \searrow 0$ as $n \to \infty$, and observe that $\epsilon_n R_n \to \infty$ as $n \to \infty$. Choose $n_0 \in \mathbb{N}_0$ sufficiently large such that $D_{10\epsilon_n}(p_n) \subset [-\sigma_n, \sigma_n] \times S^1$ for all $n \geq n_0$. By Hofer's topologial lemma there exist $\epsilon'_n \in (0, \epsilon_n]$ and $p'_n \in [-\sigma_n, \sigma_n] \times S^1$ satisfying:

1) $\epsilon'_n R'_n \geq \epsilon_n R_n$;
2) $p'_n \in D_{2\epsilon_n}(p_n) \subset D_{10\epsilon_n}(p_n)$;
3) $|\nabla G_n(p)| \leq 2R'_n$, for all $p \in D_{\epsilon'_n}(p'_n) \subset D_{10\epsilon_n}(p_n)$,

where $R'_n := |du_n(p'_n)|$. Via rescaling consider the maps $\tilde{G}_n : D_{\epsilon'_n R'_n}(0) \to \mathbb{C}$, defined by

$$\tilde{G}_n(w) := G_n\left(p'_n + \frac{w}{R'_n}\right)$$

for $w \in D_{\epsilon'_n R'_n}(0)$. Observe that $p'_n + (w/R'_n) \in D_{\epsilon'_n}(p'_n)$ for $w \in D_{\epsilon'_n R'_n}(0)$, and that for $\tilde{G}_n$ we have:

1) $|\nabla \tilde{G}_n(0)| = 1$;
2) $|\nabla \tilde{G}_n(w)| \leq 2$ for $w \in D_{\epsilon'_n R'_n}(0)$;
3) $\tilde{G}_n$ is holomorphic on $D_{\epsilon'_n R'_n}(0)$;
4) $\tilde{G}_n$ is uniformly bounded on $[-\sigma_n + \delta, \sigma_n - \delta] \times S^1$ (by Assertion 1).

By the usual regularity theory for pseudoholomorphic maps and Arzelá-Ascoli theorem, $\tilde{G}_n$ converge in $C^\infty_{\text{loc}}(\mathbb{C})$ to a bounded holomorphic map $\tilde{G} : \mathbb{C} \to \mathbb{C}$ with $|\nabla \tilde{G}(0)| = 1$. By Liouville theorem this map can be only the constant map, and so, we arrive at a contradiction with $|\nabla \tilde{G}(0)| = 1$. □

Thus for $\delta > 0$ we can replace the cylinder $[-\sigma_n + \delta, \sigma_n - \delta] \times S^1$ by $[-\sigma_n, \sigma_n] \times S^1$ if we consider $\text{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ for a smaller $\delta > 0$. We come now to the decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1.

Consider the parameter-dependent function with parameter $h \in [-\sigma_n, \sigma_n]$ defined by

$$F_{n,h} : [h, \sigma_n] \to \mathbb{R},\ s \mapsto \int_{[h,s]\times S^1} f_n^* d\alpha.$$

As $f_n^* d\alpha$ is non-negative, $F_{n,h}$ is positive and monotone. For the constant $\hbar$ defined in (1.7), we set $h_n^{(0)} = -\sigma_n$, and define

$$h_n^{(m)} := \sup\left(F^{-1}_{n,h_n^{(m-1)}}\left[0, \frac{\hbar}{4}\right]\right).$$

Since $E_{d\alpha}(u_n; [-\sigma_n, \sigma_n] \times S^1) < E_0$, the sequence $\{h_n^{(m)}\}_{m\in\mathbb{N}_0}$ has to end after N_n steps, where $h_n^{(N_n)} = \sigma_n$. On the cylinder $[h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1$, the $d\alpha-$energy of u_n can be smaller than $\hbar/4$. Obviously, we have $-\sigma_n = h_n^{(0)} < h_n^{(1)} < \cdots < h_n^{(m)} < \cdots < h_n^{(N_n)} = \sigma_n$ giving $E_{d\alpha}(u_n; [h_n^{(m-1)}, h_n^{(m)}] \times S^1) = \hbar/4$ for $m = 1, \ldots, N_n - 1$ and $E_{d\alpha}(u_n; [h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1) \leq \hbar/4$. Hence the $d\alpha-$energy can be written as

$$E_{d\alpha}(u_n; [-\sigma_n, \sigma_n] \times S^1) = (N_n - 1)\frac{\hbar}{4} + E_{d\alpha}(u_n; [h_n^{(N_n-1)}, h_n^{(N_n)}] \times S^1),$$

which implies the following bound on N_n:

$$0 \leq N_n \leq \frac{4E_0}{\hbar} + 1.$$

After going over to a subsequence, we can further assume that N_n is also independent of n; for this reason, we set $N_n = N$. Thus the cylinders $[-\sigma_n, \sigma_n] \times S^1$ have been decomposed into N smaller subcylinders $[h_n^{(0)}, h_n^{(1)}] \times S^1, \ldots, [h_n^{(N-1)}, h_n^{(N)}] \times S^1$ on which we have $E_{d\alpha}(u_n; [h_n^{(m-1)}, h_n^{(m)}] \times S^1) = \hbar/4$ for $m \in \{1, \ldots, N-1\}$ and $E_{d\alpha}(u_n; [h_n^{(N-1)}, h_n^{(N)}] \times S^1) \leq \hbar/4$.

A sequence of cylinders $[a_n, b_n] \times S^1$, where $a_n, b_n \in \mathbb{R}$ and $a_n < b_n$ is called of type b_1 if $b_n - a_n$ is bounded from above, and of type ∞ if $b_n - a_n \to \infty$ as $n \to \infty$. This is illustrated in Figure 3.2.

Lemma 25. *Let $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ be a cylinder of type ∞ and let $h > 0$ be chosen small enough such that $h_n^{(m)} - h_n^{(m-1)} - 2h = (h_n^{(m)} - h) - (h_n^{(m-1)} + h) > 0$ for all $n \in \mathbb{N}$. Then there exists a constant $C_h > 0$ such that*

$$\|du_n(z)\|_{C^0} = \sup_{\|v\|_{eucl}=1} \|du_n(z)v\| < C_h$$

for all $z \in [h_n^{(m-1)} + h, h_n^{(m)} - h] \times S^1$ and $n \in \mathbb{N}$.

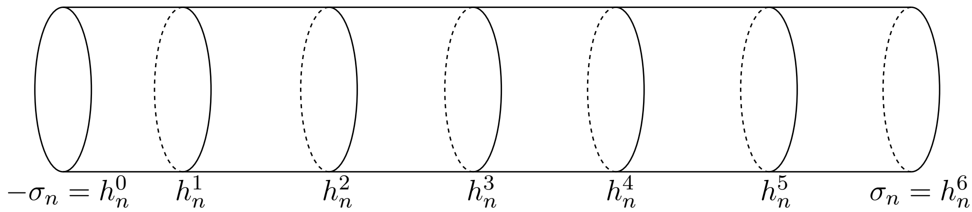

Figure 3.2: Decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into smaller cylinders $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ having $d\alpha-$energy $\hbar/4$ or less.

Proof. The proof makes use of bubbling-off analysis. Assume that there exists $h > 0$ such that $h_n^{(m)} - h_n^{(m-1)} - 2h > 0$ and

$$\sup_{z \in [h_n^{(m-1)}+h, h_n^{(m)}-h] \times S^1} \|du_n(z)\|_{C^0} = \infty. \tag{3.1}$$

Then there exists a sequence $z_n \in (h_n^{(m-1)} + h, h_n^{(m)} - h) \times S^1$ with the property $R_n := \|du_n(z_n)\|_{C^0} \to \infty$ as $n \to \infty$. Let $\epsilon_n = R_n^{-\frac{1}{2}} \searrow 0$ as $n \to \infty$, and observe that $\epsilon_n R_n \to \infty$ as $n \to \infty$. Choose $n_0 \in \mathbb{N}$ sufficiently large such that $D_{10\epsilon_n}(z_n) \subset [h_n^{(m-1)}, h_n^{(m)}] \times S^1$ for all $n \geq n_0$. By Hofer's topological lemma, there exist $\epsilon'_n \in (0, \epsilon_n]$ and $z'_n \in [h_n^{(m-1)}, h_n^{(m)}] \times S^1$ satisfying:

1) $\epsilon'_n R'_n \geq \epsilon_n R_n$;
2) $z'_n \in D_{2\epsilon_n}(z_n) \subset D_{10\epsilon_n}(z_n)$;
3) $\|du_n(z)\|_{C^0} \leq 2R'_n$, for all $z \in D_{\epsilon'_n}(z'_n) \subset D_{10\epsilon_n}(z_n)$,

where $R'_n := \|du_n(z'_n)\|_{C^0}$. Applying rescaling consider the map $v_n : D_{\epsilon'_n R'_n}(0) \to \mathbb{R} \times M$, defined by

$$v_n(w) = (b_n(w), g_n(w)) := u_n\left(z'_n + \frac{w}{R'_n}\right) - a_n(z'_n)$$

for $w \in D_{\epsilon'_n R'_n}(0)$. Note that $z'_n + (w/R'_n) \in D_{\epsilon'_n}(z'_n)$ for $w \in D_{\epsilon'_n R'_n}(0)$, and that for v_n we have

1) $\|dv_n(0)\|_{C^0} = 1$;
2) $\|dv_n(w)\|_{C^0} \leq 2$ for $w \in D_{\epsilon'_n R'_n}(0)$;
3) $E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \leq \hbar/4$ (straightforward calculation shows that the $\alpha-$energy is also uniformly bounded);

4) v_n solves

$$\pi_\alpha dg_n \circ i = J \circ \pi_\alpha dg_n,$$
$$(g_n^*\alpha) \circ i = db_n + \frac{\gamma_n}{R'_n}$$

on $D_{\epsilon'_n R'_n}(0)$.

As the gradients of v_n are uniformly bounded, v_n converge in $C^\infty_{\text{loc}}(\mathbb{C})$ to a finite energy plane $v = (b, g) : \mathbb{C} \to \mathbb{R} \times M$ characterized by:

1) $\|dv(0)\|_{C^0} = 1$;
2) $\|dv(w)\|_{C^0} \leq 2$ for $w \in \mathbb{C}$;
3) $E_{d\alpha}(v; \mathbb{C}) \leq \hbar/4$;
4) v is a finite energy holomorphic plane.

Assertion 3 follows from the fact that for an arbitray $R > 0$ we have

$$E_{d\alpha}(v, D_R(0)) = \lim_{n\to\infty} E_{d\alpha}(v_n; D_R(0)) \leq \lim_{n\to\infty} E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \leq \frac{\hbar}{4},$$

while Assertion 4 follows from the fact that γ_n has a uniformly bounded L^2-norm. Note that by employing the above argument, a bound for the $\alpha-$energy can be also obtained. Now, as v is non-constant, Theorem 31 of [2] gives $E_{d\alpha}(v; \mathbb{C}) \geq \hbar$, which is a contradiction to Assertion 3. Thus Assumption (3.1) does not hold, and the gradient of u_n on cylinders of type ∞ is uniformly bounded. □

Now we change the above decomposition so that the lengths of the cylinders of type b_1 are also bounded by below and describe the alternating appearance of cylinders of types ∞ and b_1. This process is necessary, because on the cylinders of type b_1 whose length tends to zero we cannot analyze the convergence behaviour of the maps u_n and cannot describe their limit object. We proceed as follows.

***Step 1.*:** We consider a cylinder $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ of type ∞, on which we apply Lemma 25. When doing this we choose a sufficiently small constant $h > 0$, so that the gradients are uniformly bounded only on $[h_n^{(m)} + h, h_n^{(m+1)} - h] \times S^1$ by the constant $C_h > 0$, which in turn, is again a cylinder of type ∞. By this procedure, a cylinder $[h_n^{(m)}, h_n^{(m+1)}] \times S^1$ of type ∞ is decomposed into three smaller cylinders: two cylinders $[h_n^{(m)}, h_n^{(m)} + h] \times S^1$, $[h_n^{(m+1)} - h, h_n^{(m+1)}] \times S^1$ of

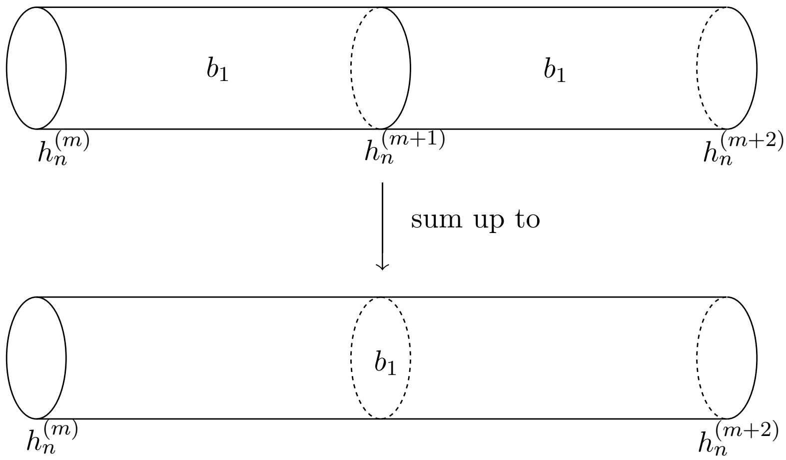

Figure 3.3: Two cylinders of type b_1 are combined to form a bigger cylinder of type b_1.

type b_1 and one cylinder $[h_n^{(m)} + h, h_n^{(m+1)} - h] \times S^1$ of type ∞. The length of these two cylinders of type b_1 is $h > 0$. To any other cylinder of type ∞ we apply the same procedure with a fixed constant $h > 0$.

Step 2.: We combine all cylinders of type b_1, which are next to each other, to form a bigger cylinder of type b_1. This can be seen in Figure 3.3. By this procedure, we guarantee that in a constellation consisting of three cylinders that lie next to each other, the type of the middle cylinder is different to the types of the left and right cylinders. Thus we got rid of the cylinders of type b_1 with length tending to zero, and made sure that the cylinders of types ∞ and b_1 appear alternately. We aditionally assume that the first and last cylinders in the decomposition are of type ∞, since otherwise, we can glue the cylinder of type b_1 to the thick part of the surface and consider $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ for a smaller $\delta > 0$. By this procedure, we decompose $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1, while the first and last cylinders in the decomposition are of type ∞.

Step 3.: For $\tilde{E}_0 = 2(E_0 + C_h)$ and in view of the non-degeneracy of the contact manifold (M, α), let the constant $\hbar_0$ be given by

$$\hbar_0 := \min\{|T_1 - T_2| \mid T_1, T_2 \in \mathcal{P}_\alpha, T_1 \neq T_2, T_1, T_2 \leq \tilde{E}_0\}. \tag{3.2}$$

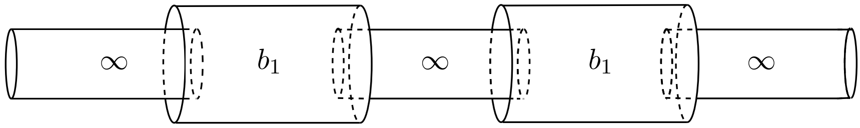

Figure 3.4: Decomposition of $[-\sigma_n, \sigma_n] \times S^1$ into cylinders of types ∞ and b_1 in an alternating order.

Observe that because of $\tilde{E}_0 \geq E_0$, $\hbar_0 \leq \hbar$. If $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is a cylinder of type ∞ for some $m \in \{1, \ldots, N\}$, we define the constant $\hbar_0$ as above and apply Step 1 and Step 2 to decompose this cylinder into cylinders of types ∞ and b_1, while the first and last cylinders in the decomposition are of type ∞. The cylinders of type ∞ have now a $d\alpha-$energy smaller than $\hbar_0/4$. We apply this procedure to all cylinders of type ∞. In summary, $[-\sigma_n, \sigma_n] \times S^1$ is decomposed into cylinders of type ∞ with a $d\alpha-$energy smaller than $\hbar_0/4$ and cylinders of type b_1, with the first and last cylinders being of type ∞.

***Step 4.*:** We enlarge the cylinders of type b_1 without changing their type. Let $h > 0$ be as in Lemma 25 and pick $m \in \{1, \ldots, N\}$ such that $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is of type b_1. For n sufficiently large, we replace the cylinder $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ by the bigger cylinder $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$, and apply this procedure to all cylinders of type b_1. As a result, neighbouring cylinders will overlap. Essentially, this means that if $[h_n^{(m-2)}, h_n^{(m-1)}] \times S^1$ is a cylinder of type ∞, which lies to the left of a cylinder $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$ of type b_1, then their intersection is $[h_n^{(m-1)} - 3h, h_n^{(m-1)}] \times S^1$. This can be seen in Figure 3.4.

By the above procedure, the cylinder $[-\sigma_n, \sigma_n] \times S^1$ is decomposed into an alternating constellation of cylinders of types ∞ and b_1. On cylinders of type ∞, the $d\alpha-$energy is smaller than $\hbar_0/4$, while on cylinders of type b_1, the $d\alpha-$energy can be larger than $\hbar_0/4$. By Lemma 25, the gradients of the $\mathcal{H}-$holomorphic curves on the cylinders of type ∞ are uniformly bounded by the constant $C_h > 0$ with respect to the Euclidean metric on the domain, and to the metric described in (1.1) on the target space $\mathbb{R} \times M$. Finally, the cylinders of types ∞ and b_1 overlap.

We are now well prepared to analyse the convergence of the $\mathcal{H}-$holomorphic curves on cylinders of types ∞ and b_1. After obtaining separate convergence results, we glue the limit objects of these cylinders on the overlaps, and obtain a limit object on the whole cylinder $[-\sigma_n, \sigma_n] \times S^1$. Sections 3.2.2 and 3.2.3 deal with the convergence and the description of the limit object on cylinders of types ∞ and b_1, while in Section 3.2.4 we carry out the gluing of these two convergence results.

3.2.2. Cylinders of type ∞. We describe the convergence and the limit object of the sequence of $\mathcal{H}$−holomorphic curves u_n, defined on cylinders of type ∞. Let $m \in \{1, \dots, N\}$ be such that $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ is a cylinder of type ∞ as described in Section 3.2.1, i.e. $h_n^{(m)} - h_n^{(m-1)} \to \infty$ as $n \to \infty$. Consider the diffeomorphism $\psi_n : [-R_n^{(m)}, R_n^{(m)}] \to [h_n^{(m)}, h_n^{(m+1)}]$ given by $\psi_n(s) = s + (h_n^{(m)} + h_n^{(m+1)})/2$ and the $\mathcal{H}$−holomorphic maps $u_n = (a_n, f_n) : [-R_n^{(m)}, R_n^{(m)}] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ_n . For deriving a C^∞_{loc}−convergence result we consider the following setting:

C1: $R_n^{(m)} \to \infty$ as $n \to \infty$.

C2: γ_n is a harmonic 1−form on $[-R_n^{(m)}, R_n^{(m)}] \times S^1$ with respect to the standard complex structure i, i.e. $d\gamma_n = d\gamma_n \circ i = 0$.

C3: The $d\alpha$−energy of u_n is uniformly small, i.e. $E_{d\alpha}(u_n; [-R_n^{(m)}, R_n^{(m)}] \times S^1) \leq \hbar_0/2$ for all n, where $\hbar_0$ is the constant defined in (3.2).

C4: The energy of u_n is uniformly bounded, i.e. for the constant $E_0 > 0$ we have $E(u_n; [-R_n^{(m)}, R_n^{(m)}] \times S^1) \leq E_0$ for all $n \in \mathbb{N}$.

C5: The map u_n together with the 1−form γ_n solve the $\mathcal{H}$−holomorphic curve equation

$$\pi_\alpha df_n \circ i = J(f_n) \circ \pi_\alpha df_n,$$
$$(f_n^*\alpha) \circ i = da_n + \gamma_n.$$

C6: The harmonic 1−form γ_n has a uniformly bounded L^2−norm, i.e. for the constant $C_0 > 0$ we have $\|\gamma_n\|^2_{L^2([-R_n^{(m)}, R_n^{(m)}] \times S^1)} \leq C_0$ for all n.

C7: The map u_n has a uniformly bounded gradient due to Lemma 25 and Step 4 of Section 3.2.1, i.e. for the constant $C_h > 0$ we have

$$\|du_n(z)\|_{C^0} = \sup_{\|v\|_{\mathrm{eucl}}=1} \|du_n(z)v\| < C_h$$

for all $z \in [-R_n^{(m)}, R_n^{(m)}] \times S^1$ and all $n \in \mathbb{N}$.

C8: If $P_n := P_{\gamma_n}(\{0\} \times S^1)$ is the period of γ_n over the closed curve $\{0\} \times S^1$, as defind in (1.4), we assume that the sequence $R_n P_n$ is bounded by the constant $C > 0$. Moreover, after going over to some subsequence, we assume that $R_n P_n$ converges to some real number τ.

C9: If $S_n := S_{\gamma_n}(\{0\} \times S^1)$ is the co-period of γ_n over the curve $\{0\} \times S^1$ as defined in (1.5), we assume that $S_n R_n \to \sigma$ as $n \to \infty$.

Remark 26. The special circles Γ_i^{nod} in Remark 1 are of two types: contractible and non-contractible. In the contractible case, Γ_i^{nod} lies in the isotopy class of $(\rho_n \circ \psi_n)(\{0\} \times S^1)$ and the conformal periods and co-periods of the harmonic $1-$forms γ_n vanish. Hence, conditions C1-C9 are satisfied on the sequence of degenerating cylinders $[-R_n^{(m)}, R_n^{(m)}] \times S^1$. In the non-contractible case, Γ_i^{nod} also lies in the isotopy class of $(\rho_n \circ \psi_n)(\{0\} \times S^1)$, and by the assumptions of Theorem 2, conditions C1-C9 are satisfied.

To simplify notation we drop the index m. We consider two cases. In Case 1, there exists a subsequence of u_n with vanishing center action, and we use Theorem 2 from [15] to describe the convergence of the $\mathcal{H}-$holomorphic curves with harmonic perturbations γ_n. In Case 2, each subsequence of u_n has a center action larger than $\hbar_0$, and we use Theorem 4 from [15] to describe the convergence.

Before stating the main results of [15] we construct a sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \times S^1 \to [-1, 1] \times S^1$ with certain properties. We first construct diffeomorphisms $\check{\theta}_n : [-R_n, R_n] \to [-1, 1]$, the θ_n are then obtained as $\check{\theta}_n \times id_{S^1}$.

Remark 27. For all sequences $R_n, h_n \in \mathbb{R}_{>0}$ with $h_n < R_n$ and $h_n, R_n/h_n \to \infty$ as $n \to \infty$, consider a sequence of diffeomorphisms $\check{\theta}_n : [-R_n, R_n] \to [-1, 1]$ with the following properties:

1) The left and right shifts $\check{\theta}_n^{\pm}(s) := \check{\theta}_n(s \pm R_n)$ restrict to maps $\check{\theta}_n^{+} : [0, h_n] \to [-1, -1/2]$ resp. $\check{\theta}_n^{-} : [-h_n, 0] \to [1/2, 1]$, which converge in C^∞_{loc} to diffeomorphisms $\check{\theta}^{-} : [0, \infty) \to [-1, -1/2)$ and $\check{\theta}^{+} : (-\infty, 0] \to (1/2, 1]$, respectively.

2) $\check{\theta}_n$ is a linear diffeomorphism on $[-R_n + h_n, R_n - h_n]$. More precisely, we require

$$\begin{aligned} \check{\theta}_n : \text{Op}([-R_n + h_n, R_n - h_n]) &\to \text{Op}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \\ s &\mapsto \frac{s}{2(R_n - h_n)}, \end{aligned}$$

where $\mathrm{Op}([-R_n+h_n, R_n-h_n])$ and $\mathrm{Op}([-1/2,1/2])$ are sufficiently small neighborhoods of the intervals $[-R_n+h_n, R_n-h_n]$ and $[-1/2,1/2]$, respectively.

3) These maps $\check{\theta}_n$ give rise to the desired diffeomorphisms $\theta_n := \check{\theta}_n \times id_{S^1} : [-R_n, R_n] \times S^1 \to [-1,1] \times S^1$. Similarly, we define $\theta_n^{\pm} := \check{\theta}_n^{\pm} \times id_{S^1}$ and $\theta^{\pm} := \check{\theta}^{\pm} \times id_{S^1}$.

Denote by $u_n^{\pm}(s,t) := u_n(s \pm R_n, t)$ the left and right shifts of the maps u_n, and by $\gamma_n^{\pm} := \gamma_n(s \pm R_n, t)$ the left and right shifts of the harmonic perturbation, which are defined on $[0,h_n] \times S^1$ and $[-h_n, 0] \times S^1$, respectively. In both cases we use the diffeomorphisms θ_n to pull the structures back to the cylinder $[-1,1] \times S^1$. Let $i_n := d\theta_n \circ i \circ d\theta_n^{-1}$ be the induced complex structure on $[-1,1] \times S^1$. Then $u_n \circ \theta_n^{-1} : [-1,1] \times S^1 \to \mathbb{R} \times M$ is a sequence of $\mathcal{H}-$holomorphic curves with harmonic perturbations $(\theta_n^{-1})^* \gamma_n$ with respect to the complex structure i_n on $[-1,1] \times S^1$ and the cylindrical almost complex structure J on the target space $\mathbb{R} \times M$. From the result $\check{\theta}_n^{-1}(s) = (\check{\theta}_n^{-})^{-1}(s) - R_n$, and the fact that $\check{\theta}_n^{-}$ and $\check{\theta}_n^{+}$ converge in $C^{\infty}_{\mathrm{loc}}$ to $\check{\theta}^{-}$ on $[-1,-1/2)$ and $\check{\theta}^{+}$ on $(1/2,1]$, respectively, it follows that the complex structures i_n converge in $C^{\infty}_{\mathrm{loc}}$ to a complex structure $\tilde{i}$ on $[-1,-1/2) \times S^1$ and $(1/2,1] \times S^1$. In what follows we reformulate Theorem 2 and 4 from [15] under the conditions that the sequence of cylinders satisfy assumptions C1-C9. These assumptions guarantee the fullfilment of the conditions of Theorem 2 and 4 from [15]. First, we formulate the convergence in the case when there exists a subsequence of u_n, still denoted by u_n, with a vanishing center action.

Theorem 28. *Let u_n be a sequence of $\mathcal{H}-$holomorphic cylinders with harmonic perturbations γ_n that satisfy C1-C9 and possessing a subsequence having vanishing center action. Then there exists a subsequence of u_n, still denoted by u_n, the $\mathcal{H}-$holomorphic cylinders $u^{\pm}$ with exact harmonic perturbations $d\Gamma^{\pm}$ defined on $(-\infty, 0] \times S^1$ and $[0,\infty) \times S^1$, respectively, $\sigma, \tau \in \mathbb{R}$ and a point $w_f \in M$ such that for every sequence $h_n \in \mathbb{R}_+$ with $h_n, R_n/h_n \to 0$ the following $C^{\infty}_{loc}-$ and C^0-convergence results hold: $\sigma_n := S_n R_n \to \sigma$, $\tau_n := P_n R_n \to \tau$ and*

$C^{\infty}_{loc}-$convergence:

1) *For any sequence $s_n \in [-R_n+h_n, R_n-h_n]$ for which s_n/R_n converges to $\kappa \in [-1,1]$, the shifted maps $u_n(s+s_n,t)$, defined on $[-R_n+h_n-s_n, R_n-h_n-s_n] \times S^1$, converge in C^{∞}_{loc} on $\mathbb{R} \times S^1$ to the constant*

map $(-\sigma\kappa, \phi^{\alpha}_{-\tau\kappa}(w_f))$. *The shifted harmonic* $1-$*forms* $\gamma_n(s+s_n,t)$ *converge in* C^{∞}_{loc} *to* 0.

2) *The left shifts* $u_n^-(s,t) := u_n(s-R_n,t)$, *defined on* $[0,h_n)\times S^1$, *converge in* C^{∞}_{loc} *to* $u^- = (a^-, f^-)$ *defined on* $[0,+\infty)\times S^1$ *and*

$$\lim_{s\to\infty} u^-(s,\cdot) = (\sigma, \phi^{\alpha}_{\tau}(w_f)) \quad in \quad C^{\infty}(S^1, \mathbb{R}\times M).$$

The left shifted harmonic $1-$*forms* γ_n^- *converge in* C^{∞}_{loc} *to* $d\Gamma^-$ *defined on* $[0,+\infty)\times S^1$ *with* $\|d\Gamma^-\|^2_{L^2([0,\infty)\times S^1)} \leq C_0$.

3) *The right shifts* $u_n^+(s,t) := u_n(s+R_n,t)$ *defined on* $(-h_n,0]\times S^1$, *converge in* C^{∞}_{loc} *to* $u^+ = (a^+, f^+)$, *defined on* $(-\infty,0]\times S^1$ *and*

$$\lim_{s\to-\infty} u^+(s,\cdot) = (-\sigma, \phi^{\alpha}_{-\tau}(w_f)) \quad in \quad C^{\infty}(S^1, \mathbb{R}\times S^1).$$

The right shifted harmonic $1-$*forms* γ_n^+ *converge in* C^{∞}_{loc} *to* $d\Gamma^+$ *defined on* $(-\infty,0]\times S^1$ *with* $\|d\Gamma^+\|^2_{L^2((-\infty,0]\times S^1)} \leq C_0$.

C^0-*convergence: For every sequence of diffeomorphisms* $\theta_n : [-R_n,R_n]\times S^1 \to [-1,1]\times S^1$ *as in Remark 27*

1) *The maps* $v_n : [-1/2,1/2]\times S^1 \to \mathbb{R}\times M$ *defined by* $v_n = u_n\circ\theta_n^{-1}$, *converge in* C^0 *to the map* $(s,t)\mapsto(-2\sigma s, \phi^{\alpha}_{-2\tau s}(w_f))$.

2) *The maps* $v_n^- : [-1,-1/2]\times S^1 \to \mathbb{R}\times M$ *defined by* $v_n^- = u_n\circ(\theta_n^-)^{-1}$, *converge in* C^0 *to a map* $v^- : [-1,-1/2]\times S^1 \to \mathbb{R}\times M$ *such that* $v^- = u^-\circ(\theta^-)^{-1}$ *and* $v^-(-1/2,t) = (\sigma, \phi^{\alpha}_{\tau}(w_f))$.

3) *The maps* $v_n^+ : [1/2,1]\times S^1 \to \mathbb{R}\times M$ *defined by* $v_n^+ = u_n\circ(\theta_n^+)^{-1}$, *converge in* C^0 *to a map* $v^+ : [1/2,1]\times S^1 \to \mathbb{R}\times M$ *such that* $v^+(s,t) = u^+\circ(\theta^+)^{-1}$ *and* $v^+(1/2,t) = (-\sigma, \phi^{\alpha}_{-\tau}(w_f))$.

Definition 29. Let $(r_-,r_+)\subset\mathbb{R}$ be an interval and $u,v : (r_-,r_+)\times S^1 \to \mathbb{R}\times M$ be smooth maps. We say u and v have *the same asymptotics as* $s\to r_{\pm}$, if for some (or equivalently, any) metric $d_{C^{\infty}}$ metrizing the Whitney C^{∞}-topology on $C^{\infty}(S^1,\mathbb{R}\times M)$, which is invariant under the $\mathbb{R}$-action on $\mathbb{R}\times M$, we have

$$\lim_{s\to r_{\pm}} d_{C^{\infty}}(u(s,\cdot), v(s,\cdot)) = 0.$$

Theorem 28 concerns $\mathcal{H}$-holomorphic curves with vanishing center action. If instead the maps u_n (after passing to a subsequence) have all positive center action, we have the following.

Theorem 30. *Let $u_n : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ be a sequence of $\mathcal{H}-$holomorphic cylinders with harmonic perturbations γ_n satisfying C1-C9. Assume that $R_n \to \infty$ and that all the u_n have positive center action.*

Then there exist a subsequence of u_n (for each n suitably shifted in the $\mathbb{R}-$coordinate), still denoted by u_n, $\mathcal{H}-$holomorphic half cylinders $u^\pm$ with exact harmonic perturbations $d\Gamma^\pm$ defined on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$ respectively, $T \in \mathbb{R} \setminus \{0\}$ and a $|T|$-periodic Reeb orbit x and $\sigma, \tau \in \mathbb{R}$ such that for every sequence $h_n \in \mathbb{R}_{>0}$ with $h_n, R_n/h_n \to \infty$, the following convergence results hold: $\sigma_n := S_n R_n \to \sigma, \tau_n := P_n R_n \to \tau$ and

$C^\infty_{loc}-$convergence:

1) *For any sequence $s_n \in [-R_n + h_n, R_n - h_n]$ for which s_n/R_n converges to $\kappa \in [-1, 1]$, the shifted maps $u_n(s + s_n, t) - Ts_n$, defined on $[-R_n + h_n - s_n, R_n - h_n - s_n] \times S^1$, converge in C^∞_{loc} on $\mathbb{R} \times S^1$ to the map $(s, t) \to (Ts - \sigma\kappa, \phi^\alpha_{-\tau\kappa}(x(Tt)) = x(Tt - \tau\kappa))$ on $\mathbb{R} \times S^1$. The shifted harmonic $1-$forms $\gamma_n(s + s_n, t)$ converge in C^∞_{loc} to 0.*
2) *The left shifts $u^-_n(s, t) := u_n(s - R_n, t) + TR_n$, defined on $[0, h_n) \times S^1$, converge in C^∞_{loc} to $u^- = (a^-, f^-)$ defined on $[0, +\infty) \times S^1$. The curve u^- has the same asymptotics as $(Ts + \sigma, \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau))$ as $s \to \infty$. The left shifted harmonic $1-$forms γ^-_n converge in C^∞_{loc} to $d\Gamma^-$ defined on $[0, +\infty) \times S^1$ with $\|d\Gamma^-\|^2_{L^2([0,\infty)\times S^1)} \leq C_0$.*
3) *The right shifts $u^+_n(s, t) := u(s + R_n, t) - TR_n$, defined on $(-h_n, 0] \times S^1$ converge in C^∞_{loc} to $u^+ = (a^+, f^+)$, defined on $(-\infty, 0] \times S^1$. The curve u^+ has the same asymptotics as $(Ts - \sigma, \phi^\alpha_{-\tau}(x(Tt)) = x(Tt - \tau))$ as $s \to -\infty$. The right shifted harmonic $1-$forms γ^+_n converge in C^∞_{loc} to $d\Gamma^+$ defined on $(-\infty, 0] \times S^1$ with $\|d\Gamma^+\|^2_{L^2((-\infty,0]\times S^1)} \leq C_0$.*

C^0-convergence: For every sequence of diffeomorphisms $\theta_n : [-R_n, R_n] \times S^1 \to [-1, 1] \times S^1$ as in Remark 27

1) *The maps $f_n \circ \theta_n^{-1} : [-1/2, 1/2] \times S^1 \to M$ converge in C^0 to*

$$\phi^\alpha_{-2\tau s}(x(Tt)) = x(Tt - 2\tau s).$$

2) *The maps $f^-_n \circ (\theta^-_n)^{-1} : [-1, -1/2] \times S^1 \to M$ converge in C^0 to a map $f^- \circ (\theta^-)^{-1} : [-1, -1/2] \times S^1 \to M$ such that*

$$f^-((\theta^-)^{-1}(-1/2), t) = \phi^\alpha_\tau(x(Tt)) = x(Tt + \tau).$$

3) *The maps* $f_n^+ \circ (\theta_n^+)^{-1} : [1/2, 1] \times S^1 \to M$ *converge in* C^0 *to a map* $f^+ \circ (\theta^+)^{-1} : [1/2, 1] \times S^1 \to M$ *such that*

$$f^+((\theta^+)^{-1}(1/2), t) = \phi^\alpha_{-\tau}(x(Tt)) = x(Tt - \tau).$$

4) *Set* $r_n^- := \inf_{t \in S^1} a_n(-sgn(T)R_n, t)$ *and* $r_n^+ := \sup_{t \in S^1} a_n(sgn(T)R_n, t)$ *where* $sgn(T) := T/|T| \in \{\pm 1\}$.

   Then $r_n^+ - r_n^- \to \infty$ *and for every* $R > 0$ *there exists* $\rho > 0$ *and* $N \in \mathbb{N}$ *such that* $a_n \circ \theta_n^{-1}(s, t) \in [r_n^- + R, r_n^+ - R]$ *for all* $n \geq N$ *and all* $(s, t) \in [-\rho, \rho] \times S^1$.

Since $\theta^- : [0, \infty) \times S^1 \to [-1, -1/2) \times S^1$ is a biholomorphism with respect to the standard complex structrue i on the domain and the pull-back structure $\tilde{i} := [(\theta^-)^{-1}]^* i$, we can identify $[-1, -1/2) \times S^1$ with the punctured disk equipped with the standard complex structure, that extends over the puncture.

We use now Theorems 28 and 30 to describe the limit object.

In Case 1, the "limit surface" in the symplectization consists of two disks which are connected by a straight line at the origin. The limit map $u = (a, f) : [-1, 1] \times S^1 \to \mathbb{R} \times M$ with the limit perturbation $1-$form γ can be described as follows (see Figure 3.5).

D1: On $[-1, -1/2) \times S^1$, u is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ such that at the puncture it is asymptotic to $(\sigma, \phi^\alpha_\tau(w_f))$, while the harmonic perturbation is asymptotic to a constant.

D2: On $(1/2, 1] \times S^1$, u is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ such that at the puncture it is asymptotic to $(-\sigma, \phi^\alpha_{-\tau}(w_f))$, while the harmonic perturbation is asymptotic to a constant.

D3: On the middle part $[-1/2, 1/2] \times S^1$, u is given by $u(s, t) = (-2\sigma s, \phi^\alpha_{-2\tau s}(w_f))$. On this part the $1-$form γ is not defined.

In Case 2, the limit surface is the disjoint union of the cylinders $[-1, -1/2) \times S^1$ and $(1/2, 1] \times S^1$. The $\mathcal{H}-$holomorphic curve $u = (a, f) : ([-1, -1/2) \coprod (1/2, 1]) \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ can be described as follows.

D1': u is asymptotic on $[-1, -1/2) \times S^1$ and $(1/2, 1] \times S^1$ to a trivial cylinder over the Reeb orbit $x(Tt + \tau)$ or $x(Tt - \tau)$, respectively, while the harmonic perturbation is asymptotic to a constant.

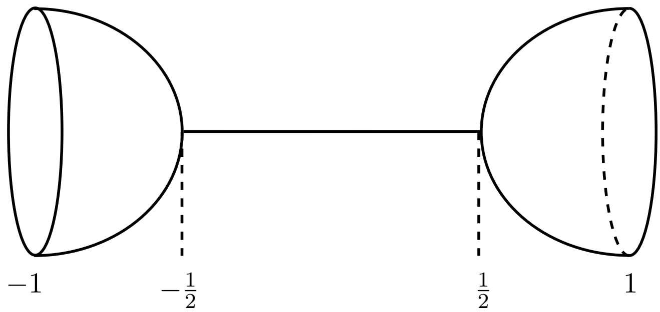

Figure 3.5: The limit surface consists of two cones connected by a straight line.

D2': On the middle part $[-1/2, 1/2] \times S^1$, the $M-$component f is given by $f(s,t) = x(Tt - 2\tau s)$.

3.2.3. Cylinders of type b_1. We analyze the convergence on cylinders of type b_1 by using the results of Appendix A. Let $m \in \{1, \dots, N\}$ be such that the cylinders $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$ are of type b_1. By the construction described in the previous section and Lemma 25, the $\mathcal{H}-$holomorphic curves have uniform gradient bounds on the two boundary cylinders $[h_n^{(m-1)} - 3h, h_n^{(m-1)}] \times S^1$ and $[h_n^{(m)}, h_n^{(m)} + 3h] \times S^1$.

The convergence analysis is organized as follows. As in Section 3.1 we apply bubbling-off analysis on the cylinder $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ to show that on any compact set in the complement of a finite number of points $\mathcal{Z}^{(m)}$ in $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1$, the gradient of u_n is uniformly bounded. The points on which the gradient might blow up are located in $(h_n^{(m-1)} - h, h_n^{(m)} + h) \times S^1$. Each resulting puncture from $\mathcal{Z}^{(m)}$ lies in a disk D_r of radius r smaller than $h/2$. For a smaller radius r, we assume that all disks D_r are pairwise disjoint and that their union lies in $(h_n^{(m-1)} - h, h_n^{(m)} + h) \times S^1$ (see Figure 3.6).

Under these assumptions, the $\mathcal{H}-$holomorphic curves converge in C^∞ on the complement of the union of these disks (centered at the punctures) to a $\mathcal{H}-$holomorphic curve. What is left to prove is the convergence in each D_r; for this we use the results of Appendix A. In the final step, we glue the convergence results on the disks to the rest of the cylinder, and obtain the desired description on the entire cylinder of type b_1.

Under the biholomorphic map $[h_n^{(m-1)} - 3h, h_n^{(m)} + 3h] \times S^1 \to [0, H_n^{(m)}] \times S^1, (s,t) \mapsto (s - h_n^{(m-1)} + 3h, t)$, where $H_n^{(m)} := h_n^{(m)} - h_n^{(m-1)} + 6h$, assume

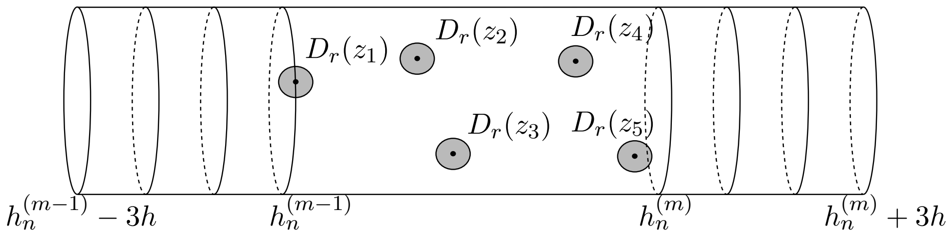

Figure 3.6: The gradient might blow up on the discs $D_r(z_i)$ contained in $(h_n^{(m-1)} - h, h_n^{(m)} + h) \times S^1$.

that the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n are defined on $[0, H_n^{(m)}] \times S^1$. By going over to a subsequence, we have $H_n^{(m)} \to H^{(m)}$ as $n \to \infty$. Consider the translated $\mathcal{H}-$holomorphic curves $u_n - a_n(0,0) = (a_n, f_n) - a_n(0,0) : [0, H_n^{(m)}] \times S^1 \to \mathbb{R} \times M$ with harmonic perturbations γ_n. In order to keep the notation simple, let the curve $u_n - a_n(0,0)$ be still denoted by u_n. The analysis is performed in the following setting:

E1: The maps $u_n = (a_n, f_n)$ are $\mathcal{H}-$holomorphic curves with harmonic perturbation γ_n on $[0, H_n^{(m)}] \times S^1$ with respect to the standard complex structure i on the domain and the almost complex structure J on ξ.

E2: The maps u_n have uniformly bounded energies, while the harmonic perturbations γ_n have uniformly bounded L^2-norms, i.e., with the constants $E_0, C_0 > 0$ we have

$$E(u_n; [0, H_n^{(m)}] \times S^1) \leq E_0$$

and

$$\|\gamma_n\|^2_{L^2([0,H_n^{(m)}]\times S^1)} \leq C_0$$

for all $n \in \mathbb{N}$.

E3: The maps u_n have uniformly bounded gradients on $[0, 3h] \times S^1$ and $[H_n^{(m)} - 3h, H_n^{(m)}] \times S^1$ with respect to the Euclidean metric on the domain and the cylindrical metric on the target space $\mathbb{R} \times M$, i.e.

$$\|du_n(z)\| = \sup_{\|v\|_{\text{eucl.}}=1} \|du_n(z)v\|_{\overline{g}} < C_h$$

for all $z \in ([0, 3h] \cup [H_n^{(m)} - 3h, H_n^{(m)}]) \times S^1$ and $n \in \mathbb{N}$.

The next lemma states the existence of a finite set $\mathcal{Z}^{(m)}$ of punctures on which the gradient of u_n blows up.

Lemma 31. *There exists a finite set of points $\mathcal{Z}^{(m)} \subset [3h, H_n^{(m)} - 3h] \times S^1$ such that for any compact subset $\mathcal{K} \subset ([0, H_n^{(m)}] \times S^1) \backslash \mathcal{Z}^{(m)}$ there exists a constant $C_{\mathcal{K}} > 0$ such that*

$$\|du_n(z)\| = \sup_{\|v\|_{eucl.}=1} \|du_n(z)v\|_{\overline{g}} < C_{\mathcal{K}}$$

for all $z \in \mathcal{K}$ and $n \in \mathbb{N}$.

Proof. The proof relies on the same arguments of bubbling-off analysis, which have been employed in Theorem 19 from Section 3.1 for the thick part. □

Pick some $r > 0$ such that $r < h/2$, and let $D_r(\mathcal{Z}^{(m)})$ consists of $|\mathcal{Z}^{(m)}|$ pairwise disjoint closed disks of radius $r > 0$, centered at the punctures of $\mathcal{Z}^{(m)}$. Obviously, $D_r(\mathcal{Z}^{(m)}) \subset (2h, H_n^{(m)} - 2h) \times S^1$. Then by Lemma 31, u_n has a uniformly bounded gradient on $([0, H_n^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})$. As $([0, H_n^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})$ is connected, we assume, after going over to some subsequence, that $u_n|_{([0,H_n^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})}$ converge in C^∞ to some smooth map $u|_{([0,H^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})} = (a, f)|_{([0,H^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})}$. Before treating the convergence of the $\mathcal{H}-$holomorphic curves in a neighbourhood of the punctures of $\mathcal{Z}^{(m)}$, we establish the convergence of the harmonic perturbations γ_n on $[0, H_n^{(m)}] \times S^1$, so that at the end

- $u_n|_{([0,H_n^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})}$ converge in C^∞ to a $\mathcal{H}-$holomorphic curve $u|_{([0,H^{(m)}] \times S^1) \backslash D_r(\mathcal{Z}^{(m)})}$, and
- the harmonic perturbations γ_n have uniformly bounded C^k-norms on the disks $D_r(\mathcal{Z}^{(m)})$ for all $k \in \mathbb{N}_0$.

The latter result is needed to describe the convergence of the harmonic perturbations γ_n on the disks $D_r(\mathcal{Z}^{(m)})$. As in the previous section, we set $\gamma_n = \zeta_n ds + \chi_n dt$, where ζ_n and χ_n are harmonic functions defined on $[0, H_n^{(m)}] \times S^1$ such that $\zeta_n + i\chi_n$ are holomorphic. By the uniform L^2-bound of γ_n it follows that

$$\|\gamma_n\|^2_{L^2([0,H_n^{(m)}] \times S^1)} = \int_{[0,H_n^{(m)}] \times S^1} \left(\zeta_n^2 + \chi_n^2\right) dsdt \leq C_0$$

for all $n \in \mathbb{N}$, and so, that the L^2-norms of the holomorphic functions $\zeta_n + i\chi_n$ are uniformly bounded. Letting $G_n = \zeta_n + i\chi_n$ we state the following

Proposition 32. *There exists a subsequence of G_n, also denoted by G_n, that converges in C^∞ to some holomorphic map G defined on $[0, H^{(m)}] \times S^1$. Moreover, the harmonic perturbations γ_n converge in C^∞ to a harmonic map γ.*

Proof. By Proposition 24, G_n has a uniformly bounded C^1-norm, while by the standard regularity results from the theory of pseudoholomorphic curves (see, for example, Section 2.2.3 of [6]), the C^k derivatives of G_n are also uniformly bounded. Hence, in view of Arzelá-Ascoli theorem, we can extract a subsequence that converges to some holomorphic function G. □

Let us analyze the convergence of the $\mathcal{H}-$holomorphic curves in a neighbourhood of the punctures of $\mathcal{Z}^{(m)}$, which are given by Lemma 31. For $r > 0$ as above and $z \in \mathcal{Z}^{(m)}$, consider the closed disks $D_r(z)$ and the $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : D_r(z) \to \mathbb{R} \times M$ with harmonic perturbations γ_n that converge in C^∞ to some harmonic $1-$form γ. According to the biholomorphism $D \to D_r(z)$, $p \mapsto rp + z$, where D is the standard closed unit disk, regard the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations as being defined on D instead of $D_r(z)$. The following setting is pertinent to our analysis:

- **F1:** The maps $u_n = (a_n, f_n) : D \to \mathbb{R} \times M$ are $\mathcal{H}-$holomorphic curves with harmonic perturbations γ_n with respect to the standard complex structure i on D and the almost complex structure J on ξ.
- **F2:** The maps $u_n = (a_n, f_n)$ and γ_n have uniformly bounded energies and L^2-norms.
- **F3:** For any constant $1 > \tau > 0$, $u_n|_{A_{1,\tau}} = (a_n, f_n)|_{A_{1,\tau}}$ converge in C^∞ to a $\mathcal{H}-$holomorphic map with harmonic perturbation γ, where $A_{1,\tau} = \{z \in D \mid \tau \leq |z| \leq 1\}$.

As the domain of definition D is simply connected, we infer that γ_n is exact, i.e. it can be written as $\gamma_n = d\tilde{\Gamma}_n$, where $\tilde{\Gamma}_n : D \to \mathbb{R}$ is a harmonic function. By Condition F2, $\tilde{\Gamma}_n$ has a uniformly bounded gradient $\nabla\tilde{\Gamma}_n$ in the L^2-norm, and it is apparent that the existence of $\tilde{\Gamma}_n$ is unique up to addition by a constant. Let us make some remarks on the choice of $\tilde{\Gamma}_n$ and discuss some of its properties. By using the mean value theorem for harmonic functions as in Proposition 24 we conclude (after eventually, shrinking D)

that the gradient $\nabla\tilde{\Gamma}_n$ are uniformly bounded in C^0. Denote by $z = s + it$ the coordinates on D, and let

$$K_n = \frac{1}{\pi}\int_D \tilde{\Gamma}_n(s,t)dsdt$$

be the mean value of $\tilde{\Gamma}_n$, so that by the mean value theorem for harmonic functions, $K_n = \tilde{\Gamma}_n(0)$. Finally, define the map $\Gamma_n(z) := \tilde{\Gamma}_n(z) - \tilde{\Gamma}_n(0)$ which obviously satisfies $\gamma_n = d\Gamma_n$.

Remark 33. From Poincaré inequality it follows that $\|\Gamma_n\|_{L^2(D)} \leq c\|\nabla\tilde{\Gamma}_n\|_{L^2(D)}$ for some constant $c > 0$ and so, that Γ_n is uniformly bounded in L^2-norm. Again, by using the mean value theorem for harmonic functions, we deduce (after maybe shrinking D) that Γ_n has a uniformly bounded C^0-norm, and consequently, that Γ_n has a uniformly bounded C^1-norm. Because $\gamma_n = d\Gamma_n$ is a harmonic $1-$form, $\partial_s\Gamma_n + i\partial_t\Gamma_n$ is a holomorphic function. In this context, by Proposition 24, Γ_n converge in C^∞ to a harmonic function $\Gamma : D \to \mathbb{R}$.

In the following we transform the $\mathcal{H}-$holomorphic curves defined on the disk to a usual pseudoholomorphic curve by encoding the harmonic perturbation $\gamma_n = d\Gamma_n$ in the $\mathbb{R}-$coordinate of the $\mathcal{H}-$holomorphic curve u_n. Specifically, we define the maps $\overline{u}_n = (\overline{a}_n, \overline{f}_n) = (a_n + \Gamma_n, f_n)$ which are obviously pseudoholomorphic. The transformation is usable if we ensure that the energy bounds are still satisfied. For an ordinary pseudoholomorphic curve, the sum of the $\alpha-$ and $d\alpha-$energies, that are both positive, yield the Hofer energy $E_{\mathrm{H}}(\overline{u}_n; D)$. A uniform bound on the Hofer energy, which ensures a uniform bound on the $\alpha-$ and $d\alpha-$energies of $\overline{u}_n$, is

$$E_{\mathrm{H}}(\overline{u}_n; D) = \sup_{\varphi\in\mathcal{A}}\int_D \overline{u}_n^* d(\varphi\alpha) = \sup_{\varphi\in\mathcal{A}}\int_{\partial D}\varphi(\overline{a}_n)\overline{f}_n^*\alpha \leq \int_{\partial D}|\overline{f}_n^*\alpha| \leq C_h.$$

Here, the last inequality follows from Condition F3, according to which, u_n converge in C^∞ in a fixed neighbourhood of ∂D. Note that the constant C_h is guaranteed by Lemma 31.

In a next step we use the results of Appendix A to establish the convergence of the maps $\overline{u}_n$ and to describe their limit object. Then we undo the transformation in the $\mathbb{R}-$coordinate (more precisely, the encoding of γ_n in the $\mathbb{R}-$coordinate of the curve u_n) and give a convergence result together with a description of the limit object for u_n. Before proceeding we state the setting corresponding to the pseudoholomorphic curves $\overline{u}_n$.

G1: The maps $\overline{u}_n = (\overline{a}_n, \overline{f}_n) : D \to \mathbb{R} \times M$ solve the pseudoholomorphic curve equation

$$\begin{aligned} \pi_\alpha d\overline{f}_n \circ i &= J(\overline{f}_n) \circ \pi_\alpha d\overline{f}_n, \\ \overline{f}_n^* \alpha \circ i &= d\overline{a}_n \end{aligned}$$

on D.

G2: The maps $\overline{u}_n$ have uniformly bounded energies.

G3: For any $\tau > 0$, $\overline{u}_n|_{A_{1,\tau}} = (\overline{a}_n, \overline{f}_n)|_{A_{1,\tau}}$ converge in C^∞ to a pseudoholomorphic map.

We consider two cases. In the first case, the $\mathbb{R}-$components of $\overline{u}_n$ are uniformly bounded, while in the second case they are not. Actually, the first case does not occur. We will prove this result in the next lemma by using standard bubbling-off analysis. Let $z_n \in D$ be the sequence choosen from the bubbling-off argument of Lemma 31, i.e. for which we have that

$$\|d\overline{u}_n(z_n)\|_{C^0} = \sup_{z \in D} \|d\overline{u}_n(z)\|_{C^0} \to \infty \tag{3.3}$$

as $n \to \infty$.

Lemma 34. *The $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are unbounded on D.*

Proof. We prove by contradiction using bubbling-off analysis. Assume that the $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are uniformly bounded. Employing the same arguments as in the proof of Lemma 25 for the sequence $R_n := \|d\overline{u}_n(z_n)\|_{C^0}$, we find that the maps $v_n : D_{\epsilon'_n R'_n}(0) \to \mathbb{R} \times M$ converge in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to a non-constant finite energy holomorphic plane v. Note that the boundedness of $E_{d\alpha}(v; \mathbb{C})$ follows from the fact that for an arbitray $R > 0$ we have

$$E_{d\alpha}(v, D_R(0)) = \lim_{n \to \infty} E_{d\alpha}(v_n; D_R(0)) \le \lim_{n \to \infty} E_{d\alpha}(v_n; D_{\epsilon'_n R'_n}(0)) \le C_h,$$

yielding $E_{d\alpha}(v; \mathbb{C}) \le C_h$. As we have assumed that the $\mathbb{R}-$coordinates of $\overline{u}_n$ are uniformly bounded it follows that the $\mathbb{R}-$coordiantes of v_n, and so, of v, are also uniformly bounded. By singularity removal, v can be extended to a pseudoholomorphic sphere. Thus the $d\alpha-$energy vanishes and by the maximum principle, the function a is constant. For this reason, v must be constant and we are led to a contradiction. ☐

We consider now the second case in which the $\mathbb{R}-$coordinates of the maps $\overline{u}_n$ are unbounded, and make extensive use of the results of Appendix A. By the maximum principle, the function $\overline{a}_n$ tends to $-\infty$, while by Proposition 42, the maps $\overline{u}_n = (\overline{a}_n, f_n) : (D, i) \to \mathbb{R} \times M$ converge to a broken holomorphic curve $\overline{u} = (\overline{a}, \overline{f}) : (Z, j) \to \mathbb{R} \times M$. Here, Z is obtained as follows. Let Z be a surface diffeomorphic to D, and let $\Delta = \Delta_n \amalg \Delta_p \subset Z$ be a collection of finitely many disjoint loops away from ∂Z. Furthermore, let $Z \backslash \Delta_p = \coprod_{\nu=0}^{N+1} Z^{(\nu)}$ for some $N \in \mathbb{N}$ as described in Appendix A. For a loop $\delta \in \Delta_p$, there exists $\nu \in \{0, \ldots, N\}$ such that δ is adjacent to $Z^{(\nu)}$ and $Z^{(\nu+1)}$. Fix an embedded annuli

$$A^{\delta,\nu} \cong [-1, 1] \times S^1 \subset Z \backslash \Delta_n$$

such that $\{0\} \times S^1 = \delta$, $\{-1\} \times S^1 \subset Z^{(\nu)}$, and $\{1\} \times S^1 \subset Z^{(\nu+1)}$. In this context, there exist a sequence of diffeomorphism $\varphi_n : D \to Z$ and a sequence of negative real numbers $\min(a_n) = r_n^{(0)} < r_n^{(1)} < \cdots < r_n^{(N+1)} = -K-2$, where $K \in \mathbb{R}$ is the constant determined in Appendix A and $r_n^{(\nu+1)} - r_n^{(\nu)} \to \infty$ as $n \to \infty$ such that the following hold:

H1: $i_n := (\varphi_n)_* i \to j$ in C^∞_{loc} on $Z \backslash \Delta$.

H2: The sequence $\overline{u}_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ converges in C^∞_{loc} on $Z^{(\nu)} \backslash \Delta_n$ to a punctured nodal pseudoholomorphic curve $\overline{u}^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$, and in C^0_{loc} on $Z^{(\nu)}$.

H3: The sequence $\overline{f}_n \circ \varphi_n^{-1} : Z \to M$ converges in C^0 to a map $f : Z \to M$, whose restriction to Δ_p parametrizes the Reeb orbits and to Δ_n parametrizes points.

H4: For any $S > 0$, there exist $\rho > 0$ and $\tilde{N} \in \mathbb{N}$ such that $\overline{a}_n \circ \varphi_n^{-1}(s, t) \in [r_n^{(\nu)} + S, r_n^{(\nu+1)} - S]$ for all $n \geq \tilde{N}$ and all $(s, t) \in A^{\delta,\nu}$ with $|s| \leq \rho$.

To establish a convergence result for the $\mathcal{H}-$holomorphic curve u_n we undo the tranformation. The maps u_n are given by $u_n = \overline{u}_n - \Gamma_n$, where $\Gamma_n : D \to \mathbb{R}$ is the harmonic function defined in Remark 33. Observe that by Remark 33, the Γ_n converge in $C^\infty(D)$ to some harmonic function and are uniformly bounded in $C^0(D)$. Via the above diffeomorphisms $\varphi_n : D \to Z$, consider the functions $\mathscr{G}_n := \Gamma_n \circ \varphi_n^{-1} : Z \to \mathbb{R}$. Since Γ_n are harmonic functions with respect to i, $\mathscr{G}_n$ are harmonic functions on Z with respect to i_n. Moreover, their gradients and absolute values are bounded in L^2- and C^0-norms, respectively, i.e.

$$\int_Z d\mathscr{G}_n \circ i_n \wedge d\mathscr{G}_n \leq C_0 \tag{3.4}$$

and

$$\|\mathscr{G}_n\|_{C^0(Z)} \leq C_1 \tag{3.5}$$

for some constant $C_1 > 0$ and for all $n \in \mathbb{N}$, respectively.

Lemma 35. *For any compact subset $\mathcal{K} \subset (Z \backslash \Delta)$ there exists a subsequence of $\mathscr{G}_n$, also denoted by $\mathscr{G}_n$, such that $\mathscr{G}_n \to \mathscr{G}$ in $C^\infty(\mathcal{K})$ as $n \to \infty$, where $\mathscr{G}$ is a harmonic function defined on a neighbourhood of $\mathcal{K}$.*

Proof. Let $\mathcal{K} \subset (Z \backslash \Delta)$ be a compact subset. By Lemma 47 there exists a finite covering of $\mathcal{K}$ by the charts $\psi_n^{(l)} : D \to U_n^{(l)}$ and $\psi^{(l)} : D \to U^{(l)}$, where $l \in \{1, \dots, N\}$ and $N \in \mathbb{N}$. For some $r \in (0,1)$, the following hold:

1) $\psi_n^{(l)}$ are $i - i_n-$biholomorphisms and $\psi^{(l)}$ is an $i - j-$biholomorphism;
2) $\psi_n^{(l)} \to \psi^{(l)}$ in $C^\infty_{\text{loc}}(D)$ as $n \to \infty$;
3) $\mathcal{K} \subset \bigcup_{l=1}^N \psi_n^{(l)}(D_r(0))$ for all $n \in \mathbb{N}$, and $\mathcal{K} \subset \bigcup_{l=1}^N \psi^{(l)}(D_r(0))$.

Consider the function $\mathscr{G}_n^{(l)} := \mathscr{G}_n \circ \psi_n^{(l)} : D \to \mathbb{R}$ for some $l \in \{1, \dots, N\}$. Because $\psi_n^{(l)}$ are $i - i_n-$biholomorphisms, $\mathscr{G}_n^{(l)}$ is a harmonic function with respect to i. From (3.4) and (3.5), $\mathscr{G}_n^{(l)}$ satisfies

$$\int_D d\mathscr{G}_n^{(l)} \circ i \wedge d\mathscr{G}_n^{(l)} \leq C_0$$

and $\left\|\mathscr{G}_n^{(l)}\right\|_{C^0(D)} \leq C_1$. Relying on the compactness result for harmonic functions we assume that $\mathscr{G}_n^{(l)}$ converges in $C^0(D_{3r/2}(0))$ to a harmonic function $\mathscr{G}^{(l)}$ defined on $D_{3r/2}(0)$. By the mean value theorem for harmonic functions, there exists a constant $c > 0$ such that $\left\|\nabla\mathscr{G}_n^{(l)}\right\|_{C^0(D_{4r/3}(0))} \leq c$ for all $n \in \mathbb{N}$. Hence $\mathscr{G}_n^{(l)}$ is uniformly bounded in $C^1(D_{4r/3}(0))$. Because $d\mathscr{G}_n^{(l)}$ defines a harmonic $1-$form, $\partial_s\mathscr{G}_n^{(l)} + i\partial_t\mathscr{G}_n^{(l)}$ is a uniformly bounded holomorphic function defined on $D_{4r/3}(0)$, where s, t are the coordinates on $D_{4r/3}(0)$. By means of the Cauchy integral formula, all derivatives of $\partial_s\mathscr{G}_n^{(l)} + i\partial_t\mathscr{G}_n^{(l)}$ are uniformly bounded on $D_{5r/4}(0)$. From this and the fact that $\mathscr{G}_n^{(l)}$ converges uniformly to $\mathscr{G}^{(l)}$ we deduce that there exists a further subsequence, also denoted by $\mathscr{G}_n^{(l)}$, that converges in $C^\infty(D_{6r/5}(0))$ to a harmonic function $\mathscr{G}^{(l)} : D_{6r/5}(0) \to \mathbb{R}$. For n sufficiently large, $\psi^{(l)}(\overline{D_r(0)}) \subset$

$\psi^{(l)}(D_{6r/5}(0))$ and $\psi^{(l)}(\overline{D_r(0)}) \subset \psi_n^{(l)}(D_{6r/5}(0))$. Hence the harmonic function $\mathscr{G}_n = \mathscr{G}_n^{(l)} \circ (\psi_n^{(l)})^{-1} : \psi^{(l)}(\overline{D_r(0)}) \to \mathbb{R}$ converges in $C^\infty(\psi^{(l)}(\overline{D_r(0)}))$ to a harmonic function $\tilde{\mathscr{G}}^{(l)} := \mathscr{G}^{(l)} \circ (\psi^{(l)})^{-1} : \psi^{(l)}(\overline{D_r(0)}) \to \mathbb{R}$. Obviously, if $l, l' \in \{1, \dots, N\}$ are such that $\psi^{(l)}(D_r(0)) \cap \psi^{(l')}(D_r(0)) \neq \emptyset$, the uniqueness of the limit yields $\tilde{\mathscr{G}}^{(l)}|_{\psi^{(l)}(D_r(0)) \cap \psi^{(l')}(D_r(0))} = \tilde{\mathscr{G}}^{(l')}|_{\psi^{(l)}(D_r(0)) \cap \psi^{(l')}(D_r(0))}$. Hence all $\tilde{\mathscr{G}}^{(l)}$ glue together to a harmonic function defined in a neighbourhood of $\mathcal{K}$. □

By Lemma 35 it is apparent that after going over to a diagonal subsequence, $\mathscr{G}_n$ converges in $C^\infty_{\text{loc}}(Z \backslash \Delta)$ to a harmonic function $\mathscr{G} : Z \backslash \Delta \to \mathbb{R}$ with respect to j. This shows that the $\mathcal{H}-$holomorphic curve $u_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ with harmonic perturbation $d\mathscr{G}_n$ converges in C^∞_{loc} on $Z^{(\nu)} \backslash \Delta_n$ to a $\mathcal{H}-$holomorphic curve $u^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$ with harmonic perturbation $d\mathcal{G}$, where $u^{(\nu)} = \overline{u}^{(\nu)} - \mathscr{G}$ for all ν. What is left is the description of the convergence of the $\mathcal{H}-$holomorphic curves $u_n \circ \varphi_n^{-1}$ with harmonic perturbation $d\mathscr{G}_n$ in a neighbourhood of the loops from Δ_n, i.e. accross the nodes from Δ_n. Observe that, from (3.5), $\mathscr{G}_n$ is uniformly bounded on Z by the constant C_1 and the L^2-norm of $d\mathscr{G}_n$ is uniformly bounded by the constant C_0. A neighbourhood C_n of a loop in Δ_n can be biholomorphically parametrized as $[-r_n, r_n] \times S^1$ by the biholomorphism $\psi_n : [-r_n, r_n] \times S^1 \to C_n$, where $r_n \to \infty$ as $n \to \infty$. From the C^0 bound of $\mathscr{G}_n$ on Z, the maps $u_n \circ \varphi_n^{-1}$ are uniformly bounded in C^0 on C_n (maybe after some shift in the $\mathbb{R}-$coordinaten). Thus we consider the $\mathcal{H}-$holomorphic cylinder $u_n \circ \varphi_n^{-1} \circ \psi_n$ with harmonic perturbation $\psi_n^* d\mathscr{G}_n$ defined on $[-r_n, r_n] \times S^1$. Note that the energy of $u_n \circ \varphi_n^{-1} \circ \psi_n$ is uniformly bounded by the constant E_0. As in Section 3.2 we divide the cylinder $[-r_n, r_n] \times S^1$ into cylinders of type ∞ with an energy less than $\hbar_0/2$ and cylinders of type b_1. We apply the result of Section 3.2.2 to cylinders of type ∞. Keep in mind that according to Remark 26), conditions C1-C9 are satisfied. For cylinders of type b_1, the maps $u_n \circ \varphi_n^{-1} \circ \psi_n$, after a specific shift in the $\mathbb{R}-$coordinate, are contained in a compact subset of $\mathbb{R} \times M$. By the usual bubbling-off analysis and the maximum principle, these maps together with the harmonic perturbation converge in C^∞ on cylinders of type b_1.

We “glue” the convergence result for the ∞-type subcylinders of $[-r_n, r_n] \times S^1$ introduced in Section 3.2.2 together with the $C^\infty-$convergence result for the cylinders of type b_1. This process is similar to that described in Section 3.2.4. However, we are faced with a more simple situation because we can choose the surface $C^{(m)}$ as the cylinder $[-1 - 4h, 1 + 4h] \times S^1$, where h is the constant defined in Section 3.2.1. By the method described in Section 3.2.4, the diffeomorphism φ_n, the surface Z, and the set of nodes Δ_n

having properties H1-H4 are replaced by some modified versions, which are still denoted by φ_n, Z and Δ_n. More precisely, the loops from Δ_n are the center loops that correspond to subcylinders of type ∞ in the decomposition of the components from the thick part which are conformal equivalent to $[-r_n, r_n] \times S^1$, where $r_n \to \infty$ as $n \to \infty$. As in the convergence description (see Section 2.3.3), we choose the nodal special cylinders A^{nod} around the elements from Δ_n.

Remark 36. Around a puncture from $Z^{(\nu)}$, the $\mathcal{H}-$holomorphic curve $u_n \circ \varphi_n$ is asymptotic to a trivial cylinder over a Reeb orbit (see Section 2.2). This result is a consequence of the uniform C^0-bound of the harmonic functions $\mathscr{G}_n$.

We are now in the position to formulate the convergence result for the $\mathcal{H}-$holomorphic curves u_n with harmonic perturbation γ_n defined on the disk D. There exist the diffeomorphisms $\varphi_n : D \to Z$ such that the following hold:

- **I1:** $i_n \to j$ in C^∞_{loc} on $Z \backslash \Delta_p \amalg A^{\text{nod}}$.
- **I2:** For every special cylinder A_{ij} of Z there exists an annulus $\overline{A}_{ij} \cong [-1,1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, i_n)$ and (A_{ij}, i_n) are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n - h_n, h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s,t) \mapsto (\kappa(s), t)$.
- **I3:** The sequence of $\mathcal{H}-$holomorphic curves (D, i, u_n, γ_n) with boundary converges to a stratified $\mathcal{H}-$holomorphic building $(Z, j, u, \mathcal{P}, D, \gamma)$ in the sense of Definition 18 from Section 2.3.3. Moreover, the curves converge in C^∞ in a neighbourhood of the boundary ∂D.

This convergence result can be applied to disks such as neighbourhoods of all points of $\mathcal{Z}^{(m)}$. To deal with the entire cylinder of type b_1, we glue the obtained convergence result on disks centered at points of $\mathcal{Z}^{(m)}$ to the complement of disk neighbourhoods of $\mathcal{Z}^{(m)}$. During the convergence description of the $\mathcal{H}-$holomorphic curves u_n restricted to disk neighbourhoods of the points of $\mathcal{Z}^{(m)}$, the diffeomorphism φ_n, describing the convergence, have the property that in a neighbourhood of ∂D they are independent of n (see Appendix A). Coming back to the puncture $z \in \mathcal{Z}^{(m)}$ we focus on the neighbourhood $D_r(z)$. Considering the translation and streching diffeomorphism $D \to D_r(z)$, $p \mapsto z + rp$, we see that $\varphi_n : D_r(z) \backslash D_{r\tau}(z) \hookrightarrow Z$ is independent of n; hereafter, we drop the index n and denote it by $\varphi : D_r(z)\backslash D_{r\tau}(z) \hookrightarrow Z$. This map is used to glue Z and $([0, H^{(m)}] \times S^1)\backslash D_{r\tau}(z)$ along the collar $D_r(z)\backslash D_{r\tau}(z)$. Consider the surface

$$C^{(m)} = ((([0, H^{(m)}] \times S^1)\backslash D_{r\tau}(z)) \coprod Z \Big/ \sim,$$

where $x \sim y$ if and only if $x \in D_r(z)\backslash D_{r\tau}(z)$, $y \in \varphi(D_r(z)\backslash D_{r\tau}(z))$ and $\varphi(x) = y$. This gives rise to the diffeomorphism $\psi_n^{(m)} : [0, H_n^{(m)}] \times S^1 \to C^{(m)}$, defined by

$$\psi_n^{(m)}(x) = \begin{cases} x, & x \in C^{(m)}\backslash D_r(z) \\ \varphi_n(x), & x \in D_r(z). \end{cases}$$

We are now able to describe the convergence on cylinders of type b_1. Let Δ_n, Δ_p and A^{nod} be the collection of loops from $C^{(m)}$ obtained by the above convergence process for each point of $\mathcal{Z}^{(m)}$. Take notice that the complex structure $j^{(m)}$ on $C^{(m)}$ is given by

$$j^{(m)}(p) := \begin{cases} i, & p \in C^{(m)}\backslash D_r(\mathcal{Z}^{(m)}) \\ j, & p \in Z \end{cases}$$

and that it is well-defined since φ is a biholomorphism. There exists a sequence of diffeomorphisms $\psi_n^{(m)} : [0, H_n^{(m)}] \times S^1 \to C^{(m)}$ such that the following hold:

J1: $(\psi_n^{(m)})_* i \to j^{(m)}$ in C^∞_{loc} on $C^{(m)}\backslash \Delta_p \amalg A^{\text{nod}}$.

J2: For every special cylinder A_{ij} of $C^{(m)}$ there exists an annulus $\overline{A}_{ij} \cong [-1, 1] \times S^1$ such that $A_{ij} \subset \overline{A}_{ij}$ and $(\overline{A}_{ij}, i_n)$ and (A_{ij}, i_n) are conformally equivalent to $([-R_n, R_n] \times S^1, i)$ and $([-R_n + h_n, R_n - h_n] \times S^1, i)$, respectively, where $R_n, R_n - h_n \to \infty$ as $n \to \infty$, i is the standard complex structure and the diffeomorphisms are of the form $(s, t) \mapsto (\kappa(s), t)$.

J3: The $\mathcal{H}-$holomorphic curves $([0, H_n^{(m)}] \times S^1, i, u_n, \gamma_n)$ with boundary converges to a stratified broken $\mathcal{H}-$holomorphic building $(C^{(m)}, j, u, \mathcal{P}, \mathcal{D}, \gamma)$ with boundary in the sense of Definition 18 from Section 2.3.3. By construction, the curves converge in C^∞ in a neighbourhood of the boundary ∂D.

3.2.4. Gluing cylinders of type ∞ with cylinders of type b_1. By a modified version of the diffeomorphisms θ_n we identify the cylinders of type ∞ with the cylinder $[-1 - 4h, 1 + 4h] \times S^1$ where $h > 0$ is the constant from Lemma 25, so that after the gluing process, we end up with a bigger cylinder of finite length and a sequence of diffeomorphisms. Let us make this precedure more precise.

Let $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ and $[h_n^{(m)} - 4h, h_n^{(m+1)} + 4h] \times S^1$ be cylinders of types ∞ and b_1, respectively. First we consider the cylinders $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$ of type ∞. With the constant $h > 0$ defined in Section 3.2.1, let $[h_n^{(m-1)} + 4h, h_n^{(m)} - 4h] \times S^1$ be a subcylinder. By the uniform gradient bounds of u_n on cylinders of type ∞, we conclude that the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n converge in C^∞ on $[h_n^{(m)} - 4h, h_n^{(m)}] \times S^1$ to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. For the subcylinders $[h_n^{(m-1)} + 4h, h_n^{(m)} - 4h] \times S^1$ we perform the same analysis as in Theorems 28 and 30. After going over to a subsequence we obtain a sequence of diffeomorphisms

$$\theta_n : [h_n^{(m-1)} + 4h, h_n^{(m)} - 4h] \times S^1 \to [-1, 1] \times S^1,$$

so that Theorems 28 and 30 hold for the cylinders $[h_n^{(m-1)} + 4h, h_n^{(m)} - 4h] \times S^1$. Next we extend the diffeomorphisms θ_n, θ^- and θ^+ to $[h_n^{(m-1)}, h_n^{(m)}] \times S^1$, $[-4h, \infty) \times S^1$ and $(-\infty, 4h] \times S^1$, respectively, such that

$$\begin{aligned} \theta_n|_{([h_n^{(m-1)}, h_n^{(m-1)}+3h] \times S^1) \amalg ([h_n^{(m)}-3h, h_n^{(m)}] \times S^1)} &= \mathrm{id}, \\ \theta^-|_{[-4h,-h] \times S^1} &= \mathrm{id}, \\ \theta^+|_{[h,4h] \times S^1} &= \mathrm{id}, \end{aligned}$$

$\theta_n^- \to \theta^-$ in $C^\infty([-4h, -h] \times S^1)$, and $\theta_n^+ \to \theta^+$in $C^\infty([h, 4h] \times S^1)$.

We consider now the cylinders of type b_1 and note that the diffomorphisms

$$\psi_n : [h_n^{(m)} - 4h, h_n^{(m+1)} + 4h] \times S^1 \to C^{(m)}$$

have the property that

$$\psi_n|_{([h_n^{(m)}-4h, h_n^{(m)}-h] \times S^1) \amalg ([h_n^{(m+1)}-3h, h_n^{(m+1)}] \times S^1)} = \mathrm{id}.$$

In this regard we consider the surface

$$\left(([-1-4h, 1+4h] \times S^1) \amalg C^{(m)}\right) /_\sim$$

where $x \sim y$ if and only if $x \in [h, 4h] \times S^1$ and $y \in [h_n^{(m)} - 4h, h_n^{(m)} - h] \times S^1$ such that $\theta_n(y) = x$.

By this procedure we glue all cylinders of types ∞ and b_1, and obtain a bigger cylinder C_n together with a sequence of diffeomorphisms $\Phi_n : [-\sigma_n, \sigma_n] \times S^1 \to C_n$, where $[-\sigma_n, \sigma_n] \times S^1$ is the parametrization of the $\delta-$thin part, i.e. of $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$. Let $\varphi_n : \mathcal{C}_n \to [-\sigma_n, \sigma_n] \times S^1$ be the conformal parametrization of the cylindrical component of $\mathrm{Thin}_\epsilon(\dot{S}^{\mathcal{D},r}, h_n)$. Since both ends of $[-\sigma_n, \sigma_n] \times S^1$ contain cylinders of type ∞, we infer by the above construction, that Φ_n is identity near the boundary. Specifically, with the constant $h > 0$ we have

$$\Phi_n|_{([-\sigma_n, -\sigma_n+3h]\times S^1)\amalg([\sigma_n-3h,\sigma_n]\times S^1)} = \mathrm{id}.$$

Then we consider the surface

$$\left(\left(\dot{S}^{D,r} \backslash \varphi_n^{-1}([-\sigma_n + 3h, \sigma_n - 3h] \times S^1)\right) \amalg C_n\right) /_\sim$$

where $x \sim y$ if and only if $x \in \dot{S}^{\mathcal{D},r} \backslash \varphi_n^{-1}([-\sigma_n + 3h, \sigma_n - 3h] \times S^1)$ and $y \in C_n$ such that $\Phi_n \circ \varphi_n(x) = y$.

In this way we handle all components of $\mathrm{Thin}_\delta(\dot{S}^{\mathcal{D},r}, h_n)$ that are conformal equivalent to hyperbolic cylinders.

3.2.5. Punctures and elements of $\mathcal{Z}$. We analyze the convergence of u_n on components of the thin part which are biholomorphic to cusps, as well as, in a neighbourhood of the points from $\mathcal{Z}$. Recall that cusps correspond to neighbourhoods of punctures. Let $p \in S^{\mathcal{D},r}$ be a puncture or an element from $\mathcal{Z}$. By Lemma 20 of Appendix C, there exist the open neighbourhoods U_n and U of p, and the biholomorphisms $\psi_n : D \to U_n$ and $\psi : D \to U$ such that ψ_n converge in C^∞ to ψ. We consider the sequence of $\mathcal{H}-$holomorphic curves u_n with harmonic perturbations γ_n restricted to U_n. By the convergence of u_n on the thick part, for every open neighbourhoods U and V of p, such that $V \Subset U$, the $\mathcal{H}-$holomorphic curves u_n together with the harmonic perturbations γ_n converge in C^∞ on $\overline{U \backslash V}$ to some $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ. Via the biholomorphisms ψ_n and ψ, we consider the $\mathcal{H}-$holomorphic curves u_n and the harmonic perturbations γ_n as being defined on $D\backslash\{0\}$. Actually, we consider the following setup: For the sequence of $\mathcal{H}-$holomorphic curves $u_n = (a_n, f_n) : D\backslash\{0\} \to \mathbb{R} \times M$ with the harmonic perturbations γ_n defined on the whole disk D, the following are satisfied:

K1: The energy of u_n is uniformly bounded, i.e. with the constant $E_0 > 0$ we have $E(u_n; D\backslash\{0\}) \leq E_0$ for all $n \in \mathbb{N}$.

K2: The L^2-norms of γ_n are uniformly bounded, i.e. with the constant $C_0 > 0$ we have $\|\gamma_n\|^2_{L^2(D\backslash\{0\})} \leq C_0$ for all $n \in \mathbb{N}$.

K3: For every open neighbourhoods U and V of p such that $V \Subset U$, the $\mathcal{H}-$holomorphic curves u_n with harmonic perturbations γ_n converge in C^∞ on $\overline{U\backslash V}$ to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ.

We consider two cases. In the first case there exists a subsequence of u_n for which the singularity at 0 is removable, i.e. the $\mathbb{R}-$coordinate a_n is bounded in a neighbourhood of 0, but not necessarily uniformly bounded. In particular, this case is typically for neighbourhoods of points from $\mathcal{Z}$. Hence the sequence of $\mathcal{H}-$holomorphic curves u_n can be defined accross the puncture 0 and we end up with a sequence of $\mathcal{H}-$holomorphic disks with fixed boundary. To describe the compactness we use the results of Section 3.2.3.

In the second case, there exists no subsequence of the u_n that has a bounded $\mathbb{R}-$coordinate a_n near 0. Since D is simply connected, there exists a harmonic function $\tilde{\Gamma}_n : D \to \mathbb{R}$ such that $\gamma_n = d\tilde{\Gamma}_n$. By the second condition from above, the gradients $\nabla\tilde{\Gamma}_n$ are uniformly bounded in L^2-norm by the constant $C_0 > 0$. Denote by

$$K_n = \frac{1}{\pi}\int_D \tilde{\Gamma}_n(x,y)dxdy$$

the mean value of $\tilde{\Gamma}_n$ on the disk D. Furthermore, define $\Gamma_n := \tilde{\Gamma}_n - K_n$; Γ_n is a harmonic function on the disk with vanishing average and satisfying $\gamma_n = d\Gamma_n$, while the gradients $\nabla\Gamma_n$ have uniformly bounded L^2-norms. By Poincaré inequality, the L^2-norm of Γ_n is uniformly bounded, i.e. with the constant $C_0 > 0$ we have $\|\Gamma_n\|_{L^2(D)} \leq C_0$ for all $n \in \mathbb{N}$. Pick $\tau \in (0,1)$ and denote by D_τ the disk around 0 of radius τ. From the mean value inequality for harmonic functions, Γ_n is uniformly bounded in $C^0(D_\tau)$. Via the biholomorphism $[0,\infty)\times S^1 \to D\backslash\{0\}$, $(s,t) \mapsto e^{-2\pi(s+it)}$ we consider the $\mathcal{H}-$holomorphic maps u_n together with the harmonic perturbations γ_n as being defined on the half open cylinder $[0,\infty)\times S^1$. Specifically we consider the following setup: For the sequence $u_n = (a_n, f_n) : [0,\infty)\times S^1 \to \mathbb{R}\times M$ of $\mathcal{H}-$holomorphic half cylinders with harmonic perturbations γ_n the following are satisfied:

L1: The energy of u_n and the L^2-norm of the harmonic pertuabtions γ_n are uniformly bounded, i.e. with the constants $E_0, C_0 > 0$ we have $E(u_n; [0,\infty)\times S^1) \leq E_0$ and $\|\gamma_n\|^2_{L^2(D\backslash\{0\})} \leq C_0$ for all $n \in \mathbb{N}$.

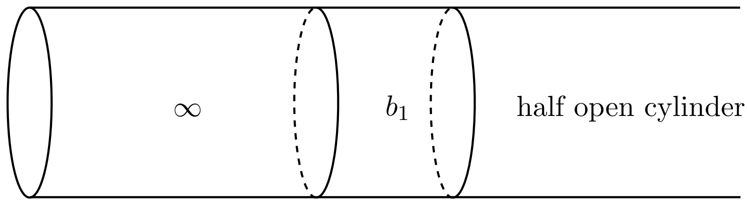

Figure 3.7: Decomposition of a punctured neighbourhood into cylinders of type ∞, b_1 and a half open cylinder.

L3: The $\mathcal{H}-$holomorphic curves u_n converge in C^∞_{loc} to a $\mathcal{H}-$holomorphic curve u with harmonic perturbation γ.

L4: The harmonic perturbations γ_n satisfy $\gamma_n = d\Gamma_n$, where $\Gamma_n : [0,\infty) \times S^1 \to \mathbb{R}$ is a harmonic function with a uniformly bounded gradient $\nabla\Gamma_n$ in L^2-norm. Furthermore, Γ_n is uniformly bounded in $C^0([0,\infty) \times S^1)$.

By using the decomposition discussed in Section 3.2.1 we split the half cylinder into smaller cylinders with $d\alpha-$energies smaller than $\hbar_0/2$. As described in Section 3.2.1 we end up with a sequence of finitely many cylinder of types ∞ and b_1, and a half cylinder with a small $d\alpha-$energy. The appearance of the cylinders of types b_1 and ∞ is alternating; the decomposition starts with a cylinder of type ∞ and ends with a cylinder of type b_1 followed by the half cylinder (see Figure 3.7).

For the cylinders of types ∞ and b_1 we formulate the convergence results as in Sections 3.2.3 and 3.2.2. Since the harmonic $1-$forms γ_n are defined over the puncture p, the period of the harmonic perturbation γ_n over each cylinder (either of type ∞ or type b_1) is 0. Hence, the convergence properties of the cylinders of type ∞ are the same as in the classical theory of Hofer (see [9]), and we are left with the half cylinder having a $d\alpha-$energy smaller than $\hbar_0/2$. We have the following setup:

M1: $u_n = (a_n, f_n) : [0,\infty) \times S^1 \to \mathbb{R} \times M$ is a $\mathcal{H}-$holomorphic curve with harmonic perturbation γ_n.

M2: The energy of u_n and the L^2-norm of γ_n are uniformly bounded by the constants E_0 and C_0, respectively, while the $d\alpha-$energy of u_n is smaller than $\hbar_0/2$.

M3: The harmonic perturbations γ_n satisfy $\gamma_n = d\Gamma_n$, where $\Gamma_n : [0,\infty) \times S^1 \to \mathbb{R}$ is a harmonic function with a uniformly bounded gradient $\nabla\Gamma_n$ in L^2-norm. Furthermore, Γ_n is uniformly bounded in $C^0([0,\infty)\times S^1)$.

M4: The gradients of u_n are uniformly bounded, i.e. there exists a constant $\tilde{C}>0$ such that

$$\|du_n(z)\| = \sup_{\|v\|_{\text{eucl.}}=1} \|du_n(z)(v)\|_{\overline{g}} \leq \tilde{C} \tag{3.6}$$

for all $z\in[0,\infty)\times S^1$ and all $n\in\mathbb{N}$.

By bubbling-off analysis and in view of the uniformly small $d\alpha-$energy, Assumption (3.6) is also valid. Moreover, by the mean value thorem for harmonic functions and the uniformly boundedness of the L^2-norms of $\nabla\Gamma_n$, the harmonic perturbation γ_n is uniformly bounded in C^0 on $[0,\infty)\times S^1$ with respect to the standard Euclidean metric. We turn the $\mathcal{H}-$holomorphic curve u_n with harmonic perturbation γ_n into a usual pseudoholomorphic curve $\overline{u}_n$ by setting $\overline{u}_n=(\overline{a}_n,\overline{f}_n)=(a_n+\Gamma_n,f_n)$ as in Section 3.2.3. In the following we show that the $\alpha-$ and $d\alpha-$ energies of $\overline{u}_n$ are uniformly bounded. As $\overline{f}_n=f_n$ we have

$$E_{d\alpha}(\overline{u}_n;[0,\infty)\times S^1)=E_{d\alpha}(u_n;[0,\infty)\times S^1)\leq\frac{\hbar_0}{2}$$

and therefore the $d\alpha-$energy is uniformly small. By definition and according to the uniform bound on the gradients (3.6) and the uniform C^0-bound of the harmonic $1-$forms γ_n, we obtain

$$\begin{aligned}
&E_\alpha(\overline{u}_n;[0,\infty)\times S^1)\\
&\leq -\sup_{\varphi\in\mathcal{A}}\int_{[0,\infty)\times S^1} d(\varphi(\overline{a}_n)d\overline{a}_n\circ i)+E_{d\alpha}(u_n)\\
&= -\sup_{\varphi\in\mathcal{A}}\left[\lim_{r\to\infty}\int_{\{r\}\times S^1}\varphi(\overline{a}_n)d\overline{a}_n\circ i-\int_{\{0\}\times S^1}\varphi(\overline{a}_n)d\overline{a}_n\circ i\right]+E_{d\alpha}(u_n)\\
&\leq 2C+\frac{\hbar_0}{2}.
\end{aligned}$$

Thus the $\alpha-$energy is uniformly bounded. From the definition of $\tilde{E}_0$ (see Section 3.2.1) we have $E(\overline{u}_n;[0,\infty)\times S^1)\leq\tilde{E}_0$ for all $n\in\mathbb{N}$. In this regard, we consider the following setup:

N1: $\overline{u}_n=(\overline{a}_n,f_n):[0,\infty)\times S^1\to\mathbb{R}\times M$ is a pseudoholomorphic curve.

N2: The energy of $\overline{u}_n$ is uniformly bounded, while the $d\alpha-$energy of $\overline{u}_n$ is uniformly smaller than $\hbar_0/2$.

Using the diffeomorphism θ defined above together with the notation (B.2), and employing Theorem 46 of Appendix B we have the following

Theorem 37. *There exists a subsequence of $\overline{u}_n$, still denoted by $\overline{u}_n$ such that the following is satisfied.*

1) *$\overline{u}_n$ is asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $T \neq 0$ with $|T| \leq \tilde{E}_0$ and a sequence $c_n \in S^1$ such that*

$$\lim_{s\to\infty} \overline{f}_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{\overline{a}_n(s,t)}{s} = T$$

for all $n \in \mathbb{N}$.

2) *$\overline{u}_n$ converge in C^∞_{loc} to a pseudoholomorphic half cylinder $\overline{u} : [0,\infty) \times S^1 \to \mathbb{R} \times M$ having a bounded energy and a $d\alpha-$energy smaller than $\hbar_0/2$. Moreover, there exists $c^* \in S^1$ such that*

$$\lim_{s\to\infty} \overline{f}(s,t) = x(T(t+c^*)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{\overline{a}(s,t)}{s} = T.$$

3) *The maps $g_n = \overline{f}_n \circ \theta^{-1} : [0,1] \times S^1 \to M$, where $g_n(1,t) = x(T(t+c_n))$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$, that satisfy $g(1,t) = x(T(t+c^*))$, where x is a Reeb orbit of period $T \neq 0$ from part 1.*

With this result we are in the position to formulate the convergence of the sequence of $\mathcal{H}-$holomorphic half cylinders u_n with harmonic perturbations γ_n.

Theorem 38. *There exists a subsequence u_n still denoted by u_n such that the following is satisfied.*

1) *u_n is asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x of period $T \neq 0$ with $|T| \leq \tilde{E}_0$ and a sequence $c_n \in S^1$ such that*

$$\lim_{s\to\infty} f_n(s,t) = x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T$$

for all $n \in \mathbb{N}$.

2) *u_n converge in C^∞_{loc} to a $\mathcal{H}-$holomorphic half cylinder $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ with harmonic perturbation γ having a bounded energy*

and a $d\alpha-$energy smaller than $\hbar_0/2$. Moreover, there exists $c^* \in S^1$ such that

$$\lim_{s\to\infty} f(s,t) = x(T(t+c^*)) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a(s,t)}{s} = T.$$

3) *The maps*

$$g_n = f_n \circ \theta^{-1} : [0,1] \times S^1 \to M,$$

where $g_n(1,t) = x(T(t+c_n))$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$, and satisfy $g(1,t) = x(T(t+c^*))$, where x is a Reeb orbit of period $T \neq 0$ from part 1.

Proof. Since the Γ_n are uniformly bounded in C^0-norm, the first assertion is obvious. Employing the same arguments as in [15], i.e. the mean value theorem for harmonic functions and Cauchy integral formula, we deduce that Γ_n have uniformly bounded derivatives, and so, converge in C^∞_{loc} on $[0,\infty) \times S^1$ to a harmonic function $\Gamma : [0,\infty) \times S^1 \to \mathbb{R}$ with a gradient bounded in L^2-norm. Let us show that $\Gamma : [0,\infty) \times S^1 \to \mathbb{R}$ is bounded in C^0. Via the conformal diffomorphism $[0,\infty)\times S^1 \to D\backslash\{0\}, (s,t) \mapsto e^{-2\pi(s+it)}$ we assume that the harmonic functions Γ_n and Γ are defined on the disk D. Then, since the Γ_n are uniformly bounded in C^0 and have gradients with uniformly bounded L^2-norms, it follows that $\Gamma_n \to \Gamma$ in $C^\infty(\overline{D_\rho(0)})$ for some $0 < \rho < 1$. This shows that Γ is uniformly bounded on D and hence, via the conformal map $[0,\infty) \times S^1 \to D\backslash\{0\}$ it is uniformly bounded on $[0,\infty) \times S^1$. Thus, the second assertion is proved, and by means of $\overline{f}_n = f_n$, the third assertion is evident. □

By cutting a smal piece of finite length from the infinite half cylinder, we can make the cylinder preceding the infinite half cylinder to be of type b_1. Assuming that the infinite half cylinder is of type ∞, we glue all cylinders of types ∞ and b_1 together (as described in the previous section). Via the map $[0,1) \times S^1 \to D\backslash\{0\}, (s,t) \mapsto (1-s)e^{2\pi it}$, we identify the cylinder $[0,1) \times S^1$, which is diffeomorphic with the infinite half open cylinder, with a punctured disk $D\backslash\{0\}$. In this way the upper half open cylinder $[0,1) \times S^1$ can be identified with a neighbourhood of a puncture.

4. Discussion on conformal period

In this section we analyze Condition C8 and C9 of Section 3.2.2 dealing with the boundedness of the sequence R_nP_n, and which can be regarded as a connection between the conformal data of the Riemann surface and the harmonic $1-$forms γ_n. Without this additional condition the convergence result from [15] can not be estabished. The reason is that the almost complex structure constructed on the contact manifold M might not vary in a compact interval. We show that this condition is not automatically satisfied by giving a counterexample. It should be pointed out that this example contradicts Lemma A.2 of [12]. Essentially, we will construct a sequence of harmonic $1-$forms γ_n on a sequence of stable Riemann surfaces, that degenerate along a single circle, have uniformly bounded L^2-norms but unbounded P_n/ℓ_n, where P_n denotes the period of γ_n along the degenerating circle and ℓ_n its length with respect to the hyperbolic metric. Observe that the quantity $1/\ell_n$ is similar to R_n .

Let $(S_n, j_n, \mathcal{M}_n)$ be a sequence of stable Riemann surfaces of genus g, where $\mathcal{M}_n \subset S_n$ are finite sets of marked points with the same cardinality. Choose a basis $c_1, \ldots, c_{2g} \in H_1(S_n; \mathbb{Z})$ which is independent of n. This choice is possible because all S_n have genus g and are closed (they are topologically the same). By the Deligne-Mumford convergence,

$$(S_n, j_n, \mathcal{M}_n) \to (S, j, \mathcal{M}, \mathcal{D}, r),$$

where $(S, j, \mathcal{M}, \mathcal{D}, r)$ is a decorated nodal Riemann surface. Again, according to the definition of the Deligne-Mumford convergence, there exist the diffeomorphisms $\varphi_n : S^{\mathcal{D},r} \to S_n$, such that $j_n \to j$ on $S^{\mathcal{D},r} \backslash \coprod_{j=1}^{l} \Gamma_j$ or equivalently, $h_n \to h$ on $\dot{S}^{\mathcal{D},r} \backslash \coprod_{j=1}^{l} \Gamma_j$ where Γ_j are special circles, and h_n and j_n are the pull-back of the complex structure and the hyperbolic metric from S_n and $\dot{S}_n$ via the diffeomorphism φ_n. Assume that $l = 1$, i.e. that there exists only one degenerating geodesic in the Deligne-Mumford convergence. Denote this geodesic by Γ. Furthermore, assume that $\Gamma = c_1$ (Γ lies in the class of c_1). The main result of this section is the following

Proposition 39. *There exists a sequence of harmonic $1-$forms $\gamma_n \in \mathcal{H}^1_{j_n}(S_n)$ with uniformly bounded L^2-norms, periods, and co-periods, but unbounded conformal periods.*

Proof. Choose a sequence of harmonic $1-$forms $\gamma_n \in \mathcal{H}^1_{j_n}(S^{\mathcal{D},r})$ with vanishing periods except on Γ (on all of c_i with $i \neq 1$ except on $c_1 = \Gamma$). By normalization, assume that $\|\gamma_n\|_{L^2(S^{\mathcal{D},r})} = 1$. The uniform bounds on the L^2-norms imply that the periods P_n of γ_n over Γ converge to 0. Thus γ_n converge in C^∞_{loc} to γ on $S^{\mathcal{D},r} \backslash \Gamma$ which can be seen as a harmonic $1-$form on S with vanishing periods. By Hodge theory, we have $\gamma = 0$. For n sufficiently large, the L^2-norms of γ_n concentrate in the collar neighbourhood around

Γ. Indeed, from

$$1 = \|\gamma_n\|^2_{L^2(S^{\mathcal{D},r})} = \|\gamma_n\|^2_{L^2(\mathcal{C}_n)} + \|\gamma_n\|^2_{L^2(S^{\mathcal{D},r}\backslash\mathcal{C}_n)},$$

where $\mathcal{C}_n$ is the cylindrical component of the $\delta-$thin part for some sufficiently small but fixed $\delta > 0$, it follows that $S^{\mathcal{D},r}\backslash\mathcal{C}_n$ is contained in a compact subset of $S^{\mathcal{D},r}\backslash\Gamma$, and so, that $\|\gamma_n\|^2_{L^2(S^{\mathcal{D},r}\backslash\mathcal{C}_n)}$ converge to 0, and for n sufficiently large we have $\|\gamma_n\|_{L^2(\mathcal{C}_n)} \leq 1$ and $\|\gamma_n\|_{L^2(\mathcal{C}_n)} \to 1$ as $n \to \infty$. If F_n is the unique holomorphic $1-$form with $\mathrm{Re}(F_n) = \gamma_n$,

$$\|F_n\|^2_{L^2(S^{\mathcal{D},r})} = \frac{i}{2}\int_{S^{\mathcal{D},r}} F_n \wedge \overline{F}_n.$$

The collar $\mathcal{C}_n$ is conformally equivalent to $[-R_n, R_n] \times S^1$, where $R_n \sim 1/\ell_n$ and ℓ_n is the length of Γ with respect to h_n. On $\mathcal{C}_n$ we write $\gamma_n = \zeta_n ds + \chi_n dt$, where ζ_n and χ_n are harmonic functions on the cylinder $[-R_n, R_n] \times S^1$ (s is the coordinate in $[-R_n, R_n]$ and t is the coordinate on S^1), express the holomorphic $1-$form F_n as $F_n = (\zeta_n - i\chi_n)dz = (\zeta_n - i\chi_n)(ds + idt)$, and note that $\|F_n\|_{L^2(\mathcal{C}_n)} = \|\gamma_n\|_{L^2(\mathcal{C}_n)}$. Consider the quantity $|\,\|F_n\|_{L^2(\mathcal{C}_n)} - |b_0|\,\|dz\|_{L^2(\mathcal{C}_n)}\,|$, where $b_0 = -\tilde{S}_n - iP_n$ and $\tilde{S}_n$ is the co-period defined by

$$\tilde{S}_n = \int_\Gamma \gamma_n \circ j_n = -\int_{\{0\}\times S^1} \zeta_n(0,t)dt.$$

Recalling that

$$P_n = \int_\Gamma \gamma_n = \int_{\{0\}\times S^1} \chi_n(0,t)dt,$$

we obtain

$$\begin{aligned}
&\left|\|F_n\|_{L^2(\mathcal{C}_n)} - |b_0|\,\|dz\|_{L^2(\mathcal{C}_n)}\right| \\
&= \left\|\left[(\zeta_n + \tilde{S}_n) - i(\chi_n - P_n)\right] dz\right\|_{L^2(\mathcal{C}_n)} \\
&\leq \left\|(\zeta_n + \tilde{S}_n)dz\right\|_{L^2(\mathcal{C}_n)} + \|(\chi_n - P_n)dz\|_{L^2(\mathcal{C}_n)}.
\end{aligned}$$

Further calculation gives

$$\|dz\|_{L^2(\mathcal{C}_n)} = \sqrt{2R_n}, \quad \left\|(\zeta_n + \tilde{S}_n)dz\right\|_{L^2(\mathcal{C}_n)} = \left\|\zeta_n + \tilde{S}_n\right\|_{L^2([-R_n,R_n]\times S^1)}$$

and similarly $\|(\chi_n - P_n)dz\|_{L^2(\mathcal{C}_n)} = \|\chi_n - P_n\|_{L^2([-R_n,R_n]\times S^1)}$. Application of Lemma 40 yields

$$\begin{aligned} &\left\|\zeta_n + \tilde{S}_n\right\|^2_{L^2([-R_n,R_n]\times S^1)} \\ &= \int_{-R_n}^{R_n} \left\|\zeta_n(s) + \tilde{S}_n\right\|^2_{L^2(S^1)} ds \\ &\leq \left(36\int_{-R_n}^{R_n} \rho^2(s)ds\right) \max\left\{\left\|\zeta_n(\pm R_n) + \tilde{S}_n\right\|^2_{L^2(S^1)}\right\} \end{aligned}$$

and

$$\begin{aligned} &\|\chi_n - P_n\|^2_{L^2([-R_n,R_n]\times S^1)} \\ &\leq \left(36\int_{-R_n}^{R_n} \rho^2(s)ds\right) \max\left\{\|\chi_n(\pm R_n) - P_n\|^2_{L^2(S^1)}\right\}. \end{aligned}$$

Using

$$\int_{-R_n}^{R_n} \rho^2(s)ds = 4(1 - e^{-4R_n}) \leq 4,$$

we obtain

$$\begin{aligned} \left\|\zeta_n + \tilde{S}_n\right\|^2_{L^2([-R_n,R_n]\times S^1)} &\leq 144 \max\left\{\left\|\zeta_n(\pm R_n) + \tilde{S}_n\right\|^2_{L^2(S^1)}\right\}, \\ \|\chi_n - P_n\|^2_{L^2([-R_n,R_n]\times S^1)} &\leq 144 \max\left\{\|\chi_n(\pm R_n) - P_n\|^2_{L^2(S^1)}\right\}. \end{aligned}$$

Because the harmonic 1−forms γ_n converge to 0 in $C^\infty_{\text{loc}}(S^{\mathcal{D},r}\backslash\Gamma)$, $\zeta_n(\pm R_n)$, $\chi_n(\pm R_n)$, $\tilde{S}_n$, and P_n converge to zero. Hence

$$\left|\|F_n\|_{L^2(\mathcal{C}_n)} - \sqrt{2}|b_0|\sqrt{R_n}\right| \to 0$$

as $n\to\infty$. As $\|F_n\|_{L^2(\mathcal{C}_n)}$ is almost 1, there exists the constants $C_0, C_1 > 0$ such that

$$C_0\frac{1}{\sqrt{2R_n}} \leq |b_0| \leq C_1\frac{1}{\sqrt{2R_n}}$$

giving

$$C_0\frac{1}{\sqrt{2R_n}} \leq \sqrt{P_n^2 + \tilde{S}_n^2} \leq C_1\frac{1}{\sqrt{2R_n}},$$

or equivalently,

$$C_0\sqrt{\frac{R_n}{2}} \leq \sqrt{(P_nR_n)^2 + (\tilde{S}_nR_n)^2} \leq C_1\sqrt{\frac{R_n}{2}}.$$

These inequalities show that either P_nR_n or $\tilde{S}_nR_n$ tend to ∞, although P_n and $\tilde{S}_n$ stay uniformly bounded (γ_n have uniformly bounded L^2-norms). If P_nR_n remains uniformly bounded then we replace γ_n by $\gamma_n \circ j_n$. $\square$

Lemma 40. *For any harmonic functions ζ and χ on the cylinder $[-R, R] \times S^1$ such that $\eta = \zeta ds + \chi dt$ is a harmonic $1-$form on $[-R, R] \times S^1$ we have*

$$\begin{aligned} \left\|\zeta(s) + \tilde{S}\right\|_{L^2(S^1)} &\leq 6\rho(s) \max\left\{\left\|\zeta(\pm R) + \tilde{S}\right\|_{L^2(S^1)}\right\} \\ \|\chi(s) - P\|_{L^2(S^1)} &\leq 6\rho(s) \max\left\{\|\chi(\pm R) - P\|_{L^2(S^1)}\right\} \end{aligned}$$

for all $s \in [-R, R]$. Here $\tilde{S}$ and P are the co-period and the period of η, respectively, and $\rho(s)^2 = 8e^{-2R}\cosh(2s)$.

Proof. Any harmonic $1-$form η defined on the cylinder $[-R, R] \times S^1$ can be written as $\eta = (-\tilde{S}ds + Pdt) + \tilde{\zeta}(s, t)ds + \tilde{\chi}(s, t)dt$ where $\tilde{\zeta}$ and $\tilde{\chi}$ are harmonic functions on $[-R, R] \times S^1$ with vanishing average. Note that the average of ζ corresponds to the co-period $\tilde{S}$ and the average of χ corresponds to $-P$. To show this, write η in the form $\eta = \zeta(s, t)ds + \chi(s, t)dt$ and compute the averages of ζ and χ as

$$\frac{1}{2R}\int_{[-R,R]\times S^1} \zeta(s, t)ds \wedge dt = \frac{1}{2R}\int_{[-R,R]\times S^1} \eta \wedge dt = \int_{\{0\}\times S^1} \eta \circ j = -\tilde{S}$$

and

$$\begin{aligned} \frac{1}{2R}\int_{[-R,R]\times S^1} \chi(s, t)ds \wedge dt &= \frac{1}{2R}\int_{[-R,R]\times S^1} ds \wedge \eta \\ &= -\frac{1}{2R}\int_{-R}^{R}\left(\int_{\{s\}\times S^1} \eta\right) ds = P, \end{aligned}$$

respectively. Hence the $1-$form $\eta - (-\tilde{S}ds + Pdt) = \tilde{\zeta}(s, t)ds + \tilde{\chi}(s, t)dt$ has vanishing average twist and vanishing periods. The Fourier series of $\tilde{\zeta}$ and

$\tilde{\chi}$ in the t variable are

$$\tilde{\zeta}(s,t) = \frac{a_0(s)}{2} + \sum_{k=1}^{\infty} a_k(s)\cos(kt) + b_k(s)\sin(kt),$$
$$\tilde{\chi}(s,t) = \frac{\alpha_0(s)}{2} + \sum_{k=1}^{\infty} \alpha_k(s)\cos(kt) + \beta_k(s)\sin(kt).$$

Since $\tilde{\zeta}$ and $\tilde{\chi}$ are harmonic, the Fourier expansion coefficients solve $a_k'' = k^2 a_k$, $b_k'' = k^2 b_k$, $\alpha_k'' = k^2\alpha_k$ and $\beta_k'' = k^2\beta_k$ for $k \in \mathbb{N}_0$. The solutions to these ordinaty differential equations are of the form

$$\begin{aligned}
a_0(s) &= c_0 + s d_0,\\
a_k(s) &= c_k\cosh(ks) + d_k\sinh(ks),\\
b_k(s) &= e_k\cosh(ks) + f_k\sinh(ks),\\
\alpha_0(s) &= \delta_0 + \epsilon_0 s,\\
\alpha_k(s) &= \delta_k\cosh(ks) + \epsilon_k\sinh(ks),\\
\beta_k(s) &= \eta_k\cosh(ks) + \theta_k\sinh(ks).
\end{aligned}$$

Since $d\eta = d(\eta\circ j) = 0$ we obtain $\partial_t\tilde{\zeta} = \partial_s\tilde{\chi}$ and $\partial_s\tilde{\zeta} = -\partial_t\tilde{\chi}$, giving $a_0(s) = c_0$ and $\alpha_0(s) = \delta_0$. As $\tilde{\zeta}ds + \tilde{\chi}dt$ has vanishing co-period and vanishing period, we find $a_0(s) = \alpha_0(s) = 0$, and the following relations relating the coefficients a_k, b_k, α_k, and β_k for $k \in \mathbb{N}$: $\delta_k = f_k$, $\epsilon_k = e_k$, $\eta_k = -d_k$ and $\theta_k = -c_k$. Consequently, a_k, b_k, α_k, and β_k can be written as

$$\begin{aligned}
a_k(s) &= c_k\cosh(ks) + d_k\sinh(ks),\\
b_k(s) &= e_k\cosh(ks) + f_k\sinh(ks),\\
\alpha_k(s) &= f_k\cosh(ks) + e_k\sinh(ks),\\
\beta_k(s) &= -d_k\cosh(ks) - c_k\sinh(ks).
\end{aligned}$$

Let us express $\tilde{\zeta}$ and $\tilde{\chi}$ as

$$\tilde{\zeta}(s,t) = \sum_{k=1}^{\infty} a_k(s)\cos(kt) + b_k(s)\sin(kt) = \sum_{k\in\mathbb{Z}\setminus\{0\}} F_k(s)e^{2\pi ikt},$$
$$\tilde{\chi}(s,t) = \sum_{k=1}^{\infty} \alpha_k(s)\cos(kt) + \beta_k(s)\sin(kt) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathit{\Gamma}_k(s)e^{2\pi ikt},$$

where $F_k = \frac{1}{2}(a_k - ib_k)$, $F_{-k} = \frac{1}{2}(a_k + ib_k)$, $\Gamma_k = \frac{1}{2}(\alpha_k - i\beta_k)$, and $\Gamma_{-k} = \frac{1}{2}(\alpha_k + i\beta_k)$ for $k \geq 1$. From

$$\frac{\cosh(ks)}{\cosh(kR)} \leq 3e^{-R}\cosh(s) \leq 3\rho(s) \text{ and } \frac{|\sinh(ks)|}{|\sinh(Rs)|} \leq 3e^{-R}\cosh(s) \leq 3\rho(s),$$

where $\rho(s)^2 = 8e^{-2R}\cosh(2s)$, it follows that

$$\cosh(ks) \leq 3\rho(s)\cosh(kR) \quad \text{and} \quad |\sinh(ks)| \leq 3\rho(s)\sinh(Rs).$$

Define the functions

$$K(k) = \begin{cases} +1, & c_k \text{ and } d_k \text{ have the same parity} \\ -1, & \text{otherwise} \end{cases}$$

and

$$G(k) = \begin{cases} +1, & e_k \text{ and } f_k \text{ have the same parity} \\ -1, & \text{otherwise.} \end{cases}$$

For $s \in [0, R] \times S^1$ we then have

$$\begin{aligned}
\left\|\tilde{\zeta}(s)\right\|^2_{L^2(S^1)} &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |F_k(s)|^2 \\
&= \frac{1}{2}\sum_{k=1}^{\infty}(c_k\cosh(ks) + d_k\sinh(ks))^2 \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty}(e_k\cosh(ks) + f_k\sinh(ks))^2 \\
&= \frac{1}{2}\sum_{k=1,K(k)=1}^{\infty}(c_k\cosh(ks) + d_k\sinh(ks))^2 \\
&\quad + \frac{1}{2}\sum_{k=1,K(k)=-1}^{\infty}(c_k\cosh(ks) + d_k\sinh(ks))^2 \\
&\quad + \frac{1}{2}\sum_{k=1,G(k)=1}^{\infty}(e_k\cosh(ks) + f_k\sinh(ks))^2 \\
&\quad + \frac{1}{2}\sum_{k=1,G(k)=-1}^{\infty}(e_k\cosh(ks) + f_k\sinh(ks))^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{k=1, K(k)=1}^{\infty} c_k^2 \cosh^2(ks) + d_k^2 \sinh^2(ks) + 2c_k d_k \cosh(ks) \sinh(ks) \\
&\quad + \frac{1}{2} \sum_{k=1, K(k)=-1}^{\infty} c_k^2 \cosh^2(ks) + d_k^2 \sinh^2(ks) - (-2c_k d_k) \cosh(ks) \sinh(ks) \\
&\quad + \frac{1}{2} \sum_{k=1, G(k)=1}^{\infty} e_k^2 \cosh^2(ks) + f_k^2 \sinh^2(ks) + 2e_k f_k \cosh(ks) \sinh(ks) \\
&\quad + \frac{1}{2} \sum_{k=1, G(k)=-1}^{\infty} e_k^2 \cosh^2(ks) + f_k^2 \sinh^2(ks) - (-2e_k f_k) \cosh(ks) \sinh(ks) \\
&\leq \frac{1}{2} 9\rho(s)^2 \sum_{k=1, K(k)=1}^{\infty} c_k^2 \cosh^2(kR) + d_k^2 \sinh^2(kR) + 2c_k d_k \cosh(kR) \sinh(kR) \\
&\quad + \frac{1}{2} 9\rho(s)^2 \sum_{k=1, K(k)=-1}^{\infty} c_k^2 \cosh^2(kR) + d_k^2 \sinh^2(kR) + 2c_k d_k \cosh(kR) \sinh(-kR) \\
&\quad + \frac{1}{2} 9\rho(s)^2 \sum_{k=1, G(k)=1}^{\infty} e_k^2 \cosh^2(kR) + f_k^2 \sinh^2(kR) + 2e_k f_k \cosh(kR) \sinh(kR) \\
&\quad + \frac{1}{2} 9\rho(s)^2 \sum_{k=1, G(k)=-1}^{\infty} e_k^2 \cosh^2(Rk) + f_k^2 \sinh^2(Rk) + 2e_k f_k \cosh(Rk) \sinh(-kR) \\
&= \frac{9}{2} \rho(s)^2 \sum_{k=1, K(k)=1}^{\infty} (c_k \cosh(kR) + d_k \sinh(kR))^2 \\
&\quad + \frac{9}{2} \rho(s)^2 \sum_{k=1, K(k)=-1}^{\infty} (c_k \cosh(-kR) + d_k \sinh(-kR))^2 \\
&\quad + \frac{9}{2} \rho(s)^2 \sum_{k=1, G(k)=1}^{\infty} (e_k \cosh(kR) + f_k \sinh(kR))^2 \\
&\quad + \frac{9}{2} \rho(s)^2 \sum_{k=1, G(k)=-1}^{\infty} (e_k \cosh(-kR) + f_k \sinh(-kR))^2 \\
&\leq \frac{9}{2} \rho(s)^2 \sum_{k=1}^{\infty} (c_k \cosh(kR) + d_k \sinh(kR))^2 \\
&\quad + \frac{9}{2} \rho(s)^2 \sum_{k=1}^{\infty} (c_k \cosh(-kR) + d_k \sinh(-kR))^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} (e_k \cosh(kR) + f_k \sinh(kR))^2 \\
&+ \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} (e_k \cosh(-kR) + f_k \sinh(-kR))^2 \\
&= \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} a_k(R)^2 + \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} a_k(-R)^2 \\
&+ \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} b_k(R)^2 + \frac{9}{2}\rho(s)^2 \sum_{k=1}^{\infty} b_k(-R)^2 \\
&= 9\rho(s)^2 \left(\left\| \tilde{\zeta}(R) \right\|^2_{L^2(S^1)} + \left\| \tilde{\zeta}(-R) \right\|^2_{L^2(S^1)} \right).
\end{aligned}
$$

The same inequality holds for negative s, and a similar estimate can be derived for the harmonic function $\tilde{\chi}$. Thus

$$
\begin{aligned}
\left\| \tilde{\zeta}(s) \right\|^2_{L^2(S^1)} &\leq 9\rho(s)^2 \left(\left\| \tilde{\zeta}(R) \right\|^2_{L^2(S^1)} + \left\| \tilde{\zeta}(-R) \right\|^2_{L^2(S^1)} \right), \\
\|\tilde{\chi}(s)\|^2_{L^2(S^1)} &\leq 9\rho(s)^2 \left(\|\tilde{\chi}(R)\|^2_{L^2(S^1)} + \|\tilde{\chi}(-R)\|^2_{L^2(S^1)} \right),
\end{aligned}
$$

and from $\tilde{\zeta}(s,t) := \zeta(s,t) + \tilde{S}$ and $\tilde{\chi}(s,t) := \chi(s,t) - P$, we end up with

$$
\begin{aligned}
\left\| \zeta(s) + \tilde{S} \right\|^2_{L^2(S^1)} &\leq 9\rho(s)^2 \left(\left\| \zeta(R) + \tilde{S} \right\|^2_{L^2(S^1)} + \left\| \zeta(-R) + \tilde{S} \right\|^2_{L^2(S^1)} \right) \\
&\leq 18\rho(s)^2 \max \left\{ \left\| \zeta(R) + \tilde{S} \right\|^2_{L^2(S^1)}, \left\| \zeta(-R) + \tilde{S} \right\|^2_{L^2(S^1)} \right\}, \\
\|\chi(s) - P\|^2_{L^2(S^1)} &\leq 9\rho(s)^2 \left(\|\chi(R) - P\|^2_{L^2(S^1)} + \|\chi(-R) - P\|^2_{L^2(S^1)} \right) \\
&\leq 18\rho(s)^2 \max \left\{ \|\chi(R) - P\|^2_{L^2(S^1)}, \|\chi(-R) - P\|^2_{L^2(S^1)} \right\}.
\end{aligned}
$$

□

Remark 41. In [12], a notion of convergence for $\mathcal{H}-$holomorphic curves is derived by using a result (Lemma A.2) which states that the conformal co-period of a harmonic $1-$form on a Riemann surface can be universally controlled by its periods. Proposition 39 gives a counterexample to this statement.

Appendix A. Holomorphic disks with fixed boundary

This appendix is devoted to the description of the convergence of pseudoholomorphic disks with fixed boundaries in symplectization, as well as, of their limit object. The results are used for proving the convergence of a cylinder of "finite length", i.e. of type b_1 as discussed in Section 3.2.3.

Let $u_n = (a_n, f_n) : D \to \mathbb{R} \times M$ be a sequence of pseudoholomorphic curves in the symplectization $\mathbb{R} \times M$ of the contact manifold (M, α), and being defined on the open unit disk D with respect to the standard complex structure i and the cylindrical almost complex structure J on $\mathbb{R} \times M$. For any $\tau > 0$ we assume that there exists a subsequence of u_n, also denoted by u_n, such that

$$u_n \to u \tag{A.1}$$

as $n \to \infty$ in $C^\infty(D \backslash \overline{D_\tau(0)})$. Furthermore, we assume that the Hofer energy $E_{\mathrm{H}}(u_n; D)$ of u_n is uniformly bounded. In the following we analyze the convergence of u_n.

The functions a_n can be supposed to be not uniformly bounded. If this is not the case, we may deduce using standard bubbling-off analysis that the gradients of u_n are uniformly bounded on all of D, which in turn, implies that u_n converge in $C^\infty(D)$ to a pseudoholomorphic disk with finite Hofer energy.

To describe the convergence and the limit object we use the arguments from [8] and [11]. However, we drop the details and explain only the strategy and mention the convergence result. As we have assumed that the $\mathbb{R}-$coordinates of u_n are unbounded, the maximum principle for subharmonic functions gives $a_n \to -\infty$. By (A.1) we have the $C^\infty-$convergence of u_n on an arbitrary neighbourhood of ∂D, and by a specific choice of this neighbourhood, we assume that the $\mathbb{R}-$components of u_n, when restricted to this neighbourhood, do not leave a fixed interval $[-K, K]$ for some $K \in \mathbb{R}$ with $K > 0$. Thus from level $-K-2$ we start with the decomposition of $a_n^{-1}((-\infty, -K-2])$ into cylindrical, essential and one "bottom" boundary components. This decomposition which is identical to the decomposition done in [8] and [11] is illustrated in Figure A1. From [8] and [11] we know that there are at most $N_0 \in \mathbb{N}$ cylindrical components.

In addition to the above decomposition, we add one more boundary components, namely the "upper" boundary component. This surface has two types of boundaries. The first one is the boundary ∂D which lies in a specific neighbourhood such that its image under u_n belongs to $[-K, K] \times M$.

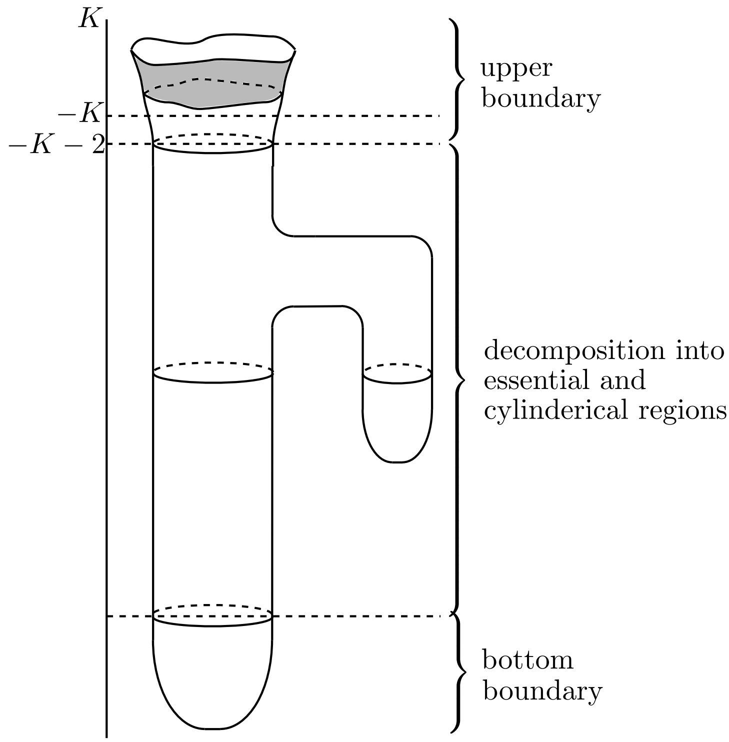

Figure A1: Decomposition of $a_n^{-1}((-\infty, -K-2])$.

The second one is the boundary which connects certain cylindrical components. For the cylindrical, essential and bottom boundary components we use the results established in [8] and [11] to describe the convergence and the limit object. For the "upper" boundary component we use Theorem 3.2 of [8], also known as "Gromov compactness with free boundary". Here the choice of the neighbourhood of ∂D, on which the $\mathbb{R}-$components of u_n lie in $[-K, K]$, plays an essential role. The existence of a special parametrization of a neighbourhood of ∂D will enable us to apply "Gromov compactness with free boundary" in the analysis of the convergence property of the upper boundary component. Essentially, the application of "Gromov compactness with free boundary", requires that the properties (A4) and (A5) under Definition 3.1 of [8] are satisfied. The following considerations ensure these conditions: Choose $L_0' \geq 1$ as in Remark 3.3 after Theorem 3.2 of [8]. More precisely, L_0' depends only on the genus g of the surface, the number of boundary components m, the number of marked points q, the uniform bound C on the area of the considered pseudoholomorphic curves, the constant ϵ_0

from Remark II.4.3 of [5], and the constant C_{ML} from Lemma 3.17 of [8] (the classical monotonicity lemma). For this L_0' we write $L_0'(g,m,q,C,\epsilon_0,C_{ML})$. Furthermore, choose L_0 as

$$L_0 := \max\big\{L_0'(0,1,2,C,\epsilon_0,C_{\mathrm{ML}}), L_0'(0,2,1,C,\epsilon_0,C_{\mathrm{ML}}),\\ L_0'(0,3,0,C,\epsilon_0,C_{\mathrm{ML}}),\dots,L_0'(0,2N_0,0,C,\epsilon_0,C_{\mathrm{ML}})\big\}.$$

Note that when determining the constant L_0' in the first two cases, we introduce one and two artificial punctures, i.e. $q=2$ or $q=1$, in order to make our surface stable. Set $\tau_0 = e^{-10\pi L_0}$ and choose $\tau < \tau_0$. In view of (A.1), assume that there exists a constant $K>0$ such that $u_n(D\backslash\overline{D_\tau(0)}) \subset [-K,K]\times M$ for all $n\in\mathbb{N}$. Hence the boundary is fixed in the symplectization. The boundary region can be conformally parametrized as follows. Consider the map $\beta_{\partial D,0} : [0,5L_0]\times S^1 \to D\backslash D_{\tau_0}(0)$, $(s,t)\mapsto e^{-2\pi(s+it)}$. This map is obviously a conformal parametrization of the boundary region. Let now $L = -\ln(\tau)/10\pi$. Obviously, $L\geq L_0$ and the map $\beta_{\partial D} : [0,5L]\times S^1\to D\backslash D_\tau(0)$, $(s,t)\mapsto e^{-2\pi(s+it)}$ is a conformal parametrization of a neighbourhood of the boundary circle ∂D. Fix this boundary. This conformal parametrization is obviously independent of n and will be used in conjunction with "Gromov compactness with free boundary". Finally, glue the upper boundary component to the rest of the surface, and obtain the resulting limit surface together with the convergence description.

To formulate the convergence result we introduce some notations. Let Z be an oriented surface diffeomorphis to the standard unit disk D and $\Delta = \Delta_n \amalg \Delta_p \subset Z$ a collection of finitely many disjoint simple loops divided into two disjoint sets. Denote by Z_{Δ_n} the surface obtained by collapsing the curves in Δ_n to points. Write

$$Z^* := Z_{\Delta_n}\backslash\Delta_p =: Z^{(0)} \amalg \coprod_{\nu=1}^{N} Z^{(\nu)} \amalg Z^{(N+1)}$$

as a disjoint union of components $Z^{(\nu)}$. Here $Z^{(0)}$ is the bottom boundary component which is the disjoint union of finetly many disks, while $Z^{(N+1)}$ is the upper boundary component whose boundary is of two types. One type is the boundary of the disk D and the other boundary components are certain loops from Δ_p. Let j be a conformal structure on $Z\backslash\Delta$ such that $(Z\backslash\Delta, j)$ is a punctured Riemann surface together with an identification of distinct pairs of punctures given by the elements of Δ. This shows that Z^* has the structure of a nodal punctured Riemann surface with a remaining identification of punctures given by the loops $\{\delta^i\}_{i\in I} = \Delta_p$, for some index set I. A broken pseudoholomorphic curve (with $N+2$ levels) is a map $F = (F^{(0)}, F^{(1)}, \dots, F^{(N+1)}) : (Z^*, j) \to X$, where $X = \coprod_{\nu=0}^{N+1}(\mathbb{R} \times M)$ such that $F^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$ is a punctured pseudoholomorphic curve with the additional property that F extends to a continous map $\overline{F} : Z \to \overline{X}$. Here $\overline{X}$ is obtained as follows. The negative end of the compactification of $\mathbb{R} \times M$ of the $\nu-$th copy is glued to the positive end of the compactification of $\mathbb{R} \times M$ of the copy $\nu + 1$. This procedure is done for $\nu = 0, \dots, N$. For a loop $\delta \in \Delta_p$, there exists $\nu \in \{0, \dots, N\}$ such that δ is adjacent to $Z^{(\nu)}$ and $Z^{(\nu+1)}$. Fix an embedded annulli $A^{\delta,\nu} \cong [-1, 1] \times S^1 \subset Z \backslash \Delta_n$ such that $\{0\} \times S^1 = \delta$, $\{-1\} \times S^1 \subset Z^{(\nu)}$ and $\{1\} \times S^1 \subset Z^{(\nu+1)}$.

In this context, we state a convergence result which has been established in [8].

Proposition 42. *The sequence of pseudoholomorphic disks $u_n = (a_n, f_n) : (D, i) \to \mathbb{R} \times M$ satisfying (A.1) and having a uniformly bounded Hofer energy has a subsequence that converges to a broken pseudoholomorphic curve $u = (a, f) : (Z, j) \to \mathbb{R} \times M$ with $N + 2$ levels in the following sense: There exists a sequence of diffomorphisms $\varphi_n : D \to Z$ and a sequence of negative real numbers $\min(a_n) = r_n^{(0)} < r_n^{(1)} < \cdots < r_n^{(N+1)} = -K - 2$ with $K \in \mathbb{R}$ and $r_n^{(\nu+1)} - r_n^{(\nu)} \to \infty$ as $n \to \infty$ such that the following hold:*

1) *Z with the circles Δ collapsed to points is a nodal Riemann surface (in the sense of the above discussion, but with boundary). $i_n := (\varphi_n)_* i \to j$ in C^∞_{loc} on $Z \backslash \Delta$. For every $i \in I$, the annulus $(A^i, (\varphi_n)_* i)$ is conformally equivalent to a standard annulus $[-R_n, R_n] \times S^1$ by a diffeomorphism of the form $(s, t) \mapsto (\kappa(s), t)$ with $R_n \to \infty$ as $n \to \infty$.*
2) *The sequence $u_n \circ \varphi_n^{-1}|_{Z^{(\nu)}} : Z^{(\nu)} \to \mathbb{R} \times M$ converges in C^∞_{loc} on $Z^{(\nu)} \backslash \Delta_n$ to a punctured nodal pseudoholomorphic curve $u^{(\nu)} : (Z^{(\nu)}, j) \to \mathbb{R} \times M$, and in C^0_{loc} on $Z^{(\nu)}$.*
3) *The sequence $f_n \circ \varphi_n^{-1} : Z \to M$ converges in C^0 to a map $f : Z \to M$, whose restriction to Δ_p parametrizes the Reeb orbits and to Δ_n parametrizes points.*
4) *For any $S > 0$, there exist $\rho > 0$ and $K \in \mathbb{N}$ such that $a_n \circ \varphi_n^{-1}(s, t) \in [r_n^{(\nu)} + S, r_n^{(\nu+1)} - S]$ for all $n \geq K$ and all $(s, t) \in A^{\delta,\nu}$ with $|s| \leq \rho$.*
5) *The diffeomorphisms $\varphi_n \circ \beta_{\partial D} : [0, 5L] \times S^1 \to Z$ are independent of n.*

Appendix B. Half cylinders with small energy

This appendix is devoted to the description of the convergence of a sequence of pseudoholomorphic half cylinders $u_n = (a_n, f_n) : [0, \infty) \times S^1 \to \mathbb{R} \times M$ with uniformly bounded $\alpha-$ and $d\alpha-$energies. More precisely, we assume that there exists a constant $E_0 > 0$ such that $E(u_n; [0, \infty) \times S^1) \le E_0$ and

$$E_{d\alpha}(u_n; [0, \infty) \times S^1) \le \frac{\hbar}{2}, \tag{B.1}$$

where $\hbar > 0$ is defined as in Section 3.2.1. Since the $d\alpha-$energy is smaller than $\hbar/2$ it follows, from the usual bubbling-off analysis, that the gradients of u_n are uniformly bounded with respect to the standard Euclidian metric on the cylinder $[0, \infty) \times S^1$ and the induced cylindrical metric on $\mathbb{R} \times M$. To analyze the convergence of such a sequence we use the results of Appendix A and [15]. As before we split the analysis of the convergence in two parts, namely the $C^\infty_{\text{loc}}-$ and the C^0-convergence. Before stating the convergence results we need some auxiliary results similar to those from [15]. We begin with a remark on the asymptotic of a pseudoholomorphic half cylinder.

Remark 43. Let $u = (a, f) : [0, \infty) \times S^1 \to \mathbb{R} \times M$ be a pseudoholomorphic half cylinder with $E(u; [0, \infty) \times S^1) \le E_0$ and $E_{d\alpha}(u; [0, \infty) \times S^1) \le \hbar/2$. To describe the behaviour of u as $s \to \infty$, we first assume that u has a bounded image in $\mathbb{R} \times M$. Consider the conformal transformation $h : [0, \infty) \times S^1 \to D \backslash \{0\}$, $(s, t) \mapsto e^{-2\pi(s+it)}$. Then $u \circ h^{-1} = (a \circ h^{-1}, f \circ h^{-1})$ is a pseudoholomorphic punctured disk satisfying the same assumption as u does. By the removal of singularity, $u \circ h^{-1}$ can be defined on the whole disk D. In this case we use the results from Appendix A to describe the convergence. If u has an unbounded image in $\mathbb{R} \times M$, then due to Proposition 5.6 from [1], there exists $T \neq 0$ and a periodic orbit x of X_α such that x is of period $|T|$ and

$$\lim_{s \to \infty} f(s, t) = x(Tt) \quad \text{and} \quad \lim_{s \to \infty} \frac{a(s, t)}{s} = T \ \text{ in } C^\infty(S^1).$$

To analyze the convergence of the sequence of pseudoholomorphic half cylinders $u_n = (a_n, f_n) : [0, \infty) \times S^1 \to \mathbb{R} \times M$ we destinguish two cases.

In the first case each element of a subsequence of u_n, still denoted by u_n, has a bounded image in the symplectization $\mathbb{R} \times M$. By Remark 43 we consider the sequence of pseudoholomorphic disks $u_n \circ h^{-1} : D \to \mathbb{R} \times M$ having uniformly bounded energies and small $d\alpha-$energies. After applying bubbling-off analysis and accounting on the uniform energy bounds as well as on the small $d\alpha-$energies, we obtain a subsequence having uniform gradient bounds with respect to the Euclidian metric on the domains and the induced metric on $\mathbb{R}\times M$. After a specific shift in the $\mathbb{R}-$coordinate, $u_n\circ h^{-1}$ converge in C^∞ to a pseudoholomorphic disk $u:D\to\mathbb{R}\times M$.

In the second case each element of a subsequence of u_n, still denoted by u_n, has an unbounded image in $\mathbb{R}\times M$. In the following we assume that after a specific shift in the $\mathbb{R}-$coordinate, $a_n(0,0)=0$. Before describing the convergence of u_n, we prove an asymptotic result for punctures which is similar to that given in [1].

Proposition 44. *After going over to a subsequence the pseudoholomorphic half cylinders u_n are asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x and $T\neq 0$ with $|T|\leq C$ and a sequence $c_n\in S^1$ such that*

$$\lim_{s\to\infty} f_n(s,t)=x(T(t+c_n)) \quad \textit{and} \quad \lim_{s\to\infty}\frac{a_n(s,t)}{s}=T.$$

Moreover, $u_n\to u$ in C^∞_{loc}, where u is a pseudoholomorphic half cylinder $u:[0,\infty)\times S^1\to\mathbb{R}\times M$ which is asymptotic to the same Reeb orbit $x(T(t+c^*))$ of period T as above. Here, $c^*\in S^1$ and $c_n\to c^*$ as $n\to\infty$.

Proof. Let the sequence u_n be asymptotic to some Reeb orbit. More precisely, for all $n\in\mathbb{N}$ there exist $T_n\neq 0$ and a periodic orbit x_n of period $|T_n|$ such that

$$\lim_{s\to\infty} f_n(s,t)=x_n(T_nt) \quad \text{and} \quad \lim_{s\to\infty}\frac{a_n(s,t)}{s}=T_n$$

in $C^\infty(S^1)$. For simplicity, choose a subsequence of T_n, also denoted by T_n, which is always positive (positive puncture). Since we are in the non-degenerate case and $T_n\leq E_0$, assume, after going to some subsequence, that $T_n=\overline{T}>0$ and $x_n(\overline{T}t)=\overline{x}(\overline{T}(t+c_n))$, where $c_n\in S^1$ for all n. Thus after going over to some subsequence we may assume that $c_n\to c^*\in S^1$. From the uniform boundedness of the gradients of u_n, the elliptic regularity, and Arzelá-Ascoli theorem, we have $u_n\to u:[0,\infty)\times S^1\to\mathbb{R}\times M$ in C^∞_{loc}. Here u is a pseuhoholomorphic half cylinder with bounded energy and a small $d\alpha-$energy which is asymptotic to some periodic orbit with period $\underline{T}$ or a point; both being denoted by $\underline{x}$. Choose the sequences $\underline{N}_n,\overline{N}_n\overset{n\to\infty}{\longrightarrow}\infty$ and $\underline{N}_n<\overline{N}_n$ such that after going over to a subsequence we have

$$\lim_{n\to\infty} f_n(\underline{N}_n,t)=\underline{x}(\underline{T}t) \quad \text{and} \quad \lim_{n\to\infty} f_n(\overline{N}_n,t)=\overline{x}(\overline{T}(t+c^*)) \ \text{ in } C^\infty(S^1),$$

and consider the maps

$$v_n = u_n|_{[\underline{N}_n, \overline{N}_n]\times S^1}.$$

which have by construction $d\alpha-$energy tending to 0. Performing the same analysis as in [1] we conclude that $\underline{x} = \overline{x}$ and $\underline{T} = \overline{T}$. □

To describe the C^0-convergence of u_n we use the results established in [15]. In view of Proposition 44, choose a sequence $R_n > 0$ such that $R_n \to \infty$ and $a_n(R_n, t) - TR_n \to 0$ as $n \to \infty$. Consider the shifted maps $\overline{u}_n(s,t) := u_n(s + R_n, t) - TR_n$ for $(s,t) \in [-R_n, R_n] \times S^1$. These are pseudoholomorphic cylinders with uniformly bounded $\alpha-$ and $d\alpha-$energies and a $d\alpha-$energy smaller than $\hbar/2$. Recall that these pseudoholomorphic cylinders are a special case of the $\mathcal{H}-$holomorphic cylinders described in [15]. We distinguish two cases corresponding to subsequences with vanishing and non-vanshing center actions. In latter case, the center action is greater than $\hbar > 0$. By Proposition 44, the first case does not appear and we are left with the case in which $A(\overline{u}_n) \geq \hbar$. By Corollary 25 of [15], for every $\epsilon > 0$ there exists $h > 0$ such that for all $n \in \mathbb{N}$ and $R_n > h$, $\text{dist}_{\overline{g}_0}(\overline{f}_n(s,t), x(Tt + c_n)) < \epsilon$ and $|\overline{a}_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [-R_n + h, R_n - h] \times S^1$. On the other hand, we have the following result: For every $\epsilon > 0$ there exists $h > 0$ such that for all $n \in \mathbb{N}$ and $R_n > h$, $\text{dist}_{\overline{g}_0}(f_n(s,t), x(Tt + c_n)) < \epsilon$ and $|a_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [h, 2R_n - h] \times S^1$. As R_n can be choosen arbitrary large the following equivalent statement readily follows:

Corollary 45. *For every $\epsilon > 0$ there exist $h > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $dist_{\overline{g}_0}(f_n(s,t), x(Tt + c_n)) < \epsilon$ and $|a_n(s,t) - Ts - a_0| < \epsilon$ for all $(s,t) \in [h, \infty) \times S^1$.*

Consider the diffeomorphism $\theta : [0, \infty) \times S^1 \to \mathbb{R} \times M$ and the maps

$$g_n := f_n \circ \theta^{-1} : [0,1) \times S^1 \to M, \tag{B.2}$$

which by Proposition 44 converge in C^∞_{loc} to a map $g := f \circ \theta^{-1} : [0,1) \times S^1 \to M$. By Corollary 45, the maps g_n and g can be continously extended to $[0,1] \times S^1$ by $g_n(1,t) = g(1,t) = x(Tt+c_n)$ for all $n \in \mathbb{N}$ and all $t \in S^1$. Hence due to Corollary 45, g_n converge in C^0 to g. As a consequence, we formulate the following compactness property of the sequence of pseudoholomorphic half cylinders $u_n : [0, \infty) \times S^1 \to \mathbb{R} \times M$ with uniformly bounded energies and $d\alpha-$energies less than $\hbar/2$:

Theorem 46. *Let u_n be a sequence of pseudoholomorphic curves having uniformly bounded energy by E_0 and satisfying condition (B.1). Then there* *exists a subsequence of u_n, still denoted by u_n, such that the following is satisfied.*

1) *u_n is asysmptotic to the same Reeb orbit, i.e. there exists a Reeb orbit x and $T \neq 0$ with $|T| \leq C$ and a sequence $c_n \in S^1$ such that*

$$\lim_{s\to\infty} f_n(s,t) = x(Tt+c_n) \quad \textit{and} \quad \lim_{s\to\infty} \frac{a_n(s,t)}{s} = T.$$

   for all $n \in \mathbb{N}$.

2) *u_n converge in C^∞_{loc} to a pseudoholomorphic half cylinder $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ having uniformly bounded energy by the constant E_0 and satisfying condition (B.1).*

3) *The maps $g_n : [0,1] \times S^1 \to M$ converge in C^0 to a map $g : [0,1] \times S^1 \to M$ and satisfy $g(1,t) = x(T(t+c^*))$, where x is a Reeb orbit of period $T \neq 0$.*

Appendix C. Special coordinates

Let S be a compact surface with boundary, and let j_n and j be complex structures on S for all $n \in \mathbb{N}$. Additionally, let h_n and h be the hyperbolic structures on S with respect to j_n and j, respectively. Assume that $j_n \to j$ and $h_n \to h$ in $C^\infty(S)$. In this appendix we construct a sequence of biholomorphic coordinates around some point in S with respect to the complex structure j_n that converges in a certain sense to the biholomorphic coordinates with respect to j. This result is used in Section 3 for proving the convergence on the thick part.

Lemma 47. *For each $z \in int(S)$ there exist the open neighbourhoods $U_n(z) = U_n$ and $U(z) = U$ of z and the diffeomorphisms*

$$\psi_n : D_1(0) \to U_n,$$
$$\psi : D_1(0) \to U$$

such that

1) *ψ_n are $i - j_n-$biholomorphisms and ψ is a $i - j-$biholomorphism;*
2) *$\psi_n \to \psi$ in $C^\infty_{loc}(D_1(0))$ as $n \to \infty$ with respect to the Euclidian metric on $D_1(0)$ and h on S;*
3) *$\psi_n(0) = z$ for every n and $\psi(0) = z$.*

Proof. Around $z \in \mathrm{int}(S)$, choose the $i-j-$holomorphic coordinates $c : D_2(0) \to U$ such that $U \subset \mathrm{int}(S)$ and $c(0) = z$, and consider the complex structures $j^{(n)} := c^* j_n$. Since $j_n \to j$ as $n \to \infty$ in C^∞, $j^{(n)} \to i$ in $C^\infty_{\mathrm{loc}}(D_2(0))$ as $n \to \infty$. Let $d^{\mathbb{C}}_n$ be the operator defined by $d^{\mathbb{C}}_n f = df \circ j^{(n)}$ and let $d^{\mathbb{C}}$ be the operator defined by $d^{\mathbb{C}} f = df \circ i$. Denote by $p_x : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$ the projection onto the first coordinate. Consider the problem of finding a smooth fumction $f : \overline{D_1(0)} \to \mathbb{R}$ such that

$$\text{(C.1)} \qquad \begin{aligned} dd^{\mathbb{C}}_n f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0) \end{aligned}$$

for all n and

$$\text{(C.2)} \qquad \begin{aligned} dd^{\mathbb{C}} f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0). \end{aligned}$$

As the second problem translates into

$$\text{(C.3)} \qquad \begin{aligned} \Delta f &= 0 \text{ on } D_1(0), \\ f &= p_x \text{ on } \partial D_1(0), \end{aligned}$$

where Δ is the standard Laplace operator in $\mathbb{R}$, the unique solution is $f(x, y) = x$ for all $(x, y) \in \overline{D_1(0)}$. To see the uniqueness observe that the difference of f with any other solution of (C.3) solves $\Delta u = 0$ with $u|_{\partial D_1(0)} = 0$. Thus from the maximum principle for harmonic functions we deduce that $u \equiv 0$, and so, that (C.3) has the unique solution f. In coordinates representation, $j^{(n)}$ can be written as

$$j^{(n)} = \begin{pmatrix} j^{(n)}_{11} & j^{(n)}_{12} \\ j^{(n)}_{21} & j^{(n)}_{22} \end{pmatrix}$$

and take notice that $j^{(n)} \to i$ in C^∞ on $D_1(0)$ as $n \to \infty$. The solutions of (C.1) are equivalent to the solutions of

$$\text{(C.4)} \qquad \begin{aligned} dd^{\mathbb{C}}_n \tilde{f} &= t_n \text{ on } D_1(0), \\ \tilde{f} &= 0 \text{ on } \partial D_1(0), \end{aligned}$$

where $t_n = -dd^{\mathbb{C}}_n p_x$. Hence $dd^{\mathbb{C}}_n$ is an elliptic and coercive operator, and thus by Proposition 5.10 from [13], the problem (C.4) has a uniquely weak solution $\tilde{f}_n \in W^{1,2}(D_1(0))$ for all n. From regularity theorem, the solutions $\tilde{f}_n$ are smooth for all n. Thus $f_n := \tilde{f}_n + p_x$ is the smooth unique solution of

(C.1). Let us show that $f_n \to f$ in $C^\infty_{\mathrm{loc}}(D_1(0))$ as $n \to \infty$. For $u_n := f_n - f$ we have

$$dd^{\mathbb{C}}_n u_n = g_n \text{ on } D_1(0),$$
$$u_n = 0 \text{ on } \partial D_1(0).$$

Here, $g_n \in C^\infty(D_1(0))$ is defined by $g_n := dd^{\mathbb{C}}_n f$, and because of $j^{(n)} \to i$ in $C^\infty(D_1(0))$ as $n \to \infty$, g_n converges to 0 in $C^\infty_{\mathrm{loc}}(D_1(0))$ as $n \to \infty$. For every $m \in \mathbb{N}_0$ we consider the bounded operator

$$dd^{\mathbb{C}}_n : W^{2+m,2}_\partial(D_1(0), \mathbb{R}) \to W^{m,2}(D_1(0), \mathbb{R}),$$

where $W^{2+m,2}_\partial(D_1(0), \mathbb{R})$ consists of maps from $W^{2+m,2}(D_1(0), \mathbb{R})$ that vanish at the boundary. By Proposition 5.10 together with Propositons 5.18 and 5.19 of [13] we deduce that the operator $dd^{\mathbb{C}}_n$ is bounded invertible; hence $u_n = (dd^{\mathbb{C}}_n)^{-1} g_n$. Since $dd^{\mathbb{C}}_n \to \Delta$ in operator norm, $(dd^{\mathbb{C}}_n)^{-1}$ is a uniformly bounded family, and so, $\|u_n\|_{W^{m+2,2}} \to 0$ as $n \to \infty$. Furthermore, as $m \in \mathbb{N}_0$ was arbitrary, the Sobolev embedding theorem yields $u_n \to 0$ in $C^\infty_{\mathrm{loc}}(D_1(0))$ as $n \to \infty$. Thus we have constructed a unique sequence of solutions $\{f_n : \overline{D_1(0)} \to \mathbb{R}\}_{n\in\mathbb{N}}$ of (C.1), and a unique solution $f : \overline{D_1(0)} \to \mathbb{R}, (x, y) \mapsto x$ of (C.2) satisfying $f_n \to f$ in $C^\infty_{\mathrm{loc}}(D_1(0))$ as $n \to \infty$.

According to Lemma 6.8.1 of [7], there exists a $j^{(n)} - i$−holomorphic function $F_n : D_1(0) \to \mathbb{C}$ and a $i - i$−holomorhic function $F : D_1(0) \to \mathbb{C}$ such that $f_n = \Re(F_n)$ and $f = \Re(F)$. Let us investigate the unique extensions of the functions F_n and F. For doing this we set $F_n = f_n + ib$ and $F = f + ib$, where $b_n, b : D_1(0) \to \mathbb{R}$ are harmonic functions. As F_n and F are $j^{(n)} - i$−holomorphic and $i - i$−holomorphic, respectively, they solve the equations

$$dF_n + i \circ dF_n \circ j^{(n)} = 0$$

and

$$dF + i \circ dF \circ i = 0,$$

respectively, which in turn, are equivalent to

$$db_n = -df_n \circ j^{(n)}$$

and

$$db = -df \circ i,$$

respectively. By the harmonicity of f_n and f, and the application of Poincaré lemma on $D_1(0)$, we find the solutions b_n and b which are unique up to addition with some constant. They can be make unique by requiring that

$b_n(0) = 0$ and $b(0) = 0$. In particular, we find $F(x, y) = x + iy$. Then we get $db_n \to db$ in $C^\infty_{\text{loc}}(D_1(0))$ as $n \to \infty$, and from $b_n(0) = 0$ and $b(0) = 0$, we actually get $b_n \to b$ in $C^\infty_{\text{loc}}(D_1(0))$ as $n \to \infty$. Hence $F_n \to F = \text{id}$ in $C^\infty_{\text{loc}}(D_1(0))$ as $n \to \infty$.

For n large, F_n is bijective onto its image (maybe after shrinking the domain). This follows from the proof of the inverse function theorem. With $\tilde{F}_n = F_n - f_n(0)$, the maps ψ_n and ψ are defined by $\psi_n = c \circ \tilde{F}_n : D_1(0) \to U_n$ and $\psi = c \circ F : D_1(0) \to U$ for sufficiently large n, respectively. □

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