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On the dynamics of some vector fields tangent to non-integrable plane fields

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Let $\mathcal{E}^3 \subset TM^n$ be a smooth 3-distribution on a smooth n-manifold, and $\mathcal{W} \subset \mathcal{E}$ a line field such that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. We give a condition for the existence of a plane field $\mathcal{D}^2$ such that $\mathcal{W} \subset \mathcal{D}$ and $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ near a closed orbit of $\mathcal{W}$. If $\mathcal{W}$ has a non-singular Morse-Smale section, we get a condition for the global existence of $\mathcal{D}$. As a corollary we obtain conditions for a non-singular vector field W on a 3-manifold to be Legendrian, and for an even contact structure $\mathcal{E} \subset TM^4$ to be induced by an Engel structure $\mathcal{D}$.

1. Introduction

The only topologically stable families of smooth distributions on smooth manifolds are line fields, contact structures, even contact structures, and Engel structures [2, 6, 8, 17]. An even contact structure is a maximally non-integrable hyperplane field on an even dimensional manifold. An Engel structure is a 2-plane field $\mathcal{D}$ on a 4-manifold M such that $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is an even contact structure. Engel structures were discovered more than a century ago [2, 6] and they have sparked big interest throughout the years [3, 14, 16, 19, 20].

We want to understand which even contact structures $(M^4, \mathcal{E})$ are induced by Engel structures $\mathcal{D}$, i.e. $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. There are some obvious topological obstructions: M admits an even contact structure if (up to a 2-cover) its Euler characteristic vanishes (see [12]), whereas it admits an Engel structure only if it is parallelizable (up to a 4-cover, see [20]). For this reason we only consider even contact structures $\mathcal{E}$ which admit a framing $\mathcal{E} = \langle W, A, B\rangle$ where W spans the characteristic foliation, i.e. the unique

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line field $\mathcal{W} \subset \mathcal{E}$ satisfying $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. In this case an orientable Engel structure compatible with $\mathcal{E}$ takes the form $\mathcal{D}_L = \langle W, L\rangle$, where $L \in \Gamma\langle A, B\rangle$ and $[\mathcal{D}_L, \mathcal{D}_L] = \mathcal{E}$.

The same framework can be used to describe different contexts. For example if M is an orientable manifold of dimension 3 and $\mathcal{E} := TM = \langle W, A, B\rangle$, then a plane field of the form $\mathcal{D}_L = \langle W, L\rangle$, where $L \in \Gamma\langle A, B\rangle$ and $[\mathcal{D}_L, \mathcal{D}_L] = \mathcal{E}$, is an orientable contact structure for which W is Legendrian. We introduce a more general family of distributions which permits to treat the above cases at once. For a given 3-distribution $\mathcal{E} \subset TM$ on a manifold M, we say that a 2-plane field $\mathcal{D} \subset \mathcal{E}$ *generates* $\mathcal{E}$ or that $\mathcal{D}$ is *maximally non-integrable within* $\mathcal{E}$ if $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$.

Definition 1.1. If $\mathcal{E} = \langle W, A, B\rangle$ and $\mathcal{W} = \langle W\rangle$ satisfy $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, we say that $\mathcal{D}_L = \langle W, L\rangle \subset \mathcal{E}$ *generates* $\mathcal{E}$ *up to homotopy* if there is a family of plane fields $\mathcal{D}_{L_s} = \langle W, L_s\rangle \subset \mathcal{E}$ continuous in $s \in [0,1]$ and such that $\mathcal{D}_{L_0} = \mathcal{D}_L$ and $\mathcal{D}_{L_1}$ generates $\mathcal{E}$ [1].

If γ is an orbit of $\mathcal{W} = \langle W\rangle$, $p \in \gamma$ and ϕ_t denotes the flow of W at time t, we introduce a rotation angle function $\theta(p;t)$ associated with L, whose derivative is non-vanishing if and only if $\mathcal{D}_L = \langle W, L\rangle$ generates $\mathcal{E}$ in a neighbourhood of γ. If γ is closed of period T, we consider the quantity $\mathrm{rot}_{\gamma,p}(L) = \theta(p;T) - \theta(p;0)$, which we call the *rotation number of* L *along* γ *at* p. The *maximal rotation number* $\mathrm{maxrot}_\gamma(L)$ *of* L *along* γ is the maximum of the rotation number under homotopies of L and $\langle A, B\rangle$. This quantity gives an obstruction to the existence of $\mathcal{D}$ generating $\mathcal{E}$ in a neighbourhood of γ. If the dynamics of W are particularly simple, we can give a necessary and sufficient condition for the global existence of $\mathcal{D}$ generating $\mathcal{E}$.

Theorem A. *Let* $\mathcal{E} = \langle W, A, B\rangle$ *be a rank* 3 *distribution on a manifold* M*, and denote by* $\mathcal{W} = \langle W\rangle$*. Suppose that* $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$*, and let* W *be a non-singular Morse-Smale vector field. There exists* $\mathcal{D} \subset \mathcal{E}$ *such that* $\mathcal{W} \subset \mathcal{D}$ *and that positively generates* $\mathcal{E}$ *on* M *if and only if there exists* $L \in \Gamma\langle A, B\rangle$ *such that* $\mathrm{maxrot}_\gamma(L) > 0$ *for all closed orbits* γ *of* W*.*

The previous theorem is new already in the special case of Legendrian vector fields. This question has already been studied in the case of Morse-Smale gradient vector fields in [7].

[1] We do not consider all possible homotopies of the plane field $\mathcal{D}_L \subset \mathcal{E}$, only those tangent to $\mathcal{W}$.

1.1. Structure of the paper

In Section 2 we introduce the rotation number, and we study its behaviour under homotopies in Section 3. In Section 4 we apply the theory to Morse-Smale vector fields. In Sections 5 we study the case of Legendrian vector fields and even contact structures.

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2. Rotation number

Suppose that $\mathcal{E} \subset TM$ is a rank 3 distribution which admits a global framing $\mathcal{E} = \langle W, A, B \rangle$, such that the flow of W preserves $\mathcal{E}$, and denote $\mathcal{W} = \langle W \rangle$. Given $L \in \Gamma\mathcal{E}$ nowhere tangent to $\mathcal{W}$, we want to determine when the distribution $\mathcal{D}_L := \langle W, L \rangle$ is homotopic within $\mathcal{E}$ to a maximally non-integrable plane field within $\mathcal{E}$. Since we have fixed a framing, $\mathcal{E}$ is oriented. Moreover every $\mathcal{D}_L$ that generates $\mathcal{E}$ uniquely defines an orientation of $\mathcal{E}$ given by $\{W, L, [W, L]\}$. We say that $\mathcal{D}_L$ *positively (resp. negatively) generates* $\mathcal{E}$ if these orientations coincide (resp. they are opposite).

In analogy with [1, 14], for a given orbit of $\mathcal{W}$ parametrized by the immersion $\gamma : [a, b] \to M$ we define the developing map

$$\begin{aligned} &\delta_\gamma : [a, b] \to \mathbb{RP}^1 \equiv \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma[0]}) \\ &\text{via} \quad \delta_\gamma(t) = \left[\mathcal{D}_L|_{\gamma(t)} \right] \in \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma[t]}) \equiv \mathbb{P}(\mathcal{E}/\mathcal{W}|_{\gamma[0]}), \end{aligned}$$

where the identification $\mathcal{E}/\mathcal{W}|_{\gamma[t]} \equiv \mathcal{E}/\mathcal{W}|_{\gamma[0]}$ is given by γ_*^{-1}. A framing $\{A, B\}$ fixes a trivialization of $\mathcal{E}/\mathcal{W}$ and permits to lift δ_γ to an angle function θ at $p = \gamma(a)$, up to choosing a lift $\theta(p; 0)$. If we fix W such that $\mathcal{W} = \langle W \rangle$, this furnishes a parametrization γ via the flow ϕ_t of W, so that the angle function θ satisfies

$$\delta_\gamma(t) = [\phi_{-t*} L(p)] = [\cos\theta(p; t) A(p) + \sin\theta(p; t) B(p)].$$

With techniques similar to the ones used in [1, 4] we can prove the following

Proposition 2.1. *The distribution* $\mathcal{D}_L = \langle W,\, L\rangle \subset \mathcal{E}$ *generates* $\mathcal{E}$ *in a neighbourhood of a orbit parametrized by an immersion* γ *if and only if* δ_γ *is an immersion.*

Definition 2.2. Let $\mathcal{E} = \langle W,\, A,\, B\rangle$ be as above and let $\phi_t : [0,T] \to M$ parametrize a closed orbit of W of period T. We call *rotation number of* L *around* γ *at* $p = \gamma(0)$ the quantity

$$\mathrm{rot}_{\gamma,\,p}\,(L) = \theta(p;T) - \theta(p;0).$$

The rotation number is not an integer. Moreover it depends on the choice of $\{A,\ B\}$, and is not invariant under homotopies of L, as the following example shows.

Example 2.3. The Lie algebra $\mathfrak{g}$ of the Lie group Sol_1^4 is generated by $\{W,\, X,\, Y,\, Z\}$ satisfying $[W,X] = -X$, $[W,Y] = Y$, $[X,Y] = Z$, and all other brackets are zero. We have a left-invariant Engel structure $\mathcal{D} = \langle W,\, X + Y\rangle$ (see [19] for more details). The left-invariant even contact structure $\langle W,\, X,\, Y\rangle$ has characteristic foliation spanned by W, whose flow preserves $\langle X\rangle$ and $\langle Y\rangle$. Hence for each compact quotient Sol_1^4/Γ such that W admits a closed orbit γ, we have $\mathrm{rot}_{\gamma,\,p}\,(X) = 0$. Notice that $L_s = X + sY$ gives a homotopy $\mathcal{D}_{L_s} = \langle W,\, L_s\rangle$ between $\mathcal{D}_{L_0} = \langle W,\, X\rangle$ and $\mathcal{D}_{L_1} = \langle W,\, X + Y\rangle$, which is an Engel structure. In particular $\mathrm{rot}_{\gamma,\,p}\,(X + Y) \neq 0$ by Proposition 2.1.

We have invariance of the rotation number under a smaller family of homotopies of L.

Lemma 2.4. *Let* L_τ *for* $\tau \in [0,1]$ *be a smooth family of vector fields tangent to* $\mathcal{E} = \langle W,\, A,\, B\rangle$ *and nowhere tangent to* $\mathcal{W}$*. If* $L_\tau(p) = L_0(p)$ *for all* $\tau \in [0,1]$ *then* $\mathrm{rot}_{\gamma,\,p}\,(L_1) = \mathrm{rot}_{\gamma,\,p}\,(L_0)$.

Proof. Parametrize γ via ϕ_t, the flow of W, and denote by θ_τ the angle function associated with L_τ. Now the angle functions θ_0 and θ_1 are homotopic relative to the end points through the family of angle functions θ_τ, which concludes the proof. □

If $L(p)$ and $\mathcal{W}$ are fixed, but the homotopy class of L is allowed to vary, the rotation number may vary by an integer multiple of 2π. By definition, $\mathrm{rot}_{\gamma,\,p}\,(L)$ is invariant under homotopies of $\{A,\ B\}$ relative to p. One can show that changing representative in the homotopy class of $\{A,\ B\}$ changes the rotation number at most π (see the proof of Proposition 3.1). This suggests to take into account all possible "initial phases" of L and choices of $\mathcal{B}(p) = \{A(p),\, B(p)\}$. More precisely, identifying L with a map $L : M \to S^1$ via the framing $\{A,\, B\}$, and denoting by $R(\eta)$ the rotation of S^1 of angle $\eta \in \mathbb{R}$, we define

$$\Phi^L_{\gamma,\,\mathcal{B}(p)} : \mathbb{R} \to \mathbb{R} \quad \text{s.t.} \quad \eta \mapsto \mathrm{rot}_{\gamma,\,p}\,(R(\eta) \circ L)\,. \tag{2.1}$$

Taking the maximum with respect to all possible initial phases and choices of $\mathcal{B}(p)$ we get

Definition 2.5. The *maximal rotation number of L along γ* is

$$\mathrm{maxrot}_\gamma\,(L) = \max\Big\{\Phi^L_{\gamma,\,\mathcal{B}(p)}(\eta)\,\Big|\,\eta \in \mathbb{R},\, \mathcal{B}(p)\Big\}.$$

Lemma 2.6. *The maximal rotation number of L along γ does not depend on p.*

Proof. Applying the linearised flow of W, we see that $r = \Phi^L_{\gamma,\,\mathcal{B}(p)}(\eta)$ coincides with the rotation number of $\phi_{t*}R(\eta) \circ L$ calculated with respect to $\phi_{t*}(\mathcal{B}(p))$. There is an angle $\eta' \in \mathbb{R}$ such that $R(\eta') \circ L$ and $\phi_{t*}R(\eta) \circ L$ coincide at $\phi_t(p)$, up to a positive rescaling. Since both $R(\eta') \circ L$ and $\phi_{t*}R(\eta) \circ L$ are homotopic to L, they must be homotopic to each other relative to $\{\phi_t(p)\}$. By Lemma 2.4 $r = \Phi^L_{\gamma,\,\phi_{t*}(\mathcal{B}(p))}(\eta') \le \mathrm{maxrot}_\gamma\,(L)$ calculated in $\phi_t(p)$. Now using transitivity of ϕ_t we conclude the proof. $\square$

Theorem 2.7. *Let $\mathcal{E} = \langle W,\, A,\, B\rangle$ be a distribution of rank 3 such that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, and let γ be a closed orbit for W. Then $\mathcal{D}_L = \langle W,\, L\rangle$ generates $\mathcal{E}$ in a neighbourhood of γ up to homotopy if and only if $|\,\mathrm{maxrot}_\gamma\,(L)\,| > 0$.*

Proof. Let L_τ for $\tau \in [0, 1]$ be a homotopy such that $L = L_0$ and $\langle W,\, L_1\rangle$ is maximally non-integrable within $\mathcal{E}$ in a neighbourhood of γ. We need to show that for some $p \in \gamma$, there is a homotopy relative to $L_1(p)$ between L_1 and $R(\eta) \circ L$ for some $\eta \in \mathbb{R}$. This is done by taking η such that $R(\eta) \circ L(p) = L_1(p)$, exactly as in the proof of Lemma 2.6. This implies the claim thanks to Lemma 2.4 and Proposition 2.1.

Conversely let $|\,\mathrm{maxrot}_\gamma\,(L)\,| > 0$. Without loss of generality we can suppose that $\mathrm{rot}_{\gamma,\,p}\,(L) > 0$. First homotope L relative to $\{p\}$ and to the boundary of $\mathcal{O}p(p)$ to a maximally non-integrable distribution within $\mathcal{E}$ near p. The rotation number does not change by Lemma 2.4. Fix $\mathcal{W} = \langle W\rangle$ and let ϕ_t denote its flow. For $\epsilon > 0$ small, take a disc $D^3 \hookrightarrow M$ centered at $\phi_\epsilon(p)$ and

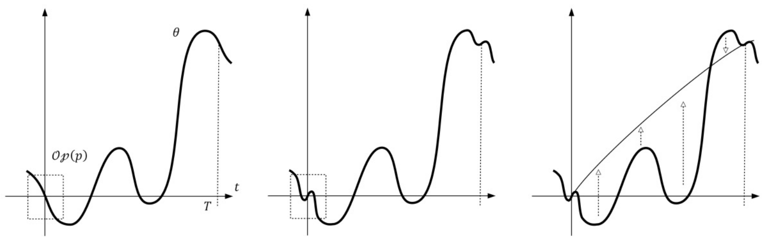

Figure 2.1: Homotopy of θ when the rotation number is positive.

everywhere transverse to $\mathcal{W}$. Up to shrinking the disc D^3, we can suppose that the map $F : D^3 \times [\epsilon, T-\epsilon] \to M$ given by the flow $(q,t) \mapsto \phi_t(q)$ is an embedding and hence a flow box for W. In this chart for $q \in D^3$ we can express

$$F^*L(\phi_t(q)) = \rho(q;t)\Big(\cos\theta(q;t)\ F^*A(\phi_t(q)) + \sin\theta(q;t)\ F^*B(\phi_t(q))\Big),$$

for some functions $\rho > 0$ and θ. Since $\mathrm{rot}_{\gamma,p}(L) > 0$, up to choosing $\epsilon > 0$ small enough, we can suppose that $\theta(0;T-\epsilon) - \theta(0;\epsilon) > 0$. Hence there exists a homotopy $\theta_\tau : D^3 \times [\epsilon, T-\epsilon] \to \mathbb{R}$ such that $\theta_0 = \theta$, the restriction of θ_τ to the boundary $\partial(D^3 \times [\epsilon, T-\epsilon])$ is θ, and

$$\theta_1(0;t) = h(t)\Big(\theta(0;T-\epsilon) - \theta(0;\epsilon)\Big) + \theta(0;\epsilon)$$

for a smooth step function h (see Figure 2.1). This defines a family of vector fields L_t such that $\mathcal{D}_{L_1}$ generates $\mathcal{E}$ on a (possibly smaller) neighbourhood of γ. □

3. Character of closed orbits of $\mathcal{W}$

We now consider the action $[\phi_{t*}] : \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p) \to \mathbb{P}(\mathcal{E}_{\phi_t(p)}/\mathcal{W}_{\phi_t(p)})$ of the flow ϕ_t of a section W of $\mathcal{W}$ on $\mathbb{P}(\mathcal{E}/\mathcal{W})$. This is discussed in detail in [13] for the case of Engel structures. If $p \in M$ is contained in a closed orbit of $\mathcal{W}$ of period T, then $P := [\phi_{T*}] \in \mathrm{PSL}(2,\mathbb{R})$, where we identify $\mathbb{RP}^1 = \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p)$. We say that a closed orbit γ is:

- *Elliptic* if $|\operatorname{tr} P| < 2$ or $P = \pm id$, in which case we can represent P by a rotation $P \equiv R(\delta)$ with $\delta \in \mathbb{R}$.

- *Parabolic* if $|\operatorname{tr} P| = 2$ and $P \neq \pm id$, in which case we can represent P by

$$P \equiv \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

- *Hyperbolic* if $|\operatorname{tr} P| > 2$, in which case we can represent P by

$$P \equiv \pm \begin{pmatrix} e^{-\mu} & 0 \\ 0 & e^{\mu} \end{pmatrix} \qquad \mu \in \mathbb{R}.$$

The following result analyses how $\Phi^L_{\gamma,p}(\eta)$ changes when we change η.

Proposition 3.1. *Let $\mathcal{E} = \langle W, A, B\rangle$ be a distribution of rank 3 such that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, $\mathcal{D}_L$ a distribution of rank 2 such that $\mathcal{D}_L = \langle W, L\rangle \subset \mathcal{E}$, γ a closed orbit for W, and p a point on γ.*

1) *If γ is hyperbolic, then for every $\eta \in \mathbb{R}$ we have $\left|\Phi^L_{\gamma,p}(\eta) - \operatorname{rot}_{\gamma,p}(L)\right| < \pi$. Moreover there exists a constant $c \in (0, \pi)$ such that $\mathcal{D}_L$ positively generates $\mathcal{E}$ in a neighbourhood of γ up to homotopy if and only if $\operatorname{rot}_{\gamma,p}(L) > -c$.*
2) *If γ is parabolic, then for every $\eta \in \mathbb{R}$ we have $\left|\Phi^L_{\gamma,p}(\eta) - \operatorname{rot}_{\gamma,p}(L)\right| < 2\pi$. Moreover there exists a constant $c \in (0, 2\pi)$ such that $\mathcal{D}_L$ positively generates $\mathcal{E}$ in a neighbourhood of γ up to homotopy if and only if $\operatorname{rot}_{\gamma,p}(L) > -c$.*
3) *If γ is elliptic, then $\Phi^L_{\gamma,p}(\eta)$ does not depend on η.*

Proof. The developing map δ^η_γ of the rotated plane field $\mathcal{D}_{R(\eta)L}$ satisfies

$$\delta^\eta_\gamma(T) = [\phi_{-T*} R(\eta) L(p)] = [\phi_{-T*} R(\eta) \phi_{T*}]\delta_\gamma(T) = P^{-1} R(\eta) P \circ \delta_\gamma(T).$$

We need to analyse the rotation induced by $M = P^{-1}R(\eta)P$ on $v = \phi_{-T*}L(p)$. P will rotate v by an angle r, $R(\eta)$ will further rotate it by an angle η, and finally P^{-1} by an angle r'. Now denoting by $\theta_{R(\eta)L}$ and θ_L the rotation angles associated with $R(\eta)L$ and L we have

$$\begin{aligned}\operatorname{rot}_{\gamma,p}(R(\eta)L) &= \theta_{R(\eta)L}(p;T) - \theta_{R(\eta)L}(p;0) \\ &= \theta_L(p;T) + r + \eta + r' - \theta_L(p;0) - \eta,\end{aligned}$$

hence it suffices to study the term $r + r'$. In the case of a hyperbolic orbit we have $|r|$, $|r'| < \pi/2$, whereas for a parabolic orbit we have $|r|$, $|r'| < \pi$. If γ is elliptic, then $M = R(\delta)R(\eta)R(-\delta) = R(\eta)$, so that $r + r' = 0$. □

Remark 3.2. The cases where $\mathrm{rot}_{\gamma,p}(L) \leq 0$ and nonetheless $\mathcal{D}_L = \langle W, L\rangle$ positively generates $\mathcal{E}$ up to homotopy on γ occur only when γ is hyperbolic or parabolic. In these cases $\mathrm{rot}_{\gamma,p}(L)$ is not allowed to be "too negative".

Corollary 3.3. *Let $\mathcal{E} = \langle W, A, B\rangle$ be a distribution of rank 3 such that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, and let γ be an unknotted elliptic closed orbit for W. Then the rotation number $r = \mathrm{rot}_{\gamma,p}(L)$ of $L \in \Gamma\langle A, B\rangle$ does not depend on L. In particular there exists an oriented plane field $\mathcal{D}$ such that $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ and which positively generates $\mathcal{E}$ on a neighbourhood of γ if and only if $r > 0$.*

Proof. Let L and L' be non-singular vector fields in $\langle A, B\rangle$; identify them with maps $L, L' : M \to S^1$. Since γ is unknotted, there is an embedded disc D^2 such that $\partial D^2 = \gamma$, hence there exists a homotopy between L and L'. Since γ is elliptic, by point (3) of Proposition 3.1 we have that $r = \mathrm{rot}_{\gamma,p}(L) = \mathrm{rot}_{\gamma,p}(L')$. The second claim now follows directly from Theorem 2.7. □

Notice that the hypothesis that γ is unknotted is equivalent to γ being null-homotopic if the dimension of M is greater than 3.

4. Morse-Smale vector fields

Since the dynamics of non-singular Morse-Smale (NMS) vector fields can be described once we understand neighbourhoods of the closed orbits, it is reasonable to expect that the rotation number will play a central role when W is NMS. For the basic theory of Morse-Smale vector fields see [10].

4.1. Morse-Smale vector fields and round handle decompositions

Recall that a NMS vector field W is a non-singular vector field which has finitely many non-degenerate closed orbits $\gamma_1, \ldots, \gamma_m$, whose union is the non-wandering set $\Omega = \gamma_1 \cup \cdots \cup \gamma_m$. Moreover for every $i, j \in \{1, \ldots, m\}$ the stable manifold $W^s(\gamma_i)$ and the unstable manifold $W^u(\gamma_j)$ intersect transversely. A round handle decomposition (RHD) of M is a filtration $M_1 \subset M_2 \subset \cdots \subset M_m = M$ where M_k is obtained from M_{k-1} by attaching a *round handle* $R_h = D^h \times D^{n-h-1} \times S^1$. We call h the index of the round handle, $\partial_+ R_h = D^h \times S^{n-h-2} \times S^1$ the enter region or the positive boundary and $\partial_- R_h = S^{h-1} \times D^{n-h-1} \times S^1$ the exit region or the negative boundary.

Theorem 4.1 [15]**.** *Let W be a non-singular Morse-Smale vector field on M. Then M admits a RHD $M_1 \subset M_2 \subset \cdots \subset M_m = M$ such that every round handle R is a neighbourhood of a closed orbit γ of W, and the index of R (as a handle) is the index of γ (as a closed orbit). The attaching procedure is performed via the flow of W, which is transverse to ∂M_k pointing outwards for every k.*

We include a sketch of the proof for completeness, since it will be relevant in the proof of Theorem 4.2.

Sketch of proof. The idea is to order the closed orbits of M via $\gamma_i \leq \gamma_j$ if $W^u(\gamma_i) \cap W^s(\gamma_j) \neq 0$, and reason by induction. In other words, $\gamma_i \leq \gamma_j$ if there is a orbit whose α-limit is γ_i and whose ω-limit is γ_j. The *no cycle condition* (see [18]) ensures that this is compatible with a total ordering of $\{\gamma_1, \ldots, \gamma_m\}$.

The first orbits in the ordering are the source orbits, i.e. the ones for which $W^u = \emptyset$, hence we construct M_1 by taking a neighbourhood of γ_1. Suppose that we have constructed inductively M_{k-1} such that $\gamma_1, \ldots, \gamma_{k-1} \subset M_{k-1}$, $\gamma_j \cap M_{k-1} = \emptyset$ for $j > k-1$, and the flow is transverse to ∂M_{k-1} pointing outwards. If γ_k is a source orbit, then we take a neighbourhood R_k disjoint from M_{k-1} and define $M_k = M_{k-1} \cup R_k$.

If γ_k is not a source orbit, this means that M_{k-1} contains all of them, so a generic point in $M \setminus \{\gamma_1, \ldots, \gamma_{k-1}\}$ has to have one of the source orbits as α-limit. We take a small tubular neighbourhood R_k of γ_k and we attach it using all flow lines of W that have α-limit in M_{k-1}. This might introduce corners and the boundary of M_k will not be transverse to W. For these reasons we smoothen it as illustrated in Figure 4.1. For further details on the proof see [15]. □

4.2. Morse-Smale flows preserving a 3-distribution

We give a necessary and sufficient condition for the existence of $\mathcal{D} \subset \mathcal{E}$ that generates $\mathcal{E}$ when W is NMS.

Theorem 4.2. *Let $\mathcal{E} = \langle W, A, B \rangle$ be a rank 3 distribution on a manifold M, and denote by $\mathcal{W} = \langle W \rangle$. Suppose that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, and let W be a non-singular Morse-Smale vector field. There exists $\mathcal{D} \subset \mathcal{E}$ such that $\mathcal{W} \subset \mathcal{D}$ and that positively generates $\mathcal{E}$ on M if and only if there exists $L \in \Gamma\langle A, B \rangle$ such that* $\operatorname{maxrot}_\gamma(L) > 0$ *for all closed orbits γ of W.*

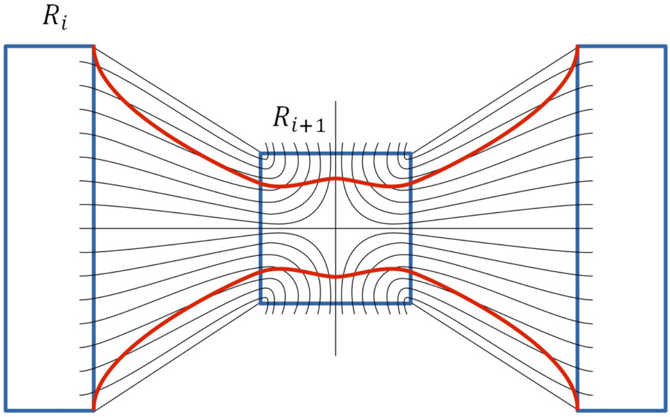

Figure 4.1: Smoothen the corners.

Proof. If such a plane field $\mathcal{D}$ exists, then we can take L to be any vector field satisfying $\mathcal{D} = \langle W, L\rangle$ and the claim follows by Proposition 2.1. The idea for the converse is to construct $\mathcal{D}$ inductively using the RHD of Theorem 4.1. First we construct $\mathcal{D}$ in a neighbourhood of the source orbit γ_1 using Theorem 2.7. Suppose that we have attached $k-1$ handles to obtain M_{k-1}, and that we want to attach the k-th handle R_k. If γ_k is a source orbit then we construct $\mathcal{D}$ on $R_k = \mathcal{O}p(\gamma_k)$ as above, and attach it by disjoint union $M_k = M_{k-1} \cup R_k$. This procedure yields a plane field $\mathcal{D}$ homotopic to $\mathcal{D}_L = \langle W, L\rangle$ which generates $\mathcal{E}$ along the core of each handle.

If γ_k is not a source orbit, the proof Theorem 4.1 ensures that R_k is a neighbourhood of γ_k, and that the attaching procedure happens via the flow of W. We first construct $\mathcal{D}$ on R_k using Theorem 2.7. The existence of L ensures that the $\mathcal{D}$ extends to a plane field on M_k which generates $\mathcal{E}$ on a neighbourhood of M_{k-1} and of γ_k.

In general we cannot homotope this plane field to a maximally non-integrable one on M_k. The problem is that the attaching region is of the form $\partial_+ R_k \times I$, where $\partial_+ R_k \times \{1\}$ is the subset of R_k where W points inwards, and W is tangent to the I-factor on $\partial_+ R_k \times I$. This means that, on the universal cover $\partial_+ \tilde{R}_k \times I$, a lift of L takes the form $\tilde{L} = \cos f_t\, \tilde{A} + \sin f_t\, \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are lifts of A and B, and $f_t : \partial_+ \tilde{R}_k \to \mathbb{R}$ is a I-family of angle functions. Hence we can homotope L transversely to ∂_t so that $\langle \partial_t, L\rangle$ generates $\mathcal{E}$ if and only if $f_1 > f_0$. There is no reason for this to happen in general.

Let $K = \max\{f_1(p) - f_0(p) | p \in \partial_+ \tilde{R}_k\}$. For any $p \in \partial M_{k-1} \times (-\epsilon, \epsilon)$ on a collared neighbourhood ∂M_{k-1}, the vector field L can be described by an embedding $h_p : (-\epsilon, \epsilon) \to S^1$. We substitute it by $\tilde{h}_p : (-\epsilon, \epsilon) \to S^1$ which coincides with h on $\mathcal{O}p(\{-\epsilon, \epsilon\})$, and such that it makes a number of turns around S^1 bigger than $K/2\pi$. In this way we obtain a new vector field L', not homotopic to L in general, and such that its associated family of angle functions f'_t satisfies $f'_1 > f'_0$. We can now homotope L' to a maximally non-integrable plane field within $\mathcal{E}$ on the attaching region. We might now need to round the corners of M_k, and this can be done exactly as in the proof of Theorem 4.1 (see Figure 4.1). □

5. Morse-Smale Legendrians and even contact structures

Theorem 4.2 gives a necessary and sufficient condition for a NMS vector field to be Legendrian. An interesting example of 3-manifold admitting NMS vector fields is S^3. However only very few 3-manifolds admit such vector fields (see [15, Theorem A]). It is interesting to know when a given vector field L is transverse to a contact structure. This question has already been studied in [9] for L tangent to the fibres of a S^1-bundle over a surface, and in [11] for L tangent to the fibres of a Seifert fibration.

If L is Legendrian for some orientable contact structure $\mathcal{D}$, then there is a contact structure $\tilde{\mathcal{D}}$ transverse to L. Indeed choose $\tilde{L}$ such that $\mathcal{D} = \langle L, \tilde{L} \rangle$ and consider $\tilde{\mathcal{D}} = \phi_{\epsilon *}\mathcal{D}$, where ϕ_ϵ denotes the flow of $\tilde{L}$ for small time ϵ. The contact condition ensures that $\tilde{\mathcal{D}}$ is transverse to $\mathcal{D}$, moreover it contains $\tilde{L}$, so it is transverse to L.

Corollary 5.1. *There exists a vector field on S^3 which is transverse to a contact structure but never Legendrian.*

Proof. Consider the vector field W normal to the canonical Reeb foliation on S^3. Using the theory of confoliations [5, Chapter 2] we can $\mathcal{C}^0$-deform the tangent bundle of the Reeb foliation to get a contact structure, so that L is transverse to a contact structure. L has two unknotted elliptic closed orbits with trivial monodromy, which obstructs the existence of a contact structure for which L is Legendrian. □

Theorem 4.2 suggests to study even contact structures which admit a NMS section of $\mathcal{W}$. It is not clear if every parallelizable 4-manifold admits such structures, and in fact many NMS flows on 4-manifolds cannot span the characteristic foliation of an even contact structure. On the other hand this property becomes true if we allow perturbations of W.

Lemma 5.2. *Near every closed orbit γ of a NMS vector field W on M there exists an even contact structure $\mathcal{E}$, whose characteristic line field is spanned by a $\mathcal{C}^0$-perturbation of W fixing γ.*

Proof. Up to a perturbation, γ has a tubular neighbourhood $\nu\gamma = S^1 \times D^3$ where

$$W|_{\nu\gamma} = \partial_\theta + 2\epsilon_1 x\partial_x + 2\epsilon_2 y\partial_y + 4\epsilon_3 z\partial_z,$$

with $\epsilon_i = \pm 1$ depending on the index of γ. This is proven using the linearised Poincaré map (see [10] for more details). If ϵ_i are all equal, then W is Liouville for the symplectic form $\omega = dx \wedge dy + dz \wedge d\theta$, so we have an even contact form $\alpha = i_W\omega$. If the ϵ_i are not all equal, then the vector field $V = 2\epsilon_1 x\partial_x + 2\epsilon_2 y\partial_y + 4\epsilon_3 z\partial_z$ preserves the contact structure defined by $\eta = dz - xdy + ydx$ on D^3, so that $\nu\gamma$ is the suspension of the time 1 flow of V (see [13] for more details on this construction). □

The methods developed in this paper are well-suited for constructing examples of even contact structures which do not admit compatible Engel structures.

Proposition 5.3. *Every even contact structure is $\mathcal{C}^0$-close to one which is not induced by an Engel structure.*

Proof. On a manifold M consider an even contact structure $\mathcal{E}$ with characteristic foliation $\mathcal{W}$. On a small neighbourhood U construct an even contact structure $\mathcal{E}'$ with a contractible characteristic closed orbit γ having trivial monodromy. Make sure that $\mathcal{E}'|_{\mathcal{O}p(\gamma)}$ extends to a formal even contact structure on M, which coincides with $\mathcal{E}$ on $M \setminus U$. We conclude the proof using the (relative) complete h-principle for even contact structures (see [12]). □

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