

Fiber Floer cohomology and conormal stops

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Let S be a closed orientable spin manifold. Let $K \subset S$ be a submanifold and denote its complement by M_K . In this paper we prove that there exists an isomorphism between partially wrapped Floer cochains of a cotangent fiber stopped by the unit conormal Λ_K and chains of a Morse theoretic model of the based loop space of M_K , which intertwines the A_∞ -structure with the Pontryagin product. As an application, we restrict to codimension 2 spheres $K \subset S^n$ where $n = 5$ or $n \geq 7$. Then we show that there is a family of knots K so that the partially wrapped Floer cohomology of a cotangent fiber is related to the Alexander invariant of K . A consequence of this relation is that the link $\Lambda_K \cup \Lambda_x$ is not Legendrian isotopic to $\Lambda_{\text{unknot}} \cup \Lambda_x$ where $x \in M_K$.

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1. Introduction

In this paper we consider the wrapped Floer cohomology of a cotangent fiber with wrapping stopped by a conormal. We relate it to chains of based loops on the complement of a submanifold. Then we show that the Legendrian conormal knows about the smooth topology of the submanifold beyond the fundamental group.

Let S be a closed orientable spin manifold. Let $K \subset S$ be a submanifold and denote its complement by M_K . Consider the disk cotangent bundle DT^*S equipped with the canonical Liouville form $\lambda = pdq$. The ideal contact boundary of the Weinstein domain DT^*S is the unit cotangent bundle ST^*S . Associated to K are the conormal bundle

$$L_K = \{(q, p) \in T^*S \mid q \in K, \langle p, T_qK \rangle = 0\} \subset DT^*S,$$

and the unit conormal $\Lambda_K = L_K \cap ST^*S$. Consider a cotangent fiber $F = DT^*_\xi S$ at $\xi \in M_K$ and let $CW^*_{\Lambda_K}(F, F)$ be the partially wrapped Floer cochains on F with wrapping stopped by Λ_K . Let BM_K denote the space of piecewise geodesic loops in M_K based at ξ . Consider the space $C^{cell}_{-*}(BM_K)$ of cellular chains of BM_K equipped with the Pontryagin product. Then we have the following result:

Theorem 1 (Theorem 5 and Theorem 6). *There exists a geometrically defined isomorphism of A_∞ -algebras $\Psi: CW^*_{\Lambda_K}(F, F) \longrightarrow C^{cell}_{-*}(BM_K)$.*

*Moreover, Ψ induces an isomorphism $HW^*_{\Lambda_K}(F, F) \longrightarrow H_{-*}(\Omega_\xi M_K)$ of $\mathbb{Z}[\pi_1(M_K)]$ -modules.*

We define $CW^*_{\Lambda_K}(F, F)$ using a surgery approach similar to [EL17, Appendix B] and [ENS16, Section 6] (see Section 3.1 for details). The outline of the surgery approach is the following. We attach a handle modeled on $D_\epsilon T^*([0, \infty) \times \Lambda_K)$ to DT^*S along a neighborhood of Λ_K . We denote the resulting Liouville sector by W_K (with terminology as in [GPS20]). Then $CW^*_{\Lambda_K}(F, F)$ is the wrapped Floer cochain complex of F in W_K . The skeleton of W_K is $L_K \cup S$ with clean intersection $L_K \cap S = K$. By performing Lagrangian surgery along the clean intersection, we obtain an exact Lagrangian submanifold $M_K \subset W_K$ which is diffeomorphic to the complement $S \setminus K$ (see Section 3.1 for details).

Let $\Omega_\xi M_K$ denote the space of loops in M_K based at ξ . Consider singular chains on the space of based loops $C_{-*}(\Omega_\xi M_K)$. We give it the structure of an A_∞ -algebra by equipping it with the Pontryagin product and all higher

products equal to zero. See Subsection 3.2 and Subsection 4.2 for a more detailed discussion about the model of the based loop space we use.

In the spirit of Cieliebak–Latschev [CL09] and Abouzaid [Abo12b], we have a geometrically defined A_∞ -homomorphism $\Psi: CW_{\Lambda_K}^*(F, F) \rightarrow C_{-*}(\Omega_\xi M_K)$. By analyzing the action filtrations, we show that Ψ is diagonal with respect to the action filtrations. A key point in proving that Ψ is an isomorphism is showing that the disks contributing to the diagonal are transversely cut out. The solutions of the linearized Floer equation are precisely those vector fields along the disk that restricts to broken Jacobi fields along γ on which the Hessian of the energy functional is negative definite.

In the surgery approach we attach a handle modeled on $D_\varepsilon T^*([0, \infty) \times \Lambda_K)$, with skeleton $[0, \infty) \times \Lambda_K$. We consider a generic product metric on $[0, \infty) \times \Lambda_K$ such that the metric on Λ_K is scaled by a positive function with strictly negative derivative (warped product metric), see (A.1). By the genericity of the metric, there is a natural one-to-one correspondence between Reeb chords and geodesics, see Lemma 15 for details.

Since W_K and M_K are non-compact we use monotonicity of J -holomorphic curves to prove that relevant moduli spaces of J -holomorphic curves are compact, see Section A for details.

1.1. Applications

Let Q be a smooth manifold and let $K \subset Q$ be a submanifold. Consider the cotangent bundle T^*Q and the unit conormal bundle Λ_K . It is known in certain cases that the symplectic topology of T^*Q knows about the smooth topology of Q [Abo12a, ES16, EKS16]. In some cases the contact topology of Λ_K knows about the smooth topology of K . For instance, it is known that conormal tori $\Lambda_K \subset ST^*\mathbb{R}^3$ of knots $K \subset \mathbb{R}^3$ are complete knot invariants [She16, ENS16]. The results of Ekholm–Ng–Shende fit nicely into the broader picture of partially wrapped Floer cohomology that we consider in this paper, and is summarized in [ENS16, Section 1.3]. Specifically, in [ENS16] it is proven that there is a ring isomorphism

$$HW_{\Lambda_K}^0(F, F) \cong \mathbb{Z}[\pi_1(M_K)],$$

which is also obtained from Theorem 1 by restricting to degree 0. Furthermore there is a relation between the knot contact homology of $K \subset \mathbb{R}^3$ and the Alexander polynomial of K [Ng08, ENS16].

Let $K \subset S^n$ be a codimension 2 sphere. In this paper we show that the partially wrapped Floer cohomology of the fiber is related to the Alexander

invariant. The Alexander invariant is $H_*(\widetilde{M}_K)$ regarded as a $\mathbb{Z}[\pi_1(M_K)]$ -module, where \widetilde{M}_K denotes the infinite cyclic cover of M_K , see Subsection 5.4 for details. Denote by Λ_{unknot} the unit conormal of the standard embedded $S^{n-2} \subset S^n$. As an application of Theorem 1 we have the following theorem.

Theorem 2 (Theorem 7). *Let $n = 5$ or $n \geq 7$. Let $x \in M_K$ be a point. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$, such that $\Lambda_K \cup \Lambda_x$ is not Legendrian isotopic to $\Lambda_{\text{unknot}} \cup \Lambda_x$.*

1.2. Relation to other results

Let Q be a closed smooth manifold and consider the exact symplectic manifold $(T^*Q, d\lambda)$ where λ is the canonical Liouville form $\lambda = pdq$. Abbondandolo–Schwarz proved that the wrapped Floer cohomology of a cotangent fiber $T^*_\xi Q$ is isomorphic to the homology of the based loop space of Q [AS06]. Abouzaid extended this to an A_∞ -quasi-isomorphism in [Abo12b] where the loop space is equipped with the Pontryagin product. Recently, Ganatra–Pardon–Shende proved that this result continues to hold even when Q is not assumed to be compact as a consequence of a deeper relationship between the wrapped Fukaya category of a Liouville sector and a certain category of sheaves [GPS18a].

In this paper, we consider a similar J -holomorphic curve setup to the one used by Abouzaid in [Abo12b], but instead we work in the context of the partially wrapped Fukaya category of T^*S stopped by the unit conormal Λ_K .

Remark 1. Another interesting geometric point of view which motivates Theorem 1 is the following. Consider the wrapped Fukaya category of T^*M_K [GPS20, BKO19]. By [GPS18a, Corollary 6.1] we have an A_∞ -quasi-isomorphism

$$CW^*(F, F) \cong C_{-*}(\Omega_\xi M_K),$$

where $F \subset T^*M_K$ is the cotangent fiber at $\xi \in M_K$. We realize W_K as the result of attaching a handle to T^*M_K as follows: Take a tubular neighborhood $N(K) \subset S$ of K and consider $N'(K) := N(K) \cap M_K$. Then remove $L_{N'(K)} \subset T^*M_K$ and replace it with $L_{N(K)}$, identifying their common boundaries $\Lambda_{N(K)} = \Lambda_{N'(K)}$.

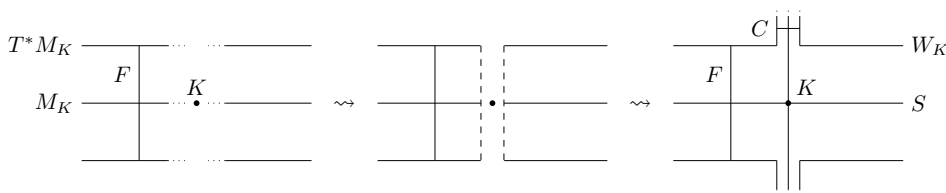


Figure 1: The figure shows the construction of W_K via handle attachment on T^*M_K .

From the point of view of handle attachment, there is a new generator of the wrapped Fukaya category, namely the cocore disk C . Because of this, the wrapped Floer cohomology of the fiber F will change on the level of chains. However, if we push C very far out in the punctured handle by a Lagrangian isotopy, we look at filtered A_∞ -algebras and yield a chain isomorphism

$$\mathcal{F}_L CW^*(F, F)_{W_K} \cong \mathcal{F}_L CW^*(F, F)_{T^*M_K},$$

where \mathcal{F}_L means we only consider generators of action less than L . A standard filtration argument then shows that the wrapped Floer cohomology of F is unaffected by this type of handle attachment and thus $HW^*(F, F)_{W_K} \cong HW^*(F, F)_{T^*M_K}$. Hence we obtain an indirect proof of the isomorphism

$$HW_{\Lambda_K}^*(F, F) = HW^*(F, F)_{W_K} \cong HW^*(F, F)_{T^*M_K} \cong H_{-*}(\Omega M_K)$$

in Theorem 1.

1.3. Organization of the paper

In Section 2 we describe the version of wrapped Floer cohomology defined without Hamiltonian perturbations which we use in this paper. In Section 3 we first discuss the surgery approach to define partially wrapped Floer cohomology. Then we define the operations $\Psi = \{\Psi_m\}_{k=1}^\infty$ between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}(\Omega_\xi M_K)$ and show that Ψ is an A_∞ -homomorphism. Section 4 is devoted to proving that Ψ is an isomorphism between $CW_{\Lambda_K}^*(F, F)$ and the Morse theoretic model of chains of based loops. Lastly, in Section 5 we equip $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}(\Omega_\xi M_K)$ with $\mathbb{Z}[\pi_1(M_K)]$ -module structures relate $HW_{\Lambda_K}^*(F, F)$ to the Alexander invariant $H_*(\widetilde{M}_K)$ for certain families of codimension 2 knots $K \subset S^n$. Then we show that this relation is used to show that $\Lambda_K \cup \Lambda_x$ is not Legendrian isotopic to $\Lambda_{\text{unknot}} \cup \Lambda_x$, where $x \in S^n \setminus K$ is a point.

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2. Wrapped Floer cohomology without Hamiltonian

In this paper, we consider a version of wrapped Floer cohomology defined without Hamiltonian perturbations. Wrapped Floer cohomology without Hamiltonian has been studied in e.g. [Ekh12, DR16, EL17] and in particular it is useful in proving various surgery formulas involving the wrapped Floer cohomology [BEE12, EL17, Ekh19]. It has also been used to study knots via knot contact homology from which there is a relationship to string topology and the cord algebra [ENS16, EENS13, CELN17].

Remark 2. The relationship between wrapped Floer cohomology defined with and without Hamiltonians has also been studied. The version without Hamiltonian is known to be quasi-isomorphic to the version defined with Hamiltonians by counting strips with a Hamiltonian term that is turned on as one goes from the positive end to the negative end [EHK16, Theorem 7.2]. Such J -holomorphic maps with a Hamiltonian term that turns on has been more systematically studied in [EO17] and it is proven in [EL17, Lemma 68, 69] that the two versions of wrapped Floer cohomology are A_∞ -quasi-isomorphic.

When working with wrapped Floer cohomology without Hamiltonian we have a priori bubbling issues. This is circumvented by considering parallel copies, which also removes the possibility of having multiply covered curves, see [EL17, Section 3.3]. Furthermore we need to count anchored curves [BEE12, Section 2.2] [EL17, Section A.1]. A specific perturbation scheme involving anchored curves is constructed in [Ekh19], and we fix such perturbation scheme so that all relevant moduli spaces are transversely cut out.

We give a brief description of the wrapped Floer cohomology without Hamiltonian by following [EL17, Appendix A-B]. We consider a Weinstein domain M together with a smooth exact Lagrangian submanifold

$(M, \omega := d\lambda, L)$. Let $Y := \partial M$ and $\Lambda := L \cap Y$ be its Legendrian boundary. The boundary $(Y, \alpha := \lambda|_Y)$ is a contact manifold. We consider the completion of M and L by attaching cylindrical ends $[0, \infty) \times Y$ to Y and $[0, \infty) \times \Lambda$ to Λ . Then we pick a system of parallel copies of L as in [EL17, Section 3.3]. Consider a family $(H_k, h_k)_{k=1}^\infty$ of pairs of Morse functions, $H_k: L \rightarrow \mathbb{R}$ and $h_k: \Lambda \rightarrow \mathbb{R}$. Let L_k be the time-1 flow of L of the Hamiltonian vector field X_{H_k} , and let $\Lambda_k := L_k \cap Y$. Then we call $\{L_k\}_{k=0}^\infty$ a *system of parallel copies of L* where $L_0 := L$. Let $\bar{L} := \bigcup_{k=0}^\infty L_k$ and $\bar{\Lambda} := \bigcup_{k=0}^\infty \Lambda_k$.

Note that in this paper, L is a cotangent fiber. Therefore we choose the Morse functions H_k in such a way that all of them only have one minimum, since $L \cong D^n$.

2.1. A_∞ -structure and moduli space of disks

Let (M, λ) be a spin Weinstein domain. Let $L \subset M$ be an orientable exact Lagrangian with vanishing Maslov class (see [Arn67] for a definition of the Maslov class). Let $\bar{L} = \{L_k\}_{k=0}^\infty$ be the corresponding system of parallel copies of L as in the previous section.

First we define $CW^*(L, L)$ as a \mathbb{Z} -graded module over \mathbb{Z} . Note that, for each Reeb chord c' starting at Λ_i and ending at Λ_j , there is a unique Reeb chord c of Λ close to c' . Similarly, for each transverse intersection point a' in $L_i \cap L_j$, there is a unique transverse intersection point $a \in L_0 \cap L_1$. We implicitly fix an identification of c' with c , and a' with a . We then define $CW^*(L, L)$ to be the \mathbb{Z} -graded module over \mathbb{Z} , which is generated by Reeb chords of Λ and intersection points $L_0 \cap L_1$. The grading is given by the Maslov index (see Remark 4 below for a more precise definition).

We now describe how we equip $CW^*(L, L)$ with a A_∞ -structure $\{\mu^i\}_{i=1}^\infty$ which is defined by J -holomorphic curve counts. Let $D_m \subset \mathbb{C}$ denote the positively oriented unit disk, with m points along the boundary removed. We denote the boundary punctures in D_m by ζ_1, \dots, ζ_m , one of which is distinguished. These boundary punctures subdivide the boundary of D_m into m arcs. We enumerate these arcs by $\kappa_1, \dots, \kappa_m$, according to the boundary orientation, starting from the distinguished boundary puncture. We call $\kappa := \{\kappa_i\}_{i=1}^m$ a *boundary numbering* of D_m . If the sequence κ is decreasing (increasing), we say that the disk D_m has decreasing (increasing) boundary numbering κ . If $\kappa_{i-1} \leq \kappa_i$ ($\kappa_{i-1} \geq \kappa_i$), we say that the puncture ζ_i is increasing (decreasing), and if $\kappa_{i-1} = \kappa_i$ we say that ζ_i is a constant puncture.

We equip the boundary punctures $\zeta_j \in \partial D_m$ with both a positive and a negative strip-like end. Namely, we pick biholomorphisms

$$\begin{cases} \varepsilon_+^i : (0, \infty) \times [0, 1] \longrightarrow N(\zeta_i) \\ \varepsilon_-^i : (-\infty, 0) \times [0, 1] \longrightarrow N(\zeta_i) \end{cases} \quad \forall i \in \{1, \dots, m\} ,$$

where $N(\zeta_i)$ is a neighborhood of the boundary puncture $\zeta_i \in \partial D_m$.

Using notation as in [EL17], we are interested in the moduli spaces of J -holomorphic disks which are denoted by $\mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$, $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ and $\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)$. These moduli spaces consist of *filling disks*, *symplectization disks* and *partial holomorphic buildings* respectively, and we define them below.

Filling disks: Consider D_m equipped with a strictly decreasing boundary numbering. Note that every puncture is strictly decreasing except for the distinguished puncture, which is strictly increasing. We let $\mathbf{c} = c_1 \cdots c_m$ be a word of generators of $CW^*(L, L)$. Then we define $\mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$ to be the moduli space of J -holomorphic maps $u : (D_m, \partial D_m) \rightarrow (M, \bar{L})$ such that

- near the boundary puncture ζ_i , u is asymptotic to the generator c_i , that is

$$\begin{cases} \lim_{s \rightarrow \pm\infty} u(\varepsilon_{\pm}^i(s, t)) = c_i, & \text{if } c_i \text{ is an intersection generator} \\ \lim_{s \rightarrow \pm\infty} u(\varepsilon_{\pm}^i(s, t)) = (\infty, c_i), & \text{if } c_i \text{ is a Reeb chord generator.} \end{cases}$$

The sign in the above formulas is equal to $-$ if $i = j$, and $+$ otherwise.

- u maps the boundary arc labeled by κ_j to the component L_{κ_j} of \bar{L} .

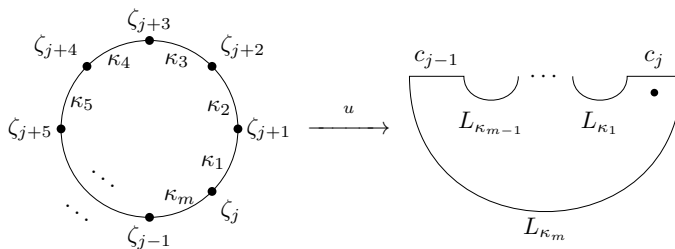


Figure 2: A J -holomorphic disk in $\mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$. The dot on the right hand side indicates that ζ_j (near which, u is asymptotic to c_j) is the distinguished puncture.

Symplectization disks: Consider D_m equipped with a decreasing boundary numbering (not necessarily strictly decreasing). Let $D_{m,k} :=$

$D_m \setminus \{ \zeta_1^{\text{in}}, \dots, \zeta_k^{\text{in}} \}$, where each ζ_i^{in} is a point in the interior of D_m . We equip each ζ_i^{in} with a negative cylinder-like end. That is a biholomorphism

$$\varphi_-^i : (0, \infty) \times S^1 \longrightarrow N(\zeta_i^{\text{in}}) \quad \forall i \in \{1, \dots, k\} .$$

We let $\mathbf{c} = c_1^{\sigma_1} \cdots c_m^{\sigma_m}$ be a word of signed Reeb chord generators of $CW^*(L, L)$, where $\sigma_i \in \{+, -\}$ for every i . We also let $\gamma = \gamma_1 \cdots \gamma_k$ be a word of Reeb orbits in Y , each of which is equipped with an *asymptotic marker*, i.e. a point $p_i \in \text{im } \gamma_i$. The distinguished boundary puncture ζ_j induces an asymptotic marker for each interior puncture ζ_i^{in} , which is a half-ray $\varphi^{-1}((-\infty, 0) \times \{x_i\})$ near ζ_i [EO17, Section 2.1]. By abuse of notation we say that $x_i \in S^1$ is the asymptotic marker of ζ_i^{in} . Then we define $\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma; \kappa)$ to be the moduli space of J -holomorphic maps $u : (D_{m,k}, \partial D_{m,k}) \longrightarrow (\mathbb{R} \times Y, \mathbb{R} \times \overline{A})$ such that

- near the boundary puncture ζ_i , u is asymptotic to the Reeb chord $c_i^{\sigma_i}$ of A at $\pm\infty$, depending on the sign σ_i , that is

$$\lim_{s \rightarrow \pm\infty} u(\varepsilon_{\pm}^i(s, t)) = (\pm\infty, c_i) .$$

- near the interior puncture ζ_i^{in} , u is asymptotic to the Reeb orbit γ_i in Y at $-\infty$ respecting the asymptotic markers, that is

$$\begin{cases} \lim_{s \rightarrow -\infty} u(\varphi_-^i(s, t)) = (-\infty, \gamma_i) \\ \lim_{s \rightarrow -\infty} u(\varphi_-^i(s, x_i)) = (-\infty, p_i) . \end{cases}$$

- u maps the boundary arc labeled by κ_j to the component $\mathbb{R} \times A_{\kappa_j}$ of $\mathbb{R} \times \overline{A}$, and
- if ζ_i is a constant puncture, we require ζ_i to be a *negative puncture* (i.e. asymptotic to a Reeb chord of A at $-\infty$).

Let γ be a Reeb orbit in Y , equipped with the asymptotic marker $p \in \text{im } \gamma$. Let S denote S^2 with one puncture $\zeta \in S^2$, with a fixed choice of asymptotic marker x at ζ . Equip ζ with a positive cylinder-like end

$$\varphi_+ : (0, \infty) \times S^1 \longrightarrow N(\zeta) .$$

Let $\mathcal{M}^\lambda(\gamma)$ be the λ -perturbed moduli space of J_λ -holomorphic maps $u : S \longrightarrow X$ with notation as in [Ekh19, Theorem 1.1], satisfying

$$\begin{cases} \lim_{s \rightarrow \infty} u(\varphi_+(s, t)) = (\infty, \gamma) \\ \lim_{s \rightarrow \infty} u(\varphi_+(s, x)) = (\infty, p) . \end{cases}$$

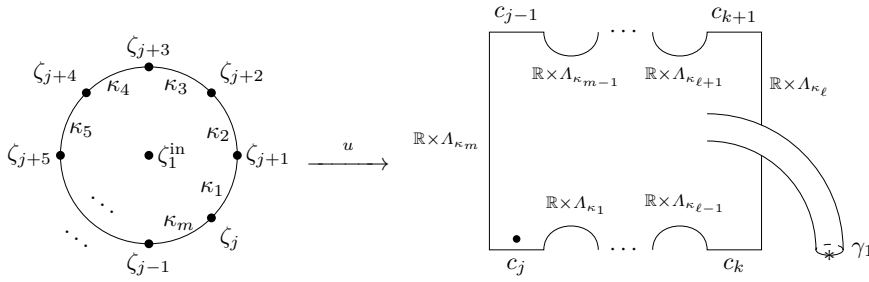


Figure 3: A J -holomorphic disk in $\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma_1; \kappa)$. The dot on the right hand side indicates that the puncture ζ_j is the distinguished puncture. The $*$ on the right hand side is the asymptotic marker $p_1 \in \text{im } \gamma_1$.

Then we define

$$\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa) := \bigcup_{\gamma} \left(\mathcal{M}^{\text{neg}}(\mathbf{c}, \gamma; \kappa) \times \prod_{\gamma_i \in \gamma} \mathcal{M}^{\lambda}(\gamma_i) \right),$$

See [Ekh19] and [EL17, Appendix A.1] for more deatils. Each curve in $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ should be interpreted as curves shown in Figure 3, but with all Reeb orbits capped off by punctured J_{λ} -holomorphic spheres.

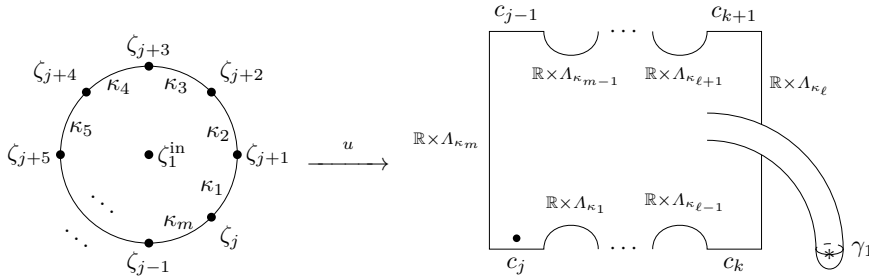


Figure 4: A J -holomorphic disk in $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$. The dot on the right hand side indicates that the puncture ζ_j is the distinguished puncture. The $*$ on the right hand side is the asymptotic marker $p_1 \in \text{im } \gamma_1$.

Partial holomorphic buildings: The domain of a partial holomorphic building is a possibly broken disk with $m + 1$ boundary punctures, see Figure 5. We denote this (possibly broken) disk by D_{m+1} and equip

it with a decreasing boundary numbering κ . In the target, the partial holomorphic building consists of a two-level J -holomorphic building, with exactly one symplectization disk (called the *primary disk*), and multiple filling disks (called *secondary disks*). We require that the distinguished puncture (which is the only increasing puncture), is a negative puncture of the primary disk. If the primary disk only has one negative puncture, the primary disk is the only component, and the disk is not broken. If the primary disk has more than 1 negative puncture, each additional negative puncture has a secondary disk attached to it, at the distinguished puncture of the secondary disks disk. If $\mathbf{c} = c_0c_1 \cdots c_m$ is a word of generators of $CW^*(L, L)$, where c_0 is the generator to which the distinguished puncture is asymptotic to, we denote the moduli space of partial holomorphic buildings by $\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)$.

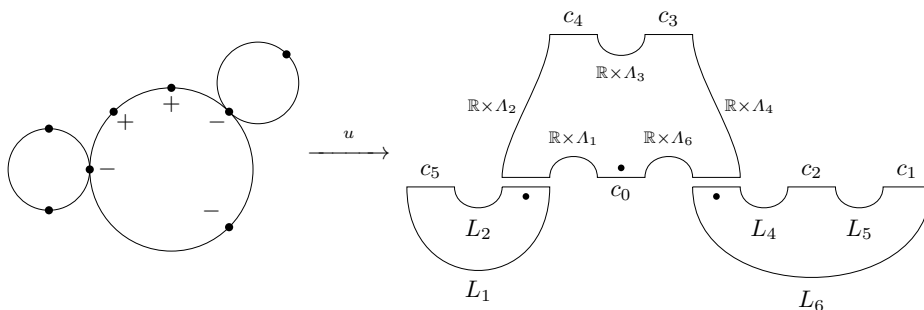


Figure 5: A partial holomorphic building in $\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)$. The dots on the right hand side indicate the distinguished punctures of the corresponding disks. The signs on the left hand side indicate the sign of the punctures of the primary disk.

Remark 3. Take note that we might have additional negative punctures of the symplectization disk, at which there are constant filling disks with only 1 positive puncture attached. We have not depicted these above, but they should nonetheless be taken into account.

By [EL17, Theorem 63,65] and [Ekh19, Theorem 1.1], $\mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$ and $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ are transversely cut out smooth manifolds that are independent of the boundary numbering κ up to diffeomorphism. This follows from the observation that disks in $\mathcal{M}^{\text{fi}}(\mathbf{c}; \kappa)$ or $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ can not be multiply covered for topological reasons. Transversality is then proved using standard techniques

as in [EES07]. Furthermore the moduli spaces admit compactifications that consists of J -holomorphic buildings of several levels.

Remark 4. For Reeb chord generators the grading $|a|$ is more explicitly described as follows. Suppose that $a: [0, \ell] \rightarrow Y$, then we first define the Conley–Zehnder index $\text{CZ}(a)$ by following [EES05, Section 2.2]. Namely, let a^- and a^+ be the start and endpoints of the Reeb chord a , respectively. Then pick a capping path $\gamma_c: [0, 1] \rightarrow \Lambda \subset Y$ so that $\gamma_c(0) = a^+$, $\gamma_c(1) = a^-$. Let $\alpha = \lambda|_{\partial M}$ and $\xi = \ker \alpha$. Then $T_{a^+}\Lambda \subset \xi_{a^+}$ is a Lagrangian submanifold. By parallel transport along γ_c and via the linearized Reeb flow we get a path of Lagrangian submanifolds in the contact planes $\xi \subset TY$. If we close this path up by *positive close-up* in the contact planes we obtain a loop of Lagrangian submanifolds in ξ denoted by Γ_a . We then define the Conley–Zehnder index of a to be the Maslov index of Γ_a (in the sense of [RS93]),

$$\text{CZ}(a) := \mu(\Gamma_a).$$

Then we define

$$|a| = -\text{CZ}(a) + (n - 1).$$

For Lagrangian intersection generators $x \in L_0 \cap L_1$ we use the choice of graded lifts of L_0 and L_1 to obtain a path starting at $T_x L_1$ and ending at $T_x L_0$. We close this path up in $T_x M$ by a positive rotation. This gives a loop of Lagrangian submanifolds denoted by Γ_x , which starts and ends at $T_x L_0 \subset T_x M$. Then define the grading of x as the Maslov index of this loop [EL17, p. 89] [CEL10, Appendix A]

$$|x| := \mu(\Gamma_x).$$

The dimension of the moduli space $\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$ is dependent on whether the distinguished puncture is a Reeb chord or a Lagrangian intersection puncture. To emphasize the differences, we introduce some more notation.

- If the distinguished puncture is a Reeb chord generator we denote it by $\mathcal{M}^{\text{fi,Reeb}}(\mathbf{a}; \kappa)$, and
- if the distinguished puncture is an intersection generator we denote it by $\mathcal{M}^{\text{fi,Lag}}(\mathbf{a}; \kappa)$.

Theorem 3. *Let $\mathbf{a} = ca_2 \cdots a_m$ be a word of generators of $CW^*(L, L)$. Assume that c is the distinguished puncture and that it is a Reeb chord*

generator. Then the dimension of the moduli space $\mathcal{M}^{\text{fi,Reeb}}(\mathbf{a}; \kappa)$ is

$$\dim \left(\mathcal{M}^{\text{fi,Reeb}}(\mathbf{a}; \kappa) \right) = (n - 3) + m - |c| - \sum_{j=2}^m |a_j| .$$

Let $\mathbf{a} = xa_2 \cdots a_m$ be a word of generators of $CW^*(L, L)$. Assume that x is the distinguished puncture and that it is a Lagrangian intersection generator. Then the dimension of the moduli space $\mathcal{M}^{\text{fi,Lag}}(\mathbf{a}; \kappa)$ is

$$\dim \left(\mathcal{M}^{\text{fi,Lag}}(\mathbf{a}; \kappa) \right) = -3 + m - |x| - \sum_{j=2}^m |a_j| .$$

For any word of Reeb chord generators $\mathbf{c} = c_1 \cdots c_m$, the dimension of the moduli space $\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)$ is

$$\dim \left(\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa) \right) = (n - 3) + m + \sum_{\sigma_j=-} (|c_j| - (n - 1)) - \sum_{\sigma_j=+} |c_j| .$$

For any word of Reeb chord generators $\mathbf{c} = c_0 \cdots c_m$, the dimension of the moduli space $\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa)$ is

$$\dim \left(\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa) \right) = -1 + m + |c_0| - \sum_{j=1}^m |c_j| .$$

Proof. The theorem follows from applying [CEL10, Theorem A.1], and the fact that the index of a several-level J -holomorphic building is the sum of the indices of the disks at each level. Let $\mathbf{a} = a_1 \cdots a_m$ be a word of generators of $CW^*(L, L)$. Let either $u \in \mathcal{M}^{\text{sy}}(\mathbf{a}; \kappa)$ or $u \in \mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$. Let \widehat{D}_m be the unit disk in \mathbb{C} together with ζ_1, \dots, ζ_m regarded as marked points (and not punctures). The boundary of \widehat{D}_m is equal to the union of closed boundary arcs C such that the interiors of all the boundary arcs C are pairwise disjoint, and only missing the marked points $\{\zeta_1, \dots, \zeta_m\}$.

- (T1) For all Reeb chord generators $a_i \in \mathbf{a}$, fix a complex trivialization Z_{a_i} of the contact structure ξ along a_i , such that the linearized Reeb flow along the chord a_i expressed in Z_{a_i} is constantly equal to the identity.
- (T2) For each boundary arc C in \widehat{D}_m , fix a complex trivialization Z_C of u^*TM (if $u \in \mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$) or $u^*T(\mathbb{R} \times Y)$ (if $u \in \mathcal{M}^{\text{sy}}(\mathbf{a}; \kappa)$) with the following properties:
 - a) If an endpoint of C is a puncture ζ_i asymptotic to a Reeb chord a_i , then $Z_C = Z_{a_i}$.

- b) If an endpoint of C is a puncture ζ_i asymptotic to an intersection generator $x_i \in L_{\kappa_i} \cap L_{\kappa_{i+1}}$, then $Z_C = Z_{C'}$ where ζ_i is the common endpoint of the boundary arcs C and C' .

Items (T1) and (T2) above give a complex trivialization $Z_{\partial_j u}$ of u^*TM (or $u^*T(\mathbb{R} \times Y)$) over the j^{th} boundary arc C_j of \widehat{D}_m . For each boundary arc C_j , let C'_j be the complement of its endpoints in C_j . The tangent planes of L along all $f(C'_j)$ expressed in the trivialization $Z_{\partial_j u}$ gives a collection of paths of Lagrangian subspaces in \mathbb{C}^n . We close up this path to a loop as follows. For each Reeb chord $a_i \in \mathbf{a}$, denote its start and endpoints by a_i^\pm respectively.

- (C1) For each positive puncture ζ_i near which u is asymptotic to the Reeb chord a_i , the tangent planes of $L = \mathbb{R} \times \Lambda$ are connected by the product of the linearized Reeb flow along a_i in ξ , and the identity in the \mathbb{R} -factor, followed by negative close-up in the contact plane in $\xi_{a_i^+} \times \mathbb{C}$ (cf. Remark 4). Denote this path of Lagrangian subspaces by $g_{a_i^+}$.
- (C2) For each negative puncture ζ_i near which u is asymptotic to the Reeb chord a_i , the tangent planes of $L = \mathbb{R} \times \Lambda$ are connected by the product of the backwards linearized Reeb flow along a_i in ξ , and the identity in the \mathbb{R} -factor, followed by negative close-up in the contact plane in $\xi_{a_i^-} \times \mathbb{C}$ (cf. Remark 4). Denote this path of Lagrangian subspaces by $g_{a_i^-}$.
- (C3) For each puncture ζ_i near which u is asymptotic to the intersection generator $x_i \in L_{\kappa_i} \cap L_{\kappa_{i+1}}$, connect the planes $T_{x_i}L_{\kappa_i}$ and $T_{x_i}L_{\kappa_{i+1}}$ by a negative rotation taking $T_{x_i}L_{\kappa_i}$ to $T_{x_i}L_{\kappa_{i+1}}$ in \mathbb{C}^n (cf. Remark 4 and [CEL10, Remark A.1]). Denote this path of Lagrangian subspaces by $g_{x_i}^\cap$.

Define $\mu(\partial u, Z_{\partial u})$ to be the Maslov index of the loop of Lagrangian subspaces in \mathbb{C}^n which is constructed by closing up paths of Lagrangian subspaces as described in (C1), (C2) and (C3). For the moduli spaces of filling disks and symplectization disks, we then have by [CEL10, Theorem A.1.] that

$$\begin{aligned} \dim \left(\mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa) \right) &= (n - 3) + m + \mu(\partial u, Z_{\partial u}) \\ \dim \left(\mathcal{M}^{\text{sy}}(\mathbf{a}; \kappa) \right) &= (n - 3) + m + \mu(\partial u, Z_{\partial u}) . \end{aligned}$$

Since L is assumed to have vanishing Maslov class, the contribution to $\mu(\partial u, Z_{\partial u})$ is equal to the sum of each contribution at every boundary puncture of D_m . Next we describe each of these contributions in terms of the grading of each generator. First let $u \in \mathcal{M}^{\text{sy}}(\mathbf{a}; \kappa)$.

(sy1) If ζ_i is a positive puncture near which u is asymptotic to the Reeb chord a_i then

$$\mu(g_{a_i}^+ \circ (\Gamma_{a_i})^{-1}) = -(n - 1) \Leftrightarrow \mu(g_{a_i}^+) = \mu(\Gamma_{a_i}) - (n - 1) = -|a_i| .$$

(sy2) If ζ_i is a negative puncture near which u is asymptotic to the Reeb chord a_i then

$$\mu(g_{a_i}^- \circ \Gamma_{a_i}) = 0 \Leftrightarrow \mu(g_{a_i}^-) = -\mu(\Gamma_{a_i}) = |a_i| - (n - 1) .$$

Then let $u \in \mathcal{M}^{\text{fi}}(\mathbf{a}; \kappa)$.

(fi1) Let ζ_i be a puncture near which u is asymptotic to the Reeb chord a_i .

a) If ζ_i is the distinguished puncture then

$$\mu(g_{a_i}^+ \circ \Gamma_{a_i}) = 0 \Leftrightarrow \mu(g_{a_i}^+) = -\mu(\Gamma_{a_i}) = |a_i| - (n - 1) .$$

b) If ζ_i is not the distinguished puncture then

$$\mu(g_{a_i}^+ \circ (\Gamma_{a_i})^{-1}) = -(n - 1) \Leftrightarrow \mu(g_{a_i}^+) = \mu(\Gamma_{a_i}) - (n - 1) = -|a_i| .$$

(fi2) Let ζ_i be a puncture near which u is asymptotic to the intersection generator x_i

a) If ζ_i is the distinguished puncture then

$$\mu(g_{x_i}^\cap \circ (\Gamma_{x_i})^{-1}) = -n \Leftrightarrow \mu(g_{x_i}^\cap) = \mu(\Gamma_{x_i}) - n = |x_i| - n .$$

b) If ζ_i is not the distinguished puncture then

$$\mu(g_{x_i}^\cap \circ \Gamma_{x_i}) = 0 \Leftrightarrow \mu(g_{x_i}^\cap) = -\mu(\Gamma_{x_i}) = -|x_i| .$$

From (sy1) and (sy2) we obtain

$$\dim(\mathcal{M}^{\text{sy}}(\mathbf{c}; \kappa)) = (n - 3) + m + \sum_{\sigma_j = -} (|c_j| - (n - 1)) - \sum_{\sigma_j = +} |c_j| .$$

From (fi1)(a), (fi1)(b) and (fi2)(b) we obtain

$$\dim(\mathcal{M}^{\text{fi,Reeb}}(\mathbf{a}; \kappa)) = (n - 3) + m - |c| - \sum_{j=2}^m |a_j| .$$

From (fi2)(a), (fi1)(b) and (fi2)(b) we obtain

$$\begin{aligned} \dim \left(\mathcal{M}^{\text{fi,Lag}}(\mathbf{a}; \kappa) \right) &= (n - 3) + m + (|x| - n) - \sum_{j=2}^m |a_j| \\ &= -3 + m + |x| - \sum_{j=2}^m |a_j|. \end{aligned}$$

For a partial holomorphic building, let a_1, \dots, a_p be the positive punctures of the primary disk, let d_0 be the distinguished negative puncture of the primary disk and let d_1, \dots, d_q be the remaining negative punctures. Let b_1, \dots, b_r be all the non-distinguished punctures of all the secondary disks, see Figure 6. Each secondary disk lies in $\mathcal{M}^{\text{fi,Reeb}}(\mathbf{a}; \kappa)$. We may then compute the dimension by taking sums, that is

$$\begin{aligned} \dim \left(\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa) \right) &= \left[(n - 3) - \sum_{j=1}^p (|a_j| - 1) + \sum_{j=0}^q (|d_j| - (n - 2)) \right] \\ &\quad + \left[\sum_{j=1}^q \left((n - 3) - (|d_j| - 1) \right) + \sum_{j=1}^r (|b_j| - 1) \right]. \end{aligned}$$

After canceling we get

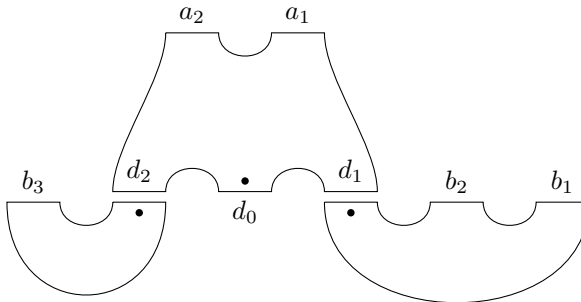


Figure 6.

$$\begin{aligned} \dim \left(\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa) \right) &= (n - 3) - \sum_{j=1}^p (|a_j| - 1) + (|d_0| - (n - 2)) - \sum_{j=1}^r (|b_j| - 1) \\ &= -1 + (p + r) + |d_0| - \sum_{j=1}^p |a_j| - \sum_{j=1}^r |b_j|. \end{aligned}$$

Now let $c_0 := d_0$ and let \mathbf{c} be the word of $m := p + r$ letters corresponding to all the generators $a_1, \dots, a_p, b_1, \dots, b_r$ in the appropriate order. Therefore

$$\dim \left(\mathcal{M}^{\text{pb}}(\mathbf{c}; \kappa) \right) = m - 1 + |c_0| - \sum_{j=1}^m |c_j| .$$

□

We now define operations, one for each $i \geq 1$,

$$\mu^i : CW^*(L_{\kappa_{i-1}}, L_{\kappa_i}) \otimes \cdots \otimes CW^*(L_{\kappa_1}, L_{\kappa_2}) \longrightarrow CW^*(L_{\kappa_1}, L_{\kappa_i}) ,$$

that counts various J -holomorphic disks discussed above. We split it as a sum $\mu^i = \mu_{\text{Lag}}^i + \mu_{\text{Reeb}}^i$, where μ_{Lag}^i takes values in Lagrangian intersection generators and μ_{Reeb}^i takes values in Reeb chord generators.

First we consider μ_{Lag}^i . Let $\mathbf{c}' = c_1 \cdots c_i$ be a word of generators of $CW^*(L, L)$. Then

$$\mu_{\text{Lag}}^i(c_i \otimes \cdots \otimes c_1) := \sum_{|c_0|=|\mathbf{c}'|+(2-i)} \left| \mathcal{M}^{\text{fi,Lag}}(c_0 \mathbf{c}'; \kappa) \right| c_0 .$$

The sum is taken over all Lagrangian intersection generators c_0 so that $\dim \left(\mathcal{M}^{\text{fi,Lag}}(c_0 \mathbf{c}'; \kappa) \right) = 0$.

To define μ_{Reeb}^i , consider a word of generators $\mathbf{c}' = c_1 \cdots c_i$. Then

$$\mu_{\text{Reeb}}^i(c_i \otimes \cdots \otimes c_1) := \sum_{|c_0|=|\mathbf{c}'|+(2-i)} \left| \mathcal{M}^{\text{pb}}(c_0 \mathbf{c}'; \kappa) \right| c_0 .$$

The sum is taken over all Reeb chords c_0 so that $\dim \left(\mathcal{M}^{\text{pb}}(c_0 \mathbf{c}'; \kappa) \right) = 0$. The total operation μ^i is then defined as

$$(2.1) \quad \mu^i(c_i \otimes \cdots \otimes c_1) = (-1)^\diamond \left(\mu_{\text{Lag}}^i(c_i \otimes \cdots \otimes c_1) + \mu_{\text{Reeb}}^i(c_i \otimes \cdots \otimes c_1) \right)$$

where

$$\diamond = \sum_{j=1}^i j |c_j| .$$

Lemma 1. *With the sign conventions as in [Sei08], $(CW^*(L, L), \{\mu^i\}_{i=1}^\infty)$ forms an A_∞ -algebra, that is*

$$\sum_{\substack{d_1+d_2=d+1 \\ 0 \leq k < d_1}} (-1)^{\boxtimes_k} \mu^{d_1}(c_d, \dots, c_{k+d_2+1}, \mu^{d_2}(c_{k+d_2}, \dots, c_{k+1}), c_k, \dots, c_1) = 0,$$

where

$$\boxtimes_k = k + \sum_{j=1}^k |c_j|.$$

Proof. See [EL17, Lemma 67]. □

3. Partially wrapped Floer cohomology and chains of based loops

Let S be any closed orientable spin manifold and $K \subset S$ any submanifold. The purpose of this section is to describe the surgery approach to compute the partially wrapped Floer cohomology of a cotangent fiber in the Weinstein domain $(DT^*S, \lambda = pdq)$ stopped by the unit conormal Λ_K . We then define a chain map relating the partially wrapped Floer cohomology of a fiber to chains of based loops on a Lagrangian submanifold M_K that is diffeomorphic to the complement $S \setminus K$.

In Section 3 we describe the surgery approach in more detail, and also construct the Lagrangian M_K . In Subsection 3.2 we describe the model we use for the chains of based loops on M_K , and equip it with the Pontryagin product. Then in Subsection 3.3 we describe the moduli space of half strips which we need in order to define an A_∞ -homomorphism between the partially wrapped Floer cocomplex and the chains of based loops on M_K . The construction of the A_∞ -homomorphism is carried out in Subsection 3.4.

3.1. Partially wrapped Floer cohomology using a surgery approach

Following [EL17, Appendix B] and [ENS16, Section 6] we will now describe the surgery approach. We consider the disk cotangent bundle DT^*S the conormal bundle of K

$$L_K = \left\{ (q, p) \in DT^*S \mid q \in K, \langle p, T_q K \rangle = 0 \right\}.$$

Let $\Lambda_K := L_K \cap ST^*S$ be the unit conormal of K . We take a tubular neighborhood U of Λ_K in ST^*S and we attach a handle modeled on $D_\varepsilon T^*([0, \infty) \times \Lambda)$ to U . After handle attachment and after smoothing out corners, the Liouville vector field is equal to $p\partial_p$ in $D_\varepsilon T^*([T, \infty) \times \Lambda)$ (for $T \geq 0$ large enough) for coordinates (q, p) in the handle. We call the resulting manifold W_K , see Figure 7.

We then consider a cotangent fiber $F \cong DT_\xi^*S$ at $\xi \in M_K$ in W_K . Denote the wrapped Floer cochains of F in W_K as described in Section 2 by $CW_{\Lambda_K}^*(F, F)$.

Remark 5. In the language of Sylvan [Syl19], we obtain a *stop* σ_{Λ_K} from Λ_K as follows. Pick a tubular neighborhood $U \supset \Lambda_K$ in ST^*S , and a strict contactomorphism $\varphi: (U, \lambda|_U) \rightarrow (V, dz - ydx)$ where V is a tubular neighborhood of $\Lambda_K \subset J^1(\Lambda_K) = T^*\Lambda_K \times \mathbb{R}$, viewed as the zero section. Then the Liouville hypersurface $\sigma_{\Lambda_K} := \varphi^{-1}(T^*\Lambda_K \cap V) \subset U$ is a stop.

Another point of view, is to remove the tubular neighborhood U from ST^*S , and take the Liouville completion of $(DT^*S) \setminus U$ to obtain a Liouville sector as defined [GPS20]. The wrapped Fukaya category of this Liouville sector coincides with the wrapped Fukaya category associated to the pair (M, σ_{Λ_K}) , and also with the Fukaya category associated to W_K [EL17, GPS20, GPS18b].

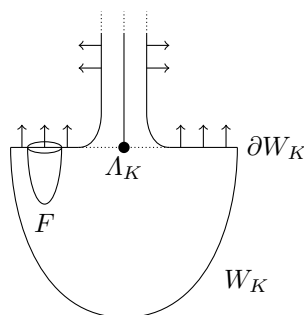


Figure 7: The Liouville sector W_K .

To construct the complement Lagrangian M_K , we perform Lagrangian surgery of L_K and S which intersect cleanly along K . Above each point of K , the intersection $L_K \cap S$ looks like the transverse intersection of two Lagrangian disks of dimension k . We perform Lagrangian surgery along K as in [MW18, Section 2.2.2] [AENV14]. We denote the result of the surgery by $M_K \cong S \setminus K$ (cf. [AENV14]).

Remark 6. Note that the Maslov class of M_K vanishes, because it is the result of surgery of $S \subset W$ and $L_K \subset W$, both of which have vanishing Maslov class. In particular, consider the following model. We pick a \mathbb{C}^n -neighborhood around $p \in K$ such that $L_K = i\mathbb{R}^n$ and $S = \mathbb{R}^n$. Following the discussion in [ES16, Section 2.2], we have a phase function $\phi: H \rightarrow \mathbb{R}$ which is unique up to an additive constant on the handle H , so that $\phi|_{S \cap H} = 0$ and $\phi|_{L_K \cap H} = n - 1$.

Any loop that is based at any point outside of the handle pass through the entire handle an even number of times, which means that the total Maslov index of the loop is zero.

3.2. Based loops on M_K

Consider the Moore loop space of M_K , based at ξ

$$\Omega_\xi M_K = \{ \gamma: [0, R] \rightarrow M_K \mid \gamma(0) = \gamma(R) = \xi \} .$$

We use a cubical model for chains of based loops as in [Abo12b, EL17].

A *singular k -cube* is a smooth map $\sigma: [0, 1]^k \rightarrow \Omega_\xi M_K$ and it is called *degenerate* if $\sigma(x_1, \dots, x_k)$ is constant in at least one of the coordinates. We define the space of cubical k -chains by

$$C_k(\Omega_\xi M_K) = \frac{\mathbb{Z}[\text{singular } k\text{-cubes}]}{\mathbb{Z}[\text{degenerate singular } k\text{-cubes}]} .$$

We equip $C_*(\Omega_\xi M_K)$ with the differential

$$(3.1) \quad \partial\sigma := \sum_{i=1}^k \sum_{\varepsilon=0}^1 (-1)^{i+\varepsilon} \sigma(\delta_{i,\varepsilon}(x_1, \dots, x_k)) ,$$

where

$$\delta_{i,\varepsilon}(x_1, \dots, x_k) = (x_1, \dots, x_{i-1}, \varepsilon, x_{i+1}, \dots, x_k), \quad \varepsilon \in \{0, 1\}$$

is the map that replaces the i -th coordinate with ε .

The Pontryagin product P is defined as the following composition:

$$(3.2) \quad C_k(\Omega_\xi M_K) \otimes C_\ell(\Omega_\xi M_K) \rightarrow C_{k+\ell} \left((\Omega_\xi M_K)^2 \right) \rightarrow C_{k+\ell}(\Omega_\xi M_K)$$

$$\sigma_2 \otimes \sigma_1 \longmapsto (-1)^{|\sigma_1|} \sigma_2 \times \sigma_1 \longmapsto (-1)^{|\sigma_1|} \sigma_1 \circ \sigma_2$$

The cross product of a singular i -cube σ_1 and a j -cube σ_2 is the $(i + j)$ -cube

$$\begin{aligned} \sigma_1 \times \sigma_2: [0, 1]^{i+j} &\longrightarrow \Omega_\xi M_K \times \Omega_\xi M_K \\ (x_1, \dots, x_{i+j}) &\longmapsto (\sigma_1(x_1, \dots, x_i), \sigma_2(x_{i+1}, \dots, x_{i+j})). \end{aligned}$$

The map \circ is pointwise concatenation of loops where we first follow $\sigma_1(x_1, \dots, x_i)$, and then $\sigma_2(x_{i+1}, \dots, x_{i+j})$. That is

$$(\sigma_1 \circ \sigma_2)(x) = \sigma_1(x_1, \dots, x_i) \circ \sigma_2(x_{i+1}, \dots, x_{i+j}),$$

where

$$\begin{aligned} &(\sigma_1(x_1, \dots, x_i) \circ \sigma_2(x_{i+1}, \dots, x_{i+j}))(t) \\ &:= \begin{cases} \sigma_1(x_1, \dots, x_i)(t), & t \in [0, R_1] \\ \sigma_2(x_{i+1}, \dots, x_{i+j})(t - R_1), & t \in [R_1, R_1 + R_2]. \end{cases} \end{aligned}$$

From the definitions of P and ∂ we see that for any two singular cubes $\sigma_1 \in C_k(\Omega_\xi M_K)$ and $\sigma_2 \in C_\ell(\Omega_\xi M_K)$ we have

$$\partial(\sigma_1 \circ \sigma_2) = (-1)^k(\sigma_1 \circ \partial\sigma_2) + \partial\sigma_1 \circ \sigma_2.$$

This leads via (3.2) to

$$\partial P(\sigma_2 \otimes \sigma_1) + P(\sigma_2 \otimes \partial\sigma_1) + (-1)^{k+1}P(\partial\sigma_2 \otimes \sigma_1) = 0.$$

Hence $(C_*(\Omega_\xi M_K), \partial, P)$ is an A_∞ -algebra with all higher operations being zero with sign conventions as in [AS10, Sei08].

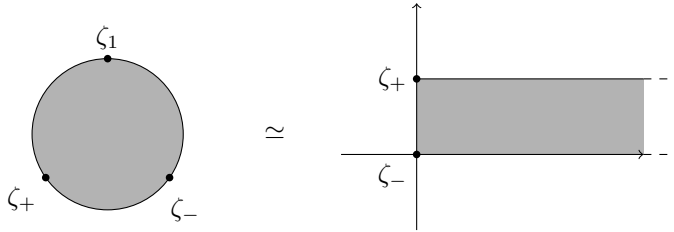
3.3. Moduli space of half strips

Consider the cotangent fiber $F \cong T_\xi^*S \subset W_K$ at $\xi \in M_K$ defined in Section 3.1 and consider a system of parallel copies of F as in Section 2. In this section we construct a moduli spaces of J -holomorphic half strips similar to [Abo12b]. This moduli space is used to define a chain map between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}(\Omega_\xi M_K)$. By non-compactness of W_K in the horizontal direction, we use monotonicity for J -holomorphic half strips to establish compactness of moduli spaces, see Section A for details.

Let $D_3 \subset \mathbb{C}$ be the positively oriented unit disk with three boundary punctures $\zeta_+, \zeta_-, \zeta_1$. Then D_3 is biholomorphic to

$$T := ([0, \infty) \times [0, 1]) \setminus \{\zeta_+, \zeta_-\} \subset \mathbb{C},$$

where $\zeta_+ = (0, 1) \in \mathbb{C}$ and $\zeta_- = (0, 0) \in \mathbb{C}$. The boundary segment between ζ_+ and ζ_- is called the *outgoing segment*.



Define

$$\begin{cases} Z_- = (-\infty, 0) \times [0, 1] \subset \mathbb{C} \\ Z_+ = (0, \infty) \times [0, 1] \subset \mathbb{C}, \end{cases}$$

equipped with the standard complex structure j on \mathbb{C} . We pick a positive strip-like end ε^+ near ζ_+ , and a negative strip-like end ε_- near ζ_- . That is, ε_{\pm} are maps

$$\begin{aligned} \varepsilon_+ : Z_+ &\longrightarrow T \\ \varepsilon_- : Z_- &\longrightarrow T \end{aligned}$$

defined in neighborhoods of ζ_+ and ζ_- respectively. Fix a family $\{J_t\}_{t \in [0,1]} \subset \mathcal{J}(W_k, \omega)$ of ω -compatible almost complex structures, parametrized by $t \in [0, 1]$. Then consider a map

$$J_T : T \longrightarrow \mathcal{J}(W_K, \omega)$$

which satisfies

$$\begin{cases} J_T(s, t) = J_t, & s > N \text{ for some } N > 0 \\ (\varepsilon_-)^* J_T = J_t, & \text{near } \zeta_- \\ (\varepsilon_+)^* J_T = J_t, & \text{near } \zeta_+. \end{cases}$$

Given a generator $a \in CW_{\Lambda_K}^*(F, F)$ we consider maps

$$u : T \longrightarrow W_K$$

that satisfies the following Floer equation:

$$(3.3) \quad \begin{cases} du + J_T \circ du \circ j = 0 \\ \lim_{s \rightarrow \infty} u(s, t) = a(t), & \forall t \in [0, 1] \\ \lim_{s \rightarrow \infty} u(\varepsilon_+(s, t)) = \xi, & \forall t \in [0, 1] \\ \lim_{s \rightarrow -\infty} u(\varepsilon_-(s, t)) = \xi, & \forall t \in [0, 1] \end{cases}$$

where the boundary conditions on u is indicated in Figure 8 below.

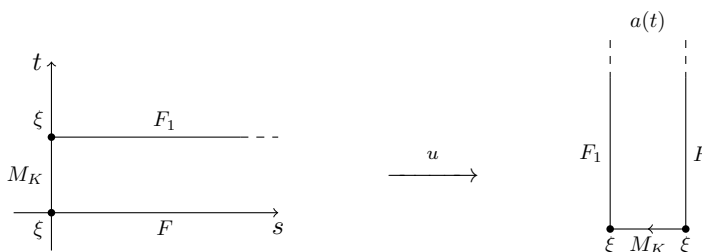


Figure 8: A J_T -holomorphic disk in $\mathcal{M}(a)$.

For a generator $a \in CW_{\Lambda_K}^*(F, F)$ we define $\mathcal{M}(a)$ to be the moduli space of J_T -holomorphic maps $u: T \rightarrow W_K$ that satisfies (3.3).

Analogous to [EL17, Theorem 63] and [Abo12b, Lemma 4.2] we have

Lemma 2. *For generic choices of almost complex structure J_T , the moduli space $\mathcal{M}(a)$ is a smooth orientable manifold of dimension*

$$\dim \mathcal{M}(a) = -|a| .$$

Proof. See the proof of Lemma 3 for the proof of the statement about the dimension. Note that, because we work with a system of parallel copies of F , J_T -holomorphic curves can not be multiply covered, and transversality for such is achieved using standard methods as in [EES07, EL17]. \square

Let $D_{m+2} \subset \mathbb{C}$ be the positively oriented unit disk with $m + 2$ boundary punctures which we denote by $\zeta_-, \zeta_1, \dots, \zeta_m, \zeta_+$. Let \mathcal{R}_m be the Deligne–Mumford space of unit disks in the complex plane with $m + 1$ boundary punctures that are oriented counterclockwise. Let $\overline{\mathcal{R}}_m$ denote the Deligne–Mumford compactification of \mathcal{R}_m as in [Abo10, Section C.1] and [Sei08, Section (9f)]. Also define \mathcal{H}_m to be the Deligne–Mumford space of unit disks in the complex plane with $m + 2$ boundary punctures that are oriented

counterclockwise. Its Deligne–Mumford compactification is denoted by $\overline{\mathcal{H}}_m$. The boundary of $\overline{\mathcal{H}}_m$ is obtained by adding broken disks and hence the codimension one boundary of $\overline{\mathcal{H}}_m$ is covered by the following spaces

$$(3.4) \quad \overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{H}}_{m_2}, \quad m_1 + m_2 = m$$

$$(3.5) \quad \overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{R}}_{m_2}, \quad m_1 + m_2 = m + 1$$

where we regard each stratum as being included in $\overline{\mathcal{H}}_m$ via the natural inclusion.

Consider a word of generators $a_k \in CW_{\Lambda_K}^*(F_{k-1}, F_k)$

$$\mathbf{a} = a_1 \cdots a_m.$$

Then we define the moduli space $\mathcal{M}(\mathbf{a})$ to be maps

$$u: T \longrightarrow W_K,$$

where $T \in \overline{\mathcal{H}}_m$, and so that u satisfies the following Floer equation

$$\begin{cases} du + J_T \circ du \circ j = 0 \\ \lim_{s \rightarrow \infty} u(\varepsilon^k(s, t)) = a_k(t), \quad \forall t \in [0, 1] \text{ and } k \in \{1, \dots, m\} \\ \lim_{s \rightarrow \infty} u(\varepsilon_+(s, t)) = \xi, \quad \forall t \in [0, 1] \\ \lim_{s \rightarrow -\infty} u(\varepsilon_-(s, t)) = \xi, \quad \forall t \in [0, 1] \end{cases}$$

where $\varepsilon_{\pm}: Z_{\pm} \rightarrow T$ and $\varepsilon^k: Z_+ \rightarrow T$ are strip-like ends near each puncture ζ^{\pm} and ζ^k for $k \in \{1, \dots, m\}$. The boundary conditions of u is indicated in Figure 9 below

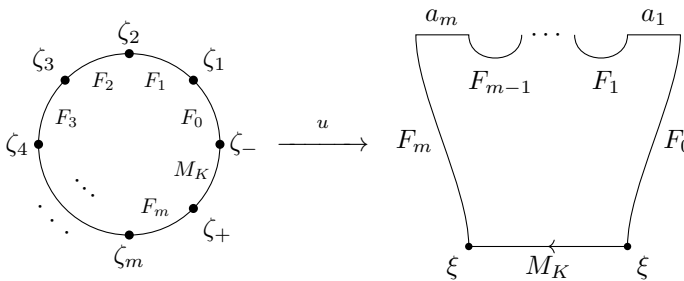


Figure 9: A J_T -holomorphic disk in $\mathcal{M}(\mathbf{a})$.

Again, analogous to [EL17, Theorem 63] and [Abo12b, Lemma 4.7] we have the following standard transversality result.

Lemma 3. *For a generic choice of almost complex structure, $\mathcal{M}(\mathbf{a})$ is a smooth orientable manifold of dimension*

$$\dim \mathcal{M}(\mathbf{a}) = -1 + m - \sum_{j=1}^m |a_j| .$$

Proof. We first observe that disks in $\mathcal{M}(\mathbf{a})$ have switching boundary condition which implies that they can not be multiply covered for topological reasons. Then transversality is proved using standard techniques as in [EES07, EL17].

We now prove the statement about the dimension. The proof is similar to the proof of Theorem 3. By [CEL10, Theorem A.1.] we have

$$\dim \mathcal{M}(\mathbf{a}) = (n - 3) + m + 2 + \mu(\partial u, Z_{\partial u}) ,$$

where $\mu(\partial u, Z_{\partial u})$ is defined as in the proof of Theorem 3. There is a new type of contribution coming from the Lagrangian intersection punctures ζ_{\pm} . By definition of $\mu(\partial u, Z_{\partial u})$ we see that the sum of the contributions from both ζ_{\pm} is equal to $-n$. The Maslov class of M_K vanishes (see Remark 6), so the only contributions to $\mu(\partial u, Z_{\partial u})$ comes from the generators \mathbf{a} and the Lagrangian intersection punctures ζ_{\pm} . Therefore

$$\dim \mathcal{M}(\mathbf{a}) = (n - 1) + m - n - \sum_{j=1}^m |a_j| = -1 + m - \sum_{j=1}^m |a_j| .$$

Furthermore, by vanishing of the Maslov class of M_K (see Remark 6) it allows us to find a coherent orientation of the moduli spaces. See Section B for a more general discussion about orientations. □

Since W_K is non-compact, we use monotonicity together with a generically chosen metric to make sure J_T -holomorphic half strips do not escape to horizontal infinity, see Section A and in particular Theorem 8. This gives that $\mathcal{M}(\mathbf{a})$ can be compactified by adding several-level curves and we denote the compactification by $\overline{\mathcal{M}}(\mathbf{a})$. Similar to [Abo12b, Lemma 4.9] and by (3.4), (3.5) the codimension one boundary of $\overline{\mathcal{M}}(\mathbf{a})$ is stratified as

$$(3.6) \quad \partial \overline{\mathcal{M}}(\mathbf{a}) = \coprod_{\substack{\tilde{\mathbf{a}} \subset \mathbf{a} \\ t+s+r=m}} \overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}}) \amalg \coprod_{\substack{\mathbf{a}' \mathbf{a}'' = \mathbf{a} \\ m_1+m_2=m}} \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'') .$$

Note that we define $\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ to mean either $\mathcal{M}^{\text{pb}}(\tilde{\mathbf{a}})$ or $\mathcal{M}^{\text{fi,Lag}}(\tilde{\mathbf{a}})$ as in Section 2, depending on whether the breaking happens at a Reeb chord or a Lagrangian intersection generator.

To be more precise, (3.6) means that the codimension one boundary of $\partial\overline{\mathcal{M}}(\mathbf{a})$ is covered by images of the natural inclusions of products $\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ for subwords $\tilde{\mathbf{a}} \subset \mathbf{a}$ and $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$ for partitions

$$\mathbf{a} = \underbrace{a_1 \cdots a_{m_1}}_{=\mathbf{a}'} \underbrace{a_{m_1+1} \cdots a_{m_1+m_2}}_{=\mathbf{a}''} .$$

Note that $\mathbf{a} \setminus \tilde{\mathbf{a}}$ is the word of generators obtained by starting with the word \mathbf{a} and replacing the subword $\tilde{\mathbf{a}}$ with an auxiliary generator y , see Figure 10 and Figure 11. If $\mathbf{a} = a_1 \cdots a_m$ and $\tilde{\mathbf{a}} = a_{t+1} \cdots a_{t+s} \subset \mathbf{a}$ then

$$(3.7) \quad \mathbf{a} \setminus \tilde{\mathbf{a}} := a_1 \cdots a_t y a_{t+s+1} \cdots a_m = (\mathbf{a} \setminus \tilde{\mathbf{a}})_1 y (\mathbf{a} \setminus \tilde{\mathbf{a}})_2 .$$

In this case, where the auxiliary generator y is placed at position $t + 1$ in (3.7) we say that $\tilde{\mathbf{a}} \subset \mathbf{a}$ is a *subword of \mathbf{a} at position $t + 1$* .

We now have a lemma of how the orientation of the different strata compares to the boundary orientation. See Section B for a general discussion about orientations of moduli spaces.

Lemma 4. *The product orientation on $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$ differs from the boundary orientation on $\partial\overline{\mathcal{M}}(\mathbf{a})$ by $(-1)^{\ddagger_1}$ where*

$$(3.8) \quad \ddagger_1 = (m_2 + 1) \left(\sum_{i=1}^{m_1} |a_i| \right) + m_1 ,$$

while the product orientation on $\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ differs from the boundary orientation on $\partial\overline{\mathcal{M}}(\mathbf{a})$ by $(-1)^{\ddagger_2}$ where

$$(3.9) \quad \ddagger_2 = s \left(|\xi| + \sum_{i=1}^{t+s} |a_i| \right) + s(m - t) + t + s ,$$

whenever $\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ is rigid. Here $\tilde{\mathbf{a}}$ is a subword of \mathbf{a} at position $t + 1$ as in (3.7).

Proof. See Section B. □

Lemma 5. *There exists a family of fundamental chains $[\overline{\mathcal{M}}(\mathbf{a})] \in C_*(\overline{\mathcal{M}}(\mathbf{a}))$ such that*

$$(3.10) \quad \begin{aligned} \partial[\overline{\mathcal{M}}(\mathbf{a})] = & \sum_{\mathbf{a}'\mathbf{a}''=\mathbf{a}} (-1)^{\ddagger_1} [\overline{\mathcal{M}}(\mathbf{a}')] \times [\overline{\mathcal{M}}(\mathbf{a}'')] \\ & + \sum_{\tilde{\mathbf{a}} \subset \mathbf{a}} (-1)^{\ddagger_2} [\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})] \times [\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})], \end{aligned}$$

where \ddagger_1 and \ddagger_2 are as in (3.8) and (3.9) respectively.

Proof. See [Abo12b, Lemma 4.11]. □

3.4. The evaluation map and construction of the A_∞ -homomorphism

In this section we construct the evaluation map used to define the A_∞ -homomorphism between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}(\Omega_\xi M_K)$.

First pick any smooth, orientation reversing map $r: \mathbb{R} \rightarrow D_{m+2}$ which parametrizes the outgoing segment. (That is, the boundary arc of D_{m+2} that lies between ζ_+ and ζ_- .)

Pick two strip-like ends

$$\varepsilon_\pm: (0, \infty) \times [0, 1] \rightarrow U_\pm,$$

where $U_\pm \subset D_{m+2}$ are neighborhoods of $\zeta_\pm \in D_{m+2}$. We pick the strip-like ends so that $\varepsilon_\pm((0, \infty) \times \{0\}) \subset U_\pm$ are the parts of the boundary of D_{m+2} that points towards ζ_\pm (according to the boundary orientation on D_{m+2}), and $\varepsilon_\pm((0, \infty) \times \{1\}) \subset U_\pm$ are the parts of the boundary of D_{m+2} that points away from ζ_\pm .

Assume that $r: \mathbb{R} \rightarrow D_{m+2}$ satisfies the following

$$(3.11) \quad \begin{cases} \lim_{t \rightarrow \pm\infty} r(t) = \zeta_\mp \\ \sup_{|t| \geq \tilde{t}} |(\varepsilon_\pm^{-1} \circ r)^{(n)}(t)| < \infty, \quad \exists \tilde{t} > 0, \forall n \geq 1. \end{cases}$$

Then $u \circ r: \mathbb{R} \rightarrow M_K$ is a map so that $\lim_{t \rightarrow \pm\infty} (u \circ r)(t) = \xi$. We reparametrize r by arc length with respect to a Riemannian metric on M_K (see (A.1)), and compactify the domain. In doing so, we get a smooth, orientation reversing map

$$\tilde{r}: [0, R] \rightarrow D_{m+2},$$

that satisfies $(u \circ \tilde{r})(0) = (u \circ \tilde{r})(R) = \xi$, which means $u \circ \tilde{r} \in \Omega_\xi M_K$. We then define the *evaluation map* as

$$(3.12) \quad \begin{aligned} \text{ev}: \mathcal{M}(\mathbf{a}) &\longrightarrow \Omega_\xi M_K \\ u &\longmapsto u \circ \tilde{r}. \end{aligned}$$

Lemma 6. *Let $u: D_{m+2} \rightarrow W_K$ be a J -holomorphic disk and take $r: \mathbb{R} \rightarrow D_{m+2}$ so that (3.11) holds. Then $\partial_s u \circ r$ decays exponentially in the C^∞ -topology.*

Proof. Pick strip-like ends

$$\varepsilon_\pm: (0, \infty) \times [0, 1] \longrightarrow U_\pm,$$

as above. By [RS01, Theorem A] we have that $\partial_s u \circ \varepsilon_\pm$ decays exponentially in the C^∞ -topology. When we say that a function decays exponentially in the C^∞ -topology we mean that there are constants $\delta, c_0, c_1, c_2, \dots > 0$ so that $\forall k \in \mathbb{N}$ and for every $t_0 \in (0, \infty)$ we have

$$(3.13) \quad \|\partial_s u \circ \varepsilon_\pm\|_{C^k([t_0, \infty) \times [0, 1])} \leq c_k e^{-\delta t_0}.$$

Next consider $r: \mathbb{R} \rightarrow D_{m+2}$ which satisfies (3.11), where $\tilde{t} > 0$ is large enough so that $r(t) \in U_\pm$ for $|t| > \tilde{t}$. This also gives

$$(u \circ r)(t) = \begin{cases} (u \circ \varepsilon_-) \circ (\varepsilon_-^{-1} \circ r)(t), & t \geq \tilde{t} \\ (u \circ \varepsilon_+) \circ (\varepsilon_+^{-1} \circ r)(t), & t \leq -\tilde{t} \end{cases}$$

where $\varepsilon_\pm^{-1} \circ r: \mathbb{R} \rightarrow (0, \infty) \times [0, 1]$ are maps so that

$$\begin{cases} (\varepsilon_-^{-1} \circ r)(t) \subset (0, \infty) \times \{1\} \\ (\varepsilon_+^{-1} \circ r)(t) \subset (0, \infty) \times \{0\}, \end{cases}$$

and

$$\begin{cases} \lim_{t \rightarrow \infty} (\varepsilon_-^{-1} \circ r)(t) = (\infty, 1) \\ \lim_{t \rightarrow -\infty} (\varepsilon_+^{-1} \circ r)(t) = (\infty, 0). \end{cases}$$

Then we have constants $\delta, c_0, c_1, c_2, \dots > 0$ so that $\forall k \geq 0$

$$\begin{aligned} \|\partial_s u \circ r\|_{C^k([\tilde{t}, \infty))} &= \sum_{|\alpha| \leq k} \sup_{|t| \geq \tilde{t}} |D^\alpha(\partial_s u \circ r)| \\ &= \sum_{|\alpha| \leq k} \sup_{|t| \geq \tilde{t}} \left| D^\alpha [\partial_s u \circ \varepsilon_\pm] (\varepsilon_\pm^{-1}(r(t))) \cdot D^\alpha [\varepsilon_\pm^{-1} \circ r] (t) \right| \\ &= \sum_{|\alpha| \leq k} \left[\sup_{[t_0, \infty) \times [0, 1]} |D^\alpha(\partial_s u \circ \varepsilon_\pm)| \right] \left[\sup_{|t| \geq \tilde{t}} |D^\alpha(\varepsilon_\pm^{-1} \circ r)| \right]. \end{aligned}$$

Here D^α denotes derivative with respect to the multi-index α . Because of (3.11) we have

$$\sup_{|t| \geq \tilde{t}} |D^\alpha(\varepsilon_\pm^{-1} \circ r)| \leq A_\alpha,$$

where A_α is some constant depending on α . We conclude

$$\|\partial_s u \circ r\|_{C^k([\tilde{t}, \infty))} \leq A_k \|\partial_s u \circ \varepsilon_\pm\|_{C^k([t_0, \infty) \times [0, 1])} \leq A_k \cdot c_k e^{-\delta t_0},$$

by (3.13), where $A_k := \max_{|\alpha| \leq k} A_\alpha$. Furthermore we note that $\tilde{t} > 0$ is large enough so that for $|t| \geq \tilde{t}$ we have

$$\begin{cases} (\varepsilon_-^{-1} \circ r)(t) \subset [t_0, \infty) \times \{1\} \\ (\varepsilon_+^{-1} \circ r)(t) \subset [t_0, \infty) \times \{0\}. \end{cases}$$

□

The previous lemma enables us to extend the evaluation map to the compactification of the moduli space of half strips.

Lemma 7. *There is an extension of the evaluation map ev to a continuous map on the compactification of $\mathcal{M}(\mathbf{a})$,*

$$ev: \overline{\mathcal{M}}(\mathbf{a}) \longrightarrow \Omega_\xi M_K,$$

such that the following diagram commutes up to an overall sign of $(-1)^{\ddagger_1}$, where \ddagger_1 is defined in (3.8).

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'') & \xrightarrow{\iota} & \overline{\mathcal{M}}(\mathbf{a}) \\ \downarrow ev \times ev & & \downarrow ev \\ \Omega_\xi M_K \times \Omega_\xi M_K & \xrightarrow{\circ} & \Omega_\xi M_K, \end{array}$$

The map ι in the top row is inclusion as in (3.6). The map in the bottom row is concatenation of loops.

Proof. For this proof, we follow the idea outlined in [Abo12b, p. 37].

Extension of ev to the compactification: It is obvious how to extend it to the boundary strata $\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{cw}(\tilde{\mathbf{a}})$; we define the evaluation map of such broken disk to be the same as the evaluation map when we forget about the factor $\mathcal{M}^{cw}(\tilde{\mathbf{a}})$. However, if we have a sequence $\{u^\nu\}_{\nu=0}^\infty \subset \mathcal{M}(\mathbf{a})$ which Gromov converges to a broken disk in any of the boundary strata $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$, then the Gromov limit is a stable J -holomorphic map (a broken disk), consisting of two J -holomorphic disks $u_i: D_{k_i} \rightarrow W_K$ where $k_1 + k_2 - 2 = m + 2$, and two boundary punctures $z_1 \in \partial D_{k_1}$, $z_2 \in \partial D_{k_2}$ so that we either have $(z_1, z_2) = (\zeta_-, \zeta_+)$ or $(z_1, z_2) = (\zeta_+, \zeta_-)$ [Fra08]. More precisely, it means that there are two families of Möbius transformations of the unit disk $D \subset \mathbb{C}$

$$\varphi_1^\nu, \varphi_2^\nu: D \rightarrow D, \text{ where } \nu \in \mathbb{N},$$

so that

$$(3.14) \quad \begin{cases} u^\nu \circ \varphi_1^\nu \rightarrow u_1 & \text{in } C_{loc}^\infty(D_{k_1} \setminus \{z_1\}) \\ u^\nu \circ \varphi_2^\nu \rightarrow u_2 & \text{in } C_{loc}^\infty(D_{k_2} \setminus \{z_2\}), \end{cases}$$

and

$$\begin{cases} (\varphi_1^\nu)^{-1} \circ \varphi_2^\nu \rightarrow z_1 & \text{in } C_{loc}^\infty(D_{k_1} \setminus \{z_1\}) \\ (\varphi_2^\nu)^{-1} \circ \varphi_1^\nu \rightarrow z_2 & \text{in } C_{loc}^\infty(D_{k_2} \setminus \{z_2\}). \end{cases}$$

Recall that convergence in $C_{loc}^\infty(X)$ means C^∞ -convergence on every compact subset $K \subset X$.

Define parametrizations

$$\begin{aligned} r_1: \mathbb{R} &\rightarrow D_{k_1} \\ r_2: \mathbb{R} &\rightarrow D_{k_2} \end{aligned}$$

so that r_1 and r_2 satisfy (3.11). Then the two maps $u_i \circ r_i: \mathbb{R} \rightarrow M_K$ are smooth maps so that $\partial_s u_i \circ r_i$ decay exponentially in the C^∞ -topology by Lemma 6. Hence the composition of two smooth loops $u_i \circ r_i$ is again a smooth loop. There are two cases, depending on whether the two components of the broken disk have the puncture ζ_+ or ζ_-

in common. That is, we either have $(z_1, z_2) = (\zeta_-, \zeta_+)$ or $(z_1, z_2) = (\zeta_+, \zeta_-)$. In the first case when $(z_1, z_2) = (\zeta_-, \zeta_+)$, we define a map $\gamma: \mathbb{R} \rightarrow M_K$ as

$$(3.15) \quad \gamma(t) := \begin{cases} (u_1 \circ r_1) \left(t - \frac{1}{t} \right), & t < 0 \\ \xi, & t = 0 \\ (u_2 \circ r_2) \left(t - \frac{1}{t} \right), & t > 0. \end{cases}$$

In the second case when $(z_1, z_2) = (\zeta_+, \zeta_-)$ we swap places of $u_1 \circ r_1$ and $u_2 \circ r_2$ in the above definition of γ .

We then claim that this map is smooth and has exponentially decaying derivatives in the C^∞ -topology as $t \rightarrow \pm\infty$. Since $u_1 \circ r_1$ and $u_2 \circ r_2$ are smooth maps with exponentially decaying derivatives in the C^∞ -topology as $t \rightarrow \pm\infty$, it suffices to show that all derivatives of γ at $t = 0$ exists. This follows from the exponential decay of every derivative of $u_1 \circ r_1$ and $u_2 \circ r_2$ in the C^∞ -topology. We may then reparametrize γ by arc length and compactify the domain to obtain a map $\tilde{\gamma}: [0, R] \rightarrow M_K$ so that $\tilde{\gamma}(0) = \tilde{\gamma}(R) = \xi$, that is $\tilde{\gamma} \in \Omega_\xi M_K$, and we define $\text{ev}((u_1, u_2)) := \tilde{\gamma}$.

Commutativity of the diagram: It follows almost immediately from the definition of the evaluation map

$$\text{ev}: \overline{\mathcal{M}}(\mathbf{a}) \rightarrow \Omega_\xi M_K$$

that the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}') & \xrightarrow{\iota} & \overline{\mathcal{M}}(\mathbf{a}) \\ \downarrow \text{ev} \times \text{ev} & & \downarrow \text{ev} \\ \Omega_\xi M_K \times \Omega_\xi M_K & \xrightarrow{\circ} & \Omega_\xi M_K \end{array}$$

commutes, since γ in (3.15) is essentially defined as the concatenation of $u_1 \circ r_1$ and $u_2 \circ r_2$. More precisely, we consider $u_i \circ r_i: \mathbb{R} \rightarrow M_K$ for $i \in \{1, 2\}$ as above. Then reparametrize r_1 and r_2 by arc length so that we obtain two maps

$$u_i \circ \tilde{r}_i: [0, R_i] \rightarrow M_K.$$

These maps are so that $(u_i \circ \tilde{r}_i)(0) = (u_i \circ \tilde{r}_i)(R_i) = \xi$ for $i \in \{1, 2\}$, and the concatenation of these maps yields a map $\psi: [0, R_1 + R_2] \rightarrow$

M_K defined by

$$\psi(t) = \begin{cases} (u_1 \circ \tilde{r}_1)(t), & t \in [0, R_1] \\ (u_2 \circ \tilde{r}_2)(t - R_1), & t \in [R_1, R_1 + R_2] \end{cases}$$

which coincides with the map $\tilde{\gamma}: [0, R] \rightarrow M_K$ obtained by parametrizing γ defined in (3.15) by arc length. The overall sign $(-1)^{\ddagger_1}$ comes from Lemma 4, see Section B for a discussion about sign and orientations.

Continuity of ev : We claim that ev is a continuous map, meaning that if $\{u^\nu\}_{\nu=0}^\infty \subset \mathcal{M}(\mathbf{a})$ is a Gromov convergent sequence of J -holomorphic disks, then the map $\tilde{\gamma}(t)$ defined in (3.15) is realized as a limit of loops in the compact-open topology of $\Omega_\xi M_K$.

Pick a family of smooth maps $\{r^\nu: \mathbb{R} \rightarrow D_{m+2}\}_{\nu=0}^\infty$ which satisfies (3.11). Then we have that $\{u^\nu \circ r^\nu\}_{\nu=0}^\infty$ is a family of smooth maps with exponentially decaying derivatives as $t \rightarrow \pm\infty$ in the C^∞ -topology by Lemma 6. From (3.14) we have two families of Möbius transformations $\{\varphi_1^\nu\}_{\nu=0}^\infty$ and $\{\varphi_2^\nu\}_{\nu=0}^\infty$ such that

$$u^\nu \circ \varphi_i^\nu \rightarrow u_i, \quad \text{in } C_{\text{loc}}^\infty(D_{k_i} \setminus \{z_i\}),$$

for $i \in \{1, 2\}$. We also have that φ_i^ν preserves the boundary of D_m and that $(\varphi_i^\nu)^{-1}$ preserves boundary marked points in the sense that $\lim_{\nu \rightarrow \infty} (\varphi_i^\nu)^{-1}(\zeta_j) = \zeta_j$. Then we have

$$(3.16) \quad \begin{cases} (\varphi_1^\nu)^{-1} \circ r^\nu \rightarrow r_1, & \text{in } C_{\text{loc}}^\infty(\mathbb{R}_{<0}) \\ (\varphi_2^\nu)^{-1} \circ r^\nu \rightarrow r_2, & \text{in } C_{\text{loc}}^\infty(\mathbb{R}_{>0}). \end{cases}$$

Hence for any multi-index α and $i \in \{1, 2\}$ we have

$$(3.17) \quad \begin{aligned} & |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(u_i \circ r_i)| \\ &= |D^\alpha(u^\nu \circ \varphi_i^\nu \circ (\varphi_i^\nu)^{-1} \circ r^\nu) - D^\alpha(u_i \circ r_i)| \\ &= \left| D^\alpha(u^\nu \circ \varphi_i^\nu) D^\alpha \left[(\varphi_i^\nu)^{-1} \circ r^\nu \right] - D^\alpha(u_i) D^\alpha(r_i) \right| \\ &\leq \left| D^\alpha(u^\nu \circ \varphi_i^\nu) \cdot \left(D^\alpha \left[(\varphi_i^\nu)^{-1} \circ r^\nu \right] - D^\alpha(r_i) \right) \right| \\ &\quad + \left| D^\alpha(r_i) \cdot [D^\alpha(u^\nu \circ \varphi_i^\nu) - D^\alpha(u_i)] \right|. \end{aligned}$$

Let $i \in \{1, 2\}$ and define $\mathbb{R}_1 := \mathbb{R}_{<0}$ and $\mathbb{R}_2 := \mathbb{R}_{>0}$. Inserting suprema over suitable compact sets $A \subset \mathbb{R}_i$ and $K \subset D_{k_i} \setminus \{z_i\}$ gives

$$\begin{aligned}
 (3.18) \quad & \sup_{A \subset \mathbb{R}_i} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(u_i \circ r_i)| \\
 & \stackrel{(3.17)}{\leq} \sup_{K \subset D_{k_i} \setminus \{z_i\}} |D^\alpha(u^\nu \circ \varphi_i^\nu)| \sup_{A \subset \mathbb{R}_i} |D^\alpha((\varphi_i^\nu)^{-1} \circ r^\nu) - D^\alpha(r_i)| \\
 & \quad + \sup_{A \subset \mathbb{R}_i} |D^\alpha(r_i)| \sup_{K \subset D_{k_i} \setminus \{z_i\}} |D^\alpha(u^\nu \circ \varphi_i^\nu) - D^\alpha(u_i)| \\
 & \leq C_1 \underbrace{\sup_{A \subset \mathbb{R}_i} |D^\alpha((\varphi_i^\nu)^{-1} \circ r^\nu) - D^\alpha(r_i)|}_{\rightarrow 0} \\
 & \quad + C_2 \underbrace{\sup_{K \subset D_{k_i} \setminus \{z_i\}} |D^\alpha(u^\nu \circ \varphi_i^\nu) - D^\alpha(u_i)|}_{\rightarrow 0},
 \end{aligned}$$

Here we have used that $u^\nu \circ \varphi_i^\nu \rightarrow u_i$ in $C_{\text{loc}}^\infty(D_{k_i} \setminus \{z_i\})$ and hence that $u^\nu \circ \varphi_i^\nu$ is also bounded in this topology. Furthermore we have used that $(\varphi_i^\nu)^{-1} \circ r^\nu \rightarrow r_i$ in $C_{\text{loc}}^\infty(\mathbb{R}_i)$ by (3.16).

Then by recalling the definition of $\gamma(t)$ in (3.15), we have

$$\begin{aligned}
 & \sup_{A \subset \mathbb{R}} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(\gamma)| \\
 & \leq \sup_{A \subset \mathbb{R}_{<0}} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(\gamma)| + \sup_{A \subset \mathbb{R}_{>0}} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(\gamma)| \\
 & = \sup_{A \subset \mathbb{R}_{<0}} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(u_1 \circ r_1)| \\
 & \quad + \sup_{A \subset \mathbb{R}_{>0}} |D^\alpha(u^\nu \circ r^\nu) - D^\alpha(u_2 \circ r_2)|
 \end{aligned}$$

By (3.18), we get $u^\nu \circ r^\nu \rightarrow \gamma$ in $C_{\text{loc}}^\infty(\mathbb{R})$, and thus by passing to arc length parametrizations we get $\text{ev}(u^\nu) \rightarrow \text{ev}(u)$ in the compact-open topology on $\Omega_\xi M_K$. \square

The evaluation map

$$\text{ev}: \overline{\mathcal{M}}(\mathbf{a}) \rightarrow \Omega_\xi M_K$$

induces a map on chains $\text{ev}_*: C_{-*}(\overline{\mathcal{M}}(\mathbf{a})) \rightarrow C_{-*}(\Omega_\xi M_K)$. We then pick a fundamental chain $[\overline{\mathcal{M}}(\mathbf{a})]$ by Lemma 5 so that (3.10) holds, and define a

family of maps $\{\Psi_m\}_{m=1}^\infty$

$$(3.19) \quad \Psi_m : CW_{\Lambda_K}^*(F_{m-1}, F_m) \otimes \cdots \otimes CW_{\Lambda_K}^*(F_0, F_1) \longrightarrow C_{-*}(\Omega_\xi M_K) \\ a_m \otimes \cdots \otimes a_1 \longmapsto (-1)^\S \text{ev}_*[\overline{\mathcal{M}}(\mathbf{a})],$$

where

$$\begin{aligned} \S &= \sum_{j=1}^m j|a_j| + (m+1)|\xi| + (|\xi| + m) \dim \overline{\mathcal{M}}(\mathbf{a}) \\ &= \sum_{j=1}^m j|a_j| + (|\xi| + m) \sum_{j=1}^m |a_j| \pmod{2}. \end{aligned}$$

Note that $|\xi|$ means the grading of ξ regarded as an intersection generator of $CW_{\Lambda_K}^*(F_0, F_1)$ as in Subsection 2.1.

Lemma 8. *The following diagram commutes,*

$$\begin{array}{ccc} C_{-k}(\overline{\mathcal{M}}(\mathbf{a})) & \xrightarrow{\partial} & C_{-k+1}(\overline{\mathcal{M}}(\mathbf{a})) \\ \downarrow \text{ev}_* & & \downarrow \text{ev}_* \\ C_{-k}(\Omega_\xi M_K) & \xrightarrow{\partial} & C_{-k+1}(\Omega_\xi M_K) \end{array}$$

and the following diagram commutes up to an overall sign of $(-1)^{\ddagger_1 + \dim \overline{\mathcal{M}}(\mathbf{a}')}$, where \ddagger_1 is defined in (3.8).

$$\begin{array}{ccc} C_{-k}(\overline{\mathcal{M}}(\mathbf{a}')) \otimes C_{-\ell}(\overline{\mathcal{M}}(\mathbf{a}'')) & \xrightarrow{\iota_* \circ \times} & C_{-(k+\ell)}(\overline{\mathcal{M}}(\mathbf{a})) \\ \downarrow \text{ev}_* \otimes \text{ev}_* & & \downarrow \text{ev}_* \\ C_{-k}(\Omega_\xi M_K) \otimes C_{-\ell}(\Omega_\xi M_K) & \xrightarrow{P} & C_{-(k+\ell)}(\Omega_\xi M_K) \end{array}$$

In the latter diagram we have the subdivision $\mathbf{a} = \mathbf{a}'\mathbf{a}''$, and the map ι_* is the composition of the map induced by the inclusion

$$\iota : \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'') \longrightarrow \overline{\mathcal{M}}(\mathbf{a}).$$

Proof. That the first diagram commutes follows more or less by definition. Namely, let $A \in C_{-k}(\overline{\mathcal{M}}(\mathbf{a}))$. Then $\text{ev}_*(A) = \text{ev} \circ A$, and by using the definition of ∂ in (3.1) and the definition of ev in (3.12) we get

$$\begin{aligned} \partial(\text{ev}_*(A)) &= \sum_{i=1}^k \sum_{\varepsilon=0}^1 (-1)^{i+\varepsilon} (\text{ev} \circ A)(\delta_{i,\varepsilon}(x)) = \sum_{i=1}^k \sum_{\varepsilon=0}^1 (-1)^{i+\varepsilon} (A(\delta_{i,\varepsilon}(x)) \circ \tilde{r}) \\ &= \left(\sum_{i=1}^k \sum_{\varepsilon=0}^1 (-1)^{i+\varepsilon} A(\delta_{i,\varepsilon}(x)) \right) \circ \tilde{r} \\ &= \text{ev}_*(\partial A). \end{aligned}$$

The second diagram is split up into the following diagram

$$\begin{array}{ccccc} C_{-k}(\overline{\mathcal{M}}(\mathbf{a}')) \otimes C_{-\ell}(\overline{\mathcal{M}}(\mathbf{a}'')) & \xrightarrow{\times} & C_{-(k+\ell)}(\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')) & \xrightarrow{\iota_*} & C_{-(k+\ell)}(\overline{\mathcal{M}}(\mathbf{a})) \\ \downarrow \text{ev}_* \otimes \text{ev}_* & & \downarrow \text{ev}_* \times \text{ev}_* & & \downarrow \text{ev}_* \\ C_{-k}(\Omega_\xi M_K) \otimes C_{-\ell}(\Omega_\xi M_K) & \xrightarrow{\times} & C_{-(k+\ell)}((\Omega_\xi M_K)^2) & \xrightarrow{\circ} & C_{-(k+\ell)}(\Omega_\xi M_K). \end{array}$$

$\xrightarrow{\quad P \quad}$

The right square commutes, since the corresponding diagram before application of C_{-*} commutes, by Lemma 7, and the maps ι_* , ev_* and \circ on chains are defined pointwise. The left square also commutes, because $\text{ev}_* \otimes \text{ev}_*$ and $\text{ev}_* \times \text{ev}_*$ act componentwise. Hence the outer square also commutes.

The overall sign $(-1)^{\sharp_1 + \dim(\overline{\mathcal{M}}(\mathbf{a}'))}$ comes from the definition of P in (3.2), and from the inclusion

$$\iota: \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'') \longrightarrow \overline{\mathcal{M}}(\mathbf{a}),$$

of $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$ as a boundary stratum of $\overline{\mathcal{M}}(\mathbf{a})$ as in (3.10). □

Lemma 9. *The maps $\{\Psi_m\}_{m=1}^\infty$ form an A_∞ -homomorphism. That is,*

$$\partial \Psi_m + \sum_{m_1+m_2=m} P(\Psi_{m_2} \otimes \Psi_{m_1}) = \sum_{r+s+t=m} (-1)^{\sharp_t} \Psi_{r+1+t}(\text{id}^{\otimes r} \otimes \mu^s \otimes \text{id}^{\otimes t}),$$

where

$$\sharp_t = t + \sum_{j=1}^t |x_j|.$$

Proof. From Lemma 3 it is clear that Ψ_m has degree $1 - m$.

We first ignore signs and prove the statement modulo 2. We look at the codimension one boundary of $\overline{\mathcal{M}}(\mathbf{a})$ of dimension d . It consists of two types of broken J -holomorphic curves as in (3.10), and we analyze each boundary term separately.

1) The first boundary term is

$$\coprod_{\tilde{\mathbf{a}} \subset \mathbf{a}} \overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}}),$$

where $\tilde{\mathbf{a}} \subset \mathbf{a}$ is a subword at position $t + 1$ of \mathbf{a} .

2) The second boundary term is

$$\coprod_{\mathbf{a}' \mathbf{a}'' = \mathbf{a}} \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}''),$$

and it consists of broken half strips that is broken at the Lagrangian intersection point ξ .

In view of (3.10), we consider the fundamental chain of $\partial \overline{\mathcal{M}}(\mathbf{a})$. Consider the natural inclusions of the boundary strata

$$\begin{aligned} \iota: \overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'') &\longrightarrow \overline{\mathcal{M}}(\mathbf{a}) \\ \iota: \overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}}) &\longrightarrow \overline{\mathcal{M}}(\mathbf{a}). \end{aligned}$$

We consider $\partial \Psi_m(a_m \otimes \cdots \otimes a_1)$ and use Lemma 8. Then

$$\begin{aligned} (3.20) \quad \partial \Psi_m(a_m \otimes \cdots \otimes a_1) &= \partial \text{ev}_* [\overline{\mathcal{M}}(\mathbf{a})] = \text{ev}_* \partial [\overline{\mathcal{M}}(\mathbf{a})] \\ &= \sum_{\mathbf{a}' \mathbf{a}'' = \mathbf{a}} \text{ev}_* \left(\iota_* \left([\overline{\mathcal{M}}(\mathbf{a}')] \times [\overline{\mathcal{M}}(\mathbf{a}'')] \right) \right) \\ &\quad + \sum_{\tilde{\mathbf{a}} \subset \mathbf{a}} \text{ev}_* \left(\iota_* \left([\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})] \times [\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})] \right) \right) \end{aligned}$$

We start by considering boundary terms of type (1). The evaluation applied to these terms is

$$\text{ev}_* \iota_* \left([\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})] \times [\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})] \right) = \text{ev}_* [\overline{\mathcal{M}}((\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \mu^s(\tilde{\mathbf{a}})(\mathbf{a} \setminus \tilde{\mathbf{a}})_2)],$$

because of the definition of ev on these boundary strata. Note that if $\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ is not rigid, then the image $\text{ev}_* \iota_* \left([\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})] \times [\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})] \right)$ would be degenerate in $C_{-*}(\Omega_\xi M_K)$, and hence does not contribute. In figures we illustrate

this equality as follows

$$\text{ev}_* \iota_* \left[\begin{array}{c} \tilde{\mathbf{a}} \\ \text{---} \\ (\mathbf{a} \setminus \tilde{\mathbf{a}})_2 \quad \text{---} \quad (\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \\ \text{---} \\ \text{---} \end{array} \right] = \text{ev}_* \left[\begin{array}{c} \mu^s(\tilde{\mathbf{a}}) \\ \text{---} \\ (\mathbf{a} \setminus \tilde{\mathbf{a}})_2 \quad \text{---} \quad (\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \\ \text{---} \\ \text{---} \end{array} \right].$$

The word $(\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \mu^s(\tilde{\mathbf{a}}) (\mathbf{a} \setminus \tilde{\mathbf{a}})_2$ is the word obtained from \mathbf{a} , by replacing the word $\tilde{\mathbf{a}}$ with $\mu^s(\tilde{\mathbf{a}})$. Therefore

$$(3.21) \quad \text{ev}_* \left(\iota_* \left([\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})] \times [\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})] \right) \right) = \Psi_{r+1+t} (a_m \otimes \cdots \otimes a_{t+s+1} \otimes \mu^s(\tilde{\mathbf{a}}) \otimes a_t \otimes \cdots \otimes a_1),$$

where

$$\begin{cases} t = \text{length of the word } (\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \\ s = \text{length of the word } \tilde{\mathbf{a}} \\ r = \text{length of the word } (\mathbf{a} \setminus \tilde{\mathbf{a}})_2. \end{cases}$$

This means that the broken disks of type (1) correspond to terms of the form $\Psi_{r+1+t}(\text{id}^{\otimes r} \otimes \mu^s \otimes \text{id}^{\otimes t})$ where $r + s + t = m$.

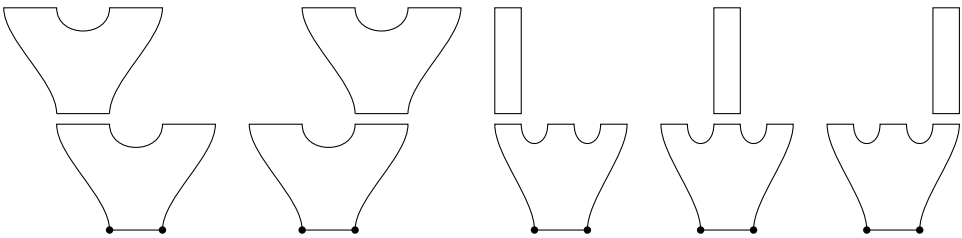


Figure 10: All broken disks of type (1) in the case of $m = 3$.

Similarly, for the first terms in (3.20) which correspond to broken disks of type (2) we apply Lemma 8 to get

$$(3.22) \quad \text{ev}_* \left(\iota_* \left([\overline{\mathcal{M}}(\mathbf{a}')] \times [\overline{\mathcal{M}}(\mathbf{a}'')] \right) \right) = P \left(\text{ev}_* [\overline{\mathcal{M}}(\mathbf{a}'')] \otimes \text{ev}_* [\overline{\mathcal{M}}(\mathbf{a}')] \right) = P(\Psi_{m_2}(\mathbf{a}'') \otimes \Psi_{m_1}(\mathbf{a}')),$$

so that the broken disks of type (2) correspond to terms of the form $P(\Psi_{m_2} \otimes \Psi_{m_1})$ where $m_1 + m_2 = m$.

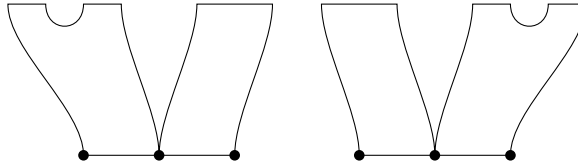


Figure 11: All broken disks of type (2) in the case of $m = 3$.

Therefore via (3.21) and (3.22), equation (3.20) becomes

$$\partial\Psi_m + \sum_{m_1+m_2=m} P(\Psi_{m_2} \otimes \Psi_{m_1}) = \sum_{r+s+t=m} \Psi_{r+1+t}(\text{id}^{\otimes r} \otimes \mu^s \otimes \text{id}^{\otimes t}),$$

and these are precisely the A_∞ -relations modulo 2. For confirmation of signs we refer the reader to Section B. □

4. The chain map is an isomorphism

This section is dedicated to the proof of Theorem 1.

Theorem 4 (Theorem 1). *There exists a geometrically defined isomorphism of A_∞ -algebras $\Psi: CW_{\Lambda_K}^*(F, F) \rightarrow C_{-*}^{cell}(BM_K)$.*

The first step is to replace the full Moore loop space with a Morse theoretic model of it. It is the space of piecewise geodesic loops and we denote it by BM_K (see Subsection 4.2). In the Morse theoretic model of the loop space, we have that the geodesics on M_K are precisely critical points of the energy functional, with finite dimensional unstable manifolds, and infinite dimensional stable manifolds. There is a one-to-one correspondence between Reeb chords and oriented geodesics. Assuming that the metric is generic gives moreover that Reeb chords of degree $-\lambda$ are in one-to-one correspondence with geodesics of index λ (see Lemma 15). We will show that the evaluation map defined in Subsection 3.4 is transverse to the infinite dimensional stable manifolds, and that the kernel of the linearized operator D_u has the same dimension as the unstable manifold.

In Subsection 4.1 we will define the action filtration on $CW_{\Lambda_K}^*(F, F)$, followed by Subsection 4.2 where we first replace the full Moore loop space with the Morse theoretic model consisting of piecewise geodesic loops, and then we filter the space of loops by length. In Subsection 4.3 we prove that Ψ_1 respects the action filtrations and in fact that Ψ_1 is diagonal with respect to the action filtrations. In Subsection 4.5 we prove that $CW_{\Lambda_K}^*(F, F)$ is

isomorphic to the Morse theoretic model of the loop space in each filtration level, which allows us to pass to colimits.

Consider $M_K \subset W_K$ and fix a generic Riemannian metric g on M_K such that in the handle $D_\varepsilon T^*([0, \infty) \times \Lambda_K)$ of W_K , the metric has the form

$$dt^2 + f(t)g,$$

where t is the coordinate in the $[0, \infty)$ -factor, and $f: [0, \infty) \rightarrow [0, \infty)$ satisfies $f'(0) = -1$, $f'(t) < 0$ and $f''(t) \geq 0$, see (A.1) for details.

4.1. Length filtration on $CW_{\Lambda_K}^*(F, F)$

For a Reeb chord generator $c \in CW_{\Lambda_K}^*(F, F)$ define its *action* by

$$\mathfrak{a}(c) := \int_0^\ell c^* \lambda.$$

In our case with $F \cong T_\xi^* S \subset W_K$ for $\xi \in M_K$ we only have a single Lagrangian intersection generator ξ , whose action we define explicitly as $\mathfrak{a}(\xi) := 0$. We then filter $CW_{\Lambda_K}^*(F, F)$ by this action, and use the notation

$$\mathcal{F}_p CW_{\Lambda_K}^*(F, F) := \{c \in CW_{\Lambda_K}^*(F, F) \mid \mathfrak{a}(c) < p\}.$$

Now, by applying Stokes' theorem to any J -holomorphic disk which contributes to $\mu^1(c)$ we get the following lemma. (Compare with e.g. [Ekh06, Lemma B.3].)

Lemma 10. *The differential $\mu^1: CW_{\Lambda_K}^*(F, F) \rightarrow CW_{\Lambda_K}^*(F, F)$ does not increase the action of generators. That is,*

$$\mathfrak{a}(c) \geq \mathfrak{a}(\mu^1(c)),$$

for any $c \in CW_{\Lambda_K}^*(F, F)$.

4.2. Length filtration on $C_{-*}(\Omega_\xi M_K)$

In this section we review basic material on the Morse theory of loop spaces from [Mil63].

One goal in this section is to replace the full Moore loop space $\Omega_\xi M_K$ with a homotopy equivalent Morse theoretic model by approximating Moore

loops by piecewise geodesic loops. The second goal is to in detail define the filtration on the model of chains of based loops we use.

By abuse of notation, we denote by $\Omega_\xi M_K$ the space of continuous based loops $\gamma: [0, 1] \rightarrow M_K$ with fixed domain $[0, 1]$. It is homotopy equivalent with the space of Moore loops as defined in Subsection 3.2. With respect to the generic Riemannian metric h on M_K as described in (A.1), equip $\Omega_\xi M_K$ with the supremum metric

$$d^*(\gamma, \beta) := \sup_{t \in [0, 1]} h(\gamma(t), \beta(t)), \quad \gamma, \beta \in \Omega_\xi M_K.$$

The metric topology on $\Omega_\xi M_K$ induced by d^* then agrees with the compact-open topology. Define $\Omega^{\text{pw}} M_K$ as the space of piecewise smooth loops, and equip it with the metric

$$d(\gamma, \beta) := d^*(\gamma, \beta) + \left(\int_0^1 |\dot{\gamma}|^2 - |\dot{\beta}|^2 dt \right)^{\frac{1}{2}}, \quad \gamma, \beta \in \Omega^{\text{pw}} M_K.$$

By [Mil63, Theorem 17.1], we have that the inclusion $i: \Omega^{\text{pw}} M_K \rightarrow \Omega_\xi M_K$ is a homotopy equivalence. We define the energy of $\gamma \in \Omega^{\text{pw}} M_K$ by

$$(4.1) \quad E(\gamma) := \int_0^1 |\dot{\gamma}|^2 dt.$$

Similarly we define the length of $\gamma \in \Omega^{\text{pw}} M_K$ as

$$L(\gamma) := \int_0^1 |\dot{\gamma}| dt.$$

Define

$$\Omega^{\text{pw}, c} M_K := \left\{ \gamma \in \Omega^{\text{pw}} M_K \mid E(\gamma) < c^2 \right\}.$$

Fix a subdivision of $[0, 1]$,

$$0 = t_0 < t_1 < t_2 < \dots < t_m = 1.$$

Then define BM_K to be the set of loops in $\Omega^{\text{pw}} M_K$ that are geodesic in the time interval $[t_i, t_{i+1}]$ for each $i \in \{0, \dots, m - 1\}$. Let

$$B^c M_K := \left\{ \gamma \in BM_K \mid E(\gamma) < c^2 \right\}.$$

Applying [Mil63, Lemma 16.1] then gives that for a sufficiently fine subdivision, $B^c M_K$ is a smooth finite dimensional manifold which is a natural

submanifold of $(M_K)^{m-1}$. Moreover by [Mil63, Theorem 16.2], $B^c M_K$ is a deformation retract of $\Omega^{\text{pw},c} M_K$, and critical points of $E|_{\Omega^{\text{pw},c} M_K}$ are the same as the critical points of $E|_{B^c M_K}$, and $E|_{B^c M_K}$ is furthermore a Morse function.

We consider another increasing filtration on BM_K by filtering by length. Namely, define the length filtration of BM_K by

$$\mathcal{F}_c BM_K = \{ \gamma \in BM_K \mid L(\gamma) < c \} ,$$

and correspondingly

$$\mathcal{F}_c \Omega^{\text{pw}} M_K = \{ \gamma \in \Omega^{\text{pw}} M_K \mid L(\gamma) < c \} .$$

By the same proof as [Mil63, Theorem 16.2], we construct an explicit deformation retract of $\mathcal{F}_c \Omega^{\text{pw}} M_K$ onto $\mathcal{F}_c BM_K$ (see Lemma 11 below).

If $\sigma \in C_{-k}(BM_K)$ is a cubical k -chain of piecewise geodesic loops, we define the action of σ as

$$\mathfrak{a}(\sigma) := \max_{x \in [0,1]^k} L(\sigma(x)) .$$

We then define

$$\mathcal{F}_c C_{-*}(BM_K) := \{ \sigma \in C_{-*}(BM_K) \mid \mathfrak{a}(\sigma) < c \} ,$$

which gives us an increasing filtration on $C_{-*}(BM_K)$. Futhermore, we see by definition that $\mathfrak{a}(\partial\sigma) \leq \mathfrak{a}(\sigma)$.

Lemma 11. *There is a deformation retract*

$$r : \mathcal{F}_c \Omega^{\text{pw}} M_K \longrightarrow \mathcal{F}_c BM_K ,$$

which therefore induces a quasi-isomorphism

$$r_* : \mathcal{F}_c C_{-*}(\Omega^{\text{pw}} M_K) \longrightarrow \mathcal{F}_c C_{-*}(BM_K) .$$

Proof. From the proof of [Mil63, Theorem 16.2], we first define a retraction

$$r : \mathcal{F}_c \Omega^{\text{pw}} M_K \longrightarrow \mathcal{F}_c BM_K ,$$

as follows. Consider the closed ball with center $\xi \in M_K$ and radius c

$$B(\xi, c) = \{ x \in M_K \mid h(x, \xi) \leq c \} .$$

For any $\gamma \in \mathcal{F}_c \Omega^{\text{pw}} M_K$, fix a fine enough subdivision of $[0, 1]$

$$0 = t_0 < t_1 < \dots < t_{k-1} < 1 = t_k,$$

so that $h(\gamma(t_{i-1}), \gamma(t_i)) < \varepsilon$ for some $\varepsilon > 0$ small enough so that there is a unique geodesic between $\gamma(t_{i-1})$ and $\gamma(t_i)$. Because γ is contained in the ball $B(\xi, c)$, we have by [Mil63, Corollary 10.8] that there is a unique minimal geodesic between $\gamma(t_{i-1})$ and $\gamma(t_i)$ of length less than ε . Define $r(\gamma)$ so that for each $i \in \{1, \dots, k-1\}$ we have

$$r(\gamma)|_{[t_{i-1}, t_i]} = \text{unique minimal geodesic of length less than } \varepsilon \\ \text{from } \gamma(t_{i-1}) \text{ to } \gamma(t_i).$$

Since geodesics are locally length minimizing, it is clear that $L(\gamma) \geq L(r(\gamma))$ and therefore that r takes values in $\mathcal{F}_c BM_K$. For each $s \in [0, 1]$ we define

$$r_s: \mathcal{F}_c \Omega^{\text{pw}} M_K \longrightarrow \mathcal{F}_c BM_K,$$

in such a way that for $s \in [t_{i-1}, t_i]$ and any $i \in \{1, \dots, k-1\}$ the map r_s is so that

$$\begin{cases} r_s(\gamma)|_{[0, t_{i-1}]} = r(\gamma)|_{[0, t_{i-1}]} \\ r_s(\gamma)|_{[t_{i-1}, s]} = \text{unique minimal geodesic from } \gamma(t_{i-1}) \text{ to } \gamma(s) \\ r_s(\gamma)|_{[s, 1]} = \gamma|_{[s, 1]}. \end{cases}$$

Then $\begin{cases} r_0(\gamma) = \gamma \\ r_1(\gamma) = r(\gamma) \end{cases}$ and it is continuous in both s and γ . Hence it shows that $\mathcal{F}_c BM_K$ is a deformation retract of $\mathcal{F}_c \Omega^{\text{pw}} M_K$.

It is now straightforward to see that this map is defined on singular chains. Namely, for any fixed $c > 0$, we pick a fine enough subdivision of $[0, 1]$

$$0 = t_0 < t_1 < \dots < t_{N-1} < 1 = t_N,$$

so that for every $i \in \{1, \dots, N-1\}$ we have

$$\max_{x \in [0, 1]^k} L\left(\sigma(x)|_{[t_{i-1}, t_i]}\right) < \varepsilon.$$

Hence for any $x \in [0, 1]^k$, there is a unique geodesic from $\sigma(x)(t_{i-1})$ to $\sigma(x)(t_i)$. Then r induces a map

$$(4.2) \quad r_*: \mathcal{F}_c C_{-*}(\Omega^{\text{pw}} M_K) \longrightarrow \mathcal{F}_c C_{-*}(BM_K) \\ \sigma \longmapsto r \circ \sigma. \quad \square$$

By [Mil63, Theorem 16.3], BM_K is a CW-complex with one cell of dimension λ for each closed geodesic on M_K of index λ . We consider the cellular chain complex $C_{-*}^{\text{cell}}(BM_K)$. We think of the generators of $C_{-\lambda}^{\text{cell}}(BM_K)$ as the unstable manifolds of geodesics of index λ with respect to the energy functional E on BM_K . We define the action of a λ -cell e_λ as

$$\mathbf{a}(e_\lambda) = \max_{x \in [0,1]^\lambda} L(e_\lambda(x)).$$

It is well known that singular chains and cellular chains on a CW-complex are homotopy equivalent. Denote the induced isomorphism on homology by

$$(4.3) \quad s: H_{-*}(BM_K) \xrightarrow{\cong} H_{-*}^{\text{cell}}(BM_K).$$

In particular by Lemma 11 the map r_* in (4.2) induces an isomorphism

$$(4.4) \quad r_*: H_{-*}(\Omega^{\text{pw}}M_K) \xrightarrow{\cong} H_{-*}(BM_K).$$

4.3. The chain map Ψ_1 respects the filtration

The goal for this section is to prove that the chain map

$$\Psi_1: CW_{\Lambda_K}^*(F, F) \longrightarrow C_{-*}(\Omega_\xi M_K),$$

respects the filtrations \mathcal{F}_c defined on $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}(\Omega_\xi M_K)$ in Subsections 4.1, 4.2 respectively. The plan is to follow and adapt the proof of [CELN17, Proposition 8.9] to the current situation. The outline of the proof is to consider any J -holomorphic disk $u \in \overline{\mathcal{M}}(a)$ contributing to $\Psi_1(a)$ and integrate the 2-form $d\lambda_\tau$ (defined in (4.7) below) over the disk. Using Stokes' theorem we show that $0 \leq \int_{u^{-1}(W_K)} u^* d\lambda_\tau = \mathbf{a}(a) - L(\gamma)$.

Consider a generator $a \in CW_{\Lambda_K}^*(F, F)$ and pick some loop $\gamma = \Psi_1(a)(x) : [0, 1] \longrightarrow M_K$. Then pick a tubular neighborhood $N(M_K)$ of M_K in W_K and a symplectomorphism

$$(4.5) \quad \varphi: N(M_K) \longrightarrow D_\delta T^*M_K,$$

by the Lagrangian neighborhood theorem for some positive constant δ . By a similar argument to that of the proof of [Wei71, Theorem 7.1], we may assume that φ sends the fiber $F \cap N(M_K)$ to a fiber of $D_\delta T^*M_K$.

Recall that we use the metric on M_K defined in (A.1). Pick coordinates (q, p) in T^*M_K , and define the canonical 1-form $\beta = pdq$. Then let β_1 be a

1-form on T^*M_K that is given by

$$\beta_1 = \frac{\delta p dq}{|p|}.$$

When we restrict to $S_\delta T^*M_K$, the Reeb vector field $R = p\partial_q$ and the contact structure $\xi = \ker \beta_1$ have the following expressions in these coordinates

$$R = \sum_{i=1}^n p_i \partial_{q_i}, \quad \xi = \ker \beta_1 \cap \ker(pdp) = \left(\text{span} \{R, p\partial_p\} \right)^{\perp_{d\beta_1}}.$$

Then we have the splitting $T_{(q,p)}T^*M_K = \text{span} \{R, p\partial_p\} \oplus \xi$. We have picked an almost complex structure J on W_K which is compatible with $d\lambda$. The almost complex structure J induces an almost complex structure J' on T^*M_K defined as

$$J' := (d\varphi) \circ J \circ (d\varphi)^{-1},$$

which satisfies the following:

- 1) J' is compatible with $dp \wedge dq$, and
- 2) J' preserves the splitting $T_{(q,p)}T^*M_K = \text{span} \{R, p\partial_p\} \oplus \xi$.

These two conditions ensure that the map

$$\varphi \circ u: (u^{-1}(N(M_K)), j) \longrightarrow (D_\delta T^*M_K, J')$$

is J' -holomorphic.

By the proof of [CELN17, Lemma 8.8] we have that $d\beta_1(v, J'v) \geq 0$. However, if we integrate $d\beta_1$ over the domain of $u \in \overline{\mathcal{M}}(a)$ we can not use Stokes' theorem directly since β_1 is singular along the zero section, so we have to make some further modifications to get rid of this singularity.

Let

$$(4.6) \quad \tau: [0, \infty) \longrightarrow [0, 1],$$

be a smooth function so that

- $\tau(s) = 0$ near $s = 0$, and
- $\tau'(s) \geq 0$ for every s ,
- $\tau(s) = 1$ for $s \geq \varepsilon$ for some small $\varepsilon < \delta$.

Then define

$$\beta_\tau := \frac{\delta\tau(|p|)}{|p|}pdq.$$

Lemma 12 ([CELN17, Lemma 8.8]). *For any $v \in T_{(q,p)}T^*M_K$ outside of the zero section we have*

$$d\beta_\tau(v, J'v) \geq 0.$$

For $\tau(|p|) > 0$ and $\tau'(|p|) > 0$ equality holds if and only if $v = 0$, whereas at points where $\tau(|p|) > 0$ and $\tau'(|p|) = 0$ equality holds if and only if v is a linear combination of the Liouville vector field $p\partial_p$ and the Reeb vector field $R = p\partial_q$.

Let $a \in CW_{\lambda_K}^*(F, F)$ be a generator and consider $u: D_3 \rightarrow W_K$ in $\overline{\mathcal{M}}(a)$. Denote by $\gamma := \text{ev}(u)$. Using the symplectomorphism φ in (4.5) we define an exact 2-form on W_K . Define

$$(4.7) \quad d\lambda_\tau := \varphi^*d\beta_\tau,$$

on $N(M_K) \subset W_K$. We may extend $d\lambda_\tau$ to the whole of W_K by defining it to be

$$d\lambda_\tau = \begin{cases} \varphi^*d\beta_\tau, & \text{in } N(M_K) \\ \omega, & \text{otherwise.} \end{cases}$$

Lemma 13. *The 2-form $d\lambda_\tau$ on W_K defined above satisfies*

$$d\lambda_\tau(v, Jv) \geq 0.$$

Proof. In $N(M_K)$ we have $d\lambda_\tau = \varphi^*d\beta_\tau$, in which case the conclusion follows from Lemma 12. Otherwise we have $d\lambda_\tau = \omega$ which is non-negative on complex lines, because J is ω -compatible. \square

Lemma 14. *Consider the exact Lagrangian fiber $F \cap N(M_K)$. Its image $F' := \varphi(F \cap N(M_K)) \subset D_\delta T^*M_K$ under φ is exact with respect to β_1 .*

Proof. This follows immediately from the assumption that φ maps $F \cap N(M_K)$ to a fiber of $D_\delta T^*M_K$, say $F' = D_\delta T_x^*M_K$ for $x \in M_K$. \square

Proposition 1. *Let $a \in CW_{A_K}^*(F, F)$ be any generator and u be any J -holomorphic half strip with positive puncture at a . Letting $\gamma := \text{ev}(u)$ we have*

$$\mathbf{a}(a) \geq L(\gamma),$$

with equality if and only if u is a branched covering of a half strip over a Reeb chord.

Proof. Since $d\lambda_\tau(u, Ju) \geq 0$ by Lemma 13 we integrate it over the disk $u: D_3 \rightarrow W_K$ and use Stokes' theorem:

$$\begin{aligned} 0 &\leq \int_{u^{-1}(W_K)} u^* d\lambda_\tau = \int_{u^{-1}(W_K \setminus N(M_K))} u^* d\lambda_\tau + \int_{u^{-1}(N(M_K))} u^* d\lambda_\tau \\ (4.8) \quad &= \underbrace{\int_{u^{-1}(W_K \setminus N(M_K))} u^* \omega}_{=: I_1} + \underbrace{\int_{(\varphi \circ u)^{-1}(D_\delta T^* M_K)} (\varphi \circ u)^* d\beta_\tau}_{=: I_2}. \end{aligned}$$

For the remainder of this proof we follow the proof of [CELN17, Proposition 8.9]. We start by computing I_2 . To do this we consider $\beta_1 = \frac{\delta p dq}{|p|}$. Then pick a biholomorphism

$$\psi: [0, \delta_0] \times [0, 1] \rightarrow U \subset D_3,$$

where $U \subset D_3$ is a neighborhood of the boundary arc between the boundary punctures ζ_\pm both of which are mapped to $\xi \in M_K \subset W_K$, so that $\psi(0, t)$ is a parametrization of the boundary arc between ζ_- and ζ_+ . We choose δ_0 small enough so that $(\varphi \circ u \circ \psi)(\delta_0, t)$ does not hit $M_K \subset T^*M_K$. Let

$$q(t) := \varphi \circ u \circ \psi(0, t).$$

Since we have a non-flat metric h on M_K (see (A.1)) we consider the splitting $T(T^*M_K) \cong V \oplus H$ and geodesic normal coordinates (q, p) on T^*M_K . The almost complex structure J then takes the vertical subspace to the horizontal and vice versa. Consider the Levi-Civita connection on $T(T^*M_K)$, and denote its associated Christoffel symbols by Γ_{ij}^k . Recall that in geodesic normal coordinates, the metric tensor at (q, p) has components $h_{ij}(q, p) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In particular the Christoffel symbols vanish at (q, p) . For any x in a neighborhood of (q, p) it follows that $\Gamma_{ij}^k(x) = O(|x|)$.

The almost complex structure in a neighborhood of (q, p) is

$$\begin{cases} J(\partial_{p_i}) = \partial_{q_i} - \Gamma_{ij}^k p^j \partial_{p_k} \\ J(\partial_{q_i}) = -\partial_{p_i} + \Gamma_{ij}^k p^j \partial_{q_k} - \Gamma_{ij}^m \Gamma_{mn}^k p^j p^n \partial_{p_k} . \end{cases}$$

Since u is J -holomorphic, we write

$$\tilde{u}(s, t) := (\varphi \circ u \circ \psi)(s, t) = (Q(s, t), P(s, t)) ,$$

where

$$\begin{cases} \partial_s Q^k - \Gamma_{ij}^k \partial_t Q^i P^j + \partial_t P^k = 0 \\ \partial_s P^k - \partial_t Q^k + \Gamma_{ij}^k \partial_t P^i P^j - \Gamma_{ij}^m \Gamma_{mn}^k \partial_t Q^i \partial_t Q^j P^m P^n = 0 . \end{cases}$$

Recall that in our geodesic normal coordinates we have $\Gamma_{ij}^k(x) = O(|x|)$ where x is in a neighborhood of (q, p) , and hence with $x = \tilde{u}(s, t)$ we have

$$(4.9) \quad \begin{cases} \partial_s Q + \partial_t P + O(|x|) = 0 \\ \partial_s P - \partial_t Q + O(|x|) = 0 . \end{cases}$$

If we write $Q(s, t) = q(t) + v(s, t)$ we get from the the second equation in (4.9) that

$$P(s, t) = s (\dot{q}(t) + O(|x|)) + w(s, t) ,$$

where $w(s, t) := \int_0^s \partial_t v(\sigma, t) d\sigma$. We now have $v(0, t) = 0 = w(0, t)$ and hence $\frac{\partial v}{\partial t}(0, t) = 0 = \frac{\partial w}{\partial t}(0, t)$. Setting $s = 0$ in (4.9) gives $\frac{\partial v}{\partial s}(0, t) = O(|\tilde{u}(0, t)|) = \frac{\partial w}{\partial s}(0, t)$. Next, from Taylor's formula we get $\frac{\partial v}{\partial t}(\delta_0, t) = O(\delta_0)$ and $w(\delta_0, t) = \delta_0 O(|\tilde{u}(0, t)|) + O(\delta_0^2)$. Then we get

$$(4.10) \quad \begin{aligned} \tilde{u}^* \beta_1|_{s=\delta_0} &= \frac{(\delta_0 \dot{q}(t) + w(\delta_0, t))(\dot{q}(t) + \frac{\partial v}{\partial t}(\delta_0, t))}{|\delta_0 \dot{q}(t) + w(\delta_0, t)|} dt \\ &= \frac{\langle \delta_0 \dot{q}(t) + \delta_0 O(|\tilde{u}(0, t)|) + O(\delta_0^2), \dot{q}(t) + O(\delta_0) \rangle}{|\delta_0 \dot{q}(t) + \delta_0 O(|\tilde{u}(0, t)|) + O(\delta_0^2)|} dt \\ &= \frac{\langle \dot{q}(t) + O(|\tilde{u}(0, t)|) + O(\delta_0), \dot{q}(t) + O(\delta_0) \rangle}{|\dot{q}(t) + O(|\tilde{u}(0, t)|) + O(\delta_0)|} dt \\ &= \left(|\dot{q}(t)| + O(\delta_0) \right) dt \end{aligned}$$

Next, pick $\varepsilon > 0$ so that it is smaller than the minimal norm of the p -components of $(\varphi \circ u \circ \psi)(\delta_0, t)$ and pick a function $\tau : [0, \infty) \rightarrow [0, 1]$ as

in (4.6). Namely, τ satisfies

$$\begin{cases} \tau'(s) \geq 0, & \forall s \in [0, \infty) \\ \tau(s) = 0, & \text{near } s = 0 \\ \tau(s) = 1, & s \geq \varepsilon. \end{cases}$$

Consider $\beta_\tau = \frac{\delta\tau(|p|)pdq}{|p|}$. By Lemma 12 we have $(\varphi \circ u)^*d\beta_\tau \geq 0$, and also that β_τ agrees with β_1 in the set $\{|p| \geq \varepsilon\} \subset T^*M_K$. Then we get

$$\begin{aligned} \lim_{\delta_0 \rightarrow 0} \int_{\{\delta_0\} \times [0,1]} (\varphi \circ u \circ \psi)^* \beta_\tau &= \lim_{\delta_0 \rightarrow 0} \int_{\{\delta_0\} \times [0,1]} |\dot{q}(t)| + O(\delta_0) dt \\ &= \lim_{\delta_0 \rightarrow 0} L(\gamma) + O(\delta_0) = L(\gamma). \end{aligned}$$

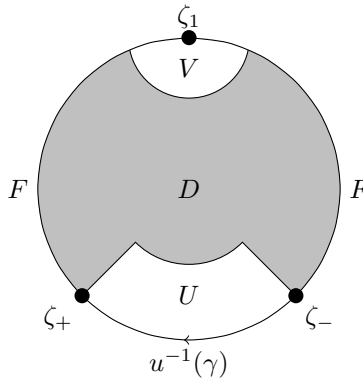


Figure 12: Domain of the J -holomorphic disk u with neighborhoods around the outgoing segment between ζ_- and ζ_+ and the positive puncture ζ_1 marked in white.

By Lemma 14 we have that $F' = \varphi(F \cap N(M_K)) \subset D_\delta T^*M_K$ is exact with respect to β_τ . Therefore we get

$$\begin{aligned} I_2 &= \int_{(\varphi \circ u)^{-1}(D_\delta T^*M_K)} (\varphi \circ u)^* d\beta_\tau \\ &= \int_{(\varphi \circ u)^{-1}(S_\delta T^*M_K)} (\varphi \circ u)^* \beta_\tau - \lim_{\delta_0 \rightarrow 0} \int_{\{\delta_0\} \times [0,1]} (\varphi \circ u \circ \psi)^* \beta_\tau \\ (4.11) \quad &= \int_{(\varphi \circ u)^{-1}(S_\delta T^*M_K)} (\varphi \circ u)^* \beta_\tau - L(\gamma). \end{aligned}$$

Finally, the integral I_1 in (4.8) is computed by using Stokes' theorem and that $d\lambda_\tau = \omega$ outside of $N(M_K)$ by definition.

$$(4.12) \quad I_1 = \int_{u^{-1}(W_K \setminus N(M_K))} u^* \omega = \mathbf{a}(a) - \int_{u^{-1}(\partial N(M_K))} u^* \lambda$$

By combining (4.12) with (4.11) we get

$$(4.13) \quad 0 \leq \int_{u^{-1}(W_K)} u^* d\lambda_\tau = - \int_{u^{-1}(\partial N(M_K))} u^* \lambda + \int_{(\varphi \circ u)^{-1}(S_\delta T^* M_K)} (\varphi \circ u)^* \beta_\tau + \mathbf{a}(a) - L(\gamma).$$

Note that along $S_\delta T^* M_K$ we have $\beta_\tau = \beta = pdq$. Furthermore φ is an exact symplectomorphism so we have $\varphi^* \beta - \lambda = d\theta$. Hence

$$\int_{u^{-1}(\partial N(M_K))} u^* (\varphi^* \beta - \lambda) = \int_{u^{-1}(\partial N(M_K))} u^* d\theta = 0,$$

and therefore (4.13) turns into

$$0 \leq \int_{u^{-1}(W_K)} u^* d\lambda_\tau = \mathbf{a}(a) - L(\gamma) \Leftrightarrow \mathbf{a}(a) \geq L(\gamma).$$

□

Corollary 1. *Let $a \in CW_{\Lambda_K}^*(F, F)$ be any generator and let $u \in \overline{\mathcal{M}}(a)$. Then*

$$\mathbf{a}(a) \geq \mathbf{a}(\Psi_1 a) \geq \mathbf{a}((r_* \circ \Psi_1)(a)).$$

Proof. Fix a generator $a \in CW_{\Lambda_K}^*(F, F)$ and consider the moduli space $\overline{\mathcal{M}}(a)$. The action of $\Psi_1(a) \in C_{-*}(\Omega_\xi M_K)$ is

$$\mathbf{a}(\Psi_1(a)) = \max_{x \in [0,1]^*} L(\Psi_1(a)(x)).$$

Note that the maximum is well-defined by the compactness of $[0, 1]^*$. Let $x_{\max} \in [0, 1]^*$ be such that $L(\Psi_1(a)(x_{\max})) = L(\Psi_1(a))$, and let $\gamma_{\max} := \Psi_1(a)(x_{\max})$. Since Proposition 1 holds for any $\gamma \in \Omega_\xi M_K$, in particular it holds for γ_{\max} . Therefore

$$\mathbf{a}(a) \geq L(\gamma_{\max}) = \max_{x \in [0,1]^*} L(\Psi_1(a)(x)) = \mathbf{a}(\Psi_1 a).$$

Moreover, the inequality $\mathbf{a}(\Psi_1 a) \geq \mathbf{a}((r_* \circ \Psi_1)(a))$ holds because r_* does not increase filtration (see the proof of Lemma 11). □

4.4. The chain map Ψ_1 is diagonal with respect to the action filtrations

In this section we prove that Ψ_1 is diagonal with respect to the action filtrations, that is $\mathbf{a}(a) = \mathbf{a}(\Psi_1 a)$.

We first give a brief outline of the proof. We consider the *trivial* J -holomorphic half strip u_0 whose image is the cone over the Reeb chord $a \in CW_{\Lambda_K}^*(F, F)$ and whose tangent space at (q, p) is spanned by $p\partial_q$ and $p\partial_p$ in geodesic normal coordinates. We show that u_0 is transversely cut out, and therefore by Lemma 12 together with the proof of Proposition 1 we get

$$0 = \int_{u_0^{-1}(W_K)} u_0^* d\lambda_\tau = \mathbf{a}(a) - L(\gamma_0).$$

To prove that u_0 is transversely cut out, we choose a generic Riemannian metric g on M_K (see (A.1)). Then we have a one-to-one correspondence between Reeb chords of degree $-\lambda$ and geodesics of index λ (see Lemma 15 below). We consider vector fields $v \in \ker D_{u_0}$ in the kernel of the linearized Cauchy–Riemann operator at u_0 . Then we show that v restricts to broken Jacobi fields along γ for which the Hessian of the energy functional (4.1) is negative definite.

The following lemma is essentially found and proven in [RS95, Prop 6.38] and [Dui76]. Recall that the degree of Reeb chords is defined via the Conley–Zehnder index, see Lemma 4 for details.

Lemma 15. *Let $a < b$ be two real numbers. There is a one-to-one correspondence between Reeb chords a of degree $-\lambda$ with action $\mathbf{a}(a) = A$ and geodesics γ in BM_K of index λ with length $L(\gamma) = A$.*

Proof. It is a consequence of the first part of Proposition 3 and in particular (4.18) (which do not depend on the current lemma) that Reeb chords with action A are in one-to-one correspondence with geodesics in BM_K with length A . What is left to show is that this one-to-one correspondence also preserves degree/index.

Let $\gamma \in BM_K$ be a (non-broken) geodesic. By Morse theory on the loop space, the index of γ is defined as the Morse index of the energy functional (4.1). Morse’s index theorem [Mil63, Theorem 15.1] says that the index of γ is equal to the number of points $\gamma(t)$ for $t \in (0, 1)$, which is conjugate to $\gamma(0)$ along γ , counted with multiplicity. Recall that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ by definition, if there is a Jacobi field K along γ so that $K(t) = K(0) = 0$.

A Jacobi field is a vector field along γ satisfying the Jacobi equation

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} K + R(\dot{\gamma}, K)\dot{\gamma} = 0,$$

where ∇ is the Levi-Civita connection on the bundle γ^*TM_K , and R is the corresponding curvature tensor. The geodesic flow on M_K lifts to the Reeb flow on ST^*M_K . Therefore Jacobi fields — which are seen as linearizations of the geodesic flow — lift to the linearized Reeb flow.

Assume that $t_1 \in (0, 1)$ so that $\gamma(t_1)$ is a point that is conjugate to $\gamma(0)$. Then let $\{e_i(t)\}_{i=2}^n$ be a parallel orthonormal frame of $\dot{\gamma}(t)^\perp \subset T_{\gamma(t)}M_K$, and let $K(t) = \sum_{i=2}^n K_i(t)e_i(t)$ be a Jacobi field so that $K(t_1) = K(0) = 0$. Defining $e_1(t) := \dot{\gamma}(t)$ we thus have that $\{e_i(t)\}_{i=1}^n$ is a parallel orthonormal frame of $T_{\gamma(t)}M_K$ along γ . Using the notation $\dot{K} := \nabla_{\frac{d}{dt}} K$, we define $L := \dot{K}$. Then we have the system

$$\begin{cases} \dot{K} = L \\ \dot{L} = -R(\dot{\gamma}, K)\dot{\gamma}. \end{cases}$$

By using $K = \sum_{i=1}^n K_i e_i$ and $L = \sum_{i=1}^n L_i e_i$ with $K_1 = L_1 = 0$, we get the system of differential equations

$$\begin{cases} \dot{K}_i(t) = L_i(t) \\ \dot{L}_i(t) = -\sum_{j=1}^n R_j^i(t)K_j(t) \end{cases} \quad \text{for all } i \in \{1, \dots, n\}$$

where $R(t) = \{R_j^i(t)\}_{i,j=1}^n = \{\langle R(\dot{\gamma}, e_j)\dot{\gamma}, e_i \rangle\}_{i,j=1}^n$ is a symmetric matrix. This is equivalent to say that

$$\frac{d}{dt} \begin{pmatrix} K \\ L \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -R & 0 \end{pmatrix} \begin{pmatrix} K \\ L \end{pmatrix},$$

An explicit fundamental solution to this system is

$$\Phi(t) = \exp \left(t \begin{pmatrix} 0 & I_n \\ -R & 0 \end{pmatrix} \right) = \begin{pmatrix} C(-t^2 R) & tS(-t^2 R) \\ (-tR)S(-t^2 R) & C(-t^2 R) \end{pmatrix},$$

where

$$C(A) = \sum_{k=0}^{\infty} \frac{A^k}{(2k)!}, \quad S(A) = \sum_{k=0}^{\infty} \frac{A^k}{(2k+1)!},$$

for $A \in \text{End}(\mathbb{R}^n)$. The fundamental solution Φ satisfies $\Phi(0) = I$ and in particular

$$\begin{pmatrix} K(t) \\ L(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} K(0) \\ L(0) \end{pmatrix}.$$

We use that the Jacobi field K vanishes at $t = 0$. Then we plug in $t = t_1 \in (0, 1)$ for which we also have $K(t_1) = 0$. Thus

$$\begin{pmatrix} 0 \\ L(t_1) \end{pmatrix} = \Phi(t) \begin{pmatrix} 0 \\ L(0) \end{pmatrix} = \begin{pmatrix} t_1 S(-t_1^2 R)L(0) \\ C(-t_1^2 R)L(0) \end{pmatrix}.$$

From this, we have that $\gamma(t_1)$ is conjugate to $\gamma(0)$ if and only if $S(-t^2 R)$ is singular at $t = t_1$.

We consider $(K, L) \in T(T_{\gamma(t)}M_K)$, and using the metric isomorphism and scaling K properly, we consider $(K, L) \in T(ST_{\gamma(t)}^*M_K)$ for $\gamma(t) \in M_K$. Since we assumed that K (and hence L) was orthogonal to γ , we regard the lift as $(K, L) \in \xi \subset T(ST^*M_K)$ along the lifted geodesic. Since $\xi_z \cong \mathbb{C}^{n-1} \cong \mathbb{R}^{n-1} \oplus i\mathbb{R}^{n-1}$ is symplectic with the standard symplectic form in these coordinates, we have that $i\mathbb{R}^{n-1} \subset \xi$ is Lagrangian. We then consider the path of Lagrangians

$$\ell(t) = \Phi(t)(i\mathbb{R}^{n-1}) = \Phi(t) \begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \begin{pmatrix} tS(-t^2 R)\zeta \\ C(-t^2 R)\zeta \end{pmatrix}, \quad \forall \zeta \in i\mathbb{R}^{n-1}.$$

Whenever $\gamma(t)$ is conjugate to $\gamma(0)$, the matrix $S(-t^2 R)$ is singular. Hence it has non-trivial kernel which contributes to the Maslov index exactly the dimension of the kernel. The dimension of the kernel also correspond to the multiplicity of $\gamma(t)$ as a conjugate point to $\gamma(0)$. By closing up the loop positively, we find an extra contribution of $n - 1$. Hence

$$\mu(\ell) = (n - 1) + \sum_{t : S(-t^2 R) \text{ singular}} \dim \ker S(-t^2 R) = (n - 1) + \text{ind}(\gamma),$$

from which we conclude $\text{ind}(\gamma) = -|a|$. □

Proposition 2. *Let $a \in CW_{\Lambda_K}^*(F, F)$ be any generator. Let u_0 be a J -holomorphic half strip as in Subsection 3.3, and let $v \in \ker D_{u_0}$, where D_{u_0} is the linearized Cauchy–Riemann operator at u_0 . Then consider the linearized*

solution

$$u_\varepsilon := \exp_{u_0}(\varepsilon v),$$

for any $\varepsilon > 0$. Then we have

$$\mathbf{a}(a) > L(\gamma_\varepsilon),$$

where $\gamma_\varepsilon := \text{ev}(u_\varepsilon)$.

Proof. We modify the proof of Proposition 1. Since u_ε is not J -holomorphic, we first prove that the estimate $0 < \int_{u_\varepsilon^{-1}(W_K)} u_\varepsilon^* d\lambda_1$ holds.

In a neighborhood of $(q, p) \in \text{im } u_\varepsilon \subset W_K$, we have a splitting

$$T_{(q,p)}W_K = \text{span} \{p\partial_p, p\partial_q\} \oplus \xi \cong \mathbb{R}^2 \times \mathbb{R}^{2n-2}.$$

We pick a small ball $B(\rho)$ of radius $\rho > 0$ around $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2}$. Let J_0 be the product almost complex structure on $\mathbb{R}^2 \times \mathbb{R}^{2n-2}$, which extends J over $B(\rho)$, and let $|\cdot|$ be the norm determined by $d\lambda_1$ and J_0 . Then we pick some coordinate $z = (s, t) \in T$ in the domain of u_ε , and define

$$q(z) := (J(u_\varepsilon(z)) + J_0(u_\varepsilon(z)))^{-1}(J(u_\varepsilon(z)) - J_0(u_\varepsilon(z))).$$

In the operator norm we have $\|q(z)\| = O(\rho)$ as $\rho \rightarrow 0$. Then we have

$$(4.14) \quad 2(J_0 + J)^{-1}J\bar{\partial}_J u_\varepsilon = \bar{\partial}_{J_0} u_\varepsilon + q(z)\partial_{J_0} u_\varepsilon =: A(z).$$

Let ρ be small enough so that $|\partial_{J_0} u_\varepsilon|^2 - 3|q(z)\partial_{J_0} u_\varepsilon|^2 > 0$. Then we have

$$\begin{aligned} |\partial_{J_0} u_\varepsilon|^2 + \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 &< |\partial_{J_0} u_\varepsilon|^2 + \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \\ &+ \left(|\partial_{J_0} u_\varepsilon|^2 - 3|q(z)\partial_{J_0} u_\varepsilon|^2 \right) + |A(z)|^2 \\ &= 2|\partial_{J_0} u_\varepsilon|^2 + \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \\ &- 3 \left(|A(z)|^2 + |q(z)\partial_{J_0} u_\varepsilon|^2 \right) + 4|A(z)|^2 \\ &\leq 2 \left(|\partial_{J_0} u_\varepsilon|^2 - \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \right) + 4|A(z)|^2. \end{aligned}$$

Next we note that

$$u_\varepsilon^* d\lambda_1 = \left(|du_\varepsilon|^2 - 2\left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \right) \text{dvol}_T = \left(|\partial_{J_0} u_\varepsilon|^2 - \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \right) \text{dvol}_T,$$

and hence

$$\begin{aligned} |du_\varepsilon|^2 \operatorname{dvol}_T &= \left(|\partial_{J_0} u_\varepsilon|^2 + \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \right) \operatorname{dvol}_T \\ &< 2 \left(|\partial_{J_0} u_\varepsilon|^2 - \left| \bar{\partial}_{J_0} u_\varepsilon \right|^2 \right) \operatorname{dvol}_T + 4|A(z)|^2 \operatorname{dvol}_T \\ &= 2u_\varepsilon^* d\lambda_1 + 4|A(z)|^2 \operatorname{dvol}_T. \end{aligned}$$

In view of the definition of $A(z)$ in (4.14), we have that $|A(z)|^2 = O(\varepsilon^4)$.

Let $\pi_\xi: TW_K \rightarrow TW_K$ be the projection onto the contact plane ξ . By Lemma 12 the only contribution to $\int_{u_\varepsilon^{-1}(B(\rho))} u_\varepsilon^* d\lambda_1$ comes from the restriction to ξ . Summing over all balls $B(\rho)$ covering the image of u_0 gives

$$\begin{aligned} (4.15) \quad 2 \int_{u_\varepsilon^{-1}(W_K)} u_\varepsilon^* d\lambda_1 &= 2 \int_{u_\varepsilon^{-1}(W_K)} \pi_\xi(u_\varepsilon^* d\lambda_1) \\ &\geq \|\pi_\xi(du_\varepsilon)\|^2 - 4\|\pi_\xi(A(z))\|^2. \end{aligned}$$

The Taylor expansion of u_ε around $\varepsilon = 0$ is

$$u_\varepsilon = u_0 + \varepsilon v + O(\varepsilon^2),$$

where $v \in \ker D_{u_0}$. Because v is a non-zero solution of the linearized equation $D_{u_0} v = 0$, we rescale v in such a way that

$$(4.16) \quad \|v\|_{W_\kappa^{2,2}}^2 = \|v\|_{L_\kappa^2}^2 + \|dv\|_{W_\kappa^{1,2}}^2 = 1$$

where $W_\kappa^{k,p}$ is the weighted Sobolev space $W_\kappa^{k,p}([0, \infty) \times [0, 1])$ with weight $e^{\kappa s}$ for some small $\kappa > 0$ and where s is the coordinate in the $[0, \infty)$ -factor. Let $Z_T = [0, T] \times [0, 1] \subset [0, \infty) \times [0, 1]$ for some $T > 0$. We use the Poincaré inequality $\|v\|_{L_\kappa^2(Z_T)}^2 \leq C_1 \|dv\|_{L_\kappa^2(Z_T)}^2$, where $C_1 > 0$ (given that $\kappa > 0$ is small enough), together with (4.16). This gives that $\|dv\|_{W_\kappa^{1,2}(Z_T)}^2 \geq C_0$ for some $C_0 > 0$ and some $T > 0$. Hence $\|dv\|_{W_\kappa^{1,2}}^2 \geq C$ for some $C > 0$.

The same argument applied to $\pi_\xi(v)$ and $\pi_\xi(dv)$ gives $\|\pi_\xi(dv)\|_{W_\kappa^{1,2}}^2 \geq C'$ for some $C' > 0$. Hence

$$\|\pi_\xi(du_\varepsilon)\|^2 - 4\|\pi_\xi(A(z))\|^2 = \|\varepsilon\pi_\xi(dv)\|^2 + O(\varepsilon^3) \geq C'\varepsilon^2 + O(\varepsilon^3) > 0,$$

for small enough $\varepsilon > 0$. By (4.15) we therefore have $\int_{u_\varepsilon^{-1}(W_K)} u_\varepsilon^* d\lambda_1 > 0$.

Next, we show $\int_{u_\varepsilon^{-1}(W_K)} u_\varepsilon^* d\lambda_1 = \mathbf{a}(a) - L(\gamma_\varepsilon)$. The proof is similar to the computation in the proof of Proposition 1. The only difference is the

computation of I_2 (with notation as in Proposition 1). Since $\bar{\partial}_J u_\varepsilon = O(\varepsilon^2)$, equation (4.9) becomes

$$(4.17) \quad \begin{cases} \partial_s Q + \partial_t P + O(|x|) = O(\varepsilon^2) \\ \partial_s P - \partial_t P + O(|x|) = O(\varepsilon^2), \end{cases}$$

where $x = \tilde{u}(s, t)$. Then from the second equation we get

$$P(s, t) = s \left(\dot{q}(t) + O(|x|) + O(\varepsilon^2) \right) + w(s, t).$$

Setting $s = 0$ in (4.17) gives

$$\frac{\partial v}{\partial s}(0, t) = O(|\tilde{u}(0, t)|) + O(\varepsilon^2) = \frac{\partial w}{\partial s}(0, t),$$

and hence

$$\begin{cases} \frac{\partial v}{\partial t}(\delta_0, t) = O(\delta_0) \\ w(\delta_0, t) = \delta_0 \left(O(|\tilde{u}(0, t)|) + O(\varepsilon^2) \right) + O(\delta_0^2). \end{cases}$$

By repeating the same calculation as in (4.10) we end up at

$$\begin{aligned} \tilde{u}^* \beta_1|_{s=\delta_0} &= \frac{\left\langle \dot{q}(t) + O(|\tilde{u}(0, t)|) + O(\varepsilon^2) + O(\delta_0), \dot{q}(t) + O(\delta_0) \right\rangle}{\left| \dot{q}(t) + O(|\tilde{u}(0, t)|) + O(\varepsilon^2) + O(\delta_0) \right|} dt \\ &= \left(|\dot{q}(t)| + O(\delta_0) \right) dt. \end{aligned}$$

The rest of the proof of Proposition 1 (which does not require holomorphicity) gives us the result. □

Proposition 3. *Let $a \in CW_{\Lambda_K}^*(F, F)$ be any generator and consider $\Psi_1(a) \in C_{-*}(\Omega_\xi M_K)$. Then*

$$\mathfrak{a}(a) = \mathfrak{a}(\Psi_1 a).$$

The same is also true for the chain map

$$r_* \circ \Psi_1 : CW_{\Lambda_K}^*(F, F) \longrightarrow C_{-*}(BM_K).$$

Proof. By Corollary 1 we have that

$$0 \leq \mathfrak{a}(a) - \mathfrak{a}(\Psi_1 a),$$

and to prove equality, it is enough to show that for any $a \in CW_{\Lambda_K}^*(F, F)$, there exists a transversely cut out J -holomorphic disk $u \in \overline{\mathcal{M}}(a)$ with

$$\int_{u^{-1}(W_K)} u^* d\lambda_\tau = 0.$$

We let

$$u_0: T \longrightarrow W_K,$$

be the J -holomorphic half strip that is the cone over the Reeb chord $a \in CW_{\Lambda_K}^*(F, F)$. In geodesic normal coordinates at $(q, p) = u_0(s, t)$, we have that the tangent space of $\text{im } u_0$ at (q, p) is $T_{u_0(s,t)} \text{im } u_0 = \text{span} \{p\partial_p, p\partial_q\}$ which means that $u_0^* d\lambda_\tau = 0$ by Lemma 12 and hence by Proposition 1 we have

$$(4.18) \quad \mathfrak{a}(a) = L(\gamma).$$

What is left to show is that u_0 is transversely cut out. Consider the following space of vector fields along u_0

$$V = \{ \eta \in \ker D_{u_0} \mid I(\pi_* \eta, \pi_* \eta) < 0 \}.$$

where π is the projection $\pi: W_K \longrightarrow M_K$ along the Liouville flow, and I is the index form (see (4.21) below for a definition). By Lemma 3 we have

$$\text{ind } D_{u_0} = \text{ind } \gamma.$$

That is, $\text{ind } D_{u_0}$ is equal to the dimension of the maximal subspace of the space of sections of $\gamma^* TM_K$ on which I is negative definite. The projection

$$\pi_*|_{\ker D_{u_0}}: \ker D_{u_0} \longrightarrow \gamma^* TM_K$$

is injective by unique continuation (cf. [Wen16, Corollary 2.27]), which implies that we have

$$(4.19) \quad \dim V \leq \text{ind } D_{u_0}.$$

For $v \in \ker D_{u_0}$ we have that $u_\varepsilon = \exp_{u_0}(\varepsilon v)$ is a disk that is near to u_0 for small $\varepsilon > 0$. In particular, it is a solution of the Floer equation (3.3) up to

the first order. By Proposition 2 and (4.18) we get

$$0 > L(\gamma_\varepsilon) - \mathbf{a}(a) = L(\gamma_\varepsilon) - L(\gamma),$$

which in turn implies

$$(4.20) \quad 0 > E(\gamma_\varepsilon) - E(\gamma),$$

where $E(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2 dt$ is the energy of the curve γ .

Now, by defining $E(s) := E(\text{ev}(\exp_{u_0}(sv)))$ we compute

$$(4.21) \quad \left. \frac{d^2}{ds^2} E(s) \right|_{s=0} = \left. \frac{d^2}{ds^2} E(\text{ev}(\exp_{u_0}(sv))) d \text{ev}_{\exp_{u_0}(sv)}(d \exp_{u_0}(sv)) \right|_{s=0} \\ = \frac{d^2}{ds^2} E(\text{ev}(u_0)) d \text{ev}_{u_0}(v) = I(\pi_* v, \pi_* v),$$

where I is the index form, see e.g. [Jos08, Section 4.1]. The Taylor expansion of $E(\varepsilon)$ around $\varepsilon = 0$ is

$$E(\varepsilon) - E(0) = \varepsilon^2 I(\pi_* v, \pi_* v) + O(\varepsilon^3) \stackrel{(4.20)}{<} 0.$$

Hence for small enough $\varepsilon > 0$, we obtain $I(\pi_* v, \pi_* v) < 0$ and consequently $v \in V$. Therefore we have

$$\dim \ker D_{u_0} \leq \dim V \stackrel{(4.19)}{\leq} \text{ind } D_{u_0},$$

which concludes $\dim \text{coker } D_{u_0} = 0$ and therefore u_0 is transversely cut out.

The same proof shows that $r_* \circ \Psi_1$ is diagonal with respect to the action and length filtrations of Reeb chords and chains of broken geodesic loops, respectively, because of [Mil63, Lemma 15.4]. Namely, the index of the Hessian E_{**} is equal to the index of E_{**} restricted to the tangent space $T_\gamma BM_K$. \square

4.5. Isomorphism between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}^{\text{cell}}(BM_K)$

The goal of this section is to show that there is a chain isomorphism between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}^{\text{cell}}(BM_K)$. The outline of the proof is the following. Given a generator $a \in CW_{\Lambda_K}^*(F, F)$ we consider the trivial J -holomorphic half strip $u_0 \in \mathcal{M}(a)$ as in Subsection 4.4. By the genericity of the metric as in (A.1) we show that the evaluation map defined in Subsection 3.4 is transverse to the infinite dimensional stable manifold of the geodesic γ in BM_K .

This gives a chain isomorphism between $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}^{\text{cell}}(BM_K)$ by identifying a neighborhood of $u_0 \in \mathcal{M}(a)$ with the unstable manifold of the geodesic $\gamma \in BM_K$ which corresponds to the generator a .

We use the notation

$$\mathcal{F}_{[x_1, x_2]}C := \mathcal{F}_{x_1}C / \mathcal{F}_{x_2}C,$$

and order the generators of $CW_{\Lambda_K}^*(F, F)$ by their action

$$0 \leq \mathbf{a}(a_1) < \mathbf{a}(a_2) < \dots$$

Pick a strictly increasing sequence of numbers $\{a_i\}_{i=1}^\infty$ so that

$$0 \leq \mathbf{a}(a_1) < A_1 < \mathbf{a}(a_2) < A_2 < \dots,$$

and define

$$\begin{aligned} \mathcal{F}_{A_i}CW_{\Lambda_K}^*(F, F) &:= \{c \in CW_{\Lambda_K}^*(F, F) \mid \mathbf{a}(c) < A_i\} \\ \mathcal{F}_{A_i}C_{-*}(BM_K) &:= \{\sigma \in C_{-*}(BM_K) \mid \mathbf{a}(\sigma) < A_i\} \\ \mathcal{F}_{A_i}C_{-*}^{\text{cell}}(BM_K) &:= \{\sigma \in C_{-*}^{\text{cell}}(BM_K) \mid \mathbf{a}(\sigma) < A_i\}. \end{aligned}$$

We extend the filtration to all of \mathbb{Z} by letting $A_i = 0$ for every $i \leq 0$.

Note that the ordering of the generators of $CW_{\Lambda_K}^*(F, F)$ gives an ordering of the generators of $C_{-*}(BM_K)$ and $C_{-*}^{\text{cell}}(BM_K)$ by Proposition 3.

Recall the definition of the retraction

$$r: \mathcal{F}_{A_i}\Omega^{\text{pw}}M_K \longrightarrow \mathcal{F}_{A_i}BM_K,$$

defined in the proof of Lemma 11: Let $\gamma \in \mathcal{F}_{A_i}\Omega^{\text{pw}}M_K$ be any loop with $L(\gamma) = \int_0^1 |\dot{\gamma}| dt < a_i$. Pick a subdivision of the domain of $[0, 1]$

$$0 = t_0 < t_1 < \dots < t_{N-1} < 1 = t_N,$$

which is fine enough so that $\rho(\gamma(t_{i-1}), \gamma(t_i)) < \varepsilon$ for some $\varepsilon > 0$ small enough. Then $r(\gamma)$ is defined so that

$$r(\gamma)|_{[t_{i-1}, t_i]} = \text{unique minimal geodesic of length} < \varepsilon \text{ from } \gamma(t_{i-1}) \text{ to } \gamma(t_i).$$

Then we define

$$\begin{aligned} r_*: \mathcal{F}_{A_i}C_{-*}^{\text{cell}}(\Omega^{\text{pw}}M_K) &\longrightarrow \mathcal{F}_{A_i}C_{-*}^{\text{cell}}(BM_K) \\ \sigma &\longmapsto r \circ \sigma. \end{aligned}$$

Theorem 5. *The map*

$$r_* \circ \Psi_1 : CW_{\Lambda_K}^*(F, F) \longrightarrow C_{-*}^{\text{cell}}(BM_K)$$

$$a \longmapsto r_* \circ \text{ev}_*[\overline{\mathcal{M}}(a)],$$

is an isomorphism

Proof. We first show that for any $i \in \mathbb{Z}$ the map

$$r_* \circ \Psi_1 : \mathcal{F}_{[A_{i-1}, A_i]} CW_{\Lambda_K}^*(F, F) \longrightarrow \mathcal{F}_{[A_{i-1}, A_i]} C_{-*}^{\text{cell}}(BM_K)$$

$$a \longmapsto r_* \circ \text{ev}_*[\overline{\mathcal{M}}(a)],$$

is an isomorphism.

By the definition of the numbers $\{A_i\}_{i=0}^\infty$ there is only one generator $a \in \mathcal{F}_{[A_{i-1}, A_i]} CW_{\Lambda_K}^*(F, F)$. Denote its degree by λ . By Lemma 15, there is exactly one generator $\sigma \in \mathcal{F}_{[A_{i-1}, A_i]} C_{-\lambda}^{\text{cell}}(BM_K)$ that corresponds to a . We think of σ as the unstable manifold of the geodesic γ corresponding to a . Since both $\mathcal{F}_{[A_{i-1}, A_i]} CW_{\Lambda_K}^*(F, F)$ and $\mathcal{F}_{[A_{i-1}, A_i]} C_{-*}^{\text{cell}}(BM_K)$ only contains one generator each, we only need to show that

$$(4.22) \quad \sigma = (r_* \circ \Psi_1)(a).$$

By Proposition 3 we already know that the trivial J -holomorphic half strip $u_0 \in \overline{\mathcal{M}}(a)$ over a is so that $\text{ev}(u_0) = \gamma$. To prove that equation (4.22) holds it is enough to consider the map

$$r \circ \text{ev} : \overline{\mathcal{M}}(a) \longrightarrow BM_K,$$

and show that it is locally surjective at $\gamma \in \text{im } \sigma \subset BM_K$. We do this by showing that it is a submersion. That is, we consider

$$d(r \circ \text{ev})_{u_0} : T_{u_0} \overline{\mathcal{M}}(a) \longrightarrow T_\gamma BM_K,$$

and we show that it is surjective onto the image of σ . As noted above, σ should be thought of as the unstable manifold of γ inside BM_K with respect to the energy functional E . The following composition

$$T_{u_0} \overline{\mathcal{M}}(a) \xrightarrow{d\text{ev}_{u_0}} T_\gamma \Omega^{\text{pw}} M_K \xrightarrow{dr_\gamma} T_\gamma BM_K$$

is described as follows. Pick a subdivision of the domain of γ

$$0 \leq t_0 < t_1 < \dots < t_N \leq 1,$$

for some $N \in \mathbb{Z}_+$. The tangent space of $\Omega^{\text{pw}}M_K$ at γ has the following splitting

$$T_\gamma \Omega^{\text{pw}}M_K = T_\gamma BM_K \oplus T',$$

by [Mil63, Lemma 15.3, 15.4]. Here $T_\gamma BM_K$ is the space of *broken Jacobi fields* vanishing at the endpoints, and T' is the space of all vector fields W along γ so that $W(t_k) = 0$ for every $k \in \{1, \dots, N\}$ (cf. [Mil63, Section 15]). Furthermore we write

$$T_\gamma BM_K = T_\gamma \sigma \oplus T^+,$$

where $T_\gamma \sigma$ is the (maximal) subspace of $T_\gamma BM_K$ on which the Hessian E_{**} is negative definite, and $T^+ \subset T_\gamma BM_K$ is the subspace on which E_{**} is positive semidefinite. We will show that for any non-zero $v \in T_{u_0} \overline{\mathcal{M}}(a)$, its image $d \text{ev}_{u_0}(v)$ does not lie in T' , and that the image $d(r \circ \text{ev})_{u_0}(v)$ does not lie in T^+ .

$d \text{ev}_{u_0}$ is transverse to T' : Consider any non-zero $v \in T_{u_0} \overline{\mathcal{M}}(a) = \ker D_{u_0}$, where

$$D_{u_0} : W_\kappa^{2,2}(D_3, u_0^*TW_K) \longrightarrow W_\kappa^{1,2}(D_3, \Lambda^{0,1} \otimes_J u_0^*TW_K)$$

is the linearization of $\overline{\partial}_J$ at u_0 . Here $W_\kappa^{2,2}$ is the Sobolev space $W^{2,2}$ with weight $e^{\kappa s}$ for some small $\kappa > 0$ at the positive punctures in the domain D_3 . The differential of the evaluation map ev is a trace operator on $W_\kappa^{2,2}(D_3, u_0^*TW_K)$, so $d \text{ev}_{u_0}(v)$ is a vector field in W_K along $\gamma \subset M_K \subset W_K$. Assume that v is such that $v' := d \text{ev}_{u_0}(v) \in T'$, that is $v'(\gamma(t_k)) = 0$ for every $k \in \{1, \dots, N\}$. Since $d \text{ev}_{u_0}$ is a restriction to γ^*TM_K , we assume that v is so that $v(\gamma(t_k)) = 0$ for every $k \in \{1, \dots, N\}$. We consider the subspace

$$A := \left\{ v \in W_\kappa^{2,2}(D_3, u_0^*TW_K) \mid v(\gamma(t_k)) = 0, k \in \{1, \dots, N\} \right\}.$$

It is closed and has codimension N . The restricted linearized operator $D_{u_0}|_A$ is therefore a Fredholm operator with index

$$\text{ind } D_{u_0}|_A = \text{ind } D_{u_0} - N.$$

If we pick N large enough by making the subdivision of the domain of loops fine enough, the index $\text{ind } D_{u_0}|_A$ is negative. Hence $\ker D_{u_0} \cap A$ is empty for generic choices of almost complex structures on W_K . This means that $\text{im } d \text{ev}_{u_0} \cap T' = \{0\}$.

$d(r \circ \text{ev})_{u_0}$ is transverse to T^+ : Next, we show that for any non-zero $v \in \ker D_{u_0}$, its projection

$$v'' := d(r \circ \text{ev})_{u_0}(v) \in T_\gamma BM_K$$

does not lie in T^+ . We consider the path $s \mapsto \exp_{u_0}(sv)$ for $s \in (0, \varepsilon)$ with ε small enough. Then by the proof of Proposition 3 we have for every $s \in (0, \varepsilon)$ that

$$E((r \circ \text{ev})(u_0)) > E((r \circ \text{ev})(\exp_{u_0}(sv))).$$

Repeating the argument in the proof of Proposition 3 gives $I(v'', v'') < 0$, which shows that $v'' = d(r \circ \text{ev})_{u_0}(\xi)$ does not lie in T^+ .

Therefore $r_* \circ \Psi_1: \mathcal{F}_{[A_{i-1}, A_i]} CW_{\Lambda_K}^*(F, F) \rightarrow \mathcal{F}_{[A_{i-1}, A_i]} C_{-*}^{\text{cell}}(BM_K)$ is an isomorphism.

The filtrations on $CW_{\Lambda_K}^*(F, F)$ and $C_{-*}^{\text{cell}}(BM_K)$ are both bounded from below which gives an isomorphism $\mathcal{F}_{A_i} CW_{\Lambda_K}^*(F, F) \cong \mathcal{F}_{A_i} C_{-*}^{\text{cell}}(BM_K)$ for every $i \in \mathbb{Z}$. Thus every square in the following diagram commutes.

$$\begin{array}{ccccccc} \cdots & \subset & \mathcal{F}_{A_i} CW_{\Lambda_K}^*(F, F) & \subset & \mathcal{F}_{A_{i+1}} CW_{\Lambda_K}^*(F, F) & \subset & \cdots \\ & & \cong \downarrow r_* \circ \Psi_1 & & \cong \downarrow r_* \circ \Psi_1 & & \\ \cdots & \subset & \mathcal{F}_{A_i} C_{-*}^{\text{cell}}(BM_K) & \subset & \mathcal{F}_{A_{i+1}} C_{-*}^{\text{cell}}(BM_K) & \subset & \cdots \end{array}$$

We then pass to colimits to obtain the isomorphism

$$CW_{\Lambda_K}^*(F, F) \cong C_{-*}^{\text{cell}}(BM_K). \quad \square$$

Proof of Theorem 4. Theorem 4 is now an immediate corollary of Theorem 5, because there is a chain homotopy equivalence $C_{-*}^{\text{cell}}(BM_K) \simeq C_{-*}(\Omega_\xi M_K)$. So in particular we have $H_{-*}(\Omega_\xi M_K) \cong H_{-*}^{\text{cell}}(BM_K)$ via $s \circ r_*$ defined in (4.3) and (4.4). Hence

$$\Psi_1: HW_{\Lambda_K}^*(F, F) \rightarrow H_{-*}(\Omega_\xi M_K),$$

is an isomorphism. □

5. Applications

The first goal of this section is to equip $HW_{\Lambda_K}^*(F, F)$ and $H_{-*}(\Omega_\xi M_K)$ with the structure of $\mathbb{Z}[\pi_1(M_K)]$ -modules. The second goal is then to consider

the case when $S = S^n$ and exhibit examples of codimension 2 knots $K \subset S^n$ where the Alexander invariant is related to $CW_{\Lambda_K}^*(F, F)$ as $\mathbb{Z}[\pi_1(M_K)]$ -modules. From this we draw the conclusion that the unit conormal of K knows about the smooth topology of K beyond the fundamental group.

After we have discussed the $\mathbb{Z}[\pi_1(M_K)]$ -module structures in Subsection 5.1, we will provide background material surrounding the Alexander invariant in Subsections 5.2, 5.3. Then, in Subsection 5.4 we use the Leray–Serre spectral sequence associated with the path-loop fibration to relate the Alexander invariant to $CW_{\Lambda_K}^*(F, F)$ as a $\mathbb{Z}[\pi_1(M_K)]$ -module.

5.1. $\mathbb{Z}[\pi_1(M_K)]$ -module structures on $HW_{\Lambda_K}^*(F, F)$ and $H_{-*}(\Omega M_K)$

Consider any homotopy class $[\gamma] \in \pi_1(M_K)$ represented by the unique minimizing geodesic γ in the given homotopy class. Via the cell structure of BM_K , we associate to γ a generator $\sigma_\gamma \in H_{-*}(\Omega M_K)$. Then consider the map

$$\begin{aligned} \pi_1(M_K) \times H_{-*}(\Omega M_K) &\longrightarrow H_{-*}(\Omega M_K) \\ (\gamma, \sigma) &\longmapsto \gamma\sigma := (-1)^{|\sigma_\gamma|} P(\sigma \otimes \sigma_\gamma), \end{aligned}$$

where P denotes the Pontryagin product as in (3.2).

Lemma 16. *The map $(\gamma, \sigma) \longmapsto \gamma\sigma$ defines a group action of $\pi_1(M_K)$ on $H_{-*}(\Omega M_K)$.*

Proof. Let $[\gamma_1], [\gamma_2] \in \pi_1(M_K)$. As above, we assign to γ_1 and γ_2 the cohomology classes $\sigma_{\gamma_1}, \sigma_{\gamma_2} \in H_{-*}(\Omega M_K)$. Assign to the composition $\gamma_1\gamma_2$ the cohomology class

$$\sigma_{\gamma_1\gamma_2} := \sigma_{\gamma_1} \circ \sigma_{\gamma_2} = (-1)^{|\sigma_{\gamma_1}|} P(\sigma_{\gamma_2} \otimes \sigma_{\gamma_1}) \in H_{-*}(\Omega M_K).$$

Since P is associative up to a sign in cohomology we have

$$\begin{aligned} \gamma_1(\gamma_2\sigma) &= P(P(\sigma \otimes \sigma_{\gamma_2}) \otimes \sigma_{\gamma_1}) = (-1)^{|\sigma_{\gamma_1}|} P(\sigma \otimes P(\sigma_{\gamma_2} \otimes \sigma_{\gamma_1})) \\ &= (-1)^{|\sigma_{\gamma_1}| + |\sigma_{\gamma_2}|} P(\sigma \otimes \sigma_{\gamma_1\gamma_2}) = (\gamma_1\gamma_2)\sigma. \end{aligned}$$

□

By linearity we extend the action to a $\mathbb{Z}[\pi_1(M_K)]$ -module structure on $H_{-*}(\Omega M_K)$.

Consider a generator $a_\gamma \in HW_{\Lambda_K}^*(F, F)$, and denote by γ the geodesic that a_γ corresponds to. Via γ , we let $\sigma_\gamma \in H_{-*}(\Omega M_K)$ be the cohomology class corresponding to $a_\gamma \in HW_{\Lambda_K}^*(F, F)$. Then define

$$\begin{aligned} \pi_1(M_K) \times HW_{\Lambda_K}^*(F, F) &\longrightarrow HW_{\Lambda_K}^*(F, F) \\ (\gamma, a) &\longmapsto \gamma a := (-1)^{|a_\gamma|} \mu^2(a \otimes a_\gamma). \end{aligned}$$

Lemma 17. *The map $(\gamma, a) \mapsto \gamma a$ defines a group action of $\pi_1(M_K)$ on $HW_{\Lambda_K}^*(F, F)$.*

Proof. Let $\sigma_{\gamma_1}, \sigma_{\gamma_2}$ and $\sigma_{\gamma_1\gamma_2}$ be as in the proof of Lemma 16 above. Let $a \in HW_{\Lambda_K}^*(F, F)$ be any generator. Because μ^2 is associative up to a sign in cohomology we have

$$\begin{aligned} (5.1) \quad \gamma_1(\gamma_2 a) &= (-1)^{|a_{\gamma_1}|+|a_{\gamma_2}|} \mu^2(\mu^2(a \otimes a_{\gamma_2}) \otimes a_{\gamma_1}) \\ &= (-1)^{|a_{\gamma_2}|} \mu^2(a \otimes \mu^2(a_{\gamma_2} \otimes a_{\gamma_1})). \end{aligned}$$

Because $\{\Psi_m\}_{k=1}^\infty$ is an A_∞ -homomorphism, we glue the two disks contributing to $\Psi_1(a_{\gamma_1}) = \sigma_{\gamma_1}$ and $\Psi_1(a_{\gamma_2}) = \sigma_{\gamma_2}$ to obtain

$$(5.2) \quad P(\Psi_1(a_{\gamma_2}) \otimes \Psi_1(a_{\gamma_1})) = \Psi_1(\mu^2(a_{\gamma_2} \otimes a_{\gamma_1})).$$

Hence there exists a J -holomorphic disk in the symplectization of ∂W_K with two positive punctures a_{γ_1} and a_{γ_2} . Define $a_{\gamma_1\gamma_2} := (-1)^{|\sigma_{\gamma_1}|} \mu^2(a_{\gamma_2}, a_{\gamma_1})$. Then (5.2) says that

$$\Psi_1(a_{\gamma_1\gamma_2}) = (-1)^{|\sigma_{\gamma_1}|} P(\sigma_{\gamma_2} \otimes \sigma_{\gamma_1}) = \sigma_{\gamma_1\gamma_2}.$$

Thus $a_{\gamma_1\gamma_2} \in HW_{\Lambda_K}^*(F, F)$ is the generator corresponding to the concatenation $[\gamma_1\gamma_2] \in \pi_1(M_K)$. Combining this with (5.1) gives

$$\begin{aligned} \gamma_1(\gamma_2 a) &= (-1)^{|a_{\gamma_1}|+|a_{\gamma_2}|} \mu^2(\mu^2(a \otimes a_{\gamma_2}) \otimes a_{\gamma_1}) \\ &= (-1)^{|a_{\gamma_2}|} \mu^2(a \otimes \mu^2(a_{\gamma_2} \otimes a_{\gamma_1})) \\ &= (-1)^{|a_{\gamma_1\gamma_2}|} \mu^2(a \otimes a_{\gamma_1\gamma_2}) = (\gamma_1\gamma_2)a \end{aligned}$$

Lastly, we need to prove that if $\gamma_{\text{const}} \in \pi_1(M_K)$ is the constant loop, then $\mu^2(a \otimes a_{\gamma_{\text{const}}}) = a$. This follows from the definition of Ψ_1 . The generator of $HW_{\Lambda_K}^*(F, F)$ which corresponds to the (0-chain of) the constant loop is the unique Lagrangian intersection generator in the compact part of W_K

which corresponds to the unique intersection point of F_i with F_j , call it $x \in HW_{\Lambda_K}^*(F, F)$. Therefore

$$\gamma_{\text{const}} a = \mu^2(a \otimes x) = a. \quad \square$$

By linearity we extend the action to a $\mathbb{Z}[\pi_1(M_K)]$ -module structure on $HW_{\Lambda_K}^*(F, F)$.

Theorem 6 (Theorem 1). *The isomorphism*

$$\Psi_1: HW_{\Lambda_K}^*(F, F) \longrightarrow H_{-*}(\Omega M_K),$$

is an isomorphism of $\mathbb{Z}[\pi_1(M_K)]$ -modules.

Proof. Let $a \in HW_{\Lambda_K}^*(F, F)$ be a generator and let $[\gamma] \in \pi_1(M_K)$ be a homotopy class represented by a unique minimizing geodesic γ . Then consider a generator $a_\gamma \in HW_{\Lambda_K}^*(F, F)$ so that $\Psi_1(a_\gamma) = \sigma_\gamma$, where $\sigma_\gamma \in H_{-*}(\Omega M_K)$ is the cohomology class corresponding to γ . Then we have

$$\begin{aligned} \Psi_1(\gamma a) &= (-1)^{|a_\gamma|} \Psi_1(\mu^2(a \otimes a_\gamma)) \stackrel{(5.2)}{=} (-1)^{|a_\gamma|} P(\Psi_1(a) \otimes \Psi_1(a_\gamma)) \\ &= (-1)^{|\sigma_\gamma|} P(\Psi_1(a) \otimes \sigma_\gamma) = \gamma \Psi_1(a). \end{aligned} \quad \square$$

Remark 7. Note that $\mathbb{Z}[\pi_1(M_K)] \cong H_0(\Omega M_K)$, and consider $C_0^{\text{cell}}(BM_K)$ as an A_∞ -algebra with operations $\{m_i\}_{i=1}^\infty$ where $m_1 = 0$, $m_2 = P$ is the Pontryagin product, and $m_i = 0$ for $i \geq 3$. We observe that $CW_{\Lambda_K}^*(F, F)$ can be equipped with the structure of a left A_3 -module over $C_0^{\text{cell}}(BM_K)$. More precisely we define this left A_3 -module structure as a sequence of maps

$$\nu_{CW}^r: (C_0^{\text{cell}}(BM_K))^{\otimes(r-1)} \otimes CW_{\Lambda_K}^*(F, F) \longrightarrow CW_{\Lambda_K}^*(F, F),$$

defined by

$$\begin{cases} \nu_{CW}^1(c) := \mu^1(c) \\ \nu_{CW}^2(x \otimes c) := \mu^2(a_x \otimes c) \\ \nu_{CW}^3(x_2 \otimes x_1 \otimes c) := \mu^3(a_{x_2} \otimes a_{x_1} \otimes c) \\ \nu_{CW}^k(x_{k-1} \otimes \cdots \otimes x_1 \otimes c) := 0, & k \geq 4, \end{cases}$$

where a_{x_i} is the unique Reeb chord corresponding to x_i via Ψ_1 . Then by computation we have that $\{\nu_{CW}^r\}_{r=1}^\infty$ satisfies the following equation for

$n \in \{1, 2, 3\}$.

$$(5.3) \quad \sum_{i=0}^{n-1} m_{n-i}(x_{n-1} \otimes \cdots \otimes x_{i+1} \otimes \nu_{CW}^{i+1}(x_i \otimes \cdots \otimes x_1 \otimes c)) \\ + \sum_{\substack{\ell+k < r \\ \ell \geq 1 \\ k \geq 0}} (-1)^{|c| + \mathfrak{A}_k} \nu_{CW}^{r-\ell+1}(x_{r-1} \otimes \cdots \otimes x_{k+\ell+1} \\ \otimes \mu^\ell(x_{k+\ell} \otimes \cdots \otimes x_{k+1}) \otimes x_k \otimes \cdots \otimes x_1 \otimes c) = 0$$

Note that this means that there is a group action up to homotopy of $C_0^{\text{cell}}(BM_K)$ on $CW_{\Lambda_K}^*(F, F)$, but there are no higher coherent homotopies. However, this is enough to directly obtain Lemma 16 and Lemma 17.

Since $C_{-*}^{\text{cell}}(BM_K)$ is an A_∞ -algebra, it can be regarded as a left A_∞ -module over itself, and therefore also as a left A_∞ -module over $C_0^{\text{cell}}(BM_K)$ via the sequence of maps

$$\nu_{\text{cell}}^r : (C_0^{\text{cell}}(BM_K))^{\otimes(r-1)} \otimes C_{-*}^{\text{cell}}(BM_K) \longrightarrow C_{-*}^{\text{cell}}(BM_K),$$

defined by

$$\begin{cases} \nu_{\text{cell}}^1(y) := m_1(y) \\ \nu_{\text{cell}}^2(x \otimes y) := P(x \otimes y) \\ \nu_{\text{cell}}^k(x_{k-1} \otimes \cdots \otimes x_1 \otimes y) := 0, \quad k \geq 3. \end{cases}$$

By a computation we see that $\{\nu_{\text{cell}}^r\}_{r=1}^\infty$ satisfies (5.3) for every $n \in \mathbb{Z}_+$. For $n \geq 4$ the equation is trivial.

Furthermore we have that the A_∞ -homomorphism $\{\Psi_k\}_{k=1}^\infty$ induces an isomorphism of A_3 -modules over $C_0^{\text{cell}}(BM_K)$ as follows. The isomorphism of A_3 -modules over $C_0^{\text{cell}}(BM_K)$ is a sequence of maps

$$\psi_r : (C_0^{\text{cell}}(BM_K))^{\otimes(r-1)} \otimes CW_{\Lambda_K}^*(F, F) \longrightarrow C_{-*}^{\text{cell}}(BM_K)$$

defined by

$$\psi_r(x_{r-1} \otimes \cdots \otimes x_1 \otimes c) := \Psi_r(a_{x_{r-1}} \otimes \cdots \otimes a_{x_1} \otimes c),$$

where a_{x_i} is the unique generator corresponding to x_i via Ψ_1 . Then by computation we have that $\{\psi_r\}_{r=1}^\infty$ satisfies the following equation for $n \in$

{1, 2, 3}.

$$\begin{aligned} & \sum_{i=0}^{r-1} \nu_{\text{cell}}^{r-i}(x_{r-1} \otimes \cdots \otimes x_{i+1} \otimes \psi_{i+1}(x_i \otimes \cdots \otimes x_1 \otimes c)) \\ &= \sum_{i=0}^{r-1} \psi_{r-i}(x_{r-1} \otimes \cdots \otimes x_{i+1} \otimes \nu_{CW}^{i+1}(x_i \otimes \cdots \otimes x_1 \otimes c)) \\ &+ \sum_{\substack{s+t+k=r \\ t, k \geq 1 \\ s \geq 0}} (-1)^{|c| + \mathbf{X}_k} \psi_{r-\ell+1}(x_{r-1} \otimes \cdots \otimes x_{k+\ell+1} \\ &\quad \otimes m_\ell(x_{k+\ell} \otimes \cdots \otimes x_{k+1}) \otimes x_k \otimes \cdots \otimes x_1 \otimes c) = 0, \end{aligned}$$

The fact that this is an A_3 -module isomorphism directly implies Theorem 6.

5.2. Plumbings and infinite cyclic covers

In this section we review standard background material from [Rol76].

Let $p, q \geq 2$ and $n = p + q + 1$. We consider the plumbing of S^p with S^q . That is, consider $S^p \times D^q$ and $S^q \times D^p$. By identifying D^p with the upper hemisphere of S^p , we have

$$\begin{aligned} D^p \times D^q &\subset S^p \times D^q \\ D^q \times D^p &\subset S^q \times D^p. \end{aligned}$$

We then take the disjoint union of $S^p \times D^q$ with $S^q \times D^p$ and identify their common submanifolds $D^p \times D^q \cong D^q \times D^p$ via $f: (x, y) \mapsto (y, x)$. We call the resulting space the plumbing of S^p and S^q , denoted by $S^p \#_{\text{plumb}} S^q$. In short we write

$$\Sigma = S^p \#_{\text{plumb}} S^q := (S^p \times D^q) \sqcup_f (S^q \times D^p).$$

We note that $S^p \vee S^q$ is the deformation retract of Σ . Let $K := \partial \Sigma$ and note that it is a $(p + q - 1)$ -dimensional sphere. Embed Σ into S^n and consider the complement of its boundary $M_K := S^n \setminus K$; denote its infinite cyclic cover by \widetilde{M}_K .

Following [Rol76, Section 5.C] we find the simplicial structure of \widetilde{M}_K by cutting along Σ . More precisely, let $\Sigma^\pm \cong \overset{\circ}{\Sigma} \times (-1, 1)$ be an open bicollar

of the interior of Σ , and let

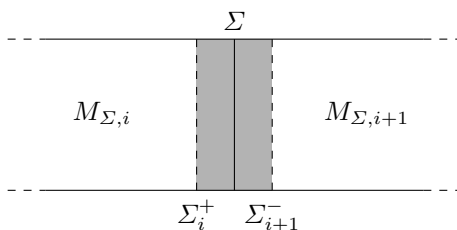
$$\begin{aligned} \Sigma^+ &:= \overset{\circ}{\Sigma} \times (0, 1) \subset S^n \\ \Sigma^- &:= \overset{\circ}{\Sigma} \times (-1, 0) \subset S^n \\ M_\Sigma &:= S^n \setminus \Sigma. \end{aligned}$$

Consider infinitely many copies of each of Σ^+ , Σ^- and M_Σ . Denote the copies by $M_{\Sigma;i}$, Σ_i^+ and Σ_i^- for $i \in \mathbb{Z}$. Then consider the disjoint union of all the $M_{\Sigma;i}$ and glue them together by identifying $\Sigma_i^+ \subset M_{\Sigma;i}$ with $\Sigma_{i+1}^- \subset M_{\Sigma;i+1}$ via the map

$$\begin{aligned} \Sigma_i^+ = \overset{\circ}{\Sigma} \times (0, 1) &\longrightarrow \overset{\circ}{\Sigma} \times (-1, 0) = \Sigma_{i+1}^- \\ (\sigma, t) &\longmapsto (\sigma, t - 1). \end{aligned}$$

Define

$$(5.4) \quad \widetilde{M}_K := \coprod_{i=-\infty}^{\infty} M_{\Sigma;i} / (\Sigma_i^+ \sim \Sigma_{i+1}^-).$$



5.3. The Alexander invariant

In this section we review standard material on the Alexander invariant from [Rol76].

Associated to the open cover $\mathcal{U} = (M_{\Sigma;i})_{i=-\infty}^{\infty}$ of \widetilde{M}_K is the sequence of inclusions

$$\coprod_{i=-\infty}^{\infty} M_{\Sigma;i} \cap M_{\Sigma;i+1} \begin{matrix} \xrightarrow{\iota_{i+1}} \\ \xleftarrow{\iota_i} \end{matrix} \coprod_{i=-\infty}^{\infty} M_{\Sigma;i} \xrightarrow{\kappa} \widetilde{M}_K$$

Since the group of deck transformations of \widetilde{M}_K is infinite cyclic, we choose a generator $\tau \in \text{Aut}(\widetilde{M}_K, \pi)$ which induces an automorphism

$$\tau_* : H_*(\widetilde{M}_K) \longrightarrow H_*(\widetilde{M}_K).$$

This gives a $\mathbb{Z}[t^{\pm 1}]$ -module structure on $H_*(\widetilde{M}_K)$ as follows. Let $p(t) = \sum_{i=-s}^r c_i t^i \in \mathbb{Z}[t^{\pm 1}]$, then for any $\alpha \in H_*(\widetilde{M}_K)$ let

$$p(t)\alpha = \sum_{i=-s}^r c_i \tau_*^i(\alpha),$$

where τ_*^i is the i -fold composition power of τ_* . The *Alexander invariant* is then defined as $H_*(\widetilde{M}_K)$ considered as a $\mathbb{Z}[t^{\pm 1}]$ -module.

Lemma 18 ([Rol76, Theorem 7.G.1]). *There exist non-trivial knots $K \subset S^{n+2}$ with infinite cyclic knot group, $\pi_1(M_K) \cong \mathbb{Z}$.*

Proof. Let $p, q \geq 2$ and let $n = p + q - 1$. We then consider any K obtained as $\partial\Sigma$, where

$$\Sigma = S^p \#_{\text{plumb}} S^q.$$

Now we have that \widetilde{M}_Σ is simply connected: Every loop in S^{n+2} shrinks missing Σ since Σ is homotopy equivalent to $S^p \vee S^q$. This is because $\text{codim}(S^p) \geq 3$ and $\text{codim}(S^q) \geq 3$ in S^{n+2} . From the construction of \widetilde{M}_K in (5.4) we thus have $\pi_1(\widetilde{M}_K) \cong 1$. Hence, because the group of deck transformations of $\widetilde{M}_K \rightarrow M_K$ is \mathbb{Z} , we have $\pi_1(M_K) \cong \mathbb{Z}$.

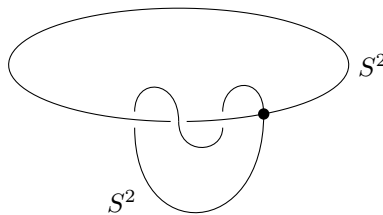


Figure 13: The core of the self plumbing of two knotted S^2 embedded in S^5 .

To see that such non-trivial K exists, we may consider

$$K = \partial(S^2 \#_{\text{plumb}} S^2) \subset S^5,$$

where the core of the plumbing is shown in Figure 13. The Alexander invariant of K is non-trivial by a computation (cf. [Rol76, Exercise 7.F.5]). \square

5.4. Using the Leray–Serre spectral sequence

Consider the knot $K = S^p \#_{\text{plumb}} S^q \subset S^n$ with $p + q = n - 1$ as in Subsection 5.2. We use the notation $\mathbb{Z}\pi := \mathbb{Z}[\pi_1(M_K)]$.

Associated to the path-loop fibration

$$\begin{array}{ccc} \Omega M_K & \hookrightarrow & PM_K \\ & & \downarrow \\ & & M_K \end{array}$$

is the Leray–Serre spectral sequence. It is first quadrant spectral sequence $\{E_{i,j}^r, d_{i,j}^r\}_{i,j \in \mathbb{N}}$ of $\mathbb{Z}\pi$ -modules which converges:

$$E_{i,j}^2 \cong H_i(M_K; H_j(\Omega M_K)) \implies H_{i+j}(PM_K) = \begin{cases} \mathbb{Z}\pi/(t-1), & i+j=0 \\ 0, & \text{otherwise,} \end{cases}$$

Note that $\pi_1(M_K)$ is Abelian and hence we can consider $H_i(M_K; H_j(\Omega M_K))$ as a $\mathbb{Z}\pi$ -module.

Since $C_*(\widetilde{M}_K)$ is only supported in degrees 0, p and q we have the following facts

- Following [Hat02, Section 3.H] and [Shu10] we have the following identification

$$H_i(M_K; H_j(\Omega M_K)) \cong H_i\left(C_*(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_j(\Omega M_K)\right).$$

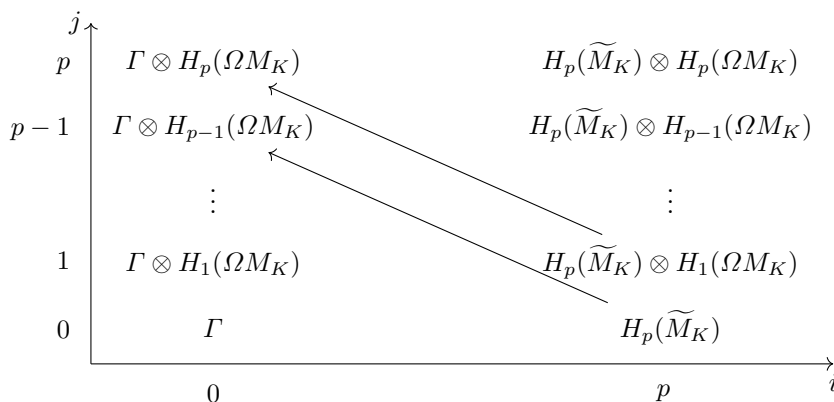
Assume that $|p - q| \neq 1$. Then we trivially have

$$H_i(M_K; H_j(\Omega M_K)) = \begin{cases} H_j(\Omega M_K), & i=0 \\ H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_j(\Omega M_K), & i=p \\ H_q(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_j(\Omega M_K), & i=q \\ 0, & \text{otherwise,} \end{cases}$$

because $C_*(\widetilde{M}_K)$ is only supported in $* \in \{0, p, q\}$.

- $E_{i,j}^2$ is only supported on the vertical lines $i \in \{0, p, q\}$.
- The bottom row is $E_{i,0}^2 = H_i(\widetilde{M}_K)$, since $H_0(\Omega M_K) \cong \mathbb{Z}\pi$.

Example 1. Consider the case when K is obtained as the boundary of $S^p \#_{\text{plumb}} S^p \subset S^{2p+1}$ where the core of the plumbing is depicted in Figure 13. In this case, $E_{i,j}^2$ is only supported at the vertical lines $i \in \{0, p\}$. For this spectral sequence, the p -th page is the first page after page 1 that has non-zero differentials. Namely, the p -th page of this spectral sequence is



Where $\Gamma = \mathbb{Z}\pi/(t-1)$ and every tensor product is taken over $\mathbb{Z}\pi$. The differentials at every page succeeding the p -th page is zero, so in particular we get

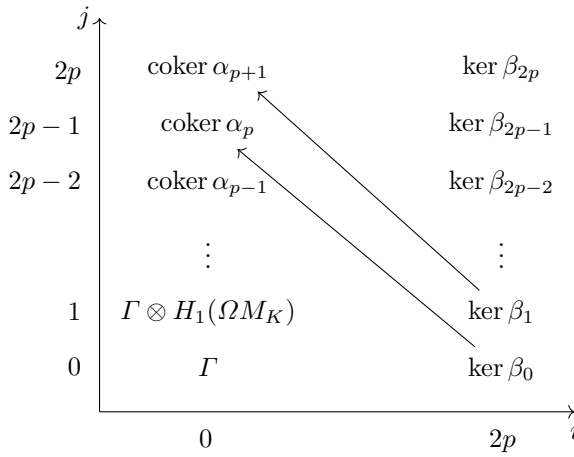
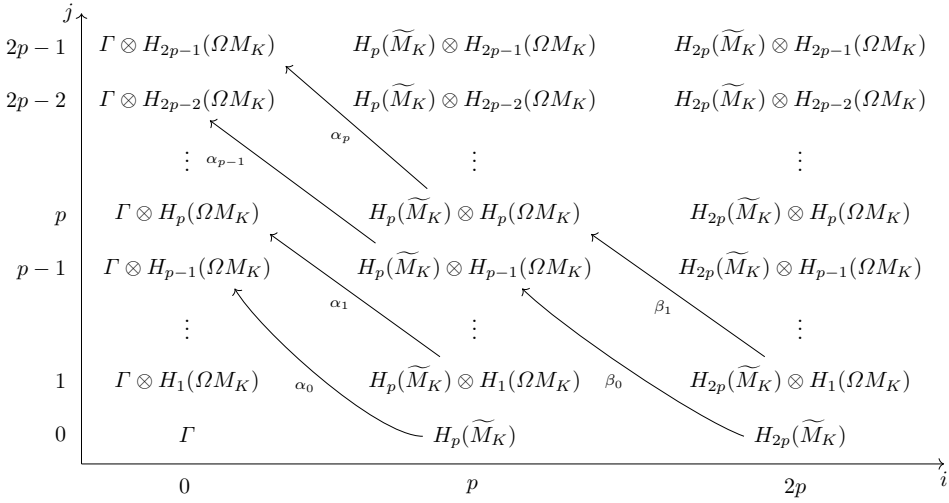
$$H_p(\widetilde{M}_K) \cong \mathbb{Z}\pi/(t-1) \otimes_{\mathbb{Z}\pi} H_{p-1}(\Omega M_K) \cong \mathbb{Z}\pi/(t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-p}(F, F).$$

Example 2. Suppose K is obtained as the boundary of $S^p \#_{\text{plumb}} S^{2p} \subset S^{3p+1}$ for $p \geq 2$ where the core of the plumbing is depicted in Figure 13. In this case the second page of the spectral sequence is only supported at the lines $i \in \{0, p, 2p\}$. The p -th page of the spectral sequence is the first page after page 1 that has non-zero differentials and it looks as follows: Immediately from this page, we get an isomorphism of $\mathbb{Z}\pi$ -modules

(5.6)

$$H_p(\widetilde{M}_K) \cong \mathbb{Z}\pi/(t-1) \otimes_{\mathbb{Z}\pi} H_{p-1}(\Omega M_K) \cong \mathbb{Z}\pi/(t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-p}(F, F).$$

Furthermore, the next page with non-zero differentials is page $2p$, which looks as follows



This is the last page with non-zero differentials, so $\text{coker } \alpha_{p-1} \cong \{0\}$ and hence from page p , we obtain an exact sequence

$$(5.7) \quad H_{2p}(\widetilde{M}_K) \xrightarrow{\beta_0} H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_{p-1}(\Omega M_K) \xrightarrow{\alpha_{p-1}} \Gamma \otimes_{\mathbb{Z}\pi} H_{2p-2}(\Omega M_K) \longrightarrow 0,$$

and for each $i \in \mathbb{Z}_+$ we have an exact sequence

$$H_{2p}(\widetilde{M}_K) \otimes H_i(\Omega M_K) \xrightarrow{\beta_i} H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_{p-1+i}(\Omega M_K) \xrightarrow{\alpha_{p-1+i}} \Gamma \otimes_{\mathbb{Z}\pi} H_{2p-2+i}(\Omega M_K).$$

Furthermore, from page $2p$ we get isomorphisms

$$\ker \beta_j \cong \operatorname{coker} \alpha_{p+j},$$

for every $j \in \mathbb{N}$. In particular, in view of (5.7), we have $\ker \beta_0 \cong \operatorname{coker} \alpha_p$ where

$$\alpha_p: H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} H_p(\Omega M_K) \longrightarrow \Gamma \otimes_{\mathbb{Z}\pi} H_{2p-1}(\Omega M_K).$$

So if $H_*(\Omega M_K)$ and α_p is known, we compute $H_p(\widetilde{M}_K)$ by (5.6) but also a quotient of $H_{2p}(\widetilde{M}_K)$ by exactness of (5.7)

$$H_{2p}(\widetilde{M}_K) / \operatorname{coker} \alpha_p \cong \ker \alpha_{p-1}.$$

Let us summarize what we have.

$$\begin{cases} H_p(\widetilde{M}_K) \cong \mathbb{Z}\pi / (t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-p}(F, F) \\ H_{2p}(\widetilde{M}_K) / \operatorname{coker} \alpha_p \cong \ker \alpha_{p-1}, \end{cases}$$

where

$$\begin{aligned} \alpha_p: H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{-p}(F, F) &\longrightarrow \mathbb{Z}\pi / (t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-2p}(F, F) \\ \alpha_{p-1}: H_p(\widetilde{M}_K) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-p}(F, F) &\longrightarrow \mathbb{Z}\pi / (t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{2-2p}(F, F). \end{aligned}$$

Example 3. For a slightly more general case, where $p \geq 2$ and $q > p + 1$ we consider again K to be the boundary of $S^p \#_{\text{plumb}} S^q \subset S^{p+q+1}$ where the core of the plumbing is depicted in Figure 13. The Leray–Serre spectral sequence is supported at the lines $i \in \{0, p, q\}$. Exactly like in Example 2, we compute

$$H_p(\widetilde{M}_K) \cong \mathbb{Z}\pi / (t-1) \otimes_{\mathbb{Z}\pi} HW_{\Lambda_K}^{1-p}(F, F).$$

Let Λ_{unknot} denote the unit conormal of the standard embedded $S^{n-2} \subset S^n$. A consequence of these computations is the following theorem.

Theorem 7 (Theorem 2). *Let $n = 5$ or $n \geq 7$. Let $x \in M_K$ be a point. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$, such that $\Lambda_K \cup \Lambda_x$ is not Legendrian isotopic to $\Lambda_{\text{unknot}} \cup \Lambda_x$.*

Proof. For the case $n = 5$, consider the knot $K = \partial(S^2 \#_{\text{plumb}} S^2) \subset S^5$ where the core of the plumbing is depicted in Figure 13. In the case $n \geq 7$ we let

$p \geq 2$ and $q > p + 1$ and consider $K = \partial(S^p \#_{\text{plumb}} S^q) \subset S^{p+q+1}$, where the core of the plumbing is again depicted in Figure 13.

We note that for dimensional reasons we have $\pi_1(M_K) \cong \mathbb{Z}$, but the Alexander invariant shows that K is non-trivial [Rol76, Section 7.G] (see also Lemma 18).

The computations in Example 1 and Example 3 show that in particular

$$H_p(\widetilde{M}_K) \cong \mathbb{Z}[\pi_1(M_K)]/(t - 1) \otimes_{\mathbb{Z}[\pi_1(M_K)]} HW_{\Lambda_K}^{1-p}(F, F).$$

Since we use classical methods to show that $H_p(\widetilde{M}_K)$ is non-trivial (see Lemma 18, it follows that $HW_{\Lambda_K}^{1-p}(F, F)$ is non-trivial. Consider the unknot $S^{n-2} \subset S^n$, then the complement M_{unknot} is homotopy equivalent to a circle, which means that $H_{-*}(\Omega M_{\text{unknot}}) \cong HW_{\Lambda_{\text{unknot}}}^*(F, F)$ is only supported in degree 0. Therefore we have $HW_{\Lambda_K}^{1-p}(F, F) \not\cong HW_{\Lambda_{\text{unknot}}}^{1-p}(F, F)$ and so $\Lambda_K \cup \Lambda_x$ is not Legendrian isotopic to $\Lambda_{\text{unknot}} \cup \Lambda_x$. \square

Appendix A. Monotonicity of J -holomorphic half strips

To establish compactness of the moduli spaces $\mathcal{M}(\mathbf{a})$ in Subsection 3.3 we need to make sure that J -holomorphic half strips in $\mathcal{M}(\mathbf{a})$ does not escape to horizontal infinity. Pick a tubular neighborhood of $K \subset M_K$ and call it $N(K)$. Then we decompose M_K as

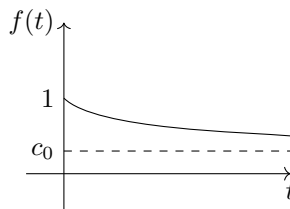
$$M_K \cong (S \setminus N(K)) \cup_{\partial N(K)} ([0, \infty) \times \partial N(K)) ,$$

where we identify $\partial(S \setminus N(K)) \cong \partial N(K)$ with $\{0\} \times \partial N(K) \cong \partial N(K)$. Pick a generic Riemannian metric g on $S \setminus N(K)$ such that geodesics are non-degenerate critical points of the length and energy functionals. Define a function

$$f: [0, \infty) \longrightarrow [0, \infty)$$

so that

$$(A.1) \quad \begin{cases} f(0) = 1 \\ f(t) > c_0 > 0, & \forall t \in [0, \infty) \\ f'(0) = -1 \\ f'(t) < 0, & \forall t \in [0, \infty) \\ f''(t) \geq 0, & \forall t \in [0, \infty) \end{cases}$$



Define a metric h on M_K as

$$h = \begin{cases} g, & \text{in } S \setminus N(K) \\ dt^2 + f(t)g|_{\partial N(K)}, & \text{in } [0, \infty) \times \partial N(K), \end{cases}$$

where t is the coordinate in the $[0, \infty)$ -factor. Similar to the situation in [EL17, Appendix C], if $x, y \in \partial N(K)$ are two points and $c: [0, \ell] \rightarrow M_K$ a geodesic with $c(s_1) = x \in \partial N(K)$ and $c(s_2) = y \in \partial N(K)$, then there is a unique geodesic $(t(s), c(s)) \in [0, \infty) \times \partial N(K)$ so that

- $(0, c(s_1)) = (0, x)$ and $(t(\ell), c(s_2)) = (0, y)$, and
- $t: [0, \ell] \rightarrow [0, \infty)$ is a Morse function with a unique maximum at some interior point $s_0 \in (0, \ell)$.

If we define

$$N_i := [0, i] \times \partial N(K),$$

then

$$N_0 \subset N_1 \subset N_2 \subset \dots \subset [0, \infty) \times \partial N(K),$$

is an exhaustion of $[0, \infty) \times \partial N(K)$ by compacts. Then given any geodesic $c: [0, \ell] \rightarrow M_K$, there exists some $m \geq 0$ so that $c(t) \in N_m$ for every $t \in [0, \ell]$. In particular, if we restrict to the present situation in this paper, where every geodesic is a loop based at $\xi \in S \setminus N(K)$. To this end fix some constant $L_0 > 0$ and assume $\gamma \in \mathcal{F}_{L_0} BM_K$, that is γ is a piecewise geodesic loop based at $\xi \in S \setminus N(K)$ with length bounded above by L_0 (for details see Subsection 4.2). Then there is some $m = m(L_0, h) > 0$ depending only on L_0 and the metric h so that $\gamma(t) \in N_m$ for every t . We prove that there exists some $m_0 > 0$ (depending on m and the metric h) so that the J -holomorphic strips lie inside of N_{m_0} by using the monotonicity lemma [Sik94, Proposition 4.7.2] (see also [CEL10, Lemma 3.4]).

Our metric h defined in (A.1) extends to a metric on W_K such that it has bounded geometry in the terminology of [Sik94, Section 4]. Furthermore, since $M_K \subset W_K$ is Lagrangian, the tuple (W_K, J, M_K, h) is tame in the sense of [Sik94, Definition 4.1.1]. Let $r_W, C_W > 0$ be constants so that for any $x, y \in M_K$

$$d_{M_K}(x, y) \leq r_W \Rightarrow d_{W_K}(x, y) \leq C_W d_{M_K}(x, y),$$

where d_{M_K} and d_{W_K} are the metrics induced by h on M_K and W_K respectively. If we denote the lower bound on the injectivity radius by ρ , we may assume $r_W \leq \rho$.

Lemma 19 ([Sik94, Proposition 4.7.2 (ii)]). *Let (V, J, W, μ) be tame. Then there exist a positive constant $C_4(W) > 0$ with the following property. Let $u: T \rightarrow V$ be a J -holomorphic curve so that $u(\partial T) \subset \partial B(x, r) \cup W$ where $x \in u(T)$ and $r < r_W$. Then*

$$\text{area}(u(T) \cap B(x, r)) \geq C_4(W)r^2.$$

We use this lemma with $V = W_K$, $W = F \cup M_K$ and $\mu = h$.

Theorem 8. *Let $A > 0$ be arbitrary and consider a generator $a \in \mathcal{F}_{ACW_{\Lambda_K}^*}(F, F)$. Then there exists $m > 0$ so that $\text{im } u \subset N_m$ for any $u \in \mathcal{M}(a)$.*

Proof. Consider a generator $a \in \mathcal{F}_{ACW_{\Lambda_K}^*}(F, F)$ and pick some $u \in \overline{\mathcal{M}}(a)$. Then by Proposition 3 we have

$$L(\text{ev}(u)) = \mathbf{a}(a) < A,$$

Because the J -holomorphic disk $u \in \mathcal{M}(a)$ has boundary on the Reeb chord a , the exact Lagrangian $F \cong DT_\xi^*S$ for $\xi \in M_K$ and the geodesic $\gamma := \text{ev}(u)$. Therefore there is some $m' > 0$ (depending only on A) so that $\partial \text{im } u \subset N_{m'}$ for any $u \in \mathcal{M}(a)$. Then pick some $m > m' > 0$ (which a priori can be equal to ∞) and assume that $\text{im } u \subset N_m$. We consider $U := \text{im } u \cap (N_m \setminus N_{m'})$ and then we prove that m is finite. Namely, fix some $r < r_W$ and let $v_1, \dots, v_\mu \in U$ be the maximal number of points so that $d_{W_K}(v_i, v_j) > 2r$. Then we apply Lemma 19 to each $U_i := U \cap B(v_i, r)$ so that $\text{area}(U_i) \geq C_4r^2$ for each $i \in \{1, \dots, \mu\}$. Therefore

$$\text{area}(U) \geq \mu \text{area}(U_1) \geq \mu C_4r^2 \Leftrightarrow \mu \leq \frac{\text{area}(U)}{C_4r^2}.$$

Since $\mathbf{a}(a)$ is bounded by A , so is the area of U . Hence

$$\mu < \frac{A}{C_4r^2} < \infty.$$

This shows that there is some finite $m > 0$ such that $\text{im } u \subset N_m$ for every $u \in \mathcal{M}(a)$. □

Appendix B. Signs, gradings and orientations of moduli spaces

In this section, we use the same conventions and setup as in [Sei08, Section (11)] and [FOOO10, Section 8]. Pick some $T \in \overline{\mathcal{H}}_m$ and consider the collection of Lagrangian branes $F_0^\#, \dots, F_m^\#$ of a cotangent fiber $F_0 \cong T_\xi^*S \subset W_K$ at $\xi \in M_K$ and a system of parallel copies \overline{F} as in Subsection 3.3 and Section 2. Pick a word of generators $\mathbf{a} = a_1 \cdots a_m$ where $a_k \in CW^*(F_{k-1}, F_k)$, and pick abstract perturbation data so that $\mathcal{M}(\mathbf{a})$ is regular. Then for some $u \in \mathcal{M}(\mathbf{a})$, denote the linearization of the operator $\overline{\partial}_{J_T}$ at the J -holomorphic disk u by D_u . Then we have the following:

Lemma 20 ([Abo12b, Lemma 6.1]). *With the choice as above there is a canonical up to homotopy isomorphism*

$$\det D_u \cong o_\xi \otimes o_{a_1}^\vee \otimes \cdots \otimes o_{a_m}^\vee \otimes o_\xi^\vee$$

and in particular

$$\bigwedge^{\text{top}}(T\overline{\mathcal{M}}(\mathbf{a})) \cong \bigwedge^{\text{top}}(T\overline{\mathcal{H}}_m) \otimes o_\xi \otimes o_{a_1}^\vee \otimes \cdots \otimes o_{a_m}^\vee \otimes o_\xi^\vee.$$

Since the orientation lines o_x are naturally graded by the indices of the linearized operators D_x , we have a natural isomorphism coming from reordering tensor products of orientation lines which produces a Koszul sign

$$o_{x_1} \otimes o_{x_2} \cong (-1)^{|x_1||x_2|} o_{x_2} \otimes o_{x_1}.$$

Furthermore there are natural non-degenerate pairings

$$o_x \otimes o_x^\vee \cong \mathbb{R}.$$

From now on we use the following abbreviation: For the word $\mathbf{a} = a_1 \cdots a_m$, we let

$$o_{\mathbf{a}} := o_{a_1} \otimes \cdots \otimes o_{a_m}.$$

As in (3.4) and (3.5) denote by \mathcal{H}_m the moduli space of abstract J -holomorphic disks with $m + 2$ boundary punctures, and its Deligne–Mumford

compactification by $\overline{\mathcal{H}}_m$. Then the codimension one boundary $\partial\overline{\mathcal{H}}_m$ is covered by the natural inclusions of the following strata

$$\begin{aligned} \text{(B.1)} \quad & \overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{H}}_{m_2}, \quad m_1 + m_2 = m \\ \text{(B.2)} \quad & \overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{R}}_{m_2}, \quad m_1 + m_2 = m + 1. \end{aligned}$$

Here \mathcal{R}_m is the Deligne–Mumford space of unit disks in the complex plane with $m + 1$ boundary punctures that are oriented counterclockwise. We would like to compare the product orientation of each of the strata with the boundary orientation on $\partial\overline{\mathcal{H}}_m$. The orientation of the boundary is determined as follows. Any orientation on a manifold X induces an orientation on its boundary via the outward normal first-rule. More precisely via the canonical isomorphism

$$\bigwedge^{\text{top}} TX \cong \nu_{\partial X} \otimes \bigwedge^{\text{top}} T\partial X,$$

where $\nu_{\partial X}$ is the normal bundle of ∂X which is canonically trivialized by the outwards normal vector along the boundary. Following the conventions in [Sei08, Abo10, Abo12b] there is a choice of coherent orientations on $\overline{\mathcal{H}}_m$ such that the boundary strata (B.1) and (B.2) differs from the boundary orientation on $\partial\overline{\mathcal{H}}_m$ by a sign $(-1)^{\dagger_1}$ and $(-1)^{\dagger_2}$ respectively where we have

$$\begin{aligned} \text{(B.3)} \quad & \dagger_1 = m_1 \\ \text{(B.4)} \quad & \dagger_2 = m_2(m - k) + k + m_2, \end{aligned}$$

and $\overline{\mathcal{R}}_{m_2}$ is attached to the $(k + 1)$ -th outgoing leaf of $\overline{\mathcal{H}}_{m_1}$ (cf. [Sei08, (12.22)]). The first sign \dagger_1 is obtained from [Sei08, (12.22)] by using $m = m_2 + 1$, $d = m_1 + m_2 + 1$ and $n = d$ since $\overline{\mathcal{H}}_{m_2}$ is attached to $\overline{\mathcal{H}}_{m_1}$ at the last outgoing leaf.

The second sign \dagger_2 is obtained from [Sei08, (12.22)] by using $m = m_2$, $d = m_1 + m_2$ and $n = k$.

Proof of Lemma 4. We consider the moduli space $\overline{\mathcal{M}}(\mathbf{a})$ and the stratification of its codimension one boundary as in (3.6). We first consider the strata of the form $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$ where $\mathbf{a}'\mathbf{a}'' = \mathbf{a}$. Then, using Lemma 20 we have

$$\begin{aligned} \bigwedge^{\text{top}} (T\overline{\mathcal{M}}(\mathbf{a}')) \otimes \bigwedge^{\text{top}} (T\overline{\mathcal{M}}(\mathbf{a}'')) &= \bigwedge^{\text{top}} (T\overline{\mathcal{H}}_{m_1}) \otimes o_\xi \otimes o_{\mathbf{a}'}^\vee \otimes o_\xi^\vee \\ &\quad \otimes \bigwedge^{\text{top}} (T\overline{\mathcal{H}}_{m_2}) \otimes o_\xi \otimes o_{\mathbf{a}''}^\vee \otimes o_\xi^\vee. \end{aligned}$$

Reordering the factors so that $\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{m_2})$ becomes adjacent to $\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{m_1})$ introduces the Koszul sign $(-1)^{\dagger_1}$ where

$$\dagger_1 = (m_2 + 1) \binom{m_1}{\sum_{i=1}^{m_1} |a_i|},$$

since $\dim \overline{\mathcal{H}}_{m_2} = m_2 + 1$. Canceling the adjacent factors o_ξ^\vee and o_ξ then gives

$$\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{m_1}) \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{m_2}) \otimes o_\xi \otimes o_{\mathbf{a}'}^\vee \otimes o_{\mathbf{a}''}^\vee \otimes o_\xi^\vee.$$

Then by (B.3) we get a sign $(-1)^{\dagger_1}$ when comparing the product orientation of $\overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{H}}_{m_2}$ with the boundary orientation of $\partial\overline{\mathcal{H}}_m$. After these reorderings we arrive at

$$\bigwedge^{\text{top}}(T\partial\overline{\mathcal{H}}_m) \otimes o_\xi \otimes o_{\mathbf{a}'}^\vee \otimes o_{\mathbf{a}''}^\vee \otimes o_\xi^\vee = \bigwedge^{\text{top}}(T\partial\overline{\mathcal{H}}_m) \otimes o_\xi \otimes o_{\mathbf{a}}^\vee \otimes o_\xi^\vee,$$

which is canonically isomorphic to $\bigwedge^{\text{top}}(T\overline{\mathcal{M}}(\mathbf{a}))$. The total sign difference between the product orientation on $\overline{\mathcal{M}}(\mathbf{a}') \times \overline{\mathcal{M}}(\mathbf{a}'')$ and the boundary orientation on $\partial\overline{\mathcal{M}}(\mathbf{a})$ is therefore

$$\dagger_1 = \dagger_1 + \dagger_1 = (m_2 + 1) \binom{m_1}{\sum_{i=1}^{m_1} |a_i|} + m_1.$$

Similarly, we compare the product orientation of $\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ with the boundary orientation on $\partial\overline{\mathcal{M}}(\mathbf{a})$. Recall from (3.7) that if $\tilde{\mathbf{a}} \subset \mathbf{a}$ is a subword at position $t + 1$, then $\mathbf{a} \setminus \tilde{\mathbf{a}}$ denotes the word \mathbf{a} with the subword $\tilde{\mathbf{a}}$ replaced by an auxiliary generator y . Again by Lemma 20 we therefore have

$$\begin{aligned} \bigwedge^{\text{top}}(T\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}})) \otimes \bigwedge^{\text{top}}(T\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})) &= \bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r}) \otimes o_\xi \otimes o_{\mathbf{a} \setminus \tilde{\mathbf{a}}}^\vee \otimes o_\xi^\vee \\ &\quad \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \otimes o_y \otimes o_{\tilde{\mathbf{a}}}^\vee \end{aligned}$$

Assuming that $\mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ is rigid means especially that $|y| = 2 - s + |y| - \sum_{i=1}^s |a_{t+i}|$ and so we move $o_{(\mathbf{a} \setminus \tilde{\mathbf{a}})_2}^\vee \otimes o_\xi^\vee$ past $\bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \otimes o_y \otimes o_{\tilde{\mathbf{a}}}^\vee$ without

introducing any sign and arrive at

$$\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r}) \otimes o_\xi \otimes o_{(\mathbf{a} \setminus \tilde{\mathbf{a}})_1}^\vee \otimes o_y^\vee \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \otimes o_y \otimes o_{\tilde{\mathbf{a}}}^\vee \otimes o_{(\mathbf{a} \setminus \tilde{\mathbf{a}})_2}^\vee \otimes o_\xi^\vee.$$

Because $\dim \overline{\mathcal{R}}_s = s$, moving $\bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s)$ to the front and adjacent to $\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r})$ gives

$$\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r}) \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \otimes o_\xi \otimes o_{(\mathbf{a} \setminus \tilde{\mathbf{a}})_1}^\vee \otimes o_y^\vee \otimes o_y \otimes o_{\tilde{\mathbf{a}}}^\vee \otimes o_{(\mathbf{a} \setminus \tilde{\mathbf{a}})_2}^\vee \otimes o_\xi^\vee,$$

with a sign difference of $(-1)^{\dagger_2}$ where

$$\dagger_2 = s \left(|\xi| + |y| + \sum_{i=1}^t |a_i| \right) = s \left(|\xi| + \sum_{i=1}^{t+s} |a_i| \right).$$

Recall from the assumptions in Lemma 4 that $\tilde{\mathbf{a}} \subset \mathbf{a}$ is a subword at position $t + 1$.

Then using $o_y^\vee \otimes o_y \cong \mathbb{R}$ and $\mathbf{a} = (\mathbf{a} \setminus \tilde{\mathbf{a}})_1 \tilde{\mathbf{a}} (\mathbf{a} \setminus \tilde{\mathbf{a}})_2$ this collapses to

$$\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r}) \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \otimes o_\xi \otimes o_{\tilde{\mathbf{a}}}^\vee \otimes o_\xi^\vee,$$

and using (B.4), $\bigwedge^{\text{top}}(T\overline{\mathcal{H}}_{t+1+r}) \otimes \bigwedge^{\text{top}}(T\overline{\mathcal{R}}_s) \cong \bigwedge^{\text{top}}(T\overline{\mathcal{H}}_m)$ with a sign difference of $(-1)^{\dagger_2}$. The total sign difference between the product orientation of $\overline{\mathcal{M}}(\mathbf{a} \setminus \tilde{\mathbf{a}}) \times \mathcal{M}^{\text{cw}}(\tilde{\mathbf{a}})$ and the boundary orientation on $\partial \overline{\mathcal{M}}(\mathbf{a})$ is therefore

$$\ddagger_2 = \dagger_2 + \dagger_2 = s \left(|\xi| + \sum_{i=1}^{t+s} |a_i| \right) + s(m - t) + t + s.$$

□

Proof of Lemma 9 (continued). To confirm that the signs match up in the A_∞ -relation

$$\partial \Psi_m + \sum_{m_1+m_2=m} P(\Psi_{m_2} \otimes \Psi_{m_1}) = \sum_{r+s+t=m} (-1)^{\mathfrak{A}_t} \Psi_{r+1+t}(\text{id}^{\otimes r} \otimes \mu^s \otimes \text{id}^{\otimes t}),$$

we look at the terms one by one and compute the sign that is in front of each term. In the first term $\partial \Psi_m$ it is only the sign from Ψ_m that is taken

into account, namely $(-1)^{\S}$ where

$$\S = \sum_{i=1}^m i|a_i| + (|\xi| + m) \sum_{i=1}^m |a_i| .$$

The second term has a sign coming from:

- 1) The definition of the Pontryagin product P in (3.2) contributes with a sign $(-1)^\circ$ where

$$\circ = |\Psi_{m_1}(a_{m_1} \otimes \cdots \otimes a_1)| = \dim \overline{\mathcal{M}}(\mathbf{a}') = -1 + m_1 - \sum_{i=1}^{m_1} |a_i| ,$$

- 2) the difference between the product orientation on $\overline{\mathcal{H}}_{m_1} \times \overline{\mathcal{H}}_{m_2}$ and the boundary orientation on $\partial \overline{\mathcal{H}}_m$ is $(-1)^{\ddagger_1}$, where $m_1 + m_2 = m$ and

$$\ddagger_1 = (m_2 + 1) \left(\sum_{i=1}^{m_1} |a_i| \right) + m_1 ,$$

as in Lemma 4,

- 3) the definition of $\Psi_{m_1}(a_{m_1} \otimes \cdots \otimes a_1)$ in (3.19) contributes with a sign $(-1)^{\S_1}$ where

$$\S_1 = \sum_{i=1}^{m_1} i|a_i| + (|\xi| + m_1) \sum_{i=1}^{m_1} |a_i| ,$$

and

- 4) the definition of $\Psi_{m_2}(a_m \otimes \cdots \otimes a_{m_1+1})$ in (3.19) contributes with a sign $(-1)^{\S_2}$ where

$$\S_2 = \sum_{i=m_1+1}^m (i - m_1)|a_i| + (|\xi| + m_2) \sum_{i=m_1+1}^m |a_i| .$$

Now it is straightforward to check that $\circ + \ddagger_1 + \S_1 + \S_2 = 1 + \S \pmod{2}$.

$$\begin{aligned}
 \circ + \ddagger_1 + \S_1 + \S_2 &= -1 + m_1 - \sum_{i=1}^{m_1} |a_i| + (m_2 + 1) \left(\sum_{i=1}^{m_1} |a_i| \right) + m_1 + \sum_{i=1}^{m_1} i |a_i| \\
 &\quad + (|\xi| + m_1) \sum_{i=1}^{m_1} |a_i| + \sum_{i=m_1+1}^m (i - m_1) |a_i| \\
 &\quad + (|\xi| + m_2) \sum_{i=m_1+1}^m |a_i| \\
 &= 1 + m_2 \sum_{i=1}^{m_1} |a_i| + \sum_{i=1}^m i |a_i| + (|\xi| + m_1) \sum_{i=1}^{m_1} |a_i| \\
 &\quad + m_1 \sum_{i=m_1+1}^m |a_i| + (|\xi| + m_2) \sum_{i=m_1+1}^m |a_i| \\
 &= 1 + \sum_{i=1}^m i |a_i| + (|\xi| + m) \sum_{i=1}^m |a_i| = 1 + \S \pmod{2}.
 \end{aligned}$$

Next we consider the term in the right hand side. Let $y := \mu^s(a_{t+s} \otimes \cdots \otimes a_{t+1})$. This sum has a sign coming from:

- 1) The difference between the product orientation on $\overline{\mathcal{H}}_{r+1+t} \times \overline{\mathcal{R}}_s$ and the boundary orientation on $\partial \overline{\mathcal{H}}_m$ is $(-1)^{\ddagger_2}$ where $r + s + t = m$ and

$$\ddagger_2 = s \left(|\xi| + \sum_{i=1}^{t+s} |a_i| \right) + s(m - t) + t + s,$$

- 2) the definition of $\mu^s(a_{t+s} \otimes \cdots \otimes a_{t+1})$ in (2.1) contributes with a sign $(-1)^\diamond$ where

$$\diamond = \sum_{i=t+1}^{t+s} (i - t) |a_i|,$$

- 3) the definition of $\Psi_{r+1+t}(a_m \otimes \cdots \otimes a_{t+s+1} \otimes y \otimes a_t \otimes \cdots \otimes a_1)$ in (3.19) contributes with a sign $(-1)^\S$ where

$$\begin{aligned} \tilde{\xi} &= \sum_{i=1}^t i|a_i| + (t+1)|y| + \sum_{i=t+s+1}^m (i-s+1)|a_i| \\ &\quad + (|\xi| + r + t + 1) \left(\sum_{i=1}^t |a_i| + |y| + \sum_{i=t+s+1}^m |a_i| \right). \end{aligned}$$

Note that since we assume that $\mathcal{M}^{\text{cw}}(a_{t+1} \cdots a_{t+s})$ is rigid, we have

$$|y| = 2 - s + \sum_{i=t+1}^{t+s} |a_i| = s + \sum_{i=t+1}^{t+s} |a_i| \pmod{2},$$

hence we get

$$\begin{aligned} \tilde{\xi} &= \sum_{i=1}^t i|a_i| + (t+1) \left(s + \sum_{i=t+1}^{t+s} |a_i| \right) + \sum_{i=t+s+1}^m (i-s+1)|a_i| \\ &\quad + (|\xi| + r + t + 1) \left(s + \sum_{i=1}^m |a_i| \right) \\ &= \sum_{i=1}^t i|a_i| + (t+1) \sum_{i=t+1}^{t+s} |a_i| + (s+1) \sum_{i=t+s+1}^m |a_i| \\ &\quad + \sum_{i=t+s+1}^m i|a_i| + (|\xi| + r)s + (|\xi| + r + t + 1) \sum_{i=1}^m |a_i| \pmod{2}. \end{aligned}$$

It is then again a straightforward calculation to show that $\ddagger_2 + \diamond + \tilde{\xi} + \blacktriangleright_t = \xi \pmod{2}$.

$$\begin{aligned} \ddagger_2 + \diamond + \tilde{\xi} + \blacktriangleright_t &= s \left(|\xi| + \sum_{i=1}^{t+s} |a_i| \right) + s(m-t) + t + s + \sum_{i=t+1}^{t+s} (i-t)|a_i| \\ &\quad + \sum_{i=1}^t i|a_i| + (t+1) \sum_{i=t+1}^{t+s} |a_i| + (s+1) \sum_{i=t+s+1}^m |a_i| \\ &\quad + \sum_{i=t+s+1}^m i|a_i| + (|\xi| + r)s + (|\xi| + r + t + 1) \sum_{i=1}^m |a_i| + \sum_{i=1}^t |a_i| + t \end{aligned}$$

$$\begin{aligned}
&= s \sum_{i=1}^{t+s} |a_i| + s(m-t) + t + s + \sum_{i=t+1}^{t+s} i|a_i| + \sum_{i=1}^t i|a_i| \\
&\quad + \sum_{i=t+1}^{t+s} |a_i| + (s+1) \sum_{i=t+s+1}^m |a_i| + \sum_{i=t+s+1}^m i|a_i| \\
&\quad + rs + (|\xi| + r + t + 1) \sum_{i=1}^m |a_i| + \sum_{i=1}^t |a_i| + t \\
&= m \sum_{i=1}^m |a_i| + rs + s^2 + s + \sum_{i=1}^m i|a_i| + rs + |\xi| \sum_{i=1}^m |a_i| \\
&= \sum_{i=1}^m i|a_i| + (|\xi| + m) \sum_{i=1}^m |a_i| = \S \pmod{2}.
\end{aligned}$$

□

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