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Asymptotic expansion of generalized Witten integrals for Hamiltonian circle actions

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We derive a complete asymptotic expansion of generalized Witten integrals for Hamiltonian circle actions on arbitrary symplectic manifolds, characterizing the coefficients in the expansion as integrals over the symplectic strata of the corresponding Marsden-Weinstein reduced space and distributions on the Lie algebra. The obtained coefficients involve singular contributions of the lower-dimensional strata related to numerical invariants of the fixed-point set.

1. Introduction

Let (M, ω) be a $2n$-dimensional connected symplectic manifold with a non-trivial Hamiltonian action of a compact connected Lie group G and momentum map $\mathcal{J} : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of

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G. If $\zeta \in \mathfrak{g}^*$ is a regular value of $\mathcal{J}$, the corresponding *Marsden-Weinstein symplectic quotient* or *reduced space* $\mathscr{M}^\zeta := \mathcal{J}^{-1}(\{\zeta\})/G$ is a symplectic orbifold, and if ζ is not a regular value, $\mathscr{M}^\zeta$ is a stratified space which can have serious singularities. The geometry and topology of $\mathscr{M}^\zeta$ have been extensively studied in the last decades [9, 13, 14, 16, 24, 26] mostly for compact M, a major tool being the Witten integral and its asymptotic expansion, which carries important geometric and topological information.

In this paper, we study Witten-type integrals in the case where $G = T := \mathrm{SO}(2) \cong S^1$ is the circle group and $\zeta \in \mathfrak{t}^* := \mathfrak{g}^*$ is not necessarily a regular value. While we do not assume M to be compact, we consider compactly supported integrands. More precisely, we derive a complete asymptotic expansion of *generalized Witten integrals* of the form

$$I^\zeta_{a,\sigma}(\varepsilon) := \int_{\mathfrak{t}} \int_M e^{i(\mathcal{J}(p)-\zeta)(x)/\varepsilon} a(p)\, dM(p) \sigma(x)\, dx \tag{1.1}$$

for arbitrary $\zeta \in \mathfrak{t}^*$ in integer powers of $\varepsilon > 0$, where $\mathfrak{t} := \mathfrak{so}(2)$ is the Lie algebra of T for which we fix an identification $\mathfrak{t} \cong \mathbb{R}$, $a \in \mathrm{C}^\infty_\mathrm{c}(M)$ a compactly supported amplitude, $\sigma \in \mathcal{S}(\mathfrak{t})$ a Schwartz function on $\mathfrak{t}$, $dM := \omega^n/n!$ the symplectic volume form on M, and dx the Lebesgue measure on $\mathfrak{t} \cong \mathbb{R}$.

We regard

$$I^\zeta_{a,\sigma}(\varepsilon) = I^\zeta(\varepsilon)(a \otimes \sigma) \tag{1.2}$$

as the evaluation of a distribution $I^\zeta(\varepsilon) \in (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$ at the test function $a \otimes \sigma \in \mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t})$. By *complete asymptotic expansion* we mean an expansion of $I^\zeta(\varepsilon)$ in $(\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$ of the form

$$I^\zeta(\varepsilon) \sim \varepsilon^{j_0(\zeta)} \sum_{j=0}^\infty \varepsilon^j A^\zeta_j, \qquad A^\zeta_j \in \mathcal{D}'(M) \otimes \mathcal{S}'(\mathfrak{t}), \quad j_0(\zeta) \in \mathbb{Z}, \tag{1.3}$$

that is, for each $J_0 \in \mathbb{N}_0$ and each compact set $K \subset M$ there is an $N_{J_0,K,\zeta} \in \mathbb{N}_0$ and a family of differential operators $\{D^l_{J_0,K,\zeta}\}_{0 \le l \le N_{J_0,K,\zeta}}$ on M such that for all $\sigma \in \mathcal{S}(\mathfrak{t})$ and all $a \in \mathrm{C}^\infty_\mathrm{c}(M)$ with $\operatorname{supp} a \subset K$ one has

$$\left| I^\zeta_{a,\sigma}(\varepsilon) - \varepsilon^{j_0(\zeta)} \sum_{j=0}^{J_0} \varepsilon^j A^\zeta_j(a \otimes \sigma) \right| \le \sum_{k,l=0}^{N_{J_0,K,\zeta}} \left\| D^l_{J_0,K,\zeta} a \right\|_\infty \left\| \sigma^{(k)} \right\|_\infty \varepsilon^{j_0(\zeta)+J_0+1}.$$

Here $\mathcal{D}(M)$ denotes the space of test functions on M, given by $\mathrm{C}_{\mathrm{c}}^{\infty}(M)$ with the test function topology; its dual $\mathcal{D}'(M)$ is the space of distributions[1] on M, and $\mathcal{D}'(M) \otimes \mathcal{S}'(\mathfrak{t})$ embeds into $(\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$. We are specially interested in the dependence of the coefficient distributions A_j^ζ and the leading order

$$\ell(\zeta) := j_0(\zeta) + \inf\left\{ j \in \mathbb{N}_0 \mid A_j^\zeta \neq 0 \right\}$$

on the parameter $\zeta \in \mathfrak{t}^*$, which may be a regular or singular value of $\mathcal{J}$.

Distributions of the form $I^\zeta(\varepsilon)$ arise in the study of the Fourier transform of *Duistermaat-Heckman-type distributions*. The latter are tempered distributions $L_\varrho \in \mathcal{S}'(\mathfrak{t})$ associated with a compactly supported equivariant differential form ϱ on M, that is a polynomial map $\varrho : \mathfrak{t} \to \Omega_c(M)^T$, by

$$(1.4) \quad L_\varrho(x) := \int_M e^{i(J(x)-\omega)} \varrho(x), \quad x \in \mathfrak{t}, \quad J(x)(p) := \mathcal{J}(p)(x), \quad p \in M.$$

The connection between the integrals (1.1) and the distributions (1.4) is explained in detail in Section 2.2, where we also describe how the original integral studied by Witten [26] arises as a special case of the generalized Witten integral (1.1). If M is compact and $\varrho = 1$, L_ϱ corresponds to the inverse Fourier transform of the pushforward $\mathcal{J}_*(dM)$ of the symplectic volume form along $\mathcal{J}$. As was discovered by Duistermaat and Heckman [5], $\mathcal{J}_*(dM)$ is a piecewise polynomial measure on $\mathfrak{t}^*$, or equivalently, L_ϱ is exactly given by the leading term in the stationary phase approximation. This can be seen as a special instance of the localization formula of Berline-Vergne [2, 3] and Atiyah-Bott [1], one of the central principles in equivariant cohomology. For general M and ϱ, one would expect that the coefficients A_j^ζ in the expansion of $I^\zeta(\varepsilon)$ are given by piecewise polynomial measures on $\mathfrak{t}^*$ as well, and our results show that this is indeed the case. Furthermore, as will be discussed below, these measures have a geometric meaning in terms of the symplectic data of $\mathscr{M}^\zeta$.

If $\zeta \in \mathfrak{t}^*$ is a regular value of $\mathcal{J}$, the phase function $\psi^\zeta(p,x) := \mathcal{J}(p)(x) - \zeta(x)$ in (1.1) is a *Morse-Bott function* and the stationary phase theorem

[1] In this paper, we shall identify distributions with distribution densities on M via the symplectic volume form dM, which defines a strictly positive smooth density on M.

yields an expansion (1.3) of $I^\zeta(\varepsilon)$ with $j_0(\zeta) = \ell(\zeta) = 1$ and

$$A_0^\zeta(a \otimes \sigma) = 2\pi\sigma(0) \int_{\mathscr{M}^\zeta} \langle a \rangle_T \, d\mathscr{M}^\zeta, \qquad (1.5)$$
$$A_j^\zeta(a \otimes \sigma) = \sigma^{(j)}(0) \int_{\mathscr{M}^\zeta} \langle D_j^\zeta a \rangle_T \, d\mathscr{M}^\zeta,$$

where D_j^ζ is a differential operator of order j defined near $\mathcal{J}^{-1}(\{\zeta\})$, D_0^ζ is just multiplication by 2π, $d\mathscr{M}^\zeta$ is the symplectic volume form on the orbifold $\mathscr{M}^\zeta = \mathcal{J}^{-1}(\{\zeta\})/T$, and $\langle f \rangle_T (T \cdot p) := \int_T f(g \cdot p)\, dg$ denotes the function on M/T defined by integrating $f \in \mathrm{C}_c^\infty(M)$ over an orbit $T \cdot p \subset M$ using the Haar measure dg on T fixed by our identification $\mathfrak{t} \cong \mathbb{R}$. Furthermore, D_j^ζ is transversal to $\mathcal{J}^{-1}(\{\zeta\})$, and the coefficients in (1.5) depend smoothly on ζ in the sense that if $\mathscr{V} \subset \mathfrak{t}^*$ is an open set consisting entirely of regular values of $\mathcal{J}$, then the function $\mathscr{V} \ni \zeta \to A_j^\zeta(a \otimes \sigma) \in \mathbb{C}$ is smooth for each j, a, and σ, see Proposition 2.5 for more details.

When ζ is a singular value of $\mathcal{J}$, serious difficulties arise in the study of the integrals (1.1), since then the stationary phase principle cannot be applied. Moreover, the behavior of the coefficients in the regular expansion (1.5) as ζ approaches a singular value is unclear a priori. In this paper, we address both of these problems. In order to state our results, consider for an arbitrary $\zeta \in \mathfrak{t}^*$ the stratification of the symplectic quotient $\mathscr{M}^\zeta = \mathcal{J}^{-1}(\{\zeta\})/T$ by infinitesimal orbit types

$$\mathscr{M}^\zeta = \mathscr{M}^\zeta_{\mathrm{top}} \sqcup \mathscr{M}^\zeta_{\mathrm{sing}}, \qquad \mathscr{M}^\zeta_{\aleph} := (\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_\aleph)})/T, \qquad (1.6)$$

where $M_{(h_\aleph)}$ denotes the stratum of M of infinitesimal orbit type $(\mathfrak{h}_\aleph)$ with $\mathfrak{h}_{\mathrm{top}} = \{0\}$ and $\mathfrak{h}_{\mathrm{sing}} = \mathfrak{t}$. $\mathscr{M}^\zeta_{\mathrm{top}}$ is an orbifold called the *top stratum*. It is either dense in $\mathscr{M}^\zeta$ or empty, which happens iff T acts trivially on $\mathcal{J}^{-1}(\{\zeta\})$. The orbifold $\mathscr{M}^\zeta_{\mathrm{top}}$ inherits a symplectic form $\omega^\zeta_{\mathrm{top}}$ uniquely characterized by $i^*\omega = \pi^*\omega^\zeta_{\mathrm{top}}$, where $i : \mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})} \to M$ is the inclusion and $\pi : \mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})} \to \mathscr{M}^\zeta_{\mathrm{top}}$ the orbit projection. Writing M^T for the space of fixed-points of the T-action on M and $\mathcal{F}$ for the set of all connected components of M^T, each $F \in \mathcal{F}$ is a symplectic submanifold of (M, ω) on which $\mathcal{J}$ is constant, and the singular values of $\mathcal{J}$ are $\{\mathcal{J}(F) : F \in \mathcal{F}\} \subset \mathfrak{t}^*$. The space $\mathscr{M}^\zeta_{\mathrm{sing}}$ can be identified with the union of all $F \in \mathcal{F}$ with $\mathcal{J}(F) = \zeta$. Each $F \in \mathcal{F}$ provides certain numerical invariants of the Hamiltonian T-space $(M, \omega, \mathcal{J})$. The simplest is the codimension of F in M, an even number denoted by $\operatorname{codim} F$, which is non-zero thanks to our assumptions that the T-action on M is non-trivial and that M is connected. Moreover, the behavior of $\mathcal{J}$ near F intrinsically determines two non-negative even integers $n_F^{\pm}$ fulfilling $n_F^+ + n_F^- = \operatorname{codim} F$. Technically, n_F^+ and n_F^- arise as the positive and negative indices of inertia of some non-degenerate quadratic form Q_F on $\mathbb{R}^{\operatorname{codim} F}$ assigned to F, see Section 2.3. We therefore call F *definite* with sign $s_F \in \{+,-\}$ if $n_F^{s_F} = \operatorname{codim} F$ and *indefinite* otherwise. With these preparations, we can state our first main result, proved in Section 5.1.

Theorem 1.1. *For each $\zeta \in \mathfrak{t}^* \cong \mathbb{R}$, the generalized Witten integral* (1.1) *has an asymptotic expansion*

$$I^\zeta(\varepsilon) \sim \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A_j^\zeta$$

in $(\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$ with coefficient distributions of the form

$$A_j^\zeta = (A_j^\zeta)_{\mathrm{top}} + (A_j^\zeta)_{\mathrm{sing}}$$

given by

$$\begin{aligned} (A_j^\zeta)_{\mathrm{top}}(a \otimes \sigma) &= \sigma^{(j)}(0) \int_{\mathscr{M}_{\mathrm{top}}^\zeta} \left\langle D_j^\zeta a \right\rangle_T d\mathscr{M}_{\mathrm{top}}^\zeta, \\ (A_j^\zeta)_{\mathrm{sing}}(a \otimes \sigma) &= \sum_{\substack{F \in \mathcal{F}: \mathcal{J}(F) = \zeta, \\ F \cap \operatorname{supp} a \neq \emptyset}} A_{j,F}(a \otimes \sigma), \end{aligned} \tag{1.7}$$

where $A_{j,F} = 0$ unless $j \geq \frac{1}{2}\operatorname{codim} F - 1$, in which case one has

$$A_{j,F}(a \otimes \sigma) = \begin{cases} \sigma_{\mp}^{[j]}(0) \displaystyle\int_F D_{j,F} a \, dF, & F \text{ definite, } s_F = \pm, \\ \sigma_+^{[j]}(0) \displaystyle\int_F D_{j,F}^+ a \, dF + \sigma_-^{[j]}(0) \int_F D_{j,F}^- a \, dF, & F \text{ indefinite.} \end{cases}$$

The objects occurring here are as follows:

- $\mathcal{S}(\mathfrak{t}) \ni \sigma \mapsto \sigma_\pm^{[j]}(0) \in \mathbb{C}$ *are tempered distributions defined by*

$$\sigma_\pm^{[j]}(0) := \frac{i^j}{2\pi} \left\langle \xi_\pm^j, \hat{\sigma} \right\rangle := \frac{(\pm i)^j}{2\pi} \int_0^\infty \hat{\sigma}(\pm\xi) \xi^j \, d\xi, \qquad j \in \mathbb{N}_0, \tag{1.8}$$

  where we use our identification $\mathbb{R} \cong \mathfrak{t}^*$ and $\hat{\sigma}$ is the Fourier transform of σ, normalized such that

$$\sigma_+^{[j]}(0) + \sigma_-^{[j]}(0) = \sigma^{(j)}(0);$$

- D_j^ζ *is a differential operator of order* j *defined on a neighborhood of* $\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})}$ *in* $M_{(\mathfrak{h}_{\mathrm{top}})}$*, transversal to* $\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})}$*, and for* $j = 0$ *one has* $D_0^\zeta = 2\pi$*;*

- $D_{j,F}$ *(in the definite case) and* $D_{j,F}^\pm$ *(in the indefinite case) are* ζ*-independent differential operators of order* $2j + 2 - \operatorname{codim} F$ *defined on a neighborhood of* F *in* M*, and for the lowest index* $j = \frac{1}{2}\operatorname{codim} F - 1$ *these operators equal the following constants:*

$$
\begin{aligned}
&D_{\operatorname{codim} F/2-1,F} = 2^{\operatorname{codim} F/2-1} C_F,\\
&C_F = (2\pi)^2 \frac{(\pi i)^{\operatorname{codim} F/2-1}}{|\lambda_1^F \cdots \lambda_{\operatorname{codim} F/2}^F|(\operatorname{codim} F/2-1)!},\\
&D^\pm_{\operatorname{codim} F/2-1,F} = N_F^\pm C_F, \quad N_F^\pm \in \mathbb{Z}\setminus\{0\},
\end{aligned}
\tag{1.9}
$$

where $\lambda_1^F, \dots, \lambda_{\operatorname{codim} F/2}^F \in \mathbb{Z}\setminus\{0\}$ *are the weights of the fiber-wise* T*-action on the symplectic normal bundle of* F *in* M*, see Section 2.3, and the non-zero integers* $N_F^\pm$ *are explicitly determined by the invariants* n_F^+ *and* n_F^-*, see* (5.4)*;*

- $d\mathscr{M}_{\mathrm{top}}^\zeta := (\omega_{\mathrm{top}}^\zeta)^{n-1}/(n-1)!$ *and* $dF := \omega^{\dim F/2}/(\dim F/2)!$ *are the symplectic volume forms.*

In particular, the leading order of the asymptotic expansion is given by

$$
\ell(\zeta) = \begin{cases} 1, & \mathscr{M}_{\mathrm{top}}^\zeta \neq \emptyset,\\ \inf\{\operatorname{codim} F/2 : F \in \mathcal{F},\ \mathcal{J}(F) = \zeta\}, & \mathscr{M}_{\mathrm{top}}^\zeta = \emptyset. \end{cases}
$$

Furthermore, the operators D_j^ζ*,* $D_{j,F}$*, and* $D_{j,F}^\pm$ *are natural in the following sense: if* $(M', \omega', \mathcal{J}')$ *is another Hamiltonian* T*-space and* $\Phi : M \to M'$ *an isomorphism of Hamiltonian* T*-spaces, then the above statements hold for* $(M', \omega', \mathcal{J}')$ *with the operators*

$$
\begin{aligned}
&(D_j^\zeta)' := \Phi_* D_j^\zeta, \qquad (D_{j,F'})' := \Phi_* D_{j,\Phi^{-1}(F')},\\
&(D_{j,F'}^\pm)' := \Phi_* D^\pm_{j,\Phi^{-1}(F')}, \qquad F' \in \mathcal{F}',
\end{aligned}
$$

where we use the notation $\Phi_* D(f) := D(f \circ \Phi)$ *for a differential operator* D *defined on an open subset* $U \subset M$ *and* $f \in \mathrm{C}^\infty(\Phi(U))$*, and* $\mathcal{F}' = \{\Phi(F) : F \in \mathcal{F}\}$ *is the set of connected components of* M'^T*.*

Remark. 1) If ζ is a regular value of $\mathcal{J}$, then Theorem 1.1 reduces to the usual asymptotic expansion (1.5) since $(A_j^\zeta)_{\mathrm{sing}} = 0$ and $\mathscr{M}_{\mathrm{top}}^\zeta = \mathscr{M}^\zeta$.

2) The constants in (1.9) are non-zero, so the singular contributions $(A_j^\zeta)_{\mathrm{sing}}$ do occur in general.

3) We emphasize that the distributions $(A_j^\zeta)_{\mathrm{sing}}$ depend on ζ only via the condition $\mathcal{J}(F) = \zeta$ in the sum in (1.7); the individual distributions $A_{j,F}$ are independent of ζ.

4) Note that the sum over all $F \in \mathcal{F}$ with $F \cap \operatorname{supp} a \neq \emptyset$ in (1.7) is finite because $\operatorname{supp} a$ is compact. Moreover, as each compact subset of M intersects only finitely many connected components $F \in \mathcal{F}$ non-trivially, one has for each $j \in \mathbb{N}_0$ the convergence of distributions

$$(A_j^\zeta)_{\mathrm{sing}} = \sum_{F \in \mathcal{F}: \mathcal{J}(F) = \zeta} A_{j,F} \qquad \text{in } \mathcal{D}'(M) \otimes \mathcal{S}'(\mathfrak{t}),$$

where the sum on the right hand side may be infinite because we do not assume M to be compact.

5) The expressions A_j^ζ in the expansion of $I^\zeta(\varepsilon)$ are given in terms of the piecewise polynomial measures $\xi_\pm^j \in \mathcal{S}'(\mathfrak{t}^*)$, $j \in \mathbb{N}_0$. This was to be expected from the Duistermaat-Heckman theorem or, more generally, from the localization principle. But since the latter only applies to equivariantly closed differential forms, while we are considering general amplitudes, we could not rely on localization. Also notice that the remainder in the expansion of the generalized Witten integral does not vanish in general – this is an exclusive phenomenon for equivariantly closed differential forms and constitutes the essence of the localization principle. In fact, localization implies that the expansion of Theorem 1.1, when applied to the original Witten integral (2.9), consists only of finitely many terms.

Theorem 1.1 shows that the coefficients in the asymptotic expansion are sums of two qualitatively different terms: for each $\zeta \in \mathfrak{t}^*$ there are *regular contributions* $(A_j^\zeta)_{\mathrm{top}}$ of the same form as the coefficients in (1.5), and there are *singular contributions* $(A_j^\zeta)_{\mathrm{sing}}$, which are tensor products of distributions on M supported in $\mathcal{J}^{-1}(\{\zeta\}) \cap M^T$ and some mildly exotic tempered distributions on $\mathfrak{t}$. In particular, there are *singularly leading* terms associated with each fixed point set component $F \in \mathcal{F}$ fulfilling $\mathcal{J}(F) = \zeta$, occurring at $j = \frac{1}{2}\operatorname{codim} F - 1$. In the latter, the obtained distribution on the manifold is simply integration over F, up to a constant determined uniquely by the numerical invariants n_F^+, n_F^- and the weights $\lambda_1^F, \dots, \lambda_{\operatorname{codim} F/2}^F$. If ζ is a regular value, the singular contributions vanish. For general singular values of ζ, both regular and singular contributions appear, and in the special case that $\mathscr{M}^{\zeta}_{\mathrm{top}} = \emptyset$, the regular contributions vanish and the singularly leading terms actually make up the leading term of the asymptotic expansion. Let us also emphasize that all coefficient distributions in the asymptotic expansion have a clear symplectic meaning given in terms of the symplectic structure of the strata of $\mathscr{M}^{\zeta}$.

Note that Theorem 1.1 gives an asymptotic expansion for each individual $\zeta \in \mathfrak{t}^*$ and makes no statement about the continuity of the obtained coefficients upon variations of ζ. This question is dealt with in Section 5.2, where we prove the following statement on the (dis)continuity of the coefficients, our second main result.

Theorem 1.2. *For every $\zeta_0 \in \mathfrak{t}^* \cong \mathbb{R}$, $a \in \mathrm{C}^{\infty}_{\mathrm{c}}(M)$, $\sigma \in \mathcal{S}(\mathfrak{t})$, and $j \in \mathbb{N}_0$, one has*

$$\lim_{\substack{\zeta \to \zeta_0 \\ \pm(\zeta - \zeta_0) > 0}} A^{\zeta}_j(a \otimes \sigma) = \sigma^{(j)}(0) \Bigg(\int_{\mathscr{M}^{\zeta_0}_{\mathrm{top}}} \langle D^{\zeta_0}_j a \rangle_T \, d\mathscr{M}^{\zeta_0}_{\mathrm{top}} + \sum_{\substack{F \in \mathcal{F}: \mathcal{J}(F) = \zeta_0, \\ \operatorname{codim} F/2 - 1 \leq j, \\ F \text{ indefinite}, \\ F \cap \operatorname{supp} a \neq \emptyset}} \int_F D^{\mp}_{j,F} a \, dF + \sum_{\substack{F \in \mathcal{F}: \mathcal{J}(F) = \zeta_0, \\ \operatorname{codim} F/2 - 1 \leq j, \\ F \text{ definite}, \, s_F = \pm, \\ F \cap \operatorname{supp} a \neq \emptyset}} \int_F D_{j,F} a \, dF \Bigg).$$

The previous theorem shows that for $j > 0$ the functions $\mathfrak{t}^* \ni \zeta \mapsto A^{\zeta}_j(a \otimes \sigma) \in \mathbb{C}$ are in general highly discontinuous at each singular value $\zeta = \zeta_0$, where the discontinuities are three-fold:

1) the family of definite fixed point sets F contributing to the limit depends on the sign in the limit. In particular, this produces discontinuities at $j = \operatorname{codim} F/2 - 1$ which can be quantitatively calculated in terms of explicit scalar multiples of $\int_F a \, dF$ using (1.9) and (5.4);
2) the operators occurring in the contributions of the indefinite fixed point sets F depend on the sign in the limit. In particular, one has

$$D^{+}_{\operatorname{codim} F/2 - 1, F} \neq D^{-}_{\operatorname{codim} F/2 - 1, F}$$

by (1.9) since $N^{+}_F \neq N^{-}_F$, see (5.4). Again, the discontinuities produced at $j = \operatorname{codim} F/2 - 1$ can be quantitatively calculated in terms of explicit scalar multiples of $\int_F a \, dF$;

3) neither when approaching ζ_0 from above or below need the limit agree with the value $A_j^{\zeta_0}(a \otimes \sigma)$. This is because $A_j^{\zeta_0}$ involves the distributions $\sigma \mapsto \sigma_{\pm}^{[j]}(0)$, which occur in neither of the limits, and also due to the fact that the distributions on the manifold occurring in $A_j^{\zeta_0}$ are different from those appearing in the limits in Theorem 1.2.

On the other hand, if ζ_0 is a regular value of $\mathcal{J}$, then the result of Theorem 1.2 reduces to the statement that the coefficients in the regular asymptotic expansion (1.5) depend continuously on ζ at ζ_0.

Methods. To overcome the problems arising in the asymptotic expansion of the generalized Witten integral at singular values of the momentum map, we do not perform a desingularization procedure, but implement a destratification process which consists of several steps. First, we linearize the phase function near each $F \in \mathcal{F}$ using the Guillemin-Sternberg-Marle local normal form and a classical result by Whitney on smooth extensions of even functions defined on half-spaces. This linearization is not the result of a monomialization of the phase function, so that no desingularization of the critical set has taken place. As a result, for each $F \in \mathcal{F}$ one obtains an oscillatory integral with a clean critical set but with an integrand which is not smooth at the singular value $\mathcal{J}(F)$, so that the stationary phase principle cannot be applied. Instead, in a second step we take the Fourier transform on the Lie algebra and split the integral at $\mathcal{J}(F)$ to obtain C^∞-amplitudes. In a third step, we Taylor expand the integrand in powers of ε, resulting in a separation into singular and regular contributions and a complete asymptotic description for the linearized integral. It is this separation of the contributions originating from the different strata that we call *destratification*. Finally, we translate the results obtained in the local model into meaningful expressions that live on the strata of the symplectic quotient and can be patched together to get the stated global results.

In this work we restricted ourselves to the simplest case of an S^1-action since the derivation of a complete asymptotic expansion of the Witten integral for arbitrary compact group actions or even torus actions is considerably more involved. In fact, restricting to circle actions has the advantage that general phenomena such as the discontinuities of the asymptotic expansion at singular values due to the contributions by lower-dimensional strata are clearly visible while the computational effort is reduced to a minimum. Another simplification (which occurs for any abelian Lie group) is that we do not have to distinguish between orbit reduction and point reduction.

Previous results. For compact Hamiltonian G-manifolds M arising in geometric invariant theory, an asymptotic expansion of the Witten integral was derived by Jeffrey, Kiem, Kirwan, and Woolf [9, Theorem 34] using Parseval's formula on the Lie algebra $\mathfrak{g}$ and the localization formula of Berline-Vergne and Atiyah-Bott on the manifold M. The latter amounts to performing an exact stationary phase analysis only in the manifold variables, the critical set in question being clean. The terms in their expansion are given in terms of piecewise polynomial functions evaluated on a Gaussian. Our approach could be regarded as a singular stationary phase analysis performed simultaneously in the manifold *and* Lie algebra variables. In particular, the vanishing of the Lie algebra derivatives enforces a localization on level sets of the momentum map, which in the case of circle actions leads to a precise description of the coefficients of the piecewise polynomial functions in the expansion [9, Theorem 34] in terms of integrals on the symplectic strata of the reduced space. In case that 0 is a regular value of the momentum map, a stationary phase expansion similar to ours was given for arbitrary G by Meinrenken [19, Theorem 3.1], generalizing a corresponding formula of Jeffrey-Kirwan [10, Proposition 8.10]. In case that 0 is a singular value, the main term in the Witten expansion was implicitly characterized in [9, Theorem 18] as an integral over a desingularization of the symplectic quotient, as well as by Lerman and Tolman [16, Theorem 5.1] in the special case of S^1-actions. As explained above, our approach is not based on a desingularization but a destratification process, which results in an intrinsic symplectic characterization of all coefficients that could not be obtained before via desingularization techniques.

For non-compact Hamiltonian G-manifolds, the generalized Witten integral was studied by the second author in [23] in the special case that $M = T^*N$ is the cotangent bundle of a smooth manifold N, equipped with the canonical symplectic form, and the action of G on M is the lift of a smooth G-action on N. Performing a stationary phase analysis in the Lie algebra and manifold variables and desingularizing partially, the leading term of the asymptotic expansion was characterized as an integral on the top stratum of the reduced space, together with a remainder estimate. Nevertheless, the employed desingularization techniques would require further development to consider higher order terms; in particular, the singular contributions $(A_j^\zeta)_{\text{sing}}$ occurring in our asymptotic expansion in Theorem 1.1 were not seen in [23]. Before, Prato and Wu [22] proved a Duistermaat-Heckman type formula in the non-compact setting under a suitable properness assumption on the momentum map.

Applications and outlook. The asymptotic behavior of the Witten integral plays an essential role in the derivation of residue formulas in equivariant cohomology, which was the main theme of [9, 10] and [23]. These formulas, in principle, allow to compute the (intersection) cohomology of reduced spaces in terms of the equivariant cohomology of the underlying Hamiltonian space and the fixed-point data of the group action. The motivation for the present work was to extend the results obtained in [23] for basic differential forms to arbitrary equivariant differential forms. This requires a complete asymptotic expansion of the Witten integral and not just a computation of the leading term with a remainder estimate, which was sufficient to deal with basic forms.

Residue formulas were also applied by Jeffrey and Kirwan [11] to Riemann-Roch numbers and the Guillemin-Sternberg conjecture under the assumption that 0 is a regular value of the momentum map. This conjecture was proved by Meinrenken [19] under similar assumptions relying directly on the stationary phase expansion of the Witten integral. In the case when 0 is a singular value, the Guillemin-Sternberg conjecture was first proved by Meinrenken and Sjamaar [18]. Another proof was given by Paradan and Vergne [21], both approaches being based on desingularizations of the reduced space.

In forthcoming papers, we intend to apply the results derived in this paper to the study of the cohomology of symplectic quotients for general Hamiltonian circle actions via residue formulas, complementing the previous works [16] and [9]. In particular, we plan to extend their work to noncompact manifolds and to interpret the residues in terms of the symplectic data of the strata of the reduced space. Our destratification approach should also yield new insights into the Guillemin-Sternberg conjecture for circle actions.

Structure of the paper. This paper is structured as follows: Section 2 contains a brief introduction to Hamiltonian actions and reduced spaces, followed by the definition of the Witten integral, and gives normal forms for the momentum map and the relevant local integrals. In Section 3, complete asymptotic expansions are derived for the local integrals via a destratification process. The coefficients in the local expansions are interpreted geometrically in Section 4. Our main results are then proved in Section 5. A notation index can be found at the end of this paper.

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2. Background and setup

We begin by introducing some concepts in the general setting of a Hamiltonian action of a general compact connected Lie group G, and then specialize to the circle case $G = T = S^1$.

2.1. Hamiltonian actions and reduced spaces

Let M be a $2n$-dimensional symplectic manifold with symplectic form ω. Assume that M carries a Hamiltonian action of a compact connected Lie group G of dimension d with Lie algebra $\mathfrak{g}$, and denote the corresponding Kostant-Souriau momentum map by

$$\mathcal{J} : M \to \mathfrak{g}^*, \quad \mathcal{J}(p)(X) = J(X)(p),$$

which is characterized by the property

$$dJ(X) = \iota_{\widetilde{X}}\omega \qquad \forall\, X \in \mathfrak{g}, \tag{2.1}$$

where $\widetilde{X}$ denotes the fundamental vector field on M associated to X, d is the de Rham differential, and ι denotes contraction. Note that $\mathcal{J}$ is G-equivariant in the sense that $\mathcal{J}(k^{-1}p) = \mathrm{Ad}^*(k)\mathcal{J}(p)$. Let $(\Omega^*_G(M)_c, D)$ be the complex of compactly supported equivariant differential forms on M. The elements in $\Omega^*_G(M)_c$ can be regarded as G-equivariant polynomial maps $\mathfrak{g} \to \Omega^*_c(M)$, where G acts on $\mathfrak{g}$ by the adjoint action $\mathrm{Ad}\,(G)$ and on the algebra $\Omega^*_c(M)$ of compactly supported differential forms by the pullbacks associated to the G-action on M. The differential D is then defined by

$$D(\alpha)(X) := d(\alpha(X)) + \iota_{\widetilde{X}}(\alpha(X)), \qquad \alpha \in \Omega^*_G(M)_c.$$

We denote the cohomology of the complex $(\Omega^*_G(M)_c, D)$, which is called the *equivariant cohomology of M*, by $H^*_G(M)_c$. Further, let

$$\overline{\omega} := \omega - \mathcal{J} \tag{2.2}$$

be the equivariantly closed extension $\overline{\omega}$ of the symplectic form ω. The approach used here is usually called the *Cartan model*.

Remark 2.1 (Sign convention). The sign convention in the definition of D (and hence $\overline{\omega}$) varies in the literature. We define D in coherence with [1], while in [10] one has $D(\alpha)(X) := d(\alpha(X)) - \iota_{\widetilde{X}}(\alpha(X))$, which leads to $\overline{\omega} = \omega + \mathcal{J}$ as opposed to our definition (2.2).

From the definition of the momentum map it is clear that the kernel of its derivative is given by

$$\ker d\mathcal{J}|_p = (\mathfrak{g} \cdot p)^\omega, \qquad p \in M, \tag{2.3}$$

where we denoted the symplectic complement of a subspace $V \subset T_pM$ by V^ω, and $\mathfrak{g} \cdot p := \{\widetilde{X}_p : X \in \mathfrak{g}\}$. Consequently, if $\zeta \in \mathcal{J}(M)$ is a regular value of $\mathcal{J}$, the level set $\mathcal{J}^{-1}(\{\zeta\})$ is a (not necessarily connected) manifold of codimension 1, and $T_p(\mathcal{J}^{-1}(\{\zeta\})) = \ker\, d\mathcal{J}|_p = (\mathfrak{g} \cdot p)^\omega$, which is equivalent to

$$\widetilde{X}_p \neq 0 \qquad \forall\, p \in \mathcal{J}^{-1}(\{\zeta\}),\, 0 \neq X \in \mathfrak{g},$$

compare [20, Chapter 8]. The latter condition means that all stabilizers of points $p \in \mathcal{J}^{-1}(\{\zeta\})$ are finite, and therefore either of exceptional or principal type, so that $\mathcal{J}^{-1}(\{\zeta\})/G^\zeta$ is an orbifold. In addition, in view of the exact sequence

$$0 \longrightarrow T_p(\mathcal{J}^{-1}(\{\zeta\})) \xrightarrow{d\iota^\zeta} T_pM \xrightarrow{d\mathcal{J}} T_\zeta\mathfrak{g}^* \longrightarrow 0, \qquad p \in \mathcal{J}^{-1}(\{\zeta\}),$$

where $\iota^\zeta : \mathcal{J}^{-1}(\{\zeta\}) \hookrightarrow M$ denotes the inclusion, and the corresponding dual sequence, $\mathcal{J}^{-1}(\{\zeta\})$ is orientable because M is orientable, compare [15, Chapter XV.6].

If ζ is not a regular value, both $\mathcal{J}^{-1}(\{\zeta\})$ and $\mathcal{J}^{-1}(\{\zeta\})/G^\zeta$ are stratified spaces. While usually the orbit type stratification [24] is more common, for our purposes it will be more convenient to consider the infinitesimal orbit stratification

$$\mathcal{J}^{-1}(\{\zeta\}) = \bigcup_{(\mathfrak{h})} \mathcal{J}^{-1}(\{\zeta\})_{(\mathfrak{h})},$$

see [18, Section 3]. Its strata consist of infinitesimal orbit type orbifolds, where an infinitesimal orbit type $(\mathfrak{h})$ of the G-action is an equivalence class of isotropy algebras $\mathfrak{h} \subset \mathfrak{g}$, and two such algebras $\mathfrak{h}, \mathfrak{h}'$ are equivalent if there is an element $g \in G^\zeta$ with $\mathfrak{h} = \mathrm{Ad}(g)\mathfrak{h}'$.

Let us now restrict to the case $G = T = S^1$. The infinitesimal orbit type stratification is then quite simple. In fact, the quotient $\mathscr{M}^\zeta = \mathcal{J}^{-1}(\{\zeta\})/T$ is stratified by infinitesimal orbit types according to

$$\mathscr{M}^\zeta = \mathscr{M}^\zeta_{\mathrm{top}} \sqcup \mathscr{M}^\zeta_{\mathrm{sing}}, \qquad \mathscr{M}^\zeta_\aleph := (\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_\aleph)})/T, \tag{2.4}$$

where $M_{(\mathfrak{h}_\aleph)}$ denotes the stratum of M of infinitesimal orbit type $(\mathfrak{h}_\aleph)$ with $\mathfrak{h}_{\mathrm{top}} = \{0\}$ and $\mathfrak{h}_{\mathrm{sing}} = \mathfrak{t}$.

Remark 2.2. It can happen that $\mathscr{M}^\zeta_{\mathrm{top}}$ is empty. For example, if $M = \mathbb{R}^2$ with S^1 acting by rotations around the origin, the zero level set $\mathcal{J}^{-1}(\{0\})$ consists only of the origin, which is a fixed point.

Since ω is non-degenerate, we see from (2.3) that

$$p \in M^T = M_{(\mathfrak{h}_{\mathrm{sing}})} \quad \Longleftrightarrow \quad d\mathcal{J}|_p = 0.$$

Since $\mathcal{J}$ is constant on each F we have

Lemma 2.3. *The momentum map $\mathcal{J} : M \to \mathfrak{t}^*$ has no critical points in $M_{(\mathfrak{h}_{\mathrm{top}})}$ and its singular values are $\{\mathcal{J}(F) : F \in \mathcal{F}\}$.* $\square$

As already mentioned in the introduction, the *top stratum* $\mathscr{M}^\zeta_{\mathrm{top}}$ is dense in $\mathscr{M}^\zeta$ if non-empty and an orbifold, while $\mathscr{M}^\zeta_{\mathrm{sing}}$ is a smooth manifold, each of its components being identical to some $F \in \mathcal{F}$. Furthermore, each F is a symplectic submanifold of M, and there is also a natural symplectic form $\omega^\zeta_{\mathrm{top}}$ on $\mathscr{M}^\zeta_{\mathrm{top}}$, as explained on page 1284.

2.2. The Witten integral

The central objects of our study are generalized Witten integrals of the form (1.1), and our main tools for their investigation will be Fourier analysis and singular stationary phase expansion. Fix an identification $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}$ and let

dx and $d\zeta$ be measures on $\mathfrak{t}$ and $\mathfrak{t}^*$ that correspond to Lebesgue measure by the fixed identifications, respectively. Denote by

$$\mathcal{F}_{\mathfrak{t}} : \mathcal{S}(\mathfrak{t}^*) \to \mathcal{S}(\mathfrak{t}), \qquad \mathcal{F}_{\mathfrak{t}} : \mathcal{S}'(\mathfrak{t}) \to \mathcal{S}'(\mathfrak{t}^*)$$

the Fourier transform on the Schwartz space and the space of tempered distributions, given by[2]

$$\begin{aligned} \widehat{\psi}(x) &:= (\mathcal{F}_{\mathfrak{t}}\psi)(x) := \int_{\mathfrak{t}^*} e^{-i\langle \zeta, x\rangle}\psi(\zeta)\, d\zeta, \\ \langle \zeta, x\rangle &:= \zeta(x), \quad x \in \mathfrak{t}, \qquad \psi \in \mathcal{S}(\mathfrak{t}^*), \end{aligned} \tag{2.5}$$

and recall that $\overline{\omega} := \omega - \mathcal{J}$. Consider now the *generalized Duistermaat-Heckman integral*

$$L_\varrho : \mathfrak{t} \to \mathbb{C}, \qquad L_\varrho(x) := \int_M e^{-i\,\overline{\omega}(x)}\varrho(x), \qquad \varrho \in \Omega^*_T(M)_c, \tag{2.6}$$

regarded as a tempered distribution in $\mathcal{S}'(\mathfrak{t})$, compare [6, 10, 26][3]. If M is compact and $\varrho = 1$, L_ϱ is the classical Duistermaat-Heckman integral, whose $\mathfrak{t}$-Fourier transform is given by the pushforward $\mathcal{J}_*(\omega^n/n!)$ of the Liouville form along $\mathcal{J}$, which is a piecewise polynomial measure on $\mathfrak{t}^*$ [5]. Motivated by this, we shall examine the behavior of the Fourier transform of L_ϱ near the origin, and for this sake consider an approximation of the Dirac δ-distribution centered at $\zeta \in \mathfrak{t}^*$ given by

$$\phi^\zeta_\varepsilon(\xi) := \phi((\xi - \zeta)/\varepsilon)/\varepsilon, \qquad \varepsilon > 0,$$

where $\phi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathfrak{t}^*)$ is a test function satisfying $\widehat{\phi}(0) = 1$. We are then interested in the limit

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \left\langle \widehat{L_\varrho}, \phi^\zeta_\varepsilon \right\rangle &= \lim_{\varepsilon\to 0^+} \int_{\mathfrak{t}} L_\varrho(x)\widehat{\phi^\zeta_\varepsilon}(x)\, dx = \lim_{\varepsilon\to 0^+} \int_{\mathfrak{t}} L_\varrho(x) e^{-i\zeta(x)}\widehat{\phi}(\varepsilon x)\, dx \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{\mathfrak{t}} \left[\int_M e^{i(J-\zeta)(x)/\varepsilon} e^{-i\omega}\varrho(x/\varepsilon)\right] \widehat{\phi}(x)\, dx, \end{aligned} \tag{2.7}$$

which does not need to exist a priori in general, and its dependence on ζ in a neighbourhood of $0 \in \mathfrak{t}^*$. Thus, we are led to the definition of the *Witten*

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[2] Regarding normalization conventions, see [12, footnotes on p. 125].

[3] Jeffrey and Kirwan use the notation $\Pi_*(\varrho e^{-i\overline{\omega}})$ for our map L_ϱ, see [10, p. 299].

integral

$$\mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon) := \int_{\mathfrak{t}} \left[\int_M e^{i(J-\zeta)(x)} e^{-i\omega} \varrho(x) \right] \hat{\phi}(\varepsilon x)\, dx, \qquad (2.8)$$
$$\varrho \in \Omega^*_T(M)_c,\ \phi \in \mathcal{S}(\mathfrak{t}^*),\ \varepsilon > 0, \zeta \in \mathfrak{t}^*,$$

and to the investigation of its asymptotic behavior as $\varepsilon \to 0^+$ and $\zeta \to 0$. Note that in this notation, $\langle \widehat{L_\varrho}, \phi^{\zeta}_{\varepsilon} \rangle = \mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon)$. Furthermore, if ϱ is equivariantly closed, $\mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon)$ actually only depends on the cohomology class of ϱ in view of [23, Lemma 1].

Remark 2.4. The original Witten integral considered in [26] reads in our setting

$$\frac{1}{(2\pi)^2 i} \int_{\mathfrak{t}} \left[\int_M \left(e^{-i\bar{\omega}} \varrho\right)(x) \right] e^{-\nu \frac{x^2}{2}}\, dx, \qquad \nu > 0,\ \varrho \in \Omega^*_T(M)_c. \qquad (2.9)$$

Writing $\varepsilon := \sqrt{\nu}$ we see that this equals $((2\pi)^2 i)^{-1}$ times $\mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon)$ with $\hat{\phi}(x) = e^{-\frac{x^2}{2}}$ and $\zeta = 0$.

To formulate (2.8) more explicitly, write ϱ as a finite linear combination

$$\varrho(x) = \sum_{k,m} \varrho_{k,m} x^k, \qquad \varrho_{k,m} \in \Omega^m(M)_c, \qquad k, m \in \mathbb{N} \cup \{0\}. \qquad (2.10)$$

For those $\varrho_{k,m}$ which are differential forms of odd degree, there is no appropriate power $N \in \mathbb{N} \cup \{0\}$ such that $\omega^N \wedge \varrho_{k,m}$ is a volume form, therefore only the $\varrho_{k,m}$ with m even contribute to $\mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon)$. Thus,

$$\mathcal{W}^{\zeta}_{\varrho,\phi}(\varepsilon) = \sum_{k,m,\ m\,\text{even}} \varepsilon^{-k-1} \int_{\mathfrak{t}} \left[\int_M e^{i(J-\zeta)(x)/\varepsilon} \frac{(-i\omega)^{n-m/2} \varrho_{k,m}}{(n-m/2)!} \right] x^k \widehat{\phi}(x)\, dx.$$

We associate to each $\varrho_{k,m}$ a T-invariant function $a_{k,m} \in \mathrm{C}^\infty_{\mathrm{c}}(M)$ by the relation

$$\frac{(-i\omega)^{n-m/2} \varrho_{k,m}}{(n-m/2)!} = a_{k,m}\, dM, \qquad (2.11)$$

where $dM := \omega^n/n!$ is the symplectic volume form on M. In this way, we are reduced to studying the asymptotic behavior of the *generalized Witten* *integrals*

$$(2.12)\quad I^{\zeta}_{a,\sigma}(\varepsilon) = \int_{\mathfrak{t}}\int_M e^{i\psi^{\zeta}(p,x)/\varepsilon} a(p)\, dM(p)\sigma(x)\, dx, \qquad \zeta \in \mathfrak{t}^*, \quad \varepsilon \to 0^+,$$

with amplitudes $a \in \mathrm{C}^{\infty}_{\mathrm{c}}(M)$, $\sigma \in \mathcal{S}(\mathfrak{t})$, where the phase function $\psi^{\zeta} \in \mathrm{C}^{\infty}(M \times \mathfrak{t})$ is given by

$$(2.13)\qquad \psi^{\zeta}(p,x) := \mathcal{J}(p)(x) - \zeta(x).$$

Now, when trying to describe the asymptotic behavior of the integral $I^{\zeta}_{a,\sigma}(\varepsilon)$ by means of the generalized stationary phase principle, one faces the serious difficulty that the critical set of the phase function ψ^{ζ} is in general not smooth. Indeed, due to the linear dependence of $J(x)$ on x we obtain

$$\partial_x\, \psi^{\zeta}(p,x) = \mathcal{J}(p) - \zeta,$$

and because of the non-degeneracy of ω,

$$dJ(x) = \iota_{\widetilde{x}}\omega = 0 \quad \Longleftrightarrow \quad \widetilde{x} = 0,$$

where $\widetilde{x}$ is the fundamental vector field on M associated to x. Hence, the critical set reads

$$(2.14)\qquad \begin{aligned} \mathrm{Crit}(\psi^{\zeta}) &:= \Big\{(p,x) \in M \times \mathfrak{t} : d\psi^{\zeta}(p,x) = 0\Big\} \\ &= \big\{(p,x) \in \mathcal{J}^{-1}(\{\zeta\}) \times \mathfrak{t} : \widetilde{x}_p = 0\big\}. \end{aligned}$$

Let us first assume that ζ is a regular value. As discussed in Section 2.1, $\mathcal{J}^{-1}(\{\zeta\})$ is an orientable manifold, and all stabilizers of points in $\mathcal{J}^{-1}(\{\zeta\})$ are finite. Consequently, $\mathrm{Crit}(\psi^{\zeta}) = \mathcal{J}^{-1}(\{\zeta\}) \times \{0\}$; in particular, it is an orientable manifold. Even further, the critical set of the phase function ψ^{ζ} is *clean* [23, Proof of Proposition 2], which means that the transversal Hessian is non-degenerate at all points in $\mathrm{Crit}(\psi^{\zeta})$, and the generalized stationary phase theorem [23, Theorem C] can be applied, yielding a complete asymptotic expansion for $I^{\zeta}_{a,\sigma}(\varepsilon)$. More precisely and generally, we have the following:

Proposition 2.5. *For each $\zeta \in \mathfrak{t}^*$, there is a family $\{\mathscr{D}^{\zeta}_j\}_{j\in\mathbb{N}_0}$ of differential operators $\mathscr{D}^{\zeta}_j$ of order j defined on a neighborhood of the submanifold $\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})}$ which are transversal to it such that the following holds:*

For each $J_0 \in \mathbb{N}_0$ and each compact set $K \subset M$ with $\mathcal{J}^{-1}(\{\zeta\}) \cap K$ containing only regular points of $\mathcal{J}$, there exists an $N_{J_0,K,\zeta} \in \mathbb{N}_0$ and a family of differential operators $\{D^l_{J_0,K,\zeta}\}_{0 \le l \le N_{J_0,K,\zeta}}$ on M such that for all $a \in \mathrm{C}^\infty_{\mathrm{c}}(M)$ with supp $a \subset K$ *and all $\sigma \in \mathcal{S}(\mathbb{R})$ one has*

(2.15)

$$\Big| I^\zeta_{a,\sigma}(\varepsilon) - \varepsilon \sum_{j=0}^{J_0} \varepsilon^j \sigma^{(j)}(0) \mathcal{I}^\zeta_j(a) \Big| \le \sum_{k,l=0}^{N_{J_0,K,\zeta}} \big\| D^l_{J_0,K,\zeta} a \big\|_\infty \big\| \sigma^{(k)} \big\|_\infty \, \varepsilon^{J_0+2} \quad \forall\, \varepsilon > 0$$

with distributions $\mathcal{I}^\zeta_j \in \mathcal{D}'(M)$ of the form

$$\mathcal{I}^\zeta_j(a) = \int_{\mathscr{M}^\zeta_{\mathrm{top}}} \langle \mathscr{D}^\zeta_j a \rangle_T \, d\mathscr{M}^\zeta_{\mathrm{top}}, \qquad \mathcal{I}^\zeta_0(a) = 2\pi \int_{\mathscr{M}^\zeta_{\mathrm{top}}} \langle a \rangle_T \, d\mathscr{M}^\zeta_{\mathrm{top}}, \tag{2.16}$$

where $d\mathscr{M}^\zeta_{\mathrm{top}}$ is the symplectic volume form on $\mathscr{M}^\zeta_{\mathrm{top}}$, and for a function f on M and a T-orbit $T \cdot p \subset M$, we put $\langle f \rangle_T (T \cdot p) := \int_T f(g \cdot p)\, dg$, where dg is the Haar measure on T fixed by our identification $\mathfrak{t} \cong \mathbb{R}$. Moreover, if $\mathscr{V} \subset \mathfrak{t}^*$ is an open set such that $\mathcal{J}^{-1}(\mathscr{V}) \cap K$ contains only regular points of $\mathcal{J}$, then the functions $\mathscr{V} \ni \zeta \mapsto \mathcal{I}^\zeta_{j,k}(a) \in \mathbb{C}$ are smooth for all $a \in \mathrm{C}^\infty_{\mathrm{c}}(M)$ with supp $a \subset K$.

Proof. It suffices to apply [23, Proposition 2 and Proposition 7], where the form of the coefficients as stated here follows from [23, Eqs. (18), (62)], taking into account that the function H appearing in [23, Eq. (62)] is linear with respect to the $\mathfrak{t}$-variable in our case, exactly as in [23, proof of Proposition 3]. □

If ζ is not a regular value of $\mathcal{J}$, there are compact subsets $K \subset M$ such that $\mathrm{Crit}(\psi^\zeta) \cap K \times \mathfrak{t}$ is not clean and the usual stationary phase theorem cannot be applied. Instead, we shall linearize the phase function ψ^ζ in suitable local coordinates to derive an asymptotic expansion of the generalized Witten integral by a careful direct analysis.

2.3. Local normal forms for the momentum map and the Witten integral

We shall now introduce suitable coordinates on M near the set of fixed-points

$$M^T := \{p \in M : t \cdot p = p \; \forall\, t \in T\}.$$

The connected components of M^T are symplectic submanifolds of M of possibly different dimensions. Recall that we denote the set of these components by $\mathcal{F}$. Let $F \in \mathcal{F}$ and consider the symplectic normal bundle $E_F := TF^\omega \subset TM$ of F in M. Since F is symplectic, one has $TM|_F = TF \oplus E_F$ and E_F carries a symplectic structure. In particular, the total space of E_F becomes a symplectic manifold. Furthermore, the group $T = S^1$ acts on E_F fiberwise, and we may choose an S^1-invariant complex structure on E_F compatible with the symplectic one. Each fiber of the so complexified bundle E_F then splits into a direct sum of complex 1-dimensional representations of S^1, so that with $\dim F = 2n_F$

$$E_F = \bigoplus_{j=1}^{n-n_F} \mathcal{E}_j^F, \tag{2.17}$$

the $\mathcal{E}_j^F$ being complex line bundles over F. The Lie algebra $\mathfrak{t}$ acts on them by

$$(\mathcal{E}_j^F)_p \ni v \mapsto i\lambda_j^F(x)v \in (\mathcal{E}_j^F)_p, \qquad p \in F,\ x \in \mathfrak{t},\ \lambda_j^F \in \mathfrak{t}^* \cong \mathbb{R},$$

where $\lambda_1^F, \dots, \lambda_{n-n_F}^F \in \mathbb{Z} \setminus \{0\}$ are the weights of the T-action on $(E_F)_p$. They do not depend on the point $p \in F$ because F is connected, and they can be grouped into positive weights $\lambda_1^F, \dots, \lambda_{\ell_F^+}^F$ and negative weights $\lambda_{\ell_F^+ +1}^F, \dots, \lambda_{\ell_F^+ + \ell_F^-}^F$. The codimension of F in M is given by $\operatorname{codim} F = 2(n - n_F) = 2(\ell_F^+ + \ell_F^-)$. We shall now make use of the local normal form theorem for the momentum map $\mathcal{J}$ due to Guillemin-Sternberg [7] and Marle [17], which in our situation reads as follows:

Proposition 2.6. *For each component $F \in \mathcal{F}$, there exist*

1) *a faithful unitary representation $\rho_F : S^1 \to (S^1)^{\ell_F^+ + \ell_F^-} \subset U(\ell_F^+) \times U(\ell_F^-) \subset U(\ell_F^+ + \ell_F^-)$ with positive weights $\lambda_1^F, \dots, \lambda_{\ell_F^+}^F \in \mathbb{N}$ and negative weights $\lambda_{\ell_F^+ +1}^F, \dots, \lambda_{\ell_F^+ + \ell_F^-}^F \in -\mathbb{N}$,*
2) *a principal K_F-bundle $P_F \to F$, where K_F is a subgroup of $U(\ell_F^+) \times U(\ell_F^-)$ commuting with $\rho_F(S^1)$,*

such that

$$E_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-},$$

where $P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \to F$ is the vector bundle associated to P_F by the K_F-action. Furthermore, there is a symplectomorphism $\Phi_F : U_F \to V_F$ from an S^1-invariant neighborhood U_F of F in M onto an S^1-invariant neighborhood

V_F of the zero section in E_F, which is equivariant with respect to the S^1-action on $E_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ given by ρ_F, and

$$\mathcal{J} \circ \Phi_F^{-1}([\wp, w]) = \frac{1}{2} \sum_{j=1}^{\ell_F^+ + \ell_F^-} \lambda_j^F |w_j|^2 + \mathcal{J}(F), \tag{2.18}$$
$$w = (w_1, \dots, w_{\ell_F^+ + \ell_F^-}),\ [\wp, w] \in P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}.$$

In particular, $2\ell_F^-$ and $2\ell_F^+$ are the dimensions of the negative and positive eigenspaces of the Hessian of $\mathcal{J}$ at a point of F, respectively.

Proof. See [16, Lemma 3.1]. □

Note that the local normal form neighborhood U_F has the property that

$$U_F \cap M^T = F. \tag{2.19}$$

By shrinking the U_F, we shall assume that each point $p \in M$ lies in only finitely many U_F. We then choose a locally finite partition of unity $\{\chi_{\mathrm{top}}, \chi_F\}_{F \in \mathcal{F}}$ on M subordinate to the open cover

$$M = M_{(\mathfrak{h}_{\mathrm{top}})} \cup \bigcup_{F \in \mathcal{F}} U_F,$$

consisting of T-invariant functions such that $\chi_F \equiv 1$ in a neighborhood of F for each $F \in \mathcal{F}$.

The generalized Witten integral (2.12) with parameter $\zeta \in \mathfrak{t}^*$ and amplitudes $a \in \mathrm{C}_c^\infty(M)$, $\sigma \in \mathcal{S}(\mathfrak{t})$ can now be written as

$$I_{a,\sigma}^\zeta(\varepsilon) = I_{a\chi_{\mathrm{top}},\sigma}^\zeta(\varepsilon) + \sum_{F \in \mathcal{F}: F \cap \operatorname{supp} a \neq \emptyset} I_{a\chi_F,\sigma}^\zeta(\varepsilon), \tag{2.20}$$

which is a finite sum because supp a is compact and our partition of unity is locally finite. More abstractly and conveniently, we can write the decomposition in terms of distributions as

$$I^\zeta(\varepsilon) = I_{\chi_{\mathrm{top}}}^\zeta(\varepsilon) + \sum_{F \in \mathcal{F}} I_{\chi_F}^\zeta(\varepsilon) \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'. \tag{2.21}$$

We shall focus our attention in the following on the localized integrals $I_{a\chi_F,\sigma}^\zeta(\varepsilon)$. In terms of the coordinates provided by Φ_F we obtain with (2.18) and the notation

$$\zeta_F := \zeta - \mathcal{J}(F)$$

for each of the localized integrals the formula

$$\begin{aligned}
(2.22) \qquad I^{\zeta}_{a\chi_F,\sigma}(\varepsilon) &= \int_{\mathfrak{t}} \int_{V_F} e^{i(\mathcal{J}\circ\Phi_F^{-1}([\wp,w])-\zeta)(x)/\varepsilon}\,(a\chi_F) \\
&\qquad \times (\Phi_F^{-1}([\wp,w]))\, d[\wp,w]\sigma(x)\,dx \\
&= \int_{\mathbb{R}} \int_{P_F\times_{K_F}\mathbb{C}^{\ell_F^+ + \ell_F^-}} e^{i\frac{x}{2\varepsilon}(\langle Q_F w,w\rangle - 2\zeta_F)}\,(a\chi_F) \\
&\qquad \times (\Phi_F^{-1}([\wp,w]))\, d[\wp,w]\sigma(x)\,dx,
\end{aligned}$$

where we identified $\mathfrak{t}$ with $\mathbb{R}$, $d[\wp,w]$ denotes the symplectic form on $P_F\times_{K_F}\mathbb{C}^{\ell_F^+ + \ell_F^-}\cong E_F$, which agrees on V_F with the pullback of the symplectic volume form $(\Phi_F^{-1})^*(dM|_{U_F})$, and we introduced on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ the non-degenerate quadratic form

$$(2.23) \qquad \langle Q_F w,w\rangle := \sum_{j=1}^{\ell_F^+ + \ell_F^-} \lambda_j^F |w_j|^2 = \sum_{j=1}^{\ell_F^+ + \ell_F^-} \lambda_j^F \big((\operatorname{Re} w_j)^2 + (\operatorname{Im} w_j)^2\big).$$

Since $\mathcal{J}\circ\Phi_F^{-1}([\wp,w]) = \langle Q_F w,w\rangle$ depends only on $w\in\mathbb{C}^{\ell_F^+ + \ell_F^-}$, we want to lift $I^{\zeta}_{a\chi_F,\sigma}$ to $P_F\times\mathbb{C}^{\ell_F^+ + \ell_F^-}$ in order to integrate independently over w in (2.22). Thus, let

$$dF := \omega^{\dim F/2}/(\dim F/2)!$$

be the symplectic volume form on F and let $d\wp$ be a smooth volume density on P_F. Then, since P_F is a smooth fiber bundle over F, there is a differential form η_F on P_F such that

$$(2.24) \qquad d\wp = |\eta_F \wedge \pi_F^* dF|,$$

where $\pi_F : P_F \to F$ is the fiber bundle projection (cf. [15, p. 430]). Let us fix a preferred volume density $d\wp$ by demanding that the fiber volume $V(p) := \int_{\pi_F^{-1}(\{p\})}\eta_F$ is equal to 1 for each $p\in F$. This can be simply achieved by normalizing some chosen $d\wp$ with the function $1/V$. Let now dw be the symplectic volume form on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ with respect to the standard symplectic structure on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$. Then we claim that the product measure $d\wp\, dw$ on

$P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-}$ fulfills

$$d\wp\, dw = |\Pi_F^* \eta_F \wedge \tilde{\pi}_F^* d[\wp, w]|, \tag{2.25}$$

where $\tilde{\pi}_F : P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-} \to P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ is the fiber bundle projection, and $\Pi_F : P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-} \to P_F$ is the projection onto the first factor. The relation (2.25) can be proved as follows. By (2.24), we have

$$d\wp\, dw = |\Pi_F^* \eta_F \wedge (\pi_F \circ \Pi_F)^* dF \wedge dw|, \tag{2.26}$$

where we identified dw with its pullback along the projection $P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-} \to \mathbb{C}^{\ell_F^+ + \ell_F^-}$ onto the second factor. Since (2.25) is a pointwise relation, it suffices to establish it locally. Let therefore $p \in F$ and let $\mathcal{U} \subset M$ be a Darboux chart around p such that $\mathcal{U} \cap F$ is the vanishing locus of the last $2(\ell_F^+ + \ell_F^-)$ coordinates in $\mathcal{U}$. Then the symplectic normal bundle $E_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ of F in TM is trivial over $\mathcal{U} \cap F$ and with respect to this trivialization $E_F|_{\mathcal{U}\cap F} \cong (\mathcal{U} \cap F) \times \mathbb{C}^{\ell_F^+ + \ell_F^-}$ one has $d[\wp, w]|_{E_F|_{\mathcal{U}\cap F}} = dF dw$. This gives us on $\tilde{\mathcal{U}} := (\pi_F \circ \Pi_F)^{-1}(\mathcal{U} \cap F) \subset P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-}$ the relation

$$(\pi_F \circ \Pi_F)^* dF \wedge dw = \tilde{\pi}_F^* d[\wp, w],$$

proving (2.25) on $\tilde{\mathcal{U}}$. Covering all of $P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-}$ with sets of the form $\tilde{\mathcal{U}}$ finally proves (2.25). Thanks to (2.25) and the fiber volume normalization $V \equiv 1$, we now have for any continuous function f on $P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ with compact support the equality

$$\int_{P_F \times \mathbb{C}^{\ell_F^+ + \ell_F^-}} \tilde{\pi}_F^*(f)\, d\wp\, dw = \int_{P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}} f([\wp, w])\, d[\wp, w]. \tag{2.27}$$

Applying this to our integral $I^\zeta_{a\chi_F,\sigma}$ yields with Fubini

$$\begin{aligned} I^\zeta_{a\chi_F,\sigma}(\varepsilon) = \int_{\mathbb{R}} \int_{\mathbb{C}^{\ell_F^+ + \ell_F^-}} & e^{i\frac{x}{2\varepsilon}(\langle Q_F w, w\rangle - 2\zeta_F)} \\ & \times \left[\int_{P_F} (a\chi_F)(\Phi_F^{-1}(\tilde{\pi}_F(\wp, w)))\, d\wp \right] dw\, \sigma(x)\, dx \\ =: \int_{\mathbb{R}} \int_{\mathbb{R}^{\operatorname{codim} F}} & e^{i\frac{x}{2\varepsilon}(\langle Q_F w, w\rangle - 2\zeta_F)}\, \tilde{a}_F(w)\, dw\, \sigma(x)\, dx, \end{aligned}$$

where we identified $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ with $\mathbb{R}^{2(\ell_F^+ + \ell_F^-)} = \mathbb{R}^{\operatorname{codim} F}$. With respect to this identification, denote by

$$n_F^+ := 2\ell_F^+, \qquad n_F^- := 2\ell_F^-$$

the real dimensions of the positive and negative eigenspaces of Q_F, and assume first that both $n_F^+ \neq 0$ and $n_F^- \neq 0$. Introducing polar coordinates $w^+ = (w_1, \dots, w_{n_F^+}) = r\theta^+ \in \mathbb{R}^{n_F^+}$ and $w^- = (w_{n_F^+ + 1}, \dots, w_{\operatorname{codim} F}) = s\theta^- \in \mathbb{R}^{n_F^-}$ in these directions with radii $r, s > 0$ and $\theta^\pm \in S^{n_F^\pm - 1} \subset \mathbb{R}^{n_F^\pm}$ and substituting $(w_j, w_{j+1}) \mapsto |\lambda_j^F|^{-1/2}(w_j, w_{j+1})$ for $1 \leq j \leq \operatorname{codim} F - 1$, $j \in 2\mathbb{N} - 1$, the integral $I^\zeta_{a\chi_F,\sigma}(\varepsilon)$ reads

$$(2.28)\quad I^\zeta_{a\chi_F,\sigma}(\varepsilon) = \Lambda_F^{-1} \int_{-\infty}^{\infty} \int_0^\infty \int_0^\infty e^{i\frac{x}{2\varepsilon}(r^2 - s^2 - 2\zeta_F)}\, \alpha_F(r,s)\; dr\, ds\, \sigma(x)\, dx,$$

where the Jacobian of the substitution is given by $\Lambda_F^{-1} r^{n_F^+ - 1} s^{n_F^- - 1}$ with the constant

$$(2.29)\qquad \Lambda_F := \prod_{j=1}^{\operatorname{codim} F/2} |\lambda_j^F| \in \mathbb{N}$$

and we put, with $d\theta^\pm$ denoting the standard round measure on the Euclidean unit sphere $S^{n_F^\pm - 1}$,

(2.30)

$$\begin{aligned}
\alpha_F(r,s) &:= r^{n_F^+ - 1} s^{n_F^- - 1} S_F(r,s),\\
S_F(r,s) &:= \int_{S^{n_F^+ - 1}} \int_{S^{n_F^- - 1}} \tilde{a}_F(r\theta^+, s\theta^-)\, d\theta^+\, d\theta^-,\\
\tilde{a}_F(w) &:= \int_{P_F} (a\chi_F)\Big(\Phi_F^{-1}\Big(\tilde{\pi}_F\Big(\wp, \tfrac{w_1}{|\lambda_1^F|^{\frac12}}, \tfrac{w_2}{|\lambda_1^F|^{\frac12}}, \dots, \tfrac{w_{\operatorname{codim} F - 1}}{|\lambda^F_{\operatorname{codim} F/2}|^{\frac12}}, \tfrac{w_{\operatorname{codim} F}}{|\lambda^F_{\operatorname{codim} F/2}|^{\frac12}}\Big)\Big)\Big)\, d\wp.
\end{aligned}$$

The function $\tilde{a}_F$ is a local cutoff of the original amplitude a which has been transformed using the normal form symplectomorphism Φ_F. Note that the double spherical mean $S_F(r,s)$ is symmetric in r and s. If $n_F^+ = 0$ and $n_F^- \neq 0$ or $n_F^+ \neq 0$ and $n_F^- = 0$, the integral (2.22) can be written as

$$(2.31)\quad I^\zeta_{a\chi_F,\sigma}(\varepsilon) = \Lambda_F^{-1} \int_{-\infty}^{\infty} \int_0^\infty e^{i\frac{x}{2\varepsilon}(\pm r^2 - 2\zeta_F)}\, \alpha_F(r)\, dr\, \sigma(x)\, dx, \quad n_F^\mp = 0,$$

where, with $\tilde{a}_F$ as in (2.30) and with $d\theta$ denoting the standard round measure on $S^{\operatorname{codim} F - 1}$, one has

$$(2.32)\qquad \alpha_F(r) := r^{\operatorname{codim} F - 1} S_F(r), \qquad S_F(r) := \int_{S^{\operatorname{codim} F - 1}} \tilde{a}_F(r\theta)\, d\theta,$$

and the spherical mean $S_F(r)$ is symmetric in r.

3. Asymptotic expansions

As before, consider a $2n$-dimensional symplectic manifold (M, ω) carrying a Hamiltonian action of $T = S^1$ with momentum map $\mathcal{J} : M \to \mathfrak{t}^*$. We are now ready to derive an asymptotic expansion of the generalized Witten integral $I^{\zeta}_{a,\sigma}(\varepsilon)$ introduced in (2.12). For this sake, we shall use the decomposition (2.20) of $I^{\zeta}_{a,\sigma}(\varepsilon)$ into a global regular part $I^{\zeta}_{a\chi_{\text{top}},\sigma}(\varepsilon)$ and a finite sum of potentially singular localized integrals $I^{\zeta}_{a\chi_F,\sigma}(\varepsilon)$ which are singular iff $J(F) = \zeta$, as can be read off from their presentation (2.28). In the following, we shall determine asymptotic expansions for each of those localized integrals, which are at the heart of our results.

3.1. Contribution of the top stratum

By Lemma 2.3 the momentum map is regular on $M_{\mathfrak{h}_{\text{top}}}$. Therefore Proposition 2.5 yields a complete stationary phase expansion for $I^{\zeta}_{a\chi_{\text{top}},\sigma}(\varepsilon)$. The coefficients $Q^{\zeta}_{j,k}(a\chi_{\text{top}})$ do have a geometric interpretation in terms of integrals over $\mathscr{M}^{\zeta}_{\text{top}}$ and are smooth in ζ. Let us next turn to the more interesting contributions localized in the neighborhoods U_F.

3.2. Contributions of the indefinite fixed point set components

Let us start by considering an $F \in \mathcal{F}$ for which Q_F is indefinite, so that $n^+_F \neq 0$ and $n^-_F \neq 0$ holds, our departing point being the integrals (2.28). While in a previous version of this paper we followed an approach of Brummelhuis, Paul, and Uribe [4], we shall now follow a simpler approach kindly pointed out to us by Michèle Vergne. The starting point is the following classical result of Whitney on extensions of even functions.

Lemma 3.1 ([25, Theorem 1 on p. 159 and Remark on p. 160]). *Given $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, let $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ be a function that is even in the i-th variable, that is, one has*

$$f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, -x_i, \dots, x_n) \qquad \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then there exists a function $g \in \mathrm{C}^\infty(\mathbb{R}^n)$ such that

$$f(x) = g(x_1, \dots, x_i^2, \dots, x_n) \qquad \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

This important result has a direct application to functions on $\mathbb{R}^2$ which are even in both variables:

Corollary 3.2. *For every function $f \in \mathrm{C}^\infty(\mathbb{R}^2)$ which is even in both variables, there is a function $g \in \mathrm{C}^\infty(\mathbb{R}^2)$ such that $f(x,y) = g(x^2, y^2)$ for all $x, y \in \mathbb{R}$.*

Proof. Given f, we apply Lemma 3.1 with $i = 1$ to get a function $\tilde{h} \in \mathrm{C}^\infty(\mathbb{R}^2)$ with $f(x,y) = \tilde{h}(x^2, y)$ for all $x, y \in \mathbb{R}$. The function $y \mapsto \tilde{h}(x,y)$ does not need to be even when $x < 0$, but it suffices to put

$$h(x,y) := \frac{1}{2}\big(\tilde{h}(x,y) + \tilde{h}(x,-y)\big), \qquad x, y \in \mathbb{R},$$

to obtain a function $h \in \mathrm{C}^\infty(\mathbb{R}^2)$ which is even in the second variable and satisfies $h(x^2, y) = f(x,y)$ for all $x, y \in \mathbb{R}$. Applying now Lemma 3.1 with $i = 2$ to h gives us the desired function $g \in \mathrm{C}^\infty(\mathbb{R}^2)$. $\square$

As a consequence, we can write the spherical mean $S_F(r,s)$, which is a compactly supported even function in both variables r and s, in the form

$$S_F(r,s) = \mathscr{S}_F(r^2, s^2) \qquad r, s \in \mathbb{R}, \tag{3.1}$$

with a function $\mathscr{S}_F \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^2)$. Indeed, Corollary 3.2 gives us a function $\mathscr{G}_F \in \mathrm{C}^\infty(\mathbb{R}^2)$ satisfying the analog of (3.1); a function $\mathscr{S}_F$ as desired can then be constructed by multiplying $\mathscr{G}_F$ with an arbitrary cutoff function equal to 1 on the compact set $\{(r,s) : (\sqrt{|r|}, \sqrt{|s|}) \in \operatorname{supp} S_F\} \subset \mathbb{R}^2$. This reduces the study of the integrals (2.28) to the general study of integrals of the form

(3.2)
$$I^\zeta_{\mathscr{S},\sigma}(\varepsilon) := \int_\mathbb{R} \int_0^\infty \int_0^\infty e^{i(r^2-s^2-\zeta)x/\varepsilon} r^{2L^+-1} s^{2L^--1} \mathscr{S}(r^2, s^2)\, dr\, ds\, \sigma(x)\, dx,$$
$$\varepsilon > 0,\ \zeta \in \mathbb{R},$$

where $L^+, L^- \geq 1$ are two natural numbers and $\mathscr{S} \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^2)$, $\sigma \in \mathcal{S}(\mathbb{R})$ are functions, $\mathcal{S}(\mathbb{R})$ denoting the space of Schwartz functions on $\mathbb{R}$.

The first crude, but central asymptotics are obtained in the following

Proposition 3.3. *If $\pm\zeta > 0$, one has for every $M \in \mathbb{N}_0$ the asymptotics*

$$\begin{aligned}
I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = \varepsilon\Bigg(& \sum_{j=0}^{M} \varepsilon^j \sigma^{(j)}(0) \sum_{k=0}^{L} \zeta^k \int_{|\zeta|}^{\infty} \Bigg[\sum_{l=k}^{\min(k+j,L)} c_{j,k,l}\, t^{L-l} (\partial_- - \partial_+)^{j-l+k} \Bigg] \\
& \qquad \times \mathscr{S}\Big(\frac{t+\zeta}{2}, \frac{t-\zeta}{2}\Big)\, dt \\
& + \sum_{k=0}^{L} |\zeta|^k \sum_{j=L-k+1}^{M} \varepsilon^j \sigma^{(j)}(0) \sum_{\max(0,j-L-1)\le p+q \le k+j-L-1} \\
& c^{\mp}_{j,k,p,q} (\mp \partial_{\pm})^p (\partial_- - \partial_+)^q \mathscr{S}((|\zeta|, |\zeta|)_{\pm}) \Bigg) \\
& + \mathcal{O}_M\Bigg(\varepsilon^{M+2} (1 + |\zeta|^L + |\zeta|^{-M-1}) \\
& \times \sum_{l=0}^{2(M+1)} \sum_{r=0}^{M+L+1} \|D_{l,M}\mathscr{S}\|_\infty \int_{\mathbb{R}} |\hat{\sigma}(u)| (1+|u|)^r \, du \Bigg),
\end{aligned}$$

where $D_{l,M}$ is a differential operator of order $\le M+1$ on $\mathbb{R}^2$. If $\zeta = 0$, one has

$$\begin{aligned}
I^{0}_{\mathscr{S},\sigma}(\varepsilon) = \varepsilon\Bigg(& \sum_{j=0}^{M} \varepsilon^j \sigma^{(j)}(0) \int_0^{\infty} \Bigg[\sum_{l=0}^{\min(j,L)} c_{j,0,l}\, t^{L-l} (\partial_- - \partial_+)^{j-l} \Bigg] \mathscr{S}\Big(\frac{t}{2}, \frac{t}{2}\Big)\, dt \\
& + \sum_{j=L+1}^{M} \varepsilon^j \sum_{p+q=j-L-1} \Big[c^{+}_{j,0,p,q} \sigma^{[j]}_{+}(0) \partial_-^p + c^{-}_{j,0,p,q} \sigma^{[j]}_{-}(0) (-\partial_+)^p \Big] \\
& \times (\partial_- - \partial_+)^q \mathscr{S}(0,0) \Bigg) \\
& + \mathcal{O}_M\Bigg(\varepsilon^{M+2} \sum_{l=0}^{2(M+1)} \sum_{r=0}^{M+L+1} \|D_{l,M}\mathscr{S}\|_\infty \int_{\mathbb{R}} |\hat{\sigma}(u)| (1+|u|)^r \, du \Bigg),
\end{aligned}$$

where $L := L^+ + L^- - 2$, $(x,y)_+ := (x,0)$, $(x,y)_- := (0,y)$, the expressions $\sigma^{[j]}_{\pm}(0)$ are as in (1.8), $\partial_+\mathscr{S}$ and $\partial_-\mathscr{S}$ are the partial derivatives of $\mathscr{S}$ with respect to the first and second variable, respectively, and $c_{j,k,l}, c^{\pm}_{j,k,p,q} \in \mathbb{C}$ are *explicitly computable in terms of* L^+, L^-. *Some particular values are*

$$
\begin{aligned}
c_{0,0,0} &= 2^{-2-L}\pi,\\
c^{\pm}_{L+1,0,0,0} &= 2^{-2-L}\pi(-i)^{L-1}\\
&\qquad \times \sum_{l=0}^{L} \frac{(\pm 1)^{L-l+1}}{L-l+1} \sum_{\substack{l^+ + l^- = l\\ 0\leq l^\pm \leq L^\pm - 1}} (-1)^{l^+} \binom{L^+-1}{l^+}\binom{L^--1}{l^-}.
\end{aligned} \tag{3.3}
$$

To begin, notice that the critical set of the phase function in (3.2), regarded as a function on $\mathbb{R}^3$, becomes singular for $\zeta = 0$. One could therefore be inclined to desingularize the critical set in some way in order to be able to apply the stationary phase theorem. This was the way followed in [23], which was sufficient to compute the leading term and an estimate for the remainder. Nevertheless, serious difficulties arise when trying to find a complete asymptotic expansion. Instead, the proof of Proposition 3.3 will be based on a *destratification process*, which we shall carry out in the following. As a first step, we linearize the phase function by means of the substitution $r^2 = T$, $s^2 = U$, yielding

$$
I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = \frac{1}{4}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty e^{i(T-U-\zeta)x/\varepsilon} T^{L^+-1}U^{L^--1}\mathscr{S}(T,U)\,dT\,dU\,\sigma(x)\,dx.
$$

Performing the substitutions $U - T = u$, $U + T = t$ we then obtain with $L := L^+ + L^- - 2$ the formula

$$
\begin{aligned}
I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = 2^{-3-L}&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(u+\zeta)x/\varepsilon}\sigma(x)\\
&\times \int_{|u|}^\infty (t-u)^{L^+-1}(t+u)^{L^--1}\mathscr{S}\Big(\frac{t-u}{2},\frac{t+u}{2}\Big)\,dt\,dx\,du,
\end{aligned}
$$

where the t-integrals correspond to integrals over the level sets $\{(T,U) \mid U - T = u\} \subset \mathbb{R}^2_+$. The critical set of the linearized phase function[4] now consists of the single point $(u,x) = (-\zeta, 0)$, but a stationary phase analysis is not possible since the amplitude is not smooth at $u = 0$ due to the integral limit $|u|$. Instead, we carry out the Fourier transform on the Lie algebra, and split the u-integral at 0 in order to obtain smooth coefficients. Expanding in

[4]Note that the linearization of the phase function is not the result of a monomialization, so that no desingularization of the critical set as in [23] has taken place. In particular, as a consequence of a desingularization, an exceptional divisor would have to appear, which is not the case here.

addition the binomial expressions $(t \mp u)^{L^\pm - 1}$ and substituting $u \mapsto \varepsilon u - \zeta$, we arrive at

$$
\begin{aligned}
(3.4) \qquad I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = 2^{-3-L}\varepsilon \sum_{l=0}^{L} c_l \Bigg[& \int_{\zeta/\varepsilon}^{\infty} \hat{\sigma}(u)(\varepsilon u - \zeta)^l \\
& \times \int_{\varepsilon u - \zeta}^{\infty} t^{L-l} \mathscr{S}\Big(\frac{t - \varepsilon u + \zeta}{2}, \frac{t + \varepsilon u - \zeta}{2}\Big)\, dt\, du \\
& + \int_{-\infty}^{\zeta/\varepsilon} \hat{\sigma}(u)(\varepsilon u - \zeta)^l \\
& \times \int_{-\varepsilon u + \zeta}^{\infty} t^{L-l} \mathscr{S}\Big(\frac{t - \varepsilon u + \zeta}{2}, \frac{t + \varepsilon u - \zeta}{2}\Big)\, dt\, du \Bigg]
\end{aligned}
$$

with

$$
(3.5) \qquad c_l := \sum_{\substack{l^+ + l^- = l \\ 0 \le l^\pm \le L^\pm - 1}} (-1)^{l^+} \binom{L^+ - 1}{l^+} \binom{L^- - 1}{l^-}, \qquad l \in \mathbb{N}_0.
$$

In order to obtain an expansion in powers of ε, it is natural to Taylor expand the t-integral at $\varepsilon = 0$ which, by the following lemma, will result in a separation into singular and regular contributions.

Lemma 3.4. *For $N \in \mathbb{N}_0$, $\zeta \in \mathbb{R}$, and $\mathscr{S} \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^2)$, define two functions $F^\pm_{N,\zeta,\mathscr{S}} \in \mathrm{C}^\infty(\mathbb{R})$ by*

$$
F^\pm_{N,\zeta,\mathscr{S}}(v) := \int_{\pm(v-\zeta)}^{\infty} t^N \mathscr{S}\Big(\frac{t - v + \zeta}{2}, \frac{t + v - \zeta}{2}\Big)\, dt.
$$

Then, for $m \in \mathbb{N}_0$, the m-th derivative of $F^\pm_{N,\zeta,\mathscr{S}}$ is of the form

$$
\begin{aligned}
(F^\pm_{N,\zeta,\mathscr{S}})^{(m)}(v) = {} & \frac{1}{2^m} \int_{\pm(v-\zeta)}^{\infty} t^N (\partial_- - \partial_+)^m \mathscr{S}\Big(\frac{t - v + \zeta}{2}, \frac{t + v - \zeta}{2}\Big)\, dt \\
& + \sum_{i=0}^{m-1} (\mp 1)^{m+i} (\pm(v-\zeta))^{\max(0, N+1-m+i)} \\
& \times \sum_{p+q=i} C_{N,m,p,q} (\pm\partial_\mp)^p (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v - \zeta)_\mp),
\end{aligned}
$$

where the notation is as in Proposition 3.3, and the constants $C_{N,m,p,q} \in \mathbb{R}$ *satisfy*

$$\begin{aligned} &C_{N,m,m-1,0} = 1, \quad C_{N,N+1,0,0} = (-1)^N N!, \\ &C_{N,m,0,m-1} = 2^{1-m}, \quad C_{N,m,p,q} = 0 \quad \text{if } p+q < \max(0, m-1-N). \end{aligned} \tag{3.6}$$

Proof. For $m = 0$ the claim is trivially true; there are no constants $C_{N,0,p,q}$ because the sum over i is empty. For $m = 1$ we get

$$\begin{aligned} (F^{\pm}_{N,\zeta,\mathscr{S}})^{(1)}(v) &= \frac{d}{dv} \int_{\pm(v-\zeta)}^{\infty} t^N \mathscr{S}\Big(\frac{t-v+\zeta}{2}, \frac{t+v-\zeta}{2}\Big)\, dt \\ &= \frac{1}{2} \int_{\pm(v-\zeta)}^{\infty} t^N (\partial_- - \partial_+) \mathscr{S}\Big(\frac{t-v+\zeta}{2}, \frac{t+v-\zeta}{2}\Big)\, dt \\ &\quad \mp (\pm(v-\zeta))^N \mathscr{S}((\zeta - v, v - \zeta)_{\mp}), \end{aligned}$$

so that the claim holds. Assuming now that it holds for some $m \geq 1$, we obtain

$$\begin{aligned} &(F^{\pm}_{N,\zeta,\mathscr{S}})^{(m+1)}(v) \\ &= \frac{d}{dv}\Big(\frac{1}{2^m} \int_{\pm(v-\zeta)}^{\infty} t^N (\partial_- - \partial_+)^m \mathscr{S}\Big(\frac{t-v+\zeta}{2}, \frac{t+v-\zeta}{2}\Big)\, dt \\ &\qquad + \sum_{i=0}^{m-1} (\mp 1)^{m+i} (\pm(v-\zeta))^{\max(0,N+1-m+i)} \\ &\qquad\quad \times \sum_{p+q=i} C_{N,m,p,q} (\pm\partial_{\mp})^p (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v-\zeta)_{\mp})\Big) \\ &= \frac{1}{2^{m+1}} \int_{\pm(v-\zeta)}^{\infty} t^N (\partial_- - \partial_+)^{m+1} \mathscr{S}\Big(\frac{t-v+\zeta}{2}, \frac{t+v-\zeta}{2}\Big)\, dt \\ &\quad \mp \frac{1}{2^m} (\pm(v-\zeta))^N (\partial_- - \partial_+)^m \mathscr{S}((\zeta - v, v-\zeta)_{\mp}) \\ &\quad + \sum_{i=\max(0,m-N)}^{m-1} (\mp 1)^{m+i} (\pm 1)(N+1-m+i)(\pm(v-\zeta))^{\max(0,N+1-m+i)-1} \\ &\qquad \times \sum_{p+q=i} C_{N,m,p,q} (\pm\partial_{\mp})^p (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v-\zeta)_{\mp}) \\ &\quad + \sum_{i=0}^{m-1} (\mp 1)^{m+i} (\pm(v-\zeta))^{\max(0,N+1-m+i)} \\ &\qquad \times \sum_{p+q=i} C_{N,m,p,q} (\pm\partial_{\mp})^{p+1} (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v-\zeta)_{\mp}). \end{aligned}$$

Taking into account that $\max(0, N+1-m+i) \geq 1$ in the summand of $\sum_{i=\max(0,m-N)}^{m-1}$ and performing the substitutions $i \mapsto i+1$, $p \mapsto p-1$ in the final sums, the expression for $(F^{\pm}_{N,\zeta,\mathscr{S}})^{(m+1)}(v)$ becomes

$$\begin{aligned}
&\frac{1}{2^{m+1}} \int_{\pm(v-\zeta)}^{\infty} t^N (\partial_- - \partial_+)^{m+1} \mathscr{S}\Big(\frac{t-v+\zeta}{2}, \frac{t+v-\zeta}{2}\Big)\, dt \\
&\quad \mp \frac{1}{2^m} (\pm(v-\zeta))^N (\partial_- - \partial_+)^m \mathscr{S}((\zeta - v, v-\zeta)_{\mp}) \\
&\quad - \sum_{i=\max(0,m-N)}^{m-1} (\mp 1)^{m+1+i} (N+1-m+i)(\pm(v-\zeta))^{\max(0,N+1-m-1+i)} \\
&\quad \times \sum_{p+q=i} C_{N,m,p,q} (\pm\partial_{\mp})^p (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v-\zeta)_{\mp}) \\
&\quad + \sum_{i=1}^{m} (\mp 1)^{m+1+i} (\pm(v-\zeta))^{\max(0,N+1-m-1+i)} \\
&\quad \times \sum_{\substack{p+q=i\\ p\geq 1}} C_{N,m,p-1,q} (\pm\partial_{\mp})^p (\partial_- - \partial_+)^q \mathscr{S}((\zeta - v, v-\zeta)_{\mp}).
\end{aligned}$$

This is of the claimed form with

$$C_{N,m+1,p,q} = \begin{cases}
2^{-m}, & p=0, q=m \\
C_{N,m,p-1,q}, & p+q=m,\ p\geq 1, \\
C_{N,m,p-1,q} - (N+1-m+p+q) C_{N,m,p,q}, & \\
\qquad m-1 \geq p+q \geq \max(0, m-N),\ p \geq 1, & \\
-(N+1-m+q) C_{N,m,0,q}, & \\
\qquad m-1 \geq q \geq \max(0, m-N),\ p=0, & \\
C_{N,m,p-1,q}, & \\
\qquad 1 \leq p+q \leq \max(0, m-N) - 1,\ p \geq 1, & \\
0, \text{else}. &
\end{cases}$$

The first two lines prove the two equations on the left in (3.6) since $C_{N,m,m-1,q} = 1$ by the induction hypothesis. In the only case above where $p+q < \max(0, m-N)$, we have $C_{N,m+1,p,q} = C_{N,m,p-1,q}$, which is zero by the induction hypothesis because $p-1+q < \max(0, m-1-N)$. Finally, inspecting the case $q=p=0$ yields $C_{N,m+1,0,0} = -(N+1-m) C_{N,m,0,0}$ for $1 \leq m \leq N$, $C_{N,1,0,0} = 1$, which gives us by iteration the desired formula $C_{N,N+1,0,0} = (-1)^N N!$, so that (3.6) is fully verified. $\square$

As an immediate consequence, we get

Corollary 3.5. *In the situation of Lemma 3.4, there are differential operators $D^{\pm}_{N,m}$ of order m on $\mathbb{R}^2$ such that*

$$|(F^{\pm}_{N,\zeta,\mathscr{S}})^{(m)}(v)| \leq (1+|v-\zeta|^N)\big\|D^{\pm}_{N,m}\mathscr{S}\big\|_\infty \qquad \forall\, v,\zeta \in \mathbb{R}. \tag{3.7}$$

□

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. We perform in (3.4) for each u a Taylor expansion with Lagrange remainder of the functions $\varepsilon \to F^{\pm}_{N,\zeta,\mathscr{S}}(\varepsilon u)$ at $\varepsilon = 0$, where $N = L - l$. With Corollary 3.5 this yields for arbitrary Taylor cutoff orders $M^+, M^- \in \mathbb{N}_0$

(3.8)
$$\begin{aligned}
I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = 2^{-3-L}\varepsilon \sum_{l=0}^{L} c_l \Bigg[&\int_{\zeta/\varepsilon}^{\infty} \hat{\sigma}(u)(\varepsilon u-\zeta)^l \\
&\times \left(\sum_{m^+=0}^{M^+} \frac{(\varepsilon u)^{m^+}(F^{+}_{L-l,\zeta,\mathscr{S}})^{(m^+)}(0)}{m^+!} \right) du + R^{+}_{\mathscr{S},\sigma,l,M^+}(\zeta,\varepsilon) \\
&+ \int_{-\infty}^{\zeta/\varepsilon} \hat{\sigma}(u)(\varepsilon u-\zeta)^l \left(\sum_{m^-=0}^{M^-} \frac{(\varepsilon u)^{m^-}(F^{-}_{L-l,\zeta,\mathscr{S}})^{(m^-)}(0)}{m^-!} \right) du \\
&+ R^{-}_{\mathscr{S},\sigma,l,M^-}(\zeta,\varepsilon) \Bigg],
\end{aligned}$$

where

$$\begin{aligned}
|R^{\pm}_{\mathscr{S},\sigma,l,M^{\pm}}(\zeta,\varepsilon)| &\leq \varepsilon^{M^{\pm}+1} \frac{\big\|D^{\pm}_{L-l,M^{\pm}+1}\mathscr{S}\big\|_\infty}{(M^{\pm}+1)!} \\
&\qquad \times \int_{\mathbb{R}} |\hat{\sigma}(u)||\varepsilon u-\zeta|^l |u|^{M^{\pm}+1}\big[1+\sup_{|t|\leq 1}|t\varepsilon u-\zeta|^{L-l}\big]\, du \\
&= \mathcal{O}_{M^{\pm}}\Bigg(\varepsilon^{M^{\pm}+1}(1+|\zeta|^L)\big\|D^{\pm}_{L-l,M^{\pm}+1}\mathscr{S}\big\|_\infty \\
&\qquad \times \sum_{r=0}^{M^{\pm}+L+1} \int_{\mathbb{R}} |\hat{\sigma}(u)|(1+|u|)^r\, du \Bigg).
\end{aligned}$$

Plugging in the derivatives at 0 from Lemma 3.4, choosing $M^+ = M^- = M$, and expanding the binomial expression $(\varepsilon u - \zeta)^l$, we arrive after some further basic manipulations at the formula

$$\begin{aligned}
I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = {} & 2^{-3-L}\varepsilon \sum_{j=0}^{M} \varepsilon^j \sum_{l=0}^{L} (-1)^l c_l \sum_{k=0}^{\min(j,l)} \binom{l}{k} \frac{(-1)^k}{(j-k)!} \\
& \times \Bigg[\int_{\zeta/\varepsilon}^{\infty} \hat{\sigma}(u) u^j \, du \Bigg(\frac{\zeta^{l-k}}{2^{j-k}} \int_{-\zeta}^{\infty} t^{L-l} (\partial_- - \partial_+)^{j-k} \mathscr{S}\Big(\frac{t+\zeta}{2}, \frac{t-\zeta}{2}\Big) \, dt \\
& + \sum_{m=0}^{j-k-1} (-1)^{m+1} (-\zeta)^{\max(l,L-m)-k} \\
& \times \sum_{p+q=j-k-m-1} C_{L-l,j-k,p,q} \, \partial_-^p (\partial_- - \partial_+)^q \mathscr{S}(0,-\zeta) \Bigg) \\
& + \int_{-\infty}^{\zeta/\varepsilon} \hat{\sigma}(u) u^j \, du \Bigg(\frac{\zeta^{l-k}}{2^{j-k}} \int_{\zeta}^{\infty} t^{L-l} (\partial_- - \partial_+)^{j-k} \mathscr{S}\Big(\frac{t+\zeta}{2}, \frac{t-\zeta}{2}\Big) \, dt \\
& + \sum_{m=0}^{j-k-1} \zeta^{\max(l,L-m)-k} \sum_{p+q=j-k-m-1} \\
& C_{L-l,j-k,p,q} \, (-\partial_+)^p (\partial_- - \partial_+)^q \mathscr{S}(\zeta,0) \Bigg) \Bigg] \\
& + \mathcal{O}_M \Bigg(\varepsilon^{M+2} (1+|\zeta|^L) \sum_{l=0}^{2(M+1)} \sum_{r=0}^{M+L+1} \big\| D^l_M \mathscr{S} \big\|_\infty \\
& \times \int_{\mathbb{R}} |\hat{\sigma}(u)| (1+|u|)^r \, du \Bigg)
\end{aligned}$$

with a new family $\{D^l_M\}_l$ of differential operators defined by $D^+_{l,M+1}$ for $0 \le l \le M+1$ and by $D^-_{l-M-1,M+1}$ for $M+2 \le l \le 2M+2$. Next, we note that for $\zeta > 0$ and each $j, N \in \mathbb{N} \cup \{0\}$ one has

$$\begin{aligned}
\int_{-\infty}^{\zeta/\varepsilon} \hat{\sigma}(u) u^j \, du &= \int_{-\infty}^{\infty} \hat{\sigma}(u) u^j \, du - \int_{\zeta/\varepsilon}^{\infty} \hat{\sigma}(u) u^j \, du \\
&= 2\pi (-i)^j \sigma^{(j)}(0) - \int_{\zeta/\varepsilon}^{\infty} \hat{\sigma}(u) u^j \, du,
\end{aligned}$$

$$\Big|\int_{\zeta/\varepsilon}^{\infty}\hat\sigma(u)u^j\,du\Big| = \Big|\int_{\zeta/\varepsilon}^{\infty}u^{-N}\hat\sigma(u)u^{j+N}\,du\Big|$$
$$\le \int_{\zeta/\varepsilon}^{\infty}|u^{-N}\hat\sigma(u)u^{j+N}|\,du \le \varepsilon^N\zeta^{-N}\int_{\mathbb{R}}|\hat\sigma(u)u^{j+N}|\,du,$$

and similarly for $\zeta < 0$. This allows us to replace $\int_{-\infty}^{\zeta/\varepsilon}\hat\sigma(u)u^j\,du$ by $2\pi(-i)^j\sigma^{(j)}(0)$ up to an error estimated by arbitrarily high powers of ε, at the cost of getting equally high negative powers of ζ. Together with the above estimates for the remainder of (3.8), we get the claimed remainder estimate.

To obtain the claimed form of the coefficients in the asymptotic expansion, we now substitute $k \mapsto l-k$, swap the sums over k and l, restrict the range of m using the vanishing relation in (3.6), and substitute $m \mapsto L-l-m+k$. This yields for $\pm\zeta > 0$

$$\begin{aligned} I^{\zeta}_{\mathscr{S},\sigma}(\varepsilon) = 2^{-2-L}\pi\varepsilon\sum_{j=0}^{M}\varepsilon^j(-i)^j\sigma^{(j)}(0)\sum_{k=0}^{L}(-1)^k\sum_{l=k}^{\min(k+j,L)}\binom{l}{k}\frac{c_l}{(j-l+k)!}\\ \times\Bigg(\frac{\zeta^k}{2^{j-l+k}}\int_{|\zeta|}^{\infty}t^{L-l}(\partial_- - \partial_+)^{j-l+k}\mathscr{S}\Big(\frac{t+\zeta}{2},\frac{t-\zeta}{2}\Big)\,dt\\ +\sum_{m=L-j+1}^{L-l+k}(\pm1)^{L-l-m+k+1}|\zeta|^{\max(k,m)}\\ \times\sum_{p+q=m+j-L-1}C_{L-l,j+k-l,p,q}\,(\mp\partial_\pm)^p(\partial_- - \partial_+)^q\mathscr{S}((|\zeta|,|\zeta|)_\pm)\Bigg)\end{aligned}$$

up to the remainder term, and for $\zeta = 0$

$$\begin{aligned} I^{0}_{\mathscr{S},\sigma}(\varepsilon) = 2^{-2-L}\pi\varepsilon\Bigg[\sum_{j=0}^{M}\varepsilon^j(-i)^j\sigma^{(j)}(0)\sum_{l=0}^{\min(j,L)}\frac{c_l}{2^{j-l}(j-l)!}\\ \times\int_0^{\infty}t^{L-l}(\partial_- - \partial_+)^{j-l}\mathscr{S}\Big(\frac{t}{2},\frac{t}{2}\Big)\,dt\\ +\sum_{j=L+1}^{M}\varepsilon^j(-i)^j\sum_{l=0}^{L}\frac{c_l}{(j-l)!}\Bigg(\sigma_-^{[j]}(0)\end{aligned} \tag{3.9}$$

$$\times \sum_{p+q=j-L-1} C_{L-l,j-l,p,q}\,(-\partial_+)^p(\partial_- - \partial_+)^q\mathscr{S}(0,0)$$

$$+ (-1)^{L-l+1}\sigma_+^{[j]}(0) \sum_{p+q=j-L-1} C_{L-l,j-l,p,q}\,\partial_-^p(\partial_- - \partial_+)^q\mathscr{S}(0,0)\Bigg)\Bigg]$$

up to the remainder term. The formula for $\pm\zeta > 0$ immediately leads us to define

$$c_{j,k,l} := 2^{-2-L}\pi(-i)^j(-1)^k\binom{l}{k}\frac{c_l}{2^{j-l+k}(j-l+k)!}.$$

In particular, we find $c_{0,0,0} = 2^{-2-L}\pi$ as claimed since $c_0 = 1$. Similarly, $c^{\pm}_{j,k,p,q}$ can be computed explicitly from the formula for $\pm\zeta > 0$ but in a much more complicated way than $c_{j,k,l}$ due to the presence of $\max(k,m)$ in the exponent of $|\zeta|$. However, we can read off $c^{\pm}_{j,0,p,q}$ from (3.9), obtaining

$$c^{\pm}_{j,0,p,q} := 2^{-2-L}\pi(-i)^j\sum_{l=0}^{L}(\mp 1)^{L-l+1}\frac{c_l C_{L-l,j-l,p,q}}{(j-l)!}, \qquad j \geq L+1.$$

Finally, for the computation of $c_{L+1,0,0,0}$ we use that $C_{L-l,L+1-l,0,0} = (-1)^{L-l}(L-l)!$ by (3.6). $\square$

3.3. Contributions of the definite fixed point set components

It remains to study the less difficult case of a fixed point set component $F \in \mathcal{F}$ for which Q_F is definite, so that one either has $n_F^+ = \operatorname{codim} F$, $n_F^- = 0$ or $n_F^- = \operatorname{codim} F$, $n_F^+ = 0$. The spherical mean S_F from (2.32) is then an even function of only one variable that we write in the form

$$S_F(r) = \mathscr{S}_F(r^2) \qquad r \in \mathbb{R},$$

with $\mathscr{S}_F \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$ by applying again Whitney's classical result [25]. This reduces the study of the integrals (2.31) to the general study of integrals of the form

(3.10)
$$I^{\pm,\zeta}_{\mathscr{S},\sigma}(\varepsilon) := \int_{\mathbb{R}}\int_0^\infty e^{i(\pm r^2-\zeta)x/\varepsilon} r^{2L-1}\mathscr{S}(r^2)\,dr\,\sigma(x)\,dx, \qquad \varepsilon > 0,\ \zeta \in \mathbb{R},$$

where $L \geq 1$ is a natural number and $\mathscr{S} \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$, $\sigma \in \mathcal{S}(\mathbb{R})$ are functions.

Proposition 3.6. *If $\pm\zeta > 0$, then one has for each $M \in \mathbb{N}_0$, $M \geq L-1$, the asymptotic estimates*

$$\begin{aligned} I^{\pm,\zeta}_{\mathscr{S},\sigma}(\varepsilon) &= \varepsilon \sum_{k=0}^{L-1} \zeta^k \sum_{j=L-1-k}^{M-k} \varepsilon^j \sigma^{(j)}(0)(\pm 1)^{j+k} c_{j,k} \mathscr{S}^{(j+k+1-L)}(\pm\zeta) \\ &\quad + \mathcal{O}_M\Bigg(\varepsilon^{M+2}(1+|\zeta|^{L-1}+|\zeta|^{-M-1}) \\ &\qquad \times \sum_{r=0}^{M+L+1} \Big(\big\| \mathscr{S}^{(r)} \big\|_\infty + \int_{\mathbb{R}} |\hat{\sigma}(u)|(1+|u|)^r \, du \Big)\Bigg), \\ I^{\mp,\zeta}_{\mathscr{S},\sigma}(\varepsilon) &= \mathcal{O}_M\Bigg(\varepsilon^{M+2}(1+|\zeta|^{L-1}+|\zeta|^{-M-1}) \\ &\qquad \times \sum_{r=0}^{M+L+1} \Big(\big\| \mathscr{S}^{(r)} \big\|_\infty + \int_{\mathbb{R}} |\hat{\sigma}(u)|(1+|u|)^r \, du \Big)\Bigg), \end{aligned}$$

where

$$c_{j,k} = \pi \binom{L-1}{k} \frac{i^j}{(j+k+1-L)!}.$$

If $\zeta = 0$, then one has for each $M \in \mathbb{N}_0$, $M \geq L-1$, the asymptotic estimates

$$\begin{aligned} I^{\pm,0}_{\mathscr{S},\sigma}(\varepsilon) &= \varepsilon \sum_{j=L-1}^{M} \varepsilon^j \sigma^{[j]}_{\mp}(0)(\pm 1)^j c_{j,0} \mathscr{S}^{(j+1-L)}(0) \\ &\quad + \mathcal{O}_M\Bigg(\varepsilon^{M+2} \sum_{r=0}^{M+L+1} \Big(\big\| \mathscr{S}^{(r)} \big\|_\infty + \int_{\mathbb{R}} |\hat{\sigma}(u)|(1+|u|)^r \, du \Big)\Bigg). \end{aligned}$$

Proof. After substituting $r^2 = u$, $u \mapsto \varepsilon u \pm \zeta$, we can write

$$I^{\pm,\zeta}_{\mathscr{S},\sigma}(\varepsilon) = \frac{\varepsilon}{2} \sum_{j=0}^{L-1} \varepsilon^j \binom{L-1}{j} (\pm\zeta)^{L-1-j} \int_{\mp\zeta/\varepsilon}^{\infty} u^j \mathscr{S}(\varepsilon u \pm \zeta) \hat{\sigma}(\mp u) \, du.$$

Performing a Taylor expansion of the function $x \mapsto \mathscr{S}(x \pm \zeta)$ at $x = 0$ yields, similarly as in (3.8), for every $M \in \mathbb{N}_0$

$$\begin{aligned} I^{\pm,\zeta}_{\mathscr{S},\sigma}(\varepsilon) = \frac{\varepsilon}{2} \sum_{m=0}^{L+M-1} \varepsilon^m \int_{\mp\zeta/\varepsilon}^{\infty} u^m \hat{\sigma}(\mp u)\, du \\ \times \sum_{j=\max(0,m-M)}^{\min(m,L-1)} \binom{L-1}{j} (\pm\zeta)^{L-1-j} \frac{\mathscr{S}^{(m-j)}(\pm\zeta)}{(m-j)!} \\ + \mathcal{O}_M\Bigg(\varepsilon^{L+M+1}(1+|\zeta|^{L-1}) \\ \times \sum_{r=0}^{M+L} \Big(\big\|\mathscr{S}^{(r)}\big\|_\infty + \int_{\mathbb{R}} |\hat{\sigma}(u)|(1+|u|)^r\, du \Big)\Bigg). \end{aligned}$$

We can now finish the proof analogously to the proof of Proposition 3.3. □

4. Geometric interpretation of the coefficients

We shall now interpret the coefficients obtained in the previous section geometrically for $\zeta = 0$.

4.1. Local geometric interpretation

As a first step, we specialize to the situation that in Propositions 3.3 and 3.6 the function $\mathscr{S} = \mathscr{S}_f$ is given in terms of a spherical mean value of an arbitrary function $f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{\operatorname{codim} F})$. Turning first to the indefinite case we thus write

(4.1)
$$S_f(r,s) := \int_{S^{n^+_F-1}} \int_{S^{n^-_F-1}} f(r\theta^+, s\theta^-)\, d\theta^+\, d\theta^- =: \mathscr{S}_f(r^2, s^2), \qquad r,s \in \mathbb{R}.$$

In order to interpret the coefficients in Proposition 3.3 for such functions $\mathscr{S} = \mathscr{S}_f$, we first observe the following fundamental relation between derivatives of S_f and $\mathscr{S}_f$. For each $k \in \mathbb{N}_0$, one has

$$\begin{aligned} \partial^k_+ \mathscr{S}_f(t,u) &= \delta^k_r S_f(\sqrt{t}, \sqrt{u}) \qquad \forall\, (t,u) \in (0,\infty)\times[0,\infty), \\ \partial^k_- \mathscr{S}_f(t,u) &= \delta^k_s S_f(\sqrt{t}, \sqrt{u}) \qquad \forall\, (t,u) \in [0,\infty)\times(0,\infty), \end{aligned} \tag{4.2}$$

where, as in Proposition 3.3, $\partial_+\mathscr{S}_f$ and $\partial_-\mathscr{S}$ are the partial derivatives of $\mathscr{S}_f$ with respect to the first and second variable, respectively, and $\delta^k_r = (\delta_r)^k$,

$\delta_s^k = (\delta_s)^k$ are powers of the operators

$$\delta_r S_f(r,s) := \frac{1}{2r}\frac{\partial S_f}{\partial r}(r,s) \quad \forall\, r>0, s\geq 0,$$
$$\delta_s S_f(r,s) := \frac{1}{2s}\frac{\partial S_f}{\partial s}(r,s) \quad \forall\, r\geq 0, s>0.$$

Definition 4.1. *For $\zeta \in \mathbb{R}$ we introduce the full, pointed, and slit quadrics*

$$\Sigma^\zeta := \big\{w \in \mathbb{R}^{\operatorname{codim} F} : \langle Q_F w, w\rangle - 2\zeta = 0\big\},$$
$$\Sigma^\zeta_\bullet := \big\{w \in \mathbb{R}^{\operatorname{codim} F}\setminus\{0\} : \langle Q_F w, w\rangle - 2\zeta = 0\big\} \subset \Sigma^\zeta,$$
$$\Sigma^\zeta_\times := \big\{w \in \mathbb{R}^{n_F^+}_\bullet \times \mathbb{R}^{n_F^-}_\bullet : \langle Q_F w, w\rangle - 2\zeta = 0\big\} \subset \Sigma^\zeta_\bullet,$$

with the notation $\mathbb{R}^{n_F^\pm}_\bullet := \mathbb{R}^{n_F^\pm}\setminus\{0\}$.

Note that

$$\Sigma^\zeta_\bullet = \Sigma^\zeta_\times \iff \zeta = 0, \qquad \Sigma^\zeta_\bullet = \Sigma^\zeta \iff \zeta \neq 0. \tag{4.3}$$

The quadric Σ^ζ is the local model for the level set $\mathcal{J}^{-1}(\zeta+\mathcal{J}(F))$ near F, with $\zeta=0$ corresponding to the level set of $\mathcal{J}(F)$. Expressing the coefficients in Proposition 3.3 in terms of objects living on Σ^ζ will be the first step towards an intrinsic geometric interpretation of the former. The pointed quadric $\Sigma^\zeta_\bullet$ corresponds to the top stratum of $\mathcal{J}^{-1}(\zeta+\mathcal{J}(F))$, which is reflected by the second relation in (4.3). Finally, the slit quadric $\Sigma^\zeta_\times$ consists of all points in Σ^ζ which can be described by the inertial polar coordinates introduced in Section 2.3.

There is a natural hypersurface measure on $\Sigma^\zeta_\bullet$ induced by the symplectic measure dw on $\mathbb{R}^{\operatorname{codim} F}$, defined by the volume form $d\Sigma^\zeta := \Theta_F|_{\Sigma^\zeta_\bullet}$, where the $(\operatorname{codim} F - 1)$-form Θ_F on $\mathbb{R}^{\operatorname{codim} F}\setminus\{0\}$ is characterized uniquely near $\Sigma^\zeta_\bullet$ by the relations

$$\begin{aligned} &dw = \Theta_F \wedge dq_F,\\ &\Theta_F|_w \in \Lambda^{\operatorname{codim} F-1}T^*_w\Sigma^\zeta_\bullet \subset \Lambda^{\operatorname{codim} F-1}T^*_w\mathbb{R}^{\operatorname{codim} F} \quad \forall\, w\in\Sigma^\zeta_\bullet, \end{aligned}\tag{4.4}$$

where we wrote $q_F(w) := \frac{1}{2}\langle Q_F w, w\rangle$. Let $T_F : \mathbb{R}^{\operatorname{codim} F}\to\mathbb{R}^{\operatorname{codim} F}$ be the isomorphism

$$\begin{aligned} w &= (w_1,\dots,w_{\operatorname{codim} F})\\ &\longmapsto \left(\frac{w_1}{|\lambda_1^F|^{\frac12}}, \frac{w_2}{|\lambda_1^F|^{\frac12}}, \dots, \frac{w_{\operatorname{codim} F-1}}{|\lambda^F_{\operatorname{codim} F/2}|^{\frac12}}, \frac{w_{\operatorname{codim} F}}{|\lambda^F_{\operatorname{codim} F/2}|^{\frac12}}\right) = T_F(w). \end{aligned}\tag{4.5}$$

Pulling back both sides of (4.4) along T_F yields, with Λ_F as in (2.29), the equation

$$\Lambda_F^{-1}\, dw = T_F^* \, dw = T_F^*\Theta_F \wedge d(q_F \circ T_F). \tag{4.6}$$

We claim that, on the subset $\mathbb{R}_\bullet^{n_F^+} \times \mathbb{R}_\bullet^{n_F^-} \subset \mathbb{R}^{\operatorname{codim} F} \setminus \{0\}$, the form $T_F^*\Theta_F$ can be explicitly expressed in terms of the inertial polar coordinates introduced in Section 2.3 as follows:

$$\begin{aligned} &T_F^*\Theta_F|_{T_F^{-1}(r\theta^+, s\theta^-)} = \Lambda_F^{-1} r^{n_F^+ -1} s^{n_F^- -1}\, d\theta^+ \wedge d\theta^- \wedge \frac{r\, ds + s\, dr}{r^2+s^2}, \\ &(\theta^+, \theta^-) \in S^{n_F^+ -1} \times S^{n_F^- -1}. \end{aligned} \tag{4.7}$$

This follows from (4.6), the equation $(q_F \circ T_F)(r\theta^+, s\theta^-) = \frac{1}{2}(r^2 - s^2)$, and the equations

$$\begin{aligned} d(q_F \circ T_F) \wedge \frac{r\, ds + s\, dr}{r^2+s^2} &= (r\, dr - s\, ds) \wedge \frac{r\, ds + s\, dr}{r^2+s^2} = dr \wedge ds, \\ dw &= r^{n_F^+ -1} s^{n_F^- -1}\, dr \wedge d\theta^+ \wedge ds \wedge d\theta^-. \end{aligned}$$

Since $n_F^+ \geq 2$, $n_F^- \geq 2$, and $T_F^* d\Sigma^\zeta = T_F^*\Theta_F|_{T_F^{-1}(\Sigma_\bullet^\zeta)}$, we deduce from (4.7) that the measure $d\Sigma^\zeta$ is locally finite. Observe that $\Sigma_\times^\zeta$ is of full measure in $\Sigma_\bullet^\zeta$. Thus, we get for every $f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^{\operatorname{codim} F})$

$$\begin{aligned} \int_{\Sigma_\bullet^\zeta} f\, d\Sigma^\zeta &= \int_{T_F^{-1}(\Sigma_\times^\zeta)} f \circ T_F \; T_F^* d\Sigma^\zeta \\ &= \Lambda_F^{-1} \int_{\{r^2 - s^2 = 2\zeta,\, r,s>0\}} \frac{r^{n_F^+} s^{n_F^- -1} S_{f\circ T_F}(r,s)\, ds + r^{n_F^+ -1} s^{n_F^-} S_{f\circ T_F}(r,s)\, dr}{r^2+s^2}. \end{aligned} \tag{4.8}$$

Note that on the integration domain in the second line we have the relation $rdr = sds$.

Next, consider for $k \in \mathbb{N}_0$ the function $W_{F,k} \in \mathrm{C}^\infty(\mathbb{R}_\bullet^{n_F^+} \times \mathbb{R}_\bullet^{n_F^-})$ defined by

$$W_{F,k}(w) := 4\Lambda_F \left\| T_F^{-1} w \right\|^{2k} \left\| T_F^{-1} w^+ \right\|^{2-n_F^+} \left\| T_F^{-1} w^- \right\|^{2-n_F^-}, \tag{4.9}$$

where we use the notation $w = (w^+, w^-)$ for an element in $\mathbb{R}^{n_F^+} \times \mathbb{R}^{n_F^-} = \mathbb{R}^{\operatorname{codim} F}$. Furthermore, we define a differential operator $D_F^+ : \mathrm{C}^\infty(\mathbb{R}_\bullet^{n_F^+} \times$

$\mathbb{R}^{n_F^-}) \to \mathrm{C}^\infty(\mathbb{R}_\bullet^{n_F^+} \times \mathbb{R}^{n_F^-})$ and a differential operator $D_F^- : \mathrm{C}^\infty(\mathbb{R}^{n_F^+} \times \mathbb{R}_\bullet^{n_F^-}) \to \mathrm{C}^\infty(\mathbb{R}^{n_F^+} \times \mathbb{R}_\bullet^{n_F^-})$ by

$$D_F^\pm(f)(w) := \frac{1}{2}\Big\langle \nabla f(w), \frac{w^\pm}{\|T_F^{-1}w^\pm\|^2}\Big\rangle, \tag{4.10}$$

where $\nabla f(w) \in \mathbb{R}^{\operatorname{codim} F}$ is the Euclidean gradient of f at w and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^{\operatorname{codim} F}$. The significance of these operators lies in the following observation, which we shall in fact only use for $\zeta = 0$, when the pointed and slit quadrics agree.

Proposition 4.1. *Let $f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^{\operatorname{codim} F})$ and let $f_\times$ be the restriction of f to $\mathbb{R}_\bullet^{n_F^+} \times \mathbb{R}_\bullet^{n_F^-}$. For each $k, l \in \mathbb{N}_0$, the function $W_{F,k}(D_F^+ - D_F^-)^l f_\times$ is integrable over $\Sigma_\times^\zeta$ with respect to $d\Sigma^\zeta$ and one has*

$$\int_{\Sigma_\times^\zeta} W_{F,k}(D_F^- - D_F^+)^l f_\times \, d\Sigma^\zeta = \int_{|2\zeta|}^\infty t^k(\partial_- - \partial_+)^l \mathscr{S}_{f\circ T_F}\left(\frac{t+2\zeta}{2}, \frac{t-2\zeta}{2}\right) dt.$$

Consequently, the integrals on the right hand side have a geometric meaning in the sense that they correspond to integrals over $\Sigma_\times^\zeta$ with respect to the natural hypersurface measure $d\Sigma^\zeta$.

Proof. Let us first assume $\zeta \geq 0$ and introduce the short-hand notation $f_{k,l} := W_{F,k}(D_F^- - D_F^+)^l f_\times$ with the function $W_{F,k}$ from (4.9) and the operators $D_F^\pm$ from (4.10). The spherical mean $S_{f_{k,l}\circ T_F}(r,s)$ is well-defined for $r, s > 0$ and can be expressed in terms of $S_{f\circ T_F}(r,s)$ according to

$$S_{f_{k,l}\circ T_F}(r,s) = 4\Lambda_F \frac{(r^2+s^2)^k}{r^{n_F^+-2}\, s^{n_F^- -2}}\Big(\frac{1}{2s}\frac{\partial}{\partial s} - \frac{1}{2r}\frac{\partial}{\partial r}\Big)^l S_{f\circ T_F}(r,s), \tag{4.11}$$

as can be seen by writing w and $w^\pm$ in terms of polar coordinates in the definitions of $W_{F,k}$ and $D_F^\pm$. Using on $\Sigma_\times^\zeta$ the relations $r = \sqrt{s^2+2\zeta}$ and $rdr = sds$, we write with (4.8)

$$\int_{\Sigma_\times^\zeta} f_{k,l}\, d\Sigma^\zeta$$
$$= \frac{1}{\Lambda_F}\int_0^\infty \frac{(s^2+2\zeta)^{n_F^+/2} s^{n_F^- -1} S_{f_{k,l}\circ T_F}\left(\sqrt{s^2+2\zeta}, s\right) ds + (s^2+2\zeta)^{n_F^+/2-1} s^{n_F^- +1} S_{f_{k,l}\circ T_F}\left(\sqrt{s^2+2\zeta}, s\right) ds}{2s^2+2\zeta}$$

and then compute with (4.11)

$$
\begin{aligned}
\int_{\Sigma^\zeta_\times} f_{k,l}\, d\Sigma^\zeta &= \frac{1}{\Lambda_F}\int_0^\infty (s^2+2\zeta)^{n_F^+/2-1} s^{n_F^- -1} S_{f_{k,l}\circ T_F}\big(\sqrt{s^2+2\zeta}, s\big)\, ds \\
&= 4\int_0^\infty s(2s^2+2\zeta)^k\Big(\frac{1}{2s}\partial_s - \frac{1}{2\sqrt{s^2+2\zeta}}\partial_r\Big)^l S_{f\circ T_F}\big(\sqrt{s^2+2\zeta}, s\big)\, ds \\
&= 2\int_0^\infty (2t+2\zeta)^k\Big(\frac{1}{2\sqrt{t}}\partial_s - \frac{1}{2\sqrt{t+2\zeta}}\partial_r\Big)^l S_{f\circ T_F}\big(\sqrt{t+2\zeta}, \sqrt{t}\big)\, dt \\
&= \int_{2\zeta}^\infty t^k\left(\frac{1}{2\sqrt{\frac{t-2\zeta}{2}}}\partial_s - \frac{1}{2\sqrt{\frac{t+2\zeta}{2}}}\partial_r\right)^l S_{f\circ T_F}\left(\sqrt{\frac{t+2\zeta}{2}}, \sqrt{\frac{t-2\zeta}{2}}\right) dt \\
&= \int_{2\zeta}^\infty t^k(\partial_- - \partial_+)^l \mathscr{S}_{f\circ T_F}\left(\frac{t+2\zeta}{2}, \frac{t-2\zeta}{2}\right) dt.
\end{aligned}
$$

Here we substituted $t := s^2$ and applied (4.2) in the final step. The integrability claim follows since the right hand side is a finite integral. The case $\zeta \le 0$ is treated analogously by writing $s = \sqrt{r^2 - 2\zeta}$. $\square$

A further important observation is the following

Proposition 4.2. *For all $k \in \mathbb{N}_0$ and $f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{\operatorname{codim} F})$, one has*

$$\partial_\pm^k \mathscr{S}_{f\circ T_F}(0,0) = (2n_F^\pm)^{-k}(\operatorname{vol} S^{n_F^+-1})(\operatorname{vol} S^{n_F^- -1})(\Delta_\pm^{Q_F})^k f(0),$$

where $\operatorname{vol}(S^{n_F^\pm-1})$ is the volume of $S^{n_F^\pm-1}$ with respect to the standard round measure and the second order differential operator $\Delta_\pm^{Q_F} : \mathrm{C}^\infty(\mathbb{R}^{\operatorname{codim} F}) \to \mathrm{C}^\infty(\mathbb{R}^{\operatorname{codim} F})$ is defined by

$$\Delta_\pm^{Q_F}(f)(w) := \operatorname{tr}(Q_F^\pm)^{-1}\mathrm{Hess}_\pm(f)(w), \qquad w \in \mathbb{R}^{\operatorname{codim} F}.$$

Here tr denotes the trace, $(Q_F^\pm)^{-1}$ is the restriction of Q_F^{-1} to the subspace $\mathbb{R}^{n_F^\pm} \subset \mathbb{R}^{\operatorname{codim} F}$ on which $\pm Q_F$ is positive, and $\mathrm{Hess}_\pm(f)(w)$ denotes the quadrant of the Euclidean Hessian matrix of f at w formed by all second derivatives with respect to the variables in $\mathbb{R}^{n_F^\pm}$.

Proof. For $k = 0$, the claim is true by (4.1). Assuming that it holds for some $k \in \mathbb{N}_0$ and also for $k = 0$, we have

$$\partial_\pm^{k+1}\mathscr{S}_{f\circ T_F}(0,0) = \partial_\pm \partial_\pm^k \mathscr{S}_{f\circ T_F}(0,0) = \partial_\pm \mathscr{S}_{(2n_F^\pm)^{-k}(\Delta_\pm^{Q_F})^k(f\circ T_F)}(0,0).$$

This reduces the proof to the case $k = 1$. To treat this case, we first recall that for any bilinear form B on any Euclidean space $\mathbb{R}^n$ one has

$$\int_{S^{n-1}} \langle B\theta, \theta \rangle \; d\theta = \frac{1}{n} \operatorname{vol} S^{n-1} \operatorname{tr} B, \tag{4.12}$$

as one verifies by diagonalizing B. For arbitrary $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{\operatorname{codim} F})$, we now compute using (4.2)

$$\begin{aligned}
\partial_+ \mathscr{S}_{f\circ T_F}(0,0) &= \lim_{\varepsilon\to 0^+} \partial_+ \mathscr{S}_{f\circ T_F}(\varepsilon^2, 0) = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \frac{\partial S_{f\circ T_F}}{\partial r}(\varepsilon, 0) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_{S^{n_F^+-1}} \int_{S^{n_F^--1}} \left\langle \nabla_+ f(\varepsilon T_F \theta^+, 0), T_F\theta^+ \right\rangle \, d\theta^+ \, d\theta^- \\
&= \lim_{\varepsilon\to 0^+} \frac{\operatorname{vol} S^{n_F^- -1}}{2\varepsilon} \Bigg(\int_{S^{n_F^+-1}} \left\langle \nabla_+ f(0,0), T_F\theta^+ \right\rangle \, d\theta^+ \\
&\qquad + \varepsilon \int_{S^{n_F^+-1}} \left\langle \mathrm{Hess}_+ f(0,0) T_F\theta^+, T_F\theta^+ \right\rangle \, d\theta^+ + \mathcal{O}(\varepsilon^2) \Bigg) \\
&= \frac{\operatorname{vol} S^{n_F^- -1}}{2} \int_{S^{n_F^+-1}} \left\langle T_F \mathrm{Hess}_+ f(0,0) T_F\theta^+, \theta^+ \right\rangle \, d\theta^+
\end{aligned}$$

because $\int_{S^{n_F^+-1}} \langle v, \theta^+ \rangle \, d\theta^+ = 0$ for each $v \in \mathbb{R}^{n_F^+}$ and T_F is self-adjoint. Applying (4.12) then yields

$$\partial_+ \mathscr{S}_{f\circ T_F}(0,0) = (2n_F^+)^{-1} (\operatorname{vol} S^{n_F^+-1})(\operatorname{vol} S^{n_F^- -1}) \operatorname{tr}(T_F^\pm \mathrm{Hess}_+ f(0,0) T_F^\pm),$$

where we put $T_F^\pm := T_F|_{\mathbb{R}^{n_F^\pm}} : \mathbb{R}^{n_F^\pm} \to \mathbb{R}^{n_F^\pm}$. To get the claimed relation, it now suffices to use the cyclic property of the trace and to observe that $(T_F^\pm)^2 = (Q_F^\pm)^{-1}$. The calculation for $\partial_- \mathscr{S}_f(0,0)$ is completely analogous. $\square$

To close this section, we briefly turn to the case of a definite quadratic form Q_F and write

$$S_f(r) := \int_{S^{\operatorname{codim} F-1}} f(r\theta) \, d\theta =: \mathscr{S}_f(r^2), \qquad r \in \mathbb{R}. \tag{4.13}$$

Proposition 4.3. *One has for all $k \in \mathbb{N}_0$ and $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{\operatorname{codim} F})$*

$$\mathscr{S}^{(k)}_{f\circ T_F}(0) = \frac{\operatorname{vol}(S^{\operatorname{codim} F-1})}{(2 \operatorname{codim} F)^k} (\Delta^{Q_F})^k f(0),$$

where $\operatorname{vol}(S^{\operatorname{codim} F-1})$ *is the volume of* $S^{\operatorname{codim} F-1}$ *with respect to the standard round measure and the second order differential operator* $\Delta^{Q_F} : \mathrm{C}^\infty(\mathbb{R}^{\operatorname{codim} F})$

$\to \mathrm{C}^\infty(\mathbb{R}^{\operatorname{codim} F})$ *is defined by*

$$\Delta^{Q_F}(f)(w) := \operatorname{tr} Q_F^{-1} \operatorname{Hess}(f)(w), \qquad w \in \mathbb{R}^{\operatorname{codim} F}.$$

Here tr *denotes the trace and* $\operatorname{Hess}(f)(w)$ *denotes the Euclidean Hessian of* f *at the point* w.

Proof. In view of the relations

$$\mathscr{S}_f^{(k)}(t) = \delta_r^k S_f(\sqrt{t}) \quad \forall\, t \in (0,\infty),\ k \in \mathbb{N}_0, \qquad \delta_r S_f(r) := \frac{1}{2r} S_f'(r), \quad r > 0,$$

the proof is completely analogous to the proof of Proposition 4.2. □

4.2. Global geometric interpretation

Let us now carry out the second step of our geometric interpretation by specializing from a general spherical mean S_f as in (4.1) and (4.13) to the particular spherical means S_F from (2.30) and (2.32) which involve our amplitude a, the local normal form symplectomorphism Φ_F from Proposition 2.6, and the cutoff function χ_F from our partition of unity. Our goal is to translate the expressions from Propositions 4.1, 4.2, and 4.3, which live in our local model of M near F, into expressions that live on subsets of the symplectic reduction $\mathscr{M}^\zeta$. The technical key ingredient to achieving our goal is the following

Lemma 4.4. *Fix* (U_F, Φ_F) *as in Proposition 2.6, let* $f \in \mathrm{C}_c(U_F)$*, and define the average*

$$\langle f \rangle_T (T \cdot p) := \int_T f(g \cdot p)\, dg, \qquad p \in M,$$

where dg *is the Haar measure on* T *fixed by our identification* $\mathfrak{t} \cong \mathbb{R}$*. Let* $\zeta \in \mathbb{R}$ *and put* $\zeta_F = \mathcal{J}(F) - \zeta$*.*

1) *Let* $d\Sigma^{\zeta_F}$ *be the hypersurface measure introduced in* (4.4) *on the pointed quadric* $\Sigma_\bullet^{\zeta_F}$*. Then, with the notation as in* (2.25)*, one has*

$$\int_{\mathscr{M}^\zeta_{\mathrm{top}}} \langle f \rangle_T \, d\mathscr{M}^\zeta_{\mathrm{top}} = \int_{P_F} \int_{\Sigma_\bullet^{\zeta_F}} f(\Phi_F^{-1}(\tilde{\pi}_F(\wp, w)))\, d\Sigma^{\zeta_F}(w)\, d\wp,$$

   where $d\mathscr{M}^\zeta_{\mathrm{top}} = (\omega^\zeta_{\mathrm{top}})^{n-1}/(n-1)!$ *denotes the symplectic volume form on the top stratum of* $\mathscr{M}^\zeta$*.*

2) Similarly, we have with $dF = \omega^{\dim F/2}/(\dim F/2)!$

$$\int_F f\; dF = \int_{P_F} f(\Phi_F^{-1}(\tilde{\pi}_F(\wp, 0)))\, d\wp.$$

Proof. We begin with Assertion (1). First, note that from (2.18) and (2.23) it is clear that

$$\Phi_F(\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}} \cap U_F) = \left\{[\wp, w] \in \Phi_F(U_F) \mid w \in \Sigma_\bullet^{\zeta_F} \setminus \{0\}\right\}.$$

By slight abuse of notation, let us write $q_F([\wp, w]) := \frac{1}{2}\langle Q_F w, w\rangle =: q_F(w)$, so that we deduce from (2.18) that

$$(\Phi_F^{-1})^* d\mathcal{J} = dq_F. \tag{4.14}$$

Furthermore, recall from (2.27) that we have the equality of smooth densities

$$(\Phi_F^{-1})^*(\, dM|_{U_F}) = \, d[\wp, w], \qquad |\tilde{\pi}_F^*(d[\wp, w]) \wedge \Pi_F^* \eta_F| = |d\wp \wedge \, dw|$$

and that by (4.4) we have $d\wp \wedge dw = d\wp \wedge \Theta_F \wedge \, dq_F$. On the other hand, the hypersurface Liouville measure $d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}$ on $\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}$ is characterized by the condition

$$dM|_p = \frac{1}{n!}\omega^n|_p = d\mathcal{J}|_p \wedge d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}|_p \in \Lambda^{2n}(T_p^* M) \tag{4.15}$$

for any $p \in \mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}$. Pulling back (4.15) along $\Phi_F^{-1} \circ \tilde{\pi}_F$ and taking (4.14) into account yields

$$|(\Phi_F^{-1} \circ \tilde{\pi}_F)^*(d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}) \wedge \Pi_F^* \eta_F| = |\, d\wp \wedge \Theta_F|.$$

Since $\Theta_F|_{\Sigma_\bullet^{\zeta_F}} = d\Sigma_\bullet^{\zeta_F}$ and $\int_{\pi_F^{-1}(\{p\})} \eta_F = 1$ for each $p \in F$, this proves that one has for each $f \in C_c(U_F)$

$$\int_{\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}} f\;\, d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}} = \int_{P_F} \int_{\Sigma_\bullet^{\zeta_F}} f(\Phi_F^{-1}(\tilde{\pi}_F(\wp, w)))\, d\Sigma_\bullet^{\zeta_F}(w)\, d\wp. \tag{4.16}$$

It remains to show that

$$\int_{\mathscr{M}_{\mathrm{top}}^\zeta} \langle f\rangle_T \; d\mathscr{M}_{\mathrm{top}}^\zeta = \int_{\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}} f\;\, d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}}. \tag{4.17}$$

To this end, recall that $\omega_{\mathrm{top}}^\zeta$ is characterized by $\pi^* \omega_{\mathrm{top}}^\zeta = i_\zeta^* \omega$, where $i_\zeta :$ $\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}} \to M$ is the inclusion and $\pi : \mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}} \to \mathscr{M}_{\mathrm{top}}^\zeta$ the orbit projection. Also notice that our identification of t with $\mathbb{R}$ corresponds to a choice of an element $x_0 \in \mathfrak{t}$ that is identified with 1, and leads to an identification of $\mathcal{J}$ with $J(x_0)$. On the top stratum $M_{(\mathfrak{h}_{\mathrm{top}})} \subset M$, the fundamental vector field $\tilde{x}_0$ is nowhere-vanishing, so it has a dual one-form ξ_0. One then computes on $M_{(\mathfrak{h}_{\mathrm{top}})}$

$$\iota_{\tilde{x}_0}(\xi_0 \wedge d\mathcal{J}) = d\mathcal{J} = dJ(x_0) = \iota_{\tilde{x}_0}\omega,$$

where the first equality uses the T-invariance of $\mathcal{J}$, the middle equality is the remark above, and the last equality is the defining property of the momentum map $\mathcal{J}$. Consequently,

$$(\xi_0 \wedge d\mathcal{J})|_{M_{(\mathfrak{h}_{\mathrm{top}})}} + \beta = \omega|_{M_{(\mathfrak{h}_{\mathrm{top}})}} \tag{4.18}$$

for some $\beta \in \Omega^2(M_{(\mathfrak{h}_{\mathrm{top}})})$ that fulfills $\iota_{\tilde{x}_0}\beta = 0$. Thus, on $M_{(\mathfrak{h}_{\mathrm{top}})}$ we have

$$\begin{aligned}
\frac{1}{n!}\omega^n &= \frac{1}{n!}\omega^{n-1} \wedge \xi_0 \wedge d\mathcal{J} + \frac{1}{n!}\omega^{n-1} \wedge \beta \\
&= \frac{1}{n!}\omega^{n-1} \wedge \xi_0 \wedge d\mathcal{J} + \frac{1}{n!}\omega^{n-2} \wedge \xi_0 \wedge d\mathcal{J} \wedge \beta + \frac{1}{n!}\omega^{n-2} \wedge \beta^2 \\
&= \frac{1}{n!}\left(\sum_{j=1}^{n} \omega^{n-j} \wedge \beta^{j-1}\right) \wedge \xi_0 \wedge d\mathcal{J} + \frac{1}{n!}\beta^n
\end{aligned}$$

but $\beta^n = 0$ because β is degenerate. Now, if we insert $\beta = \omega - \xi_0 \wedge d\mathcal{J}$, then all non-zero powers of $\xi_0 \wedge d\mathcal{J}$ get killed by the wedge product with $\xi_0 \wedge d\mathcal{J}$, and we arrive at

$$\frac{1}{n!}\omega^n = \frac{1}{n!}\left(\sum_{j=1}^{n} \omega^{n-1}\right) \wedge \xi_0 \wedge d\mathcal{J} = \frac{1}{(n-1)!}\omega^{n-1} \wedge \xi_0 \wedge d\mathcal{J}.$$

Inserting this in (4.15) gives us

$$d\mathcal{J}^{-1}(\{\zeta\})_{\mathrm{top}} = \frac{1}{(n-1)!} i_\zeta^* \omega^{n-1} \wedge i_\zeta^*(\xi_0).$$

On the other hand, we compute

$$\pi^*\, d\mathscr{M}_{\mathrm{top}}^{\zeta} = \pi^*\big((\omega_{\mathrm{top}}^{\zeta})^{n-1}/(n-1)!\big) = \frac{1}{(n-1)!}\pi^*(\omega_{\mathrm{top}}^{\zeta})^{n-1} = \frac{1}{(n-1)!} i_\zeta^* \omega^{n-1},$$

so that we find

$$d\mathcal{J}^{-1}(\{\zeta\})_{\text{top}} = \pi^*(\,d\mathscr{M}^{\zeta}_{\text{top}}) \wedge i^*_{\zeta}(\xi_0),$$
$$\int_{\mathcal{J}^{-1}(\{\zeta\})_{\text{top}}} f \;\, d\mathcal{J}^{-1}(\{\zeta\})_{\text{top}} = \int_{\mathscr{M}^0_{\text{top}}} \left(\int_{T\cdot p} f\xi_0 \right) d\mathscr{M}^{\zeta}_{\text{top}}(T\cdot p).$$

We are left with comparing $\int_{T\cdot p} f\xi_0$ with $\langle f\rangle_T\,(T\cdot p)$ for any orbit $T\cdot p$, where $p\in\mathcal{J}^{-1}(\{\zeta\})_{\text{top}}$. The map $\Psi_p : S^1 \ni g \mapsto g\cdot p \in T\cdot p$ is an S^1-equivariant diffeomorphism, so $\Psi_p^*\xi_0$ is a Haar measure on S^1 and thus a constant multiple of dg. To determine the constant, we note that the derivative of Ψ_p at the identity fulfills $D\Psi_p|_1(x) = \tilde{x}|_p$, $x\in\mathfrak{t}=T_1S^1$. Consequently,

$$\Psi_p^*\xi_0|_1(x_0) = \xi_0(D\Psi_p|_1(x_0)) = \xi_0(\tilde{x}_0) = 1,$$

proving $\Psi_p^*\xi_0 = dg$. This finishes the proof of (4.17) and Assertion (1). Assertion (2) follows along the same arguments taking into account (2.19), (2.24), and the relation $\Phi_F(F) = \{[\wp, w]\in\Phi_F(U_F)\mid w=0\}$. □

Corollary 4.5. *Consider a spherical mean S_F as in* (2.30) *and* (2.32)*. Depending on whether the bilinear form Q_F is indefinite or definite, write*

$$S_F(r,s) =: \mathscr{S}_{\tilde{a}_F}(r^2,s^2) \qquad \textit{or} \qquad S_F(r) =: \mathscr{S}_{\tilde{a}_F}(r^2), \qquad r,s\in[0,\infty),$$

where $\mathscr{S}_{\tilde{a}_F}$ is a smooth, compactly supported function on $\mathbb{R}^2$ or $\mathbb{R}$ as in (4.1) *or* (4.13)*, respectively, associated with the function $\tilde{a}_F$ from* (2.30) *which is related to our amplitude $a\in \mathrm{C}^\infty_{\mathrm{c}}(M)$ by*

$$\tilde{a}_F(w) = \int_{P_F} (a\chi_F)(\Phi_F^{-1}(\tilde{\pi}_F(\wp, T_F w)))\,d\wp, \qquad w\in\mathbb{R}^{\operatorname{codim} F}.$$

1) *Suppose that Q_F is indefinite. Then, for each $k,l\in\mathbb{N}_0$, there are differential operators*

$$\mathscr{D}_{F,k,l} : \mathrm{C}^\infty(U_F\setminus F)\to \mathrm{C}^\infty(U_F\setminus F), \qquad \mathscr{Q}^{\pm}_{F,k} : \mathrm{C}^\infty(U_F)\to\mathrm{C}^\infty(U_F)$$

of orders k and $2k$, respectively, such that

$$\int_0^\infty t^l(\partial_- - \partial_+)^k \mathscr{S}_{\tilde{a}_F}\Big(\frac{t}{2},\frac{t}{2}\Big)\,dt = \int_{\mathscr{M}^{\mathcal{J}(F)}_{\text{top}}} \langle \mathscr{D}_{F,k,l}(\chi_F a)\rangle_T \; d\mathscr{M}^{\mathcal{J}(F)}_{\text{top}},$$
$$\partial^k_{\pm}\mathscr{S}_{\tilde{a}_F}(0,0) = \int_F \mathscr{Q}^{\pm}_{F,k} a\, dF. \tag{4.19}$$

2) *Suppose that Q_F is definite. Then, for each $k \in \mathbb{N}_0$, there is a differential operator $\mathscr{Q}_{F,k} : \mathrm{C}^\infty(U_F) \to \mathrm{C}^\infty(U_F)$ of order $2k$ such that*

$$\mathscr{S}^{(k)}_{\tilde{a}_F}(0) = \int_F \mathscr{Q}_{F,k} a \, dF. \tag{4.20}$$

Proof. Applying Lemma 4.4 and Propositions 4.1, 4.2, 4.3 yields the claim with integrands of the form

$$\langle \mathscr{D}_{F,k,l}(a\chi_F)\rangle_T\,, \quad \mathscr{Q}^{\pm}_{F,k}(a\chi_F), \quad \mathscr{Q}_{F,k}(a\chi_F),$$

where the operators $\mathscr{D}_{F,k,l}$, $\mathscr{Q}^{\pm}_{F,k}$, and $\mathscr{Q}_{F,k}$ are defined as follows: first, we extend the operators $D^{\pm}_F$, Δ^{Q_F}, $\Delta^{Q_F}_{\pm}$ and the functions $W_{F,k}$ from $V \subset \mathbb{R}^{\operatorname{codim} F}$ (here V is $\mathbb{R}^{\operatorname{codim} F}$ or $\mathbb{R}^{n_F^+}_{\bullet} \times \mathbb{R}^{n_F^-}_{\bullet}$) to the trivial bundle $P_F \times V$ by pulling back along the projection $P_F \times V \to V$. Then these extended operators and functions induce differential operators $\tilde{D}^{\pm}_F$, $\tilde{\Delta}^{Q_F}$, $\tilde{\Delta}^{Q_F}_{\pm}$ and functions $\tilde{W}_{F,k}$ on $P_F \times_{K_F} V$ by identifying smooth functions on $P_F \times_{K_F} V$ with K_F-invariant smooth functions on $P_F \times V$. By conjugating $\tilde{D}^-_F - \tilde{D}^+_F$ with Φ_F, taking the l-th power, and multiplying with $\tilde{W}_{F,k}$ and appropriate constants as prescribed by Proposition 3.3, we obtain the operator $\mathscr{D}_{F,k,l}$. Similarly, we define the operators $\mathscr{Q}_{F,k}$ and $\mathscr{Q}^{\pm}_{F,k}$ using $\tilde{\Delta}^{Q_F}$ and $\tilde{\Delta}^{Q_F}_{\pm}$. Finally, since $\chi_F \equiv 1$ near F, we can remove χ_F from the integrals over F. □

5. Proof of the main results

We are now ready to prove our main results, Theorems 1.1 and 1.2.

5.1. Proof of Theorem 1.1

Fix a given $\zeta \in \mathbb{R} \cong \mathfrak{t}^*$, and recall from (2.21) the equality

$$I^\zeta(\varepsilon) = I^\zeta_{\chi_{\mathrm{top}}}(\varepsilon) + \sum_{F \in \mathcal{F}} I^\zeta_{\chi_F}(\varepsilon) \tag{5.1}$$

which is equivalent to the statement (2.20). We shall derive the desired asymptotic expansion of $I^\zeta(\varepsilon)$ by deriving an asymptotic expansion of each summand in (5.1). Let us begin with the summands associated with fixed point set components $F \in \mathcal{F}$ satisfying $\mathcal{J}(F) = \zeta$. Fix such an F, put $j_F := \operatorname{codim} F/2 - 1$, and recall the notation (3.2) as well as the expression (2.28). Applying Corollary 4.5 to the statements of Propositions 3.3 and 3.6, taking $\mathscr{S} = \mathscr{S}_{\tilde{a}_F}$, $\zeta = 0$, 2ε as asymptotic parameter, and $L = \operatorname{codim} F/2 - 2$ (in

Proposition 3.3) or $L = \operatorname{codim} F/2$ (in Proposition 3.6) there, we obtain the following results:

1) Suppose that F is indefinite. Then, there are differential operators $D^{\mathrm{top}}_{j,F} : \mathrm{C}^\infty(U_F \setminus F) \to \mathrm{C}^\infty(U_F \setminus F)$, $j \in \mathbb{N}_0$, and $D^{\pm}_{j,F} : \mathrm{C}^\infty(U_F) \to \mathrm{C}^\infty(U_F)$, $j \geq j_F$, of orders j and $2(j+1) - \operatorname{codim} F$ respectively, such that one has

$$I^{\mathcal{J}(F)}_{\chi_F}(\varepsilon) \sim \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A^{\mathcal{J}(F)}_{j,F}, \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$$

with $A^{\mathcal{J}(F)}_{j,F} = A'_{j,F} + A_{j,F}$ given by

$$A'_{j,F}(a \otimes \sigma) = \sigma^{(j)}(0) \int_{\mathscr{M}^{\mathcal{J}(F)}_{\mathrm{top}}} \left\langle D^{\mathrm{top}}_{j,F}(\chi_F a) \right\rangle_T d\mathscr{M}^{\mathcal{J}(F)}_{\mathrm{top}},$$

$$A_{j,F}(a \otimes \sigma) = \begin{cases} 0, & j < j_F, \\ \sigma^{[k]}_{+}(0) \int_F D^{+}_{j,F} a \, dF + \sigma^{[k]}_{-}(0) \int_F D^{-}_{j,F} a \, dF, & j \geq j_F. \end{cases}$$

2) Suppose that F is definite, $s_F = \pm$. Then, there are differential operators $D_{j,F} : \mathrm{C}^\infty(U_F) \to \mathrm{C}^\infty(U_F)$, $j \geq \frac{1}{2}\operatorname{codim} F - 1$, of orders $2j + 2 - \operatorname{codim} F$ such that one has

$$I^{\mathcal{J}(F)}_{\chi_F}(\varepsilon) \sim \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A^{\mathcal{J}(F)}_{j,F} \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))',$$

where $A^{\mathcal{J}(F)}_{j,F}$ vanishes unless $j \geq j_F$, in which case one has $A^{\mathcal{J}(F)}_{j,F}(a \otimes \sigma) = A_{j,F}(a \otimes \sigma) = \sigma^{[j]}_{\mp}(0) \int_F D_{j,F} a \, dF$.

To prove that the operators $D^{\pm}_{j,F}$, $D_{j,F}$ equal constants for $j = \frac{1}{2}\operatorname{codim} F - 1$ and to determine the constants $C_{\mathrm{indef}}, C_{\mathrm{def}} \in \mathbb{C}$ such that $D^{\pm}_{\frac{1}{2}\operatorname{codim} F - 1, F} = C^{\pm}_{\mathrm{indef}}$, $D_{\frac{1}{2}\operatorname{codim} F - 1, F} = C_{\mathrm{def}}$, we inspect the differential operators and the constants occurring in Propositions 3.3, 3.6, 4.2, and 4.3, and recall that (2.28) and (2.31) involve an overall factor of Λ_F^{-1}, which gives us

$$\begin{aligned} C^{\pm}_{\mathrm{indef}} &= \Lambda_F^{-1} (\operatorname{vol} S^{n^+_F - 1})(\operatorname{vol} S^{n^-_F - 1}) 2^{\operatorname{codim} F/2} c^{\pm}_{\operatorname{codim} F/2 - 1, 0, 0, 0} \\ &= \frac{2^{\operatorname{codim} F/2 + 2} \pi^{\operatorname{codim} F/2}}{\Lambda_F (n^+_F/2 - 1)! (n^-_F/2 - 1)!} c^{\pm}_{\operatorname{codim} F/2 - 1, 0, 0, 0}, \end{aligned}$$

$$\begin{aligned} C_{\mathrm{def}} &= \Lambda_F^{-1}\mathrm{vol}(S^{\mathrm{codim}\,F-1})2^{\mathrm{codim}\,F/2}c_{\mathrm{codim}\,F/2-1,0} \\ &= \frac{2^{\mathrm{codim}\,F/2+1}\pi^{\mathrm{codim}\,F/2}}{\Lambda_F(\mathrm{codim}\,F/2-1)!}c_{\mathrm{codim}\,F/2-1,0}. \end{aligned}$$

Here the additional factors $2^{\mathrm{codim}\,F/2}$ occur because we apply Propositions 3.3 and 3.6 with ε replaced by 2ε. Similarly, one finds $D_{0,F}^{\mathrm{top}} = 2\pi$ by taking into account that the function $W_{F,k}$ occurring in Proposition 4.1 is constant when $k = \mathrm{codim}\,F/2 - 2$, its value being given by $W_{F,\mathrm{codim}\,F/2-2} = 2^{\mathrm{codim}\,F/2}\Lambda_F$ since $\|w\|^2 = 2\,\|w^\pm\|^2$ for $w \in T_F^{-1}(\Sigma^0)$. Plugging in the values of the constants $c^\pm_{\mathrm{codim}\,F/2-1,0,0,0}$, and $c_{\mathrm{codim}\,F/2-1,0}$, we arrive at

$$C_{\mathrm{def}} = (2\pi)^2\frac{(2\pi i)^{\mathrm{codim}\,F/2-1}}{\Lambda_F(\mathrm{codim}\,F/2-1)!}, \quad C^\pm_{\mathrm{indef}} = (2\pi)^2\frac{(\pi i)^{\mathrm{codim}\,F/2-1}}{\Lambda_F(\mathrm{codim}\,F/2-1)!}N_F^\pm,$$

and we also find that the constants $N_F^\pm$ from (1.9) are given by

$$(5.2)\quad N_F^\pm = \frac{(-1)^{\mathrm{codim}\,F/2}(\mathrm{codim}\,F/2-1)!}{(n_F^+/2-1)!(n_F^-/2-1)!}\sum_{l=0}^{\mathrm{codim}\,F/2-2}\frac{(\pm 1)^{\mathrm{codim}\,F/2-l-1}}{\mathrm{codim}\,F/2-l-1} \times \sum_{\substack{l^++l^-=l \\ 0\le l^\pm\le n_F^\pm/2-1}}(-1)^{l^+}\binom{n_F^+/2-1}{l^+}\binom{n_F^-/2-1}{l^-}.$$

To simplify the formula for $N_F^\pm$, we perform a computation kindly suggested by Iosif Pinelis [8]. One computes for arbitrary $p, q \in \mathbb{N}_0$ with the convention that $\binom{a}{b} = 0$ whenever $a < b$

$$\begin{aligned} (5.3)\quad & \sum_{l\ge 0}\frac{1}{p+q-l+1}\sum_{\substack{j+k=l \\ j\ge 0,k\ge 0}}(-1)^j\binom{p}{j}\binom{q}{k} \\ &= \sum_{l\ge 0}\int_0^1 x^{p+q-l}\sum_{\substack{j+k=l \\ j\ge 0,k\ge 0}}(-1)^j\binom{p}{j}\binom{q}{k}\,dx \\ &= \int_0^1 x^{p+q}\sum_{l\ge 0}\sum_{\substack{j+k=l \\ j\ge 0,k\ge 0}}(-1)^j\binom{p}{j}\binom{q}{k}x^{-j}x^{-k}\,dx \\ &= \int_0^1 x^{p+q}\sum_{j\ge 0}\binom{p}{j}(-x^{-1})^j\sum_{k\ge 0}\binom{q}{k}x^{-k}\,dx \\ &= \int_0^1 x^{p+q}(1-x^{-1})^p(1+x^{-1})^q\,dx = \int_0^1 (x-1)^p(1+x)^q\,dx, \end{aligned}$$

and the latter expression can be further rewritten into

$$\begin{aligned}\int_0^1 (x-1)^p \sum_{k=0}^{q} \binom{q}{k} x^k \, dx &= (-1)^p \sum_{k=0}^{q} \binom{q}{k} \int_0^1 (1-x)^p x^k \, dx \\ &= (-1)^p \sum_{k=0}^{q} \frac{q!}{k!(q-k)!} \frac{k!p!}{(k+p+1)!} \\ &= (-1)^p \frac{q!p!}{(q+p+1)!} \sum_{j=0}^{q} \binom{q+p+1}{j}.\end{aligned}$$

Applying this to (5.2) gives us the result[5]

$$N_F^{\pm} = \pm(-1)^{n_F^-/2-1} \sum_{j=0}^{n_F^{\mp}/2-1} \binom{\operatorname{codim} F/2-1}{j}. \tag{5.4}$$

In particular, we see that $N_F^{\pm}$ is a non-zero integer. Let us now turn to the remaining summands in (5.1):

(3) Assume that $F \in \mathcal{F}$ is such that $\mathcal{J}(F) \neq \zeta$. Then ζ is a regular value of $\mathcal{J}|_{U_F}$, and by Proposition 2.5 there is an asymptotic expansion

$$I^{\zeta}_{\chi_F}(\varepsilon) \;\sim\; \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A^{\zeta}_{j,F} \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$$

with

$$A^{\zeta}_{j,F}(a \otimes \sigma) = \sigma^{(j)}(0) \int_{\mathscr{M}^{\zeta}_{\mathrm{top}}} \left\langle \mathscr{D}^{\zeta}_j(\chi_F a) \right\rangle_T d\mathscr{M}^{\zeta}_{\mathrm{top}}, \qquad \mathscr{D}^{\zeta}_0 = 2\pi.$$

Of course, instead of applying Proposition 2.5 we could have also treated this case as in (1) or (2) by applying Corollary 4.5 and Propositions 3.3 or 3.6, respectively. By the uniqueness of the coefficients in the asymptotic expansion, the results of the two approaches agree. However, the form of the coefficients obtained in Proposition 2.5 is much more simple.

[5] The appearance of n_F^- rather than $n_F^{\mp}$ in the exponent determining the overall sign of $N_F^{\pm}$ is not a misprint. This subtle and seemingly peculiar asymmetry is a consequence of the fact that interchanging n_F^+ and n_F^- corresponds to replacing the momentum map $\mathcal{J}$ by $-\mathcal{J}$, which changes the phase function in the generalized Witten integral (1.1).

(4) The function $\mathcal{J}|_{M_{\mathfrak{h}_{\text{top}}}}$ has only regular values. In particular, ζ is a regular value and Proposition 2.5 gives us an asymptotic expansion

$$I^{\zeta}_{\chi_{\text{top}}}(\varepsilon) \sim \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A^{\zeta}_{j,\text{top}} \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$$

with

$$A^{\zeta}_{j,\text{top}}(a \otimes \sigma) = \sigma^{(j)}(0) \int_{\mathscr{M}^{\zeta}_{\text{top}}} \left\langle \mathscr{D}^{\zeta}_j(\chi_{\text{top}} a) \right\rangle_T d\mathscr{M}^{\zeta}_{\text{top}}, \qquad \mathscr{D}^{\zeta}_0 = 2\pi.$$

Taking (1)–(4) together we deduce with (5.1) that

$$I^{\zeta}(\varepsilon) \sim \varepsilon \sum_{j=0}^{\infty} \varepsilon^j A^{\zeta}_j \qquad \text{in} \quad (\mathcal{D}(M) \otimes \mathcal{S}(\mathfrak{t}))'$$

with

$$A^{\zeta}_j(a \otimes \sigma) = \sigma^{(j)}(0) \int_{\mathscr{M}^{\zeta}_{\text{top}}} \langle D^{\zeta}_j a \rangle_T \, d\mathscr{M}^{\zeta}_{\text{top}} + \sum_{\substack{F \in \mathcal{F}: \mathcal{J}(F)=\zeta, \\ F \cap \operatorname{supp} a \neq \emptyset}} A_{j,F}(a \otimes \sigma),$$

where the differential operator D^{ζ}_j is defined on the neighborhood $\mathscr{U}^{\zeta}_j \subset M_{(\mathfrak{h}_{\text{top}})}$ of $\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\text{top}})}$ on which the operator $\mathscr{D}^{\zeta}_j$ from Proposition 2.5 is defined and acts on a function $f \in \mathrm{C}^{\infty}(\mathscr{U}^{\zeta}_j)$ by

$$\begin{aligned} D^{\zeta}_j(f) := \mathscr{D}^{\zeta}_j(\chi_{\text{top}}|_{\mathscr{U}^{\zeta}_j} f) + \sum_{F \in \mathcal{F}: \mathcal{J}(F) \neq \zeta} \mathscr{D}^{\zeta}_j(\chi_F|_{\mathscr{U}^{\zeta}_j} f) \\ + \sum_{\substack{F \in \mathcal{F}: \mathcal{J}(F)=\zeta, \\ F \text{ indefinite}}} D^{\text{top}}_{j,F}(\chi_F f|_{(U_F \setminus F) \cap \mathscr{U}^{\zeta}_j}). \end{aligned} \tag{5.5}$$

The sum in (5.5) is locally finite and $\chi_F f|_{(U_F \setminus F) \cap \mathscr{U}^{\zeta}_j}$ extends smoothly by zero to $\mathscr{U}^{\zeta}_j$, so that D^{ζ}_j is well-defined. To see that

$$\int_{\mathscr{M}^{\zeta}_{\text{top}}} \langle D^{\zeta}_0 a \rangle_T \, d\mathscr{M}^{\zeta}_{\text{top}} = 2\pi \int_{\mathscr{M}^{\zeta}_{\text{top}}} \langle a \rangle_T \, d\mathscr{M}^{\zeta}_{\text{top}} \qquad \forall\, a \in \mathrm{C}^{\infty}_{\mathrm{c}}(M), \tag{5.6}$$

we observe that for each definite fixed point $F \in \mathcal{F}$ one has

$$U_F \cap \mathcal{J}^{-1}(\{\mathcal{J}(F)\}) = F$$

and consequently the intersection $\mathcal{J}^{-1}(\{\mathcal{J}(F)\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})}$ is disjoint from U_F. This implies that for each $\zeta \in \mathfrak{t}^*$ the family of cutoff functions

$$\{\chi_{\mathrm{top}}, \chi_F\}_{F\in\mathcal{F}:\mathcal{J}(F)=\zeta, F\,\text{indefinite}}$$

restricts to a partition of unity on the set $\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(\mathfrak{h}_{\mathrm{top}})}$. Since $\mathscr{D}_0^\zeta = 2\pi$ and $D_{0,F}^{\mathrm{top}} = 2\pi$ for all indefinite $F \in \mathcal{F}$, we get (5.6).

Finally, to prove the naturality claim, let $(M', \omega', \mathcal{J}')$ be another Hamiltonian T-space and $\Phi : M \to M'$ an isomorphism of Hamiltonian T-spaces. Then $\mathcal{F}' = \{\Phi(F) : F \in \mathcal{F}\}$ is the set of connected components of M'^T, and for each $F' = \Phi(F) \in \mathcal{F}'$ the map $\Phi_{F'} := \Phi_F \circ \Phi^{-1} : U'_F := \Phi(U_F) \to P_F$ serves as the local normal form symplectomorphism in Proposition 2.6. Furthermore, the partition of unity $\{\chi_{\mathrm{top}} \circ \Phi^{-1}, \chi_{\Phi^{-1}(F')} \circ \Phi^{-1}\}_{F'\in\mathcal{F}'}$ is subordinate to the cover $M' = M'_{(\mathfrak{h}_{\mathrm{top}})} \cup \bigcup_{F'\in\mathcal{F}'} U_{F'}$, and one has $n^{\pm}_{F'} = n^{\pm}_{\Phi^{-1}(F')}$, $\lambda_j^{F'} = \lambda_j^{\Phi^{-1}(F')}$ for each $F' \in \mathcal{F}'$, $1 \le j \le \operatorname{codim} F'/2 = \operatorname{codim} \Phi^{-1}(F')/2$. The claim now follows from the construction of the operators in the proofs of Theorem 1.1 and Corollary 4.5, which is carried out locally either using Proposition 2.5, for which the naturality property holds since the phase function on M' is given by composition with Φ in the manifold variable, or by composing operators in a Euclidean space which are uniquely determined by the numbers $n^{\pm}_{F'}$ and $\lambda_j^{F'}$ with the local normal form symplectomorphism $\Phi_{F'}$ and gluing them together using the partition of unity. This concludes the proof of Theorem 1.1. $\square$

The construction of D_j^ζ in (5.5) raises the question whether the coefficients $(A_j^\zeta)_{\mathrm{top}}$ in (1.7) depend on the choice of partition of unity when $j > 0$. That this is not the case is shown in the following

Lemma 5.1. *For each $j \in \mathbb{N}_0$, the distribution $\mathcal{I}_j^\zeta \in \mathcal{D}'(M)$ defined by*

$$\mathcal{I}_j^\zeta(a) := \int_{\mathscr{M}^\zeta_{\mathrm{top}}} \langle D_j^\zeta a\rangle_T \, d\mathscr{M}^\zeta_{\mathrm{top}},$$

and consequently the coefficient $(A_j^\zeta)_{\mathrm{top}}$ in (1.7), is independent of the choice of partition of unity.

Proof. Applying Theorem 1.1 and Proposition 2.5 immediately yields that the restriction of $\mathcal{I}_j^\zeta$ to $M_{(\mathfrak{h}_{\mathrm{top}})}$ is independent of the choice of partition of unity by the uniqueness of the coefficients in the respective asymptotic expansions. Now, given $j \in \mathbb{N}_0$, $\zeta \in \mathfrak{t}^*$, and $a \in \mathrm{C}^\infty_{\mathrm{c}}(M)$, choose $\varepsilon > 0$ and a

T-invariant cutoff function $\chi \in \mathrm{C}^\infty_{\mathrm{c}}(M)$ with $\mathrm{vol}_{\mathscr{M}^\zeta_{\mathrm{top}}}((\mathrm{supp}\ \chi)/T \cap \mathscr{M}^\zeta_{\mathrm{top}}) < \varepsilon/(\|D^\zeta_j a\|_\infty \mathrm{vol}\, T)$ and $\chi \equiv 1$ near $\mathrm{supp}\ a \cap M^T$. Then we have

$$\begin{aligned}\mathcal{I}^\zeta_j(a) &= \mathcal{I}^\zeta_j(\chi a) + \mathcal{I}^\zeta_j((1-\chi)a)\\ &= \int_{\mathscr{M}^\zeta_{\mathrm{top}}} \langle \chi D^\zeta_j a\rangle_T \, d\mathscr{M}^\zeta_{\mathrm{top}} + \mathcal{I}^\zeta_j(\chi^\zeta_j a) + \mathcal{I}^\zeta_j((1-\chi)a),\end{aligned}$$

where $\chi^\zeta_j \in \mathrm{C}^\infty_{\mathrm{c}}(M)$ is such that $D^\zeta_j(\chi a) = \chi D^\zeta_j a + \chi^\zeta_j a$. In particular, χ^ζ_j vanishes near $\mathrm{supp}\ a \cap M^T$. Since $\chi^\zeta_j a$ and $(1-\chi)a$ are supported in $M_{(\mathfrak{h}_{\mathrm{top}})}$, the terms $\mathcal{I}^\zeta_j(\chi^\zeta_j a)$ and $\mathcal{I}^\zeta_j((1-\chi)a)$ are independent of the choice of partition of unity. Finally, the integral of $\langle \chi D^\zeta_j a\rangle_T$ over $\mathscr{M}^\zeta_{\mathrm{top}}$ is bounded in absolute value by ε. Since $\varepsilon > 0$ was arbitrary, $\mathcal{I}^\zeta_j(a)$ does not depend on the choice of partition of unity. $\square$

5.2. Proof of Theorem 1.2

Fix a given $\zeta_0 \in \mathbb{R} \cong \mathfrak{t}^*$. To determine the limit behavior of each individual term in (1.7) as $\zeta \to \zeta_0$ under the two conditions $\pm(\zeta - \zeta_0) > 0$, we begin with the summands $A^\zeta_{j,F}(a \otimes \sigma)$, using the same notation as in the proof of Theorem 1.1. Thus, fix an $F \in \mathcal{F}$. As long as $\zeta \neq \mathcal{J}(F)$, we have two options how to express the coefficient $A^\zeta_{j,F}(a \otimes \sigma)$: Either we can observe that ζ is a regular value of $\mathcal{J}|_{U_F}$, and invoke the general regular stationary phase asymptotics from Proposition 2.5, as we did in (3) in the proof of Theorem 1.1. Or we can apply Proposition 3.3 (if F is indefinite) or 3.6 (if F is definite) and Corollary 4.5, as we did in (1) and (2) in the proof of Theorem 1.1. By uniqueness of the coefficients in asymptotic expansions, the two approaches do describe the same coefficients. However, only the latter approach is useful when ζ approaches the singular value $\mathcal{J}(F)$ of $\mathcal{J}|_{U_F}$ because the coefficients featured in Propositions 3.3 and 3.6 do have a clearly visible limit behavior in this case, in contrast to the less explicit but simpler terms appearing in Proposition 2.5, where no statement on limits towards singular values is made. We thus distinguish three cases:

1) If $\mathcal{J}(F) = \zeta_0$ and F is indefinite, we apply Proposition 3.3 with ε replaced by 2ε, ζ by $2\zeta_F = 2(\zeta - \mathcal{J}(F))$, $\mathscr{S} = \mathscr{S}_{\tilde{a}_F}$, and $L = \operatorname{codim} F/2 - 2$, together with Corollary 4.5, yielding

$$\begin{aligned}\lim_{\substack{\zeta\to\mathcal{J}(F)\\ \pm(\zeta-\mathcal{J}(F))>0}} & A^{\zeta}_{j,F}(a\otimes\sigma)\\ &= \begin{cases} \sigma^{(j)}(0)\displaystyle\int_{\mathscr{M}^{\mathcal{J}(F)}_{\text{top}}} \big\langle D^{\text{top}}_{j,F}(\chi_F a)\big\rangle_T \, d\mathscr{M}^{\mathcal{J}(F)}_{\text{top}}, & j < j_F,\\ \sigma^{(j)}(0)\Big(\displaystyle\int_{\mathscr{M}^{\mathcal{J}(F)}_{\text{top}}} \big\langle D^{\text{top}}_{j,F}(\chi_F a)\big\rangle_T \, d\mathscr{M}^{\mathcal{J}(F)}_{\text{top}} + \int_F D^{\mp}_{j,F} a \, dF\Big), & j \geq j_F.\end{cases}\end{aligned}$$

2) If $\mathcal{J}(F) = \zeta_0$ and F is definite, we similarly get from Proposition 3.6 and Corollary 4.5 the result

$$\lim_{\substack{\zeta\to\mathcal{J}(F)\\ \pm(\zeta-\mathcal{J}(F))>0}} A^{\zeta}_{j,F}(a\otimes\sigma) = \begin{cases} 0, & j < j_F \;\text{ or }\; s_F = \mp,\\ \sigma^{(j)}(0)\displaystyle\int_F D_{j,F} a, & j \geq j_F \;\text{ and }\; s_F = \pm.\end{cases}$$

3) If $\mathcal{J}(F) \neq \zeta_0$, then all ζ close to ζ_0 are regular values of $\mathcal{J}|_{U_F}$, and by Proposition 2.5 one has

$$\lim_{\substack{\zeta\to\zeta_0\\ \pm(\zeta-\zeta_0)>0}} A^{\zeta}_{j,F}(a\otimes\sigma) = A^{\zeta_0}_{j,F}(a\otimes\sigma) = \sigma^{(j)}(0)\int_{\mathscr{M}^{\zeta_0}_{\text{top}}} \big\langle \mathscr{D}^{\zeta_0}_j(\chi_F a)\big\rangle_T \, d\mathscr{M}^{\zeta_0}_{\text{top}}.$$

Finally, it remains to consider the limits of the contributions $A^{\zeta}_{j,\text{top}}(a\otimes\sigma)$ of the top stratum:

(4) Since the function $\mathcal{J}|_{M_{\mathfrak{h}_{\text{top}}}}$ has only regular values, Proposition 2.5 gives

$$\lim_{\substack{\zeta\to\zeta_0\\ \pm(\zeta-\zeta_0)>0}} A^{\zeta}_{j,\text{top}}(a\otimes\sigma) = A^{\zeta_0}_{j,\text{top}}(a\otimes\sigma) = \sigma^{(j)}(0)\int_{\mathscr{M}^{\zeta_0}_{\text{top}}} \big\langle \mathscr{D}^{\zeta_0}_j(\chi_{\text{top}} a)\big\rangle_T \, d\mathscr{M}^{\zeta_0}_{\text{top}}.$$

We are now ready to describe the limit behavior of

$$A^{\zeta}_j(a\otimes\sigma) = A^{\zeta}_{j,\text{top}}(a\otimes\sigma) + \sum_{F\in\mathcal{F}:F\cap\operatorname{supp} a\neq\emptyset} A^{\zeta}_{j,F}(a\otimes\sigma)$$

for each $\zeta \in \mathfrak{t}^*$. Recalling the definition (5.5) of the operators D_j^ζ, a combination of (1)–(4) yields for each $j \in \mathbb{N}_0$

$$\begin{aligned}
\lim_{\substack{\zeta\to\zeta_0\\ \pm(\zeta-\zeta_0)>0}} A_j^\zeta(a\otimes\sigma) &= \lim_{\substack{\zeta\to\zeta_0\\ \pm(\zeta-\zeta_0)>0}} A_{j,\mathrm{top}}^\zeta(a\otimes\sigma) \\
&\quad + \sum_{\substack{F\in\mathcal{F}:\mathcal{J}(F)\neq\zeta_0,\\ F\cap\operatorname{supp} a\neq\emptyset}} \lim_{\substack{\zeta\to\zeta_0\\ \pm(\zeta-\zeta_0)>0}} A_{j,F}^\zeta(a\otimes\sigma) \\
&\quad + \sum_{\substack{F\in\mathcal{F}:\mathcal{J}(F)=\zeta_0,\\ F\cap\operatorname{supp} a\neq\emptyset}} \lim_{\substack{\zeta\to\mathcal{J}(F)\\ \pm(\zeta-\mathcal{J}(F))>0}} A_{j,F}^\zeta(a\otimes\sigma) \\
&= \sigma^{(j)}(0)\Bigg(\int_{\mathscr{M}_{\mathrm{top}}^{\zeta_0}} \left\langle D_j^{\zeta_0} a\right\rangle_T d\mathscr{M}_{\mathrm{top}}^{\zeta_0} + \sum_{\substack{F\in\mathcal{F}:\mathcal{J}(F)=\zeta_0,\\ \operatorname{codim} F/2-1\leq j,\\ F \text{ indefinite},\\ F\cap\operatorname{supp} a\neq\emptyset}} \int_F D_{j,F}^{\mp} a\, dF \\
&\qquad + \sum_{\substack{F\in\mathcal{F}:\mathcal{J}(F)=\zeta_0,\\ \operatorname{codim} F/2-1\leq j,\\ F \text{ definite},\ s_F=\pm,\\ F\cap\operatorname{supp} a\neq\emptyset}} \int_F D_{j,F} a\, dF\Bigg).
\end{aligned}$$

This concludes the proof of Theorem 1.2. $\square$

Index of Notation

In what follows we include a list with the main notation used in this paper, explaining its meaning and specifying the place where it is used first.

M	A symplectic manifold with symplectic form ω	p. 1281
T	The circle group, acting on M in a Hamiltonian fashion	p. 1281
$\mathfrak{t}$	The Lie algebra of T, identified with $\mathbb{R}$ by fixing a Lebesgue measure	p. 1281
$\mathcal{J}$	The momentum map $M \to \mathfrak{t}^*$	p. 1281
$J(x)$	The map $M \to \mathbb{R}$ given by $\mathcal{J}(p)(x) = J(x)(p)$	Eq. (1.4)
$\mathcal{S}(V)$	The space of Schwartz functions on V	Eq. (1.1)
$\mathcal{D}(M)$	The space of test functions $\mathrm{C}_\mathrm{c}^\infty(M)$ with the test function topology	Eq. (1.2)

$\mathcal{D}'(M)$	The space of distributions on M, identified with the space of distribution densities on M via the symplectic volume form dM	Eq. (1.3)
$\mathscr{M}^\zeta$	The symplectic reduction $\mathscr{M}^\zeta = \mathcal{J}^{-1}(\{\zeta\})/T$	Eq. (1.6)
$\mathscr{M}^\zeta_{\mathrm{top}}$	The top stratum of $\mathscr{M}^\zeta$	Eq. (1.6)
$\mathscr{M}^\zeta_{\mathrm{sing}}$	The singular stratum of $\mathscr{M}^\zeta$	Eq. (1.6)
$M_{(h_\aleph)}$	The stratum of M of infinitesimal orbit type $(\mathfrak{h}_\aleph)$	Eq. (1.6) f.
F	A connected component of the fixed point set M^T	p. 1284
$\mathcal{F}$	The set of all connected components F of M^T	p. 1284
$I^\zeta(\varepsilon)$	The generalized Witten integral	Eq. (1.2)
a	A function in $\mathrm{C}^\infty_\mathrm{c}(M)$	Eq. (1.2)
σ	A function in $\mathcal{S}(\mathfrak{t})$	Eq. (1.2)
$I^\zeta_{a,\sigma}(\varepsilon)$	The generalized Witten integral evaluated on $a \otimes \sigma$	Eq. (1.1)
$I^\zeta_{\chi_{\mathrm{top}}}(\varepsilon)$	A component of the generalized Witten integral	Eq. (2.21)
$I^\zeta_{\chi_F}(\varepsilon)$	A component of the generalized Witten integral	Eq. (2.21)
A^ζ_j	The j-th coefficient in the expansion of $I^\zeta(\varepsilon)$	Thm. 1.1
Σ^ζ	The local model for the level set $\mathcal{J}^{-1}(\zeta + \mathcal{J}(F))$	Def. 4.1
$\Sigma^\zeta_\bullet$	A subset of Σ^ζ	Def. 4.1
$\Sigma^\zeta_\times$	A subset of $\Sigma^\zeta_\bullet$	Def. 4.1
$A^\zeta_{j,\mathrm{top}}$	A contribution to the coefficient A^ζ_j	Sec. 5.1
$A^\zeta_{j,F}$	A contribution to the coefficient A^ζ_j	Sec. 5.1
$a = \mathcal{O}(r)$	There is a constant $C > 0$ such that $\lvert a\rvert \leq Cr$	Eq. (4.12) f.
$a = \mathcal{O}_x(r)$	There is a constant $C > 0$, depending on x, such that $\lvert a\rvert \leq Cr$	Prop. 3.3

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