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On a systolic inequality for closed magnetic geodesics on surfaces

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We apply a local systolic-diastolic inequality for contact forms and odd-symplectic forms on three-manifolds to bound the magnetic length of closed curves with prescribed geodesic curvature (also known as magnetic geodesics) on an oriented closed surface. Our results hold when the prescribed curvature is either close to a Zoll one or large enough.

1. Introduction

In this paper, we apply the systolic-diastolic inequality established in [6, 7] for contact forms and odd-symplectic forms on three-manifolds, respectively, to the study of immersed closed curves with prescribed geodesic curvature on a connected oriented closed surface $(M, \mathfrak{o}_M)$ endowed with a Riemannian metric g. The Riemannian metric g and the orientation $\mathfrak{o}_M$ yield a well-defined way of measuring angles in each tangent plane and an area form μ on M. If $c : I \to M$ is a smooth curve parametrised by arc-length on some interval I, we define its geodesic curvature $\kappa_c : I \to \mathbb{R}$ to be the unique function satisfying the relation

$$\nabla_{\dot c}\dot c = \kappa_c \dot c^{\perp},$$

where ∇ is the Levi-Civita connection, and $\dot c^{\perp}$ is the unit vector with the property that the angle from $\dot c$ to $\dot c^{\perp}$ is $\frac{\pi}{2}$.

Let $f : M \to \mathbb{R}$ be a smooth function. A curve $c : \mathbb{R} \to M$ is said to be a **magnetic geodesic**, or an **f-magnetic geodesics** when we want to mention the function f explicitly, if it is parametrised by arc-length and satisfies the equation

$$\kappa_c(t) = -f(c(t)), \qquad \forall\, t \in \mathbb{R}. \tag{1.1}$$

The magnetic geodesics of f and of $-f$ are in one-to-one correspondence through time reversal. This means that $t \mapsto c(t)$ is an f-magnetic geodesic if and only if $t \mapsto c(-t)$ is a $-f$-magnetic geodesic. The study of periodic solutions of (1.1), which we refer to as closed f-magnetic geodesics, is by now a problem with a rich history and we refer the reader to [3, 4, 11, 14, 22] and the references therein for an account of the most remarkable developments and a generalization to higher dimensional manifolds M.

A crucial ingredient in our work will be to use that the tangent lifts $(c, \dot c)$ of f-magnetic geodesics are the integral curves of a vector field X_f defined on the unit tangent bundle $\mathrm{T}^1 M$, whose elements are the tangent vectors of unit norm. The foot-point projection

$$\mathfrak{p}_\infty : \mathrm{T}^1 M \to M$$

is an orientable S^1-bundle, whose fibres we orient by the $\mathfrak{o}_M$-*negative* direction. If $e \in H^2_{\mathrm{dR}}(M)$ is *minus* the real Euler class of $\mathfrak{p}_\infty$, then

$$\langle e, [M] \rangle = \chi(M), \tag{1.2}$$

where $\chi(M)$ is the Euler characteristic of M. We write $\mathfrak{h}_\infty \in [S^1, \mathrm{T}^1 M]$ for the free-homotopy class of $\mathfrak{p}_\infty$-fibres. Throughout the paper $\mathrm{T}^1 M$ will be given the orientation

$$\mathfrak{o}_{\mathrm{T}^1 M} = \mathfrak{o}_M \oplus \mathfrak{o}_{\mathfrak{p}_\infty} \tag{1.3}$$

obtained combining the orientation of M with the orientation of the $\mathfrak{p}_\infty$-fibres given above.

Definition 1.1. We say that a function $f : M \to \mathbb{R}$ is **Zoll** with respect to a given metric g if there exists an oriented S^1-bundle

$$\mathfrak{p}_f : \mathrm{T}^1 M \to M_f$$

such that the integral curves of X_f are fibres of $\mathfrak{p}_f$. We write e_f for minus the Euler class of $\mathfrak{p}_f$, and $\mathfrak{h}_f$ for the free-homotopy class of the $\mathfrak{p}_f$-fibres, and $\mathfrak{o}_{\mathfrak{p}_f}$ for the orientation of the fibres. We endow M_f with the unique orientation $\mathfrak{o}_{M_f}$ such that $\mathfrak{o}_{\mathrm{T}^1 M} = \mathfrak{o}_{M_f} \oplus \mathfrak{o}_{\mathfrak{p}_f}$.

If we take $f \equiv 0$, we recover the notion of Zoll Riemannian metric and M must be the two-sphere. We refer the reader to [9] for a thorough discussion of such metrics. A classical example of a Zoll function $f_* : M \to \mathbb{R}$ can be given when $g = g_*$ is a metric of constant Gaussian curvature K_*. We take f_* to be any constant function satisfying

$$f_*^2 + K_* > 0. \tag{1.4}$$

If c is a prime closed magnetic geodesic, its lift $\widetilde{c}$ to the universal cover $\widetilde{M}$ of M parametrises the boundary of a geodesic ball of radius

$$R = \begin{cases} \frac{1}{\sqrt{K_*}} \arctan\left(\frac{\sqrt{K_*}}{|f_*|}\right), & \text{if } K_* > 0; \\ \frac{1}{|f_*|}, & \text{if } K_* = 0; \\ \frac{1}{\sqrt{-K_*}} \operatorname{arctanh}\left(\frac{\sqrt{-K_*}}{|f_*|}\right), & \text{if } K_* < 0. \end{cases}$$

According to our sign convention, the curve rotates clockwise, if $f_* > 0$. Therefore, all f_*-magnetic geodesics are closed, and actually f_* is Zoll. Here the map $\mathfrak{p}_{f_*} : \mathrm{T}^1 M \to M_{f_*} = M$ in Definition 1.1 associates to a tangent vector the projection on M of the center of the corresponding ball in $\widetilde{M}$. In general, it is unknown whether every Riemannian metric admits a Zoll function.

Remark 1.2. If we take $f_* = 0$ and $K_* = 0$ for the two-torus or $f_*^2 + K_* < 0$ for higher genus surfaces, all the closed magnetic geodesics are not contractible. If we take $f_*^2 + K_* = 0$ on higher genus surfaces, then there are no closed magnetic geodesics at all.

Let us go back to the case of an arbitrary function $f : M \to \mathbb{R}$ and attach two quantities to it. The former is the **average** of f:

$$f_{\mathrm{avg}} := \frac{1}{\mathrm{area}(M)} \int_M f\mu\,, \qquad\qquad \mathrm{area}(M) := \int_M \mu\,.$$

The latter is the **average curvature** of f, which generalises the left-hand side of (1.4):

$$K_f := (f_{\mathrm{avg}})^2 + \frac{2\pi\chi(M)}{\mathrm{area}(M)}.$$

Indeed, by the Gauss-Bonnet theorem,

$$\frac{2\pi\chi(M)}{\mathrm{area}(M)} = \frac{1}{\mathrm{area}(M)} \int_M K\mu,$$

where K is the Gaussian curvature of g. The average curvature of f is always positive for the two-sphere $M = S^2$. For the two-torus $M = \mathbb{T}^2$, it is always non-negative and equality holds exactly when $f_{\mathrm{avg}} = 0$.

In the next proposition, we collect the first properties of Zoll functions and their closed magnetic geodesics.

Proposition 1.3. *If $f : M \to \mathbb{R}$ is a Zoll function, the following statements hold:*

(a) The surface M_f is diffeomorphic to M and there is a path of oriented S^1-bundles $\{\mathfrak{p}_r\}_{r\in[0,1]}$ with total space T^1M and a sign $\epsilon(f) \in \{-1, +1\}$ such that

$$\mathfrak{p}_0 = \epsilon(f)\mathfrak{p}_\infty, \qquad \mathfrak{p}_1 = \mathfrak{p}_f$$

where $-\mathfrak{p}_\infty$ is the bundle $\mathfrak{p}_\infty$ with opposite orientation. In particular,

$$\mathfrak{h}_f = \epsilon(f)\mathfrak{h}_\infty, \qquad \langle e_f, [M_f]\rangle = \chi(M)$$

and there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{T}^1M & \xrightarrow{F} & \mathrm{T}^1M \\ \downarrow{\scriptstyle \mathfrak{p}_\infty} & & \downarrow{\scriptstyle \mathfrak{p}_f} \\ M & \xrightarrow{\bar{F}} & M_f \end{array} \tag{1.5}$$

where F is a diffeomorphism isotopic to the identity and $\bar{F} : M \to M_f$ is a diffeomorphism which preserves the orientation if and only if $\epsilon(f) = 1$.

(b) If M is not the two-sphere, then $f_{\mathrm{avg}} \neq 0$ and $\epsilon(f) = \mathrm{sign}(f_{\mathrm{avg}})$.

(c) The average curvature is positive, namely $K_f > 0$.

Remark 1.4. If M is the two-sphere, the oriented bundle $\mathfrak{p}_\infty$ is homotopic to its opposite $-\mathfrak{p}_\infty$. Therefore, both $\epsilon(f) = -1$ and $\epsilon(f) = +1$ are good in this case.

In view of this proposition, given a function $f : M \to \mathbb{R}$ which is not necessarily Zoll, we are motivated to look for closed f-magnetic geodesics whose tangent lift belongs either to the free-homotopy class $\mathfrak{h}_\infty$ or to $-\mathfrak{h}_\infty$. Up to substituting f with $-f$, we focus on f-magnetic geodesics in the former class of curves, namely in the set

(1.6)
$$\Lambda(M;\mathfrak{h}_\infty) := \Big\{ c : \mathbb{R}/T\mathbb{Z} \to M \text{ for some } T > 0 \;\Big|\; |\dot c| \equiv 1,\ [(c,\dot c)] = \mathfrak{h}_\infty \Big\}.$$

Lemma 2.3 and Lemma 2.6 explain in more detail which curves belong to $\Lambda(M;\mathfrak{h}_\infty)$. We denote by $\Lambda(f;\mathfrak{h}_\infty)$ the subset of closed f-magnetic geodesics in $\Lambda(M;\mathfrak{h}_\infty)$.

We now describe a function $\ell_f : \Lambda(M;\mathfrak{h}_\infty) \to \mathbb{R}$ called **f-magnetic length functional**, whose critical set is exactly $\Lambda(f;\mathfrak{h}_\infty)$. To this purpose, let $c \in \Lambda(M;\mathfrak{h}_\infty)$. There exists a cylinder $\Gamma : [0,1] \times S^1 \to \mathrm{T}^1M$ such that $\Gamma(0,\cdot)$ is an oriented $\mathfrak{p}_\infty$-fibre and $\Gamma(1,\cdot)$ coincides with $(c,\dot c)$, up to reparametrisation. We regard the projection $\mathfrak{p}_\infty \circ \Gamma$ as a disc $C : D^2 \to M$ bounding c. Any disc arising in this way is called an **admissible capping disc** for c. We set

$$\ell_f : \Lambda(M;\mathfrak{h}_\infty) \to \mathbb{R}, \qquad \ell_f(c) := \ell(c) + \int_{D^2} C^*(f\mu),$$

where $\ell(c)$ is the Riemannian length of c, and C is an admissible capping disc for c. As will be shown in Section 2.1, the value of $\ell(c)$ is independent of the choice of C. The systolic-diastolic inequality will give bounds for the quantities

$$\ell_{\min}(f) := \inf_{\substack{c \in \Lambda(f;\mathfrak{h}_\infty) \\ c \text{ prime}}} \ell_f(c), \qquad \ell_{\max}(f) := \sup_{\substack{c \in \Lambda(f;\mathfrak{h}_\infty) \\ c \text{ prime}}} \ell_f(c),$$

in terms of the **average length** of f which is defined by

$$\bar\ell(f) := \frac{2\pi}{f_{\mathrm{avg}} + \sqrt{K_f}}. \tag{1.7}$$

Remark 1.5. For $M = S^2$, we automatically have $\bar{\ell}(f) > 0$. If $M = \mathbb{T}^2$, then $\bar{\ell}(f)$ is a real number if and only if $f_{\mathrm{avg}} > 0$ and in this case $\bar{\ell}(f) = \pi / f_{\mathrm{avg}} > 0$. If M has higher genus, $\bar{\ell}(f)$ is a real number, if and only if $K_f \geq 0$. In this case, f_{avg} and $\bar{\ell}(f)$ are both non-zero and have the same sign.

Definition 1.6. We say that $f : M \to \mathbb{R}$ satisfies the **magnetic systolic-diastolic inequality** if $\bar{\ell}(f)$ is a well-defined real number and

$$\ell_{\min}(f) \leq \bar{\ell}(f) \leq \ell_{\max}(f),$$

with any of the two equalities holding if and only if f is a Zoll function, whose magnetic geodesics lie in $\Lambda(M; \mathfrak{h}_\infty)$.

Remark 1.7. According to Proposition 1.3, if $M \neq S^2$, the magnetic geodesics of a Zoll function f lie in $\Lambda(M; \mathfrak{h}_\infty)$ if and only if $f_{\mathrm{avg}} > 0$. In this case K_f and $\bar{\ell}(f)$ are also positive.

Remark 1.8. One could define *mutatis mutandis* the analogous space $\Lambda(M; -\mathfrak{h}_\infty)$ and give a corresponding variational principle and a systolic-diastolic inequality for closed f-magnetic geodesics contained therein. The only difference is that one has to substitute $\bar{\ell}(f)$ with $\bar{\ell}'(f) := 2\pi(-f_{\mathrm{avg}} + \sqrt{K_f})^{-1}$.

We prove the inequality in two cases. First, we show it for functions close to a Zoll one.

Theorem 1.9. *Let M be a connected oriented closed surface endowed with a Riemannian metric, and let $f_* : M \to \mathbb{R}$ be a Zoll function, whose magnetic geodesics lie in $\Lambda(f; \mathfrak{h}_\infty)$. Then there exists a C^2-neighbourhood $\mathcal{F}$ of f_* in the space of functions such that every f in $\mathcal{F}$ satisfies the magnetic systolic-diastolic inequality.*

Next, we establish the magnetic systolic-diastolic inequality for positive functions with large average. To make this concept precise, we need a definition.

Definition 1.10. For every $k \in \mathbb{N}$ and every $f : M \to (0,\infty)$, we set

$$\langle f \rangle_k := \frac{\|f\|_{C^k}}{\min f} \in [1,\infty).$$

For a constant $C > 0$, we say that $f : M \to (0,\infty)$ is C**-strong**, if there holds

$$f_{\mathrm{avg}} > \Big(\langle f \rangle_3^4 + \langle f \rangle_2^6 \Big) e^{C\langle f \rangle_1^2}.$$

Theorem 1.11. *Let M be a connected oriented closed surface endowed with a Riemannian metric g. There exists a constant $C_g > 0$ with the property that, if $f : M \to \mathbb{R}$ is C_g-strong, then the function f satisfies the magnetic systolic-diastolic inequality.*

Remark 1.12. It is plausible that Theorem 1.9 still holds if we let the metric g also vary. To be precise, if f_* is Zoll with respect to a metric g_*, then there should exist a C^3-neighbourhood $\mathcal{G}$ of g_* and a C^2-neighbourhood $\mathcal{F}$ of f_* such that for every $(g, f) \in \mathcal{G} \times \mathcal{F}$, f satisfies the magnetic systolic-diastolic inequality with respect to g. Actually, in the purely Riemannian case (namely, when $f = 0$), the systolic-diastolic inequality holds true for metrics g on S^2, whose curvature is suitably pinched, see [1] and also [2, Corollary 4]. We also expect Theorem 1.11 to be true if we let g vary in a C^3-bounded set.

For all positive real numbers s, we have

$$(sf)_{\mathrm{avg}} = s(f_{\mathrm{avg}}), \qquad \langle sf \rangle_k = \langle f \rangle_k, \quad \forall\, k \in \mathbb{N}. \tag{1.8}$$

Thus Theorem 1.11 applies to large rescalings of any positive function.

Corollary 1.13. *Let M be a connected oriented closed surface endowed with a Riemannian metric g. For every $f : M \to (0,\infty)$, there exists a positive number $s(g,f) > 0$ such that if $s > s(g,f)$, then the function sf satisfies the magnetic systolic-diastolic inequality.* □

Theorem 1.9 and Theorem 1.11 are consequences of the contact systolic-diastolic inequality established in [7] when M is different from the two-torus, as in this case the tangent lifts of magnetic geodesics are the trajectory of a Reeb flow on the unit tangent bundle, up to reparametrisation. If M is the two-torus, its unit tangent bundle is trivial, and results in [7] are not applicable. Instead, in this case, the theorems follow from the systolic-diastolic inequality for odd-symplectic forms explored in [6].

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2. Preliminaries

2.1. The unit tangent bundle

As mentioned in the introduction, f-magnetic geodesics c yield trajectories $(c, \dot{c})$ of a flow Φ^{X_f} on the unit tangent bundle T^1M. The generating vector field X_f can be explicitly written as

$$X_f = X + \frac{1}{2\pi}(f \circ \mathfrak{p}_\infty)V,$$

where X is the geodesic vector field of g, and V is the vector field whose flow rotates the fibres of the map $\mathfrak{p}_\infty$ in the $\mathfrak{o}_M$-*negative* direction with constant angular speed $\frac{1}{2\pi}$. Thus the vector field V generates a free S^1-action on T^1M (our convention is $S^1 = \mathbb{R}/\mathbb{Z}$) and we denote by $\mathfrak{h}_\infty \in [S^1, \mathrm{T}^1M]$ the free-homotopy class of the orbits of V, namely of the oriented $\mathfrak{p}_\infty$-fibres. The Levi-Civita one-form $\eta \in \Omega^1(\mathrm{T}^1M)$ is the connection for $\mathfrak{p}_\infty$ satisfying

$$\eta(V) = 1, \qquad \mathrm{d}\eta = \frac{1}{2\pi}\mathfrak{p}_\infty^*(K\mu), \tag{2.1}$$

where K is the Gaussian curvature of g. This implies that $e = \frac{1}{2\pi}[K\mu]$, where e is minus the real Euler class of $\mathfrak{p}_\infty$.

Let α_{can} be the canonical one-form on T^1M given by

$$(\alpha_{\mathrm{can}})_v \cdot Y := g(v, \mathrm{d}_v\mathfrak{p}_\infty \cdot Y), \qquad \forall\, Y \in \mathrm{T}_v(\mathrm{T}^1M).$$

It is a contact form and its Reeb vector field coincides with the geodesic vector field X. There holds (see [8, V.2.5 and (5.2.12)])

$$\alpha_{\mathrm{can}} \wedge \mathrm{d}\alpha_{\mathrm{can}} = 2\pi\eta \wedge \mathfrak{p}_\infty^*\mu \tag{2.2}$$

so that $\alpha_{\mathrm{can}} \wedge \mathrm{d}\alpha_{\mathrm{can}}$ is a positive form with respect to the orientation $\mathfrak{o}_{\mathrm{T}^1M}$ given in (1.3).

Definition 2.1. We call a two-form Ω on T^1M **odd-symplectic** if it is closed and maximally non-degenerate. An odd-symplectic form is called **Zoll** if there exists an oriented S^1-bundle $\mathfrak{p}_\Omega : \mathrm{T}^1M \to M_\Omega$ such that the oriented leaves of the distribution $\ker\Omega$ are fibres of $\mathfrak{p}_\Omega$. In this case, Ω descends to a symplectic form ω on M_Ω, i.e. $\mathfrak{p}_\Omega^*\omega = \Omega$. We endow M_Ω with the orientation induced by T^1M and $\mathfrak{p}_\Omega$, or, equivalently, the orientation given by ω.

Remark 2.2. The two-form

$$\Omega_\infty := \mathfrak{p}_\infty^*\mu$$

is an example of a Zoll odd-symplectic form. Its associated oriented bundle is $\mathfrak{p}_\infty$.

The two-form

$$\Omega_f := \mathrm{d}\alpha_{\mathrm{can}} + \mathfrak{p}_\infty^*(f\mu),$$

is odd-symplectic, and the vector field X_f is a nowhere vanishing section of the characteristic distribution $\ker\Omega_f$. Indeed, from the equations above, we have $\Omega_f = \iota_{X_f}(\alpha_{\mathrm{can}} \wedge \mathrm{d}\alpha_{\mathrm{can}})$. This also shows that

$$\mathfrak{o}_{\mathrm{T}^1M} = \mathfrak{o}_{X_f} \oplus \mathfrak{o}_{\Omega_f}, \tag{2.3}$$

where $\mathfrak{o}_{\Omega_f}$ is the co-orientation of the characteristic distribution of Ω_f. We readily see that f is Zoll in the sense of Definition 1.1 if and only if Ω_f is Zoll in the sense of Definition 2.1.

To determine the cohomology class $[\Omega_f] = f_{\mathrm{avg}}[\Omega_\infty] \in H^2_{\mathrm{dR}}(\mathrm{T}^1M)$, we observe that the map $\mathfrak{p}_\infty^* : H^2_{\mathrm{dR}}(M) \to H^2_{\mathrm{dR}}(\mathrm{T}^1M)$ is zero if $M \neq \mathbb{T}^2$ and is injective if $M = \mathbb{T}^2$. This follows from the Gysin sequence

$$H^0_{\mathrm{dR}}(M) \xrightarrow{\cup e} H^2_{\mathrm{dR}}(M) \xrightarrow{\mathfrak{p}^*_\infty} H^2_{\mathrm{dR}}(\mathrm{T}^1 M)$$

and (1.2). Therefore, Ω_∞ is exact if and only if $M \neq \mathbb{T}^2$.

Let us now write the f-magnetic length of some $c \in \Lambda(M; \mathfrak{h}_\infty)$ in term of Ω_f. The Riemannian length of c can be expressed as

$$\ell(c) = \int_{\mathbb{R}/T\mathbb{Z}} (c, \dot c)^* \alpha_{\mathrm{can}}.$$

If $\Gamma : [0,1] \times S^1 \to \mathrm{T}^1 M$ is the cylinder lifting an admissible capping disc C, we have

$$\ell_f(c) = \ell(c) + \int_{D^2} C^*(f\mu) = \int_{[0,1]\times S^1} \Gamma^* \Omega_f, \tag{2.4}$$

due to $\int_0^1 \Gamma(0,\cdot)^* \alpha_{\mathrm{can}} = 0$ and Stokes' Theorem. From this formula, we deduce that the value of $\ell_f(c)$ does not depend on the choice of the admissible disc C. Let C' be another admissible capping disc for c, which is the projection of another cylinder Γ' in $\mathrm{T}^1 M$ such that $\Gamma'(0,\cdot)$ is an oriented $\mathfrak{p}_\infty$-fibre and $\Gamma'(1,\cdot)$ coincides with $(c, \dot c)$, up to reparametrisation. The cylinder Γ'' obtained concatenating $s \mapsto \Gamma'(s,\cdot)$ with the reversed cylinder $s \mapsto \Gamma(1-s,\cdot)$ projects to a sphere $\sigma : S^2 \to M$. The value of $\ell_f(c)$ obtained using C is the same as the one obtained using C' if and only if the following integral vanishes

$$\int_{[0,1]\times S^1} (\Gamma'')^* \Omega_f = \int_{[0,1]\times S^1} (\Gamma'')^* \big(\mathfrak{p}^*_\infty(f\mu)\big) = \langle [f\mu], [\sigma] \rangle.$$

Therefore, it is enough to show that $[\sigma] = 0$. If $M \neq S^2$, this is clear since $\pi_2(M)$ vanishes. If $M = S^2$, then $[\sigma] = 0$ if and only if $\langle e, [\sigma]\rangle = 0$. By (2.1) and Stokes' Theorem, we get

$$\langle e, [\sigma] \rangle = \langle [\tfrac{1}{2\pi} K\mu], [\sigma] \rangle = \int_{[0,1]\times S^1} (\Gamma'')^* \eta = \int_{S^1} (\gamma_0')^* \eta - \int_{S^1} \gamma_0^* \eta = 1 - 1 = 0.$$

2.2. Proof of Proposition 1.3

Case $M = \mathbb{T}^2$. Since $\mathfrak{p}_\infty$ is a trivial bundle, [6, Proposition 1.11] implies that $e_f = 0$ and that Ω_f is not exact. Moreover, by [6, Lemma 4.5], if $\mathfrak{p}$:

$\mathrm{T}^1\mathbb{T}^2 \to M_{\mathfrak{p}}$ is an oriented S^1-bundle over a closed surface $M_{\mathfrak{p}}$, for any $c \in H^2_{\mathrm{dR}}(M_{\mathfrak{p}})$ and $\mathrm{pt} \in M_{\mathfrak{p}}$ there holds

$$\mathrm{PD}(\mathfrak{p}^* c) = \langle c, [M_{\mathfrak{p}}]\rangle \cdot [\mathfrak{p}^{-1}(\mathrm{pt})] \in H_1(\mathrm{T}^1\mathbb{T}^2; \mathbb{R})$$

where PD denotes Poincaré duality. Applying this identity for $\mathfrak{p} = \mathfrak{p}_\infty$, we get

$$\mathrm{PD}([\Omega_f]) = \mathrm{PD}([\mathfrak{p}_\infty^*(f\mu)]) = \mathrm{area}(\mathbb{T}^2) \cdot f_{\mathrm{avg}} \cdot [\mathfrak{p}_\infty^{-1}(\mathrm{pt})] \in H_1(\mathrm{T}^1\mathbb{T}^2; \mathbb{R}),$$

which implies $f_{\mathrm{avg}} \neq 0$ and, as a consequence, $K_f > 0$. Applying again the identity for $\mathfrak{p} = \mathfrak{p}_f$, we deduce

$$\mathrm{PD}([\Omega_f]) = \langle [\omega_f], [\mathbb{T}^2_f]\rangle \cdot [\mathfrak{p}_f^{-1}(\mathrm{pt})],$$

where $\langle [\omega_f], [\mathbb{T}^2_f]\rangle > 0$, as ω_f is a positive symplectic form. By comparing the two formulae for $\mathrm{PD}([\Omega_f])$, we derive $[\mathfrak{p}_f^{-1}(\mathrm{pt})] = \mathrm{sign}(f_{\mathrm{avg}}) \cdot [\mathfrak{p}_\infty^{-1}(\mathrm{pt})]$, as the homology classes of the fibres of $\mathfrak{p}_\infty$ and of $\mathfrak{p}_f$ are both primitive. Since $\mathrm{T}^1\mathbb{T}^2$ is diffeomorphic to the three-torus, we have an isomorphism between the set of free-homotopy classes and the set of first homology classes, so that $\mathfrak{h}_f = \mathrm{sign}(f_{\mathrm{avg}})\mathfrak{h}_\infty$ holds, as well. Finally, as $[\Omega_f] = f_{\mathrm{avg}}[\Omega_\infty]$, the existence of a path $\{\mathfrak{p}_r\}$ connecting $\mathrm{sign}(f_{\mathrm{avg}})\mathfrak{p}_\infty$ to $\mathfrak{p}_f$ and of the commuting diagram (1.5) follows from [6, Proposition 1.11, Remark 1.12]. □

Case $M \neq \mathbb{T}^2$. The existence of a path $\{\mathfrak{p}_r\}$ connecting $\pm\mathfrak{p}_\infty$ to $\mathfrak{p}_f$ and of the commuting diagram (1.5) is a consequence of [7, Proposition 1.2] and [6, Proposition 1.11, Remark 1.12]. It implies at once that $\mathfrak{h}_f = \pm\mathfrak{h}_\infty$, and by continuity also that $\langle e_f, [M_f]\rangle = \chi(M)$, since (1.2) holds. Notice indeed that the Euler number of $\mathfrak{p}_\infty$ is also equal to $\chi(M)$, since the Euler class of $-\mathfrak{p}_\infty$ is minus the Euler class of $\mathfrak{p}_\infty$ and $-\mathfrak{p}_\infty$ induces the opposite orientation on M, so the two minus signs cancel out when computing the Euler number. When $M = S^2$, there is nothing else to prove, so let us assume for the rest of the proof that $\chi(M) < 0$. In this case, the inequality $K_f > 0$ is proven in Corollary 2.13 and we are left to establish (b). We will show, namely, that $f_{\mathrm{avg}} > 0$, provided the magnetic geodesics of f lie in $\Lambda(M; \mathfrak{h}_\infty)$. We assume by contradiction that $f_{\mathrm{avg}} < 0$. Thanks to Remark 1.5, this is equivalent to assuming that $\bar{\ell}(f) < 0$. By Theorem 1.9, we have $\ell_f(c) = \bar{\ell}(f)$ for every prime closed f-magnetic geodesic c. Notice that we are allowed to use Theorem 1.9, since, when $M \neq \mathbb{T}^2$, such a result depends only on part (a) and (c) of Proposition 1.3. Let $L_0 : \mathrm{T}M \to \mathbb{R}$ be the energy density $L_0(q, v) = \frac{1}{2} g_q(v, v)$ and let $\Lambda_0(M)$ be the set of contractible loops on M with arbitrary period. We define the Lagrangian free-period action functional $S_k^{L_0} : \Lambda_0(M) \to \mathbb{R}$ with parameter $k \in \mathbb{R}$ by

$$S_k^{L_0}(c) := \int_0^T \Big[L_0 + k\Big](c(t), \dot{c}(t))\mathrm{d}t + \int_{D^2} C^*(f\mu), \quad \forall\, c \in \Lambda_0(M),$$

where $C : D^2 \to M$ is a capping disc for $c \in \Lambda_0(M)$. The definition does not depend on C since $\pi_2(M) = 0$. Moreover, observe that if $c^m : \mathbb{R}/mT\mathbb{Z} \to M$ is the m-th iteration of c, we have

$$S_k^{L_0}(c^m) = mS_k^{L_0}(c). \tag{2.5}$$

Let $\mathcal{L}^{L_0} : \mathrm{T}M \to \mathrm{T}^*M$ be the Legendre transform associated with the Lagrangian L_0 and let $H_0 : \mathrm{T}^*M \to \mathbb{R}$ be the kinetic energy function with respect to the dual metric. The function H_0 is Legendre dual to L_0, namely

$$L_0(q, v) = \mathcal{L}^{L_0}(q, v) \cdot v - H_0\big(\mathcal{L}^{L_0}(q, v)\big), \quad \forall\, (q, v) \in \mathrm{T}M.$$

Let $\widehat{\Omega}_f$ be the twisted symplectic form on T^*M, which is defined by

$$\widehat{\Omega}_f := \mathrm{d}\widehat{\alpha}_{\mathrm{can}} + \widehat{\mathfrak{p}}_\infty^*(f\mu),$$

where $\widehat{\alpha}_{\mathrm{can}}$ is the canonical one-form on T^*M and $\widehat{\mathfrak{p}}_\infty : \mathrm{T}^*M \to M$ is the foot-point projection. We also define the Hamiltonian free-period action functional $A_k^{H_0} : \Lambda_0(\mathrm{T}^*M) \to \mathbb{R}$ on the set of contractible loops in T^*M by

$$\begin{aligned} A_k^{H_0}(q, p) := &\int_{D^2} (Q, P)^*\widehat{\Omega}_f \\ &+ \int_0^T \Big[k - H_0\Big](q(t), p(t))\mathrm{d}t, \quad \forall\, (q, p) \in \Lambda_0(\mathrm{T}^*M), \end{aligned}$$

where $(Q, P) : D^2 \to \mathrm{T}^*M$ is any capping disc for $(q, p) \in \Lambda_0(\mathrm{T}^*M)$. Since $\pi_2(\mathrm{T}^*M) = 0$, the definition does not depend on (Q, P). For all $c \in \Lambda_0(M)$, there holds

$$S_k^{L_0}(c) = A_k^{H_0}\big(\mathcal{L}^{L_0}(c, \dot{c})\big).$$

It is a classical result that the Hamiltonian flow lines of H_0 with respect to $\widehat{\Omega}_f$ on the energy hypersurface $\{H_0 = \frac{1}{2}\}$ are exactly the curves $\mathcal{L}^{L_0}(c, \dot{c})$, where c is an f-magnetic geodesic. Therefore, if c is a prime closed f-magnetic geodesic, we have

$$0 > \ell_f(c) = S^{L_0}_{\frac{1}{2}}(c) = A^{H_0}_{\frac{1}{2}}(\mathcal{L}^{L_0}(c,\dot{c})) = \int_{D^2} (Q^{L_0}_c, P^{L_0}_c)^* \widehat{\Omega}_f, \tag{2.6}$$

where $(Q^{L_0}_c, P^{L_0}_c)$ is a capping disc for $\mathcal{L}^{L_0}(c,\dot{c})$.

The two-form Ω_f on T^1M is exact since it is Zoll and $\mathfrak{p}_f$ is non-trivial. Thus there exists a contact form λ_f on T^1M such that $\mathrm{d}\lambda_f = \Omega_f$. This implies that $\{H_0 = 1/2\}$ is a stable hypersurface inside $(\mathrm{T}^*M, \widehat{\Omega}_f)$ in the sense of [17, Section 4.3] and [10, Section 2]. By [18, Lemma 2.1], there exist $k_* \in \mathbb{R}$, an open interval I containing k_*, a function $h : I \to (0,\infty)$, a Tonelli Hamiltonian $H : \mathrm{T}^*M \to \mathbb{R}$ and a diffeomorphism $\Psi : \{H = k_*\} \times I \to \{H \in I\}$ such that the following identities hold

(*i*) $\{H_0 \le 1/2\} = \{H \le k_*\}$,

(*ii*) $H \circ \Psi(p,r) = r, \quad \forall\,(p,r) \in \{H = k_*\} \times I$,

(*iii*) $\Psi\Big(\Phi^{h(r)t}_H(p), r\Big) = \Phi^t_H(\Psi(p,r)), \quad \forall\, t \in \mathbb{R},\ (p,r) \in \{H = k_*\} \times I$,

where Φ_H is the Hamiltonian flow of H. Actually, the result in [18] is stated for Hamiltonians on the standard cotangent bundle $(T^*M, \widehat{\Omega}_0)$ but a careful inspection of the proof reveals that the statement holds also on the twisted cotangent bundle $(T^*M, \widehat{\Omega}_f)$. Finally, let $L : \mathrm{T}M \to \mathbb{R}$ be the Legendre dual of H. From (i), it follows that Φ_H and Φ_{H_0} have the same oriented orbits on $\{H_0 = 1/2\} = \{H = k_*\}$. Therefore, if c_1 is a prime periodic solution of the Euler-Lagrange flow of L with energy k_*, we have that

$$\int_{D^2} (Q^{L_0}_c, P^{L_0}_c)^* \widehat{\Omega}_f = \int_{D^2} (Q^{L}_{c_1}, P^{L}_{c_1})^* \widehat{\Omega}_f, \tag{2.7}$$

where $(Q^L_{c_1}, P^L_{c_1})$ is a capping disc for $\mathcal{L}^L(c_1,\dot{c}_1)$. The left-hand side of (2.7) is negative by (2.6). Moreover, using the last two passages in (2.6) backwards with k_*, c_1, H and L instead of $\frac{1}{2}$, c, H_0 and L_0, we see that the right-hand side of (2.7) is equal to $S^L_{k_*}(c_1)$. Summing up, we have shown that $S^L_{k_*}(c_1) < 0$. By (ii),(iii) and (2.5), we see that, up to shrinking the interval I, we can assume that all periodic solutions (prime and iterated) of the Euler-Lagrange flow of L with energy $k \in I$ have negative action S^L_k. However, since $\mathfrak{p}_\infty : \{H = k_*\} \to M$ is an S^1-bundle and there exist contractible curves with negative S^L_k action for all $k \in I$, we conclude that $I \subset (c_*(L), c_u(L))$, where $c_*(L) = -\min_{q\in M} L(q,0)$ and $c_u(L) \in \mathbb{R}$ is the Mañé critical value of the universal cover. However, by [19, Theorem 1.1(2)] or [3, Theorem 1.3], there exists for almost every $k \in (c_*(L), c_u(L))$ a period orbit of the Euler-Lagrange flow of L with energy k and positive S^L_k-action. This contradiction shows that $\bar{\ell}(f) > 0$ and finishes the proof. $\square$

2.3. The space of curves $\Lambda(M; \mathfrak{h}_\infty)$

In this subsection, we will study the set $\Lambda(M; \mathfrak{h}_\infty)$ in more detail. We start with a characterisation of this space by means of the turning number of an immersed curve $b : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ that is the winding number of its velocity curve $\dot{b} : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ with respect to $0 \in \mathbb{R}^2$.

Lemma 2.3. *Let c be an immersed closed curve in M that is contractible. The curve c belongs to $\Lambda(M; \mathfrak{h}_\infty)$ if and only if the following condition holds.*

- *Case $M = S^2$.: The turning number of $\psi \circ c$ is odd, where $\psi : S^2 \setminus \{q\} \to \mathbb{R}^2$ is a diffeomorphism and $q \in M$ lies outside the support of c.*
- *Case $M \neq S^2$.: The turning number of $\widetilde{c}$ is equal to -1, where $\widetilde{c} : \mathbb{R}/T\mathbb{Z} \to \widetilde{M} \subset \mathbb{R}^2$ is a lift of c to the universal cover of M. In this case, the curve c is prime.* $\square$

A somewhat more geometrical sufficient condition for a curve to be in $\Lambda(M; \mathfrak{h}_\infty)$ is given by the notion of Alexandrov embeddedness.

Definition 2.4. A closed and arc-length parametrised curve c in M is called **negatively Alexandrov embedded** if it admits a negatively immersed capping disc $C : D^2 \to M$.

Remark 2.5. By the Schönflies Theorem, a closed curve c in M is negatively Alexandrov embedded, if:

- $M = S^2$ and c is embedded;
- $M \neq S^2$ and the lift $\widetilde{c}$ to the universal cover $\widetilde{M}$ bounds a compact region in the clock-wise direction.

Lemma 2.6. *If a closed curve c in M is negatively Alexandrov embedded, $c \in \Lambda(M; \mathfrak{h}_\infty)$ and any of its immersed capping discs is admissible. In particular, the curves from Remark 2.5 belong to $\Lambda(M; \mathfrak{h}_\infty)$.*

Proof. Let $C : D^2 \to M$ be a negatively immersed capping disc for c. Then we can define

$$(0,1] \times S^1 \ni (s,t) \longmapsto \left(C(se^{2\pi i t}), \frac{\partial_t C(se^{2\pi i t})}{|\partial_t C(se^{2\pi i t})|} \right) \in \mathrm{T}^1 M.$$

Since C is a local embedding around $0 \in D^2$, this map extends to $s = 0$ and yields a cylinder $\Gamma : [0,1] \times S^1 \to \mathrm{T}^1 M$ such that

$$\begin{aligned} &(i) \quad \mathfrak{p}_\infty(\Gamma(s,t)) = C(se^{2\pi i t}), \quad \forall\, (s,t) \in [0,1] \times S^1, \\ &(ii) \quad \Gamma(0,t) = \Phi^V_{a(t)}(z), \quad \Gamma(1,t) = \big(c(tT), \dot{c}(tT)\big), \quad \forall\, t \in S^1, \end{aligned}$$

for some orientation-preserving diffeomorphism $a : S^1 \to S^1$ and element $z \in \mathrm{T}^1 M$. This shows that C is admissible. □

We finish this subsection by providing a partial answer to the following natural question. If all the f-magnetic geodesics are closed, is the function f (or, equivalently, the odd-symplectic two-form Ω_f) Zoll? We collect the result in a lemma, which is a magnetic counterpart of the Gromoll-Grove Theorem [15].

Lemma 2.7. *Suppose that every f-magnetic geodesic is closed. The function f is Zoll in the following two cases:*

(i) There holds $M \neq S^2$ and all the prime geodesics lie in $\Lambda(M; \mathfrak{h}_\infty)$.

(ii) There holds $M = S^2$ and either all prime closed magnetic geodesics are embedded or the function f is positive and all prime closed magnetic geodesics are negatively Alexandrov embedded.

Proof. A theorem of Epstein [12] yields an S^1-action $\Phi_t : \mathrm{T}^1 M \to \mathrm{T}^1 M$, $t \in S^1$, whose orbits coincide with the tangent lifts of magnetic geodesics (up to reparametrisation) and such that the set

$$N := \big\{ z \in \mathrm{T}^1 M \mid \Phi_t(z) \neq z, \ \forall\, t \in S^1 \setminus 0 \big\}$$

is non-empty. The lemma follows once we show that $N = \mathrm{T}^1 M$. The set N is open, so that, by the connectedness of $\mathrm{T}^1 M$, we just have to prove that N is also closed. Let $(z_m) \subset N$ be a sequence such that $z_m \to z \in \mathrm{T}^1 M$. Let (c_m) be the corresponding sequence of magnetic geodesics and c the magnetic geodesic corresponding to z. Since $z_m \to z$, there exists $k \in \mathbb{N}^*$ such that (c_m) converges in the C^∞-topology to the k-th iteration of c. It suffices to show that $k = 1$. This would give that $z \in N$, and hence, that N is closed.

Let us suppose that $M \neq S^2$. The lifts $(\widetilde{c}_m)$ and $\widetilde{c}$ to $\widetilde{M}$ are such that $(\widetilde{c}_m)$ converges to the k-th iteration of $\widetilde{c}$. From Lemma 2.3, we conclude that k-times the turning number of $\widetilde{c}$ is equal to -1, which forces $k = 1$.

Let us suppose that $M = S^2$. If all prime closed magnetic geodesics are embedded, then all the curves c_m are embedded. Since S^2 is an oriented surface, it follows that c is also embedded, which forces $k = 1$. If f is everywhere positive and the curves c_m are negatively Alexandrov embedded, then by [21, Lemma 3.2], c is also negatively Alexandrov embedded. From [21, Lemma 3.1], it follows that c is prime, i.e. $k = 1$. □

Remark 2.8. In the previous lemma, we need extra conditions when $M = S^2$ since there exists a sequence of prime Alexandrov embedded curves (c_m) which converges in the C^∞-topology to a curve c, which is not prime. In particular, the set $\{c \in \Lambda(S^2; \mathfrak{h}_\infty) \mid c \text{ is prime}\}$ is not closed in the C^∞-topology. Furthermore, there are examples of positive magnetic functions on the two-sphere all of whose magnetic geodesics are closed but their lifts to the unit tangent bundle are the orbits of a non-free S^1-action [5].

2.4. Strong magnetic functions

When $f : M \to (0, \infty)$ is large, then f-magnetic geodesics stay close to the fibres of $\mathfrak{p}_\infty$. In this case, we expect Ω_f to approximate the Zoll form $\Omega_\infty = \mathfrak{p}_\infty^* \mu$. Using the notion of C-strong function given in Definition 1.10, we make this observation precise in the next lemma. This result will be employed in Section 3.2 to establish the magnetic systolic-diastolic inequality for C-strong functions.

Lemma 2.9. *Let $\mathcal{U}$ be a C^2-neighbourhood of Ω_∞ in the space of two-forms on $\mathrm{T}^1 M$. There exists a constant $C_{\mathcal{U}} > 0$ with the following property: For every $C_{\mathcal{U}}$-strong $f : M \to (0, \infty)$, there is a diffeomorphism $\Psi : \mathrm{T}^1 M \to \mathrm{T}^1 M$ isotopic to the identity such that $\frac{1}{f_{\mathrm{avg}}} \Psi^* \Omega_f \in \mathcal{U}$.*

Proof. We define

$$f_{\mathrm{norm}} := \frac{f}{f_{\mathrm{avg}}}$$

and observe that there holds

$$\min f_{\mathrm{norm}} \leq 1 \leq \max f_{\mathrm{norm}}.$$

The two-form $(f_{\mathrm{norm}} - 1)\mu$ is exact. By standard elliptic arguments (see for instance [20, Chapter 10]), we can choose a primitive one-form $\zeta \in \Omega^1(M)$

of $(f_{\mathrm{norm}} - 1)\mu$ such that

$$\|\zeta\|_{C^k} \le C_k \|f_{\mathrm{norm}} - 1\|_{C^k} \le C_k \|f_{\mathrm{norm}}\|_{C^k}, \qquad \forall\, k \in \mathbb{N} \tag{2.8}$$

for some constant $C_k > 0$ depending solely on g and $k \in \mathbb{N}$. For $s \in [0,1]$, let μ_s be the two-form given by $\mu_s := q(f,s)\mu$, where $q(f,s) := s f_{\mathrm{norm}} + (1-s)$, and Y_s be the time-dependent vector field defined through

$$\iota_{Y_s}\mu_s = -\zeta.$$

If $\psi : M \to M$ is the time-one map of Y_s, an application of Moser's trick yields

$$\psi^*(f_{\mathrm{norm}}\mu) = \mu. \tag{2.9}$$

If $\sharp : \mathrm{T}^*M \to \mathrm{T}M$ is the metric duality and $* : \mathrm{T}^*M \to \mathrm{T}^*M$ the Hodge star operator, we can write Y_s explicitly as

$$Y_s = \frac{\sharp * \zeta}{q(f,s)}.$$

Since $*$ and $\sharp$ are smooth bundle maps, we have (possibly with bigger $C_k > 0$)

$$\|Y_s\|_{C^k} \le C_k \max_{s\in[0,1]} \left\| \frac{\zeta}{q(f,s)} \right\|_{C^k}, \qquad \forall\, k \in \mathbb{N}. \tag{2.10}$$

We claim that the following bound holds (possibly with bigger $C_k > 0$):

$$\max_{s\in[0,1]} \left\| \frac{\zeta}{q(f,s)} \right\|_{C^k} \le C_k \langle f_{\mathrm{norm}} \rangle_k^{k+1} = C_k \langle f \rangle_k^{k+1}, \qquad \forall\, k \in \mathbb{N}. \tag{2.11}$$

where the last equality is due to (1.8). We prove the claim by induction and observe preliminarily that $q(f,s) \ge \min f_{\mathrm{norm}}$. For $k=0$, the estimate follows directly from (2.8). Suppose now that the estimate holds for all $k' \le k-1$. Since

$$\left\| \frac{\zeta}{q(f,s)} \right\|_{C^k} = \left\| \frac{\zeta}{q(f,s)} \right\|_{C^{k-1}} + \left\| \nabla^k \frac{\zeta}{q(f,s)} \right\|_{C^0},$$

we just have to bound the second term. We apply the Leibniz rule to the k-th derivative of the product $q(f,s) \cdot \frac{\zeta}{q(f,s)} = \zeta$ and obtain

$$\nabla^k \Big(\frac{\zeta}{q(f,s)} \Big) = \frac{1}{q(f,s)} \bigg[\nabla^k \zeta - s \sum_{k'=0}^{k-1} \binom{k}{k'} \nabla^{k-k'} f_{\mathrm{norm}} \cdot \nabla^{k'} \frac{\zeta}{q(f,s)} \bigg],$$

where we have used that $\nabla^{k-k'} q(f,s) = s\nabla^{k-k'} f_{\mathrm{norm}}$, since $k-k' \geq 1$. Consequently, we estimate using (2.8) and (2.11)

$$\begin{aligned}\Big\|\nabla^k \frac{\zeta}{q(f,s)}\Big\|_{C^0} &\leq \frac{1}{\min f_{\mathrm{norm}}}\Big[C_k\|f_{\mathrm{norm}}\|_{C^k} \\ &\qquad + \sum_{k'=0}^{k-1}\binom{k}{k'}\|f_{\mathrm{norm}}\|_{C^k} C_{k-1}\langle f_{\mathrm{norm}}\rangle_{k-1}^k\Big] \\ &\leq \frac{1}{\min f_{\mathrm{norm}}} C'_k\bigg(\|f_{\mathrm{norm}}\|_{C^k} + \frac{\|f_{\mathrm{norm}}\|_{C^k}^{k+1}}{(\min f_{\mathrm{norm}})^k}\bigg) \\ &\leq C'_k\frac{\|f_{\mathrm{norm}}\|_{C^k}}{\min f_{\mathrm{norm}}} + C'_k\bigg(\frac{\|f_{\mathrm{norm}}\|_{C^k}}{\min f_{\mathrm{norm}}}\bigg)^{k+1} \\ &\leq (C'_k+1)\bigg(\frac{\|f_{\mathrm{norm}}\|_{C^k}}{\min f_{\mathrm{norm}}}\bigg)^{k+1}\end{aligned}$$

with some $C'_k > 0$ depending only on g and k. The claim is therefore established.

Using the Levi-Civita connection for $\mathfrak{p}_\infty$, we lift Y_s horizontally to Z_s on T^1M, so that $\mathrm{d}\mathfrak{p}_\infty(Z_s) = Y_s$. Since the lifting map $Y_s \mapsto Z_s$ is smooth and depends only on g, but not on f, there is a constant $C''_k > 0$ depending on k and g such that

$$\|Z_s\|_{C^k} \leq C''_k\|Y_s\|_{C^k}, \qquad \forall\, k \in \mathbb{N}. \tag{2.12}$$

The time-one map $\Psi : \mathrm{T}^1M \to \mathrm{T}^1M$ of Z_s lifts the time-one map ψ of Y_s, so that

$$\Psi^*(\mathfrak{p}_\infty^*(f_{\mathrm{norm}}\mu)) = \mathfrak{p}_\infty^*\mu,$$

by (2.9). Putting together (2.10), (2.11), (2.12), and Lemma A.1, we get

$$B_{2,2}\big(\|\mathrm{d}\Psi\|\big) \leq \Big(\langle f\rangle_3^4 + \langle f\rangle_2^6\Big)e^{C_3\langle f\rangle_1^2},$$

for a (possibly bigger) constant $C_3 > 0$. Hence, using (A.1) we estimate

$$\|\Psi^*(\mathrm{d}\alpha_{\mathrm{can}})\|_{C^2} \leq C'''_3 B_{2,2}\big(\|\mathrm{d}\Psi\|\big)\|\mathrm{d}\alpha_{\mathrm{can}}\|_{C^2} \leq \Big(\langle f\rangle_3^4 + \langle f\rangle_2^6\Big)e^{C_3\langle f\rangle_1^2},$$

where $C'''_3 > 0$ depends only on g and where we take a bigger constant $C_3 > 0$ if necessary to incorporate $\|\mathrm{d}\alpha_{\mathrm{can}}\|_{C^2}$ and it is possible to bring the constant to the exponent since $\langle f\rangle_1^2 \geq 1$.

Let us suppose now that f is C-strong for some positive number $C > 0$. We compute

$$\tfrac{1}{f_{\mathrm{avg}}}\Psi^*\Omega_f - \Omega_\infty = \tfrac{1}{f_{\mathrm{avg}}}\Psi^*(\mathrm{d}\alpha_{\mathrm{can}}) + \Psi^*(\mathfrak{p}_\infty^*(f_{\mathrm{norm}}\mu)) - \mathfrak{p}_\infty^*\mu = \tfrac{1}{f_{\mathrm{avg}}}\Psi^*(\mathrm{d}\alpha_{\mathrm{can}}).$$

Combining this identity with the bound for $\|\Psi^*(\mathrm{d}\alpha)\|_{C^2}$ found above, we arrive at

$$\begin{aligned}\left\|\tfrac{1}{f_{\mathrm{avg}}}\Psi^*\Omega_f - \Omega_\infty\right\|_{C^2} &= \tfrac{1}{f_{\mathrm{avg}}}\|\Psi^*(\mathrm{d}\alpha_{\mathrm{can}})\|_{C^2} \\ &\leq \frac{\big(\langle f\rangle_3^4 + \langle f\rangle_2^6\big)e^{C_3\langle f\rangle_1^2}}{\big(\langle f\rangle_3^4 + \langle f\rangle_2^6\big)e^{C\langle f\rangle_1^2}} = e^{(C_3-C)\langle f\rangle_1^2} \leq e^{C_3-C},\end{aligned}$$

which can be made arbitrarily small, if C is arbitrarily large. In particular, $\tfrac{1}{f_{\mathrm{avg}}}\Psi^*\Omega_f$ belongs to the given C^2-neighbourhood $\mathcal{U}$. $\square$

2.5. A systolic-diastolic inequality for odd-symplectic forms

The aim of this subsection is twofold. First we give definitions and properties of the volume and the action of odd-symplectic forms. Then we recall a local systolic-diastolic inequality for odd-symplectic forms on closed three-manifolds established in [6].

Weakly Zoll pairs. Consider the space of all oriented S^1-bundles $\mathfrak{p} : \mathrm{T}^1M \to M_{\mathfrak{p}}$ with total space T^1M, where $M_{\mathfrak{p}}$ is some closed oriented surface (diffeomorphic to M). Let $\mathfrak{P}^0(\mathrm{T}^1M)$ be the connected component of such a space containing $\mathfrak{p}_\infty : \mathrm{T}^1M \to M$. A pair $(\mathfrak{p}, c)$, where $\mathfrak{p} \in \mathfrak{P}^0(\mathrm{T}^1M)$ and $c \in H^2_{\mathrm{dR}}(M_{\mathfrak{p}})$ is called a weakly Zoll pair. A closed two-form Ω on T^1M is said to be associated with $(\mathfrak{p}, c)$, if $\Omega = \mathfrak{p}^*\omega$ for some closed two-form ω on $M_{\mathfrak{p}}$ satisfying $[\omega] = c$. As discussed above, every Zoll form Ω canonically defines a weakly Zoll pair $(\mathfrak{p}_\Omega, [\omega])$. For example, the Zoll form $\Omega_\infty = \mathfrak{p}_\infty^*\mu$ is associated with the weakly Zoll pair $(\mathfrak{p}_\infty, [\mu])$.

Let $\mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1M)$ be the set of all weakly Zoll pairs $(\mathfrak{p}, c)$ such that

$$\mathfrak{p} \in \mathfrak{P}^0(\mathrm{T}^1M), \qquad \mathfrak{p}^*c = [\Omega_\infty] \in H^2_{\mathrm{dR}}(\mathrm{T}^1M).$$

Below, we define and compute volume, action, and Zoll polynomial with respect to some fixed reference weakly Zoll pair

$$(\mathfrak{p}_\infty, c_0) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1M).$$

As we specify in the next subsection, we take different reference pairs for $M \neq \mathbb{T}^2$ and for $M = \mathbb{T}^2$. This will enable us to simplify computations. However, as observed in [6, Remark 1.19], a different choice results in different volume, action, and Zoll polynomial but in an equivalent systolic-diastolic inequality.

Volume. We pick any closed form ω_0 on M with $[\omega_0] = c_0$ and set

$$\Omega_0 = \mathfrak{p}_\infty^* \omega_0.$$

Let Ω be a closed two-form on $\mathrm{T}^1 M$ with the same cohomology class as Ω_0. We choose a one-form α on $\mathrm{T}^1 M$ such that $\Omega = \Omega_0 + \mathrm{d}\alpha$. The volume of α is defined by

$$\mathrm{Vol}(\alpha) = \frac{1}{2} \int_{\mathrm{T}^1 M} \alpha \wedge \mathrm{d}\alpha + \int_{\mathrm{T}^1 M} \alpha \wedge \Omega_0$$

and depends only on c_0 and not on the chosen ω_0 by [6, Lemma 5.2.(iii)]. As seen in Section 2.1, $\mathfrak{p}_\infty^* : H^2_{\mathrm{dR}}(M) \to H^2_{\mathrm{dR}}(\mathrm{T}^1 M)$ vanishes when $M \neq \mathbb{T}^2$, and thus $[\Omega_0] = 0$. In this case $\mathrm{Vol}(\alpha') = \mathrm{Vol}(\alpha)$ for any α' satisfying $\mathrm{d}\alpha' = \mathrm{d}\alpha$. Therefore, we define the volume by

$$\mathfrak{Vol}(\Omega) = \mathrm{Vol}(\alpha).$$

By [6, Proposition 2.8], if Ψ is a diffeomorphism on $\mathrm{T}^1 M$ isotopic to the identity, then

$$\mathfrak{Vol}(\Psi^* \Omega) = \mathfrak{Vol}(\Omega). \tag{2.13}$$

If $M = \mathbb{T}^2$, then it can happen that $\mathrm{d}\alpha' = \mathrm{d}\alpha$ but $\mathrm{Vol}(\alpha') \neq \mathrm{Vol}(\alpha)$. In this case, we can choose α such that $\mathrm{Vol}(\alpha) = 0$. Such a one-form is called **normalised** and we declare

$$\mathfrak{Vol}(\Omega) = 0.$$

Action. We define the action on the space $\Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1 M)$ of one-periodic curves in the free homotopy class $\mathfrak{h}_\infty \in [S^1, \mathrm{T}^1 M]$ of $\mathfrak{p}_\infty$-fibres by

$$\mathcal{A}_\alpha : \Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1 M) \to \mathbb{R}, \qquad \gamma \mapsto \int_{S^1} \gamma_0^* \alpha + \int_{[0,1] \times S^1} \Gamma^* \Omega.$$

where $\Gamma : [0,1] \times S^1 \to \mathrm{T}^1 M$ is any cylinder such that $\Gamma(1, \cdot) = \gamma$ and $\Gamma(0, \cdot) = \gamma_0$ is any oriented $\mathfrak{p}_\infty$-fibre. This action does not depend on the choice of ω_0 nor of Γ. Moreover, a critical point of $\mathcal{A}_\alpha$ is a closed characteristic of Ω, i.e. a closed curve tangent to the distribution $\ker\Omega$. We denote by $\mathcal{X}(\Omega)$ the set of embedded closed characteristics of Ω.

In order to define the action with respect to Ω, we observe that if α' is another one-form on T^1M such that $\Omega = \Omega_0 + \mathrm{d}\alpha'$, then

$$\mathcal{A}_{\alpha'} = \mathcal{A}_\alpha + \int_{\mathfrak{p}_\infty^{-1}(\mathrm{pt})} (\alpha' - \alpha)$$

where $\mathfrak{p}_\infty^{-1}(\mathrm{pt})$ is any fibre of $\mathfrak{p}_\infty$. When $M \neq \mathbb{T}^2$, the homology class of $\mathfrak{p}_\infty^{-1}(\mathrm{pt})$ is zero, and therefore we can simply set

$$\mathcal{A}_\Omega := \mathcal{A}_\alpha.$$

If $M = \mathbb{T}^2$, the actions $\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha'}$ might be different. Nevertheless it turns out that if α and α' have the same volume, they have the same action. In this case, we choose a normalised one-form α, i.e. $\mathrm{Vol}(\alpha) = 0$ and set

$$\mathcal{A}_\Omega := \mathcal{A}_\alpha.$$

In both cases, by [6, Proposition 6.10], if Ψ is a diffeomorphism on T^1M isotopic to the identity, then

$$\mathcal{A}_{\Psi^*\Omega}(\gamma) = \mathcal{A}_\Omega(\Psi(\gamma)). \tag{2.14}$$

Zoll polynomial. The Zoll polynomial $P : \mathbb{R} \to \mathbb{R}$ is defined by

$$P(A) = \langle e, [M]\rangle \frac{A^2}{2} + \langle c_0, [M]\rangle A. \tag{2.15}$$

For $(\mathfrak{p}, c) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1M)$, we choose any closed two-form ω on $M_\mathfrak{p}$ with $[\omega] = c$ and define the volume and the action of $(\mathfrak{p}, c)$ by

$$\mathfrak{Vol}(\mathfrak{p}, c) = \mathfrak{Vol}(\mathfrak{p}^*\omega), \qquad \mathcal{A}(\mathfrak{p}, c) = \mathcal{A}_{\mathfrak{p}^*\omega}(\mathfrak{p}^{-1}(\mathrm{pt})). \tag{2.16}$$

Note that since $\mathcal{A}(\mathfrak{p}_\infty, c_0) = 0$, there holds $\frac{\mathrm{d}P}{\mathrm{d}A}(\mathcal{A}(\mathfrak{p}_\infty, c_0)) = \langle c_0, [M]\rangle$. More generally it is shown in [6, Proposition 6.18] that for any weakly Zoll pair $(\mathfrak{p}, c) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1M)$

$$\frac{\mathrm{d}P}{\mathrm{d}A}(\mathcal{A}(\mathfrak{p}, c)) = \langle c, [M_\mathfrak{p}]\rangle. \tag{2.17}$$

The following result relates the action and the volume of a weakly Zoll pair through the Zoll polynomial. It can be thought as the equality case of the local systolic-diastolic inequality presented below.

Theorem 2.10. *[6, Theorem 1.16] There holds*

$$P(\mathcal{A}(\mathfrak{p}, c)) = \mathfrak{Vol}(\mathfrak{p}, c), \qquad \forall\,(\mathfrak{p}, c) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1 M).$$

When $M = \mathbb{T}^2$, this is equivalent to $\mathcal{A}(\mathfrak{p}, c) = 0$, $\forall\,(\mathfrak{p}, c) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1 M)$.

The general inequality. Let Ω_* be a Zoll form, which is associated with a weakly Zoll pair $(\mathfrak{p}_1, c_1) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1 M)$. In our applications Ω_* will be either Ω_{f_*} for some Zoll function f_* or Ω_∞. We fix a finite open covering $\{B_i\}$ of M_1 by balls so that all their pairwise intersections are contractible. Let $\Lambda(\mathfrak{p}_1)$ be the space of curves $\gamma \in \Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1 M)$ such that $\mathfrak{p}_1(\gamma)$ is contained in some B_i. Abbreviating $\mathcal{X}(\Omega; \mathfrak{p}_1) := \mathcal{X}(\Omega) \cap \Lambda(\mathfrak{p}_1)$, we define

$$\mathcal{A}_{\min}(\Omega) := \inf_{\gamma \in \mathcal{X}(\Omega;\mathfrak{p}_1)} \mathcal{A}_\Omega(\gamma), \qquad \mathcal{A}_{\max}(\Omega) := \sup_{\gamma \in \mathcal{X}(\Omega;\mathfrak{p}_1)} \mathcal{A}_\Omega(\gamma).$$

By [13, Section III], if an odd-symplectic form Ω is such that $\Omega - \Omega_*$ is an exact C^1-close two-form, the set $\mathcal{X}(\Omega) \cap \Lambda(\mathfrak{p}_1)$ is compact and non-empty. Therefore, the numbers $\mathcal{A}_{\min}(\Omega)$ and $\mathcal{A}_{\max}(\Omega)$ are finite and they can be shown to vary C^1-continuously with Ω. We finally state the local systolic-diastolic inequality for odd-symplectic forms.

Theorem 2.11. *[6, Corollary 1.23] There exists a C^2-neighbourhood $\mathcal{U}$ of Ω_* in the set of odd-symplectic forms on $\mathrm{T}^1 M$ with cohomology class $[\Omega_*]$ such that*

$$P(\mathcal{A}_{\min}(\Omega)) \le \mathfrak{Vol}(\Omega) \le P(\mathcal{A}_{\max}(\Omega)), \qquad \forall\,\Omega \in \mathcal{U}.$$

Moreover the equality holds in any of the two inequalities exactly when Ω is Zoll. When $M = \mathbb{T}^2$, the inequality simplifies to

$$\mathcal{A}_{\min}(\Omega) \le 0 \le \mathcal{A}_{\max}(\Omega).$$

2.6. Volume and action of magnetic functions

Case $M \neq \mathbb{T}^2$. As observed in Section 2.1, the two-forms Ω_∞ and Ω_f, where $f : M \to \mathbb{R}$ is any function, are exact. Explicit primitives are given by

$$\alpha_\infty := \frac{\mathrm{area}(M)}{\chi(M)}(\eta + \mathfrak{p}_\infty^* \zeta_\infty), \qquad \alpha_f := \alpha_{\mathrm{can}} + \frac{\mathrm{area}(M)}{\chi(M)}(f_{\mathrm{avg}}\eta + \mathfrak{p}_\infty^* \zeta),$$

where ζ and ζ_∞ are one-forms on M with differential

$$\mathrm{d}\zeta_\infty = \Big(\frac{\chi(M)}{\mathrm{area}(M)} - \frac{K}{2\pi}\Big)\mu, \qquad \mathrm{d}\zeta = \Big(\frac{\chi(M)}{\mathrm{area}(M)} f - \frac{f_{\mathrm{avg}} \cdot K}{2\pi}\Big)\mu.$$

We choose as reference weakly Zoll pair

$$(\mathfrak{p}_\infty, c_0) = (\mathfrak{p}_\infty, 0) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1 M).$$

From formula (2.15) and identity (1.2), we have the Zoll polynomial

$$P(A) = \frac{\chi(M)}{2} A^2. \tag{2.18}$$

Let Ω be an exact two-form on $\mathrm{T}^1 M$, and let α be an arbitrary primitive one-form of Ω. In this case the volume of Ω is reduced to

$$\mathfrak{Vol}(\Omega) = \frac{1}{2} \int_{\mathrm{T}^1 M} \alpha \wedge \Omega.$$

and the action of Ω is given by

$$\mathcal{A}_\Omega(\gamma) = \int_{S^1} \gamma^* \alpha, \qquad \forall\, \gamma \in \Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1 M).$$

Therefore, if α is a contact form, $\mathfrak{Vol}(\Omega)$ is (up to a sign given by the orientation) half of the contact volume and $\mathcal{A}_\Omega$ is the contact action which coincides with the period on Reeb orbits (see also [6, Remark 1.20])

We note that the volume is two-homogeneous while the action is one-homogeneous. Namely,

$$\mathfrak{Vol}(s\Omega) = s^2 \mathfrak{Vol}(\Omega), \qquad \mathcal{A}_{s\Omega} = s\mathcal{A}_\Omega, \qquad \forall\, s \in \mathbb{R}. \tag{2.19}$$

Lemma 2.12. *If $M \neq \mathbb{T}^2$ and $f : M \to \mathbb{R}$ is a function, we have*

$$\mathfrak{Vol}(\Omega_\infty) = \frac{\mathrm{area}(M)^2}{2\chi(M)}, \qquad \mathfrak{Vol}(\Omega_f) = \frac{\mathrm{area}(M)^2}{2\chi(M)} K_f.$$

If $\gamma_0 : S^1 \to \mathrm{T}^1 M$ is an oriented fibre of $\mathfrak{p}_\infty$ and $c \in \Lambda(M; \mathfrak{h}_\infty)$, then

$$\mathcal{A}_{\Omega_\infty}(\gamma_0) = \frac{\mathrm{area}(M)}{\chi(M)}, \qquad \mathcal{A}_{\Omega_f}(c, \dot{c}) = \ell_f(c) + \frac{\mathrm{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)}.$$

Proof. We compute the volume of Ω_∞ as

$$\begin{aligned}\mathfrak{Vol}(\Omega_\infty) &= \frac{1}{2}\int_{\mathrm{T}^1M}\alpha_\infty\wedge\Omega_\infty = \frac{\operatorname{area}(M)}{2\chi(M)}\int_{\mathrm{T}^1M}\eta\wedge\mathfrak{p}_\infty^*\mu \\ &= \frac{\operatorname{area}(M)}{2\chi(M)}\int_M((\mathfrak{p}_\infty)_*\eta)\mu = \frac{\operatorname{area}(M)^2}{2\chi(M)}.\end{aligned}$$

To determine the volume of Ω_f, we perform first the preliminary computation

$$\alpha_f\wedge\Omega_f = \alpha_{\mathrm{can}}\wedge\mathrm{d}\alpha_{\mathrm{can}} + \frac{\operatorname{area}(M)}{\chi(M)}\big(f_{\mathrm{avg}}\eta\wedge\mathfrak{p}_\infty^*(f\mu) + \mathfrak{p}_\infty^*\zeta\wedge\mathrm{d}\alpha_{\mathrm{can}}\big),$$

using the fact that X annihilates $\eta\wedge\mathrm{d}\alpha$ and V annihilates $\alpha\wedge\mathfrak{p}_\infty^*(f\mu)$. Then

$$\begin{aligned}2\,\mathfrak{Vol}(\Omega_f) &= \int_{\mathrm{T}^1M}\alpha_f\wedge\Omega_f \\ &= \int_{\mathrm{T}^1M}\alpha_{\mathrm{can}}\wedge\mathrm{d}\alpha_{\mathrm{can}} \\ &\quad + \frac{\operatorname{area}(M)}{\chi(M)}\left[\int_{\mathrm{T}^1M} f_{\mathrm{avg}}\eta\wedge\mathfrak{p}_\infty^*(f\mu) + \int_{\mathrm{T}^1M}\mathfrak{p}_\infty^*\zeta\wedge\mathrm{d}\alpha_{\mathrm{can}}\right] \\ &= 2\pi\int_{\mathrm{T}^1M}\eta\wedge\mathfrak{p}_\infty^*\mu \\ &\quad + \frac{\operatorname{area}(M)}{\chi(M)}\left[f_{\mathrm{avg}}\int_M f\mu + \int_{\mathrm{T}^1M}\mathfrak{p}_\infty^*(\mathrm{d}\zeta)\wedge\alpha_{\mathrm{can}}\right] \\ &= 2\pi\cdot\operatorname{area}(M) + \frac{\big(\operatorname{area}(M)\cdot f_{\mathrm{avg}}\big)^2}{\chi(M)} \\ &= \frac{\operatorname{area}(M)^2}{\chi(M)}K_f,\end{aligned}$$

where we used (2.2) and the fact that V annihilates $\mathfrak{p}_\infty^*(\mathrm{d}\zeta)\wedge\alpha_{\mathrm{can}}$.

Next we compute the actions. For the Ω_∞-action of γ_0 we find

$$\mathcal{A}_{\Omega_\infty}(\gamma_0) = \int_{S^1}\gamma_0^*\alpha_\infty = \frac{\operatorname{area}(M)}{\chi(M)}\left(\int_{S^1}\gamma_0^*(\eta+\mathfrak{p}_\infty^*\zeta_\infty)\right) = \frac{\operatorname{area}(M)}{\chi(M)}.$$

To compute the Ω_f-action of $(c,\dot c)$, let $\Gamma:[0,1]\times S^1\to\mathrm{T}^1M$ be a cylinder connecting an oriented $\mathfrak{p}_\infty$-fibre to $(c,\dot c)$ and recall the formula for the magnetic length (2.4). Using Stokes' theorem we compute

$$\mathcal{A}_{\Omega_f}(c,\dot c) = \int_{\mathbb{R}/T\mathbb{Z}} (c,\dot c)^*\alpha_f = \int_{[0,1]\times S^1} \Gamma^*\Omega_f + \int_{S^1} \Gamma(0,\cdot)^*\alpha_f$$
$$= \ell_f(c) + \frac{\mathrm{area}(M)\cdot f_{\mathrm{avg}}}{\chi(M)},$$

where in the last passage we used that $\int_{S^1}\Gamma(0,\cdot)^*\alpha_{\mathrm{can}} = 0$. □

Let $f_* : M \to \mathbb{R}$ be a Zoll function, whose magnetic geodesics lie in $\Lambda(M;\mathfrak{h}_\infty)$, and let $(\mathfrak{p}_{f_*},[\omega_{f_*}])$ be the weakly Zoll pair associated with the Zoll odd-symplectic form Ω_{f_*}. Due to Proposition 1.3.(a), there holds

$$(\mathfrak{p}_{f_*},[\omega_{f_*}]) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1M). \tag{2.20}$$

Therefore, from (2.17) and Theorem 2.10, we have

$$0 < \langle [\omega_{f_*}],[M_{f_*}]\rangle = \frac{\mathrm{d}P}{\mathrm{d}A}(\mathcal{A}(\Omega_{f_*})), \qquad P(\mathcal{A}(\Omega_{f_*})) = \mathfrak{Vol}(\Omega_{f_*}), \tag{2.21}$$

where $\mathcal{A}(\Omega_{f_*}) := \mathcal{A}(\mathfrak{p}_{f_*},[\omega_{f_*}])$ and $\mathfrak{Vol}(\Omega_{f_*}) := \mathfrak{Vol}(\mathfrak{p}_{f_*},[\omega_{f_*}])$ are the action and the volume defined in (2.16). In our case, it reads

$$\mathcal{A}(\Omega_{f_*}) = \int_{S^1} (c_{f_*},\dot c_{f_*})^*\alpha_{f_*},$$

where c_{f_*} is a prime closed f_*-magnetic geodesic.

Corollary 2.13. *If $f_* : M \to \mathbb{R}$ is a Zoll function and $M \neq \mathbb{T}^2$, then*

$$\mathcal{A}(\Omega_{f_*}) = \frac{\langle [\omega_{f_*}],[M_{f_*}]\rangle}{\chi(M)}, \quad K_{f_*} = \left(\frac{\chi(M)\mathcal{A}(\Omega_{f_*})}{\mathrm{area}(M)}\right)^2 = \left(\frac{\langle [\omega_{f_*}],[M_{f_*}]\rangle}{\mathrm{area}(M)}\right)^2.$$

In particular, $\mathcal{A}(\Omega_{f_*})$ and $\mathfrak{Vol}(\Omega_{f_*})$ have the same sign as $\chi(M)$, and K_{f_*} is positive.

Proof. From (2.18) we get $\frac{\mathrm{d}P}{\mathrm{d}A}(\mathcal{A}(\Omega_{f_*})) = \chi(M)\mathcal{A}(\Omega_{f_*})$, which together with the first relation in (2.21) yields the statement about $\mathcal{A}(\Omega_{f_*})$. Putting the second relation in (2.21), equation (2.18), and Lemma 2.12 together, we have

$$\frac{\chi(M)}{2}\mathcal{A}(\Omega_{f_*})^2 = P(\mathcal{A}(\Omega_{f_*})) = \mathfrak{Vol}(\Omega_{f_*}) = \frac{\mathrm{area}(M)^2}{2\chi(M)}K_{f_*}.$$

This proves the rest of the corollary. □

Case $M = \mathbb{T}^2$. We work with the reference weakly Zoll pair

$$(\mathfrak{p}_\infty, c_0) = (\mathfrak{p}_\infty, [\mu]) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1\mathbb{T}^2),$$

so that $\Omega_0 = \mathfrak{p}_\infty^*\mu = \Omega_\infty$. This form is not exact by the discussion in Section 2.1. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be an arbitrary function with $f_{\mathrm{avg}} > 0$, so that

$$\bar{\ell}(f) = \frac{\pi}{f_{\mathrm{avg}}} > 0.$$

We consider the normalised form

$$\bar{\Omega}_f := \frac{1}{f_{\mathrm{avg}}}\Omega_f$$

so that $\bar{\Omega}_f$ and Ω_∞ are cohomologous. More precisely,

$$\bar{\Omega}_f = \Omega_\infty + \mathrm{d}\big(\tfrac{1}{f_{\mathrm{avg}}}\alpha_f\big), \qquad \alpha_f := \alpha_{\mathrm{can}} + \mathfrak{p}_\infty^*\zeta - \bar{\ell}(f)\mathrm{d}\phi,$$

where ζ is a one-form on $\mathbb{T}^2$ is such that $\mathrm{d}\zeta = (f - f_{\mathrm{avg}})\mu$ and $\phi : \mathrm{T}^1\mathbb{T}^2 \to S^1$ is a global angular function for the bundle $\mathfrak{p}_\infty$, namely $\mathrm{d}\phi(V) \equiv 1$. As we see in the next lemma, the term $-\bar{\ell}(f)\mathrm{d}\phi$ is added in order to normalise $\frac{1}{f_{\mathrm{avg}}}\alpha_f$.

Lemma 2.14. *Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a function with $f_{\mathrm{avg}} > 0$. Then the one-form $\frac{1}{f_{\mathrm{avg}}}\alpha_f$ is normalised, i.e.* $\mathrm{Vol}(\frac{1}{f_{\mathrm{avg}}}\alpha_f) = 0$.

Proof. Using (2.2), we compute

$$\begin{aligned}
(f_{\mathrm{avg}})^2\mathrm{Vol}(\tfrac{1}{f_{\mathrm{avg}}}\alpha_f) &= f_{\mathrm{avg}}\int_{\mathrm{T}^1\mathbb{T}^2} \alpha_f \wedge \Big(\Omega_\infty + \frac{1}{2}\mathrm{d}\big(\tfrac{1}{f_{\mathrm{avg}}}\alpha_f\big)\Big) \\
&= \int_{\mathrm{T}^1\mathbb{T}^2} \big(\alpha_{\mathrm{can}} + \mathfrak{p}_\infty^*\zeta - \bar{\ell}(f)\mathrm{d}\phi\big) \\
&\qquad \wedge \Big(f_{\mathrm{avg}}\,\mathfrak{p}_\infty^*\mu + \frac{1}{2}\big(\mathrm{d}\alpha_{\mathrm{can}} + \mathfrak{p}_\infty^*(\mathrm{d}\zeta)\big)\Big) \\
&= \frac{1}{2}\int_{\mathrm{T}^1\mathbb{T}^2} \alpha_{\mathrm{can}} \wedge \mathrm{d}\alpha_{\mathrm{can}} - \bar{\ell}(f)f_{\mathrm{avg}} \cdot \mathrm{area}(\mathbb{T}^2) \\
&= \pi\int_{\mathrm{T}^1\mathbb{T}^2} \eta \wedge \mathfrak{p}_\infty^*\mu - \pi \cdot \mathrm{area}(\mathbb{T}^2) \\
&= 0.
\end{aligned}$$

□

By Lemma 2.14, we can use the one-form $\frac{1}{f_{\mathrm{avg}}}\alpha_f$ to compute the $\bar{\Omega}_f$-action of loops:

$$\mathcal{A}_{\bar{\Omega}_f}(\gamma) = \frac{1}{f_{\mathrm{avg}}}\int_{S^1}\Gamma(0,\cdot)^*\alpha_f + \frac{1}{f_{\mathrm{avg}}}\int_{[0,1]\times S^1}\Gamma^*\Omega_f, \tag{2.22}$$

where $\Gamma : [0,1]\times S^1 \to \mathrm{T}^1\mathbb{T}^2$ is a homotopy between an oriented $\mathfrak{p}_\infty$-fibre and $\gamma \in \Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1\mathbb{T}^2)$.

Lemma 2.15. *Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a function with $f_{\mathrm{avg}} > 0$. There holds*

$$\mathcal{A}_{\bar{\Omega}_f}(c,\dot{c}) = \frac{1}{f_{\mathrm{avg}}}\big(\ell_f(c) - \bar{\ell}(f)\big), \qquad \forall\, c \in \Lambda(\mathbb{T}^2;\mathfrak{h}_\infty).$$

Proof. The claim follows from substituting identity (2.4) in (2.22) and the computation

$$\int_{S^1}\Gamma(0,\cdot)^*\alpha_f = \int_{S^1}\Gamma(0,\cdot)^*\big(\alpha + \mathfrak{p}_\infty^*\zeta - \bar{\ell}(f)\mathrm{d}\phi\big) = -\bar{\ell}(f).$$

□

Finally, we observe that if $f_* : \mathbb{T}^2 \to \mathbb{R}$ is a Zoll function whose magnetic geodesics lie in $\Lambda(\mathbb{T}^2;\mathfrak{h}_\infty)$, then, by Proposition 1.3.(b), we have

$$(f_*)_{\mathrm{avg}} > 0 \tag{2.23}$$

and, setting $\bar{\omega}_{f_*} := \frac{1}{f_{\mathrm{avg}}}\omega_{f_*}$, we see that $(\mathfrak{p}_{f_*}, [\bar{\omega}_{f_*}])$ is the weakly Zoll pair associated with the Zoll odd-symplectic form $\bar{\Omega}_{f_*}$. By Proposition 1.3.(a), there holds

$$(\mathfrak{p}_{f_*}, [\bar{\omega}_{f_*}]) \in \mathfrak{Z}^0_{[\Omega_\infty]}(\mathrm{T}^1\mathbb{T}^2). \tag{2.24}$$

3. The proof of the magnetic systolic-diastolic inequality

3.1. The inequality in a neighbourhood of a Zoll function

In this subsection we give a proof of Theorem 1.9, which states that the magnetic systolic-diastolic inequality holds in a C^2-neighbourhood $\mathcal{F}$ of a Zoll function $f_* : M \to \mathbb{R}$. As before we deal with the cases $M \neq \mathbb{T}^2$ and $M = \mathbb{T}^2$ separately.

Proof of Theorem 1.9 for $M \neq \mathbb{T}^2$. In view of (2.20) and Theorem 2.11, there exist a C^1-neighbourhood $\mathcal{W}$ of the set $\Lambda(f_*; \mathfrak{h}_\infty)$ in $\Lambda(M; \mathfrak{h}_\infty)$ and a C^2-neighbourhood $\mathcal{F}$ of the function f_* in $C^\infty(M)$ such that

$$\mathcal{A}_{\min}(\Omega_f) = \min_{\substack{c \in \mathcal{W} \cap \Lambda(f; \mathfrak{h}_\infty) \\ c\,\text{prime}}} \mathcal{A}_{\Omega_f}(c, \dot{c}), \qquad \mathcal{A}_{\max}(\Omega_f) = \max_{\substack{c \in \mathcal{W} \cap \Lambda(f; \mathfrak{h}_\infty) \\ c\,\text{prime}}} \mathcal{A}_{\Omega_f}(c, \dot{c}).$$

and

$$P(\mathcal{A}_{\min}(\Omega_f)) \leq \mathfrak{Vol}(\Omega_f) \leq P(\mathcal{A}_{\max}(\Omega_f)), \qquad \forall\, f \in \mathcal{F} \tag{3.1}$$

with equality signs if and only if Ω_f is Zoll. Since $\mathcal{A}_{\min}(\Omega_f)$ and $\mathcal{A}_{\max}(\Omega_f)$ vary continuously in $f \in \mathcal{F}$, shrinking $\mathcal{F}$ if necessary, we deduce from Corollary 2.13 that for all $f \in \mathcal{F}$:

$$K_f > 0, \qquad \operatorname{sign}\big(\mathcal{A}_{\min}(\Omega_f)\big) = \operatorname{sign}\big(\chi(M)\big) = \operatorname{sign}\big(\mathcal{A}_{\max}(\Omega_f)\big). \tag{3.2}$$

We show that the magnetic systolic-diastolic inequality holds on $\mathcal{F}$. Let $f : M \to \mathbb{R}$ be a function in $\mathcal{F}$. According to Lemma 2.12 and equation (2.18), formula (3.1) becomes

$$\chi(M)\frac{\mathcal{A}_{\min}(\Omega_f)^2}{2} \leq \frac{\operatorname{area}(M)^2}{2\chi(M)} K_f \leq \chi(M)\frac{\mathcal{A}_{\max}(\Omega_f)^2}{2}.$$

The identities in (3.2) simplify this inequality to

$$\mathcal{A}_{\min}(\Omega_f) \leq \frac{\operatorname{area}(M)}{\chi(M)}\sqrt{K_f} \leq \mathcal{A}_{\max}(\Omega_f).$$

The formula for the action in Lemma 2.12 and the definition of $\ell_{\min}(f)$, $\ell_{\max}(f)$ yield

$$\mathcal{A}_{\min}(\Omega_f) \geq \ell_{\min}(f) + \frac{\operatorname{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)},$$
$$\mathcal{A}_{\max}(\Omega_f) \leq \ell_{\max}(f) + \frac{\operatorname{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)},$$

where the equalities hold when f is Zoll. Combining the inequalities above, we get

$$\ell_{\min}(f) \leq \frac{\operatorname{area}(M)}{\chi(M)}\big(\sqrt{K_f} - f_{\mathrm{avg}}\big) \leq \ell_{\max}(f),$$

and using the definition of the average curvature, we rewrite the term in the middle as

$$\frac{\mathrm{area}(M)}{\chi(M)}\big(\sqrt{K_f}-f_{\mathrm{avg}}\big)=\frac{\mathrm{area}(M)}{\chi(M)}\frac{K_f-(f_{\mathrm{avg}})^2}{\sqrt{K_f}+f_{\mathrm{avg}}}=\frac{2\pi}{\sqrt{K_f}+f_{\mathrm{avg}}}=\bar{\ell}(f).$$

This shows the magnetic systolic-diastolic inequality $\ell_{\min}(f)\le\bar{\ell}(f)\le\ell_{\max}(f)$. Moreover, if f is Zoll, we actually have equalities. Conversely, if one of the two inequalities is an equality, we also have an equality in (3.1). This implies that Ω_f, and thus f, is Zoll. □

Proof of Theorem 1.9 for $M=\mathbb{T}^2$. Thanks to (2.23), (2.24) and Theorem 2.11, there exists a C^1-neighbourhood $\mathcal{W}$ of $\Lambda(f_*;\mathfrak{h}_\infty)$ inside $\Lambda(\mathbb{T}^2;\mathfrak{h}_\infty)$ and a C^2-neighbourhood $\mathcal{F}$ of f_* in $C^\infty(\mathbb{T}^2)$ with the following properties. If $f\in\mathcal{F}$, then $f_{\mathrm{avg}}>0$ and

$$\mathcal{A}_{\min}(\bar{\Omega}_f)\le 0\le\mathcal{A}_{\max}(\bar{\Omega}_f),\qquad\forall\, f\in\mathcal{F},\tag{3.3}$$

where any of the two equalities holds if and only if $\bar{\Omega}_f$ is Zoll. Since Ω_f and $\bar{\Omega}_f$ have the same closed characteristics, we have

$$\mathcal{A}_{\min}(\bar{\Omega}_f):=\min_{\substack{c\in\mathcal{W}\cap\Lambda(f;\mathfrak{h}_\infty)\\ c\,\mathrm{prime}}}\mathcal{A}_{\bar{\Omega}_f}(c,\dot{c}),\qquad\mathcal{A}_{\max}(\bar{\Omega}_f):=\max_{\substack{c\in\mathcal{W}\cap\Lambda(f;\mathfrak{h}_\infty)\\ c\,\mathrm{prime}}}\mathcal{A}_{\bar{\Omega}_f}(c,\dot{c}).$$

From the definition of $\ell_{\min}(f)$ and $\ell_{\max}(f)$ and Lemma 2.15, we get

$$\mathcal{A}_{\min}(\bar{\Omega}_f)\ge\frac{1}{f_{\mathrm{avg}}}\big(\ell_{\min}(f)-\bar{\ell}(f)\big),\qquad\mathcal{A}_{\max}(\bar{\Omega}_f)\le\frac{1}{f_{\mathrm{avg}}}\big(\ell_{\max}(f)-\bar{\ell}(f)\big)$$

where any of the two equalities holds, if f is Zoll. Plugging these relations into (3.3), and using that f_{avg} is positive, we derive the desired inequality:

$$\ell_{\min}(f)\le\bar{\ell}(f)\le\ell_{\max}(f),$$

If any of the equalities holds, then there is an equality also in (3.3) and f is Zoll. The converse is also readily seen to be true. □

3.2. The inequality for strong magnetic functions

In this subsection we prove Theorem 1.11, which states that the magnetic systolic-diastolic inequality holds for C_g-strong functions (see Definition 1.10), where $C_g>0$ is a constant depending only on g that we will determine.

Proof of Theorem 1.11 for $M \neq \mathbb{T}^2$. By Theorem 2.11, there exists a C^0-neighbourhood $\Lambda(\mathfrak{p}_\infty) \subset \Lambda_{\mathfrak{h}_\infty}(\mathrm{T}^1 M)$ of the $\mathfrak{p}_\infty$-fibres and a C^2-neighbourhood $\mathcal{U}$ of Ω_∞ in the space of exact odd-symplectic forms on $\mathrm{T}^1 M$ such that

$$P(\mathcal{A}_{\min}(\Omega)) \leq \mathfrak{Vol}(\Omega) \leq P(\mathcal{A}_{\max}(\Omega)), \qquad \forall\, \Omega \in \mathcal{U} \tag{3.4}$$

with equality signs if and only if Ω is Zoll. Here, $\mathcal{A}_{\min}(\Omega)$ and $\mathcal{A}_{\max}(\Omega)$ are the minimal and maximal action among the closed characteristics in the set $\mathcal{X}(\Omega; \mathfrak{p}_\infty)$. Since $\mathcal{A}_{\Omega_\infty}(\gamma_0)$ and $\chi(M)$ have the same sign by Lemma 2.12, and $\mathcal{A}_{\min}$, $\mathcal{A}_{\max}$ vary continuously in $\mathcal{U}$, we have, up to shrinking $\mathcal{U}$,

$$\operatorname{sign}(\mathcal{A}_{\min}(\Omega)) = \operatorname{sign}(\chi(M)) = \operatorname{sign}(\mathcal{A}_{\max}(\Omega)), \qquad \forall\, \Omega \in \mathcal{U}.$$

In particular, from (3.4) and the formula for P, we also have

$$\operatorname{sign}(\mathfrak{Vol}(\Omega)) = \operatorname{sign}(\chi(M)). \tag{3.5}$$

We prove the theorem with $C_g := C_{\mathcal{U}}$, the constant given by Lemma 2.9. Let us consider a C_g-strong function $f : M \to (0, \infty)$, and let $\Psi : \mathrm{T}^1 M \to \mathrm{T}^1 M$ be a diffeomorphism isotopic to the identity such that $\frac{1}{f_{\mathrm{avg}}} \Psi^* \Omega_f \in \mathcal{U}$, whose existence is ensured by Lemma 2.9. From the homogeneity (2.19) of the volume and its invariance property (2.13), we have

$$\mathfrak{Vol}\big(\tfrac{1}{f_{\mathrm{avg}}} \Psi^* \Omega_f\big) = \big(\tfrac{1}{f_{\mathrm{avg}}}\big)^2 \mathfrak{Vol}(\Omega_f).$$

From the formula for the volume in Lemma 2.12 and the relation (3.5), we see that $K_f > 0$. Using the homogeneity of the action and formula (2.18) for P, we can rewrite (3.4) as

$$\mathcal{A}_{\min}(\Psi^* \Omega_f) \leq \frac{\operatorname{area}(M)}{\chi(M)} \sqrt{K_f} \leq \mathcal{A}_{\max}(\Psi^* \Omega_f).$$

Since Ψ is isotopic to $\mathrm{id}_{\mathrm{T}^1 M}$, we also see that

$$c_\gamma := \mathfrak{p}_\infty(\Psi(\gamma)) \in \Lambda(f; \mathfrak{h}_\infty), \qquad \forall\, \gamma \in \mathcal{X}(\Psi^* \Omega_f; \mathfrak{p}_\infty).$$

From Lemma 2.12 and the invariance property (2.14), we conclude that

$$\mathcal{A}_{\Psi^* \Omega_f}(\gamma) = \mathcal{A}_{\Omega_f}(c_\gamma, \dot{c}_\gamma) = \ell_f(c_\gamma) + \frac{\operatorname{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)}, \qquad \forall\, \gamma \in \mathcal{X}(\Psi^* \Omega_f; \mathfrak{p}_\infty).$$

From the definition of $\ell_{\min}(f)$ and $\ell_{\max}(f)$, we have

$$\ell_{\min}(f) + \frac{\mathrm{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)} \leq \mathcal{A}_{\min}(\Psi^*\Omega_f),$$
$$\mathcal{A}_{\max}(\Psi^*\Omega_f) \leq \ell_{\max}(f) + \frac{\mathrm{area}(M) \cdot f_{\mathrm{avg}}}{\chi(M)}$$

and equalities hold if f is Zoll. The rest of the proof goes along the same line as in the proof of Theorem 1.9 for $M \neq \mathbb{T}^2$ in Section 3.1 above. $\square$

Proof of Theorem 1.11 for $M = \mathbb{T}^2$. Theorem 2.11 yields a C^0-neighbourhood $\Lambda(\mathfrak{p}_\infty)$ of the $\mathfrak{p}_\infty$-fibres and a C^2-neighbourhood $\mathcal{U}$ of Ω_∞ in the space of odd-symplectic forms cohomologous to Ω_∞ such that

$$\mathcal{A}_{\min}(\Omega) \leq 0 \leq \mathcal{A}_{\max}(\Omega), \qquad \forall\, \Omega \in \mathcal{U}, \tag{3.6}$$

and any of the equalities holds if and only if Ω is Zoll. We prove the theorem with $C_g := C_{\mathcal{U}}$, the constant in Lemma 2.9. Let $f : \mathbb{T}^2 \to (0, \infty)$ be a C_g-strong function. In particular we have $f_{\mathrm{avg}} > 0$. Let Ψ be the diffeomorphism isotopic to the identity constructed in Lemma 2.9 with the property that $\Psi^*\bar{\Omega}_f \in \mathcal{U}$. Since Ψ is isotopic to the identity, we see that $c_\gamma := \mathfrak{p}_\infty(\Psi(\gamma)) \in \Lambda(f; \mathfrak{h}_\infty)$ for all $\gamma \in \mathcal{X}(\Psi^*\bar{\Omega}_f; \mathfrak{p}_\infty)$. From (2.14) and Lemma 2.15, we get

$$\mathcal{A}_{\Psi^*\bar{\Omega}_f}(\gamma) = \mathcal{A}_{\bar{\Omega}_f}(c_\gamma, \dot{c}_\gamma) = \frac{1}{f_{\mathrm{avg}}}(\ell_f(c_\gamma) - \bar{\ell}(f)), \qquad \forall\, \gamma \in \mathcal{X}(\Psi^*\bar{\Omega}_f; \mathfrak{p}_\infty).$$

This relation together with (3.6) yields

$$\begin{aligned} \ell_{\min}(f) - \bar{\ell}(f) \leq f_{\mathrm{avg}} \cdot \mathcal{A}_{\min}(\Psi^*\bar{\Omega}_f) &\leq 0 \\ \leq f_{\mathrm{avg}} \cdot \mathcal{A}_{\max}(\Psi^*\bar{\Omega}_f) &\leq \ell_{\max}(f) - \bar{\ell}(f) \end{aligned}$$

which in turn implies

$$\ell_{\min}(f) \leq \bar{\ell}(f) \leq \ell_{\max}(f).$$

If f is Zoll, the equalities hold. Conversely if $\ell_{\min}(f)$ or $\ell_{\max}(f)$ are equal to $\bar{\ell}(f)$, then there is an equality also in (3.6), which yields that $\Psi^*\bar{\Omega}_f$, and hence f, is Zoll. $\square$

Appendix A. C^k-estimate on the time-one map of a flow

For $h, k \in \mathbb{N}$, we define the polynomial

$$B_{h,k} : \mathbb{R}^{k+1} \to \mathbb{R}, \qquad B_{h,k}(x) = \sum_{a \in I_{h,k}} x^a,$$

where $x = (x_0, \cdots, x_k)$, $x^a := x_0^{a_0} \cdot \ldots \cdot x_k^{a_k}$, and $I_{h,k}$ is the following set of multi-indices

$$I_{h,k} := \Big\{ a = (a_0, \cdots, a_k) \in \mathbb{N}^{k+1} \;\Big|\; 0 < \sum_{j=0}^{k} (j+1) a_j \leq h + k \Big\}.$$

If $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is a map between two Riemannian manifolds, we use the short-hand

$$B_{h,k}\big(\|\mathrm{d}\Psi\|\big) := B_{h,k}\big(\|\mathrm{d}\Psi\|_{C^0}, \cdots, \|\mathrm{d}\Psi\|_{C^k}\big).$$

A straightforward computation shows that there is a constant $C_k > 0$ (depending only on $\mathcal{M}_1$ and $\mathcal{M}_2$) such that for an h-form η on $\mathcal{M}_2$,

$$\|\Psi^* \eta\|_{C^k} \leq C_k B_{h,k}(\|\mathrm{d}\Psi\|) \|\eta\|_{C^k}. \tag{A.1}$$

Let $\Psi = \Phi_1$ be the time-one map of the flow of a (time-dependent) vector field. Then using Gronwall's Lemma inductively, one can estimate $B_{h,k}(\|\mathrm{d}\Phi_1\|)$ in terms of the vector field. Here, we give only the bound for $(h,k) = (2,2)$ which is what is needed in Lemma 2.9.

Lemma A.1. *For every compact manifold $\mathcal{M}$, there exists a constant $C_k > 0$ with the following property. For every time-dependent vector field $X = \{X_s\}_{s \in [0,1]}$ on $\mathcal{M}$ such that the corresponding flow $\{\Phi_s\}$ is defined up to time 1, there holds*

$$B_{2,2}\big(\|\mathrm{d}\Phi_1\|\big) \leq \Big(\langle \nabla X \rangle_{C^2} + \langle \nabla X \rangle_{C^1}^2 \Big) e^{C \langle \nabla X \rangle_{C^0}},$$

where we have set $\langle \nabla X \rangle_{C^k} := 1 + \max_{s \in [0,1]} \|\nabla X_s\|_{C^k}, \ \forall\, k \in \mathbb{N}.$

Proof. We preliminarily observe that if V is a finite-dimensional vector space endowed with a norm coming from a scalar product, then for every $s \mapsto$

$v(s) \in V$, there holds

$$\frac{\mathrm{d}|v|}{\mathrm{d}s} \leq \Big|\frac{\mathrm{d}v}{\mathrm{d}s}\Big|.$$

By the compactness of $\mathcal{M}$, we just need to prove the lemma in local coordinates. Recalling that $\partial_s \Phi_s = X_s \circ \Phi_s$ by definition, we compute

$$\begin{aligned}
\partial_s\big|\mathrm{d}\Phi_s\big| &\leq \big|\mathrm{d}(X_s \circ \Phi_s)\big| \leq \big|\nabla X_s\big| \cdot \big|\mathrm{d}\Phi_s\big|,\\
\partial_s\big|\nabla\mathrm{d}\Phi_s\big| &\leq \Big|\nabla\Big(\big(\nabla_{\mathrm{d}\Phi_s} X_s\big)_{\Phi_s}\Big)\Big| \leq \big|\nabla^2 X_s\big| \cdot \big|\mathrm{d}\Phi_s\big|^2 + \big|\nabla X_s\big| \cdot \big|\nabla \mathrm{d}\Phi_s\big|,\\
\partial_s\big|\nabla^2\mathrm{d}\Phi_s\big| &\leq \Big|\nabla\Big(\big(\nabla_{\mathrm{d}\Phi_s}\nabla_{\mathrm{d}\Phi_s} X_s\big)_{\Phi_s} + \big(\nabla_{\nabla\mathrm{d}\Phi_s} X_s\big)_{\Phi_s}\Big)\Big|\\
&\leq \big|\nabla^3 X_s\big| \cdot \big|\mathrm{d}\Phi_s\big|^3 + 3\big|\nabla^2 X_s\big| \cdot \big|\nabla\mathrm{d}\Phi_s\big| \cdot \big|\mathrm{d}\Phi_s\big| + \big|\nabla X_s\big| \cdot \big|\nabla^2\mathrm{d}\Phi_s\big|.
\end{aligned}$$

We now apply Gronwall's Lemma [16] and indicate with $C > 0$ a constant depending on $\mathcal{M}$ but not on X. Below, we can always bring the constant to the exponent because, by definition, $\langle \nabla X \rangle_{C^0} \geq 1$. Thus we find that

$$\begin{aligned}
\|\max_s \mathrm{d}\Phi_s\| &\leq e^{C\langle\nabla X\rangle_{C^0}},\\
\|\max_s \nabla\mathrm{d}\Phi_s\| &\leq \langle\nabla X\rangle_{C^1}\|\max_s \mathrm{d}\Phi_s\|^2_{C^0} e^{C\langle\nabla X\rangle_{C^0}} \leq \langle\nabla X\rangle_{C^1} e^{C\langle\nabla X\rangle_{C^0}},\\
\|\max_s \nabla^2\mathrm{d}\Phi_s\| &\leq \Big(\langle\nabla X\rangle_{C^2}\|\max_s \mathrm{d}\Phi_s\|^3_{C^0}\\
&\qquad + \langle\nabla X\rangle_{C^1}\|\max_s \nabla\mathrm{d}\Phi_s\|_{C^0}\|\max_s \mathrm{d}\Phi_s\|_{C^0}\Big) e^{C\langle\nabla X\rangle_{C^0}}\\
&\leq \big(\langle\nabla X\rangle_{C^2} + \langle\nabla X\rangle^2_{C^1}\big) e^{C\langle\nabla X\rangle_{C^0}}.
\end{aligned}$$

Finally, from the definition of $B_{2,2}$, we get

$$\begin{aligned}
B_{2,2}\big(\|\mathrm{d}\Phi_1\|\big) &\leq \Big[\sum_{a_1+2a_2+3a_3\leq 4} \langle\nabla X\rangle^{a_2}_{C^1}\big(\langle\nabla X\rangle_{C^2} + \langle\nabla X\rangle^2_{C^1}\big)^{a_3}\Big] e^{C\langle\nabla X\rangle_{C^0}}\\
&\leq \big(\langle\nabla X\rangle_{C^2} + \langle\nabla X\rangle^2_{C^1}\big) e^{C\langle\nabla X\rangle_{C^0}}.
\end{aligned}$$

□

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