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# The log symplectic geometry of Poisson slices

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Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group $G$. An $\mathfrak{sl}_2$-triple $\tau = (\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ and a Poisson Hamiltonian $G$-variety $X$ together yield a distinguished Poisson transversal $X_\tau := \nu^{-1}(\mathcal{S}_\tau)$, where $\nu : X \longrightarrow \mathfrak{g}$ is the moment map and $\mathcal{S}_\tau := \xi + \mathfrak{g}_\eta$ is the Slodowy slice associated to $\tau$. We refer to $X_\tau$ as the *Poisson slice* determined by $X$ and $\tau$. Prominent examples include the universal centralizer $\mathcal{Z}_{\mathfrak{g}}^\tau$ and hyperkähler slice $G \times \mathcal{S}_\tau$. These have natural log symplectic completions $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ and $\overline{G \times \mathcal{S}_\tau}$ arising from the wonderful compactification $\overline{G}$. The variety $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ partially compactifies $\mathcal{Z}_{\mathfrak{g}}^\tau$, while $\overline{G \times \mathcal{S}_\tau}$ partially compactifies $G \times \mathcal{S}_\tau$ if $\tau$ is a principal $\mathfrak{sl}_2$-triple.

Our paper develops a theory of Poisson slices and a uniform approach to their partial compactifications. The theory in question is loosely comparable to that of symplectic cross-sections in real symplectic geometry. To address the partial compactification aspect, we associate to each Hamiltonian $G$-variety $X$ and $\mathfrak{sl}_2$-triple $\tau$ the Hamiltonian reduction $\overline{X}_\tau := (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G$. Assuming that $\overline{X}_\tau$ exists as a geometric quotient, we establish its Poisson-geometric features. We also show $\overline{X}_\tau$ to have an open log symplectic stratum if $X$ is symplectic and $X_\tau$ is irreducible. If $\tau$ is a principal $\mathfrak{sl}_2$-triple and the geometric quotient $X/G$ exists, we realize $\overline{X}_\tau$ as a partial compactification of $X_\tau$ over $X/G$. Our constructions specialize to yield $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ and $\overline{G \times \mathcal{S}_\tau}$ as partial compactifications of $\mathcal{Z}_{\mathfrak{g}}^\tau$ and $G \times \mathcal{S}_\tau$, respectively.

# 1. Introduction

## 1.1. Motivation and context

The Poisson slice construction yields a number of varieties relevant to geometric representation theory and symplectic geometry. One begins with a complex semisimple linear algebraic group with Lie algebra $\mathfrak{g}$. Let us also consider a Hamiltonian $G$-variety $X$, i.e. a smooth Poisson variety with a Hamiltonian action of $G$ and moment map $\nu : X \longrightarrow \mathfrak{g}$. Each $\mathfrak{sl}_2$-triple $\tau = (\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ determines a Slodowy slice

$$\mathcal{S}_\tau := \xi + \mathfrak{g}_\eta \subseteq \mathfrak{g},$$

and the preimage

$$X_\tau := \nu^{-1}(\mathcal{S}_\tau)$$

is a Poisson transversal in $X$. The variety $X_\tau$ is thereby Poisson, and we call it the *Poisson slice* determined by $X$ and $\tau$. To a certain extent, Poisson slices are complex Poisson-geometric counterparts of symplectic cross-sections [26, 27, 30, 37] in real symplectic geometry.

Noteworthy examples of Poisson slices include the product $G \times \mathcal{S}_\tau$, a hyperkähler and Hamiltonian $G$-variety studied by Bielawski [5, 6], Moore–Tachikawa [40], and several others [1, 11, 13–15]. A second example is a Coulomb branch [7] called the *universal centralizer*

$$\mathcal{Z}_{\mathfrak{g}}^{\tau} := \{(g, y) \in G \times \mathcal{S}_\tau : \mathrm{Ad}_g(y) = y\},$$

where $\tau$ is a fixed principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. This hyperkähler variety has received considerable attention in the literature [4, 7, 12, 39, 49, 53, 54], and it features prominently in Bălibanu's recent paper [2]. Bălibanu assumes $G$ to be of adjoint type. She harnesses the geometry of the wonderful compactification $\overline{G}$ and constructs a fibrewise compactification $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} \longrightarrow \mathcal{S}_\tau$ of $\mathcal{Z}_{\mathfrak{g}}^{\tau} \longrightarrow \mathcal{S}_\tau$, where the latter map is projection onto the $\mathcal{S}_\tau$-factor. She subsequently endows $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}}$ with a log symplectic structure.

The preceding discussion gives rise to the following rough questions.

- Is there a coherent and systematic approach to the partial compactification of Poisson slices that is related to $\overline{G}$ and specializes to yield $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} \longrightarrow \mathcal{S}_{\tau}$ as a fibrewise compactification of $\mathcal{Z}_{\mathfrak{g}}^{\tau} \longrightarrow \mathcal{S}_{\tau}$?
- If the previous question has an affirmative answer and $X_{\tau}$ is symplectic, does the partial compactification of $X_{\tau}$ carry a log symplectic structure?

Our inquiry stands to benefit from two observations. One first notes the universal or atomic nature of $G \times \mathcal{S}_{\tau}$ as a Poisson slice, i.e. the existence of a canonical Poisson variety isomorphism

$$X_{\tau} \cong (X \times (G \times \mathcal{S}_{\tau})) \mathbin{/\!\!/} G$$

for each Hamiltonian $G$-variety $X$ and $\mathfrak{sl}_2$-triple $\tau$ in $\mathfrak{g}$. These atomic Poisson slices have counterparts in the theories of symplectic cross-sections [30], symplectic implosion [26], symplectic contraction [33], hyperkähler implosion [16, 17], and Kronheimer's hyperkähler quotient with momentum [36]. A second observation is that $G \times \mathcal{S}_{\tau}$ sits inside of a larger log symplectic variety $\overline{G \times \mathcal{S}_{\tau}}$ as the unique open dense symplectic leaf; the construction of $\overline{G \times \mathcal{S}_{\tau}}$ assumes $G$ to be of adjoint type and exploits the geometry of $\overline{G}$.

The preceding considerations motivate us to define

$$\overline{X}_{\tau} := (X \times (\overline{G \times \mathcal{S}_{\tau}})) \mathbin{/\!\!/} G$$

and conjecture that $\overline{X}_{\tau}$ is the desired partial compactification of $X_{\tau}$. While this naive conjecture needs to be refined and made more precise, it inspires many of the results in our paper.

## 1.2. Summary of results

Our paper develops a detailed theory of Poisson slices and addresses the questions posed above. The following is a summary of our results. We work exclusively over $\mathbb{C}$ and take all Poisson varieties to be smooth. We use the Killing form to freely identify $\mathfrak{g}^*$ with $\mathfrak{g}$, as well as the left trivialization and Killing form to freely identify $T^*G$ with $G \times \mathfrak{g}$.

Suppose that $X$ is a Poisson Hamiltonian $G$-variety with moment map $\nu : X \longrightarrow \mathfrak{g}$. Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and consider the Poisson transversal

$$X_{\tau} := \nu^{-1}(\mathcal{S}_{\tau}) \subseteq X.$$

The following are some first properties of the Poisson slice $X_\tau$. Such properties are well-known in the case of a symplectic variety $X$ (see [5]).

**Proposition 1.1.** *Let $X$ be a Poisson variety endowed with a Hamiltonian $G$-action and moment map $\nu : X \longrightarrow \mathfrak{g}$. Suppose that $\tau = (\xi, h, \eta)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The following statements hold.*

(i) *The Poisson slice $X_\tau$ is transverse to the $G$-orbits in $X$.*

(ii) *There are canonical Poisson variety isomorphisms*

$$(X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \cong X_\tau \cong X /\!\!/_\xi U_\tau.$$

*The Hamiltonian $G$-variety structure on $G \times \mathcal{S}_\tau$ and meaning of the unipotent subgroup $U_\tau \subseteq G$ are given in Subsection 3.2.*

We also consider some special cases of the Poisson slice construction, including the following well-known result in the symplectic category.

**Observation 1.2.** Let $X$ be a symplectic variety endowed with a Hamiltonian action of $G$ and a moment map $\nu : X \longrightarrow \mathfrak{g}$. Suppose that $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The Poisson structure on $X_\tau$ makes it a symplectic subvariety of $X$.

Now suppose that the above-mentioned Poisson variety $X$ is *log symplectic* [22, 24, 45, 48], by which the following is meant: $X$ has a unique open dense symplectic leaf, and the degeneracy locus of the Poisson bivector is a reduced normal crossing divisor. We establish the following log symplectic counterpart of Observation 1.2.

**Proposition 1.3.** *Let $X$ be a log symplectic variety endowed with a Hamiltonian $G$-action and moment map $\nu : X \longrightarrow \mathfrak{g}$. Suppose that $\tau$ is any $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Each irreducible component of $X_\tau$ is then a Poisson subvariety of $X_\tau$. The resulting Poisson structure on each component makes the component a log symplectic subvariety of $X$.*

Now assume $G$ to be of adjoint type. One may consider the De Concini–Procesi wonderful compactification $\overline{G}$ of $G$ [18], along with the divisor $D := \overline{G} \setminus G$. The data $(\overline{G}, D)$ determine a *log cotangent bundle* $T^*\overline{G}(\log(D))$, which is known to have a canonical log symplectic structure. Its unique open dense symplectic leaf is $T^*G$, and the canonical Hamiltonian $(G \times G)$-action on $T^*G$ extends to such an action on $T^*\overline{G}(\log(D))$. The moment maps

$$\rho = (\rho_L, \rho_R) : T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \quad \text{and} \quad \overline{\rho} = (\overline{\rho}_L, \overline{\rho}_R) : T^*\overline{G}(\log(D)) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$$

can be written in explicit terms. This leads to the following straightforward observations, whose proofs use Observation 1.2 and Proposition 1.3. To this end, recall that a principal $\mathfrak{sl}_2$-triple is an $\mathfrak{sl}_2$-triple consisting of regular elements.

**Observation 1.4.** Let $\tau = (\xi, h, \eta)$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and consider the principal $\mathfrak{sl}_2$-triple $(\tau,\tau) := ((\xi,\xi),(h,h),(\eta,\eta))$ in $\mathfrak{g}\oplus\mathfrak{g}$. One then has

$$\begin{aligned} &(T^*G)_{(\tau,\tau)} = \rho^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) = \mathcal{Z}_{\mathfrak{g}}^{\tau}, \\ \text{and} \quad &(T^*\overline{G}(\log D))_{(\tau,\tau)} = \overline{\rho}^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) = \overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}}. \end{aligned}$$

The first Poisson slice is symplectic, while the second is log symplectic.

**Observation 1.5.** Consider the Hamiltonian action of $G = \{e\}\times G \subseteq G \times G$ on $T^*G$. If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then

$$(T^*G)_\tau = \rho_R^{-1}(\mathcal{S}_\tau) = G \times \mathcal{S}_\tau.$$

This Poisson slice is symplectic.

In light of these observations, it is natural to consider the Poisson slice

$$\overline{G\times\mathcal{S}_\tau} := \overline{\rho}_R^{-1}(\mathcal{S}_\tau) \subseteq T^*\overline{G}(\log D).$$

One has an inclusion $G\times\mathcal{S}_\tau \subseteq \overline{G\times\mathcal{S}_\tau}$, while $G\times\mathcal{S}_\tau$ and $\overline{G\times\mathcal{S}_\tau}$ carry residual Hamiltonian actions of $G = G\times\{e\}\subseteq G\times G$. The respective moment maps are

$$\rho_\tau := \rho_L\big|_{G\times\mathcal{S}_\tau} \quad \text{and} \quad \overline{\rho}_\tau := \overline{\rho}_L\big|_{\overline{G\times\mathcal{S}_\tau}},$$

and they feature in the following result.

**Theorem 1.6.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.*

(i) *The Poisson slice $\overline{G\times\mathcal{S}_\tau}$ is irreducible and log symplectic.*

(ii) *The inclusion $G\times\mathcal{S}_\tau \longrightarrow \overline{G\times\mathcal{S}_\tau}$ is a $G$-equivariant symplectomorphism onto the unique open dense symplectic leaf in $\overline{G\times\mathcal{S}_\tau}$.*

(iii) *The diagram*

(1.1)
$$\begin{array}{ccc} G \times \mathcal{S}_\tau & \longrightarrow & \overline{G \times \mathcal{S}_\tau} \\ & {}_{\rho_\tau}\searrow \quad \swarrow_{\overline{\rho}_\tau} & \\ & \mathfrak{g} & \end{array}$$

*commutes.*

(iv) *If $\tau$ is a principal $\mathfrak{sl}_2$-triple, then* (1.1) *realizes $\overline{\rho}_\tau$ as a fibrewise compactification of $\rho_\tau$.*

Our paper subsequently discusses the relation of (1.1) to Bălibanu's fibrewise compactification

(1.2)
$$\begin{array}{ccc} \mathcal{Z}_{\mathfrak{g}}^{\tau} & \longrightarrow & \overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} \\ & \searrow \quad \swarrow & \\ & \mathcal{S}_\tau & \end{array} .$$

We next study Hamiltonian reductions of the form

$$\overline{X}_\tau := (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G,$$

where $X$ is a Hamiltonian $G$-variety and $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The special case $\tau = 0$ features prominently in our analysis, and we write $\overline{X}$ for $\overline{X}_\tau$ if $\tau = 0$. This amounts to setting

$$\overline{X} := (X \times T^*\overline{G}(\log D)) /\!\!/ G,$$

with $G$ acting as $G = G \times \{e\} \subseteq G \times G$ on $T^*\overline{G}(\log D)$.

The variety $\overline{X}_\tau$ enjoys certain Poisson-geometric features. A first step in this direction is to set

$$\overline{X}_\tau^\circ := (X \times (\overline{G \times \mathcal{S}_\tau}))^\circ /\!\!/ G,$$

where $(X \times \overline{G \times \mathcal{S}_\tau})^\circ$ is the open set of points in $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$ with trivial $G$-stabilizers. The variety $\overline{X}_\tau^\circ$ exists as a geometric quotient if $\overline{X}_\tau$ exists as a geometric quotient, in which case one has inclusions

$$X_\tau \subseteq \overline{X}_\tau^\circ \subseteq \overline{X}_\tau$$

**Theorem 1.7.** *Let $X$ be a Hamiltonian $G$-variety, and suppose that $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Assume that $\overline{X}_\tau$ exists as a geometric quotient.*

(i) *The coordinate ring $\mathbb{C}[\overline{X}_\tau]$ carries a natural Poisson bracket for which restriction $\mathbb{C}[\overline{X}_\tau] \longrightarrow \mathbb{C}[X_\tau]$ is a Poisson algebra morphism.*

(ii) *The variety $\overline{X}_\tau^\circ$ is smooth and Poisson, and it contains $X_\tau$ as an open Poisson subvariety.*

(iii) *If $X$ is symplectic, then each irreducible component of $\overline{X}_\tau^\circ$ is log symplectic.*

(iv) *If $X$ is symplectic and $X_\tau$ is irreducible, then $X_\tau$ is the open dense symplectic leaf in a unique irreducible component of $\overline{X}_\tau^\circ$.*

Our final main result addresses the extent to which $\overline{X}_\tau$ partially compactifies $X_\tau$. We begin by assuming that both $\overline{X}_\tau$ and $X/G$ exist as geometric quotients. This allows us to construct canonical maps

$$\pi_\tau : X_\tau \longrightarrow X/G \quad \text{and} \quad \overline{\pi}_\tau : \overline{X}_\tau \longrightarrow X/G.$$

It is then straightforward to deduce that

$$\begin{array}{ccc} X_\tau & \longrightarrow & \overline{X}_\tau \\ & {\scriptstyle \pi_\tau}\searrow \quad \swarrow{\scriptstyle \overline{\pi}_\tau} & \\ & X/G & \end{array} \tag{1.3}$$

commutes, where the horizontal arrow is inclusion. This leads to the following theorem.

**Theorem 1.8.** *Let $X$ be a Hamiltonian $G$-variety, and suppose that $\tau$ is a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ and $X/G$ exist as geometric quotients, then* (4.10) *realizes $\overline{\pi}_\tau$ as a fibrewise compactification of $\pi_\tau$.*

In the case of a principal $\mathfrak{sl}_2$-triple $\tau$, we realize the fibrewise compactifications (1.1) and (1.2) as special instances of Theorem 1.8.

### 1.3. Organization

In Section 2, we introduce the concepts from Lie theory and Poisson geometry that form the foundation for our work. Section 3 details the theory of

Poisson slices and provides complete proofs of Propositions 1.1 and 1.3. Section 4 subsequently considers the Poisson slice enlargements $\overline{X}_\tau$ mentioned above, and it contains the proofs of Theorems 1.6, 1.7 and 1.8. This section concludes with a few illustrative examples. A list of recurring notation appears after Section 4.

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## 2. Preliminaries

This section provides some of the notation, conventions, and basic results used throughout our paper.

### 2.1. Fundamental conventions

We work exclusively over $\mathbb{C}$ and understand all group actions as being left group actions. We also write $\mathcal{O}_X$ for the structure sheaf of an algebraic variety $X$, as well as $\mathbb{C}[X]$ for the coordinate ring $\mathcal{O}_X(X)$. The dimension of $X$ is understood to be the supremum of the dimensions of the irreducible components. We understand $X$ to be smooth if $\dim(T_xX) = \dim(X)$ for all $x \in X$. Note that this convention forces a smooth variety to be pure-dimensional.

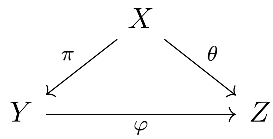
### 2.2. Quotients of $K$-varieties

Let $K$ be a linear algebraic group. We adopt the term *$K$-variety* in reference to a variety $X$ endowed with an algebraic $K$-action.

**Definition 2.1.** Suppose that $X$ is a $K$-variety. A variety morphism $\pi : X \longrightarrow Y$ is called a *categorical quotient* of $X$ if the following conditions are satisfied:

(i) $\pi$ is $K$-invariant;

(ii) if $\theta : X \longrightarrow Z$ is a $K$-invariant variety morphism, then there exists a unique morphism $\varphi : Y \longrightarrow Z$ for which




commutes.

**Definition 2.2.** Suppose that $X$ is a $K$-variety. A variety morphism $\pi : X \longrightarrow Y$ is called a *good quotient* of $X$ if the following conditions are satisfied:

(i) $\pi$ is surjective, affine, and $K$-invariant;

(ii) if $U \subseteq Y$ is open, then the comorphism $\pi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\pi^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X(\pi^{-1}(U))^K$;

(iii) if $Z \subseteq X$ is closed and $K$-invariant, then $\pi(Z)$ is closed in $Y$;

(iv) if $Z_1, Z_2 \subseteq X$ are closed, $K$-invariant, and disjoint, then $\pi(Z_1)$ and $\pi(Z_2)$ are disjoint.

One calls $\pi : X \longrightarrow Y$ a *geometric quotient* of $X$ if $\pi$ is a good quotient and $\pi^{-1}(y)$ is a $K$-orbit for each $y \in Y$.

**Remark 2.3.** A good quotient is necessarily categorical, e.g. by [50, Lemma 1.4.1.1].

Let $X$ be a $K$-variety admitting a geometric quotient $\pi : X \longrightarrow Y$, and write $X/K$ for the set of $K$-orbits in $X$. One then has a canonical bijection $Y \cong X/K$, through which $X/K$ inherits a variety structure. Any two geometric quotients $\pi : X \longrightarrow Y$ and $\pi' : X \longrightarrow Y'$ induce the same variety structure on $X/K$, and this structure makes the set-theoretic quotient map $X \longrightarrow X/K$ a geometric quotient. With this in mind, we shall sometimes write "$X/K$ exists" or "the geometric quotient $X \longrightarrow X/K$ exists" to mean that $X$ admits a geometric quotient.

We will need the following algebro-geometric notion of a principal bundle appearing in [8, Definition 2.3.1].

**Definition 2.4.** Suppose that $X$ is a $K$-variety. A $K$-invariant variety morphism $\pi : X \longrightarrow Y$ is called a *principal $K$-bundle* if the following conditions hold:

(i) $\pi$ is faithfully flat, i.e. flat and surjective;

(ii) the natural map

$$\sigma : K \times X \longrightarrow X \times_Y X, \quad \sigma(k, x) = (x, k \cdot x)$$

is an isomorphism.

A principal $K$-bundle is necessarily a geometric quotient (e.g. by [8, Proposition 2.3.3]). We understand "$X$ is a principal $K$-bundle" as meaning that $X$ admits a geometric quotient $\pi : X \longrightarrow X/K$, and that $\pi$ is a principal $K$-bundle.

## 2.3. Poisson varieties

Let $X$ be a smooth variety. Suppose that $P$ is a global section of $\Lambda^2(TX)$, and consider the bracket operation defined by

$$\{f_1, f_2\} := P(df_1 \wedge df_2) \in \mathcal{O}_X$$

for all $f_1, f_2 \in \mathcal{O}_X$. One calls $P$ a *Poisson bivector* if this bracket renders $\mathcal{O}_X$ a sheaf of Poisson algebras. We use the term *Poisson variety* in reference to a smooth variety $X$ equipped with a Poisson bivector $P$. In this case, $\{\cdot, \cdot\}$ is called the *Poisson bracket*. Let us also recall that a variety morphism $\phi : X_1 \longrightarrow X_2$ between Poisson varieties $(X_1, P_1)$ and $(X_2, P_2)$ is called a *Poisson morphism* if

$$d\phi(P_1(\phi^* \alpha)) = P_2(\alpha)$$

for all one-forms $\alpha$ defined on any open subset of $X_2$. Our convention is to have $(X_1 \times X_2, P_1 \oplus (-P_2))$ be the Poisson variety product of $(X_1, P_1)$ and $(X_2, P_2)$.

Let $(X, P)$ be a Poisson variety. Contracting the bivector with cotangent vectors allows one to view $P$ as a bundle morphism

$$P : T^*X \longrightarrow TX,$$

whose image is a holomorphic distribution on $X$. One refers to the maximal integral submanifolds of this distribution as the *symplectic leaves* of $X$. The symplectic form $\omega_L$ on a symplectic leaf $L \subseteq X$ is constructed as follows. One defines the *Hamiltonian vector field* of a locally defined function $f \in \mathcal{O}_X$ by

$$H_f := -P(df). \tag{2.1}$$

This gives rise to the tangent space description

$$T_xL = \{(H_f)_x : f \in \mathcal{O}_X\}$$

for all $x \in L$, and one has

$$(\omega_L)_x((H_{f_1})_x, (H_{f_2})_x) = \{f_1, f_2\}(x)$$

for all $x \in L$ and $f_1, f_2 \in \mathcal{O}_X$ defined near $x$.

We conclude by discussing *log symplectic varieties*, which have received considerable attention in recent years (e.g. [2, 10, 22–25, 28, 29, 31, 41, 42, 45, 47, 48]). To this end, one calls a Poisson variety $(X, P)$ *log symplectic* if the following conditions hold:

(i) $(X, P)$ has a unique open dense symplectic leaf $X_0 \subseteq X$;

(ii) the vanishing locus of $P^n$ is a reduced normal crossing divisor $D \subseteq X$, where $2n = \dim(X_0)$ and $P^n \in H^0(X, \Lambda^{2n}(TX))$ is the top exterior power of $P$.

In this case, we call $D$ the *divisor* of $(X, P)$. One immediate observation is that $D = X \setminus X_0$.

**Remark 2.5.** Since symplectic leaves are connected, Condition (i) implies that log symplectic varieties are irreducible.

## 2.4. Hamiltonian reduction

We now review the salient aspects of Hamiltonian actions in the Poisson category. To this end, let $K$ be a linear algebraic group with Lie algebra $\mathfrak{k}$. Let $(X, P)$ be a Poisson variety, and assume that $X$ is also a $K$-variety. Each $y \in \mathfrak{k}$ then determines a *fundamental vector field* $V_y$ on $X$ via

$$(V_y)_x = \frac{d}{dt}\bigg|_{t=0} (\exp(ty) \cdot x) \in T_xX$$

for all $x \in X$. The $K$-action on $X$ is called *Hamiltonian* if $P$ is $K$-invariant and there exists a $K$-equivariant morphism $\nu : X \longrightarrow \mathfrak{k}^*$ satisfying the following condition:

$$H_{\nu^y} = -V_y \tag{2.2}$$

for all $y \in \mathfrak{k}$, where $\nu^y \in \mathbb{C}[X]$ is defined by

$$\nu^y(x) = \nu(x)(y), \quad x \in X. \tag{2.3}$$

One then refers to $\nu$ as a moment map and calls $(X, P, \nu)$ a *Hamiltonian $K$-variety*. The moment map $\nu$ is known to be a Poisson morphism with respect to the Lie–Poisson structure on $\mathfrak{k}^*$ (e.g. [9, Proposition 7.1]).

We now briefly recall the process of Hamiltonian reduction for a Hamiltonian $K$-variety $(X, P, \nu)$. One begins by observing that $\nu^{-1}(0)$ is a $K$-invariant closed subvariety of $X$. Let us assume $X /\!\!/ K$ exists, by which we mean that the geometric quotient

$$\pi : \nu^{-1}(0) \longrightarrow \nu^{-1}(0)/K \tag{2.4}$$

exists. Write

$$X /\!\!/ K := \nu^{-1}(0)/K,$$

and note that the comorphism $\pi^* : \mathbb{C}[X /\!\!/ K] \longrightarrow \mathbb{C}[\nu^{-1}(0)]$ induces an algebra isomorphism

$$\mathbb{C}[X /\!\!/ K] \xrightarrow{\cong} \mathbb{C}[\nu^{-1}(0)]^K. \tag{2.5}$$

At the same time, the canonical surjection $\mathbb{C}[X] \longrightarrow \mathbb{C}[\nu^{-1}(0)]$ restricts to a surjection

$$\mathbb{C}[X]^K \longrightarrow \mathbb{C}[\nu^{-1}(0)]^K \tag{2.6}$$

if $K$ is connected and reductive. One also knows that $\mathbb{C}[X]^K$ is a Poisson subalgebra of $\mathbb{C}[X]$, and that the kernel of (2.6) is a Poisson ideal $I \subseteq \mathbb{C}[X]^K$. It follows that $\mathbb{C}[\nu^{-1}(0)]^K$ inherits the structure of a Poisson algebra. One may therefore endow $\mathbb{C}[X /\!\!/ K]$ with the unique Poisson bracket for which (2.5) is an isomorphism of Poisson algebras. We refer to the data of the variety $X /\!\!/ K$ and the Poisson algebra $\mathbb{C}[X /\!\!/ K]$ as the *Hamiltonian reduction* of $(X, P, \nu)$ if (2.4) exists and $K$ is connected and reductive.

The Hamiltonian reduction process will yield a richer geometric object in the presence of certain assumptions about the $K$-action on $X$. To this end, let $K$ be a linear algebraic group and suppose that $(X, P, \nu)$ is a Hamiltonian $K$-variety. Assume that the geometric quotient (2.4) exists, that $K$ acts freely on $\nu^{-1}(0)$, and that $X /\!\!/ K$ is a smooth variety. One may define a Poisson bivector $P_{X/\!\!/K}$ on $X /\!\!/ K$ as follows. Suppose that $x \in \nu^{-1}(0)$ and let

$$d\pi_x^* : T^*_{\pi(x)}(X /\!\!/ K) \longrightarrow T^*_x(\nu^{-1}(0))$$

be the dual of the differential $d\pi_x : T_x(\nu^{-1}(0)) \longrightarrow T_{\pi(x)}(X /\!\!/ K)$. Set

$$P_{\pi(x)}(\alpha) := d\pi_x(P_x(\tilde{\alpha}))$$

for all $\alpha \in T^*_{\pi(x)}(X /\!\!/ K)$, where $\tilde{\alpha} \in T^*_x X$ is any element that annihilates $T_x(Kx)$ and coincides with $d\pi_x^*(\alpha)$ on $T_x(\nu^{-1}(0))$. The bivector $P_{X/\!\!/K}$ renders $\mathcal{O}_{X/\!\!/K}$ a sheaf of Poisson algebras, recovering the above-described Poisson bracket on $\mathbb{C}[X /\!\!/ K]$. We call the Poisson variety $(X /\!\!/ K, P_{X/\!\!/K})$ the *Hamiltonian reduction* of $(X, P, \nu)$ *at level* 0, provided that (2.4) exists, $K$ acts freely on $\nu^{-1}(\zeta)$, and $X /\!\!/ K$ is a smooth variety.

The preceding construction generalizes to allow for Hamiltonian reduction at an arbitrary level $\zeta \in \mathfrak{k}^*$. To this end, let $K_\zeta$ denote the $K$-stabilizer of $\zeta$ with respect to the coadjoint action. One simply sets

$$X /\!\!/_\zeta K := \nu^{-1}(\zeta)/K_\zeta.$$

The definitions of the Poisson bracket on $\mathbb{C}[X /\!\!/_\zeta K]$ and Poisson bivector $P_{X/\!\!/_\zeta K}$ are analogous to their counterparts above.

## 2.5. Lie-theoretic conventions

Let $G$ be a connected semisimple linear algebraic group with Lie algebra $\mathfrak{g}$. Note that $\mathfrak{g}$ is a $G$-module via the adjoint representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}), \quad g \longrightarrow \mathrm{Ad}_g,$$

and a $\mathfrak{g}$-module via the other adjoint representation

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), \quad y \longrightarrow \mathrm{ad}_y = [y, \cdot].$$

One obtains an induced action of $G$ on the coordinate ring $\mathbb{C}[\mathfrak{g}] = \mathrm{Sym}(\mathfrak{g}^*)$, and we write $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]$ for the subalgebra of all functions fixed by $G$.

The inclusion $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]$ then determines a morphism of affine varieties

$$\chi : \mathfrak{g} \longrightarrow \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G),$$

often called the *adjoint quotient*.

Define the centralizer subalgebra

$$\mathfrak{g}_y := \{z \in \mathfrak{g} : [y,z] = 0\} \subseteq \mathfrak{g}$$

for each $y \in \mathfrak{g}$. An element $y \in \mathfrak{g}$ is called *regular* if the dimension of $\mathfrak{g}_y$ coincides with the rank of $\mathfrak{g}$. The set of all regular elements is a $G$-invariant open dense subvariety of $\mathfrak{g}$ that we denote by $\mathfrak{g}^{\mathrm{r}}$.

Recall that $(\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ is an $\mathfrak{sl}_2$-triple if the identities

$$[\xi, \eta] = h, \quad [h, \xi] = 2\xi, \quad \text{and} \quad [h, \eta] = -2\eta$$

hold in $\mathfrak{g}$, and that the associated *Slodowy slice* is defined by

$$\mathcal{S}_\tau := \xi + \mathfrak{g}_\eta \subseteq \mathfrak{g}.$$

Now assume that $\tau$ is a *principal* $\mathfrak{sl}_2$-triple, i.e. an $\mathfrak{sl}_2$-triple for which $\xi, h, \eta \in \mathfrak{g}^{\mathrm{r}}$. The slice $\mathcal{S}_\tau$ then lies in $\mathfrak{g}^{\mathrm{r}}$ and is a fundamental domain for the $G$-action on $\mathfrak{g}^{\mathrm{r}}$ [34, Theorem 8]. This slice is also known to be a section of the adjoint quotient, meaning that the restriction

$$\chi\big|_{\mathcal{S}_\tau} : \mathcal{S}_\tau \longrightarrow \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$$

is a variety isomorphism [34, Theorem 7]. Let us write

$$y_\tau := (\chi\big|_{\mathcal{S}_\tau})^{-1}(\chi(y)) \in \mathcal{S}_\tau$$

for each $y \in \mathfrak{g}$. In other words, $y_\tau$ is the unique point at which $\mathcal{S}_\tau$ meets $\chi^{-1}(\chi(y))$.

Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow \mathbb{C}$ denote the Killing form on $\mathfrak{g}$. This bilinear form is non-degenerate and $G$-invariant, i.e. the map

$$\mathfrak{g} \longrightarrow \mathfrak{g}^*, \quad y \longrightarrow \langle y, \cdot \rangle \tag{2.7}$$

is a $G$-module isomorphism. The canonical Poisson structure on $\mathfrak{g}^*$ thereby corresponds to a Poisson structure on $\mathfrak{g}$, determined by the following condition:

$$\{f_1, f_2\}(y) = \langle y, [(df_1)_y, (df_2)_y] \rangle$$

for all $f_1, f_2 \in \mathbb{C}[\mathfrak{g}]$ and $y \in \mathfrak{g}$, where the right-hand side uses (2.7) to regard $(df_1)_y, (df_2)_y \in \mathfrak{g}^*$ as elements of $\mathfrak{g}$. By means of (2.7), we shall make no further distinction between $\mathfrak{g}$ and $\mathfrak{g}^*$. One also has the $(G \times G)$-module isomorphism

$$\mathfrak{g} \oplus \mathfrak{g} \longrightarrow (\mathfrak{g} \oplus \mathfrak{g})^*, \quad (x_1, x_2) \longrightarrow (\langle x_1, \cdot \rangle, -\langle x_2, \cdot \rangle),$$

through which we shall identify $\mathfrak{g} \oplus \mathfrak{g}$ with $(\mathfrak{g} \oplus \mathfrak{g})^*$.

## 2.6. The wonderful compactification

In this subsection, we assume that $G$ is the adjoint group of $\mathfrak{g}$. Let $n = \dim \mathfrak{g}$ and write $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ for the Grassmannian of all $n$-dimensional subspaces in $\mathfrak{g} \oplus \mathfrak{g}$. Note that $G \times G$ acts on $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ by

$$(g_1, g_2) \cdot \gamma := \{(\mathrm{Ad}_{g_1}(y_1), \mathrm{Ad}_{g_2}(y_2)) : (y_1, y_2) \in \gamma\},$$

and on $G$ itself by

$$(g_1, g_2) \cdot h := g_1 h g_2^{-1}.$$

Let $\mathfrak{g}_\Delta \subseteq \mathfrak{g} \oplus \mathfrak{g}$ denote the diagonally embedded copy of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}$, and consider the $(G \times G)$-equivariant locally closed immersion

$$\varphi : G \longrightarrow \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}), \quad g \longrightarrow (g, e) \cdot \mathfrak{g}_\Delta. \tag{2.8}$$

We thereby view $G$ as a subvariety of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ and write $\overline{G}$ for its closure in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. The closed subvariety $\overline{G}$ is $(G \times G)$-invariant, smooth, and called the *wonderful compactification* of $G$ [18]. The complement $D := \overline{G} \setminus G$ is known to be a normal crossing divisor in $\overline{G}$.

The pair $(G, D)$ determines a so-called *log cotangent bundle* $T^*\overline{G}(\log D) \longrightarrow \overline{G}$. One may realize this vector bundle as the pullback of the tautological bundle $\mathcal{T} \longrightarrow \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ along the inclusion $\overline{G} \hookrightarrow \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. This amounts to setting

$$T^*\overline{G}(\log D) := \{(\gamma, (y_1, y_2)) \in \overline{G} \times (\mathfrak{g} \oplus \mathfrak{g}) : (y_1, y_2) \in \gamma\}$$

and defining the bundle projection to be

$$T^*\overline{G}(\log D) \longrightarrow \overline{G}, \quad (\gamma, (y_1, y_2)) \longrightarrow \gamma.$$

The action of $(G \times G)$ on $\overline{G}$ then lifts to the following $(G \times G)$-action on $T^*\overline{G}(\log D)$:

$$(g_1, g_2) \cdot (\gamma, (y_1, y_2)) := ((g_1, g_2) \cdot \gamma, (\mathrm{Ad}_{g_1}(y_1), \mathrm{Ad}_{g_2}(y_2))). \tag{2.9}$$

## 2.7. Poisson geometry on $T^*G$ and $T^*\overline{G}(\log D)$

Let all objects and notation be as set in 2.5. Note that the left trivialization and Killing form combine to yield a variety isomorphism

$$T^*G \cong G \times \mathfrak{g}.$$

We shall thereby make no further distinction between $T^*G$ and $G \times \mathfrak{g}$. The canonical symplectic form $\omega$ on $T^*G$ is then defined as follows on each tangent space $T_{(g,x)}(G \times \mathfrak{g}) = T_gG \oplus \mathfrak{g}$:

$$\omega_{(g,x)}\Big(\big((dL_g)_e(y_1), z_1\big), \big((dL_g)_e(y_2), z_2\big)\Big) = \langle y_1, z_2\rangle - \langle y_2, z_1\rangle + \langle x, [y_1, y_2]\rangle$$

for all $y_1, y_2, z_1, z_2 \in \mathfrak{g}$, where $L_g : G \longrightarrow G$ denotes left translation by $g$ and $(dL_g)_e : \mathfrak{g} \longrightarrow T_gG$ is the differential of $L_g$ at $e \in G$ [38, Section 5, Equation (14L)].

Now consider the identifications

$$T_{(e,x)}(G \times \mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g} \quad \text{and} \quad T^*_{(e,x)}(G \times \mathfrak{g}) = (\mathfrak{g} \oplus \mathfrak{g})^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$$

for each $x \in \mathfrak{g}$. Write $P_\omega$ for the Poisson bivector on $T^*G$ determined by $\omega$, noting that $(P_\omega)_{(e,x)}$ is a vector space isomorphism

$$(P_\omega)_{(e,x)} : \mathfrak{g}^* \oplus \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g} \oplus \mathfrak{g}$$

for each $x \in \mathfrak{g}$. To compute $(P_\omega)_{(e,x)}$, let

$$\kappa : \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}$$

denote the inverse of (2.7). This leads to the following lemma, which will be needed later.

**Lemma 2.6.** *If $x \in \mathfrak{g}$, then*

$$(P_\omega)_{(e,x)}(\alpha, \beta) = (\kappa(\beta), [x, \kappa(\beta)] - \kappa(\alpha))$$

*for all $(\alpha, \beta) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$.*

*Proof.* Write $P_\omega(\alpha,\beta) = (y,z) \in \mathfrak{g}\oplus\mathfrak{g}$ and note that

$$\begin{aligned}\alpha(v)+\beta(w) &= \omega_{(e,x)}((P_\omega)_{(e,x)}(\alpha,\beta),(v,w))\\ &= \omega_{(e,x)}((y,z),(v,w))\\ &= \langle y,w\rangle - \langle z,v\rangle + \langle x,[y,v]\rangle\\ &= \langle y,w\rangle + \langle [x,y]-z,v\rangle.\end{aligned}$$

for all $v,w\in\mathfrak{g}$. It follows that $\kappa(\alpha) = [x,y]-z$ and $\kappa(\beta)=y$, or equivalently

$$y = \kappa(\beta) \quad \text{and} \quad z = [x,\kappa(\beta)] - \kappa(\alpha).$$

□

Now assume that $G$ is the adjoint group of $\mathfrak{g}$. The variety $T^*\overline{G}(\log D)$ admits a distinguished log symplectic structure (e.g. [2]), some aspects of which we now describe. We begin by noting that

$$\tilde{\varphi} : T^*G \longrightarrow T^*\overline{G}(\log D), \quad (g,x)\longrightarrow ((g,e)\cdot\mathfrak{g}_\Delta, (\mathrm{Ad}_g(x),x)). \tag{2.10}$$

is a symplectomorphism onto the unique open dense symplectic leaf in $T^*\overline{G}(\log D)$. This yields the commutative diagram

$$\begin{array}{ccc} T^*G & \xrightarrow{\ \tilde{\varphi}\ } & T^*\overline{G}(\log D) \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow{\ \varphi\ } & \overline{G} \end{array},$$

where $\varphi : G \longrightarrow \overline{G}$ is the map (2.8). One also observes $\tilde{\varphi}$ to be equivariant with respect to (2.9) and the following $(G\times G)$-action on $T^*G$:

$$(g_1,g_2)\cdot(h,y) := (g_1hg_2^{-1}, \mathrm{Ad}_g(y)). \tag{2.11}$$

The $(G\times G)$-actions (2.9) and (2.11) are Hamiltonian with respective moment maps

$$\overline{\rho} = (\overline{\rho}_L,\overline{\rho}_R) : T^*\overline{G}(\log D)\longrightarrow \mathfrak{g}\oplus\mathfrak{g}, \quad (\gamma,(y_1,y_2))\longrightarrow (y_1,y_2) \tag{2.12}$$

and

$$\rho = (\rho_L,\rho_R) : T^*G \longrightarrow \mathfrak{g}\oplus\mathfrak{g}, \quad (g,y)\longrightarrow (\mathrm{Ad}_g(y),y). \tag{2.13}$$

Now suppose that $(X, P, \nu)$ is a Hamiltonian $G$-variety. Endow $X$ with the Hamiltonian $(G \times G)$-variety structure for which

$$G_R := \{e\} \times G$$

acts trivially and

$$G_L := G \times \{e\}$$

acts via the original Hamiltonian $G$-action and the identification $G = G_L$. It follows that the product Poisson varieties $X \times T^*G$ and $X \times T^*\overline{G}(\log D)$ are Hamiltonian $(G \times G)$-varieties with respective moment maps

$$\begin{aligned} \mu = (\mu_L, \mu_R) &: X \times T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g}, \\ (x, (g, y)) &\longrightarrow (\nu(x) - \mathrm{Ad}_g(y), -y) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) &: X \times T^*\overline{G}(\log D) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}, \\ (x, (\gamma, (y_1, y_2))) &\longrightarrow (\nu(x) - y_1, -y_2). \end{aligned} \tag{2.15}$$

We also have a commutative diagram

$$\begin{array}{ccc} X \times T^*G & \xrightarrow{\quad i \quad} & X \times T^*\overline{G}(\log D) \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \overline{\mu}} & \\ & \mathfrak{g} \oplus \mathfrak{g} & \end{array}, \tag{2.16}$$

where

$$\begin{aligned} i &: X \times T^*G \longrightarrow X \times T^*\overline{G}(\log D), \\ (x, (g, y)) &\longrightarrow (x, ((g, e) \cdot \mathfrak{g}_\Delta, (\mathrm{Ad}_g(y), y))). \end{aligned} \tag{2.17}$$

is the $(G \times G)$-equivariant open Poisson embedding given by the product of (2.10) with the identity map $X \longrightarrow X$.

The Hamiltonian $(G \times G)$-variety $X \times T^*G$ warrants some further discussion. One knows that the geometric quotient

$$\mu_L^{-1}(0) \longrightarrow (X \times T^*G) /\!\!/ G_L$$

exists, and that the action of $G_R$ on $\mu_L^{-1}(0)$ descends to a Hamiltonian action of $G$ on $(X \times T^*G) /\!\!/ G_L$. An associated moment map is obtained by descending

$$-\mu_R\big|_{\mu_L^{-1}(0)} : \mu_L^{-1}(0) \longrightarrow \mathfrak{g}$$

to the quotient variety $(X \times T^*G) /\!\!/ G_L$. It is then not difficult to verify that

$$\psi : X \xrightarrow{\cong} (X \times T^*G) /\!\!/ G_L, \quad x \longrightarrow [x : (e, \nu(x))], \quad x \in X \tag{2.18}$$

is an isomorphism of Hamiltonian $G$-varieties.

## 3. Poisson slices

This section develops the general theory of Poisson slices. Some emphasis is placed on properties of the Poisson slice $G \times \mathcal{S}_\tau$ and a larger log symplectic variety $\overline{G \times \mathcal{S}_\tau}$.

### 3.1. Poisson transversals and Poisson slices

Let $(X, P)$ be a Poisson variety. Given $x \in X$ and a subspace $V \subseteq T_xX$, we write $V^\dagger$ for the annihilator of $V$ in $T_x^*X$. Our notation suppresses the dependence of $V^\dagger$ on $T_xX$, as the ambient tangent space will always be clear from context. We will use an analogous notation for vector subbundles of $TX$.

Recall that a smooth locally closed subvariety $Y \subseteq X$ is called a *Poisson transversal* (or *cosymplectic subvariety*) if

$$TX|_Y = TY \oplus P(TY^\dagger). \tag{3.1}$$

This has the following straightforward implication for every symplectic leaf $L \subseteq X$: $L$ and $Y$ have a transverse intersection in $X$, and $L \cap Y$ is a symplectic submanifold of $L$.

The Poisson transversal $Y$ inherits a Poisson bivector $P_Y$ from $(X, P)$. To define it, note that the decomposition (3.1) gives rise to an inclusion $T^*Y \subseteq T^*X$. One can verify that

$$P(T^*Y) \subseteq TY,$$

and $P_Y$ is then defined to be the restriction

$$P_Y := P\big|_{T^*Y} : T^*Y \longrightarrow TY.$$

Note that $Y$ need not be a Poisson subvariety of $X$ in the usual sense; restricting functions need not define a morphism $\mathcal{O}_X \longrightarrow j_*\mathcal{O}_Y$ of sheaves of Poisson algebras, where $j : Y \hookrightarrow X$ is the inclusion. This is particularly apparent if $X$ is symplectic; the Poisson transversals are the symplectic subvarieties, while the Poisson subvarieties are the open subvarieties.

We record the following well-known fact for future reference (cf. [20, Example 4]).

**Proposition 3.1.** *Let $X$ be a symplectic variety. If $Y \subseteq X$ is a Poisson transversal, then $Y$ is a symplectic subvariety of $X$. The resulting symplectic structure on $Y$ coincides with the Poisson structure $Y$ inherits as a transversal.*

We need the following refinement in the case of log symplectic varieties.

**Proposition 3.2.** *Suppose that $(X, P)$ is a log symplectic variety with divisor $Z$. Let $Y \subseteq X$ be an irreducible Poisson transversal, and write $P_Y$ for the resulting Poisson bivector on $Y$. The following statements hold.*

(i) *The Poisson variety $(Y, P_Y)$ is log symplectic with divisor $Z \cap Y$.*

(ii) *If one equips $Y \setminus Z$ and $X \setminus Z$ with the symplectic structures inherited as symplectic leaves of $(Y, P_Y)$ and $(X, P)$, respectively, then $Y \setminus Z$ is a symplectic subvariety of $X \setminus Z$.*

*Proof.* We begin by proving that $Y$ is a log symplectic subvariety of $X$ in the sense of [24, Definition 7.16]. To this end, consider the unique open dense symplectic leaf $X_0 := X \setminus Z \subseteq X$. Since $Y$ is a Poisson transversal in $X$, Proposition 3.1 forces $Y_0 := Y \cap X_0$ to be a symplectic subvariety of $X_0$.

Now let $Z_1, \ldots, Z_k$ be the irreducible components of $Z$, and set

$$Z_I := \bigcap_{i \in I} Z_i$$

for each subset $I \subseteq \{1, \ldots, k\}$. Each irreducible component of $Z$ is a union of symplectic leaves in $X$ (cf. [46, Exercise 5.2]), implying that $Z_I$ is a union of symplectic leaves for each $I \subseteq \{1, \ldots, k\}$. On the other hand, the Poisson transversal $Y$ is necessarily transverse to the symplectic leaves in $X$. These last two sentences imply that $Y$ is transverse to $Z_I$ for all $I \subseteq \{1, \ldots, k\}$.

The previous two paragraphs show $Y$ to be a log symplectic subvariety of $X$, and we let $P_{\log}$ denote the resulting Poisson bivector on $Y$. It follows that $Y_0$ is the unique open dense symplectic leaf of $(Y, P_{\log})$, and that its symplectic form is the pullback of the symplectic form on $X_0$. We also know that $P_Y$ is non-degenerate on $Y_0$, and that it coincides with the pullback of the symplectic structure from $X_0$ to $Y_0$ (see Proposition 3.1). One concludes that $P_{\log}$ and $P_Y$ coincide on $Y_0$. Since $Y_0$ is dense in $Y$, it follows that $P_{\log} = P_Y$. This establishes (i) and (ii). □

The following well-known result concerns the behaviour of Poisson transversals with respect to Poisson morphisms (cf. [20, Lemma 7]).

**Proposition 3.3.** *Let $\phi : X_1 \longrightarrow X_2$ be a Poisson morphism between Poisson varieties $X_1$ and $X_2$. If $Y \subseteq X_2$ is a Poisson transversal, then $\phi^{-1}(Y)$ is a Poisson transversal in $X_1$. The codimension of $\phi^{-1}(Y)$ in $X_1$ is equal to the codimension of $Y$ in $X_2$.*

We now consider a concrete application of Proposition 3.3. To this end, recall the Lie-theoretic notation and setup established in 2.5.

**Corollary 3.4.** *Suppose that $(X, P, \nu)$ is a Hamiltonian $G$-variety. If $\tau = (\xi, h, \eta)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $\nu^{-1}(\mathcal{S}_\tau)$ is a Poisson transversal in $X$. This transversal has codimension* $\dim \mathfrak{g} - \dim(\mathfrak{g}_\eta)$ *in $X$.*

*Proof.* The moment map $\nu : X \longrightarrow \mathfrak{g}$ is necessarily a morphism of Poisson varieties (e.g. [9, Proposition 7.1]). At the same time, [21, Section 3.1] explains that $\mathcal{S}_\tau$ is a Poisson transversal in $\mathfrak{g}$. The desired now result now follows immediately from Proposition 3.3. □

A consequence of Corollary 3.4 is that $\nu^{-1}(\mathcal{S}_\tau)$ inherits a Poisson bivector $P_\tau$ from $(X, P)$. This gives rise to our notion of a *Poisson slice.*

**Definition 3.5.** Suppose that $(X, P, \nu)$ is a Hamiltonian $G$-variety, and let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We call $X_\tau := (\nu^{-1}(\mathcal{S}_\tau), P_\tau)$ the *Poisson slice* of $(X, P, \nu)$ with respect to $\tau$.

This next proposition explains why we call $X_\tau$ a Poisson slice; it is a slice for the $G$-action on $X$ in the following sense.

**Proposition 3.6.** *Let $(X, P, \nu)$ be a Hamiltonian $G$-variety. If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $X_\tau$ is transverse to the $G$-orbits in $X$.*

*Proof.* Fix $x \in \nu^{-1}(\mathcal{S}_\tau)$ and set $y := \nu(x) \in \mathcal{S}_\tau$. Consider the differential $d\nu_x : T_xX \longrightarrow \mathfrak{g}$ and its dual $d\nu_x^* : \mathfrak{g}^* \longrightarrow T_x^*X$, and let $P_\mathfrak{g}$ be the Poisson bivector on $\mathfrak{g}$. Since $\nu$ is a morphism of Poisson varieties, we have

$$(P_{\mathfrak{g}})_y = d\nu_x \circ P_x \circ d\nu_x^*.$$

We also know $\mathcal{S}_\tau \subseteq \mathfrak{g}$ to be a Poisson transversal (e.g. by Corollary 3.4), so that

$$\mathfrak{g} = T_y\mathcal{S}_\tau \oplus (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger) = T_y\mathcal{S}_\tau \oplus d\nu_x(P_x(d\nu_x^*((T_y\mathcal{S}_\tau)^\dagger))).$$

One immediate conclusion is that $\nu$ is transverse to $\mathcal{S}_\tau$. We also conclude that

$$T_x(\nu^{-1}(\mathcal{S}_\tau)) = \ker\left(\mathrm{pr}_2 \circ d\nu_x : T_xX \longrightarrow (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger)\right),$$

where

$$\mathrm{pr}_2 : \mathfrak{g} = T_y\mathcal{S}_\tau \oplus (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger) \longrightarrow (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger)$$

is the natural projection. It follows that

$$T_x(\nu^{-1}(\mathcal{S}_\tau))^\dagger = \mathrm{image}\left(d\nu_x^* \circ \mathrm{pr}_2^* : (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger)^* \longrightarrow T_x^*X\right),$$

where

$$\mathrm{pr}_2^* : (P_{\mathfrak{g}})_y((T_y\mathcal{S}_\tau)^\dagger)^* \longrightarrow \mathfrak{g}^*$$

is the dual of $\mathrm{pr}_2$. This amounts to the statement that

$$T_x(\nu^{-1}(\mathcal{S}_\tau))^\dagger = d\nu_x^*(\mathfrak{g}_\eta^\dagger),$$

because we know that the Killing form identifies $\mathfrak{g}_\eta^\dagger \subseteq \mathfrak{g}^*$ with $\mathfrak{g}_\eta^\perp = [\mathfrak{g}, \eta] \subseteq \mathfrak{g}$. We conclude that

$$T_x(\nu^{-1}(\mathcal{S}_\tau))^\dagger = \mathrm{span}\{(d\nu^{[\eta,b]})_x \colon b \in \mathfrak{g}\},$$

where $\nu^{[\eta,b]} : X \longrightarrow \mathbb{C}$ is defined by

$$\nu^{[\eta,b]}(z) = \langle \nu(z), [\eta, b]\rangle.$$

Equations (2.1) and (2.2) now imply that

$$\begin{aligned} P_x(T_x(\nu^{-1}(\mathcal{S}_\tau))^\dagger) &= \mathrm{span}\{P_x((d\nu^{[\eta,b]})_x) \colon b \in \mathfrak{g}\} \\ &= \mathrm{span}\{V_x^{[\eta,b]} \colon b \in \mathfrak{g}\} \subseteq T_x(Gx). \end{aligned}$$

This combines with $\nu^{-1}(\mathcal{S}_\tau)$ being a Poisson transversal to yield

$$T_xX = T_x(\nu^{-1}(\mathcal{S}_\tau)) \oplus P_x(T_x(\nu^{-1}(\mathcal{S}_\tau))^\dagger) = T_x(\nu^{-1}(\mathcal{S}_\tau)) + T_x(Gx),$$

completing the proof. □

Let $Y$ be an irreducible component of $X_\tau$. The bivector $P_\tau$ then restricts to a Poisson bivector $P_{Y,\tau}$ on $Y$. This leads to the following observation.

**Corollary 3.7.** *Suppose that $(X, P, \nu)$ is a Hamiltonian $G$-variety. Assume that $(X, P)$ is log symplectic with divisor $Z$, and let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $Y$ be an irreducible component of the Poisson slice $X_\tau$.*

(i) *The Poisson variety $(Y, P_{Y,\tau})$ is log symplectic with divisor $Y \cap Z$.*

(ii) *If one equips $Y \setminus Z$ and $X \setminus Z$ with the symplectic structures inherited as symplectic leaves of $(Y, P_{Y,\tau})$ and $(X, P)$, respectively, then $Y \setminus Z$ is a symplectic subvariety of $X \setminus Z$.*

(iii) *If $(X, P)$ is symplectic, then $(X_\tau, P_\tau)$ is symplectic and the symplectic form on $(X, P)$ pulls back to the symplectic form on $(X_\tau, P_\tau)$.*

*Proof.* This follows immediately from Proposition 3.1, Proposition 3.2, and Corollary 3.4. □

The following immediate consequence is used extensively in later sections.

**Corollary 3.8.** *If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $G \times \mathcal{S}_\tau$ is a symplectic subvariety of $T^*G = G \times \mathfrak{g}$.*

*Proof.* Apply Corollary 3.7(iii) to $X = T^*G$ with the Hamiltonian action of $G_R = \{e\} \times G \subseteq G \times G$. □

### 3.2. Poisson slices via Hamiltonian reduction

Recall the Hamiltonian action of $G \times G$ on $T^*G = G \times \mathfrak{g}$ discussed in Subsection 2.7. The symplectic subvariety $G \times \mathcal{S}_\tau$ is invariant under $G_L = G \times$

$\{e\} \subseteq G \times G$, and

(3.2) $$\rho_\tau := \rho_L\big|_{G\times\mathcal{S}_\tau} : G \times \mathcal{S}_\tau \longrightarrow \mathfrak{g}, \quad (g,x) \longrightarrow \mathrm{Ad}_g(x)$$

is a corresponding moment map. Now let $(X, P, \nu)$ be a Hamiltonian $G$-variety, and consider the product Poisson variety $X \times (G \times \mathcal{S}_\tau)$. The diagonal action of $G$ on $X \times (G \times \mathcal{S}_\tau)$ is then Hamiltonian with moment map

$$\mu_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow \mathfrak{g}, \quad (x,(g,y)) \longrightarrow \nu(x) - \mathrm{Ad}_g(y).$$

These considerations allow us to realize Poisson slices via Hamiltonian reduction.

**Proposition 3.9.** *Let $(X, P, \nu)$ be a Hamiltonian $G$-variety, and let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If we endow $X \times (G \times \mathcal{S}_\tau)$ with the Poisson structure and Hamiltonian $G$-action described above, then there is a Poisson variety isomorphism*

(3.3) $$\psi_\tau : X_\tau \xrightarrow{\cong} (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G, \quad x \longrightarrow [x : (e, \nu(x))].$$

*Proof.* We begin by noting that

$$\begin{aligned}\mu_\tau^{-1}(0) &= \{(x,(g,y)) \in X \times (G \times \mathcal{S}_\tau) \colon \nu(x) = \mathrm{Ad}_g(y)\} \\ &= \{(x,(g,y)) \in X \times (G \times \mathcal{S}_\tau) \colon \nu(g^{-1}\cdot x) = y\}.\end{aligned}$$

It follows that the $G$-invariant map

$$J : X \times (G \times \mathcal{S}_\tau) \longrightarrow X, \quad (x,(g,y)) \longrightarrow g^{-1}\cdot x$$

satisfies $J(\mu_\tau^{-1}(0)) \subseteq \nu^{-1}(\mathcal{S}_\tau) = X_\tau$, thereby inducing a map

$$\pi := J\big|_{\mu_\tau^{-1}(0)} : \mu_\tau^{-1}(0) \longrightarrow X_\tau.$$

One then verifies that

$$\pi^{-1}(x) = G\cdot(x, e, \nu(x)) \subseteq X \times (G \times \mathcal{S}_\tau)$$

for all $x \in X_\tau$, where $G\cdot(x,e,\nu(x))$ is the $G$-orbit of $(x,e,\nu(x))$ in $X \times (G \times \mathcal{S}_\tau)$. This forces $\pi$ to be the geometric quotient of $\mu_\tau^{-1}(0)$ by $G$ (e.g. by [52, Proposition 25.3.5]), i.e.

$$(X \times (G \times \mathcal{S}_\tau)) /\!\!/ G = X_\tau.$$

We now have two Poisson structures on $X_\tau$: the Poisson structure $P_{\text{red}}$ from Hamiltonian reduction, and the structure $P_{\text{tr}}$ obtained from $X_\tau$ being a Poisson slice in $X$. It suffices to show that these Poisson structures coincide.

Fix $x \in X_\tau$ and $\alpha \in T^*_x X_\tau$. Since $X_\tau$ is a Poisson transversal in $X$, there is a unique extension of $\alpha$ to an element

$$\tilde{\alpha} \in \left(P_x\big((T_x X_\tau)^\dagger\big)\right)^\dagger \subseteq T^*_x X.$$

The discussion of Poisson transversals in Subsection 3.1 then implies that

$$(P_{\text{tr}})_x(\alpha) = P_x(\tilde{\alpha}). \tag{3.4}$$

We also have

$$(P_{\text{red}})_x(\alpha) = d\pi_z((P_\tau)_z(\tilde{\alpha}')), \tag{3.5}$$

where $z = (x, e, \nu(x))$,

$$\tilde{\alpha}' \in T_z(Gz)^\dagger \subseteq T^*_z(X \times (G \times \mathcal{S}_\tau))$$

is an extension of $d\pi^*_z(\alpha)$, and

$$d\pi^*_z : T^*_x X_\tau \longrightarrow T^*_z(\mu_\tau^{-1}(0))$$

is the dual of

$$d\pi_z : T_z(\mu_\tau^{-1}(0)) \longrightarrow T_x X_\tau.$$

Since $J$ is $G$-invariant, we may take

$$\tilde{\alpha}' := dJ^*_z(\tilde{\alpha}).$$

We also observe that

$$dJ_z(a, b, c) = a - (V^b)_x$$

for all $(a, b, c) \in T_z(X \times (G \times \mathcal{S}_\tau)) = T_x X \oplus \mathfrak{g} \oplus \mathfrak{g}_\eta$, where $V^b$ is the fundamental vector field on $X$ associated to $b \in \mathfrak{g}$. It follows that

$$\begin{aligned}(dJ^*_z(\tilde{\alpha}))(a, b, c) &= \tilde{\alpha}(a) - \tilde{\alpha}((V^b)_x) \\ &= \tilde{\alpha}(a) - \tilde{\alpha}(P_x((d\nu^b)_x)) = \tilde{\alpha}(a) + (d\nu^b)_x(P_x(\tilde{\alpha})),\end{aligned}$$

yielding

$$\tilde{\alpha}' = (\tilde{\alpha}, d\nu_x(P_x(\tilde{\alpha})), 0) \in T_z^*(X \times (G \times \mathcal{S}_\tau)) \\ = T_x^*X \oplus \mathfrak{g}^* \oplus \mathfrak{g}_\eta^* = T_x^*X \oplus \mathfrak{g} \oplus \mathfrak{g}_\xi, \tag{3.6}$$

where we have made the identifications $\mathfrak{g}_\eta^* = (\mathfrak{g}/[\mathfrak{g},\xi])^* = [\mathfrak{g},\xi]^\perp = \mathfrak{g}_\xi$. Now set $w = (e, \nu(x)) \in G \times \mathcal{S}_\tau$ and let $Q_\tau$ be the Poisson bivector on $G \times \mathcal{S}_\tau$. Lemma 2.6 then gives

$$(Q_\tau)_w(d\nu_x(P_x(\tilde{\alpha})), 0) = (0, -d\nu_x(P_x(\tilde{\alpha}))).$$

This combines with (3.4), (3.5), and (3.6) to yield

$$\begin{aligned}(P_{\mathrm{red}})_x(\alpha) &= d\pi_z(P_x(\tilde{\alpha}), -(Q_\tau)_w(d\nu_x(P_x(\tilde{\alpha})), 0)) \\ &= d\pi_z(P_x(\tilde{\alpha}), 0, d\nu_x(P_x(\tilde{\alpha}))) \\ &= P_x(\tilde{\alpha}) \\ &= (P_{\mathrm{tr}})_x(\alpha),\end{aligned}$$

as desired. □

**Remark 3.10.** In the special case $\tau = 0$, we have $\mathcal{S}_\tau = \mathfrak{g}$ and $G \times \mathcal{S}_\tau = G \times \mathfrak{g} = T^*G$. Proposition 3.9 is then seen to recover the isomorphism (2.18).

Our next result is that Poisson slices can be realized via Hamiltonian reduction with respect to unipotent radicals of parabolic subgroups. To formulate this result, let $\tau = (\xi, h, \eta)$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and write $\mathfrak{g}_\lambda \subseteq \mathfrak{g}$ for the eigenspace of $\mathrm{ad}_h$ with eigenvalue $\lambda \in \mathbb{Z}$. The parabolic subalgebra

$$\mathfrak{p}_\tau := \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda$$

then has

$$\mathfrak{u}_\tau := \bigoplus_{\lambda < 0} \mathfrak{g}_\lambda$$

as its nilradical. Now consider the identifications

$$\mathfrak{u}_\tau^* \cong \mathfrak{g}/\mathfrak{u}_\tau^\perp = \mathfrak{g}/\mathfrak{p}_\tau \cong \mathfrak{u}_\tau^- := \bigoplus_{\lambda > 0} \mathfrak{g}_\lambda,$$

and thereby regard $\xi \in \mathfrak{u}_\tau^-$ as an element of $\mathfrak{u}_\tau^*$. Write $U_\tau \subseteq G$ for the unipotent subgroup with Lie algebra $\mathfrak{u}_\tau$, and let $(U_\tau)_\xi$ be the $U_\tau$-stabilizer of $\xi$ under the coadjoint action.

**Remark 3.11.** The Lie algebra of $(U_\tau)_\xi$ is given by

$$(\mathfrak{u}_\tau)_\xi = \bigoplus_{\lambda \leq -2} \mathfrak{g}_\lambda.$$

It follows that $(U_\tau)_\xi = U_\tau$ if and only if $\tau$ is an even $\mathfrak{sl}_2$-triple, i.e. $\mathfrak{g}_{-1} = \{0\}$. If $\tau$ is a principal triple, then $\tau$ is even and $(U_\tau)_\xi = U_\tau$ is a maximal unipotent subgroup of $G$.

Let $(X, P, \nu)$ be a Hamiltonian $G$-variety. The action of $U_\tau$ is also Hamiltonian with moment map $\nu_\tau := p_\tau \circ \mu$, where

$$\mathfrak{g} = \mathfrak{p}_\tau \oplus \mathfrak{u}_\tau^- \xrightarrow{p_\tau} \mathfrak{u}_\tau^- = \mathfrak{u}_\tau^*$$

is the projection. One has

$$\nu_\tau^{-1}(\xi) = \nu^{-1}(\xi + \mathfrak{p}_\tau),$$

while the proof of [5, Lemma 3.2] shows the stabilizer $(U_\tau)_\xi$ to act freely on $\xi + \mathfrak{p}_\tau$. It follows that $(U_\tau)_\xi$ acts freely on $\nu_\tau^{-1}(\xi)$. This leads us to prove Proposition 3.13, i.e. that the geometric quotient

$$X /\!\!/_\xi U_\tau = \nu_\tau^{-1}(\xi)/(U_\tau)_\xi \tag{3.7}$$

exists and is Poisson-isomorphic to $X_\tau$.

**Remark 3.12.** The type of Hamiltonian reduction performed in (3.7) is particularly well-studied in the case of a principal triple $\tau$. In this case, one sometimes calls the Poisson variety $X /\!\!/_\xi U_\tau$ a *Whittaker reduction* (e.g. [3, 19]). The nomenclature reflects Kostant's result [35, Theorem 1.2].

**Proposition 3.13.** *Let $(X, P, \nu)$ be a Hamiltonian $G$-variety. If $\tau = (\xi, h, \eta)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then there is a canonical isomorphism*

$$X /\!\!/_\xi U_\tau \cong X_\tau$$

*of Poisson varieties.*

*Proof.* We begin by exhibiting $X_\tau$ as the geometric quotient of $\nu_\tau^{-1}(\xi)$ by $(U_\tau)_\xi$. To this end, the proof of [5, Lemma 3.2] explains that

$$(U_\tau)_\xi \times \mathcal{S}_\tau \longrightarrow \xi + \mathfrak{p}_\tau, \quad (u, x) \longrightarrow \mathrm{Ad}_u(x)$$

defines a variety isomorphism. Composing the inverse of this isomorphism with the projection

$$(U_\tau)_\xi \times \mathcal{S}_\tau \longrightarrow (U_\tau)_\xi$$

then yields a map

$$\phi : \xi + \mathfrak{p}_\tau \longrightarrow (U_\tau)_\xi.$$

Note that for $y \in \xi + \mathfrak{p}_\tau$, $\phi(y)$ is the unique element of $(U_\tau)_\xi$ satisfying

$$\mathrm{Ad}_{\phi(y)^{-1}}(y) \in \mathcal{S}_\tau.$$

We may therefore define the map

$$\nu_\tau^{-1}(\xi) = \nu^{-1}(\xi + \mathfrak{p}_\tau) \xrightarrow{\theta} X_\tau, \quad x \longrightarrow (\phi(\nu(x)))^{-1} \cdot x.$$

One has

$$\theta^{-1}(x) = (U_\tau)_\xi \cdot x$$

for all $x \in \nu_\tau^{-1}(\xi)$, and we deduce that $\theta$ is the geometric quotient of $\nu_\tau^{-1}(\xi)$ by $(U_\tau)_\xi$ (e.g. by [52, Proposition 25.3.5]).

The previous paragraph establishes the following fact: Hamiltonian reductions of Hamiltonian $G$-varieties by $U_\tau$ at level $\xi$ always exist as geometric quotients. We implicitly use this observation in several places below.

To see that the Poisson structures on $X_\tau$ and $X \mathbin{/\!\!/}_\xi U_\tau$ coincide, we argue as follows. One has a canonical isomorphism

$$T^*G \mathbin{/\!\!/}_\xi U_\tau \cong G \times \mathcal{S}_\tau \tag{3.8}$$

of symplectic varieties, where $U_\tau$ acts on $T^*G$ via (2.11) as the subgroup $U_\tau = \{e\} \times U_\tau \subseteq G \times G$ (see [5, Lemma 3.2]). Note also that $T^*G \mathbin{/\!\!/}_\xi U_\tau$ and $G \times \mathcal{S}_\tau$ come with Hamiltonian actions of $G$ induced by the action of $G_L = G \times \{e\}$ on $T^*G \cong G \times \mathfrak{g}$. One then readily verifies that (3.8) is an isomorphism of Hamiltonian $G$-varieties.

Proposition 3.9 gives a canonical isomorphism of Poisson varieties

$$X_\tau \cong (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G.$$

The previous paragraph allows us to write this isomorphism as

$$X_\tau \cong (X \times (T^*G /\!\!/_\xi U_\tau)) /\!\!/ G = ((X \times T^*G) /\!\!/_\xi U_\tau) /\!\!/ G,$$

where $U_\tau$ acts trivially on $X$. Since the actions of $G$ and $U_\tau$ on $X \times T^*G$ commute with one another, it follows that

$$X_\tau \cong ((X \times T^*G) /\!\!/ G) /\!\!/_\xi U_\tau.$$

An application of Remark 3.10 then yields

$$X_\tau \cong X /\!\!/_\xi U_\tau,$$

completing the proof. □

### 3.3. Poisson slices in the log cotangent bundle of $\overline{G}$

Fix an $\mathfrak{sl}_2$-triple $\tau$ in $\mathfrak{g}$ and recall the notation in Subsection 2.7. Let $G$ be the adjoint group of $\mathfrak{g}$. In what follows, we study the Poisson slice

$$\overline{G \times \mathcal{S}_\tau} := \overline{\rho}_R^{-1}(\mathcal{S}_\tau) \subseteq T^*\overline{G}(\log D)$$

and its properties. We begin by observing that

$$\overline{G \times \mathcal{S}_\tau} = \{(\gamma, (x, y)) \in \overline{G} \times (\mathfrak{g} \oplus \mathfrak{g})\colon (x, y) \in \gamma \text{ and } y \in \mathcal{S}_\tau\}. \tag{3.9}$$

A few simplifications arise if $\tau$ is a principal $\mathfrak{sl}_2$-triple. To this end, recall the adjoint quotient

$$\chi : \mathfrak{g} \longrightarrow \operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G)$$

and the associated concepts and notation discussed in Subsection 2.5. The image of $\overline{\rho} : T^*\overline{G}(\log D) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ is known to be

$$\operatorname{image}(\overline{\rho}) = \{(x, y) \in \mathfrak{g} \oplus \mathfrak{g}\colon \chi(x) = \chi(y)\} \tag{3.10}$$

(see [2, Proposition 3.4]). One consequence is that $x, y \in \mathfrak{g}$ lie in the same fibre of $\chi$ whenever $(x, y) \in \gamma$ for some $\gamma \in \overline{G}$. Since $\mathcal{S}_\tau$ is a section of $\chi$, this fact combines with (3.9) to yield

$$\overline{G\times\mathcal{S}_\tau}=\{(\gamma,(x,x_\tau)):\gamma\in\overline{G},\ x\in\mathfrak{g},\ \text{and}\ (x,x_\tau)\in\gamma\}. \tag{3.11}$$

We now develop some more manifestly geometric properties of $\overline{G\times\mathcal{S}_\tau}$, beginning with the following result.

**Theorem 3.14.** *If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $\overline{G\times\mathcal{S}_\tau}$ is irreducible.*

*Proof.* Consider the closed subvariety

$$Y:=\{(x,y)\in\mathfrak{g}\times\mathcal{S}_\tau:\chi(x)=\chi_\tau(y)\}\subseteq\mathfrak{g}\oplus\mathfrak{g},$$

where $\chi_\tau:\mathcal{S}_\tau\longrightarrow\operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ denotes the restriction of $\chi$ to $\mathcal{S}_\tau$. It follows from (3.9) and (3.10) that

$$\overline{G\times\mathcal{S}_\tau}\longrightarrow Y,\quad (\gamma,(x,y))\longrightarrow(x,y) \tag{3.12}$$

is the pullback of $\overline{\rho}:T^*\overline{G}(\log D)\longrightarrow\mathfrak{g}\oplus\mathfrak{g}$ along the inclusion $Y\hookrightarrow\mathfrak{g}\oplus\mathfrak{g}$, and that (3.12) is surjective. One also knows that $\overline{\rho}$ is proper, as it results from restricting the natural projection $\overline{G}\times(\mathfrak{g}\oplus\mathfrak{g})\longrightarrow\mathfrak{g}\oplus\mathfrak{g}$ to $T^*\overline{G}(\log D)$ $\subseteq\overline{G}\times(\mathfrak{g}\oplus\mathfrak{g})$. The surjection (3.12) is therefore proper, while the proof of [2, Proposition 3.11] shows (3.12) to have connected fibres. If $Y$ were connected, then the previous sentence would force $\overline{G\times\mathcal{S}_\tau}$ to be connected as well. This would in turn force $\overline{G\times\mathcal{S}_\tau}$ to be irreducible, as Poisson slices are smooth.

In light of the previous paragraph, it suffices to prove that $Y$ is irreducible. We begin by decomposing $\mathfrak{g}$ into its simple factors, i.e.

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_N$$

with each $\mathfrak{g}_i$ a simple Lie algebra. Our $\mathfrak{sl}_2$-triple $\tau$ then amounts to having an $\mathfrak{sl}_2$-triple $\tau_i$ in $\mathfrak{g}_i$ for each $i=1,\ldots,N$, yielding

$$\mathcal{S}_\tau=\mathcal{S}_{\tau_1}\times\cdots\times\mathcal{S}_{\tau_N}\subseteq\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_N.$$

It also follows that $\chi_\tau$ decomposes as a product

$$\chi_\tau=(\chi_1)_{\tau_1}\times\cdots\times(\chi_N)_{\tau_N},$$

where $\chi_i$ is the adjoint quotient map on $\mathfrak{g}_i$ and $(\chi_i)_{\tau_i}$ is its restriction to $\mathcal{S}_{\tau_i}$. The results [51, Corollary 7.4.1] and [44, Theorem 5.4] then imply that each $(\chi_i)_{\tau_i}$ is faithfully flat with irreducible fibres of dimension

$\dim(\mathcal{S}_{\tau_i}) - \mathrm{rank}(\mathfrak{g}_i)$. These last two sentences imply that $\chi_\tau$ is faithfully flat with irreducible, equidimensional fibres, and the same argument forces $\chi$ to be faithfully flat with irreducible, equidimensional fibres. Since fibred products of faithfully flat morphisms are faithfully flat, we conclude that

$$\tilde{\chi}\colon Y \longrightarrow \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G), \quad (x,y) \mapsto \chi(x)$$

is faithfully flat. We also conclude that

$$\tilde{\chi}^{-1}(t) = \chi^{-1}(t) \times \chi_\tau^{-1}(t)$$

must be irreducible for all $t \in \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$, and that its dimension must be independent of $t$. In other words, $\tilde{\chi}$ is a faithfully flat morphism with irreducible, equidimensional fibres. This combines with the irreducibility of $\mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ and [32, Corollary 9.6] to imply that $Y$ is pure-dimensional. We may now apply the result in [43] and deduce that $Y$ is irreducible. This completes the proof. □

**Corollary 3.15.** *If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $\overline{G \times \mathcal{S}_\tau}$ is log symplectic.*

*Proof.* This is an immediate consequence of Corollary 3.7(i) and Theorem 3.14. □

Now observe that the Hamiltonian action of $G_L = G \times \{e\} \subseteq G \times G$ on $T^*\overline{G}(\log D)$ restricts to a Hamiltonian action of $G$ on $\overline{G \times \mathcal{S}_\tau}$. An associated moment map is given by

$$\overline{\rho}_\tau := \overline{\rho}_L\Big|_{\overline{G\times\mathcal{S}_\tau}} : \overline{G \times \mathcal{S}_\tau} \longrightarrow \mathfrak{g}, \quad (\gamma,(x,y)) \mapsto x.$$

At the same time, recall the Hamiltonian $G$-variety structure on $G \times \mathcal{S}_\tau$ and the moment map $\rho_\tau : G \times \mathcal{S}_\tau \longrightarrow \mathfrak{g}$ discussed in Subsection 3.2. Let us also recall the map $\tilde{\varphi} : T^*G \longrightarrow T^*\overline{G}(\log D)$ from (2.10).

**Proposition 3.16.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.*

(i) *The map $\tilde{\varphi} : T^*G \longrightarrow T^*\overline{G}(\log D)$ restricts to a $G$-equivariant symplectomorphism from $G \times \mathcal{S}_\tau$ to the unique open dense symplectic leaf in $\overline{G \times \mathcal{S}_\tau}$.*

(ii) *The diagram*

$$\begin{array}{ccc} G\times\mathcal{S}_\tau & \xrightarrow{\tilde{\varphi}\big|_{G\times\mathcal{S}_\tau}} & \overline{G\times\mathcal{S}_\tau} \\ & {\scriptstyle\rho_\tau}\searrow \quad \swarrow{\scriptstyle\overline{\rho}_\tau} & \\ & \mathfrak{g} & \end{array} \tag{3.13}$$

*commutes.*

*Proof.* By Corollary 3.7, the open dense symplectic leaf in $\overline{G\times\mathcal{S}_\tau}$ is obtained by intersecting $\overline{G\times\mathcal{S}_\tau}$ with the open dense symplectic leaf in $T^*\overline{G}(\log D)$. The latter leaf is $\tilde{\varphi}(T^*G)$, as is explained in Subsection 2.7. It is also straightforward to establish that

$$\tilde{\varphi}(G\times\mathcal{S}_\tau)=\overline{G\times\mathcal{S}_\tau}\cap\tilde{\varphi}(T^*G).$$

These last two sentences show $\tilde{\varphi}(G\times\mathcal{S}_\tau)$ to be the unique open dense symplectic leaf in $\overline{G\times\mathcal{S}_\tau}$. We also know that $\tilde{\varphi}$ restricts to a symplectomorphism from $G\times\mathcal{S}_\tau$ to $\tilde{\varphi}(G\times\mathcal{S}_\tau)$, where the symplectic form on $\tilde{\varphi}(G\times\mathcal{S}_\tau)$ is the pullback of the symplectic form on the leaf in $T^*\overline{G}(\log D)$ (see Corollary 3.8). It now follows from Corollary 3.7(ii) that

$$\tilde{\varphi}\big|_{G\times\mathcal{S}_\tau}:G\times\mathcal{S}_\tau\longrightarrow\tilde{\varphi}(G\times\mathcal{S}_\tau)$$

is a symplectomorphism with respect to this symplectic structure $\tilde{\varphi}(G\times\mathcal{S}_\tau)$ inherits as a leaf in $\overline{G\times\mathcal{S}_\tau}$. This symplectomorphism is $G$-equivariant, as $\tilde{\varphi}:T^*G\longrightarrow T^*\overline{G}(\log D)$ is $(G\times G)$-equivariant. The proof of (i) is therefore complete, while a straightforward calculation yields (ii). □

**Remark 3.17.** Let $\tau$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The description (3.11) allows one to define a closed embedding

$$\overline{G\times\mathcal{S}_\tau}\longrightarrow\overline{G}\times\mathfrak{g},\quad(\gamma,(x,x_\tau))\longrightarrow(\gamma,x).$$

We thereby obtain a commutative diagram

$$\begin{array}{ccc} \overline{G\times\mathcal{S}_\tau} & \longrightarrow & \overline{G}\times\mathfrak{g} \\ & {\scriptstyle\overline{\rho}_\tau}\searrow \quad \swarrow & \\ & \mathfrak{g} & \end{array},$$

where $\overline{G} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is projection to the second factor. One immediate consequence is that $\overline{\rho}_\tau$ has projective fibres, so that (3.13) realizes $\overline{\rho}_\tau$ as a fibrewise compactification of $\rho_\tau$. It also follows that

$$\overline{\rho}_\tau^{-1}(x) \longrightarrow \{\gamma \in \overline{G} : (x, x_\tau) \in \gamma\}, \quad (\gamma, (x, x_\tau)) \longrightarrow \gamma$$

is a variety isomorphism for each $x \in \mathfrak{g}$.

### 3.4. Relation to the universal centralizer and its fibrewise compactification

Let $G$ be the adjoint group of $\mathfrak{g}$, and let $\tau$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. It is instructive to examine the relationship between $G \times \mathcal{S}_\tau$ and $\overline{G \times \mathcal{S}_\tau}$ in the context of Balibănu's paper [2]. We begin by recalling that the *universal centralizer* of $\mathfrak{g}$ is the closed subvariety of $T^*G = G \times \mathfrak{g}$ defined by

$$\mathcal{Z}_{\mathfrak{g}}^\tau := \{(g, x) \in G \times \mathfrak{g} : x \in \mathcal{S}_\tau \text{ and } g \in G_x\},$$

where $G_x$ is the $G$-stabilizer of $x \in \mathfrak{g}$. At the same time, recall the Hamiltonian action of $G \times G$ on $T^*G$ and moment map $\rho : T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ discussed in Subsection 2.7. Consider the product $\mathcal{S}_\tau \times \mathcal{S}_\tau \subseteq \mathfrak{g} \oplus \mathfrak{g}$ and observe that

$$\mathcal{Z}_{\mathfrak{g}}^\tau = \rho^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau).$$

Note also that $\mathcal{S}_\tau \times \mathcal{S}_\tau$ is the Slodowy slice associated to the $\mathfrak{sl}_2$-triple $((\xi, \xi), (h, h), (\eta, \eta))$. It follows that $\mathcal{Z}_{\mathfrak{g}}^\tau$ is a Poisson slice in $T^*G$. Corollary 3.7(iii) then forces this Poisson slice to be a symplectic subvariety of $T^*G$.

**Remark 3.18.** Some papers realize the symplectic structure on $\mathcal{Z}_{\mathfrak{g}}^\tau$ via a Whittaker reduction of $T^*G$ (e.g. [2]). To this end, let $\mathfrak{u}$ be the nilpotent radical of the unique Borel subalgebra of $\mathfrak{g}$ containing $\eta$. Let us also write $U \subseteq G$ for the unipotent subgroup with Lie algebra $\mathfrak{u}$. Proposition 3.13 then gives a canonical isomorphism

$$\mathcal{Z}_{\mathfrak{g}}^\tau = \rho^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) \cong T^*G /\!\!/_{(\xi,\xi)} U \times U$$

of symplectic varieties, where the symplectic structure on $\mathcal{Z}_{\mathfrak{g}}^\tau$ is as defined in the previous paragraph.

One may replace $\rho : T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ with $\overline{\rho} : T^*\overline{G}(\log D) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ and proceeed analogously. In the interest of being more precise, consider the Poisson slice

$$\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} := \overline{\rho}^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) = \{(\gamma, (x, x)) : \gamma \in \overline{G},\ x \in \mathcal{S}_\tau,\ \text{and } (x, x) \in \gamma\}$$

in $T^*\overline{G}(\log D)$.

**Remark 3.19.** A counterpart of Remark 3.18 is that Proposition 3.13 gives a canonical isomorphism

$$\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} = \overline{\rho}^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) \cong T^*\overline{G}(\log D) /\!\!/_{(\xi,\xi)} U \times U$$

of Poisson varieties. This realization of $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}}$ via Whittaker reduction is used to great effect in [2].

Now recall the embedding $\tilde{\varphi} : T^*G \longrightarrow T^*\overline{G}(\log D)$ discussed in Subsection 2.7. Balibănu [2] shows $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}}$ to be log symplectic (cf. Corollary 3.7), and that $\tilde{\varphi}$ restricts to a symplectomorphism from $\mathcal{Z}_{\mathfrak{g}}^{\tau}$ to the unique open dense symplectic leaf in $\overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}}$. One also has a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathfrak{g}}^{\tau} & \xrightarrow{\ \tilde{\varphi}\big|_{\mathcal{Z}_{\mathfrak{g}}^{\tau}}\ } & \overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} \\ & {\scriptstyle q_\tau} \searrow \quad \swarrow {\scriptstyle \overline{q}_\tau} & \\ & \mathcal{S}_\tau & \end{array}, \tag{3.14}$$

where

$$q_\tau(g, x) = x \quad \text{and} \quad \overline{q}_\tau(\gamma, (x, x)) = x.$$

This diagram is seen to be the pullback of (3.13) along the inclusion $\mathcal{S}_\tau \hookrightarrow \mathfrak{g}$, and it thereby exhibits $\overline{q_\tau}$ as a fibrewise compactification of $q_\tau$ (see Remark 3.17 and cf. [2, Section 3]). This amounts to (3.14) being the restriction of (3.13) to a morphism between the Poisson slices

$$\mathcal{Z}_{\mathfrak{g}}^{\tau} = \rho^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) = \rho_\tau^{-1}(\mathcal{S}_\tau) \quad \text{and} \quad \overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}} = \overline{\rho}^{-1}(\mathcal{S}_\tau \times \mathcal{S}_\tau) = \overline{\rho}_\tau^{-1}(\mathcal{S}_\tau).$$

This present section combines with Subsection 3.3 to yield the following informal comparisons between $(\mathcal{Z}_{\mathfrak{g}}^{\tau}, \overline{\mathcal{Z}_{\mathfrak{g}}^{\tau}})$ and $(G \times \mathcal{S}_\tau, \overline{G \times \mathcal{S}_\tau})$:

- $\overline{q_\tau}$ (resp. $\overline{\rho}_\tau$) is a fibrewise compactification of $\pi_\tau$ (resp. $q_\tau$);

- (3.14) is obtained by pulling (3.13) back along the inclusion $\mathcal{S}_\tau \hookrightarrow \mathfrak{g}$;
- $\mathcal{Z}_{\mathfrak{g}}^\tau$ and $G \times \mathcal{S}_\tau$ are symplectic;
- $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ and $\overline{G \times \mathcal{S}_\tau}$ are log symplectic;
- $\tilde{\varphi}$ restricts to a symplectomorphism from $\mathcal{Z}_{\mathfrak{g}}^\tau$ (resp. $G \times \mathcal{S}_\tau$) to the unique open dense symplectic leaf in $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ (resp. $\overline{G \times \mathcal{S}_\tau}$).

## 4. The geometries of $\overline{X}$ and $\overline{X}_\tau$

This section is concerned with constructing partial compactifications of Poisson slices, an issue motivated in the introduction of our paper. Our approach is to replace a Poisson slice $X_\tau$ with a slightly larger variety $\overline{X}_\tau$, provided that the latter makes sense. If $\overline{X}_\tau$ is well-defined, we show it to enjoy certain Poisson-geometric features and discuss the extent to which it partially compactifies $X_\tau$.

Throughout Section 4, we require $G$ to be the adjoint group of $\mathfrak{g}$.

### 4.1. Definitions and first properties

Fix a Hamiltonian $G$-variety $(X, P, \nu)$ and an $\mathfrak{sl}_2$-triple $\tau$ in $\mathfrak{g}$. The product Hamiltonian $G$-varieties $X \times (G \times \mathcal{S}_\tau)$ and $X \times (\overline{G \times \mathcal{S}_\tau})$ then have respective moment maps

$$\mu_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow \mathfrak{g}, \quad (x, (g, y)) \longrightarrow \nu(x) - \mathrm{Ad}_g(y)$$

and

$$\overline{\mu}_\tau : X \times (\overline{G \times \mathcal{S}_\tau}) \longrightarrow \mathfrak{g}, \quad (x, (\gamma, (y_1, y_2))) \longrightarrow \nu(x) - y_1.$$

Note also that taking the product of

$$\tilde{\varphi}\big|_{G\times\mathcal{S}_\tau} : G \times \mathcal{S}_\tau \longrightarrow \overline{G \times \mathcal{S}_\tau}$$

with the identity $X \longrightarrow X$ produces a $G$-equivariant open Poisson embedding

$$\begin{aligned} i_\tau : X \times (G \times \mathcal{S}_\tau) &\longrightarrow X \times (\overline{G \times \mathcal{S}_\tau}) \\ (x, (g, y)) &\longrightarrow (x, ((g, e) \cdot \mathfrak{g}_\Delta, (\mathrm{Ad}_g(y), y))). \end{aligned} \tag{4.1}$$

(see Proposition 3.16). One readily verifies that the diagram

$$\begin{array}{ccc} X \times (G \times \mathcal{S}_\tau) & \xrightarrow{\;i_\tau\;} & X \times (\overline{G \times \mathcal{S}_\tau}) \\ & {}_{\mu_\tau}\searrow \quad \swarrow_{\overline{\mu}_\tau} & \\ & \mathfrak{g} & \end{array} \tag{4.2}$$

commutes.

Now recall the Hamiltonian $(G \times G)$-variety $X \times T^*\overline{G}(\log D)$ and moment map

$$\overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : X \times T^*\overline{G}(\log D) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$$

from Subsection 2.7. Let us write

$$\overline{X} := (X \times T^*\overline{G}(\log D)) /\!\!/ G_L \quad \text{and} \quad \overline{X}_\tau := (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G,$$

and understand "$\overline{X}$ exists" (resp. "$\overline{X}_\tau$ exists") to mean that

$$(X \times T^*\overline{G}(\log D)) /\!\!/ G_L \text{ (resp. } (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G)$$

exists as a geometric quotient.

**Remark 4.1.** If $\tau = 0$, then $X \times (\overline{G \times \mathcal{S}_\tau}) = X \times T^*\overline{G}(\log D)$, $\overline{\mu}_\tau = \overline{\mu}_L$, and the $G$-action on $X \times (\overline{G \times \mathcal{S}_\tau})$ is the $G_L$-action $X \times T^*\overline{G}(\log D)$. One immediate consequence is that $\overline{X} = \overline{X}_0$.

**Remark 4.2.** The action of $G_R$ on $X \times T^*\overline{G}(\log D)$ induces a residual $G$-action on $\overline{X}$, provided that $\overline{X}$ exists. This $G$-action features prominently in what follows.

Assume that $\overline{X}_\tau$ exists and recall the map

$$i : X \times T^*G \longrightarrow X \times T^*\overline{G}(\log D)$$

from (2.17). This map restricts to a $G$-equivariant open embedding

$$i\big|_{\mu_\tau^{-1}(0)} : \mu_\tau^{-1}(0) \hookrightarrow \overline{\mu}_\tau^{-1}(0), \tag{4.3}$$

which in turn descends to a morphism

$$j_\tau : (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \longrightarrow \overline{X}_\tau. \tag{4.4}$$

Let us consider the composition

$$k_\tau := j_\tau \circ \psi_\tau : X_\tau \longrightarrow \overline{X}_\tau, \tag{4.5}$$

where $\psi_\tau : X_\tau \longrightarrow (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G$ is the Poisson variety isomorphism from (3.3). It is straightforward to verify that

$$k_\tau(x) = [x : (\mathfrak{g}_\Delta, (\nu(x), \nu(x)))] \tag{4.6}$$

for all $x \in X_\tau$

**Proposition 4.3.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ exists, then $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ is an open embedding.*

*Proof.* Since $\psi_\tau$ is a variety isomorphism, it suffices to prove that $j_\tau$ is an open embedding. We achieve this by first considering the commutative square

$$\begin{array}{ccc} \mu_\tau^{-1}(0) & \xrightarrow{\ i\big|_{\mu_\tau^{-1}(0)}\ } & \overline{\mu}_\tau^{-1}(0) \\ \big\downarrow & & \big\downarrow \\ (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G & \xrightarrow{\ j_\tau\ } & \overline{X}_\tau \end{array} . \tag{4.7}$$

The vertical morphisms are open maps by virtue of being geometric quotients [52, Lemma 25.3.2], and we have explained that the upper horizontal map is open. It follows that $j_\tau$ is also an open map. Together with the observation that $j_\tau$ is injective, this implies that $j_\tau$ is an open embedding. Our proof is complete. ☐

The inclusion $X_\tau \longrightarrow X$ composes with the quotient map $X \longrightarrow X/G$ to yield

$$\pi_\tau : X_\tau \longrightarrow X/G, \tag{4.8}$$

provided that $X/G$ exists. We may also consider the morphism

$$\overline{\pi}_\tau : \overline{X}_\tau \longrightarrow X/G, \quad [x : (\gamma, (y_1, y_2))] \longrightarrow [x] \tag{4.9}$$

if both $\overline{X}_\tau$ and $X/G$ exist. The following is then an immediate consequence of (4.6).

**Proposition 4.4.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ and $X/G$ exist, then the diagram*

(4.10)
$$\begin{array}{ccc} X_\tau & \xrightarrow{\;k_\tau\;} & \overline{X}_\tau \\ & {\scriptstyle\pi_\tau}\searrow \quad \swarrow{\scriptstyle\overline{\pi}_\tau} & \\ & X/G & \end{array}$$

*commutes.*

This diagram is particularly noteworthy if $\tau$ is a principal $\mathfrak{sl}_2$-triple.

**Theorem 4.5.** *Let $\tau$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ and $X/G$ exist, then the diagram* (4.10) *realizes $\overline{\pi}_\tau$ as a fibrewise compactification of $\pi_\tau$.*

*Proof.* Our objective is to prove that $\overline{\pi}_\tau$ has projective fibres. Let us begin by fixing a point $x \in X$. We then have

(4.11) $$\overline{\pi}_\tau^{-1}([x]) = \{[x : (\gamma, (\nu(x), y))] : \gamma \in \overline{G},\ y \in \mathcal{S}_\tau,\ \text{and}\ (\nu(x), y) \in \gamma\}.$$

On the other hand, it is known that $y_1, y_2 \in \mathfrak{g}$ belong to the same fibre of the adjoint quotient $\chi : \mathfrak{g} \longrightarrow \operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ whenever $(y_1, y_2) \in \gamma$ for some $\gamma \in \overline{G}$ (see Subsection 3.3). The discussion and notation in Subsection 2.5 associated with principal $\mathfrak{sl}_2$-triples then imply the following: if $y_1 \in \mathfrak{g}$ and $y_2 \in \mathcal{S}_\tau$ are such that $(y_1, y_2) \in \gamma$ for some $\gamma \in \overline{G}$, then $y_2 = (y_1)_\tau$. We may therefore present (4.11) as the statement

$$\overline{\pi}_\tau^{-1}([x]) = \{[x : (\gamma, (\nu(x), \nu(x)_\tau))] : \gamma \in \overline{G} \text{ and } (\nu(x), \nu(x)_\tau) \in \gamma\}.$$

In other words, $\overline{\pi}_\tau^{-1}([x])$ is the image of the closed subvariety

$$\{\gamma \in \overline{G} : (\nu(x), \nu(x)_\tau) \in \gamma\} \subseteq \overline{G}$$

under the morphism

$$\overline{G} \longrightarrow \overline{X}_\tau, \quad \gamma \longrightarrow [x : (\gamma, (\nu(x), \nu(x)_\tau))].$$

This subvariety is projective by virtue of being closed in $\overline{G}$, and we conclude that $\overline{\pi}_\tau^{-1}([x])$ is projective. This completes the proof. $\square$

Let us also examine the case $\tau = 0$ in some detail. To this end, assume that $\overline{X}_0 = \overline{X}$ exists and consider the geometric quotient map

$$\overline{\pi}_L : \overline{\mu}_L^{-1}(0) \longrightarrow \overline{X}.$$

The $G_R$-action on $\overline{\mu}_L^{-1}(0)$ then descends under $\overline{\pi}_L$ to a $G$-action $\overline{X}$. On the other hand, note that the restriction of

$$-\overline{\mu}_R : X \times T^*\overline{G}(\log D) \longrightarrow \mathfrak{g}, \quad (x, (\gamma, (y_1, y_2))) \longrightarrow y_2$$

to $\overline{\mu}_L^{-1}(0)$ is $G_R$-equivariant and $G_L$-invariant. This restriction therefore descends under $\overline{\pi}_L$ to the $G$-equivariant morphism

$$\overline{\nu} : \overline{X} \longrightarrow \mathfrak{g}, \quad [x : (\gamma, (\nu(x), y))] \longrightarrow y. \tag{4.12}$$

Let us write $k : X \longrightarrow \overline{X}$, $\pi : X \longrightarrow X/G$, and $\overline{\pi} : \overline{X} \longrightarrow X/G$ for (4.5), (4.8), and (4.9), respectively, in the case $\tau = 0$.

**Proposition 4.6.** *If $\overline{X}$ exists, then $k : X \longrightarrow \overline{X}$ is a $G$-equivariant open embedding and*

$$\begin{array}{ccc} X & \xrightarrow{\quad k \quad} & \overline{X} \\ & {\scriptstyle \nu}\searrow \quad \swarrow{\scriptstyle \overline{\nu}} & \\ & \mathfrak{g} & \end{array} \tag{4.13}$$

*commutes. If $X/G$ also exists, then*

$$\begin{array}{ccc} X & \xrightarrow{\quad k \quad} & \overline{X} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \overline{\pi}} & \\ & X/G & \end{array} \tag{4.14}$$

*commutes.*

*Proof.* The commutativity of (4.13) follows immediately from (4.12) and (4.6), while Proposition 4.4 forces (4.14) to commute. Proposition 4.3 implies that $k$ is an open embedding. Our equivariance claim follows from (4.6), the above-given definition of the $G$-action on $\overline{X}$, and a direct calculation. This completes the proof. □

### 4.2. The Poisson geometries of $\overline{X}$ and $\overline{X}_\tau$

Let $(X, P, \nu)$ be a Hamiltonian $G$-variety and suppose that $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. In what follows, we show that the Poisson slice $X_\tau$ endows $\overline{X}_\tau$ with certain Poisson-geometric qualities. The most basic such feature is as follows.

**Proposition 4.7.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ exists, then $\mathbb{C}[\overline{X}_\tau]$ carries a natural Poisson bracket for which $k_\tau^* : \mathbb{C}[\overline{X}_\tau] \longrightarrow \mathbb{C}[X_\tau]$ is a Poisson algebra morphism.*

*Proof.* The definition

$$\overline{X}_\tau := (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G$$

combines with the discussion in Subsection 2.4 to yield a Poisson bracket on $\mathbb{C}[\overline{X}_\tau]$, as well as the following facts:

(i) $\mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})]^G$ is a Poisson subalgebra of $\mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})]$;

(ii) $\mathbb{C}[\overline{\mu}_\tau^{-1}(0)]^G$ has a unique Poisson bracket for which restriction

$$\beta : \mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})]^G \longrightarrow \mathbb{C}[\overline{\mu}_\tau^{-1}(0)]^G$$

is a Poisson algebra morphism;

(iii) the geometric quotient map $\overline{\mu}_\tau^{-1}(0) \longrightarrow \overline{X}_\tau$ induces a Poisson algebra isomorphism

$$\delta : \mathbb{C}[\overline{X}_\tau] \xrightarrow{\cong} \mathbb{C}[\overline{\mu}_\tau^{-1}(0)]^G.$$

We have a row of Poisson algebra morphisms

$$\mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})] \xleftarrow{\alpha} \mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})]^G \xrightarrow{\beta} \mathbb{C}[\overline{\mu}_\tau^{-1}(0)]^G \xleftarrow{\delta} \mathbb{C}[\overline{X}_\tau],$$

where $\alpha$ is the inclusion. An analogous procedure yields a second row

$$\mathbb{C}[X \times (G \times \mathcal{S}_\tau)] \xleftarrow{\alpha'} \mathbb{C}[X \times (G \times \mathcal{S}_\tau)]^G \xrightarrow{\beta'} \mathbb{C}[\mu_\tau^{-1}(0)]^G \xleftarrow{\delta'} \mathbb{C}[X_\tau]$$

of Poisson algebra morphisms. Now recall the $G$-equivariant open Poisson embedding

$$i_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow X \times (\overline{G \times \mathcal{S}_\tau})$$

from (4.1), as well as the commutative diagram (4.2). It follows that $i_\tau$ induces the first three vertical arrows in the commutative diagram

$$\begin{array}{ccccccc} \mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})] & \xleftarrow{\alpha} & \mathbb{C}[X \times (\overline{G \times \mathcal{S}_\tau})]^G & \xrightarrow{\beta} & \mathbb{C}[\overline{\mu}_\tau^{-1}(0)]^G & \xleftarrow{\delta} & \mathbb{C}[\overline{X}_\tau] \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \beta''} & & \downarrow{\scriptstyle \delta''} & & \downarrow{\scriptstyle k_\tau^*} \\ \mathbb{C}[X \times (G \times \mathcal{S}_\tau)] & \xleftarrow{\alpha'} & \mathbb{C}[X \times (G \times \mathcal{S}_\tau)]^G & \xrightarrow{\beta'} & \mathbb{C}[\mu_\tau^{-1}(0)]^G & \xleftarrow{\delta'} & \mathbb{C}[X_\tau] \end{array} .$$

Observe that $\alpha''$ is a Poisson algebra morphism, as follows from $i_\tau$ being a Poisson morphism. One deduces that $\beta''$ must also be a Poisson algebra morphism. This combines with the commutativity of the middle square and the fact that $\beta'$ and $\beta''$ are surjective Poisson algebra morphisms to imply that $\delta''$ is a Poisson algebra morphism. Since $\delta$ and $\delta'$ are Poisson algebra isomorphisms, this forces $k_\tau^*$ to be a Poisson algebra morphism. $\square$

Some more manifestly geometric features of $\overline{X}_\tau$ may be developed as follows. Write $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$ for the $G$-invariant open subvariety of points in $X \times (\overline{G \times \mathcal{S}_\tau})$ whose $G$-stabilizers are trivial.

The $G$-action on $(X \times \overline{G \times \mathcal{S}_\tau})^\circ$ is Hamiltonian with respect to the Poisson structure that $(X \times \overline{G \times \mathcal{S}_\tau})^\circ$ inherits from $(X \times (\overline{G \times \mathcal{S}_\tau}))$, and

$$\overline{\mu}_\tau^\circ := \overline{\mu}_\tau\Big|_{(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ} : (X \times (\overline{G \times \mathcal{S}_\tau}))^\circ \longrightarrow \mathfrak{g} \tag{4.15}$$

is a moment map.

Now assume that $\overline{X}_\tau$ exists and consider the geometric quotient map

$$\overline{\theta}_\tau : \overline{\mu}_\tau^{-1}(0) \longrightarrow \overline{X}_\tau.$$

The variety $(\overline{\mu}_\tau^\circ)^{-1}(0)$ is $G$-invariant and open in $\overline{\mu}_\tau^{-1}(0)$, and we set

$$X_\tau^\circ := \overline{\theta}_\tau((\overline{\mu}_\tau^\circ)^{-1}(0)) \subseteq \overline{X}_\tau.$$

We also let

$$\overline{\theta}_\tau^\circ : (\overline{\mu}_\tau^\circ)^{-1}(0) \longrightarrow \overline{X}_\tau^\circ$$

denote the restriction of $\overline{\theta}_\tau$ to $(\overline{\mu}_\tau^\circ)^{-1}(0)$.

**Lemma 4.8.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ exists, then $\overline{X}_\tau^\circ$ is an open subvariety of $\overline{X}_\tau$ and $\overline{\theta}_\tau^\circ : (\overline{\mu}_\tau^\circ)^{-1}(0) \longrightarrow \overline{X}_\tau^\circ$ is the geometric quotient of $(\overline{\mu}_\tau^\circ)^{-1}(0)$ by $G$.*

*Proof.* The geometric quotient map $\overline{\theta}_\tau : \overline{\mu}_\tau^{-1}(0) \longrightarrow \overline{X}_\tau$ is open [52, Lemma 25.3.2]. It follows that $\overline{X}_\tau^\circ = \overline{\theta}_\tau((\overline{\mu}_\tau^\circ)^{-1}(0))$ is an open subvariety of $\overline{X}_\tau$. At the same time, $\overline{\theta}_\tau^\circ$ is obtained by restricting the geometric quotient map $\overline{\theta}_\tau$ to the open, $G$-invariant subvariety $(\overline{\mu}_\tau^\circ)^{-1}(0) \subseteq \overline{\mu}_\tau^{-1}(0)$. This implies that $\overline{\theta}_\tau^\circ$ is itself a geometric quotient map. □

**Proposition 4.9.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. If $\overline{X}_\tau$ exists, then $\overline{X}_\tau^\circ$ is smooth and Poisson.*

*Proof.* Recall that $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$ is a Hamiltonian $G$-variety with moment map (4.15). Lemma 4.8 then implies that $\overline{X}_\tau^\circ$ is the Hamiltonian reduction of $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$ at level zero. The proposition now follows from generalities about Hamiltonian reductions by free actions, the relevant parts of which are discussed in Subsection 2.4. □

Now recall the open embedding $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ defined in (4.5).

**Proposition 4.10.** *Let $\tau$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and assume that $\overline{X}_\tau$ exists. The image of $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ then lies in $\overline{X}_\tau^\circ$, and $k_\tau$ defines an open embedding of Poisson varieties $X_\tau \longrightarrow \overline{X}_\tau^\circ$.*

*Proof.* Recall that $k_\tau = j_\tau \circ \psi_\tau$, and that $\psi_\tau$ is a Poisson variety isomorphism. It therefore suffices to prove the following:

(i) the image of $j_\tau : (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \longrightarrow \overline{X}_\tau$ lies in $\overline{X}_\tau^\circ$;

(ii) $j_\tau$ defines an open embedding of Poisson varieties $(X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \longrightarrow \overline{X}_\tau^\circ$.

Since $G$ acts freely on $X \times (G \times \mathcal{S}_\tau)$, the image of (4.1) lies in $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$. We may therefore interpret (4.1) as a $G$-equivariant open Poisson embedding

$$i_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow (X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$$

and (4.2) as a commutative diagram

$$\begin{array}{ccc} X \times (G \times \mathcal{S}_\tau) & \xrightarrow{\quad i_\tau \quad} & (X \times (\overline{G \times \mathcal{S}_\tau}))^\circ \\ & \searrow^{\mu_\tau} \quad \mathfrak{g} \quad {}^{\overline{\mu}_\tau^\circ}\swarrow & \end{array}.$$

Such considerations allow one to regard (4.3) and (4.4) as maps

$$i_\tau\big|_{\mu_\tau^{-1}(0)} : \mu_\tau^{-1}(0) \hookrightarrow (\overline{\mu}_\tau^\circ)^{-1}(0) \tag{4.16}$$

and

$$j_\tau : (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \longrightarrow \overline{X}_\tau^\circ, \tag{4.17}$$

respectively. This verifies (i) and yields the commutative square

$$\begin{array}{ccc} \mu_\tau^{-1}(0) & \xrightarrow{\; i_\tau|_{\mu_\tau^{-1}(0)} \;} & (\overline{\mu}_\tau^\circ)^{-1}(0) \\ \big\downarrow & & \big\downarrow \\ (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G & \xrightarrow{\; j_\tau \;} & \overline{X}_\tau^\circ \end{array} . \tag{4.18}$$

By combining this square with the description of the Poisson structure on a Hamiltonian reduction, we deduce that (4.17) is a Poisson morphism. This morphism is also an open embedding, as follows easily from Proposition 4.3. Our proof is therefore complete. □

Let us write $\overline{X}^\circ$ for $\overline{X}_\tau^\circ$ if $\tau = 0$. This variety turns out to enjoy some Poisson geometric features beyond those of a general $\overline{X}_\tau^\circ$. To develop these features, assume that $\overline{X}$ exists and let

$$\overline{\pi}_L : \overline{\mu}_L^{-1}(0) \longrightarrow \overline{X}$$

be the geometric quotient map. Write $(X \times T^*\overline{G}(\log D))^\circ$ for the $(G \times G)$-invariant open subvariety of points in $X \times T^*\overline{G}(\log D)$ whose $G_L$-stabilizers are trivial. The $(G \times G)$-action on $(X \times T^*\overline{G}(\log D))^\circ$ is Hamiltonian with respect to the Poisson structure that $(X \times T^*\overline{G}(\log D))^\circ$ inherits from $X \times T^*\overline{G}(\log D)$, and

$$(\overline{\mu}_L^\circ, \overline{\mu}_R^\circ) := \left(\overline{\mu}_L\Big|_{(X \times T^*\overline{G}(\log D))^\circ}, \overline{\mu}_R\Big|_{(X \times T^*\overline{G}(\log D))^\circ}\right) : (X \times T^*\overline{G}(\log D))^\circ \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \tag{4.19}$$

is a moment map.

Now consider the $(G \times G)$-invariant open subvariety of $(\overline{\mu}_L^\circ)^{-1}(0)$ of $\overline{\mu}_L^{-1}(0)$, and observe that

$$\overline{X}^\circ := \overline{\pi}_L((\overline{\mu}_L^\circ)^{-1}(0)).$$

Let

$$\overline{\pi}_L^\circ : (\overline{\mu}_L^\circ)^{-1}(0) \longrightarrow \overline{X}^\circ$$

denote the restriction of $\overline{\pi}_L$ to $(\overline{\mu}_L^\circ)^{-1}(0)$. At the same time, recall the definition of the $G$-action on $\overline{X}$.

**Lemma 4.11.** *Assume that $\overline{X}$ exists. The subset $\overline{X}^\circ$ is then a $G$-invariant open subvariety of $\overline{X}$, and $\overline{\pi}_L^\circ : (\overline{\mu}_L^\circ)^{-1}(0) \longrightarrow \overline{X}^\circ$ is the geometric quotient of $(\overline{\mu}_L^\circ)^{-1}(0)$ by $G_L$.*

*Proof.* Observe that $\overline{\pi}_L$ is equivariant with respect to the action of $G_R$ on $\overline{\mu}_L^{-1}(0)$ and the above-discussed $G$-action on $\overline{X}$. Since $(\overline{\mu}_L^\circ)^{-1}(0)$ is $G_R$-invariant in $\overline{\mu}_L^{-1}(0)$, this implies that $\overline{X}^\circ = \overline{\pi}_L((\overline{\mu}_L^\circ)^{-1}(0))$ is $G$-invariant in $\overline{X}$. The rest of this lemma is an immediate consequence of Lemma 4.8. □

The $G$-action that $\overline{X}^\circ$ inherits from $\overline{X}$ is compatible with the Poisson variety structure referenced in Proposition 4.9. To formulate this more precisely, recall the map $\nu : \overline{X} \longrightarrow \mathfrak{g}$ in (4.12) and set

$$\overline{\nu}^\circ := \overline{\nu}\big|_{\overline{X}^\circ} : \overline{X}^\circ \longrightarrow \mathfrak{g}.$$

**Proposition 4.12.** *If $\overline{X}$ exists, then the action of $G$ on $\overline{X}^\circ$ is Hamiltonian with moment map $\overline{\nu}^\circ : \overline{X}^\circ \longrightarrow \mathfrak{g}$.*

*Proof.* Recall that $(X \times T^*\overline{G}(\log D))^\circ$ is a Hamiltonian $(G \times G)$-variety with moment map (4.19). One deduces that $\overline{X}^\circ = (\overline{\mu}_L^\circ)^{-1}(0)/G_L$ is a Hamiltonian $G$-variety, and that the corresponding moment map is obtained by letting

$$-\overline{\mu}_R^\circ\Big|_{(\overline{\mu}_L^\circ)^{-1}(0)} : (\overline{\mu}_L^\circ)^{-1}(0) \longrightarrow \mathfrak{g}$$

descend to $\overline{X}^\circ$. It remains only to observe that this descended moment map and the $G$-action on $\overline{X}^\circ$ are restrictions of $\overline{\nu} : \overline{X} \longrightarrow \mathfrak{g}$ and the $G$-action on $\overline{X}$, respectively. □

**Proposition 4.13.** *Assume that $\overline{X}$ exists. The image of $k : X \longrightarrow \overline{X}$ then lies in $\overline{X}^\circ$, and $k$ defines an open embedding of Hamiltonian $G$-varieties $X \longrightarrow \overline{X}^\circ$.*

*Proof.* This is a direct consequence of Propositions 4.6 and Proposition 4.10. □

## 4.3. The log symplectic geometries of $\overline{X}$ and $\overline{X}_\tau$

We now examine the Poisson geometries of $\overline{X}$ and $\overline{X}_\tau$ in the special case of a symplectic Hamiltonian $G$-variety $(X, P, \nu)$. These Poisson geometries essentially become log symplectic geometries, as is consistent with the following result. Recall the map $i_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow X \times (\overline{G \times \mathcal{S}_\tau})$ defined in (4.1).

**Lemma 4.14.** *Let $(X, P, \nu)$ be an irreducible symplectic Hamiltonian $G$-variety. If $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then the following statements then hold:*

(i) *$X \times (\overline{G \times \mathcal{S}_\tau})$ is log symplectic;*

(ii) *$i_\tau$ is a $G$-equivariant symplectomorphism onto the unique open dense symplectic leaf in $X \times (\overline{G \times \mathcal{S}_\tau})$.*

*Proof.* Proposition 3.16 tells us that

$$\tilde{\varphi}\big|_{G\times\mathcal{S}_\tau} : G \times \mathcal{S}_\tau \longrightarrow \overline{G \times \mathcal{S}_\tau}$$

is a $G$-equivariant symplectomorphism onto the open dense symplectic leaf in the log symplectic variety $\overline{G \times \mathcal{S}_\tau}$. We also recall that $i_\tau$ is the product of $\tilde{\varphi}\big|_{G\times\mathcal{S}_\tau}$ with the identity $X \longrightarrow X$. These last two sentences imply that $i_\tau$ is a $G$-equivariant symplectomorphism onto the complement of the degeneracy locus in $X \times (\overline{G \times \mathcal{S}_\tau})$. Since $X \times (G \times \mathcal{S}_\tau)$ and $X \times (\overline{G \times \mathcal{S}_\tau})$ are irreducible, this implies that the image of $i_\tau$ is the unique open dense symplectic leaf in $X \times (\overline{G \times \mathcal{S}_\tau})$.

Now consider the closed subvariety

$$D_\tau := (X \times (\overline{G \times \mathcal{S}_\tau})) \setminus i_\tau(X \times (G \times \mathcal{S}_\tau)) \tag{4.20}$$

of $X \times (\overline{G \times \mathcal{S}_\tau})$ It remains only to prove the following things:

(a) $D_\tau$ is a normal crossing divisor;

(b) the top exterior power of the Poisson bivector on $X \times (\overline{G \times \mathcal{S}_\tau})$ has a reduced vanishing locus;

(c) the vanishing locus in (b) coincides with $D_\tau$.

To this end, we note that

$$D_\tau = X \times (\overline{G \times \mathcal{S}_\tau} \setminus \tilde{\varphi}(G \times \mathcal{S}_\tau)).$$

We also observe that $\overline{G \times \mathcal{S}_\tau} \setminus \tilde{\varphi}(G \times \mathcal{S}_\tau)$ is a normal crossing divisor in $\overline{G \times \mathcal{S}_\tau}$, as $\tilde{\varphi}(G \times \mathcal{S}_\tau)$ is the unique open dense symplectic leaf in the log symplectic variety $\overline{G \times \mathcal{S}_\tau}$ (see Proposition 3.16). The previous two sentences then force $D_\tau$ to be a normal crossing divisor in $\overline{G \times \mathcal{S}_\tau}$, i.e. (a) holds. The assertion (b) follows immediately from $X$ being symplectic and $\overline{G \times \mathcal{S}_\tau}$ being log symplectic. The assertion (c) follows from our description of the degeneracy locus in $X \times (\overline{G \times \mathcal{S}_\tau})$, as provided in the first paragraph of the proof. Our proof is therefore complete. □

Now recall the open embedding $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ in (4.5), as well as the fact that $k_\tau(X_\tau) \subseteq \overline{X}_\tau^\circ$ (see Proposition 4.10). If $X_\tau$ is irreducible, then $k_\tau(X_\tau)$ lies in a unique irreducible component $(\overline{X}_\tau^\circ)_{\mathrm{irr}}$ of the Poisson variety $\overline{X}_\tau^\circ$. The log symplectic nature of $\overline{X}_\tau$ is then captured by the following result, which relies heavily on the notation of Subsection 4.2.

**Theorem 4.15.** *Let $(X, P, \nu)$ be an irreducible symplectic Hamiltonian $G$-variety. Suppose that $\tau$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and that $X_\tau$ is irreducible. If $\overline{X}_\tau$ exists, then the following statements hold.*

(i) *The Poisson variety $(\overline{X}_\tau^\circ)_{\mathrm{irr}}$ is log symplectic.*

(ii) *The morphism $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ is a symplectomorphism onto the unique open dense symplectic leaf in $(\overline{X}_\tau^\circ)_{\mathrm{irr}}$.*

*Proof.* Since $G$ acts freely on $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$, the variety $(\overline{\mu}_\tau^\circ)^{-1}(0)$ is smooth. The irreducible components of $(\overline{\mu}_\tau^\circ)^{-1}(0)$ are therefore pairwise disjoint, while the connectedness of $G$ forces these components to be $G$-invariant. It follows that the irreducible components of $\overline{X}_\tau^\circ$ are precisely the images of the irreducible components of $(\overline{\mu}_\tau^\circ)^{-1}(0)$ under the quotient map

$$\overline{\theta}_\tau^\circ : (\overline{\mu}_L^\circ)^{-1}(0) \longrightarrow \overline{X}_\tau^\circ.$$

This implies that $(\overline{X}_\tau^\circ)_{\mathrm{irr}} = \overline{\theta}_\tau^\circ(Y)$ for some unique irreducible component $Y \subseteq (\overline{\mu}_\tau^\circ)^{-1}(0)$.

Now note that the image of (4.3) lies in a unique irreducible component $Z$ of the smooth variety $(\overline{\mu}_L^\circ)^{-1}(0)$, as $X_\tau$ and $\mu_\tau^{-1}(0)$ are irreducible. We also note that $\overline{\theta}_\tau^\circ(Z)$ contains the image of $k_\tau$, as follows from the commutativity of (4.7). We conclude that $\overline{\theta}_\tau^\circ(Z) = (\overline{X}_\tau^\circ)_{\mathrm{irr}}$, and the previous paragraph then implies that $Z = Y$.

In light of the above, (4.3) may be interpreted as an open embedding

$$i\big|_{\mu_\tau^{-1}(0)} : \mu_\tau^{-1}(0) \longrightarrow Y. \tag{4.21}$$

The irreducibility of $Y$ forces the complement of the image to have positive codimension in $Y$. This complement is easily checked to be $Y \cap D_\tau$, where $D_\tau \subseteq X \times (\overline{G \times \mathcal{S}_\tau})$ is defined in (4.20). We also observe that $Y \cap D_\tau$ has codimension at most one in $Y$, as $D_\tau$ is a divisor in $X \times (\overline{G \times \mathcal{S}_\tau})$. These last three sentences imply that $Y \cap D_\tau$ is a divisor in $Y$. By [2, Proposition 3.6], the Poisson structure on $\overline{\theta}^\circ_\tau(Y) = (\overline{X}^\circ_\tau)_{\text{irr}}$ is log symplectic with divisor $\overline{\theta}^\circ_\tau(Y \cap D_\tau)$. This completes the proof of (i).

Now consider the commutative diagram

$$\begin{array}{ccc} \mu_\tau^{-1}(0) & \xrightarrow{\;i|_{\mu_\tau^{-1}(0)}\;} & Y \\ \big\downarrow & & \big\downarrow \\ X_\tau & \xrightarrow{\;k_\tau\;} & (\overline{X}^\circ_\tau)_{\text{irr}} \end{array},$$

where the right vertical map is the restriction of $\overline{\theta}^\circ_\tau$. Since $Y \cap D_\tau$ is the complement of the image of (4.21), we deduce that the image of $k_\tau$ has a complement of $\overline{\theta}^\circ_\tau(Y \cap D_\tau)$. This amounts to the image of $k_\tau$ being the unique open dense symplectic leaf in $\overline{X}^\circ_{\text{irr}}$. Proposition 4.10 then implies that $k_\tau$ is a symplectomorphism onto this leaf. This establishes (ii), completing the proof. ☐

It is worth examining this result in the case $\tau = 0$. To this end, recall the open embedding $k : X \longrightarrow \overline{X}^\circ$ from Subsection 4.1 and the fact that $k(X) \subseteq \overline{X}^\circ$ (see Proposition 4.13). If $X$ is irreducible, then $k(X)$ lies in a unique irreducible component $\overline{X}^\circ_{\text{irr}}$ of $\overline{X}^\circ$. On the other hand, recall the $G$-actions on $\overline{X}$ and $\overline{X}^\circ$ discussed in Subsection 4.2. Let us also recall the map $\overline{\nu} : \overline{X} \longrightarrow \mathfrak{g}$ from (4.12).

**Corollary 4.16.** *Let $(X, P, \nu)$ be an irreducible symplectic Hamiltonian $G$-variety. If $\overline{X}$ exists, then the following statements hold.*

(i) *The Poisson variety $\overline{X}^\circ_{\text{irr}}$ is log symplectic.*

(ii) *The $G$-action on $\overline{X}$ restricts to a Hamiltonian $G$-action on $\overline{X}^\circ_{\text{irr}}$ with moment map*

$$\overline{\nu}\Big|_{\overline{X}^\circ_{\text{irr}}} : \overline{X}^\circ_{\text{irr}} \longrightarrow \mathfrak{g}.$$

(iii) *The morphism* $k : X \longrightarrow \overline{X}$ *is a* $G$*-equivariant symplectomorphism onto the unique open dense symplectic leaf in* $\overline{X}^{\circ}_{\mathrm{irr}}$.

(iv) *The symplectomorphism in* (iii) *is a embedding of Hamiltonian* $G$*-varieties.*

*Proof.* Note that $\mu_\tau = \mu_L$ if $\tau = 0$, where $\mu_L : X \times T^*G \longrightarrow \mathfrak{g}$ is the moment map for the Hamiltonian action of $G_L = G \times \{e\} \subseteq G \times G$ on $T^*G$. We also observe that the map

$$X \times G \longrightarrow \mu_L^{-1}(0), \quad (x, g) \longrightarrow (x, (g, \mathrm{Ad}_{g^{-1}}(\nu(x)))), \quad (x, g) \in X \times G$$

is a variety isomorphism. It follows that $\mu_\tau^{-1}(0)$ is irreducible if $\tau = 0$. Theorem 4.15 now implies that $\overline{X}^{\circ}_{\mathrm{irr}}$ is log symplectic, and that $k : X \longrightarrow \overline{X}$ is a symplectomorphism onto the unique open dense symplectic leaf in $\overline{X}^{\circ}_{\mathrm{irr}}$. One also knows that $k$ defines an embedding of Hamiltonian $G$-varieties $X \longrightarrow \overline{X}^{\circ}$ (see Proposition 4.13), and that the $G$-action on $\overline{X}^{\circ}$ must preserve the component $\overline{X}^{\circ}_{\mathrm{irr}}$. These last two sentences serve to verify (i)–(iv). □

### 4.4. The main examples

We now discuss some of the examples that motivate and best exhibit the results in this paper.

**Example 4.17.** Suppose that $G$ is endowed with the $G$-action defined by

$$g \cdot h := hg^{-1}, \quad g, h \in G.$$

The induced Hamiltonian $G$-action on $X = T^*G$ then satisfies

$$X_\tau \cong G \times \mathcal{S}_\tau \quad \text{and} \quad \overline{X}_\tau = (T^*G \times (\overline{G \times \mathcal{S}_\tau})) \mathbin{/\!\!/} G \cong \overline{G \times \mathcal{S}_\tau}$$

for any $\mathfrak{sl}_2$-triple $\tau$ in $\mathfrak{g}$. The fibrewise compactification in Theorem 4.5 becomes the one mentioned in Remark 3.17.

**Example 4.18.** Let $\tau$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and recall the notation used in Subsection 3.3. Consider the Hamiltonian $G$-varieties $X = G \times \mathcal{S}_\tau$

and $\overline{G \times \mathcal{S}_\tau}$, as well as the moment maps

$$\rho_\tau : G \times \mathcal{S}_\tau \longrightarrow \mathfrak{g} \quad \text{and} \quad \overline{\rho}_\tau : \overline{G \times \mathcal{S}_\tau} \longrightarrow \mathfrak{g}.$$

The discussion of $\mathcal{Z}_{\mathfrak{g}}^\tau$ and $\overline{\mathcal{Z}_{\mathfrak{g}}^\tau}$ in Subsection 3.4 combines with Proposition 3.9 to imply that

$$\begin{aligned} X_\tau &= \rho_\tau^{-1}(\mathcal{S}_\tau) = \mathcal{Z}_{\mathfrak{g}}^\tau \quad \text{and} \\ \overline{X}_\tau &= ((G \times \mathcal{S}_\tau) \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G \cong \overline{\rho}_\tau^{-1}(\mathcal{S}_\tau) = \overline{\mathcal{Z}_{\mathfrak{g}}^\tau}. \end{aligned}$$

The fibrewise compactification in Theorem 4.5 becomes Bălibanu's fibrewise compactification (3.14).

## Notation

- $\mathcal{O}_Y$ — structure sheaf of an algebraic variety $Y$
- $\mathbb{C}[Y]$ — coordinate ring of an algebraic variety $Y$
- $G$ — complex semisimple linear algebraic group
- $G_L$ — the subgroup $G \times \{e\} \subseteq G \times G$
- $G_R$ — the subgroup $\{e\} \times G \subseteq G \times G$
- $\mathfrak{g}$ — Lie algebra of $G$
- $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ — adjoint representation
- $\mathfrak{g}_\Delta$ — diagonal in $\mathfrak{g} \oplus \mathfrak{g}$
- $n$ — dimension of $\mathfrak{g}$
- $\langle \cdot, \cdot \rangle$ — Killing form on $\mathfrak{g}$
- $\tau$ — $\mathfrak{sl}_2$-triple in $\mathfrak{g}$
- $\mathcal{S}_\tau$ — Slodowy slice associated to $\tau$.
- $\chi : \mathfrak{g} \longrightarrow \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ — adjoint quotient
- $y_\tau$ — unique point at which $\mathcal{S}_\tau$ meets $\chi^{-1}(\chi(y))$, if $\tau$ is a principal $\mathfrak{sl}_2$-triple
- $\rho = (\rho_L, \rho_R) : T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ — moment map for the $(G \times G)$-action on $T^*G$
- $\rho_\tau : G \times \mathcal{S}_\tau \longrightarrow \mathfrak{g}$ — moment map for the $G$-action on $G \times \mathcal{S}_\tau$

- $X$ — Hamiltonian $G$-variety
- $\nu : X \longrightarrow \mathfrak{g}$ — moment map for the $G$-action on $X$
- $X_\tau$ — the Poisson slice $\nu^{-1}(\mathcal{S}_\tau)$
- $X/G$ — geometric quotient of $X$ by $G$
- $\mu = (\mu_L, \mu_R) : X \times T^*G \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ — moment map for the $(G \times G)$-action on $X \times T^*G$
- $\mu_\tau : X \times (G \times \mathcal{S}_\tau) \longrightarrow \mathfrak{g}$ moment map for the $G$-action on $X \times (G \times \mathcal{S}_\tau)$
- $\psi_\tau : X_\tau \longrightarrow (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G$ — canonical Poisson variety isomorphism
- $\overline{G}$ — De Concini–Procesi wonderful compactification of $G$
- $D$ — the divisor $\overline{G} \setminus G$
- $T^*\overline{G}(\log(D))$ — log cotangent bundle of $(\overline{G}, D)$
- $\overline{\rho} = (\overline{\rho}_L, \overline{\rho}_R) : T^*\overline{G}(\log(D)) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ — moment map for the $(G \times G)$-action on $T^*\overline{G}(\log(D))$
- $\overline{G \times \mathcal{S}_\tau}$ — the Poisson slice $\overline{\rho}_R^{-1}(\mathcal{S}_\tau)$
- $\mathcal{Z}_\mathfrak{g}^\tau$ — universal centralizer of $\mathfrak{g}$
- $\overline{\mathcal{Z}_\mathfrak{g}^\tau}$ — Bălibanu's partial compactification of $\mathcal{Z}_\mathfrak{g}^\tau$
- $\overline{\rho}_\tau : \overline{G \times \mathcal{S}_\tau} \longrightarrow \mathfrak{g}$ — moment map for the $G$-action on $\overline{G \times \mathcal{S}_\tau}$
- $\overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : X \times T^*\overline{G}(\log D) \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$ — moment map for the $(G \times G)$-action on $X \times T^*\overline{G}(\log D)$
- $\overline{\mu}_\tau : X \times (\overline{G \times \mathcal{S}_\tau}) \longrightarrow \mathfrak{g}$ — moment map for the $G$-action on $X \times (\overline{G \times \mathcal{S}_\tau})$
- $\overline{X}$ — the Hamiltonian reduction $(X \times T^*\overline{G}(\log D)) /\!\!/ G_L$
- $k : X \longrightarrow \overline{X}$ — canonical $G$-equivariant open embedding
- $\overline{\nu} : \overline{X} \longrightarrow \mathfrak{g}$ — equivariant extension of $\nu$ to $\overline{X}$
- $\overline{X}_\tau$ — the Hamiltonian reduction $(X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G$
- $k_\tau : X_\tau \longrightarrow \overline{X}_\tau$ — canonical open embedding
- $(X \times (\overline{G \times \mathcal{S}_\tau}))^\circ$ — set of points in $X \times (\overline{G \times \mathcal{S}_\tau})$ with trivial $G$-stabilizers

- $\overline{X}^{\circ}_{\tau}$ — the Hamiltonian reduction $(X \times (\overline{G \times \mathcal{S}_{\tau}}))^{\circ} /\!\!/ G$

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