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Monopoles and foliations without holonomy-invariant transverse measure

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This article proves a uniform exponential decay estimate for Seiberg-Witten equations on non-compact 4-manifolds with exact symplectic ends of bounded geometry. This is an extension of the analysis for asymptotically flat almost Kähler (AFAK) structures by Kronheimer and Mrowka [17]. As an application, we construct an invariant for smooth foliations without holonomy-invariant transverse measure, which takes value in the boundary-stable version of the monopole Floer homology group introduced by Kronheimer and Mrowka [18], without invoking the Eliashberg-Thurston perturbation.

1. Introduction

1.1. Seiberg-Witten equations

Let (M, g) be an oriented Riemannian 4-manifold possibly with boundary. The manifold M is allowed to be empty. We say that M is *cylindrical*, if M is isometric to $(-\infty, 0] \times Y$ for a closed, oriented Riemannian 3-manifold Y. We say that M is endowed with an *exact symplectic structure with bounded geometry* (or an *ESBG structure*), if there is an exact symplectic form $\omega = d\theta$ on M, such that the following conditions hold:

1) ω is compatible with g. In other words, ω is a self-dual 2-form with norm $\sqrt{2}$ under the metric g;
2) ∂M is compact, (M, g) is complete as a metric space;
3) The injectivity radius of M has a positive lower bound on $M - N(\partial M)$, where $N(\partial M)$ is a tubular neighborhood of ∂M with compact closure;
4) Let R be the Riemann curvature tensor of M, let ∇ be the Levi-Civita connection, then $\sup_M |\nabla^k R| < +\infty$ for every $k \geq 0$;
5) $\sup_M |\nabla^k \theta| < +\infty$ for every $k \geq 0$.

If M is empty, then by definition, M is both cylindrical and has an (empty) ESBG structure.

Suppose (M, g) is endowed with an *ESBG* structure by the symplectic form $\omega = d\theta$, then (ω, g) induces an almost complex structure on M, and there is a *canonical spin^c structure* over M such that

$$\begin{aligned}\mathbb{S}^+ &= T^{0,0}\, M \oplus T^{0,2}\, M,\\ \mathbb{S}^- &= T^{0,1}\, M.\end{aligned}$$

We will denote the canonical spin^c structure by $\mathfrak{s}_{M,\omega}$. Let Φ_0 be the section of $\mathbb{S}^+$ given by $1 \in \Gamma(M, T^{0,0}\, M)$, let D be the Dirac operator, then there exists a unique spin^c connection A_0 on $\mathfrak{s}_{M,\omega}$ such that $D_{A_0}\Phi_0 = 0$. We call A_0 the *canonical spin^c connection* on $\mathfrak{s}_{M,\omega}$. For more details on the canonical spin^c structure and the canonical spin^c connection, see, for example, [14, Sections 4.2-4.3].

Definition 1.1. *Suppose X is a complete oriented Riemannian 4-manifold. We say that X* has cylindrical and ESBG ends*, if there are two 4-dimensional* *submanifolds with boundary $M_c, M_s \subset X$ and an exact symplectic form $\omega = d\theta$ on M_s, such that the following holds.*

1) *M_s and M_c are disjoint closed subsets of X, and the closure of $X - M_c - M_s$ is compact. M_c and M_s are allowed to be empty.*
2) *M_c is cylindrical, and $\omega = d\theta$ is an ESBG structure on M_s.*
3) *If M_s is nonempty, then the ESBG structure on M_s can be extended to a neighborhood of M_s.*
4) *If M_c is nonempty, then the cylindrical structure on M_c can be extended to a neighborhood of M_c. Namely, there exists a neighborhood $N(M_c)$ of M_c and a closed oriented Riemannian 3-manifold Y, such that $N(M_c)$ is isometric to $(-\infty, \epsilon] \times Y$ for $\epsilon > 0$, where $M_c \subset N(M_c)$ is mapped to $(-\infty, 0] \times Y$.*

Definition 1.2. *Let X be a Riemannian 4-manifold with cylindrical and ESBG ends, where M_c is the cylindrical end, and $(M_s, \omega = d\theta)$ is the ESBG end. We say that a spin^c structure $\mathfrak{s}$ on X is* admissible*, if there is an isomorphism from $\mathfrak{s}|_{M_s}$ to the canonical spin^c structure $\mathfrak{s}_{M_s,\omega}$.*

Remark 1.3. To simplify notation, if $\mathfrak{s}$ is an admissible spin^c structure on X, we will always assume that there is a fixed isomorphism from $\mathfrak{s}|_{M_s}$ to $\mathfrak{s}_{M_s,\omega}$, and we will identify the positive and negative spinor bundles of $\mathfrak{s}|_{M_s}$ with $T^{0,0}M_s \oplus T^{0,2}M_s$ and $T^{1,1}M_s$ respectively.

Let $\mathfrak{s}$ be an admissible spin^c structure on X, let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ be the corresponding spinor bundle, and let ρ be the Clifford multiplications. Let $r > 0$ be a constant; later we will require r to be sufficiently large. This article studies a system of perturbed Seiberg-Witten equations on X given by

$$\begin{cases} D_A\phi = \eta_1, \\ F_A^+ = (\phi\phi^*)_0 + \eta_2, \end{cases} \tag{1.1}$$

where $(\eta_1, \eta_2) = (0, -ir\omega/4 + F_{A_0}^+)$ on M_s, and (η_1, η_2) is given by a tame perturbation introduced by [18, Section 10] on M_c. The precise definition of (1.1) will be given in Section 2.

Let A be spin^c connection of $\mathfrak{s}$, and let ϕ be a section of $\mathbb{S}^+$. Recall that we have chosen a fixed isomorphism from $\mathfrak{s}|_{M_s}$ to $\mathfrak{s}_{M_s,\omega}$ and use it to identify their spinor bundles, so we have $\mathbb{S}^+|_{M_s} = T^{0,0}M_s \oplus T^{0,2}M_s$, where the almost complex structure on M_s is induced by ω. Decompose $\phi|_{M_s}$ as

$$\phi|_{M_s} = \sqrt{r}(\alpha + \beta),$$

where $\alpha \in T^{0,0}M_s$, $\beta \in T^{0,2}M_s$. Let ∇'_A be the projection of $\nabla_A|_{M_s}$ to $T^{0,2}M_s$. More precisely, ∇'_A is a connection of $T^{0,2}M_s$ such that for every section $s \in \Gamma(M_s, T^{0,2}M_s)$, the section $\nabla'_A s$ is equal to the projection of $\nabla_A s$ to $T^{0,2}M_s$. Let A_0 be the canonical spin^c connection on $\mathfrak{s}|_{M_s}$, then there exists a unique function a on M_s which takes values in $i\mathbb{R}$ such that $A|_{M_s} - A_0 = a \cdot \mathrm{id}_{\mathbb{S}^+ \oplus \mathbb{S}^-}$. To simplify notation, we will write $a = A|_{M_s} - A_0$ for the rest of the article. Notice that a defines a unitary connection on the trivial bundle $T^{0,0}M_s$. We define the *energy density function* of (A, ϕ) on M_s as:

$$E_r(A, \phi) = |1 - |\alpha|^2 - |\beta|^2|^2 + |\beta|^2 + |\nabla_a \alpha|^2 + |\nabla'_A \beta|^2 + |F_a|^2. \tag{1.2}$$

Define a function d on M_s as follows. For each connected component $M_s^{(k)}$ of M_s, if $\partial M_s^{(k)}$ is nonempty, let d on $M_s^{(k)}$ be the distance function to $\partial M_s^{(k)}$. Otherwise, fix a point $x^{(k)} \in M_s^{(k)}$, and let d on $M_s^{(k)}$ be the distance function to $x^{(k)}$. The main analytic result of this article is the following estimate.

Theorem 1.4. *Let X be a Riemannian 4-manifold with cylindrical and ESBG ends, where M_c is the cylindrical end, and $(M_s, \omega = d\theta)$ is the ESBG end. Then there exists a constant $r_0 > 0$ and a constant z, such that for every admissible spin^c structure $\mathfrak{s}$ and every $r > r_0$, there exists a constant C depending on r, with the following significance. If (A, ϕ) is a solution to* (1.1) *on X such that*

$$\int_{M_s} E_r(A, \phi) < +\infty,$$

then

$$E_r(A, \phi) < C\, e^{-\sqrt{r} \cdot d/z} \tag{1.3}$$

pointwise on M_s.

Theorem 1.4 will be re-stated and proved in Section 4.7. The theorem is an extension of the analysis on asymptotically flat almost Kähler (AFAK) structures by Kronheimer and Mrowka [17, Section 3(iii)]. This estimate implies that the zero-dimensional component of a relevant moduli space of solutions to Seiberg-Witten equations on X is compact, therefore one can define topological invariants for X by counting the solutions of the

Seiberg-Witten equations. In Section 5, we will follow the strategies of [20] and construct an invariant for X which takes value in the monopole Floer homology group.

1.2. Taut foliations

Theorem 1.4 and the construction in Section 5 can be applied to the study of taut foliations.

Let Y be a smooth, closed, oriented three-manifold, let $\mathcal{F}$ be an oriented foliation on Y. By definition, $\mathcal{F}$ is called *taut* if the following condition is satisfied: for every point $p \in Y$, there exists an embedded circle in Y that contains p and is transverse to $\mathcal{F}$.

One of the fundamental problems in the study of taut foliations is their existence on a given 3-manifold. By the Roussarie-Thurston theorem, if Y supports a taut foliation, then every embedded sphere in Y is either nullhomotopic or is isotopic to a leaf. Reeb's stability theorem then implies that if Y supports a taut foliation, then Y is either irreducible, or is homeomorphic to $S^2 \times S^1$ with the product foliation. Gabai [12] proved that every irreducible, closed, oriented three-manifold with $b_1 \geq 1$ supports a taut foliation. The existence problem for taut foliations on irreducible manifolds with $b_1 = 0$ is still unsolved. It was proved in [20] that if Y is a rational homology sphere supporting a smooth taut foliation, then the reduced monopole Floer homology group $HM_\bullet(Y)$ must be nontrivial. This implies, for example, that the lens spaces do not support any smooth taut foliations except for $S^1 \times S^2$. The theorem was generalized to C^0-taut foliations by Bowden [3].

The flexibility of taut foliations has also been studied for years. Eynard-Bontemps [11] proved that if two taut foliations can be homotoped to each other via plane fields, then they can be homotoped to each other via foliations. On the other hand, Vogel [28] and Bowden [5] have constructed examples of taut foliations that are homotopic as plane fields but cannot be homotoped to each other via taut foliations.

The proofs of the non-vanishing and non-flexibility results above rely on the following perturbation theorem, which is due to Eliashberg and Thurston for C^2 foliations and is generalized by Bowden to the C^0 case:

Theorem (Eliashberg-Thurston [10], Bowden [3]). *Let $\mathcal{F}$ be an orientable C^0 foliation on an oriented 3-manifold Y, and assume $(Y, \mathcal{F})$ is not homeomorphic to the product foliation on $S^2 \times S^1$. Then $\mathcal{F}$ can be C^0 approximated both by a sequence of positive contact structures and a sequence* *of negative contact structures. If $\mathcal{F}$ is taut, then the positive contact approximations are weakly semi-fillable.*

If Y supports taut foliations, then the non-vanishing of $HM_\bullet(Y)$ was proved by the theorem above and the non-vanishing property of semi-fillable contact structures [20, Section 6.4]. The examples in [5, 28] were proved by first showing that the perturbed contact structures are unique up to isotopy, and then showing that the isotopy classes of the corresponding contact structures are different.

As an application of Theorem 1.4, if $\mathcal{F}$ is a smooth foliation on Y that does not admit holonomy-invariant transverse measures, we will construct two invariants $c_+(\mathcal{F}) \in \widetilde{HM}_\bullet(Y)$ and $c_-(\mathcal{F}) \in \widetilde{HM}_\bullet(-Y)$ without invoking the Eliashberg-Thurston perturbation. Here, $\widetilde{HM}_\bullet(\cdot)$ is the boundary-stable version of the monopole Floer homology introduced by [18]. We will then apply the invariants to the study of the existence and flexibility of taut foliations.

Notice that $\mathcal{F}$ is taut if and only if there exists a *closed* 2-form $\hat{\omega}$ on Y, such that $\hat{\omega}$ is everywhere positive on the tangent plane field of $\mathcal{F}$ [7, Proposition 10.4.1]. On the other hand, $\mathcal{F}$ has no holonomy-invariant transverse measure if and only if there exists an *exact* 2-form $\hat{\omega}$ on Y, such that $\hat{\omega}$ is everywhere positive on the tangent plane field of $\mathcal{F}$ [26, Theorem II.2]. Therefore, if Y is a rational homology sphere, then a foliation $\mathcal{F}$ is taut if and only if it has no holonomy-invariant transverse measure.

In Section 6, we will construct the invariants $c_+(\mathcal{F})$ and $c_-(\mathcal{F})$, and show that the gradings of $c_+(\mathcal{F}) \in \widetilde{HM}(Y)$ and $c_-(\mathcal{F}) \in \widetilde{HM}(-Y)$ are given by the homotopy classes of $\mathcal{F}$ as plane fields on Y and $-Y$ respectively. We will also show that $c_\pm(\mathcal{F})$ have nonzero images in the reduced monopole Floer homology groups under the map j_* defined by [18, Proposition 22.2.1]. Therefore, the existence of $c_+(\mathcal{F})$ gives an alternative proof for the nonvanishing theorem of $HM_{[\mathcal{F}]}(Y)$ for smooth taut foliations on rational homology spheres [20, Theorem 2.1].

The invariants $c_\pm(\mathcal{F})$ can also be used to study the flexibility of foliations. In Section 7, we will construct smooth foliations without holonomy-invariant transverse measure that are homotopic as plane fields but have different invariants c_+. Since $c_\pm(\mathcal{F})$ are invariant under smooth deformations, this gives examples of smooth foliations that are homotopic as plane fields but cannot be smoothly deformed to each other via foliations without holonomy-invariant transverse measure.

It should be pointed out that if $\mathcal{F}$ is a smooth foliation without holonomy-invariant transverse measure, then there exist linear deformations of $\mathcal{F}$ to both positive and negative contact structures [10, Theorem 2.1.2]. It is straightforward to verify that the space of all positive (negative) linear deformations is convex, so the contact structures obtained by linear deformations are unique up to isotopy. Therefore, the contact elements of the linearly deformed contact structures also give two invariants for $\mathcal{F}$ in the monopole Floer homology groups. The relation between $c_\pm(\mathcal{F})$ and the corresponding contact invariants is not clear to the author.

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2. The Seiberg-Witten equations

This section briefly reviews the definition of Seiberg-Witten equations, and introduces a perturbation on manifolds with cylindrical and ESBG ends. We will follow the notations from [18]. The reader may refer to [14, 23] for more details.

2.1. Spin^c Structures

For $n \geq 2$, let $\mathrm{Spin}(n)$ be the connected double cover of $\mathrm{SO}(n)$. Let $\mathrm{Spin}^c(n) = \big(\mathrm{U}(1) \times \mathrm{Spin}(n)\big)/\{\pm 1\}$, where $1 \in \mathrm{U}(1) \times \mathrm{Spin}(n)$ is the unit element, and the two coordinates of $-1 \in \mathrm{U}(1) \times \mathrm{Spin}(n)$ are given by $-1 \in \mathrm{U}(1)$ and the non-trivial element in the preimage of $1 \in \mathrm{SO}(n)$. Let X be an oriented Riemannian 4-manifold. By definition, a spin^c structure $\mathfrak{s}$ on X is a principal $\mathrm{Spin}^c(4)$–bundle which is a lift of the oriented orthonormal frame bundle via the surjection

$$\mathrm{Spin}^c(4) = \big(\mathrm{U}(1) \times \mathrm{Spin}(4)\big)/\{\pm 1\} \to \mathrm{Spin}(4)/\{\pm 1\} \cong \mathrm{SO}(4).$$

The group $\mathrm{Spin}^c(4)$ has a standard unitary representation on $\mathbb{C}^4$. Suppose $\mathfrak{s}$ is a spin^c structure on X, then the spinor bundle of $\mathfrak{s}$ is defined as $\mathbb{S} = \mathfrak{s} \times_{\mathrm{Spin}^c(4)} \mathbb{C}^4$. There is a Clifford multiplication $\rho : T^*X \to \mathrm{Hom}(\mathbb{S}, \mathbb{S})$ which satisfies $\rho(v)^2 = -\|v\|^2$ for all $v \in T^*X$, and the action ρ extends to $\wedge^* T^*M$. Let $d\,\mathrm{Vol}$ be the volume form of X, then $\rho(d\,\mathrm{Vol})^2 = \mathrm{id}|_{\mathbb{S}}$. Let $\mathbb{S}^+$ and $\mathbb{S}^-$ be the eigenspaces of $\rho(d\,\mathrm{Vol})$ with eigenvalues -1 and 1 respectively, then both $\mathbb{S}^+$ and $\mathbb{S}^-$ have rank 2. Let $\Lambda^+(X)$ be the vector bundle of self-dual 2-forms on X, and let $\mathrm{End}_0(\mathbb{S}^+)$ be the traceless endomorphisms of $\mathbb{S}$, then ρ maps $\Lambda^+(X) \otimes \mathbb{C}$ isomorphically to $\mathrm{End}_0(\mathbb{S}^+)$.

A unitary connection A on $\mathbb{S}$ is called a *spin^c connection* if $\nabla_A \rho = 0$, where ∇_A is the coupled connection of A and the Levi-Civita connection on $TX \otimes \mathrm{Hom}(\mathbb{S}, \mathbb{S})$. Every spin^c connection decomposes as two unitary connections on $\mathbb{S}^+$ and $\mathbb{S}^-$, and the connection on $\mathbb{S}^+$ induces a connection on $\det(\mathbb{S}^+)$. We use A^t to denote the connection on $\det(\mathbb{S}^+)$ induced by A, and use D_A to denote the Dirac operator defined by A.

The definition of spin^c structures on 3-manifolds is similar. A spin^c structure on an oriented Riemannian 3-manifold Y is a principal $\mathrm{Spin}^c(3)$–bundle which is a lift of the oriented orthonormal frame bundle. Notice that $\mathrm{Spin}^c(3) = \mathrm{SU}(2) \times \mathrm{U}(1)/\{\pm 1\} \cong \mathrm{U}(2)$. If $\mathfrak{t}$ is a Spin^c structure on a 3-manifold Y, then the spinor bundle of $\mathfrak{t}$ is defined as $\mathbb{S} = \mathfrak{t} \times_{\mathrm{U}(2)} \mathbb{C}^2$, and there is a Clifford multiplication $\rho : T^*M \to \mathrm{Hom}(\mathbb{S}, \mathbb{S})$. A unitary connection B on the spinor bundle $\mathbb{S}$ is called a *spin^c connection* if $\nabla_B \rho = 0$.

2.2. Configuration spaces

For a smooth vector bundle V over a smooth manifold M, we say that a section s of V is *locally L^p_k*, if for every $p \in M$ there exists a neighborhood U of p and a (smooth) trivialization of $V|_U$, such that $s|_U$ is L^p_k under this trivialization. We say that a connection A of V is *locally L^p_k*, if there exists a smooth connection $\hat{A}$, such that $A - \hat{A}$ is a locally L^p_k section of $T^*M \otimes V$.

We recall the following definitions of configuration spaces from [18].

Definition 2.1. *Let $\mathfrak{t}$ be a spin^c structure on a closed 3-manifold Y, let $\mathbb{S}$ be the spinor bundle. Define $\mathcal{C}_k(Y, \mathfrak{t})$ to be the set of pairs (B, ψ), where B is a locally L^2_k spin^c connection of $\mathfrak{t}$, and ψ is a locally L^2_k section of $\mathbb{S}$. Define $\mathcal{C}(Y, \mathfrak{t}) = \cap_{k \ge 1} \mathcal{C}_k(Y, \mathfrak{t})$.*

Definition 2.2. *Let $\mathfrak{s}$ be a spin^c structure on a compact 4-manifold X possibly with boundary, let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ be the spinor bundle. Define $\mathcal{C}_k(X, \mathfrak{s})$*

to be the set of pairs (A, ϕ) such that A is a locally L^2_k spin^c connection of $\mathfrak{s}$, and ϕ is a locally L^2_k section of $\mathbb{S}^+$. Define $\mathcal{C}(X, \mathfrak{s}) = \cap_{k\geq 1}\mathcal{C}_k(X, \mathfrak{s})$.

Now let X be a Riemannian 4-manifold with cylindrical and ESBG ends as given by Definition 1.1. Suppose M_c is the cylindrical end, and $(M_s, \omega = d\theta)$ is the ESBG end. Let $\mathfrak{s}$ be an admissible spin^c structure on X as in Definition 1.2. We define the configuration space for $(X, \mathfrak{s})$ as follows. Recall that for an oriented closed 3-manifold Y, the spin^c structures on Y are in one-to-one correspondence with the spin^c structures on $(-\infty, 0] \times Y$ ([18, Section 4.3]).

Definition 2.3. *Let $X, M_c, M_s, \omega = d\theta, \mathfrak{s}$ be as above, let $r > 0$ be a constant. For $k \geq 1$, define $\mathcal{C}_k(X, \mathfrak{s})$ to be the set of pairs (A, ϕ) such that:*

1) *A is a locally L^2_k Spin^c-connection of $\mathfrak{s}$, and ϕ is a locally L^2_k section of $\mathbb{S}^+$;*
2) *$\int_{M_s} E_r(A, \phi) < +\infty$, where $E_r(A, \phi)$ is defined by (1.2);*
3) *On the cylindrical end $M_c = (-\infty, 0] \times Y$, let $\mathfrak{t}$ be the spin^c structure on Y induced by $\mathfrak{s}|_{M_c}$. Then the restriction of (A, ϕ) on M_c gives a path $(-\infty, 0] \to \mathcal{C}_{k-1}(Y, \mathfrak{t})$ that is convergent at $-\infty$, in the L^2_{k-1} topology of $\mathcal{C}_{k-1}(Y, \mathfrak{t})$.*

Define $\mathcal{C}(X, \mathfrak{s}) = \cap_{k\geq 1}\mathcal{C}_k(X, \mathfrak{s})$.

2.3. Strongly tame perturbations

Let Y be an oriented closed three-manifold, let $\mathfrak{t}$ be a spin^c structure on Y, and let B_0 be a smooth spin^c connection of $\mathfrak{t}$. Let $\mathcal{L}$ be the Chern-Simons-Dirac functional on $\mathcal{C}(Y, \mathfrak{t})$ defined by [18, Definition 4.1.1] with respect to B_0. A Banach space of "tame" perturbations of $\mathcal{L}$ was introduced and studied in [18, Sections 10, 11]. For the purpose of this article, we need to introduce a stronger condition on the perturbations.

Recall that if $\mathfrak{q}$ is a perturbation of the Chern-Simons-Dirac functional, then the formal gradient of $\mathfrak{q}$ defines a perturbation $(\hat{\mathfrak{q}}^0, \hat{\mathfrak{q}}^1)$ for the Seiberg-Witten equations on the cylinder $[0, 1] \times Y$ [18, Secion 10.1].

Definition 2.4. *Let $\mathfrak{s}$ be the spin^c structure on $[0, 1] \times Y$ induced by $\mathfrak{t}$. A perturbation $\mathfrak{q}$ of $\mathcal{L}$ is called* strongly tame *if*

1) *It is a tame perturbation as defined by [18, Definition 10.5.1].*

2) *There is a constant* m_0 *such that*

$$\|\hat{\mathfrak{q}}^0(A,\phi)\|_{C^0} \le m_0(\|\phi\|_{C^0}+1)$$

for all $(A,\phi)\in\mathcal{C}([0,1]\times Y,\mathfrak{s})$.

3) *There is a constant* m_1 *such that*

$$\|\hat{\mathfrak{q}}^1(A,\phi)\|_{C^0} \le m_1$$

for all $(A,\phi)\in\mathcal{C}([0,1]\times Y,\mathfrak{s})$.

Using the calculations in [18, p. 176], it is straight forward to verify that the cylindrical functions constructed in [18, Section 11.1] are strongly tame. The proof of [18, Theorem 11.6.1] defined a norm $\|\cdot\|_{\mathcal{P}}$ on the linear space generated by a seqeuence cylindrical functions, and proved that the completion with respect to $\|\cdot\|_{\mathcal{P}}$ gives a Banach space of tame perturbations. We consider a modified norm defined by

$$\|\mathfrak{q}\|_{\hat{\mathcal{P}}} = \|\mathfrak{q}\|_{\mathcal{P}} + \sup_{(A,\phi)\in\mathcal{C}([0,1]\times Y,\mathfrak{s})} \Big(\frac{\|\hat{\mathfrak{q}}^0(A,\phi)\|_{C^0}}{\|\phi\|_{C^0}+1} + \|\hat{\mathfrak{q}}^1(A,\phi)\|_{C^0}\Big). \tag{2.1}$$

By the same argument as in [18, Theorem 11.6.1], the completion with respect to $\|\cdot\|_{\hat{\mathcal{P}}}$ gives a Banach space of strongly tame perturbations that contains the given sequence of cylindrical functions. As a consequence, the transversality property [18, Theorem 15.1.1] still holds with respect to strongly tame perturbations.

2.4. Perturbed Seiberg-Witten equations

Let X be a Riemannian 4-manifold with cylindrical and ESBG ends, where the cylindrical end is M_c and the ESBG end is $(M_s,\omega=d\theta)$. Let $\mathfrak{s}$ be an admissible spin^c structure on X, let $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$ be the spinor bundle. Let $r>0$ be a constant. This section introduces a family of perturbations of Seiberg-Witten equations on $(X,\mathfrak{s})$ parametrized by the constant r. Similar perturbations were used in [17, 27] and many other related works.

For $(A,\phi)\in\mathcal{C}_k(X,\mathfrak{s})$ with $k\ge 2$, define

$$\mathfrak{F}(A,\phi) = (\rho(F^+_{A^t}) - (\phi\phi^*)_0, D_A\Phi),$$

where $(\phi\phi^*)_0$ is the traceless part of $\phi\phi^*$. By definition, $\mathfrak{F}(A,\phi)$ is a section of $i\mathfrak{su}(\mathbb{S}^+)\oplus\mathbb{S}^-$.

By Conditions (3), (4) of Definition 1.1, the cylindrical and ESBG structures extend to neighborhoods of M_c and M_s. Let M_c' and $(M_s', \omega' = d\theta')$ be the respective neighborhoods of M_c and M_s on which the cylindrical and ESBG structures extend. If $M_c = \emptyset$, then we take $M_c' = \emptyset$. By shrinking the neighborhoods, we may assume that the closures of M_c' and M_s' are disjoint. We also assume that M_s' deformation retracts to M_s, therefore the isomorphism from $\mathfrak{s}|_{M_s}$ to $\mathfrak{s}_{M,\omega}$ extends to an isomorphism from $\mathfrak{s}|_{M_s'}$ to $\mathfrak{s}_{M_s',\omega'}$. In the following, we will fix such an isomorphism from $\mathfrak{s}|_{M_s'}$ to $\mathfrak{s}_{M_s',\tilde{\omega}}$. To simplify notation, we will use the fixed isomorphism to identify the spinor bundles of $\mathfrak{s}|_{M_s'}$ and $\mathfrak{s}_{M_s',\omega'}$.

Suppose M_c is isometric to $(-\infty, 0] \times Y$, let $\mathfrak{t}$ be the spin^c structure on Y induced by $\mathfrak{s}|_{M_c}$. Let $\mathfrak{q}$ be a strongly tame perturbation on $(Y, \mathfrak{t})$, then the flow line equation of the perturbed Chern-Simons-Dirac functional $\mathcal{L} + \mathfrak{q}$ can be written as $\mathfrak{F}(A, \phi) = \hat{\mathfrak{q}}(A, \phi)$, where $\hat{\mathfrak{q}}$ is the formal gradient of $\mathfrak{q}$. For the rest of this article, we will assume $\mathfrak{q}$ is strongly tame in the sense of Definition 2.4 and is admissible in the sense of [18, Definition 22.1.1]. Moreover, assume that $\|\mathfrak{q}\|_{\hat{\mathcal{P}}} \le 1$, where $\|\cdot\|_{\hat{\mathcal{P}}}$ is the norm defined by (2.1).

Recall that we have fixed an isomorphism from $\mathfrak{s}|_{M_s'}$ to $\mathfrak{s}_{M_s',\omega'}$ and use it to identify the spinor bundles of $\mathfrak{s}|_{M_s'}$ and $\mathfrak{s}_{M_s',\omega'}$. There is a canonical section Φ_0 of $\mathbb{S}^+|_{M_s'}$ given by $1 \in \Gamma(M_s', T^{0,0}M_s')$, and a canonical spin^c connection A_0 on $\mathfrak{s}|_{M_s'}$ characterized by $D_{A_0}\Phi_0 = 0$. Define a section $\hat{u} \in C^\infty(M_s', i\mathfrak{su}(\mathbb{S}^+) \oplus \mathbb{S}^-)$ on M_s' by

$$\begin{aligned} \hat{u} &= (-r(\Phi_0\Phi_0^*)_0 + \rho(F^+_{A_0^t}), 0) \\ &= \big(-\frac{ir}{4}\rho(\omega') + \rho(F^+_{A_0^t}), 0\big). \end{aligned} \tag{2.2}$$

Let $\hat{\tau} \in C_0^\infty(Z - M_c{}' - M_s{}', i\mathfrak{su}(\mathbb{S}^+))$. Let $\eta \ge 0$ be a smooth cut-off function on X such that $\operatorname{supp}\eta \subset M_c{}' \cup M_s{}'$, and $\eta = 1$ on $M_c \cup M_s$. Define

$$\hat{\mu} = \eta\hat{\mathfrak{q}} + \eta\hat{u} + (\hat{\tau}, 0). \tag{2.3}$$

The Seiberg-Witten equation that will be studied in this article is the equation for $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ given by:

$$\mathfrak{F}(A, \phi) = \hat{\mu}(A, \phi). \tag{2.4}$$

2.5. Convergence on different manifolds

This subsection defines a version of convergence for a sequence of connections and spinors on different manifolds, and gives a sufficient condition for the existence of a convergent subsequence. For a Riemannian manifold X, a point $p \in X$, and $d > 0$, we use $B_p(d)$ to denote the set of points $x \in X$ such that the distance from x to p is no greater than d.

Definition 2.5. *A sequence of pointed Riemannian manifolds possibly with boundary*

$$\{(X_n, g_n, p_n)\}_{n\geq 1}$$

is said to have uniformly bounded geometry*, if there exists a sequence of positive real numbers $\{r_n\}_{n\geq 1}$ such that the following conditions hold:*

1) $\lim_{n\to\infty} r_n = +\infty$;
2) *The exponential map of X_n at p_n is defined on the closed ball of radius r_n for each n;*
3) *There exists $\epsilon_0 > 0$, such that for all n, the injectivity radius of X_n is greater than ϵ_0 for every point in $B_{p_n}(r_n)$;*
4) *For every integer $n \geq 0$, let $R^{(n)}$ be the Riemann curvature tensor of X_n, then the sequence*

$$\Big\{ \sup_{B_{p_n}(r_n)} |\nabla^k R^{(n)}| \Big\}_{n\geq 1}$$

is bounded for each k.

Remark 2.6. Suppose $\{(X_n, g_n, p_n)\}_{n\geq 1}$ is a sequence of pointed Riemannian manifolds with uniformly bounded geometry, then for each $N > 0$, there exists a constant $C_N > 0$ with the following property. For every n, suppose $x \in B_{p_n}(r_n)$, let $\varphi_x : B(\epsilon_0) \to X_n$ be the normal coordinate of (X_n, g_n) centered at x with radius ϵ_0. Then $\|\varphi_x^* g_n\|_{C^N(B(\epsilon_0))} \leq C_N$. As a consequence, for each $k \in \mathbb{Z}^+, \alpha \in (0,1), r > 0$, there exists a constant Q such that $\|(B_{p_n}(r_n), g_n)\|_{k+\alpha,r} \leq Q$ for all n, where $\|\cdot\|_{k+\alpha,r}$ is the norm defined in [25, Section 2]. This observation will be used in the proof of Proposition 2.9.

Definition 2.7. *Suppose $\{(X_n, g_n, p_n)\}_{n\geq 1}$ is a sequence of oriented pointed Riemannian 4-manifolds with uniformly bounded geometry. For each n, let $\mathfrak{s}_n$ be a spin^c structure on X_n, let $\mathbb{S}_n = \mathbb{S}_n^+ \oplus \mathbb{S}_n^-$ be the corresponding spinor bundle, and let $\rho_n : T^*X_n \to \mathrm{Hom}(\mathbb{S}_n, \mathbb{S}_n)$ the Clifford multiplications. Let A_n be a locally L_k^2 spin^c connection of $\mathfrak{s}_n$, let ϕ_n be a locally L_k^2 section of $\mathbb{S}_n^+$.*

The sequence $\{(X_n, g_n, p_n, \mathfrak{s}_n, A_n, \phi_n)\}_{n\geq 1}$ *is said to be convergent to*

$$(X, g, p, \mathfrak{s}, A, \phi)$$

up to gauge transformations, if there exists a sequence

$$\{(d_n, U_n, V_n, \varphi_n, \tilde{\varphi}_n, u_n)\}_{n\geq 1}$$

such that the following conditions hold:

1) (X, g) *is a connected complete Riemannian 4-manifold, and* $p \in X$. $\{d_n\}_{n\geq 1}$ *is a sequence of positive real numbers such that* $\lim_{n\to\infty} d_n = +\infty$. *The element* V_n *is an open neighborhood of* p_n *in* X_n*, and* U_n *is an open neighborhood of* p *in* X. *Both* V_n *and* U_n *have compact closures in* X_n *and* X *respectively.*

2) *The exponential map of* X_n *at* p_n *is defined on the closed ball of radius* d_n *for each* n*, and* $B_{p_n}(d_n) \subset V_n$ *in* X_n*,* $B_p(d_n) \subset U_n$ *in* X. *The element* φ_n *is a diffeomorphism from* U_n *to* V_n *mapping* p *to* p_n. *Moreover, for every compact subset* K *of* X*, we have*

$$\lim_{n\to\infty} \|\varphi_n^*(g_n) - g\|_{C^m(K\cap U_n)} = 0, \text{ for all } m \in \mathbb{N}.$$

3) *Let* $\mathbb{S}$ *be the spinor bundle of* $\mathfrak{s}$ *and let* $\rho : T^*X \to \mathrm{Hom}(\mathbb{S}, \mathbb{S})$ *be the Clifford multiplication. The element* $\tilde{\varphi}_n$ *is a smooth unitary isomorphism from* $\mathbb{S}_n|_{U_n}$ *to* $\mathbb{S}|_{V_n}$ *lifting* φ_n. *Let* $\tilde{\varphi}_n^*(\rho_n) : T^*X \to \mathrm{Hom}(\mathbb{S}, \mathbb{S})$ *be the pull-back of* ρ_n *via* $\tilde{\varphi}$ *and the tangent map of* φ_n. *For every compact subset* K *of* X*, we have*

$$\lim_{n\to\infty} \|\tilde{\varphi}_n^*(\rho_n) - \rho\|_{C^m(K\cap U_n)} = 0, \text{ for all } m \in \mathbb{N}.$$

4) *The element* u_n *is a gauge transformation of* $\mathfrak{s}_n$ *on* V_n*, such that for every compact subset* K *of* X*, we have*

$$\lim_{n\to\infty} \|\tilde{\varphi}_n^*(u_n(A_n, \phi_n)) - (A, \phi)\|_{C^m(K\cap U_n)} = 0, \text{ for all } m \in \mathbb{N}.$$

Remark 2.8. By our definition, when X_n's are not connected, the convergence of

$$\{(X_n, g_n, p_n, \mathfrak{s}_n, A_n, \phi_n)\}_{n\geq 1}$$

only depends on the connected components containing p_n.

Proposition 2.9. *Let $\{(X_n, g_n, p_n)\}_{n\geq 1}$ be a sequence of pointed oriented Riemannian 4-manifolds with uniformly bounded geometry, let ϵ_0, $\{r_n\}_{n\geq 1}$ be the constants given by Definition 2.5. For each n, let $\mathfrak{s}_n$ be a spin^c structure on X_n, let $\mathbb{S}_n$ be the spinor bundle, let A_n be a locally L^2_1 Spin^c-connection for $\mathfrak{s}_n$, and let ϕ_n be a locally L^2_1 section of $\mathbb{S}^+_n$. Assume that there exists a constant $C > 0$ such that for every n and every point $x \in B_{p_n}(r_n)$,*

$$\int_{B_x(\epsilon_0)} |F_{A_n}|^2 < C, \tag{2.5}$$

$$|\phi_n(x)| < C, \tag{2.6}$$

and

$$D_{A_n}(\phi_n) = 0. \tag{2.7}$$

Moreover, assume that $\mathfrak{F}(A_n, \phi_n)$ is smooth for each n, and

$$\sup_{n\geq 1} \|\mathfrak{F}(A_n, \phi_n)\|_{C^k} < +\infty, \quad \text{for all } k \geq 1. \tag{2.8}$$

Then there exists a subsequence of $\{(X_n, g_n, p_n, \mathfrak{s}_n, A_n, \phi_n)\}_{n\geq 1}$ and a configuration $(X, g, p, \mathfrak{s}, A, \phi)$, such that the subsequence converges to $(X, g, p, \mathfrak{s}, A, \phi)$ in the sense of Definition 2.7.

Proof. Since $\{(X_n, g_n, p_n)\}_{n\geq 1}$ have uniformly bounded geometry, it follows from [25, Theorem 2.2] and Remark 2.6 that after taking a subsequence, there exists a complete, connected, pointed Riemannian manifold (X, g, p) and a sequence $\{(d_n, U_n, V_n, \varphi_n)\}_{n\geq 1}$, such that Conditions (1), (2) of Definition 2.7 are satisfied. Although [25, Theorem 2.2] requires (X_n, g_n, p_n) to be complete, the proof also works for non-complete manifolds as long as Conditions (1), (2) of Definition 2.5 holds. By taking a further subsequence, we may assume that $U_n \subset U_m \subset X$ for all $n \leq m$.

Now we construct a spin^c structure $\mathfrak{s}$ on X. By (2.5), the sequence

$$\|\varphi_n^*(F_{A_n})\|_{L^2(U_n\cap K)}$$

is bounded for every compact subset K of X. Take an embedded closed oriented surface Σ in X, let $N(\Sigma)$ be a tubular neighborhood of Σ. Then

$$\sup_{\{n|N(\Sigma)\subset U_n\}} \int_{N(\Sigma)} |\varphi_n^*(F_{A_n})| < +\infty,$$

As a consequence,

$$\sup_{\{n|N(\Sigma)\subset U_n\}} |\langle \varphi_n^*(c_1(\mathfrak{s}_n)), [\Sigma]\rangle| < +\infty$$

Let $\mathrm{Spin}^c(U_n)$ be the set of isomorphism classes of spin^c structures on U_n. Since the first Chern class determines the spin^c structure up to torsion, there exists a finite set $\Lambda_n \subset \mathrm{Spin}^c(U_n)$, such that $\varphi_m^*(\mathfrak{s}_m)|_{U_n} \in \Lambda_n$ for all $m \geq n$. Therefore, after taking a further subsequence, we may assume that $\varphi_m^*(\mathfrak{s}_m)|_{U_n}$ is isomorphic to $\varphi_n^*(\mathfrak{s}_n)$ for all $m \geq n$. For each n, let $\zeta_n : \varphi_n^*(\mathfrak{s}_n) \to \varphi_{n+1}^*(\mathfrak{s}_{n+1})|_{U_n}$ be an isomorphism of spin^c structures on U_n, let $\iota_n : U_n \to U_{n+1}$ be the inclusion map, then $\{(\varphi_n^*(\mathfrak{s}_n), U_n, \zeta_n, \iota_n)\}_{n\geq 1}$ generates a direct system. Taking the limit of this direct system yields a spin^c structure $\mathfrak{s}$ on X and isomorphisms $\tilde{\varphi}_n : \mathfrak{s}|_{U_n} \to \mathfrak{s}_n$ that are lifts of φ_n.

The only thing remaining to prove is the existence of (A, ϕ) and the gauge transformations u_n satisfying Condition (4) of Definition 2.7. Without loss of generality, we may assume that the closures of $V_n \subset X_n$ and $U_n \subset X$ are compact manifolds with boundary. For a pair of positive integers $n \geq m$, let $\mathcal{E}^{an}(A_n|_{V_m}, \phi_n|_{V_m})$ be the analytic energy of (A_n, ϕ_n) on V_m as defined by [18, Definition 4.5.4]. We will show that for every m,

$$\sup_{n\geq m+1} \mathcal{E}^{an}(A_n|_{V_m}, \phi_n|_{V_m}) < +\infty.$$

Since the closure of U_m in X is compact for every m, by taking a further subsequence if necessary, we may assume that $\overline{U_{m-1}} \subset U_m$ for all m. Moreover, since X is complete, we may take a further subsequence, such that for each $m \geq 1$, there is a cut-off function $\chi_m \geq 0$ on X such that $\operatorname{supp}\chi_m \subset U_{m+1}$, $\chi_m|_{U_m} = 1$, and $|\nabla\chi_m| \leq 1$ for all m.

For $n \geq m+1$, let $\phi_n^{(m)} = (\chi_m \circ \varphi_n^{-1}) \cdot \phi_n$ be a spinor on V_n. By (2.6), (2.8), and Condition (2) of Definition 2.7, we have

$$\begin{aligned} \|\mathfrak{F}(A_n, \phi_n^{(m)})\|_{L^2(V_{m+1})} &\leq C_1 \left(\|\mathfrak{F}(A_n, \phi_n)\|_{L^2(V_{m+1})} + \|\phi_n\|_{C^0} + \|\phi_n\|_{C^0}^2\right) \\ &\leq C_2 \end{aligned} \tag{2.9}$$

for constants C_1, C_2 depending on m.

Let $\mathcal{E}^{top}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}})$ be the topological energy of $(A_n, \phi_n^{(m)})$ on V_{m+1} as defined by [18, Definition 4.5.4]. Since $\phi_n^{(m)}$ is compactly supported on V_{m+1}, we have

$$\mathcal{E}^{top}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}}) = \frac{1}{4}\int_{V_{m+1}} F_{A_n^t} \wedge F_{A_n^t}, \tag{2.10}$$

which is bounded by a constant depending on m because of (2.5).

By [18, (4.16)],

$$\mathcal{E}^{an}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}}) = \mathcal{E}^{top}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}}) + \|\mathfrak{F}(A_n, \phi_n^{(m)})\|^2_{L^2(V_{m+1})},$$

therefore by (2.9) and (2.10), $\mathcal{E}^{an}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}})$ is bounded by a constant depending on m. Since

$$\mathcal{E}^{an}(A_n|_{V_{m+1}}, \phi_n^{(m)}|_{V_{m+1}}) \geq \mathcal{E}^{an}(A_n|_{V_m}, \phi_n|_{V_m}) - C_3,$$

for a constant C_3 depending on m, we conclude that

$$\sup_{n \geq m+1} \mathcal{E}^{an}(A_n|_{V_m}, \phi_n|_{V_m}) < +\infty.$$

By Condition (2) of Definition 2.7, the statement above implies that

$$\{\mathcal{E}^{an}(\tilde{\varphi}^*(A_n)|_{U_m}, \tilde{\varphi}^*(\phi_n)|_{U_m})\}_{n \geq m+1}$$

is bounded for every m. Therefore, by a diagonal argument, the existence of (A, ϕ) and u_n satisfying Condition (4) of Definition 2.7 follows from [18, Theorem 5.2.1]. $\square$

3. Exponential decay of $E_r(A, \phi)$

This section proves a weak version of Theorem 1.4, which will be stated as Proposition 3.3.

Let X be a Riemannian 4-manifold with cylindrical and ESBG ends, and suppose the cylindrical end is M_c and the ESBG end is $(M_s, \omega = d\theta)$. Let $\mathfrak{s}$ be an admissible spin^c structure on X, let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ be the spinor bundle of $\mathfrak{s}$, let Φ_0 be the canonical section of $\mathbb{S}^+|M_s$, and let A_0 be the spin^c connection on $\mathfrak{s}|_{M_s}$ such that $D_{A_0}\Phi_0 = 0$.

Let $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ be a solution to (2.4). By the standard elliptic regularity arguments, (A, ϕ) is locally C^∞ on X after suitable gauge transformations. Since the perturbation $\mathfrak{q}$ in (2.3) is assumed to be admissible, it follows from [18, Proposition 13.4.1] that Condition (3) of Defintion 2.3 implies the C^0 convergence of (A, ϕ) on M_c after gauge transformations. As a consequence,

$$\|\phi\|_{C^0(M_c)} < +\infty. \tag{3.1}$$

Let $\mathrm{inj}(X)$ be the injectivity radius of X. By Definition 1.1, $\mathrm{inj}(X) > 0$. Let

$$\epsilon_0 = \min\{\frac{\mathrm{inj}(X)}{2}, 1\}. \tag{3.2}$$

The following convention will be adopted for the rest of this article unless otherwise stated: the notations z or z_i will denote positive real numbers that only depend on X, M_s, M_c, θ, and the terms $\hat{\tau}$ and η in (2.3). The notation r_0 will denote a positive real number that depends only on the same set of data, and we will assume that $r > r_0$ for the constant r in (2.2). The value of r_0 may increase as the proof proceeds.

3.1. C^0 bound

This subsection proves the following C^0 estimate. Recall that A_0 is the canonical connection on $\mathfrak{s}|_{M_s}$, and let $a = A|_{M_s} - A_0$

Proposition 3.1. *There exist constants z, r_0, such that for all $r > r_0$ and $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ satisfying (2.4), we have $\|\phi\|_{C^0(X)} \le z \cdot \sqrt{r}$, and $\|F_a^+\|_{C^0(X)} \le z \cdot r$.*

The proof starts with the following C^0 estimate on M_s, which is adapted from [17, Lemma 3.23]. Recall that ϵ_0 is the constant defined by (3.2).

Lemma 3.2. *Let $N(\partial M_s)$ be the ϵ_0-neighborhood of ∂M_s. There exist constants z, r_0, such that if $r > r_0$ and $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ solves (2.4), we have*

$$\|\phi\|_{C^0(M_s - N(\partial M_s))} \le z \cdot \sqrt{r}.$$

Proof. By (2.2) of [27], the following inequality holds on M_s for a constant z_1:

$$\frac{1}{2} d^* d|\phi|^2 + |\nabla_A \phi|^2 + \frac{1}{4}|\phi|^2(|\phi|^2 - r) - z_1 \cdot |\phi|^2 \le 0.$$

Take $r_0 > 4z_1$. For $r > r_0$, we have

$$\frac{1}{2} d^* d|\phi|^2 + \frac{1}{4}|\phi|^2(|\phi|^2 - 2r) \le 0. \tag{3.3}$$

For $x \in M_s - N(\partial M_s)$, let ρ be the distance function to x on $B_x(\epsilon_0)$. Let f be the function on the interior of $B_x(\epsilon_0)$ defined by $f = 1/(\epsilon_0^2 - \rho^2)^2$. Since ϵ_0 is less than the injectivity radius of X, let $(g_{ij})_{1 \le i,j \le 4}$ be the metric matrix of the normal coordinates of $B_x(\epsilon_0)$ centered at x, and let g be the determinant of $(g_{ij})_{1\le i,j\le 4}$. We have

$$d^*df = -\frac{1}{\rho^{n-1}\sqrt{g}}\frac{\partial}{\partial\rho}\big(\rho^{n-1}\sqrt{g}\cdot\frac{\partial f}{\partial\rho}\big).$$

Notice that since X has bounded geometry, $\|g\|_{C^0}$ and $\|\nabla g\|_{C^0}$ are both bounded by constants independent of x, and g is bounded away from 0 on $B_x(\epsilon_0)$. A straightforward calculation shows that for some large constant $z_2>0$,

$$\frac{1}{2}d^*d\big((z_2)^2rf\big)+\frac{1}{4}\big((z_2)^2rf\big)\big((z_2)^2rf-2r\big)\ge 0. \tag{3.4}$$

Let x' be a point in $B_x(\epsilon_0)$ where the function $(z_2)^2rf-|\phi|^2$ achieves the minimum value. Since the limit of f on $\partial B_x(\epsilon_0)$ is $+\infty$, such a point x' exists in the interior of $B_x(\epsilon_0)$. By (3.3) and (3.4), we have

$$|\phi(x')|^2(|\phi(x')|^2-2r)\le\big((z_2)^2rf(x')\big)\big((z_2)^2rf(x')-2r\big),$$

therefore $|\phi(x')|^2\le\max\{2r,(z_2)^2rf(x')\}\le 2r+(z_2)^2rf(x')$. This implies $|\phi|^2\le 2r+(z_2)^2rf$, hence $|\phi(x)|\le z\cdot\sqrt{r}$ for $z=\sqrt{2+(z_2)^2}$. □

Now we prove Proposition 3.1.

Proof of Proposition 3.1. Let r_0 be the constant given by Lemma 3.2. Increase the value of r_0 if necessary such that $r_0\ge 1$, and assume $r>r_0$.

By (3.1) and Lemma 3.2, we have $\sup_X|\phi|<+\infty$. Let $x_0\in X$ be a point such that $|\phi(x_0)|\ge\frac{1}{2}\sup_X|\phi|$. Let $\epsilon<\epsilon_0$ be a positive constant that will be determined later. Notice that (A,ϕ) satisfies the following equations on $B_{x_0}(\epsilon)$:

$$\rho(F^+_{A^t})=(\phi\,\phi^*)_0+\hat{\mu}^0(A,\phi), \tag{3.5}$$
$$D_A\phi=\hat{\mu}^1(A,\phi), \tag{3.6}$$

where $\hat{\mu}$ is given by (2.3). Since the perturbation $\mathfrak{q}$ is strongly tame and $|\phi(x_0)|\ge\frac{1}{2}\sup_X|\phi|$, there exists a constant z_0 such that the following holds on $B_{x_0}(\epsilon)$:

$$\|\hat{\mu}^0(A,\phi)\|_{C^0} \le z_0\,(1+|\phi(x_0)|) + z_0\, r, \tag{3.7}$$

$$\|\hat{\mu}^1(A,\phi)\|_{C^0} \le z_0. \tag{3.8}$$

In the following, we will write $\hat{\mu}^0(A,\phi)$ as $\hat{\mu}^0$, and $\hat{\mu}^1(A,\phi)$ as $\hat{\mu}^1$. Applying D_A to both sides of (3.6) yields

$$D_A^2\phi = D_A(\hat{\mu}^1).$$

By the Weitzenböck formula, this implies

$$\nabla_A^*\nabla_A\phi + \frac{1}{2}\rho(F_{A^t}^+)\phi + \frac{1}{4}s\phi = D_A(\hat{\mu}^1), \tag{3.9}$$

where s is the scalar curvature of X. Plug in (3.5) to (3.9), and take the inner product with ϕ, we obtain

$$\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{4}|\phi|^4 + \frac{1}{4}\langle s\phi,\phi\rangle + \frac{1}{2}\langle\hat{\mu}^0\phi,\phi\rangle = \langle D_A(\hat{\mu}^1),\phi\rangle.$$

Recall that $r > r_0 \ge 1$, hence by (3.7), there exists a constant z_1 such that

$$\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{4}|\phi|^4 \le \langle D_A(\hat{\mu}^1),\phi\rangle + z_1\, r|\phi|^2 + z_1|\phi(x_0)|\,|\phi|^2.$$

By the arithmetic-geometric mean inequality,

$$-\frac{1}{16}|\phi|^4 - 4\,z_1^2\,r^2 \le -z_1\,r\,|\phi|^2,$$

$$-\frac{1}{16}|\phi|^4 - 4\,z_1^2\,|\phi(x_0)|^2 \le -z_1\,|\phi(x_0)|\,|\phi|^2.$$

Adding the above three inequalities, we obtain

$$\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{8}|\phi|^4 - 4z_1^2(\,r^2 + |\phi(x_0)|^2) \le \langle D_A(\hat{\mu}^1),\phi\rangle.$$

Let $h \ge 0$ be a smooth function on $B_{x_0}(\epsilon)$ such that $h = 1$ on $B_{x_0}(\epsilon/4)$ and $\operatorname{supp} h \subset B_{x_0}(\epsilon/2)$. Let $\chi = h^4$. Let $G_{x_0} \ge 0$ be the Green's function on

$B_{x_0}(\epsilon)$ that has a pole at x_0 and equals zero on $\partial B_{x_0}(\epsilon)$. Then:

$$\int_{B_{x_0}(\epsilon)} \Big(\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{8}|\phi|^4 - 4z_1^2(\,r^2 + |\phi(x_0)|^2)\Big)\cdot G_{x_0}\cdot\chi \\ \leq \int_{B_{x_0}(\epsilon)} \langle D_A(\hat{\mu}^1), \phi\cdot G_{x_0}\chi\rangle,$$

which gives

$$\begin{aligned}(3.10)\quad \int_{B_{x_0}(\epsilon)} \Big(\big(-\frac{1}{2}\Delta(|\phi|^2\chi) + \frac{1}{2}|\phi|^2\Delta\chi + \nabla|\phi|^2\cdot\nabla\chi\big)\cdot G_{x_0} \\ + |\nabla_A\phi|^2\, G_{x_0}\,\chi + \frac{1}{8}|\phi|^4\, G_{x_0}\,\chi - 4z_1^2(\,r^2 + |\phi(x_0)|^2)\, G_{x_0}\,\chi\Big) \\ \leq \int_{B_{x_0}(\epsilon)} \langle\hat{\mu}^1, D_A(\phi\, G_{x_0}\,\chi)\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}(3.11)\quad \frac{1}{2}|\phi(x_0)|^2 \leq \int_{B_{x_0}(\epsilon)} \Big((-\frac{1}{2}|\phi|^2\Delta\chi - \nabla|\phi|^2\cdot\nabla\chi)\cdot G_{x_0} - |\nabla_A\phi|^2\, G_{x_0}\,\chi \\ - \frac{1}{8}|\phi|^4\, G_{x_0}\,\chi + 4z_1^2(\,r^2 + |\phi(x_0)|^2)\, G_{x_0}\,\chi + \langle\hat{\mu}^1, D_A(\phi\, G_{x_0}\,\chi)\rangle\Big).\end{aligned}$$

Recall that $\chi = h^4$, hence $|\Delta\chi| \leq 4h^3|\Delta h| + 12h^2|\nabla h|^2$. By the arithmetic-geometric mean inequality, there exists a constant z_2 such that

$$\begin{aligned}-\frac{1}{2}|\phi|^2\Delta\chi \leq& |\phi|^2(2h^3|\Delta h| + 6h^2|\nabla h|^2) \\ \leq& z_2(|\Delta h|^2h^2 + |\nabla h|^4) + \frac{1}{16}|\phi|^4\, h^4 \\ =& z_2(|\Delta h|^2h^2 + |\nabla h|^4) + \frac{1}{16}|\phi|^4\, \chi.\end{aligned}$$

Similarly, there exists a constant z_3 such that

$$\begin{aligned}|\nabla|\phi|^2|\cdot|\nabla\chi| \leq& 2|\phi|\cdot|\nabla_A\phi|\cdot(4h^3|\nabla h|) \\ \leq& 32\,|\phi|^2h^2|\nabla h|^2 + \frac{1}{2}|\nabla_A\phi|^2h^4 \\ \leq& z_3|\nabla h|^4 + \frac{1}{16}|\phi|^4\,h^4 + \frac{1}{2}|\nabla_A\phi|^2h^4 \\ =& z_3|\nabla h|^4 + \frac{1}{2}|\nabla_A\phi|^2\chi + \frac{1}{16}|\phi|^4\,\chi.\end{aligned}$$

By (3.8), there exists a constant z_4 such that

$$\begin{aligned}\int_{B_{x_0}(\epsilon)} \langle \hat\mu^1, D_A\phi\rangle\, G_{x_0}\,\chi &\le \int_{B_{x_0}(\epsilon)} z_0\,|D_A\phi|\,G_{x_0}\,\chi\\ &\le \int_{B_{x_0}(\epsilon)} \frac{1}{2}|\nabla_A\phi|^2 G_{x_0}\chi + z_4\int_{B_{x_0}(\epsilon)} G_{x_0}\,\chi.\end{aligned}$$

Therefore (3.11) and the three estimates above yield

$$\begin{aligned}\frac{1}{2}|\phi(x_0)|^2 \le &\int_{B_{x_0}(\epsilon)} \Big(z_2(|\Delta h|^2h^2 + |\nabla h|^4) + z_3|\nabla h|^4\Big)\,G_{x_0} + z_4\int_{B_{x_0}(\epsilon)} G_{x_0}\,\chi\\ &+ \int_{B_{x_0}(\epsilon)} |\hat\mu^1||\phi||\nabla(G_{x_0}\,\chi)| + 4z_1^2(\,r^2 + |\phi(x_0)|^2)\int_{B_{x_0}(\epsilon)} G_{x_0}\,\chi.\end{aligned} \tag{3.12}$$

Recall that $|\phi(x_0)| \ge \frac{1}{2}\sup_X|\phi|$, therefore by (3.8),

$$\begin{aligned}\int_{B_{x_0}(\epsilon)} |\hat\mu^1||\phi||\nabla(G_{x_0}\,\chi)| &\le 2\,|\phi(x_0)|\int_{B_{x_0}(\epsilon)} |\hat\mu^1||\nabla(G_{x_0}\,\chi)|\\ &\le \frac{1}{8}|\phi(x_0)|^2 + 8\left(\int_{B_{x_0}(\epsilon)} |\hat\mu^1||\nabla(G_{x_0}\,\chi)|\right)^2\\ &\le \frac{1}{8}|\phi(x_0)|^2 + 8\,z_0^2\left(\int_{B_{x_0}(\epsilon)} |\nabla(G_{x_0}\,\chi)|\right)^2.\end{aligned} \tag{3.13}$$

Notice that the constants z_i do not depend on the choice of ϵ, and there exist constants z_5, z_6 such that

$$z_5\,\epsilon^2 \le \int_{B_{x_0}(\epsilon)} G_{x_0} \le z_6\,\epsilon^2.$$

Take $\epsilon = 1/(z_7\sqrt{r})$, with z_7 sufficiently large such that

$$\int_{B_{x_0}(\epsilon)} G_{x_0} \le z_6\,\epsilon^2 \le \min\left\{\frac{1}{r}, \frac{1}{32\,z_1^2}\right\}. \tag{3.14}$$

Plug in (3.13) and (3.14) to (3.12), and rearrange, we have

$$\begin{aligned}\frac{1}{4}|\phi(x_0)|^2 \le &\int_{B_{x_0}(\epsilon)} \Big(z_2(|\Delta h|^2h^2 + |\nabla h|^4) + z_3|\nabla h|^4\Big)\,G_{x_0}\\ &+ z_4\int_{B_{x_0}(\epsilon)} G_{x_0}\,\chi + 8\,z_0^2\left(\int_{B_{x_0}(\epsilon)} |\nabla(G_{x_0}\,\chi)|\right)^2 + 4\,z_1^2\,r^2\int_{B_{x_0}(\epsilon)} G_{x_0}\,\chi.\end{aligned}$$

Since $\epsilon = 1/(z_7\sqrt{r})$, one can choose the function h such that the right-hand side of the above inequality is bounded by $z_8 \cdot r$ for some constant z_8, hence the estimate on $\|\phi\|_{C^0(X)}$ is proved. The upper bound on $\|F_a^+\|_{C^0(X)}$ then follows from (3.5) and (3.7). $\square$

3.2. Exponential decay on M_s

Recall that the spinor $\phi|_{M_s}$ decomposes as $\phi = \sqrt{r}(\alpha+\beta)$, with $\alpha \in \Gamma(M_s, T^{0,0}M_s)$, $\beta \in \Gamma(M_s, T^{0,2}M_s)$. The spinor bundle $\mathbb{S}^+$ has a canonical section Φ_0 on M_s given by $1 \in \Gamma(M_s, T^{0,0}M_s)$, and there is a unique spin^c connection A_0 on $\mathfrak{s}|_{M_s}$ such that $D_{A_0}\Phi_0 = 0$. Take $a = A|_{M_s} - A_0$, and take ∇'_A to be the projection of $\nabla_A|_{M_s}$ to $T^{0,2}M_s$. The energy density function $E_r(A,\phi)$ is defined on M_s by (1.2). If $(A,\phi) \in \mathcal{C}_k(X,\mathfrak{s})$, then we have

$$\int_{M_s} E_r(A,\phi) < +\infty.$$

Recall that the function d on M_s is defined as follows. For each connected component $M_s^{(k)}$ of M_s, if $\partial M_s^{(k)}$ is nonempty, then d is the distance function to $\partial M_s^{(k)}$ on $M_s^{(k)}$. Otherwise, fix a point $x^{(k)} \in M_s^{(k)}$, and d is the distance function to $x^{(k)}$ on $M_s^{(k)}$. The main result of this section is the following proposition.

Proposition 3.3. *There exist constants z, z', r_0 such that the following holds. Suppose $r > r_0$ and $(A,\phi) \in \mathcal{C}_k(X,\mathfrak{s})$ solves (2.4), then there is a constant d_0, which may depend on r and (A,ϕ), such that*

$$E_r(A,\phi)(x) < z e^{-\sqrt{r}\cdot(d(x)-d_0)/z'} \tag{3.15}$$

for every $x \in M_s$ with $d(x) > d_0$.

We start the proof with the following lemma, which is adapted from [17, Lemma 3.21].

Lemma 3.4. *Let (A,ϕ) be as in Proposition 3.3, then given $\delta > 0$, there exists $d(\delta) > 0$ depending on (A,ϕ), r and δ, such that for all $x \in M_s$ with $d(x) > d(\delta)$, we have*

$$E_r(A,\phi)(x) < \delta.$$

Proof. Assume the contrary, then there is a sequence $\{x_n\}_{n\geq 1} \subset M_s$ and a constant $\delta > 0$, such that $d(x_n) \to +\infty$ and $E_r(A,\phi)(x_n) \geq \delta$ for all n.

Let $\epsilon_0 > 0$ be given by (3.2). After taking a subsequence of $\{x_n\}$ if necessary, we may assume that the balls $B_{x_n}(\epsilon_0)$ are pairwise disjoint and are all included in M_s. Let g be the metric of X. Consider the sequence $(M_s, g, x_n, \mathfrak{s}, A, \phi)$. By Proposition 2.9 and Lemma 3.2, a subsequence converges to a limit $(\widetilde{M_s}, \tilde{g}, \tilde{x}, \tilde{\mathfrak{s}}, \tilde{A}, \tilde{\phi})$. Recall that $\nabla^k\omega$ is bounded for all k. By a diagonal argument and the Arzelà-Ascoli theorem, after taking a further subsequence, the symplectic form ω converges to a limit symplectic form $\tilde{\omega}$ on $\tilde{M}_s$, in the C^∞ topology on compact subsets. The symplectic form $\tilde{\omega}$ is compatible with $\tilde{g}$, and hence it defines an energy density function $\widetilde{E_r}(\tilde{A}, \tilde{\phi})$ on $\tilde{M}_s$. By the assumptions on x_n, we have $\widetilde{E_r}(\tilde{A}, \tilde{\phi})(\tilde{x}) \geq \delta$, thus

$$\int_{B_{\tilde{x}}(\epsilon_0)} \tilde{E}_r(\tilde{A}, \tilde{\phi}) > 0.$$

Therefore, there exists a positive constant $\delta' > 0$, such that

$$\int_{B_{x_n}(\epsilon_0)} E_r(A, \phi) > \delta'$$

for sufficiently large n. This contradicts the assumption that

$$\int_{M_s} E_r(A, \phi) < +\infty. \qquad \square$$

The following lemma is an extension of [17, Lemma 3.24]. Recall that for a point p in a complete Riemannian manifold M, we use $B_p(r)$ to denote the set of points in M whose distance to p is no greater than r, and r is allowed to be greater than the injectivity radius of M at p.

Lemma 3.5. *Let $K, v_0, R > 0$, $r \geq 1$ be constants. Let M be an n-dimensional complete Riemannian manifold with $\operatorname{Ric} \geq -K$, let $x_0 \in M$. Let s be a C^2 function on $B_{x_0}(R)$. Suppose s satisfies:*

$$\frac{1}{2} d^* d\, s + rV s \leq h,$$

where h, V are C^0 functions, and $V \geq v_0$ on $B_{x_0}(R)$. Then there exists a positive constant ϵ depending only on n, K, R, and v_0, such that the following inequality holds:

$$s(x_0) \leq \Big(\sup_{B_{x_0}(R)} \Big|\frac{h}{rV}\Big|\Big) + \Big(\sup_{\partial B_{x_0}(R)} |s| \Big) e^{-\epsilon R \sqrt{r}}.$$

If $\partial B_{x_0}(R) = \varnothing$, then $\sup_{\partial B_{x_0}(R)} |s|$ in the above inequality is defined to be 0.

Proof. Let ρ be the distance function to x_0, let $k = \sqrt{K/(n-1)}$. By the distributional Laplacian comparison theorem, the following inequality holds on X in the sense of distributions:

$$\Delta\rho \le \frac{n-1}{\rho}(1+k\,\rho). \tag{3.16}$$

In other words, for every non-negative function $\varphi \in C_0^\infty(X)$, we have

$$\int_X \rho\,\Delta\varphi \le \int \frac{n-1}{\rho}(1+k\,\rho)\varphi.$$

The reader may refer to [22, Corollary 2.12] for the proof of (3.16) (see also [15] and [6, Theorem 3]).

Let $f(u)$ be a smooth, non-decreasing function on $\mathbb{R}$ such that $f(u)=0$ when $u \le R/4$ and $f(u)=u$ when $u \ge R/2$. Let $g = e^{\epsilon\sqrt{r}f(\rho)}$ be a function on M, where ϵ is a small positive constant that will be determined later. Notice that in the sense of distributions,

$$\begin{aligned} d^*dg = -\Delta g &= -(\epsilon\sqrt{r}f''(\rho) + \epsilon^2 r(f'(\rho))^2 + \epsilon\sqrt{r}f'(\rho)\Delta\rho)\,g \\ &\ge -\Big(\epsilon\sqrt{r}f''(\rho) + \epsilon^2 r(f'(\rho))^2 + \epsilon\sqrt{r}f'(\rho)\big(\frac{n-1}{\rho} + k(n-1)\big)\Big)\,g. \end{aligned}$$

Therefore, there exists a constant ϵ depending only on n, K, R, and v_0, such that

$$\frac{1}{2}d^*dg + rVg \ge 0$$

in the sense of distributrions. Let

$$\tilde g = \sup_{B_{x_0}(R)}\Big|\frac{h}{rV}\Big| + \Big(\sup_{\partial B_{x_0}(R)}|s|\Big)\cdot g/e^{\epsilon\sqrt{r}\,R},$$

then $\frac{1}{2}d^*ds + rVs \le \frac{1}{2}d^*d\tilde g + rV\tilde g$ in the sense of distributions, and $s|_{\partial B_{x_0}(R)} \le \tilde g|_{\partial B_{x_0}(R)}$. By the maximum principle for weak solutions [13, Theorem 8.1], we have $s \le g$ on the ball $B_{x_0}(R)$, hence the lemma is proved. □

Proof of Proposition 3.3. Recall that we use the notation z_i to denote constants that only depend on X, M_s, M_c, θ, and the terms $\hat\tau$ and η in (2.3). In the following we will require $r_0 \ge 1$. The proof follows the strategy of [17, Section 3], and is divided into 7 steps:

Step 1. By Lemma 3.4, there exists $d_1 > 0$ depending on (A, ϕ) and r, such that if $x \in M_s$ satisfies $d(x) > d_1$ then

$$|\alpha(x)| > \frac{1}{2}, \quad E_r(A,\phi)(x) < 1. \tag{3.17}$$

Step 2: Pointwise estimates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. By [27, Lemma 2.2], there exist constants z_1, z_2, $z_3 \geq 1$, such that if $\zeta \in (0, \frac{r}{2z_1 z_2})$, $r > z_1$, and $\delta > z_3$, let

$$u = (1 - |\alpha|^2) - \zeta|\beta|^2 + \frac{\delta}{\zeta r}, \tag{3.18}$$

then the following inequality holds:

$$\frac{1}{2}d^* du + \frac{r}{4}|\alpha|^2 u \geq 0.$$

Notice that the C^0 norm of u is bounded by (3.17), hence by Lemma 3.5 and (3.17), there are constants z_5, z_6 such that

$$u \geq -z_5 e^{-\sqrt{r}\cdot(d-d_1)/z_6}, \tag{3.19}$$

on $\{x \in M_s | d(x) > d_1 + 1\}$. Therefore there exists a constant z_7 such that

$$|\alpha|^2 \leq 1 + \frac{z_7}{r^2}, \tag{3.20}$$

$$|\beta|^2 \leq \frac{z_7}{r}\big(1 - |\alpha|^2 + \frac{z_7}{r^2}\big), \tag{3.21}$$

on $\{x \in M_s | d(x) > d_1 + 1\}$.

Step 3: Pointwise estimates of $\boldsymbol{F_a}$. On M_s, the curvature part of (2.4) can be rewritten as (cf. [16, (8),(9)])

$$F_a^+ = -\frac{i}{8}r \cdot \big(1 - |\alpha|^2 + |\beta|^2)\omega + \frac{r}{4}(\alpha^*\beta - \alpha\beta^*). \tag{3.22}$$

By (3.20) and (3.21), there exists a constant z_{11} such that

$$|F_a^+| \leq \frac{r}{4\sqrt{2}}(1 + \frac{z_{11}}{r})(1 - |\alpha|^2) + z_{11}.$$

Now we estimate $|F_a^-|$. By [27, Lemma 2.5], there exist constants z_{12}, z_{13}, z_{14}, z_{15} such that if $r > z_{15}$, then for

$$\begin{aligned} q_0 &= \frac{r}{4\sqrt{2}}(1 + \frac{z_{12}}{r})(1 - |\alpha|^2) - z_{13} \cdot r|\beta|^2 + z_{14}, \\ s &= |F_a^-|, \end{aligned}$$

we have
$$\frac{1}{2}d^*d(s-q_0)+\frac{r}{4}|\alpha|^2(s-q_0)\le |\mathcal{R}|s,$$
where $\mathcal{R}$ is a curvature term that is uniformly bounded on M_s.

Therefore, if $r>8\sup|\mathcal{R}|$, we have
$$\frac{1}{2}d^*d(s-q_0)+\frac{r}{8}|\alpha|^2(s-q_0)\le |\mathcal{R}|\cdot|q_0|.$$
By Lemma 3.5 and (3.17), there exists a constant z_{16} such that on $\{x\in M_s|d(x)>d_1+1\}$, we have
$$|F_a^-|\le \frac{r}{4\sqrt{2}}(1+\frac{z_{16}}{r})(1-|\alpha|^2)+z_{16}.$$
In conclusion, there is a constant z_{17} such that on $\{x\in M_s|d(x)>d_1+1\}$,
$$|F_a^\pm|\le \frac{r}{4\sqrt{2}}(1+\frac{z_{17}}{r})(1-|\alpha|^2)+z_{17}.$$
Step 4: Pointwise estimates of $|\nabla_a\alpha|$ and $|\nabla'_A\beta|$. Let
$$y=|\nabla_a\alpha|^2+r|\nabla'_A\beta|^2.$$
Recall that the function u is defined by (3.18). By [27, (2.43)], there exists a constant z_{18} such that
$$\frac{1}{2}d^*d(y-z_{18}\cdot r\cdot u)+\frac{r}{4}|\alpha|^2(y-z_{18}\cdot r\cdot u)\le 0.$$
By (3.20), (3.21), and Lemma 3.5, therer exists a constant z_{19} such that
$$|\nabla_a\alpha|^2+r|\nabla'_A\beta|^2=y\le z_{19}\cdot r\cdot(1-|\alpha|^2)+z_{19}.$$
Step 5: Exponential decay of $|\nabla_a\alpha|$, $|\nabla'_A\beta|$, and $|\beta|$. Let
$$y_1=|\nabla_a\alpha|^2+\frac{r}{32}|\nabla'_A\beta|^2+\frac{r^2}{16\,z_{20}}|\beta|^2.$$
By [27, (4.15)], one can choose z_{20} sufficiently large, such that there exists a constant z_{21}, such that[1]
$$\frac{1}{2}d^*dy_1+\frac{r}{4}|\alpha|^2y_1\le \big(z_{21}\cdot r\cdot(1-|\alpha|^2)+\frac{r}{8}\big)y_1. \tag{3.23}$$

[1] The derivation of [27, (4.15)] only used the pointwise estimates of α, β, F_a, $\nabla_a\alpha$ and $\nabla'_A\beta$ from [27, Section 2], and it does not depend on the refined pointwise estimate of F_a^- developed in [27, Section 3d]. Therefore, the inequalities obtained from Step 2 to Step 4 are sufficient for deriving (3.23).

By Lemma 3.4, there exists a constant d_2 such that on $\{x \in M_s | d(x) > d_2\}$,

$$|1 - |\alpha|^2| < \min\Big\{\frac{1}{16\, z_{21}}, \frac{1}{8}\Big\}.$$

Then (3.23) implies that on $\{x \in M_s | d(x) > d_2\}$,

$$\frac{1}{2} d^* dy_1 + \frac{r}{32} y_1 \leq 0.$$

By Lemma 3.5, there are constants z_{22}, z_{23} such that on $\{x \in M_s | d(x) > d_2 + 1\}$,

$$y_1 < z_{22} \cdot e^{\sqrt{r}\cdot(d-d_2)/z_{23}}, \tag{3.24}$$

Step 6: Exponential decay of $|1 - |\alpha|^2|$. By [27, (2.3)],

$$\frac{1}{2} d^* d|\alpha|^2 + |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^2(|\alpha|^2 - 1 + |\beta|^2) + \alpha \boxtimes \nabla'_A \beta + \alpha \boxtimes \beta = 0,$$

where $\boxtimes$ are pointwise bilinear operators defined by the metric and the symplectic form. A straight forward calculation shows

$$\begin{aligned}
\frac{1}{4} d^* d|1 - |\alpha|^2|^2 =& \big(\frac{1}{2} d^* d(1 - |\alpha|^2)\big) \cdot (1 - |\alpha|^2) - \frac{1}{2}|\nabla_a |\alpha|^2|^2 \\
=& -\frac{r}{4}|\alpha|^2 |1 - |\alpha|^2|^2 + |\nabla_a \alpha|^2 \cdot (1 - |\alpha|^2) \\
&+ \frac{r}{4}|\alpha|^2 |\beta|^2 (1 - |\alpha|^2) + (1 - |\alpha|^2) \cdot (\alpha \boxtimes \nabla'_A \beta + \alpha \boxtimes \beta).
\end{aligned}$$

The equation above and (3.24) imply there are constants z_{24}, z_{25}, such that on $\{x \in M_s | d(x) > d_2 + 1\}$,

$$\frac{1}{4} d^* d|1 - |\alpha|^2|^2 + \frac{r}{4}|\alpha|^2 |1 - |\alpha|^2|^2 \leq z_{24} \cdot e^{(d-d_2)\sqrt{r}/z_{25}}.$$

By Lemma 3.5, there exist constants z_{26}, z_{27} such that

$$|1 - |\alpha|^2|^2 < z_{26} \cdot e^{(d-d_2)\sqrt{r}/z_{27}}, \tag{3.25}$$

on $\{x \in M_s | d(x) > d_2 + 1\}$.

Step 7: Exponential decay of $|F_a|$. The exponential decay for $|F_a^+|$ follows from (3.22), (3.24) and (3.25). Recall that $s = |F_a^-|$. By [27, (2.19)], there exists a constant z_{28} such that

$$\frac{1}{2}d^*ds + \frac{r}{4}(|\alpha|^2 + |\beta|^2)s \le |\mathcal{R}|s + \frac{r}{4\sqrt{2}}(|\nabla_a\alpha|^2 + |\nabla'_A\beta|^2) + z_{28}\cdot r(|\alpha||\beta| + |\alpha||\nabla'_A\beta| + |\beta||\nabla_a\alpha| + |\beta|^2).$$

Therefore (3.24) shows that there exist constants z_{29}, z_{30}, such that if $|\alpha| > 7/8$, $r > 16\sup|\mathcal{R}|$, and $d(x) > d_2 + 1$, then

$$\frac{1}{2}d^*ds + \frac{r}{8}|\alpha|^2 s \le z_{29}\cdot e^{(d-d_2)\sqrt{r}/z_{30}},$$

By Lemma 3.5, this implies there are constants z_{31}, z_{32}, and a positive real number d_3 which may depend on (A,ϕ), such that

$$s < z_{31}\cdot e^{(d-d_3)\sqrt{r}/z_{32}} \tag{3.26}$$

on $\{x\in M_s | d(x) > d_3\}$.

The proposition then follows from (3.24), (3.25), and (3.26). □

4. Uniform exponential decay of $E_r(A,\phi)$

This section shows that the constant d_0 in Proposition 3.3 can be chosen to depend only on r, not on the solution (A,ϕ). Let $X, Z, M_s, M_c, \mathfrak{s}, \mathbb{S}, \omega, \theta, A$, ϕ, A_0 be as in Section 3. Recall that z_i denotes constants that only depend on X, M_s, M_c, θ, and the terms $\hat{\tau}$ and η in (2.3). The constant r_0 is a positive real number that depends on the same set of data, and the value of r_0 may increase as the proof proceeds. We will require $r > r_0$ in (2.2).

4.1. An energy identity on M_s

Recall that $\phi|_{M_s}$ decomposes as $\phi = \sqrt{r}(\alpha+\beta)$, and $a = A|_{M_s} - A_0$. For $F \in \Lambda^2T^*M_s\otimes\mathbb{C}$, define $F^\omega = \frac{1}{2}\langle\omega, F\rangle \in \mathbb{C}$. The following lemma is a rescaled version of [17, Equation (18)].

Lemma 4.1. *Let χ be a smooth cut-off function on M_s such that* $\operatorname{supp}\chi$ *is contained in the interior of M_s, and $\chi = 1$ for all $x\in M_s$ with $d(x) > 1$.*

Then we have

$$
\begin{aligned}
(4.1)\quad & \int_{M_s} \Big(\frac{r}{2}|\bar\partial_a(\chi\alpha)+\bar\partial_a^*(\chi\beta)|^2+2|iF_a^\omega-\frac{r}{8}(1-|\chi\alpha|^2+|\chi\beta|^2)|^2+ \\
& 2|F_a^{0,2}-\frac{r}{4}(\chi\alpha)^*(\chi\beta)|^2+\frac{r}{2}iF_a^\omega-2|iF_a^\omega|^2-2|F_a^{0,2}|^2\Big) \\
& =\int_{M_s}\Big(\frac{r}{4}|\nabla_a(\chi\alpha)|^2+\frac{r}{4}|\nabla_{A_1+a}(\chi\beta)|^2+\frac{r}{2}(iF_{A_1}^\omega)|\chi\beta|^2 \\
& +\frac{r^2}{32}(1-|\chi\alpha|^2-|\chi\beta|^2)^2+\frac{r^2}{8}|\chi\beta|^2-rRe\langle N\circ\partial_a(\chi\alpha),\chi\beta\rangle\Big).
\end{aligned}
$$

Where A_1 is the unique unitary connection on $T^{0,2}M_s$ such that $\nabla_{A_1}^{1,0}=\partial$, and $N:T^{1,0}M_s\to T^{0,2}M_s$ is the Nijenhuis tensor.

Remark 4.2. If $\partial M_s=\emptyset$, we may take $\chi=1$ on M_s.

Proof. The identity follows from and integration by parts Weitzenböck formulas.

For a constant $d_0>1$, let χ_{d_0} be a smooth function on M_s such that $\chi_{d_0}=1$ for all x with $d(x)\le d_0$, and $\chi_{d_0}=0$ for all x with $d(x)\ge d_0+2$, and $|\nabla\chi_{d_0}|\le 1$, $|\chi_{d_0}|\le 1$. Then integration by parts yields

$$\int_{M_s}\langle\chi_{d_0}\bar\partial_a(\chi\alpha),\bar\partial_a^*(\chi\beta)\rangle=\int_{M_s}\langle\bar\partial_a\big(\chi_{d_0}\bar\partial_a(\chi\alpha)\big),\chi\beta\rangle.$$

On the other hand, there exists a constant z_1 such that

$$
\begin{aligned}
&\left|\int_{M_s}\langle\chi_{d_0}\bar\partial_a(\chi\alpha),\bar\partial_a^*(\chi\beta)\rangle-\int_{\{x\in M_s|d(x)\le d_0+2\}}\langle\bar\partial_a(\chi\alpha),\bar\partial_a^*(\chi\beta)\rangle\right| \\
\le&\int_{\{x\in M_s|d_0\le d(x)\le d_0+2\}}|\bar\partial_a(\chi\alpha)|\cdot|\bar\partial_a^*(\chi\beta)| \\
\le& z_1\int_{\{x\in M_s|d_0\le d(x)\le d_0+2\}}E_r(A,\phi),
\end{aligned}
$$

and a constant z_2 such that

$$
\begin{aligned}
&\left|\int_{M_s}\langle\bar\partial_a\big(\chi_{d_0}\bar\partial_a(\chi\alpha)\big),\chi\beta\rangle-\int_{\{x\in M_s|d(x)\le d_0+2\}}\langle\bar\partial_a\bar\partial_a(\chi\alpha),\chi\beta\rangle\right|\\
\le&\int_{\{x\in M_s|d_0\le d(x)\le d_0+2\}}|\nabla\chi|\cdot|\bar\partial_a(\chi\alpha)|\cdot|\chi\beta|+\langle\bar\partial_a\bar\partial_a(\chi\alpha),\chi\beta\rangle\\
\le& z_2\int_{\{x\in M_s|d_0\le d(x)\le d_0+2\}}E_r(A,\phi)+E_r(A,\phi)^{1/2}\cdot|\bar\partial_a\bar\partial_a(\chi\alpha)|.
\end{aligned}
$$

Let ϵ_0 be the constant defined by (3.2). By elliptic bootstrapping, there exist constants z_3, z_4 such that for all x with $d(x)>\epsilon_0$,

$$
|\bar\partial_a\bar\partial_a(\chi\alpha)(x)|\le z_3\Big(\int_{B_x(\epsilon_0)}E_r(A,\phi)+r+1\Big)^{z_4}.
$$

Let $d_0\to+\infty$, and suppose r_0 is sufficiently large. It then follows from Proposition 3.3, the Bishop-Gromov volume comparison theorem, and the estimates above that

$$
\int_{M_s}\langle\bar\partial_a(\chi\alpha),\bar\partial_a^*(\chi\beta)\rangle=\int_{M_s}\langle\bar\partial_a\bar\partial_a(\chi\alpha),\chi\beta\rangle. \tag{4.2}
$$

Similarly, for r_0 sufficiently large, we have the following identities:

$$
\begin{aligned}
&\int_{M_s}\langle\bar\partial_a(\chi\alpha),\bar\partial_a^*(\chi\beta)\rangle=\int_{M_s}\langle\bar\partial_a\bar\partial_a(\chi\alpha),\chi\beta\rangle,\\
&\int_{M_s}\langle\bar\partial_a(\chi\alpha),\bar\partial_a(\chi\alpha)\rangle=\int_{M_s}\langle\bar\partial_a^*\bar\partial_a(\chi\alpha),\chi\alpha\rangle,\\
&\int_{M_s}\langle\bar\partial_a^*(\chi\beta),\bar\partial_a^*(\chi\beta)\rangle=\int_{M_s}\langle\bar\partial_a\bar\partial_a^*(\chi\beta),\chi\beta\rangle,\\
&\int_{M_s}\langle\nabla_a(\chi\alpha),\nabla_a(\chi\alpha)\rangle=\int_{M_s}\langle\nabla_a^*\nabla_a(\chi\alpha),\chi\alpha\rangle,\\
&\int_{M_s}\langle\nabla_{A_1+a}(\chi\beta),\nabla_{A_1+a}(\chi\beta)\rangle=\int_{M_s}\langle\nabla_{A_1+a}^*\nabla_{A_1+a}(\chi\beta),\chi\beta\rangle.
\end{aligned}
$$

On the other hand, by the Weitzenböck formulas [16, (12), (13)],

$$
\begin{aligned}
\bar\partial_a^*\bar\partial_a(\chi\alpha)&=\frac{1}{2}(\nabla_a^*\nabla_a(\chi\alpha)-2iF_a^\omega(\chi\alpha)),\\
\bar\partial_a\bar\partial_a^*(\chi\beta)&=\frac{1}{2}(\nabla_{A_1+a}^*\nabla_{A_1+a}(\chi\beta)+2iF_{A_1+a}^\omega(\chi\beta)).
\end{aligned}
$$

The lemma is then proved by a straightforward computation using the identities above and $\bar\partial_a^2(\chi\alpha) = F_a^{0,2}(\chi\alpha) - N \circ \partial_a(\chi\alpha)$. $\square$

4.2. Uniform energy bound

Proposition 4.3. *There exists a constant r_0, such that for all $r > r_0$, there exists a constant C which may depend on r with the following property. For all $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ that solves (2.4), we have*

$$\int_{M_s} E_r(A, \phi) < C. \tag{4.3}$$

Remark 4.4. A similar energy estimate was proved by [17, Lemma 3.17] for AFAK ends. The last paragraph of the proof of [17, Lemma 3.17] claimed that

$$\int_{\partial K_3} a \wedge \omega$$

has a uniform bound without detailed explanation, and the detailed proof of this estimate was given by [24] after the proof of Lemma 2.2.7. However, the constant C in the argument of [24] depends on the volume of the complement of the AFAK end. If one applies the same argument from [24] to Proposition 4.3 above, then the constant C would be given by the volume of the complement of M_s, which is infinity when $M_c \neq \emptyset$. Therefore, the arguments in [17] and [24] do not suffice in the context of this article.

If M_c is non-empty, suppose $M_c = (-\infty, 0] \times Y$, let t be the function on X which is equal to the projection to $(-\infty, 0]$ on M_c, and is equal to zero on $M - M_c$. Let $\mathfrak{t}$ be the spin^c structure on Y induced by $\mathfrak{s}|_{M_c}$.

Suppose $\mathfrak{a} \in \mathcal{C}(Y, \mathfrak{t})$ is a critical point of the perturbed Chern-Simons-Dirac functional $\not{\mathcal{L}} = \mathcal{L} + \mathfrak{q}$ on $\mathcal{C}(Y, \mathfrak{t})$, let $\gamma_{\mathfrak{a}} = (A_{\mathfrak{a}}, \phi_{\mathfrak{a}}) \in \mathcal{C}([-1, 0] \times Y, \mathfrak{s}|_{[-1,0]\times Y})$ be the configuration on $[-1, 0] \times Y$ which is in temporal gauge and represents the constant path at $\mathfrak{a}$. Recall that in Section 2.4, the perturbation $\mathfrak{q}$ on M_c is required to satisfy $\|\mathfrak{q}\|_{\hat{\mathcal{P}}} \le 1$, where $\|\cdot\|_{\hat{\mathcal{P}}}$ is defined by (2.1). By [18, Section 10.7], there is a constant z_0 such that

$$\|F_{A_{\mathfrak{a}}^t}\|_{L^2}^2 < z_0 \tag{4.4}$$

for all critical points $\mathfrak{a}$.

Choose a gauge representative of (A, ϕ) that is in temporal gauge on the cylindrical end M_c. Recall that A_0 is the canonical spin^c connection on M_s. Extend A_0 to a smooth spin^c connection on $(X, \mathfrak{s})$, such that A_0 is in temporal gauge and is translation invariant on M_c. Let $a = A - A_0$, then $F_a = \frac{1}{2}(F_{A^t} - F_{A_0^t})$. By (4.4), there exists $R_0 > 1$ depending on (A, ϕ), such that

$$\int_{t(x)\in[-R_0-1,-R_0]} |F_a(x)|^2 < \frac{1}{2}\Big(z_0 + \int_{t(x)\in[-R_0-1,-R_0]} |F_{A_0^t}|^2 + 1\Big). \tag{4.5}$$

Lemma 4.5. *There are constants $z, r_0 > 0$ and a function $T : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ which depends on X, M_c, M_s, θ and the terms $\hat{\tau}$ and η in* (2.3)*, with the following property. Suppose $r > r_0$, and suppose there are constants $R > 0, \kappa > 0$ such that*

$$\int_{t(x)\in[-R-1,-R]} |F_a|^2 \le \kappa, \tag{4.6}$$

then the following inequalities hold:

$$\int_{M_s} E_r(A, \phi) < T(\kappa, r) + z\, r^2 \cdot R, \tag{4.7}$$

$$\int_{[-R,0]\times Y} |F_a|^2 < T(\kappa, r) + z\, r^2 \cdot R. \tag{4.8}$$

Proof. We use T_i to denote the constants that may depend on κ, r but are independent of (A, ϕ).

Recall that $F_a^\omega = \frac{1}{2}\langle \omega, F_a\rangle \in i\mathbb{R}$. On M_s, equation (2.4) decomposes as

$$\begin{aligned} &\bar{\partial}_a \alpha + \bar{\partial}_a^* \beta = 0, \\ &F_a^\omega = -\frac{ir}{8}(1 - |\alpha|^2 + |\beta|^2), \\ &F_a^{0,2} = \frac{r}{4}\alpha^* \beta. \end{aligned}$$

By Proposition 3.1, Lemma 4.1, and the equations above, there exists a constant T_1 depending on r such that

(4.9)
$$\begin{aligned} T_1 + \int_{M_s} \big(\frac{r}{2} iF_a^\omega - 2|iF_a^\omega|^2 - 2|F_a^{0,2}|^2\big) \ge \int_{M_s} \Big(\frac{r}{4}|\nabla_a \alpha|^2 + \frac{r}{4}|\nabla_{A_1+a}\beta|^2 \\ + \frac{r}{2}(iF_{A_1}^\omega)|\beta|^2 + \frac{r^2}{32}(1 - |\alpha|^2 - |\beta|^2)^2 + \frac{r^2}{8}|\beta|^2 - rRe\langle N \circ \partial_a \alpha, \beta\rangle\Big). \end{aligned}$$

Suppose r_0 is sufficiently large, then for all $r > r_0$, we have

$$\frac{r}{8}|\nabla_a\alpha|^2 + \frac{r^2}{32}|\beta|^2 \geq |rRe\langle N \circ \partial_a\alpha, \beta\rangle|, \tag{4.10}$$

$$\frac{r}{4}|\nabla_{A_1+a}\beta|^2 + \frac{r^2}{32}|\beta|^2 \geq |\nabla'_A\beta|^2 + |\frac{r}{2}(iF^{\omega}_{A_1})| \cdot |\beta|^2. \tag{4.11}$$

Therefore by (4.9), for r_0 sufficiently large, we have

(4.12)

$$\int_{M_s} |1-|\alpha|^2-|\beta|^2|^2 + |\beta|^2 + |\nabla_a\alpha|^2 + |\nabla'_A\beta|^2 + |F_a^+|^2 \leq T_1 + \int_{M_s} \frac{r}{2} iF_a^{\omega}.$$

By (4.6), [18, Lemma 5.1.2], and Coulomb gauge fixing, there exists a unitary connection a' of the trivial $\mathbb{C}$–bundle on $\{x|t(x) \in [-R-1, -R]\}$, such that:

1) $\|a - a'\|_{L^2_1([-R-1,-R]\times Y)} < T_2$, for some constant T_2 depending on κ,
2) $a' = a$ when $t \in [-R - \frac{1}{3}, -R]$,
3) $F_{a'} = 0$, when $t \in [-R-1, -R-\frac{2}{3}]$.

Extend a' to $\{x|t(x) > -R\}$ by taking $a' = a$ when $t > -R$.

Recall that by Lemma 3.4, there exists a constant d_0, which may depend on (A, ϕ), such that $|\alpha(x)| \geq \frac{1}{2}$ when $d(x) \geq d_0$. Also recall that Φ_0 is the canonical section of $\mathbb{S}^+|_{M_s}$ given by $1 \in \Gamma(M_s, T^{0,0}M_s)$. Therefore we can take a gauge representative of (A, ϕ) such that $\alpha \in \mathbb{R} \cdot \Phi_0$ when $d(x) \geq d_0$. Without loss of generality, assume (A, ϕ) satisfies the above property. Notice that $|\alpha(x)| \geq \frac{1}{2}$ and $\alpha \in \mathbb{R} \cdot \Phi_0$ imply $|\nabla_a\alpha| \geq \frac{1}{2}|a|$. Therefore by Proposition 3.3, there exist constants z_1, z_2, $d_1 > 0$, where d_1 may depend on (A, ϕ), such that

$$|a'| = |a| \leq 2|\nabla_a\alpha| \leq z_1 e^{-\sqrt{r}\cdot(d-d_1)/z_2} \tag{4.13}$$

for all $x \in M_s$ with $d(x) > d_1$.

Recall that by Condition (3) of Definition 1.1, the ESBG structure can be extended to a neighborhood of M_s, therefore we can smoothly extend θ to a smooth 1-form on X such that $\theta = 0$ outside an open neighborhood of M_s. Extend ω to X by taking $\omega = d\theta$. The extensions of θ and ω do not depend on r or (A, ϕ).

The region $\{x|t(x) \geq -R-1\}$ is the union of the cylinder $[-R-1, 0] \times Y$ and $X - M_c$. For r sufficiently large, (4.13) implies the following identities via the same argument as the proof of (4.2).

$$\int_{t\geq -R-1} F_{a'}\wedge\omega=\int_{t\geq -R-1} F_{a'}\wedge d\theta=-\int_{t\geq -R-1} dF_{a'}\wedge\theta=0, \tag{4.14}$$

$$\int_{t\geq -R-1} F_{a'}\wedge F_{a'}=\int_{t\geq -R-1} F_{a'}\wedge da'=-\int_{t\geq -R-1} dF_{a'}\wedge a'=0. \tag{4.15}$$

Let $Z_R=\{x\in X|t(x)\geq -R-1\}-M_s$. Then there exists a constant z_3 such that

$$\mathrm{Vol}(Z_R)\leq z_3+R\cdot\mathrm{Vol}(Y). \tag{4.16}$$

By (4.14),

$$\int_{M_s} F_{a'}\wedge\omega+\int_{Z_R} F_{a'}\wedge\omega=0,$$

by (4.15),

$$\int_{M_s\cup Z_R}|F_{a'}^+|^2=\int_{M_s\cup Z_R}|F_{a'}^-|^2,$$

and by (4.12),

$$\int_{M_s}(E_r(A,\phi)-|F_a^-|^2)\leq T_1+\frac{r}{4}\Big|\int_{M_s}(iF_a)\wedge\omega\Big|=T_1+\frac{r}{4}\Big|\int_{M_s}(iF_{a'})\wedge\omega\Big|.$$

Therefore, there exists a constant z_4, and constants T_3,T_4 that may depend on r, such that

$$\begin{aligned}\int_{M_s}(E_r(A,\phi)-|F_a^-|^2)\leq& T_1+\frac{r}{4}\Big|\int_{M_s}F_{a'}\wedge\omega\Big|\\ =&T_1+\frac{r}{4}\Big|\int_{Z_R}F_{a'}\wedge\omega\Big|\\ \leq& T_1+\frac{r^2}{16}\int_{Z_R}|\omega|^2+\frac{1}{4}\int_{Z_R}|F_{a'}|^2\\ \leq& T_3+\frac{1}{4}\int_{Z_R}|F_{a'}|^2\\ \leq& T_3+\frac{1}{4}\int_{Z_R\cup M_s}|F_{a'}|^2\\ =&T_3+\frac{1}{2}\int_{Z_R\cup M_s}|F_{a'}^+|^2\\ \leq& T_3+\frac{1}{2}\int_{Z_R}|F_a^+|^2+\frac{1}{2}\int_{M_s}(E_r(A,\phi)-|F_a^-|^2)\\ \leq& T_4+z_4r^2\cdot R+\frac{1}{2}\int_{M_s}(E_r(A,\phi)-|F_a^-|^2),\end{aligned}$$

where the last inequality follows from Proposition 3.1 and (4.16). Hence

$$\int_{M_s} (E_r(A,\phi) - |F_a^-|^2) \leq 2\,T_4 + 2\,z_4 r^2\,R. \tag{4.17}$$

Recall that by the definition of a', we have $\|a - a'\|_{L_1^2} < T_2$, where T_2 is a constant depending on κ. Therefore, there is a constant z_5, and a constant T_5 that depends on r, such that

$$\begin{aligned}\int_{Z_R\cup M_s} |F_a|^2 &\leq T_2 + \int_{Z_R\cup M_s} |F_{a'}|^2\\ &= T_2 + 2\int_{Z_R\cup M_s} |F_{a'}^+|^2\\ &\leq T_2 + 2\int_{Z_R} |F_a^+|^2 + 2\int_{M_s} (E_r(A,\phi) - |F_a^-|^2)\\ &\leq T_5 + z_5 r^2\cdot R, \qquad (4.18)\end{aligned}$$

where the last inequality follows from Proposition 3.1, (4.16), and (4.17). The lemma then follows immediately from (4.17) and(4.18). $\square$

We also have the following estimate on M_c:

Lemma 4.6. *There are constants z_1, z_2, and a function*

$$R_0 : (\mathbb{R}^+)^2 \to \{x \in \mathbb{R} | x > 1\}$$

which depends only on Y, such that the following holds. If $A, B, R > 0$ satisfy

$$\int_{[-R,0]\times Y} |F_a|^2 \leq A + B\cdot R, \tag{4.19}$$

$$|F_a^+|^2 \leq B \text{ pointwise on } [-R,0]\times Y, \tag{4.20}$$

and

$$R > R_0(A,B),$$

then

$$\int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} |F_a|^2 < z_1\cdot B + z_2. \tag{4.21}$$

Proof. Put a in temporal gauge on $[-R,0]\times Y$, and write a as a function $a(t)$ of t which takes value in $\Gamma(Y, iT^*Y)$. Then

$$|F_a^+| = \frac{\sqrt{2}}{2}\left|\dot{a}(t) + *da(t)\right|, \tag{4.22}$$

$$|F_a^-| = \frac{\sqrt{2}}{2}\left|\dot{a}(t) - *da(t)\right|. \tag{4.23}$$

Since Y is closed, it follows from standard Hodge theory that the operator $*d$ is a closed, self-adjoint operator with a discrete spectrum on $L^2(Y, iT^*Y)$. Let

$$\cdots < \lambda_{-3} < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

be the eigenvalues of $*d$. Let

$$k_0 = \max\left\{\frac{1}{|\lambda_{-1}|}, \frac{1}{|\lambda_1|}, 1\right\}. \tag{4.24}$$

Decompose a as

$$a(t) = \sum_{n=-\infty}^{+\infty} a_n(t),$$

where $*da_n(t) = \lambda_n a_n(t)$. Let

$$b_n(t) = \dot{a}_n(t) + \lambda_n a_n(t). \tag{4.25}$$

By (4.20), for all $t\in[-R,0]$ we have

$$\int_{\{t\}\times Y} |F_a^+|^2 \le B\cdot \mathrm{Vol}(Y),$$

hence by (4.22),

$$\sum_{n=-\infty}^{\infty} \|b_n(t)\|_{L^2}^2 \le 2B\cdot \mathrm{Vol}(Y). \tag{4.26}$$

By (4.19),

$$\begin{aligned}\int_{-R}^0 \|*da(t)\|_{L^2}^2\,dt &\le \frac{1}{2}\int_{-R}^0 \left(\|\dot{a}(t) + *da(t)\|_{L^2}^2 + \|\dot{a}(t) - *da(t)\|_{L^2}^2\right)dt \\ &= \int_{[-R,0]\times Y} |F_a|^2 \le A + B\cdot R.\end{aligned}$$

For $R > 1$, the above inequality implies

$$\int_{-R}^{-R+1} \| * da(t)\|_{L^2}^2 \leq A + B \cdot R,$$

hence there exits $t_1 \in [-R, -R+1]$ such that

$$\sum_n \lambda_n^2 \, \|a_n(t_1)\|_{L^2}^2 = \| * da(t_1)\|_{L^2}^2 \leq A + B \cdot R. \tag{4.27}$$

It follows from (4.25) that

$$e^{\lambda_n t} \, b_n(t) = \frac{d}{dt}(e^{\lambda_n t} a_n(t)),$$

therefore

$$\lambda_n a_n(t) = a_n(t_1) \cdot e^{(t_1 - t)\lambda_n} \cdot \lambda_n + \int_{t_1}^{t} e^{\lambda_n (s-t)} \cdot \lambda_n \cdot b_n(s) \, ds.$$

Recall that k_0 is the constant defined by (4.24). If $t > k_0 + t_1$, we have

$$\|\lambda_n a_n(t)\|_{L^2}^2 \leq 3\big(X_n^2(t) + Y_n^2(t) + Z_n^2(t)\big),$$

where

$$\begin{aligned} X_n(t) &= \|a_n(t_1) \, e^{(t_1-t)\lambda_n} \lambda_n\|_{L^2(Y)}, \\ Y_n(t) &= \| \int_{t_1}^{t-k} b_n(s) \, e^{(s-t)\lambda_n} \lambda_n \, ds \|_{L^2(Y)}, \\ Z_n(t) &= \| \int_{t-k}^{t} b_n(s) \, e^{(s-t)\lambda_n} \lambda_n \, ds \|_{L^2(Y)} \\ &= \| \int_{0}^{k} b_n(t-s) \, e^{-s\lambda_n} \lambda_n \, ds \|_{L^2(Y)}. \end{aligned}$$

If $n > 0$, then $\lambda_n \geq 1/k_0$, hence

$$X_n(t) \leq \|\lambda_n a_n(t_1)\|_{L^2} \cdot e^{(t_1-t)/k_0}.$$

If $R \geq 2k_0 + 3$, then $-R/2 - 1/2 > k_0 + 1$, hence (4.27) and the above inequality imply

$$\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n \geq 1} X_n(t)^2 \, dt \leq (A + B \cdot R) \cdot e^{-(R-3)/k_0}.$$

By the Minkowski inequality, when $t > k_0 + t_1$,

$$Y_n(t) \le \int_{t_1}^{t-k_0} \|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_n} \lambda_n \, ds.$$

Notice that if $t - s > k_0$, then $e^{(s-t)\lambda} \cdot \lambda$ is decreasing with respect to λ for all $\lambda \ge 1/k_0$, hence

$$\begin{aligned}\Big(\sum_{n\ge 1} Y_n(t)^2\Big)^{1/2} &\le \Big(\sum_{n\ge 1}\big(\int_{t_1}^{t-k_0} \|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_n}\lambda_n \, ds\big)^2\Big)^{1/2} \\ &\le \Big(\sum_{n\ge 1}\big(\int_{t_1}^{t-k_0} \|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_1}\lambda_1 \, ds\big)^2\Big)^{1/2}.\end{aligned}$$

By the Minkowski inequality again and (4.26), we have

$$\begin{aligned}&\Big(\sum_{n\ge 1}\big(\int_{t_1}^{t-k_0} \|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_1}\lambda_1 \, ds\big)^2\Big)^{1/2} \\ \le& \int_{t_1}^{t-k_0} \Big(\sum_{n\ge 1} \|b_n(s)\|_{L^2}^2\Big)^{1/2} e^{(s-t)\lambda_1}\lambda_1 \, ds \\ \le& \sqrt{2B\cdot \mathrm{Vol}(Y)} \int_{t_1}^{t-k_0} e^{(s-t)\lambda_1}\lambda_1 \, ds \\ \le& \sqrt{2B\cdot \mathrm{Vol}(Y)} \int_{-\infty}^{0} e^{s\lambda_1}\lambda_1 \, ds = \sqrt{2B\cdot \mathrm{Vol}(Y)}.\end{aligned}$$

Therefore when $R > 2k_0 + 3$, we have

$$\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\ge 1} Y_n(t)^2 \, dt \le 2B \cdot \mathrm{Vol}(Y).$$

As for Z_n, the Minkowski inequality gives the following estimates when $R > 2k_0 + 3$:

$$\begin{aligned}\Big(\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} Z_n(t)^2\,dt\Big)^{1/2} &\leq \Big(\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \big(\int_0^{k_0} \|b_n(t-s)\|_{L^2}\cdot e^{-s\lambda_n}\lambda_n\,ds\big)^2\,dt\Big)^{1/2}\\ &\leq \int_0^{k_0}\Big(\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \|b_n(t-s)\|_{L^2}^2\,dt\Big)^{1/2} e^{-s\lambda_n}\lambda_n\,ds\\ &\leq \int_0^{k_0}\Big(\int_{-\frac{R}{2}-\frac{1}{2}-k_0}^{-\frac{R}{2}+\frac{1}{2}} \|b_n(t)\|_{L^2}^2\,dt\Big)^{1/2} e^{-s\lambda_n}\lambda_n\,ds\\ &\leq \Big(\int_{-\frac{R}{2}-\frac{1}{2}-k_0}^{-\frac{R}{2}+\frac{1}{2}} \|b_n(t)\|_{L^2}^2\,dt\Big)^{1/2}\int_0^{+\infty} e^{-s\lambda_n}\lambda_n\,ds\\ &= \Big(\int_{-\frac{R}{2}-\frac{1}{2}-k_0}^{-\frac{R}{2}+\frac{1}{2}} \|b_n(t)\|_{L^2}^2\,dt\Big)^{1/2}.\end{aligned}$$

Therefore, when $R > 2k_0 + 3$,

$$\begin{aligned}\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\geq 1} Z_n(t)^2\,dt &\leq \int_{-\frac{R}{2}-\frac{1}{2}-k_0}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\geq 1}\|b_n(t)\|_{L^2}^2\,dt\\ &\leq (k_0+1)2B\cdot \mathrm{Vol}(Y).\end{aligned}$$

Combining the estimates above, when $R > 2k_0 + 3$, we have

$$\begin{aligned}\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\geq 1}\|\lambda_n a_n(t)\|_{L^2}^2\,dt &\leq \int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\geq 1} 3\big(X_n^2(t)+Y_n^2(t)+Z_n^2(t)\big)\\ &\leq 6(k_0+2)B\ \mathrm{Vol}(Y) + 3(A+B\cdot R)\cdot e^{-(R-3)/k_0}.\end{aligned}$$

On the other hand, there exists a $t_2 \in [-1, 0]$ such that

$$\sum_n \lambda_n^2\|a_n(t_2)\|_{L^2}^2 \leq A + B\cdot R.$$

If $n < 0$, we have the identity

$$\lambda_n a_n(t) = a_n(t_2)\cdot e^{(t-t_2)(-\lambda_n)}\cdot\lambda_n - \int_t^{t_2} e^{(-\lambda_n)(t-s)}\cdot\lambda_n\cdot b_n(s)\,ds.$$

When $R > 2k_0 + 3$, a similar argument gives

$$\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n\le -1} \|\lambda_n a_n(t)\|_{L^2}^2 \, dt \\ \le 6(k_0+2)B \operatorname{Vol}(Y) + 3(A + B\cdot R)\cdot e^{-(R-3)/k_0}.$$

Therefore, when $R > 2k_0 + 3$

$$\int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \| * da\|_{L^2}^2 \, dt = \int_{-\frac{R}{2}-\frac{1}{2}}^{-\frac{R}{2}+\frac{1}{2}} \sum_{n=-\infty}^{+\infty} \|\lambda_n a_n(t)\|_{L^2}^2 \, dt \\ \le 12(k_0+2)B \operatorname{Vol}(Y) + 6(A + B\cdot R)\cdot e^{-(R-3)/k_0}.$$

Notice that by (4.22), (4.23),

$$\int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} \big||F_a^-| - |F_a^+|\big|^2 \, dt \le \int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} 2\| * da(t)\|_{L^2}^2 \, dt.$$

Hence

$$\int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} |F_a|^2 \\ \le 2\int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} |2F_a^+|^2 + 2\int_{[-\frac{R}{2}-\frac{1}{2}, -\frac{R}{2}+\frac{1}{2}]\times Y} \big||F_a^-| - |F_a^+|\big|^2 \\ \le 8B \operatorname{Vol}(Y) + 48(k_0+2)B \operatorname{Vol}(Y) + 24(A + B\cdot R)\cdot e^{-(R-3)/k_0},$$

and the lemma follows from the inequality above by taking $R_0(A,B)$ sufficiently large such that $R_0(A,B) > 2k_0 + 3$, and

$$24(A + B\cdot R)\cdot e^{-(R_0(A,B)-3)/k_0} \le 1.$$

□

Proof of Proposition 4.3. Pick r_0 sufficiently large such that Lemma 4.5 is valid for all $r > r_0$. Let the function $T : \mathbb{R}^2 \to \mathbb{R}^+$ and the constant z be as in Lemma 4.5. Let z' be the right-hand side of (4.5). Let z_0 be the constant z in Proposition 3.1. Let the function R_0 and the constants z_1, z_2 be as in Lemma 4.6.

Let $C_1 = \max\{zr^2, z_0^2 r^2\}$, let $\kappa = \max\{z', z_1 \cdot C_1 + z_2\}$.

Let

$$R_{\min} = \inf\{R \geq 1 | \int_{t\in[-R,-R+1]} |F_a|^2 < \kappa\}.$$

Since $\kappa \geq z'$, it follows from (4.5) that $R_{\min}$ exists and is finite. By (4.8),

$$\begin{aligned}\int_{[-R,0]\times Y} |F_a|^2 &< T(\kappa, r) + zr^2 \cdot R \\ &\leq T(\kappa, r) + C_1 \cdot R.\end{aligned}$$

By the definitions of z_0 and C_1, we have the following pointwise estimate

$$|F_a^+|^2 \leq z_0^2\, r^2 \leq C_1.$$

If $R_{\min} > R_0(T(\kappa, r), C_1)$ and $R_{\min} > 1$, take $R' = (R_{\min} + 1)/2$, then Lemma 4.6 gives

$$\int_{t\in[-R',-R'+1]} |F_a|^2 < z_1 \cdot C_1 + z_2 \leq \kappa.$$

Since $R' < R$, this contradicts the definition of $R_{\min}$. Therefore,

$$R_{\min} \leq \max\{R_0(T(\kappa, r), C_1), 1\}, \tag{4.28}$$

hence by (4.7),

$$\begin{aligned}\int_{M_s} E_r(A, \phi) &< T(\kappa, r) + z\, r^2 \cdot R_{\min} \\ &\leq T(\kappa, r) + z\, r^2 \cdot \max\{R_0(T(\kappa, r), C_1), 1\},\end{aligned}$$

and Proposition 4.3 is proved. $\square$

4.3. Uniform exponential decay of $E_r(A, \phi)$

We can now prove the uniform exponential decay of $E_r(A, \phi)$. Recall that the function d is defined on M_s as follows. For each connected component $M_s^{(k)}$ of M_s, if $\partial M_s^{(k)}$ is nonempty, then d is the distance function to $\partial M_s^{(k)}$ on $M_s^{(k)}$. Otherwise, fix a point $x^{(k)} \in M_s^{(k)}$, and d is the distance function to $x^{(k)}$ on $M_s^{(k)}$. The following is a re-statement of Theorem 1.4.

Theorem 4.7. *There exist constants $r_0, z > 0$ with the following significance. For every $r > r_0$, there is a constant C depending on r, such that if $r > r_0$ and $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ solves (2.4), then the inequality*

$$E_r(A, \phi) < C\, e^{-\sqrt{r}\cdot d/z} \tag{4.29}$$

holds on M_s. □

Proof. First, we prove that the constant $d(\delta)$ given by Lemma 3.4 is uniform for all (A, ϕ). More precisely, for all $r > r_0$ and $\delta > 0$, there exists a constant $d(\delta)$ depending on r and δ, such that for all $x \in M_s$ with $d(x) > d(\delta)$ and $(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s})$ solving (2.4), we have

$$E_r(A, \phi)(x) \le \delta.$$

Assume the contrary, then there exist $\delta_0 > 0$, a sequence of solutions (A_n, ϕ_n), and a sequence of points $x_n \in M_s$ with $\lim_{n\to\infty} d(x_n) = +\infty$, such that

$$E_r(A_n, \phi_n)(x_n) \ge \delta_0$$

for all n.

Let g be the metric on M_s. By Proposition 2.9 and the C^0 bound of $|\phi_n|$ given by Lemma 3.2, a subsequence of $\{(M_s, g, \mathfrak{s}, x_n, A_n, \phi_n)\}_{n\ge 1}$ converges to a limit $(\widetilde{M}, \tilde{g}, \tilde{\mathfrak{s}}, \tilde{x}, \tilde{A}, \tilde{\phi})$. The limit manifold $(\widetilde{M}, \tilde{g})$ is complete and has bounded geometry. Since $\nabla^k\theta$ is bounded for all k, by a diagonal argument and the Arzelà–Ascoli theorem, after taking a further subsequence, we may assume that θ converges to a 1-form $\widetilde{\theta}$ on $\widetilde{M}$, in the C^∞ topology on compact subsets. Therefore, $\widetilde{\omega} = d\widetilde{\theta}$ is a compatible symplectic structure on $\widetilde{M}$, and $\tilde{\mathfrak{s}}$ is the canonical spin^c structure on $\widetilde{M}$ induced by $\widetilde{\omega}$. In conclusion, $\widetilde{M}$ is an ESBG end with empty boundary, and $(\tilde{A}, \tilde{\phi})$ is a solution to (2.4) on $\widetilde{M}$.

Let $\widetilde{E}_r(\tilde{A}, \tilde{\phi})$ be the energy density function of $(\tilde{A}, \tilde{\phi})$ on $\widetilde{M}$. By the previous assumptions, we have $\widetilde{E}_r(\tilde{A}, \tilde{\phi})(\tilde{x}) \ge \delta_0$. The uniform energy bound given by Proposition 4.3 implies

$$\int_{\widetilde{M}} \widetilde{E}_r(\tilde{A}, \tilde{\phi}) < +\infty. \tag{4.30}$$

Apply Lemma 4.1 to $\widetilde{M}, \tilde{A}, \tilde{\phi}$. Since $\widetilde{M}$ is a symplectic end itself with empty boundary, we can take $\chi = 1$ on $\widetilde{M}$. Therefore when r_0 is sufficiently large and $r > r_0$, we have

$$\begin{aligned}(4.31)\qquad &\int_{\widetilde{M}} \Big(\frac{r}{2}|\bar{\partial}_{\tilde{a}}\tilde{\alpha} + \bar{\partial}^*_{\tilde{a}}\tilde{\beta}|^2 + 2|iF^{\tilde{\omega}}_{\tilde{a}} - \frac{r}{8}(1-|\tilde{\alpha}|^2+|\tilde{\beta}|^2)|^2 \\ &\quad + 2|F^{0,2}_{\tilde{a}} - \frac{r}{4}\tilde{\alpha}^*\tilde{\beta}|^2 + \frac{r}{2}iF^{\tilde{\omega}}_{\tilde{a}} - 2|iF^{\tilde{\omega}}_{\tilde{a}}|^2 - 2|F^{0,2}_{\tilde{a}}|^2\Big) \\ &= \int_{\widetilde{M}} \Big(\frac{r}{4}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{r}{4}|\nabla_{\tilde{A}_1+\tilde{a}}\tilde{\beta}|^2 + \frac{r}{2}(iF^{\tilde{\omega}}_{\tilde{A}_1})|\tilde{\beta}|^2 \\ &\quad + \frac{r^2}{32}(1-|\tilde{\alpha}|^2-|\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - rRe\langle \widetilde{N}\circ\partial_{\tilde{a}}\tilde{\alpha},\tilde{\beta}\rangle\Big),\end{aligned}$$

where $\tilde{A}_1$ is the unique unitary connection on $T^{0,2}\widetilde{M}$ such that $\nabla^{1,0}_{\tilde{A}_1} = \partial$, and $\widetilde{N} : T^{1,0}\widetilde{M} \to T^{0,2}\widetilde{M}$ is the Nijenhuis tensor.

Since $(\tilde{A},\tilde{\phi})$ solves (2.4) on $\widetilde{M}$, we have

$$\begin{aligned}&\bar{\partial}_{\tilde{a}}\tilde{\alpha} + \bar{\partial}^*_{\tilde{a}}\tilde{\beta} = 0,\\ &F^{\tilde{\omega}}_{\tilde{a}} = -\frac{ir}{8}(1-|\tilde{\alpha}|^2+|\tilde{\beta}|^2),\\ &F^{0,2}_{\tilde{a}} = \frac{r}{4}\tilde{\alpha}^*\tilde{\beta}.\end{aligned}$$

Therefore (4.31) gives

$$\begin{aligned}(4.32)\quad &\int_{\widetilde{M}} \big(\frac{r}{2}iF^{\tilde{\omega}}_{\tilde{a}} - 2|iF^{\tilde{\omega}}_{\tilde{a}}|^2 - 2|F^{0,2}_{\tilde{a}}|^2\big) = \int_{\widetilde{M}} \Big(\frac{r}{4}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{r}{4}|\nabla_{\tilde{A}_1+\tilde{a}}\tilde{\beta}|^2 \\ &\quad + \frac{r}{2}(iF^{\tilde{\omega}}_{\tilde{A}_1})|\tilde{\beta}|^2 + \frac{r^2}{32}(1-|\tilde{\alpha}|^2-|\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - rRe\langle \tilde{N}\circ\partial_{\tilde{a}}\tilde{\alpha},\tilde{\beta}\rangle\Big).\end{aligned}$$

When r_0 is sufficiently large, by (4.30) and the same proof of (4.2), we have,

$$(4.33)\qquad \int_{\widetilde{M}} F^{\tilde{\omega}}_{\tilde{a}} = \frac{1}{2}\int_{\widetilde{M}} F_{\tilde{a}}\wedge\widetilde{\omega} = \frac{1}{2}\int_{\widetilde{M}} F_{\tilde{a}}\wedge d\widetilde{\theta} = -\frac{1}{2}\int_{\widetilde{M}} d(F_{\tilde{a}})\wedge\widetilde{\theta} = 0,$$

$$(4.34)\qquad \int_{\widetilde{M}} |F^+_{\tilde{a}}|^2 - \int_{\widetilde{M}} |F^-_{\tilde{a}}|^2 = \int_{\widetilde{M}} F_{\tilde{a}}\wedge F_{\tilde{a}} = 0.$$

Therefore equation (4.32) gives

$$\begin{aligned}(4.35)\quad 0 = \int_{\widetilde{M}} &\Big(2|iF^{\tilde{\omega}}_{\tilde{a}}|^2 + 2|F^{0,2}_{\tilde{a}}|^2 + \frac{r}{4}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{r}{4}|\nabla_{\tilde{A}_1+\tilde{a}}\tilde{\beta}|^2 + \frac{r}{2}(iF^{\tilde{\omega}}_{\tilde{A}_1})|\tilde{\beta}|^2 \\ &+ \frac{r^2}{32}(1-|\tilde{\alpha}|^2-|\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - rRe\langle \widetilde{N}\circ\partial_{\tilde{a}}\tilde{\alpha},\tilde{\beta}\rangle\Big).\end{aligned}$$

When r_0 is sufficiently large, by (4.9) and (4.11), and (4.35), we have

$$0 \geq \int_{\widetilde{M}} |\nabla_{\tilde{a}}\tilde{\alpha}|^2 + |\nabla'_{\tilde{A}}\tilde{\beta}|^2 + (1-|\tilde{\alpha}|^2-|\tilde{\beta}|^2)^2 + |\tilde{\beta}|^2 + |F_{\tilde{a}}^+|^2. \tag{4.36}$$

As a consequence, the integrand of the right-hand side of (4.36) is identically 0 on $\widetilde{M}$. By (4.34), $|F_{\tilde{a}}^-|$ is also identically zero on $\widetilde{M}$, hence $\widetilde{E}_r(\tilde{A},\tilde{\phi})$ is identically zero. This contradicts the assumption that $\widetilde{E}_r(\tilde{A},\tilde{\phi})(\tilde{x}) \geq \delta_0$. In conclusion, the constant $d(\delta)$ given by Lemma 3.4 is uniform for all (A,ϕ).

Now we finish the proof of Theorem 4.7. Since the constant $d(\delta)$ given by Lemma 3.4 is uniform for all (A,ϕ), the constants d_1, d_2, d_3 in the proof of Proposition 3.3 depend only on r but not on (A,ϕ). Therefore the constant d_0 in the statement of Proposition 3.3 depends only on r but not on (A,ϕ). This proves (4.29) when $d > d_0$. By standard elliptic bootstrapping, there exists a constant C_0 depending on r such that $E_r(A,\phi) < C_0$ pointwise. This proves the estimate for (4.29) when $d \leq d_0$. Hence the theorem is proved. $\square$

4.4. Uniform decay with neck stretching

For $i = 1, 2$, suppose $X^{(i)}$ is a manifold with cylindrical and ESBG ends, where the cylindrical end is $M_c^{(i)}$, and the ESBG end is given by $(M_s^{(i)}, \omega^{(i)} = d\theta^{(i)})$. Let $Z^{(i)} = X^{(i)} - M_c^{(i)} - M_s^{(i)}$. Moreover, suppose there exists a non-empty oriented closed Riemannian 3-manifold Y, such that $M_c^{(1)}$ is given by $(-\infty, 0] \times Y$, and $M_c^{(2)}$ is given by $(-\infty, 0] \times (-Y)$.

For each constant $R > 0$, we can define a Riemannian manifold X_R as follows. Let $X_R^{(1)}$ be the subset of $X^{(1)}$ given by $Z^{(1)} \cup M_s^{(1)} \cup [-R, 0] \times Y$, let $X_R^{(2)}$ be the subset of $X^{(2)}$ given by $Z^{(2)} \cup M_s^{(2)} \cup [-R, 0] \times (-Y)$. Let X_R be the manifold obtained from $X_R^{(1)} \sqcup X_R^{(2)}$ by gluing $[-R, 0] \times Y$ with $[-R, 0] \times (-Y)$ via $(t, x) \sim (-R-t, x)$. Then X_R is a manifold with ESBG end $M_s^{(1)} \cup M_s^{(2)}$. This subsection proves that the exponential decay estimate on X_R given by Theorem 4.7 is uniform for all R.

Let $M_s = M_s^{(1)} \cup M_s^{(2)}$, let θ be the union of $\theta^{(1)}$ and $\theta^{(2)}$ on M_s, and extend θ to a smooth 1-form on X_R such that the support of θ is contained in $M_s \cup Z^{(1)} \cup Z^{(2)}$. Let $\omega = d\theta$ be a 2-form on X_R.

Let $\mathfrak{s}$ be an admissible spin^c structure on X_R, and let $(A,\phi) \in \mathcal{C}_k(X_R, \mathfrak{s})$ be a solution to (2.4). Let A_0 be the canonical connection of $\mathfrak{s}|_{M_s}$, and extend A_0 to a smooth connection of $\mathfrak{s}$ that is translation-invariant on the glued image of $[-R, 0] \times Y \subset M_c^{(1)}$ and $[-R, 0] \times (-Y) \subset M_c^{(2)}$.

Let $a = A - A_0$. Decompose $\phi = \sqrt{r}(\alpha + \beta)$ on M_s such that $\alpha \in \Gamma(M_s, T^{0,0}M_s)$, $\beta \in \Gamma(M_s, T^{0,2}M_s)$ as before, and let $E_r(A, \phi)$ be defined by (1.2).

In this subsection, we will use r_0 z, and z_i to denote the constants that depend on $X^{(1)}, M_c^{(1)}, M_s^{(1)}, \theta^{(1)}, X^{(2)}, M_c^{(2)}, M_s^{(2)}, \theta^{(2)}, \mathfrak{s}$, and the perturbation terms of the Seiberg-Witten equations. We will use C_i to denote the constants that depend on the same set of data above and also r, but are independent of R.

Lemma 4.8. *There is a constant z, such that $|\phi| < z \cdot \sqrt{r}$, $|F_a^+| < z \cdot r$.*

Proof. The proof is the same as Proposition 3.1. □

Lemma 4.9. *There exist constants r_0 and z with the following property. Suppose $r > r_0$, $R \geq 1$, then there is a constant C_0 depending on r, such that*

$$\int_{M_s} E_r(A, \phi) < C_0 + z\, r^2 \cdot R, \tag{4.37}$$

$$\int_{X_R - M_s} |F_a|^2 < C_0 + z\, r^2 \cdot R. \tag{4.38}$$

Proof. By the same argument as in (4.12), there is a constant C_1 depending on r, such that

$$\int_{M_s} (E_r(A, \phi) - |F_a^-|^2) \leq C_1 + \frac{r}{2} \int_{M_s} iF_a^\omega. \tag{4.39}$$

Similar to (4.14) and (4.15), we have

$$c_0 := -\int_{X_R} F_a \wedge F_a = -\int_{X_R} |F_a^+|^2 + \int_{X_R} |F_a^-|^2$$

is a topological invariant that only depends on $\mathfrak{s}$, and

$$\int_{X_R} F_a \wedge \omega = 0.$$

Therefore, there exists a constant z_1, and C_2, C_3 depending on r, such that

$$\begin{aligned}
\int_{M_s}(E_r(A,\phi)-|F_a^-|^2) &\le C_1+\frac{r}{2}\Big|\int_{M_s} iF_a^\omega\Big| \\
&= C_1+\frac{r}{4}\Big|\int_{M_s} F_a\wedge\omega\Big| \\
&= C_1+\frac{r}{4}\Big|\int_{X_R-M_s} F_a\wedge\omega\Big| \\
&\le C_1+\frac{r^2}{16}\int_{X_R-M_s}|\omega|^2+\frac{1}{4}\int_{X_R-M_s}|F_a|^2 \\
&\le C_2+\frac{1}{4}\int_{X_R-M_s}|F_a|^2 \\
&\le C_2+\frac{1}{4}\int_{X_R}|F_a|^2 \\
&= C_2+\frac{c_0}{4}+\frac{1}{2}\int_{X_R}|F_a^+|^2 \\
&= C_2+\frac{c_0}{4}+\frac{1}{2}\int_{X_R-M_s}|F_a^+|^2+\frac{1}{2}\int_{M_s}|F_a^+|^2 \\
&\le C_3+z_1\,r^2\cdot R+\frac{1}{2}\int_{M_s}(E_r(A,\phi)-|F_a^-|^2),
\end{aligned}$$

where the last inequality follows from Lemma 4.8. Therefore

$$\int_{M_s}(E_r(A,\phi)-|F_a^-|^2)\le 2C_3+2z_1\,r^2\cdot R. \tag{4.40}$$

On the other hand, by Lemma 4.8 again and (4.40), there exist a constant z_2, and a constant C_4 depending on r, such that

$$\begin{aligned}
\int_{X_R}|F_a|^2 &\le |c_0|+2\int_{X_R}|F_a^+|^2 \\
&\le |c_0|+2\int_{X_R-M_s}|F_a^+|^2+2\int_{M_s}(E-|F_a^-|^2) \\
&\le C_4+z_2\,r^2\cdot R.
\end{aligned}$$

The lemma is then proved by combining the two inequalities above. □

Lemma 4.10. *There exists a constant $r_0>0$, such that for every $r>r_0$ there is a constant $C>0$ depending on r but independent of R, such that if*

$(A,\phi)\in\mathcal{C}_k(X_R,\mathfrak{s})$ solves (2.4), then we have

$$\int_{M_s} E_r(A,\phi) < C. \tag{4.41}$$

Proof. The proof follows from Lemma 4.9 and an argument similar to the proof of Proposition 4.3.

Let the constants z_1, z_2 and the function R_0 be as in Lemma 4.6. Let C_0 and z be the constants in Lemma 4.9. Let z' be the constant given by Lemma 4.8. Let $C_1=\max\{zr^2, z'r^2\}$, let $\kappa = z_1C_1+z_2+1$. If $R\le R_0(C_0,C_1)$ then by Lemma 4.9,

$$\int_{M_s} E_r(A,\phi) < C_0 + zr^2\, R_0(C_0,C_1),$$

hence (4.41) holds when $C > C_0 + zr^2\, R_0(C_0,C_1)$.

If $R > R_0(C_0,C_1)$, recall that $[-R,0]\times Y\subset M_c^{(1)}$ is a subset of X_R. By Lemma 4.9,

$$\int_{[-R,0]\times Y} |F_a|^2 \le C_0 + zr^2\, R \le C_0 + C_1\cdot R,$$

by Lemma 4.8,

$$|F_a^+| \le z'r^2 \le C_1,$$

hence Lemma 4.6 gives

$$\int_{[-R/2-1/2,-R/2+1/2]\times Y} |F_a|^2 \le z_1C_1 + z_2 < \kappa. \tag{4.42}$$

Take

$$R_{\min} = \inf\{\hat{R}|\hat{R}\ge 0, \int_{[-\hat{R}-1,-\hat{R}]\times Y} |F_a|^2 < \kappa\},$$

then (4.42) implies that $R_{\min}$ exists. Let $z^{(1)}>0$ and $T^{(1)}:(\mathbb{R}^+)^2\to\mathbb{R}^+$ be the constant and the function given by Lemma 4.5 when applied to $X^{(1)}$, let $C_1^{(1)} = \max\{z^{(1)}r^2, z'r^2\}$. Then by the same proof of (4.28), we have

$$R_{\min} \le \max\left\{R_0(T^{(1)}(\kappa,r), C_1^{(1)}), 1\right\}.$$

Now apply Lemma 4.5 to $X^{(1)}$, we obtain

$$\begin{aligned}\int_{M_s^{(1)}} E_r(A,\phi) &< T^{(1)}(\kappa,r) + z^{(1)}r^2\cdot R_{\min} \\ &\le T^{(1)}(\kappa,r) + z^{(1)}r^2\cdot\max\Big\{R_0\big(T^{(1)}(\kappa,r),C_1^{(1)}\big),1\Big\},\end{aligned}$$

which yields a uniform upper bound for the integration of $E_r(A,\phi)$ on $M_s^{(1)}$. The upper bound for $M_s^{(2)}$ follows from a similar argument. ☐

Theorem 4.11. *There exist constants $r_0, z > 0$ with the following significance. For every $r > r_0$, there is a constant $C > 0$ depending on r but independent of R, such that if $(A,\phi)\in\mathcal{C}_k(X_R,\mathfrak{s})$ solves (2.4) on X_R, then the followsing inequality holds on M_s:*

$$E_r(A,\phi) < Ce^{-\sqrt{r}\cdot d/z}.$$

Proof. This follows from Lemma 4.10 and the same argument as the proof of Theorem 4.7. ☐

5. Floer chains from ESBG structures

Let X be a Riemannian 4-manifold with cylindrical and ESBG ends, such that $M_c \cong (-\infty,0]\times Y$ is the cylindrical end, and $(M_s,\omega = d\theta)$ is the ESBG end. If M_s is compact and $b_2^+(X)\ge 2$, after removing a small ball from X, we can view $X - B^4$ as a cobordism from S^3 to $-Y$. By the construction of [18, Section 25], the cobordism $X - B^4$ induces a map

$$\overrightarrow{HM}(X-B^4): \widehat{HM}(S^3)\to \widecheck{HM}(-Y).$$

The map $\overrightarrow{HM}(X-B^4)$ is only well-defined up to a sign, which can be fixed by a choice of the homology orientation of X. The construction above defines an element

$$\overrightarrow{HM}(X-B^4)(\hat{1})\in\widecheck{HM}(-Y)/\{\pm 1\},$$

where $\hat{1}\in\widehat{HM}(S^3)$ is the generator of $\widehat{HM}(S^3)$ as a $\mathbb{Z}[U_\dagger]$–module. The condition $b_2^+(X)\ge 2$ is necessary to gurantee that under a generic perturbation, the solutions of the Seiberg-Witten equations on X are all irreducible, namely the spinor part is not identically zero.

In this section, we will show that when M_s is not compact, it is still possible to define an element in $\widecheck{HM}(-Y)/\{\pm 1\}$ by counting solutions of

(2.4) on X. This is a straightforward generalization of [20, Section 6.3]. Theorem 4.7 implies that the moduli space analogous to the space $M(Z^+, \mathfrak{a})$ in [20, Section 6.3] compact. Since M_s is non-compact, Condition (2) of Definition (2.3) implies that every element in $\mathcal{C}_k(X, \mathfrak{s})$ is irreducible. The sign ambiguity of $\widetilde{HM}(-Y)/\{\pm 1\}$ is essential and cannot be resolved by a choice of homology orientation. This will be explained in Remark 5.6.

For each spin^c structure $\mathfrak{t}$ on Y, fix a stongly tame perturbation $\mathfrak{q}_\mathfrak{t}$ that is admissible in the sense of [18, Definition 22.1.1] and satisfies $\|\mathfrak{q}_\mathfrak{t}\|_{\hat{\mathcal{P}}} \le 1$, where $\|\cdot\|_{\hat{\mathcal{P}}}$ is the norm defined by (2.1). Let $\mathfrak{q}$ be the set of all $\mathfrak{q}_\mathfrak{t}$. Let $\mathfrak{C}$ be set of isomorphism classes of critical points of the Chern-Simons-Dirac functional perturbed by $\mathfrak{q}_\mathfrak{t}$ in the blown-up configuration space $\mathcal{B}^\sigma(-Y, \mathfrak{t})$ as defined in [20, Section 4.1] for all $\mathfrak{t}$ (see also [18, Section 6]).

Definition 5.1. *Suppose $[\mathfrak{a}] \in \mathfrak{C}$, and $\mathfrak{s}$ is an admissible spin^c structure on X. Let $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})$ be the moduli space of $(A, \phi) \in \mathcal{C}(X, \mathfrak{s})$ that solves* (2.4) *and is asymptotic to $[\mathfrak{a}]$ on the cylindrical end M_c after lifting to the blown-up configuration space.*

Remark 5.2. For the definition of asymptoticity in the blown-up configuration space, see [18, Definition 13.1.1] and the paragraph above it.

For a generic choice of $\hat{\tau}$ in (2.3), the moduli space $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})$ is regular for all $\mathfrak{a}$ and $\mathfrak{s}$. By the construction of [18, Sections 20, 25.2], there are two systems of compatible orientations for the moduli spaces $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})$ that are different by an overall sign. Let $\mathfrak{o}$ and $-\mathfrak{o}$ be the two systems of compatible orientations. Let $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})$ be the zero-dimensional components of $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})$. By Theorem 4.7 and the compactness results of Seiberg-Witten equations [18, Lemma 25.3.1], there are only finitely many $[\mathfrak{a}]$ and $\mathfrak{s}$ such that $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})$ is nonempty.

Definition 5.3. *Let*

$$\check{\psi}_\mathfrak{o}(X) = \sum_{\mathfrak{s}} \sum_{[\mathfrak{a}]\in\mathfrak{C}} \#\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s}) \cdot [\mathfrak{a}] \in \mathbb{Z}[\mathfrak{C}],$$

where the elements of $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})$ are counted with signs using the orientation $\mathfrak{o}$, and the summation of $\mathfrak{s}$ goes over the isomorphism classes of admissible spin^c structures over X relative to M_s.

If M_s is non-compact, by Condition (2) of Definition (2.3), all the elements of $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})$ are irreducible. Therefore, in order for $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})$

to be non-empty, the critical point $\mathfrak{a}$ has to be either irreducible or boundary-stable. As a consequence, $\check{\psi}_{\mathfrak{o}}(X)$ is an element of the boundary-stable monopole Floer chain group $\check{C}_*(-Y)$ defined by [18, (22.3)], and we have the following lemma.

Lemma 5.4. *The element $\check{\psi}_{\mathfrak{o}}(X) \in \check{C}_*(-Y)$ satisfies $\check{\partial}(\check{\psi}_{\mathfrak{o}}(X)) = 0$.*

Proof. This follows from the same proof as [20, Lemma 6.6]. □

Therefore, the homology class of $\check{\psi}_{\mathfrak{o}}(X)$ defines an element in the monopole Floer homology group $\widetilde{HM}_\bullet(-Y)$. Define

$$c(X) = [\check{\psi}_{\mathfrak{o}}(X)] \in \widetilde{HM}_\bullet(-Y)/\{\pm 1\}, \tag{5.1}$$

then $c(X)$ does not depend on the choice of the orientation $\mathfrak{o}$.

Proposition 5.5. *$c(X) \in \widetilde{HM}_\bullet(-Y)/\{\pm 1\}$ does not depend on the choices of $r > r_0$, the perturbations $\mathfrak{q}$, $\hat{\tau}$, η in (2.3), or the metric on $X - M_s$. Moreover, $c(X)$ is invariant under smooth deformations of the ESBG structures on M_s with uniformly bounded geometry.*

Proof. For $i = 1, 2$, let g_i be a metric on X that is cylindrical on M_c, let $\omega_i = d\theta_i$ be an exact symplectic form on M_s, suppose (X, g_i) is a manifold with cylindrical end M_c and ESBG end $(M_s, \omega_i = d\theta_i)$. Assume $(X, g_0, \omega_0 = d\theta_0)$ can be smoothly deformed to $(X, g_1, \omega_1 = d\theta_1)$ via manifolds with cylindrical end M_c and ESBG end M_s, such that the deformation has uniformly bounded geometry.

For $i = 0, 1$, let $\hat{\tau}_i, \eta_i$ be a choice of perturbation terms in (2.3). Let r_i be a sufficiently large constact such that Theorem 4.7 holds for $(g_1, \omega_i = d\theta_i, \hat{\tau}, \hat{\eta})$. Let g_i^Y be the metric on Y induced by the restriction of g_i to M_c. Let $\mathfrak{q}_i$ be a collection of strongly tame, admissible perturbations on (Y, g_i^Y) for all isomorphism classes of spin^c structures, and let $\check{C}_*(-Y, g_i^Y, \mathfrak{q}_i)$ be the corresponding boundary-stable Floer chain.

Let $\mathfrak{o}$ be a choice of the orientation, let $\check{\psi}_{\mathfrak{o}}(X)(i) \in \check{C}_*(-Y, g_i^Y, \mathfrak{q}_i)$ be the element defined by Definition 5.3 with respect to $g_i, \omega_i = d\theta_i, \mathfrak{q}_i, \hat{\tau}_i, \eta_i, r_i$. Let $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})(i)$ be the moduli space given by Definition 5.1 with respect to the same choice of geometric data, and let $\mathcal{M}_0(X, [\mathfrak{a}], \mathfrak{s})(i)$ be the zero-dimensional components of $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})(i)$.

Let $\hat{g}$ be a metric on $\mathbb{R} \times (-Y)$, and $\hat{\mathfrak{q}}$ be a collection of perturbations of the Seiberg-Witten equations on $(\mathbb{R} \times (-Y), \hat{g})$ for all isomorphism classes of spin^c structures, such that

1) $\hat{g}$ is the cylindrical metric given by g_1^Y on $(-\infty, 1] \times (-Y)$, and is the cylindrical metric given by g_2^Y on $[2, +\infty) \times (-Y)$;
2) $\hat{\mathfrak{q}}$ is given by the formal gradient of $\mathfrak{q}_1$ with respect to g_1^Y on $(-\infty, 1] \times (-Y)$, and is given by the formal gradient of $\mathfrak{q}_2$ with respect to g_2^Y on $[2, +\infty) \times (-Y)$.

For a generic choice of $\hat{\mathfrak{q}}|_{[1,2]\times(-Y)}$, the Seiberg-Witten equations on $(\mathbb{R} \times (-Y), \hat{g})$ with perturbation $\hat{\mathfrak{q}}$ defines a chain map

$$\check{C}_*\big(\mathbb{R} \times (-Y), \hat{\mathfrak{q}}\big) : \check{C}_*(-Y, g_1^Y, \mathfrak{q}_1) \to \check{C}_*(-Y, g_2^Y, \mathfrak{q}_2).$$

We only need to prove that

$$[\check{C}_*\big(\mathbb{R} \times (-Y), \mathfrak{q}\big)\big(\check{\psi}_{\mathfrak{o}}(X)(1)\big)] = \pm[\check{\psi}_{\mathfrak{o}}(X)(2)]$$

in the homology of $(\check{C}_*(-Y, g_2^Y, \mathfrak{q}_2), \check{\partial})$.

For $t \geq 1$, consider the Seiberg-Witten equations on X where the metric and the perturbation are given by $g_1, \omega_1, \hat{\tau}_1, \eta_1, \mathfrak{q}_1$ on $X - M_c$, are given by g_1^Y and $\mathfrak{q}_1$ on $[0, t] \times (-Y)$, are given by g_2^Y and $\mathfrak{q}_2$ on $[t+1, +\infty) \times (-Y)$, and are given by $(\hat{g}, \hat{\mathfrak{q}})|_{[1,2]\times(-Y)}$ on $[t, t+1] \times (-Y)$. Concatenate this family of equations with a smooth family of equations parametrized by $t \in [0, 1]$, such that at $t = 1$ the two equations coincide, at $t = 0$ the equation coincides with the equation defined by $g_2, \omega_2, \mathfrak{q}_2, \hat{\tau}_2, \eta_2, r_2$ on X. The family of equations can be chosen to be independent of t on $[2, +\infty) \times (-Y)$ for $t \in [0, 1]$. Moreover, by the assumptions on the ESBG structures, we may choose the family such that for each $t \in [0, 1]$, the ESBG end of X is M_s, and the family of metrics on X for $t \geq 0$ has uniformly bounded geometry.

For $t \geq 0$, let $g(t)$, $\omega(t)$, $\mathfrak{q}(t)$, $\hat{\tau}(t)$, $\eta(t)$, $r(t)$ be the corresponding geometric data as given above. Let $\mathfrak{C}(\mathfrak{q}_1, g_1^Y)$ be the set of critical points in the blown-up configuration space give by $(\mathfrak{q}_1, g_1^Y)$. For $\mathfrak{a} \in \mathfrak{C}$ and $\mathfrak{s}$ an admissible spin^c structure on X, define $\mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})(t)$ to be the moduli space given by Definition 5.1 with respect to $g(t)$, $\omega(t)$, $\mathfrak{q}(t)$, $\hat{\tau}(t)$, $\eta(t)$, $r(t)$. Define

$$\widetilde{\mathcal{M}}(X, [\mathfrak{a}], \mathfrak{s}) = \bigcup_{t \geq -1} \mathcal{M}(X, [\mathfrak{a}], \mathfrak{s})(t).$$

For a generic choice of $\hat{\tau}(t)$, the moduli space $\widetilde{\mathcal{M}}(X, [\mathfrak{a}], \mathfrak{s})$ is regular. Let $\widetilde{\mathcal{M}}_0(X, [\mathfrak{a}], \mathfrak{s})$ be the zero-dimensional components of $\widetilde{\mathcal{M}}(X, [\mathfrak{a}], \mathfrak{s})$, and let $\widetilde{\mathcal{M}}_1(X, [\mathfrak{a}], \mathfrak{s})$ be the one-dimensional components of $\widetilde{\mathcal{M}}(X, [\mathfrak{a}], \mathfrak{s})$. Then by increasing $r(t)$ if necessary, we have that $\widetilde{\mathcal{M}}_0(X, [\mathfrak{a}], \mathfrak{s})$ is compact, and so we can define an element

$$h_{\mathfrak{o}} = \sum_{\mathfrak{s}} \sum_{[\mathfrak{a}]\in\mathfrak{C}} \#\widetilde{\mathcal{M}}_0(X,[\mathfrak{a}],\mathfrak{s})\cdot\mathfrak{a} \in \check{C}_*(-Y, g_1^Y, \mathfrak{q}_1),$$

where the elements of $\widetilde{\mathcal{M}}_0(X,[\mathfrak{a}],\mathfrak{s})$ are counted with signs using the orientation $\mathfrak{o}$, and the summation of $\mathfrak{s}$ goes over the isomorphism classes of admissible spin^c structures over X relative to M_s.

The boundary of the compactification of $\widetilde{M}_1(X,[\mathfrak{a}],\mathfrak{s})$ consists of three parts: (i) elements of $\mathcal{M}_0(X,[\mathfrak{a}],\mathfrak{s})(2)$, (ii) broken trajectories given by an element of $\mathcal{M}_0(X,[\mathfrak{a}],\mathfrak{s})(1)$ and a solution of the blown-up Seiberg-Witten equations on $(\mathbb{R}\times(-Y),\hat{g})$ with respect to the perturbation $\hat{\mathfrak{q}}$, (iii) broken trajectories given by an element of $\widetilde{M}_0(X,[\mathfrak{a}],\mathfrak{s})$ and a solution of the blown-up Seiberg-Witten equations on $\mathbb{R}\times(-Y)$ with respect to $g_2^Y, \mathfrak{q}_2$. This implies

$$\check{C}_*\big(\mathbb{R}\times(-Y),\mathfrak{q}\big)\big(\check{\psi}_{\mathfrak{o}}(X)(1)\big) \pm \check{\psi}_{\mathfrak{o}}(X)(2) \pm \check{\partial} h_{\mathfrak{o}} = 0,$$

and the proposition is proved. □

Remark 5.6. Suppose Y is endowed with a contact structure ξ, let $X = \mathbb{R}\times Y$. Let $M_s = [1,+\infty)\times Y$ be given by the symplectization of ξ (cf. [17, (1)]), and let $M_c = (-\infty,0]\times Y$ be endowed with a cylindrical metric. Then the invariant

$$c(X) \in \widetilde{HM}(-Y)/\{\pm 1\}$$

coincides with the contact element of ξ defined by [20, Section 6.3]. In this case, it was proved by [21, Theorem H] that it is impossible to lift the contact class to $\widetilde{HM}(-Y)$ such that it is still an isotopy invariant.

If $Y = \emptyset$, then for each admissible $\mathfrak{s}$, counting the elements of zero-dimensional moduli space of solutions $(A,\phi)\in\mathcal{C}(X,\mathfrak{s})$ to (2.4) as in [17, Definition 2.5] gives a numerical invariant $SW(X,\mathfrak{s})\in\mathbb{Z}/\{\pm1\}$. The sign of $SW(X,\mathfrak{s})$ can be fixed by a choice of homology orientation of X following the same argument as in [17, Appendix]. By the compactness properties, there are only finitely many isomorphism classes of $\mathfrak{s}$ such that $SW(X,\mathfrak{s})\neq 0$. If ∂M_s is a contact manifold and (M_s,ω) is the symplectization, then $SW(X,\mathfrak{s})$ coincides with the monopole invariant defined by [17].

Lemma 5.7. *Suppose $(X,\omega=d\theta)$ is an ESBG end without boundary[2], let $Z\subset X$ be a 4-dimensional compact submanifold with boundary, let $M_s =$*

---

[2]In other words, X is a complete manifold without boundary that satisfies all the conditions in the definition of ESBG ends (with $\partial M = \emptyset$).

$X - Z$. View X as a manifold with an ESBG end $(M_s, \omega|_{M_s})$. Then there exists $r_0 > 0$ with the following property. Suppose $r > r_0$, $\mathfrak{s}$ is an admissilble spin^c structure relative to M_s, and

$$(A, \phi) \in \mathcal{C}(X, \mathfrak{s})$$

is a solution to (2.4). Then $\mathfrak{s}$ is isomorhphic to $\mathfrak{s}_{X,\omega}$, and (A, ϕ) is gauge equivalent to $(A_0, \sqrt{r}\Phi_0)$ over X. Moreover, the moduli space of solutions, which is a point, is regular.

Proof. Recall that by our convention, not only $\mathfrak{s}$ is isomorphic to $\mathfrak{s}_{X,\omega}$ on M_s, but there is also a fixed isomorphism from $\mathfrak{s}|_{M_s}$ to $\mathfrak{s}_{X,\omega}|_{M_s}$. Therefore, there is a complex line bundle E over X with a hermitian metric and a fixed isomorphism from E to $\underline{\mathbb{C}}$ on M_s, such that $\mathfrak{s} = \mathfrak{s}_{M,\omega} \otimes E$. To simplify the notation, we will identify $\mathfrak{s}$ with $\mathfrak{s}_{X,\omega}$ over M_s, and identify E with $\underline{\mathbb{C}}$ over M_s, using the fixed isomorphisms.

There is a unitary connection a on E, which is equal to the trivial connection of $\underline{\mathbb{C}}$ on M_s, such that A is equal to the coupling of A_0 and a. Decompose ϕ as $\sqrt{r}(\alpha + \beta)$ such that $\alpha \in T^{0,0}X \otimes E$, $\beta \in T^{0,2}X \otimes E$, where $T^{*,*}X$ is defined with respect to the almost complex structure induced by (X, ω). The same integration by parts as Lemma 4.1 gives

$$\begin{aligned}(5.2)\quad \int_X \Big(\frac{r}{2}|\bar{\partial}_a\alpha + \bar{\partial}_a^*\beta|^2 + 2|iF_a^\omega - \frac{r}{8}(1 - |\alpha|^2 + |\beta|^2)|^2 + 2|F_a^{0,2} - \frac{r}{4}\bar{\alpha}\beta|^2 \\ + \frac{r}{2}iF_a^\omega - 2|iF_a^\omega|^2 - 2|F_a^{0,2}|^2\Big) \\ = \int_X \Big(\frac{r}{4}|\nabla_a\alpha|^2 + \frac{r}{4}|\nabla_{A_1+a}\beta|^2 + \frac{r}{2}(iF^\omega_{A_1+a})|\beta|^2 \\ + \frac{r^2}{32}(1 - |\alpha|^2 - |\beta|^2)^2 + \frac{r^2}{8}|\beta|^2 - rRe\langle N \circ \partial_a\alpha, \beta\rangle\Big).\end{aligned}$$

For r_0 sufficiently large, the same argument as in (4.12) then gives

$$\int_X |1 - |\alpha|^2 - |\beta|^2|^2 + |\beta|^2 + |\nabla_a\alpha|^2 + |\nabla'_A\beta|^2 + |F_a^+|^2 \le \int_X \frac{r}{2}iF_a^\omega.$$

On the other hand, by the same argument that leads to (4.2),

$$\int_X F_a^\omega = \frac{1}{2}\int_X F_a \wedge \omega = \int_X F_a \wedge d\theta = -\int_X dF_a \wedge \theta = 0.$$

Therefore

$$|1-|\alpha|^2-|\beta|^2|^2+|\beta|^2+|\nabla_a\alpha|^2+|\nabla'_A\beta|^2+|F_a^+|^2=0$$

on X, hence E is the trivial bundle, a is the trivial connection, and (A,ϕ) is gauge equivalent to $(A_0,\sqrt{r}\,\Phi_0)$ over X. The regularity of the moduli space follows from a straightforward generalization of [17, Lemma 3.11]. $\square$

Corollary 5.8. *Suppose $(X,\omega=d\theta)$ is an ESBG end without boundary, let $Z\subset X$ be a 4-dimensional compact submanifold with boundary, let $M_s=X-Z$. View X as a manifold with an ESBG end $(M_s,\omega|_{M_s})$. Then*

$$\sum_{\mathfrak{s}} SW(X,\mathfrak{s})=\pm 1,$$

where the summation of $\mathfrak{s}$ goes over the isomorphism classes of admissible spin^c structures over X relative to M_s. $\square$

Now let $(X^{(1)},Z^{(1)},M_c^{(1)},M_s^{(1)},\theta^{(1)})$ and $(X^{(2)},Z^{(2)},M_c^{(2)},M_s^{(2)},\theta^{(2)})$ be as in Section 4.4. Assume that both $M_s^{(1)}$ and $M_s^{(2)}$ are non-compact, and that $M_c^{(1)}$ is given by $(-\infty,0]\times Y$, and $M_c^{(2)}$ is given by $(-\infty,0]\times(-Y)$. For each constant $R>0$, define X_R as in Section 4.4. Let

$$c(X^{(1)})\in \widecheck{HM}_\bullet(-Y)/\{\pm1\},\ c(X^{(2)})\in \widecheck{HM}_\bullet(Y)/\{\pm1\}$$

be given by (5.1). Then we have the following gluing result.

Proposition 5.9. *Let $j_*:\widecheck{HM}_\bullet(Y)\to\widehat{HM}_\bullet(Y)$ be the map defined by [18, Proposition 22.2.1], let $\langle\cdot,\cdot\rangle$ be the pairing of $\widehat{HM}_\bullet(-Y)$ and $\widehat{HM}_\bullet(Y)$ as given by [18, Corollary 22.5.11]. Then*

$$\langle c(X^{(1)}), j_*c(X^{(2)})\rangle=\pm\sum_{\mathfrak{s}} SW(X_R,\mathfrak{s}),$$

where the summation of $\mathfrak{s}$ goes over the isomorphism classes of admissible spin^c structures over X_R relative to $M_s^{(1)}\cup M_s^{(2)}$.

Proof. The proposition follows from Theorem 4.11 and the gluing argument of [18, Section 27]. $\square$

6. Monopoles Floer invariants of foliations

This section defines the invariants $c_\pm(\mathcal{F})$ for a smooth oriented foliation $\mathcal{F}$ on a closed oriented 3-manifold Y, where $\mathcal{F}$ does not admit holonomy-invariant transverse measure.

6.1. Symplectizations of smooth taut foliations

Let Y be a smooth closed oriented 3-manifold, let $\mathcal{F}$ be a smooth oriented foliation on Y. The orientations of Y and $\mathcal{F}$ induce a co-orientation of $\mathcal{F}$. Take a smooth non-zero 1-form λ such that $\mathcal{F} = \ker\hat{\lambda}$ and $\hat{\lambda}$ is positive on the positive side of $\mathcal{F}$. By Frobenius theorem, $\hat{\lambda}\wedge d\hat{\lambda} = 0$. Since $\mathcal{F}$ has no holonomy-invariant transverse measure, by Sullivan [26], there exists an exact 2-form $\hat{\omega}$ such that $\hat{\omega}\wedge\hat{\lambda} > 0$ everywhere on Y. Take a smooth 1-form $\hat{\theta}$ such that $d\hat{\theta} = \hat{\omega}$.

Consider the cylinder $\mathbb{R}\times Y$, let t be the coordinate of the $\mathbb{R}$-component. Let $\pi_Y : \mathbb{R}\times Y \to Y$ be the projection onto Y. Let $\omega = \pi_Y^*(\hat{\omega}) + d(t\pi_Y^*(\hat{\lambda}))$, let $\theta = \pi_Y^*(\hat{\theta}) + t\pi_Y^*(\hat{\lambda})$, then ω is a symplectic form on $\mathbb{R}\times Y$, and $\omega = d\theta$. Let $\lambda = \pi_Y^*(\hat{\lambda})$.

Fix a metric g_0 on Y such that $|\hat{\lambda}|_{g_0} = 1$ and $\hat{\lambda} = *\hat{\omega}$. Locally $\hat{\omega}$ can be written as $\hat{\omega} = e^1\wedge e^2$ where e^1 and e^2 are orthonormal cotangent vector fields on Y. Since $\hat{\lambda}\wedge d\hat{\lambda} = 0$, there is a unique 1-form μ_1 such that $d\hat{\lambda} = \mu_1\wedge\hat{\lambda}$ and $\langle\mu_1,\hat{\lambda}\rangle_{g_0} = 0$.

We have $d\mu_1\wedge\hat{\lambda} = d(\mu_1\wedge\hat{\lambda}) = d(d\hat{\lambda}) = 0$, hence there is a unique 1-form μ_2 such that $d\mu_1 = \mu_2\wedge\hat{\lambda}$ and $\langle\mu_2,\hat{\lambda}\rangle_{g_0} = 0$.

Now we define a Riemannian metric on $\mathbb{R}\times Y$ that is compatible with ω as follows. Notice that locally $\omega = e^1\wedge e^2 + dt\wedge\lambda + t\mu_1\wedge\lambda$. Take

$$g = e^1\otimes e^1 + e^2\otimes e^2 + (1+t^2)\lambda\otimes\lambda + \frac{1}{1+t^2}(dt + t\mu_1)\otimes(dt + t\mu_1). \tag{6.1}$$

It is easy to verify that g does not depend on the choice of e^1 and e^2, and that it is compatible with ω. Denote $\mathbb{R}\times Y$ by X.

Lemma 6.1. *The manifold $(X, g, \omega = d\theta)$ has the following properties:*

1) *X is complete.*
2) *The injectivity radius of X is bounded from below by a positive number.*
3) *Let R be the curvature tensor of X, and ∇ be the Levi-Civita connection, then $\sup_X|\nabla^k R| < +\infty$ for each k.*

4) $\sup_X |\nabla^k \theta| < +\infty$ *for each* k.

Proof. Suppose x is a real number and u is a vector tangent to the Y component of X, let $v = x \cdot \frac{\partial}{\partial t} + u$ be a tangent vector of X. By the definition of g and Cauchy's inequality:

$$
\begin{aligned}
&|v| \cdot \sqrt{t^2|\mu_1|^2 + t^2 + 1} \\
\geq &\sqrt{|u|^2 + \frac{1}{1+t^2}\big(x + t\cdot\mu_1(u)\big)^2} \cdot \sqrt{t^2|\mu_1|^2 + (1+t^2)} \\
\geq &|t||\mu_1||u| + |x + t\cdot\mu_1(u)| \\
\geq &|x|.
\end{aligned}
$$

Therefore $|v| \geq |x|/\sqrt{1 + z\cdot t^2}$, where $z = \sup|\mu_1|^2 + 1$. The length of a curve from the slice $t = -T$ to $t = T$ is therefore at least

$$\int_{-T}^{T} 1/\sqrt{1 + z\cdot t^2}\, dt.$$

Since

$$\int_{-\infty}^{\infty} 1/\sqrt{1 + z\cdot t^2}\, dt = +\infty,$$

this implies the completeness of X.

To prove the boundedness of $|\nabla^k R|$ and $|\nabla^k \theta|$, we use the moving frame method. Take an arbitrary point q on Y, choose local chart U_q of q, and fix a choice of e^1 and e^2 on U_q. Let

$$
\begin{aligned}
e^3 &= \sqrt{1+t^2}\cdot\lambda, \\
e^4 &= \frac{1}{\sqrt{1+t^2}}(dt + t\cdot\mu_1).
\end{aligned}
$$

Then $\{e^1, e^2, e^3, e^4\}$ form an orthonormal basis of the cotangent bundle on $U_q \times \mathbb{R}$. There exist smooth functions ν_i on U_q $(i = 1, 2, ..., 10)$, such that

$$
\begin{aligned}
d\,e^1 &= \nu_1\, e^1 \wedge e^2 + \nu_2\, e^1 \wedge \lambda + \nu_3\, e^2 \wedge \lambda, \\
d\,e^2 &= \nu_4\, e^1 \wedge e^2 + \nu_5\, e^1 \wedge \lambda + \nu_6\, e^2 \wedge \lambda, \\
\mu_1 &= \nu_7\, e^1 + \nu_8\, e^2, \\
\mu_2 &= \nu_9\, e^1 + \nu_{10}\, e^2.
\end{aligned}
$$

By shrinking U_q if necessary and identifying U_q with a subset of $\mathbb{R}^3$, we have $\|\nu_i\|_{C^m(U_q)} < +\infty$ for all m. A straightforward calculation shows:

$$
\begin{cases}
d\,e^1 &= \nu_1\, e^1\wedge e^2 + \frac{\nu_2}{\sqrt{1+t^2}}\, e^1\wedge e^3 + \frac{\nu_3}{\sqrt{1+t^2}}\, e^2\wedge e^3,\\
d\,e^2 &= \nu_4\, e^1\wedge e^2 + \frac{\nu_5}{\sqrt{1+t^2}}\, e^1\wedge e^3 + \frac{\nu_6}{\sqrt{1+t^2}}\, e^2\wedge e^3,\\
d\,e^3 &= \frac{t}{\sqrt{1+t^2}}\, e^4\wedge e^3 + \frac{\nu_7}{1+t^2}\, e^1\wedge e^3 + \frac{\nu_8}{1+t^2}\, e^2\wedge e^3,\\
d\,e^4 &= \frac{1}{1+t^2} e^4\wedge(\nu_7\, e^1+\nu_8\, e^2) - \frac{t}{1+t^2} e^3\wedge(\nu_9\, e^1+\nu_{10}\, e^2).
\end{cases}
\tag{6.2}
$$

Write

$$de^i = \sum_{j\neq k} a^i_{jk}\, e^j\wedge e^k,$$

such that $a^i_{jk} = -a^i_{jk}$, then the equations above imply that $\|a^i_{jk}\|_{C^m(\mathbb{R}\times U_q)} < +\infty$ for each m.

Suppose $\nabla e^i = \omega^i_j \otimes e^j$, where ∇ is the Levi-Civita connection. Then the connection matrix $\{\omega^j_i\}$ can be calculated from $\{a^i_{jk}\}$ by the formula

$$\omega^i_j = \sum_k (-a^k_{ji} + a^i_{kj} + a^j_{ik}) e^k,$$

and the curvature matrix under the basis $\{e^i\}$ is given by $d\omega^j_i - \omega^k_i\wedge\omega^j_k$. Since a^i_{jk} and their exterior derivatives are bounded, it follows that under the basis $\{e^i\}$, every component of $\nabla^m R$ is bounded on $\mathbb{R}\times U_q$ for all $m\geq 1$. This proves the boundedness of $|\nabla^m R|$ on $\mathbb{R}\times U_q$. Since Y is compact, it can be covered by finitely many such U_q's, therefore $|\nabla^m R|$ is bounded on $X = \mathbb{R}\times U_q$ for every m.

For the estimates on θ, write θ as

$$\theta = \nu_{11}e^1 + \nu_{12}e^2 + \nu_{13}\lambda,$$

then

$$\theta = \nu_{11}e^1 + \nu_{12}e^2 + \frac{t+\nu_{13}}{\sqrt{1+t^2}}e^3,$$

and the same calculation proves the boundedness of $|\nabla^m\theta|$.

For the lower bound on injectivity radius, we need the following theorem:

Theorem ([8, Theorem 4.7(i)]). *Let (M^n, g) be a complete Riemannian manifold, let R be the Riemannian curvature tensor, let $K > 0$ be a constant* *such that $|R| \le K$ on M. Let $0 < r < \frac{\pi}{4\sqrt{K}}$. Then the injectivity radius at each point $p \in M$ satisfies the following inequality:*

$$\operatorname{inj}(p) \ge \frac{r}{2} \cdot \frac{1}{1 + V_{2r}^{-K} / \operatorname{Vol}(B_p(r))}, \tag{6.3}$$

where V_{2r}^{-K} is the volume of a geodesic ball of radius $2r$ on the hyperbolic n–space with constant curvature $-K$.

Proof. This is a special case of [8, Theorem 4.7(i)] with $H = -K$, $x = p$, and $r_0 = s = r$. □

Back to the proof of Lemma 6.1. Let $p = (t, q) \in X$. The following argument will show that $\operatorname{Vol}(B_p(r))$ is bounded from below by a positive constant independent of p. Without loss of generality, assume $|t| > 1$.

Let $K > 0$ be an upper bound of $|R|$. For each point $q \in Y$, let L_q be the leaf of $\mathcal{F}$ through q, the metric on L_q is taken to be the restriction from g_0. Let $\epsilon = \inf_{q \in Y} \operatorname{inj}(L_q)$. Since Y is compact, ϵ is positive. Let $r = \frac{1}{2} \min\{\frac{\pi}{4\sqrt{K}}, \epsilon\}$.

Let $D(q, r/3)$ be the open disk of radius $r/3$ on L_q centered at q, Let

$$U = \{x \in Y | \operatorname{dist}_{g_0}(x, D(q, r/3)) < \frac{r}{3\sqrt{1+t^2}}\}.$$

Then the distance from each point in U to $D(q, r/3)$ under the metric $g|_{Y \times \{t\}}$ is less than $r/3$, thus the distance from each point of U to q is less than $2r/3$. Therefore,

$$B_p(r) \supseteq (e^{-r/3}\, t, e^{r/3}\, t) \times U.$$

The volume of U under the metric g_0 is bounded from below by a constant multiple of $r/(3\sqrt{1+t^2})$, where the constant depends only on g_0 and $\mathcal{F}$. Therefore the volume of $U \times (e^{-r/3}\, t, e^{r/3}\, t)$ under the product metric $\mathbb{R} \times (Y, g_0)$ is bounded from below by a positive constant. Notice that the volume form of the product metric on $\mathbb{R} \times (Y, g_0)$ is the same as the volume form of g. Therefore

$$\operatorname{Vol}(B_p(r)) > \frac{1}{z_2} \tag{6.4}$$

for some positive constant z_2 depending on $\mathcal{F}$ and g_0. The lower bound of injectivity radius of X then follows immediately from (6.3). □

Remark 6.2. The fact that the injectivity radius of X is bounded from below could be counter intuitive because of the factor $\frac{1}{1+t^2}$ in the definition of g. In fact, by the proof of Lemma 6.1, one can visualize the geometry of X as follows. First consider the three manifold Y with the metric g_0. For any $x \in Y$, $r, \epsilon > 0$, let L_x be the leaf of $\mathcal{F}$ containing x with the induced metric from g_0, let D_r be the r-neighborhood of x in L_x, and let $D_r(\epsilon)$ be the ϵ neighborhood of D_r in Y. When r is fixed and ϵ is small, $D_r(\epsilon)$ is a thin slice near D_r. Now let $r_0 > 0$ be a lower bound of the injectivity radius, then a normal neighborhood of X centering at (t, q) with radius r_0 contains the set $D_{r_0/3}\big(\frac{r_0}{3\sqrt{1+t^2}}\big) \times (e^{-r_0/3}t, e^{r_0/3}t)$. When t is large, this is (a much thinner slice near $D_{r_0/3}$) $\times$ (a long interval).

6.2. The definition of $c_\pm(\mathcal{F})$

This subsection defines the monopole Floer invariants $c_\pm(\mathcal{F})$ for the smooth foliation $\mathcal{F}$.

Let $X = \mathbb{R} \times Y$, and let g be the metric on X defined by (6.1). Let $\omega = d\theta$ be the compatible symplectic form on X as defined in Section 6.1.

Let g^+ be a Riemannian metric on X that is equal to g on $(-\infty, -1] \times Y$, and is cylindrical on $[1, +\infty) \times Y$. Let g^- be a Riemannian metric on X that is equal to g on $[1, +\infty) \times Y$, and is cylindrical on $(-\infty, -1] \times Y$. Let X_{g^+} be the Riemannian manifold (X, g^+), and let X_{g^-} be the Riemannian manifold (X, g^-). By Lemma 6.1, X_{g^+} is a manifold with cylindrical and ESBG ends, where the ESBG structure is given by $\omega = d\theta$ on $(-\infty, -1] \times Y$. Similarly, X_{g^-} is a manifold with cylindrical and ESBG ends, where the ESBG structure is given by $\omega = d\theta$ on $[1, +\infty) \times Y$.

Definition 6.3. *Define*

$$c_+(\mathcal{F}) = c(X_{g^+}) \in \widecheck{HM}_\bullet(Y)/\{\pm 1\},$$
$$c_-(\mathcal{F}) = c(X_{g^-}) \in \widecheck{HM}_\bullet(-Y)/\{\pm 1\},$$

where $c(\cdot)$ is given by (5.1).

By Proposition 5.5, $c(X_{g^\pm})$ are invariant under deformations of the ESBG structures, it follows that $c_\pm(\mathcal{F})$ are independent of the choice of g_0 and $\hat{\lambda}$, and are invariant under smooth deformations of $\mathcal{F}$ via foliations without holonomy-invariant transverse measure.

Let $j_* : \widetilde{HM}_\bullet(Y) \to \widehat{HM}_\bullet(Y)$ be the map in the long exact sequence of monopole Floer homologies introduced by [18, Proposition 22.2.1]. The next theorem proves the nonvanishing of $j_* c_\pm(\mathcal{F})$.

Theorem 6.4. *Let $\mathcal{F}$ be a smooth foliation on Y with no holonomy-invariant transverse measure, then*

$$j_* c_+(\mathcal{F}) \neq 0 \in \widehat{HM}_\bullet(Y)/\{\pm 1\},$$
$$j_* c_-(\mathcal{F}) \neq 0 \in \widehat{HM}_\bullet(-Y)/\{\pm 1\}.$$

Proof. By Proposition 5.9 and Corollary 5.8, we have

$$\langle c_\mp(\mathcal{F}), j_* c_\pm(\mathcal{F})\rangle = \pm \sum_{\mathfrak{s}} SW(X, \mathfrak{s}) = \pm 1. \tag{6.5}$$

Hence $j_* c_\pm(\mathcal{F}) \neq 0$. □

Theorem 6.5. *The grading of $c_\pm(\mathcal{F}) \in \widetilde{HM}_\bullet(\pm Y)$ is represented by the homotopy class of the tangent plane field of $\mathcal{F}$.*

Proof. The grading of $c_+(\mathcal{F})$ is represented by a nowhere vanishing section $\psi \in \Gamma(Y \times \{0\}, \mathbb{S}^+)$ such that it extends to a nowhere vanishing section of $\mathbb{S}^+|_{(-\infty,0]\times Y}$ that is asymptotic to the canonical section Ψ_0 at $t \to -\infty$. Therefore, we can take ψ to be $\Phi_0|_{Y\times\{0\}}$. A straightforward calculation then shows that the plane field corresponding to $(\mathbb{S}^+, \psi)$ is homotopic to $\ker\alpha = \mathcal{F}$. □

7. Topological applications

Corollary 7.1 ([20, Theorem 2.1], [18, Theorem 41.4.1]). *Let Y be an oriented three-manifold. If $\mathcal{F}$ is a smooth foliation on Y without holonomy-invariant transverse measure, let $[\mathcal{F}]$ be the homotopy class of the tangent plane field of $\mathcal{F}$, let $HM_{[\mathcal{F}]}(Y)$ be the reduced monopole Floer homology at the degree represented by $[\mathcal{F}]$. Then $HM_{[\mathcal{F}]}(Y) \neq 0$.*

Proof. This is an immediate consequence of Theorem 6.4 and Theorem 6.5. □

Corollary 7.2 ([17, Corollary 1.5]). *There are only finitely many homotopy classes of plane fields on Y that can be realized by the tangent plane field of a smooth foliation without holonomy-invariant transverse measure.*

Proof. By [18, Proposition 22.2.3], $HM_\bullet(Y)$ has finite rank, hence the result follows from Corollary 7.1. $\square$

Since every foliation without holonomy-invariant transverse measure is a taut foliation, the corollaries above are special cases of the non-vanishing and finiteness results in [17, 20]. On the other hand, by the discussions in Section 1.2, on a rational homology sphere every foliation without holonomy-invariant transverse measure is a taut foliation. Therefore, Corollary 7.1 and Corollary 7.2 yield alternative proofs for the non-vanishing and finiteness results of smooth taut foliations on rational homology spheres, without making reference to the Eliashberg-Thurston perturbation.

We can improve Corollary 7.1 to a more general class of three-manifolds. The following lemma shows that in many cases, smooth folaitions without holonomy-invariant transverse measure are "generic" among smooth taut foliations. The result was explained to the author by Jonathan Bowden.

Lemma 7.3 ([4]). *Let Y be an atoroidal manifold and $\mathcal{F}$ a smooth taut foliation on Y. Then either $\mathcal{F}$ can be C^0 isotoped to smooth folaition $\mathcal{F}'$ without holonomy-invariant transverse measure, or Y is diffeomorphic to a surface bundle over S^1.*

Proof. By [2], the foliation $\mathcal{F}$ can be C^0 approximated by a smooth taut folaition $\mathcal{F}_1$, such that every closed leaf of $\mathcal{F}_1$ has genus 0 or 1. If $Y \cong S^2 \times S^1$, then $\mathcal{F}$ is homeomorphic to the product foliation, and the statement of the lemma is verified. If $Y \ncong S^2 \times S^1$, by Reeb's stability theorem the foliation $\mathcal{F}_1$ has no closed leaf with genus 0. Since every closed leaf of a taut foliation is incompressible and Y is assumed to be atoroidal, the foliation $\mathcal{F}_1$ has no torus leaf. This proves that $\mathcal{F}_1$ has no closed leaf.

If $\mathcal{F}_1$ has a holonomy-invariant transverse measure μ, let A be a minimal set contained in the support of μ. The existence of A follows from [7, Corollary 4.1.13]. Since $\mathcal{F}_1$ has no closed leaf, the minimal set A is either equal to Y or is exceptional as defined in [7, Definition 4.1.4]. If A is exceptional, by Sacksteder's theorem [7, Theorem 8.2.1], there exists a leaf L in A containing a curve of contracting linear holonomy. Since L is in the support of μ, on a neighborhood of L the measure μ has to be a constant multiple of the delta measure of L. This implies that L is a closed leaf, which is a contradiction. Therefore $A = Y$. By [7, Proposition 9.5.8], in this case Y is diffeomorphic to a surface bundle over S^1. $\square$

Corollary 7.4 ([20, Theorem 2.1]). *Suppose $Y \neq S^1 \times S^2$, Y is atoroidal, and Y supports a smooth taut foliation, then $HM_\bullet(Y) \neq 0$.*

Proof. If $\mathcal{F}$ can be C^0 approximated by a smooth taut folaition $\mathcal{F}'$ such that $\mathcal{F}'$ has no holonomy-invariant transverse measure, then the result follows from Corollary 7.1. Otherwise, by Lemma 7.3, $\mathcal{F}$ can be C^0 approximated by a smooth taut folaition $\mathcal{F}'$ such that $(Y, \mathcal{F}')$ is homeomorphic to a surface bundle over S^1 foliated by the fibers. Since Y is atoroidal and $Y \neq S^1 \times S^2$, the genus of the fiber is at least 2. In this case, the desired result follows from [19, Theorem 3.1] and [19, Lemma 2.2]. $\square$

Recall that by Theorem 6.4 and Theorem 6.5, $c_\pm(\mathcal{F})$ are non-zero and are graded by the homotopy class of $\mathcal{F}$. It turns out that the invariants $c_\pm(\mathcal{F})$ are stronger than the homotopy class itself. The rest of this section constructs examples of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ such that they are homotopic as plane fields but $c_+(\mathcal{F}_1) \neq c_+(\mathcal{F}_2)$, $c_+(\mathcal{F}_1) \neq c_+(\mathcal{F}_2)$. Since $c_\pm(\mathcal{F})$ are invariant under smooth deformations, this gives examples of smooth foliations without holonomy-invariant transverse measure that are homotopic as plane fields, but cannot be smoothly deformed to each other via foliations without holonomy-invariant transverse measure.

Proposition 7.5. *Suppose M is a compact oriented 4-manifold with boundary, and let $Y = \partial M$, where Y is oriented such that there is an orientation-preserving diffeomorphim from $[0,1) \times Y$ to a neighborhood of Y in M. Let $\mathcal{F}$ be a smooth co-oriented foliation on Y that has no holonomy-invariant transverse measure. Assume there is an exact symplectic form ω on M such that $\omega|_Y$ is positive on $\mathcal{F}$. Assume further that $2\,c_1(\omega) \neq 0$. Let $-\mathcal{F}$ be the same foliation as $\mathcal{F}$ but with reversed orientation. Then $c_+(\mathcal{F})$ and $c_+(-\mathcal{F})$ are linearly independent in $\widetilde{HM}_\bullet(Y) \otimes \mathbb{Q}$, and $c_-(\mathcal{F})$ and $c_-(-\mathcal{F})$ are linearly independent in $\widetilde{HM}_\bullet(-Y) \otimes \mathbb{Q}$.*

Proof. Remove a small ball in M, the remaining part of M forms a cobordism from Y to S^3. For any Spin^c structure $\mathfrak{s}$ on M, it induces a map $\widehat{HM}(M - B^3, \mathfrak{s}) : \widehat{HM}_*(S^3) \to \widehat{HM}_*(Y)$. Let $\hat{1} \in \widehat{HM}_*(S^3) \cong \mathbb{Z}[U_\dagger]$ be a generator as $\mathbb{Z}[U_\dagger]$ module, then $\widehat{HM}(M - B^3, \mathfrak{s})(\hat{1}) \in \widehat{HM}_\bullet(-Y)$.

Write $\mathcal{F} = \ker \hat{\lambda}$, where $\hat{\lambda}$ is a 1-form on Y such that $\hat{\lambda}$ is positive on the positive side ot $\mathcal{F}$. Let λ be the pull-back of $\hat{\lambda}$ to $\mathbb{R} \times Y$.

Let $\widetilde{M} = (-\infty, 0] \times Y \cup_{\partial M} M$. We can define an exact symplectic form on $\widetilde{M}$ as follows. Let Let $M_1 = [-1, 0] \times Y \cup_{\partial M} M$. Let $\eta : (-\infty, 0] \to [-1, 0]$ be a smooth non-decreasing function such that $\eta(t) = -1$ when $t \leq -1$, $\eta(t) = t$ when $t \geq -1/2$. Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a non-decreasing function such that $\chi(t) = t$ when $t \leq -1/2$, and $\chi(t) = 0$ when $t \geq -1/4$. Let $\pi : \widetilde{M} \to M_1$ be the map defined by $\pi = \eta \times \mathrm{id}_Y$ on $(-\infty, 0] \times Y$, and

$\pi = \mathrm{id}_M$ on M. Let $\varphi : M_1 \to M$ be a diffeomorphism that is maps $(-1, x)$ to $(0, x)$ for all $x \in Y$ on the boundary. Let $\epsilon > 0$ be a constant. Define $\tilde{\omega} = (\varphi \circ \pi_1)^*\omega + d(\chi(\epsilon\lambda))$, where $\chi(\epsilon\lambda)$ is defined to be zero on M. It is straightforward to verify that when ϵ is sufficiently small, $\tilde{\omega}$ is symplectic on $\widetilde{M}$. If we endow $\widetilde{M}$ with a compatible metric such that it is equal to the metric given by (6.1) on $(-\infty, -1]$, then $\widetilde{M}$ is a ESBG end without boundary.

By the gluing property and Lemma 5.7,

$$\langle \widehat{HM}(M - B^3, \mathfrak{s})(\hat{1}), c_+(\mathcal{F})\rangle = \begin{cases} \pm 1 & \text{if } \mathfrak{s} \cong \mathfrak{s}_{\mathrm{M},\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

If we change $\mathcal{F}$ to $-\mathcal{F}$ and change the symplectic form on M from ω to $-\omega$, the canonical Spin^c structure is then changed to the conjugation of $\mathfrak{s}_0$, hence we have,

$$\langle \widehat{HM}(M - B^3, \mathfrak{s})(\hat{1}), c_+(-\mathcal{F})\rangle = \begin{cases} \pm 1 & \text{if } \mathfrak{s} \cong \mathfrak{s}_{\mathrm{M},-\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $2\, c_1(\omega) \neq 0$, the Spin^c structures $\mathfrak{s}_{M,\omega}$ and $\mathfrak{s}_{M,-\omega}$ are not isomorhpic, therefore $c_+(\mathcal{F})$ and $c_+(-\mathcal{F})$ are linearly independent. The proof for c_- follows from a similar argument. □

The next lemma provides examples that satisfy the conditions of Proposition 7.5. The result was explained to the author by Cheuk-Yu Mak. Recall that a contact form α on Y is said to have a strong symplectic filling if Y bounds a compact symplectic 4-manifold (M, ω), such that there is a vector field v near ∂M with $(\iota_v\omega)|_Y = \alpha$.

Lemma 7.6. *Let Y be an S^1 bundle over a compact surface of genus g with Euler number $e < 0$ and $e \neq 2 - 2g$. Then there exists a contact form α on Y, such that α has an exact strong symplectic filling with a non-torsion first Chern class, and such that the Reeb vector field of α is the positive unit tangent vector field of the S^1-fibers.*

Proof. Let E be a holomorphic line bundle with Euler number e over a Riemann surface of genus g, and let J be the complex structure on E. Let h be an Hermittian metric on E such that its Chern connection has negative curvature. Let E_1 be the unit disk bundle of E with respect to the metric h, then E_1 is a complex manifold with a J-convex boundary as defined in [9, Section 2.3]. The circle bundle ∂E_1 is a principal $U(1)$-bundle and the Chern connection of E induces a connection on ∂E_1. Let α_0 be the connection form on ∂E_1, then $\ker \alpha_0 = T\partial E_1 \cap J(T\partial E_1)$ is a contact structure on ∂E_1, and the Reeb vector field of α_0 is the positive unit tangent vector field of the S^1-fibers. For more details of this computation, the reader may refer to [9, Section 2.5]

By [1, Theorem (2')], there exists a smooth family of integrable almost complex structures J_t, $t \in (0,1)$ on E_1, such that $J_0 = J$ and (E_1, J_t) is Stein when $t > 0$.

Let f be a J_0-convex function defined near ∂E_1, such that $\partial E_1 = f^{-1}(1)$, the value 1 is a regular value of f, and that $f < 1$ in the interiori of E_1. Then there exists $\epsilon_0 > 0$, such that for all $0 < \delta < \epsilon_0$, the function f is J_δ-convex. Let $\alpha_\delta := df \circ J_\delta$ be a 1-form on $f^{-1}(1) = \partial E_1$, then $\alpha_{1-\delta}$ is a contact form on ∂E_1.

For sufficiently small δ, the contact structure $\ker \alpha_\delta$ is C^∞ close to $\ker \alpha_0$, hence by Gray's stability theorem there exists a diffeomorphism $\iota : \partial E_1 \to \partial E_1$ which is isotopic to the identity, and a positive function u on ∂E_1, such that $\iota^*(u \cdot \alpha_\delta) = \alpha_0$. The Reeb vector field of $\iota^*(u \cdot \alpha_\delta)$ is therefore the positive unit tangent vector field of the S^1-fibers. Notice that for a sufficiently large constant C, there exists a strong symplectic cobordism from $(\partial E_1, u \cdot \alpha_\delta)$ to $(\partial E_1, \alpha_\delta / C)$. Since $(\partial E_1, \alpha_\delta)$ is Stein fillable, this implies that the contact form $u \cdot \alpha_\delta$ is has a strong exact filling, therefore $\iota^*(u \cdot \alpha_\delta)$ has a strong exact filling. The first Chern class of the filling is equal to the first Chern class of the complex manifold (E, J), which is not torsion when $e \neq 2 - 2g$. Since $Y \cong \partial E_1$, this proves the lemma. $\square$

Let Y be an S^1 bundle over a compact surface of genus $g > 1$ with Euler number e, such that $2 - 2g < e < 0$. By [29], there exists an oriented smooth foliation $\mathcal{F}$ on Y which is transverse to the S^1 fibers. Let $-\mathcal{F}$ be the same foliation as $\mathcal{F}$ but with the opposite orientation.

Proposition 7.7. *Let Y, e, $\mathcal{F}$, and $-\mathcal{F}$ be as above, and assume $e | 2g - 2$. Then $\mathcal{F}$, $-\mathcal{F}$ are foliations without holonomy-invariant transverse measure, then $\mathcal{F}$ and $-\mathcal{F}$ are homotopic as oriented plane fields, but $c_+(\mathcal{F}) \neq c_+(-\mathcal{F})$, and $c_-(\mathcal{F}) \neq c_-(-\mathcal{F})$.*

Proof. By Lemma 7.6, there exists a contact form α on Y with a strong exact symplectic filling (M, ω), such that $c_1(\omega)$ is not torsion on M, and the Reeb vector field of α is positively transverse to $\mathcal{F}$. Notice that the Reeb vector field being positively transverse to $\mathcal{F}$ is equivalent to the form ω being positive on $\mathcal{F}$. Since ω is exact, this implies that $\mathcal{F}$ and $-\mathcal{F}$ have no holonomy-invariant transverse measure. Moreover, by Proposition 7.5,

$c_+(\mathcal{F})$ and $c_+(-\mathcal{F})$ are linearly independent, $c_-(\mathcal{F})$ and $c_-(-\mathcal{F})$ are linearly independent.

It remains to prove that $\mathcal{F}$ and $-\mathcal{F}$ are homotopic as plane fields. Let $S^1 \to Y \xrightarrow{\pi} \Sigma$ be the bundle structure of Y, let $e(Y) \in H^2(\Sigma)$ be the Euler class of the bundle. By the Gysin exact equence,

$$H^0(\Sigma) \xrightarrow{\cup e(Y)} H^2(\Sigma) \xrightarrow{\pi^*} H^2(Y)$$

is exact. Notice that $\mathcal{F}$ is isomorphic to $\pi^*(T\Sigma)$ as a plane bundle, therefore the assumption $e|2g-2$ implies that the Euler class of $\mathcal{F}$ is zero, hence $\mathcal{F}$ has a global basis $\{e_1, e_2\}$. Let e_3 be the positively oriented normal vector field of $\mathcal{F}$, then for $t \in [0,1]$ the family of plane fields $\mathcal{F}_t = \mathrm{span}\big\{\mathrm{e}_1, \cos(\pi \mathrm{t})\,\mathrm{e}_2 + \sin(\pi \mathrm{t})\,\mathrm{e}_3\big\}$ defines a homotopy from $\mathcal{F}$ to $-\mathcal{F}$. $\square$

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