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Augmentations are sheaves for Legendrian graphs

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In this article, associated to a (bordered) Legendrian graph, we study and show the equivalence between two categorical Legendrian isotopy invariants: the augmentation category, a unital A_∞-category, which lifts the set of augmentations of the associated Chekanov-Eliashberg DGA, and a DG category of constructible sheaves on the front plane, with micro-support at contact infinity controlled by the (bordered) Legendrian graph. In other words, generalizing [21], we prove "augmentations are sheaves" in the singular case.

1. Introduction

Nowadays, it has been an increasingly rich subject, to study the microlocal sheaf theoretic aspects of symplectic topology, or vice versa. This has already been of interest in the case of a cotangent bundle T^*M. Along this direction, one fundamental result is the microlocalization equivalence [19, 20], between the DG category of constructible sheaves on the base M and the infinitesimally wrapped Fukaya category of the cotangent bundle T^*M:

$$\mathrm{NZ} : \mathcal{S}\mathsf{h}(M; \mathbb{K}) \xrightarrow{\sim} \mathcal{F}\mathsf{uk}^{\epsilon}(T^*M; \mathbb{K}).$$

A more refined version of the correspondence involves introducing a Legendrian Λ contained in $T^\infty M$, the co-sphere bundle (equivalently, the set of points at contact infinity) of M. By replacing $\mathcal{S}\mathsf{h}(M;\mathbb{K})$ with a full subcategory $\mathcal{S}\mathsf{h}_\Lambda(M;\mathbb{K})$ of constructible sheaves with micro-support at infinity contained in Λ, and replacing $\mathcal{F}\mathsf{uk}^\epsilon(T^*M;\mathbb{K})$ with a full subcategory of Lagrangian branes asymptotic to Λ at infinity, the refined equivalence from the Nadler-Zaslow correspondence becomes as follows:

$$\mathrm{NZ} : \mathcal{S}\mathsf{h}_\Lambda(M;\mathbb{K}) \xrightarrow{\sim} \mathcal{F}\mathsf{uk}^\epsilon_\Lambda(T^*M;\mathbb{K}),$$

Here, both sides are categorical Legendrian isotopy invariants for $\Lambda \subset T^\infty M$. More recently, a correspondence [12–14] has also been established, between the DG category $\mathcal{S}\mathsf{h}_\Lambda(M)^c$ of compact objects in the larger category of (weakly) constructible sheaves on M, with micro-support contained in Λ, and a wrapped Fukaya category $\mathrm{Perf}\,\mathcal{W}(T^*M,\Lambda)^{\mathsf{op}}$ of T^*M, whose Lagrangians are disjoint from Λ at infinity:

$$\mathrm{GPS} : \mathcal{S}\mathsf{h}_\Lambda(M)^c \simeq \mathrm{Perf}\,\mathcal{W}(T^*M,\Lambda)^{\mathsf{op}}.$$

Notice that the latter is of central interest in Homological Mirror Symmetry.

On the other hand, there is another powerful modern Legendrian isotopy invariant for a Legendrian submanifold Λ in any contact manifold V. It is called the Legendrian contact homology DGA $A^{\mathsf{CE}}(V,\Lambda)$, obtained by counting of holomorphic disks. More specifically, the generators of this algebra are the Reeb chords of Λ, and the differential counts holomorphic disks with punctures on the boundary, in the symplectization $\mathbb{R}_t \times V$, with boundary living on the Lagrangian $\mathbb{R}_t \times \Lambda$ and approaching the Reeb chords at the punctures. The DGA $A^{\mathsf{CE}}(V,\Lambda)$ is a Legendrian isotopy invariant up to homotopy equivalence.

Motivated by the Nadler-Zaslow correspondence, it is expected that certain representation category of $A^{\mathsf{CE}}(\Lambda)$ is equivalent to the sheaf category $\mathcal{S}\mathsf{h}_\Lambda(N \times \mathbb{R}_z;\mathbb{K})$, when $V = J^1N \cong T^{\infty,-}(N \times \mathbb{R}_z)$ is a one-jet bundle naturally identified with an open contact submanifold of $T^\infty(M = N \times \mathbb{R}_z)$.

In this article, the case of the major interest to us is when $N = \mathbb{R}_x$, in which there is a rich interaction with Legendrian knot theory. That is, $\Lambda \subset J^1\mathbb{R}_x = \mathbb{R}^3_{\mathrm{std}}$ with a Maslov potential μ is now a Legendrian knot in the standard contact three space. In this case, "augmentations are sheaves" holds.

Theorem 1.1 ([21]). *There is an A_∞-equivalence:*

$$\mathrm{NRSSZ} : \mathcal{A}\mathsf{ug}_+(\Lambda,\mu;\mathbb{K}) \xrightarrow{\sim} \mathcal{C}_1(\Lambda,\mu;\mathbb{K})$$

Here, $\mathcal{A}\mathsf{ug}_+(\Lambda,\mu;\mathbb{K})$ is a unital A_∞-category whose objects are the set of augmentations of the DGA $A^{\mathsf{CE}}(\Lambda,\mu)$, a representation category of rank 1 representations of $A^{\mathsf{CE}}(\Lambda,\mu)$, and $\mathcal{C}_1(\Lambda,\mu;\mathbb{K})$ is the full subcategory of $\mathcal{S}\mathsf{h}_\Lambda(\mathbb{R}^2_{x,z};\mathbb{K})$ whose objects are microlocal rank 1 sheaves with acyclic stalks when $z \ll 0$. Concerning "representations are sheaves", or the correspondence between higher rank representations and higher microlocal rank sheaves, an equivalence in the cohomological level [8] was shown for $(2,m)$-torus knots. In the case when M is of higher dimension, the correspondence in question is widely open. However, see [6, 7, 23, 24] and [12, Sec.6.4] for related results along this direction.

Our input originates from the following speculation in the perspective of microlocal sheaf theory: given a constructible sheaf $\mathcal{F}$ on $M=\mathbb{R}^2_{x,z}$, the micro-support of $\mathcal{F}$ at infinity is in general a singular Legendrian, typically a Legendrian graph in $T^\infty M$. So it is natural to seek a contact-geometric interpretation of the sheaf category $\mathcal{S}\mathsf{h}_\Lambda(M;\mathbb{K})$ when Λ is a Legendrian graph. More generally, it is natural to ask the same question with essentially no more difficulty, when $N=I_x\subset\mathbb{R}_x$ is an open interval, and $T\subset J^1N$ is a bordered Legendrian graph. Our main result gives a positive answer to this question.

Theorem 1.2 (Theorems 4.19, 4.21 and 6.31). *Let $(\mathcal{T},\boldsymbol{\mu})$ be a bordered Legendrian graph $\mathcal{T} := (T_\mathsf{L}\to T\leftarrow T_\mathsf{R})$ in $J^1I_x\cong T^{\infty,-}(I_x\times\mathbb{R}_z)$ with a Maslov potential $\boldsymbol{\mu}=(\mu_\mathsf{L},\mu,\mu_\mathsf{R})$. There is a well-defined diagram of unital A_∞-categories:*

$$\mathcal{A}\mathsf{ug}_+(\mathcal{T},\boldsymbol{\mu};\mathbb{K}) := (\mathcal{A}\mathsf{ug}_+(T_\mathsf{L},\mu_\mathsf{L};\mathbb{K})\leftarrow\mathcal{A}\mathsf{ug}_+(T,\mu;\mathbb{K})\to\mathcal{A}\mathsf{ug}_+(T_\mathsf{R},\mu_\mathsf{R};\mathbb{K})),$$

where the objects of $\mathcal{A}\mathsf{ug}_+(T,\mu;\mathbb{K})$ are the augmentations of the LCH DGA $A^{\mathsf{CE}}(T,\mu)$ defined as in [1]. The diagram is invariant under Legendrian isotopy and basepoint moves up to A_∞-equivalence. Moreover, there is an A_∞-equivalence between two diagrams of unital A_∞-categories:

$$\mathcal{A}\mathsf{ug}_+(\mathcal{T},\boldsymbol{\mu};\mathbb{K})\xrightarrow{\sim}\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K}),$$

where

$$\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K}) := (\mathcal{C}_1(T_\mathsf{L},\mu_\mathsf{L};\mathbb{K})\leftarrow\mathcal{C}_1(T,\mu;\mathbb{K})\to\mathcal{C}_1(T_\mathsf{R},\mu_\mathsf{R};\mathbb{K}))$$

is a digram of DG categories of constructible sheaves.

The DG category $\mathcal{C}_1(T,\mu;\mathbb{K})$ is the full subcategory of $\mathcal{S}\mathsf{h}_T(I_x\times\mathbb{R}_z;\mathbb{K})$ whose objects are microlocal rank 1 sheaves with acyclic stalks for $z\ll 0$. In particular, when $T = \Lambda \subset T^{\infty,-}\mathbb{R}^2_{x,z}$ is a Legendrian graph, we get an A_∞-equivalence $\mathcal{A}\mathsf{ug}_+(\Lambda,\mu;\mathbb{K}) \xrightarrow{\sim} \mathcal{C}_1(\Lambda,\mu;\mathbb{K})$, i.e. "augmentations are sheaves" holds in the singular case.

As a consequence of Theorem 1.2, up to a normalization, the point-counting of sheaves in $\mathcal{C}_1(T,\mu;\mathbb{K})$ (with boundary conditions) over a finite field $\mathbb{K} = \mathbb{F}_q$, is equivalent to that of augmentations in $\mathcal{A}\mathsf{ug}_+(T,\mu;\mathbb{K})$ (with boundary conditions), called augmentation number and denoted by $\mathrm{aug}(\mathcal{T},\boldsymbol{\mu};\rho_{\mathsf{L}},\rho_{\mathsf{R}};\mathbb{F}_q)$. Generalizing the results in [16, 27], our preceding paper [3] solves this counting problem. More precisely, augmentation numbers are computed by ruling polynomials of T, defined via the combinatorics of decompositions of the front projection T:

Theorem 1.3. *[3] Let $(\mathcal{T},\boldsymbol{\mu})$ be as above. Let $\rho_{\mathsf{L}} \in \mathrm{NR}(T_{\mathsf{L}},\mu_{\mathsf{L}})$ and $\rho_{\mathsf{R}} \in \mathrm{NR}(T_{\mathsf{R}},\mu_{\mathsf{R}})$ be two boundary conditions (i.e. normal rulings). Then the following two Legendrian isotopy invariants are the same:*

$$\mathrm{aug}(\mathcal{T},\boldsymbol{\mu};\rho_{\mathsf{L}},\rho_{\mathsf{R}};\mathbb{F}_q) = q^{-\frac{d+\widehat{B}}{2}} z^{\widehat{B}} \langle \rho_{\mathsf{L}} | R(\mathcal{T},\boldsymbol{\mu};q,z) | \rho_{\mathsf{R}} \rangle$$

Here, $\langle \rho_{\mathsf{L}} | R(\mathcal{T},\boldsymbol{\mu};q,z) | \rho_{\mathsf{R}} \rangle \in \mathbb{Z}[q^{\pm\frac{1}{2}}, z^{\pm 1}]$ is the ruling polynomial for $(\mathcal{T},\boldsymbol{\mu})$ with boundary conditions $(\rho_{\mathsf{L}},\rho_{\mathsf{R}})$, $d := \max\deg_z \langle \rho_{\mathsf{L}} | R(\mathcal{T},\boldsymbol{\mu};z^2,z) | \rho_{\mathsf{R}} \rangle$. In the formula, we take $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, and

$$\widehat{B} := B + \sum_{v \in V(\mathcal{T})} \frac{val(v)}{2}$$

counts the number of "generalized" basepoints in T.

Moreover, the ruling polynomials satisfy the gluing property*: If $(\mathcal{T},\boldsymbol{\mu}) = (\mathcal{T}^1,\boldsymbol{\mu}^1) \circ (\mathcal{T}^2,\boldsymbol{\mu}^2)$ is a composition of two bordered Legendrian graphs, that is, $(T^1_{\mathsf{R}},\mu^1_{\mathsf{R}}) = (T^2_{\mathsf{L}},\mu^2_{\mathsf{L}})$, then*

$$\begin{aligned}&\langle \rho_{\mathsf{L}} | R(\mathcal{T},\boldsymbol{\mu};q,z) | \rho_{\mathsf{R}} \rangle \\ &\qquad = \sum_{\rho_I \in \mathrm{NR}(T^1_{\mathsf{R}},\mu^1_{\mathsf{R}})} \langle \rho_{\mathsf{L}} | R(\mathcal{T}^1,\boldsymbol{\mu}^1;q,z) | \rho_I \rangle \langle \rho_I | R(\mathcal{T}^2,\boldsymbol{\mu}^2;q,z) | \rho_{\mathsf{R}} \rangle.\end{aligned}$$

Organization

The article is organized as follows. In Section 2, we review the basic backgrounds from [3] on Legendrian graphs and bordered Legendrian graphs $(\mathcal{T},\boldsymbol{\mu})$ with a Maslov potential. The main ingredients include the Legendrian contact homology (LCH) DGAs $A^{\mathsf{CE}}(\mathcal{T},\boldsymbol{\mu}) = (A^{\mathsf{CE}}(T_{\mathsf{L}},\mu_{\mathsf{L}}) \to A^{\mathsf{CE}}(T,\mu) \leftarrow A^{\mathsf{CE}}(T_{\mathsf{R}},\mu_{\mathsf{R}}))$ for bordered Legendrian graphs.

In Section 3, we give the algebraic preliminaries before defining augmentation categories. In particular, we introduce two categories of consistent sequences: consistent sequences of bordered Legendrian graphs, and consistent sequences of DGAs. In Section 4, we define augmentation categories for bordered Legendrian graphs, and then show their unitarity and invariance.

In Section 5, we firstly give the preliminaries on the microlocal theory of sheaves. Then we construct the necessary combinatorial tools, which we call legible models for $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K}) = (\mathcal{S}\mathsf{h}(T_{\mathsf{L}};\mathbb{K}) \leftarrow \mathcal{S}\mathsf{h}(T;\mathbb{K}) \to \mathcal{S}\mathsf{h}(T_{\mathsf{R}};\mathbb{K}))$, the diagram of sheaf categories for a bordered Legendrian graph $\mathcal{T}$ in $T^{\infty,-}(I_x \times \mathbb{R}_z)$. As an application, we prove the invariance of $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})$ via combinatorics, hence the invariance of $\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K})$, the diagram of full-subcategories of $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})$ whose objects are microlocal rank 1 objects with acyclic stalks for $z \ll 0$.

In Section 6, we prove our main result "augmentations are sheaves" for bordered Legendrian graphs. The basic idea is as follows: By the invariance results in Section 4 and Section 5, we can assume the vertices in the front projection T are all of type $(0,r)$ for some r, all the left cusps and vertices are to the left of the crossings of T, and all the right cusps are to the right of the crossings of T. Moreover, we can assume all right cusps are marked. Then both of the two diagrams of A_∞-categories satisfy the sheaf property over I_x. Hence, by decomposing the front diagram T into the composition of elementary pieces, it suffices to show the theorem for each elementary piece. By the results for Legendrian knots in [21], it suffices to show the case of an elementary bordered Legendrian graph $(\mathcal{V},\boldsymbol{\mu})$ involving only a vertex. This is done by an explicit description for both of the two diagrams $\mathcal{A}\mathsf{ug}_+(\mathcal{V},\boldsymbol{\mu};\mathbb{K})$ and $\mathcal{C}_1(\mathcal{V},\boldsymbol{\mu};\mathbb{K})$. The augmentation side is done in Lemma 6.17. The sheaf side is a direct application of the legible model in Section 5.3 for $\mathcal{C}_1(\mathcal{V},\boldsymbol{\mu};\mathbb{K})$. Then we are done.

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2. Setup

Notation 2.1. For each $m \geq 0$, we denote the set $\{1, \dots, m\}$ equipped with the natural order by $[m]$.

2.1. Bordered Legendrian graphs

In this section, we briefly review the definition of *bordered Legendrian graphs* defined in [3, §2]. A graph is a finite regular one dimensional CW complex, whose 0-cells and closed 1-cells are called vertices and edges. For each vertex $\mathtt{v}$, a *half-edge* at $\mathtt{v}$ is a small enough restriction of an edge adjacent to $\mathtt{v}$. Then as usual, the *valency* of $\mathtt{v}$ is the number of half-edges at v and denoted by $\mathrm{val}(\mathtt{v})$.

A (*based*) *bordered graph* $\Gamma = (\mathtt{V}, \mathtt{V}_\mathsf{L}, \mathtt{V}_\mathsf{R}, \mathtt{B}, \mathtt{E})$ of type $(n_\mathsf{L}, n_\mathsf{R})$ consists of the following data:

- the pair $(\mathtt{V} \amalg \mathtt{V}_\mathsf{L} \amalg \mathtt{V}_\mathsf{R} \amalg \mathtt{B}, \mathtt{E})$ defines a graph $|\Gamma|$,
- each $b \in \mathtt{B}$ of $|\Gamma|$ is bivalent, and
- two disjoint subsets $\mathtt{V}_\mathsf{L}$ and $\mathtt{V}_\mathsf{R}$ consist of n_L and n_R univalent vertices of $|\Gamma|$.

Elements in $\mathtt{V}, \mathtt{V}_\mathsf{L}, \mathtt{V}_\mathsf{R}, \mathtt{B}$ and $\mathtt{E}$ will be called *vertices*, *left borders*, *right borders*, *basepoints* and *edges*, respectively. The *interior* $\mathring{\Gamma}$ of Γ is define to be the complement of $\mathtt{V}_\mathsf{L} \amalg \mathtt{V}_\mathsf{R}$.

Notation 2.2. In order to emphasize the border structures, we will denote the bordered graph Γ as

$$\boldsymbol{\Gamma} = (\Gamma_\mathsf{L} \to \Gamma \leftarrow \Gamma_\mathsf{R}),$$

where Γ_L and Γ_R are defined by $\mathtt{V}_\mathsf{L}$ and $\mathtt{V}_\mathsf{R}$, respectively, and both arrows are inclusions.

From now on, we mean by a *graph* a bordered graph with empty borders ($\varnothing \to \Gamma \leftarrow \varnothing$), which will be denoted simply by Γ.

For a closed interval $U = [x_{\mathsf{L}}, x_{\mathsf{R}}] \subset \mathbb{R}_x$, let the *bordered one-jet space* $J^1\mathbf{U} = (J^1U_{\mathsf{L}} \to J^1U \leftarrow J^1U_{\mathsf{R}})$ be the one-jet space $J^1U \coloneqq (U \times \mathbb{R}_{yz}, dz - ydx) \subset J^1\mathbb{R}_x = (\mathbb{R}^3_{xyz}, dz - ydx)$ together with two contact submanifolds

$$J^1U_{\mathsf{L}} \coloneqq (\{x_{\mathsf{L}}\} \times \mathbb{R}_z, dz) \quad \text{and} \quad J^1U_{\mathsf{R}} \coloneqq (\{x_{\mathsf{R}}\} \times \mathbb{R}_z, dz)\,.$$

Definition 2.3 (bordered Legendrian graphs). A *bordered Legendrian graph* $\mathcal{T} = (\mathsf{T}_{\mathsf{L}} \to \mathsf{T} \leftarrow \mathsf{T}_{\mathsf{R}})$ of a bordered graph $\mathbf{\Gamma} = (\Gamma_{\mathsf{L}} \to \Gamma \leftarrow \Gamma_{\mathsf{R}})$ of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$ in $J^1\mathbf{U}$ where $\mathsf{T} : \Gamma \to J^1U$ is an embedding such that

1) T is transverse to the boundary $\partial J^1U = \partial U \times \mathbb{R}_{yz}$ and the restrictions on the interior $\mathring{\Gamma}$ and both borders Γ_{L} and Γ_{R} are contained in $J^1\mathring{U}$, J^1U_{L} and J^1U_{R}, respectively.

$$\mathsf{T} \pitchfork \partial J^1U,\ \mathsf{T}_{\mathsf{L}} \coloneqq \mathsf{T}(\Gamma_{\mathsf{L}}) \subset J^1U_{\mathsf{L}},\ \mathsf{T}_{\mathsf{R}} \coloneqq \mathsf{T}(\Gamma_{\mathsf{R}}) \subset J^1U_{\mathsf{R}},\ \mathring{\mathsf{T}} \coloneqq \mathsf{T}(\mathring{\Gamma}) \subset J^1\mathring{U}.$$

2) T on edges are smooth Legendrian with boundary and pairwise non-tangent to each other at all vertices, and two edges adjacent to each basepoint form a smooth arc.

By labeling borders in T_{L} and T_{R} in top-to-bottom ways with respect to z-coordinates, we identify the left and right borders T_{L} and T_{R} with the set $[n_{\mathsf{L}}] = \{1, \dots, n_{\mathsf{L}}\}$ and $[n_{\mathsf{R}}] = \{1, \dots, n_{\mathsf{R}}\}$.

There are two projections for $J^1\mathbb{R}_x \cong \mathbb{R}^3_{xyz}$, called the *front* and *Lagrangian* projections $\pi_{\mathsf{fr}} : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xz}$ and $\pi_{\mathsf{Lag}} : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$, respectively.

Definition 2.4 (Regular projections). For a bordered Legendrian graph $\mathcal{T} = (\mathsf{T}_{\mathsf{L}} \to \mathsf{T} \leftarrow \mathsf{T}_{\mathsf{R}})$, the *front* and *Lagrangian projections* $\mathcal{T} = (T_{\mathsf{L}} \to T \leftarrow T_{\mathsf{R}}) \coloneqq \pi_{\mathsf{fr}}(\mathcal{T})$ and $\mathcal{T}_{\mathsf{Lag}} = (T_{\mathsf{Lag,L}} \to T_{\mathsf{Lag}} \leftarrow T_{\mathsf{Lag,R}}) \coloneqq \pi_{\mathsf{Lag}}(\mathcal{T})$ are said to be *regular* if in their interiors,

1) there are only finitely many transverse double points, called *crossings*, and
2) no vertices, basepoints or *x-extreme points* are crossings,
3) each edge containing a x-maximal point must involve at least one vertex or a basepoint,

where a point in the interior $\mathring{\mathsf{T}}$ is said to be *x-maximal* or *x-minimal* if it is maximal or minimal with respect to the x-coordinate, and *x-extreme* if it is either x-maximal or x-minimal.[1]

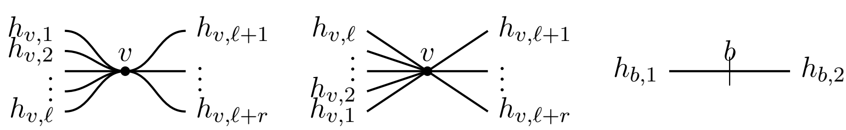
A bordered Legendrian graph of type $(0,0)$ is called a *Legendrian graph* and we denote the sets of regular front and Lagrangian projections of non-bordered and bordered Legendrian graphs by $\mathcal{LG}, \mathcal{LG}_{\mathsf{Lag}}$ and $\mathcal{BLG}, \mathcal{BLG}_{\mathsf{Lag}}$, respectively.

Remark 2.5. Due to the Legendrian property, there are no vertical tangencies and no non-transverse double points in the front projection. Instead, it contains *cusps*, which is obviously, x-extreme.

Notation 2.6. The front and Lagrangian projection of $\mathsf{T} = \mathsf{T}(\Gamma)$ with $\Gamma = (\mathtt{V}, \mathtt{V}_{\mathsf{L}}, \mathtt{V}_{\mathsf{R}}, \mathtt{B}, \mathtt{E})$ will be denoted by $T = (V, V_{\mathsf{L}}, V_{\mathsf{R}}, B, E)$ and $T_{\mathsf{Lag}} = (V_{\mathsf{Lag}}, V_{\mathsf{Lag},\mathsf{L}}, V_{\mathsf{Lag},\mathsf{R}}, B_{\mathsf{Lag}}, E_{\mathsf{Lag}})$, respectively.

For examples of regular and non-regular projections, see Figure 1. To avoid any confusion, we denote vertices and basepoints by small dots and bars, respectively.

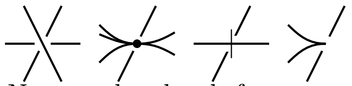
Definition 2.7 (Types and orientations). For a vertex v or a basepoint b of a bordered Legendrian graph, we say that it is of *type* (ℓ, r) if there are ℓ and r half-edges adjacent to v or b on the left and right, respectively. We label the set $H_v := \{h_{v,1}, \dots, h_{v,n}\}$ of (small enough) half-edges in front and Lagrangian projections as follows:



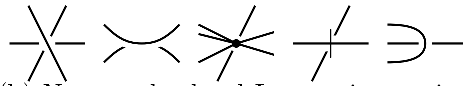
In particular, each basepoint $b \in B$ is assumed to be *oriented* from the half-edge $h_{b,1}$ to $h_{b,2}$ in the above convention.

Example/Definition 2.8 (The trivial and vertex bordered Legendrian graphs). Let $n \geq 1$. The front projections of the trivial bordered Legendrian graph $\mathcal{T}_n = (T_{n,\mathsf{L}} \to T_n \leftarrow T_{n,\mathsf{R}})$ of type (n,n) and the vertex bordered graphs $\mathbf{0}_n = (\varnothing \to 0_n \leftarrow 0_{n,\mathsf{R}})$ and $\boldsymbol{\infty}_n = (\infty_{n,\mathsf{L}} \to \infty_n \leftarrow \varnothing)$ of

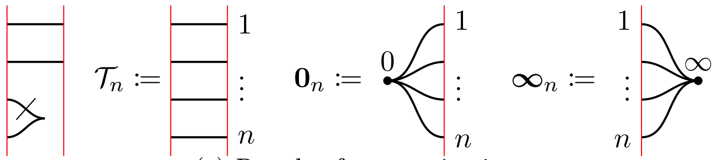
[1] In the front projection, an x-extreme point is either a cusp or a vertex of type $(0,n)$ or $(n,0)$.



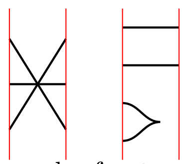
(a) Non-regular local front projections



(b) Non-regular local Lagrangian projections



(c) Regular front projections



(d) Non-regular front projections

Figure 1: Regular and non-regular projections of bordered Legendrian graphs.

type $(0, n)$ and $(n, 0)$ as depicted in Figure 1(c), whose left and right borders are points lying on the red lines at the left and right, respectively.

For convenience's sake, we define $\mathcal{T}_0 = \mathbf{0}_0 = \infty_0 = (\varnothing \to \varnothing \leftarrow \varnothing)$.

Definition 2.9 (Equivalences and isomorphisms). We say that two bordered Legendrian graphs $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* if there exists a family of bordered Legendrian graphs

$$\mathcal{T}_t : \Gamma \times [0,1] \to J^1\mathbf{U}_t \subset J^1\mathbb{R}_x, \qquad \mathcal{T}_0 = \mathcal{T}, \qquad \mathcal{T}_1 = \mathcal{T}'.$$

Two regular front (or Lagrangian) projections $\mathcal{T}$ (or $\mathcal{T}_{\mathsf{Lag}}$) and $\mathcal{T}'$ (or $\mathcal{T}'_{\mathsf{Lag}}$) of bordered Legendrian graphs $\mathcal{T}$ and $\mathcal{T}'$ are said to be *isomorphic* if there is a family of Legendrians $\mathcal{T}_t$, $t \in [0,1]$ such that $\mathcal{T}$ (or $\mathcal{T}_{\mathsf{Lag}}$) and $\mathcal{T}'$ (or $\mathcal{T}'_{\mathsf{Lag}}$) are Lagrangian (or front) projections of $\mathcal{T}_0$ and $\mathcal{T}_1$ and Lagrangian (or front) projections $\mathcal{T}_t := \pi_{\mathsf{fr}}(\mathcal{T}_t)$ (or $\mathcal{T}_{\mathsf{Lag},t} := \pi_{\mathsf{Lag}}(\mathcal{T}_t)$) are regular for all t.

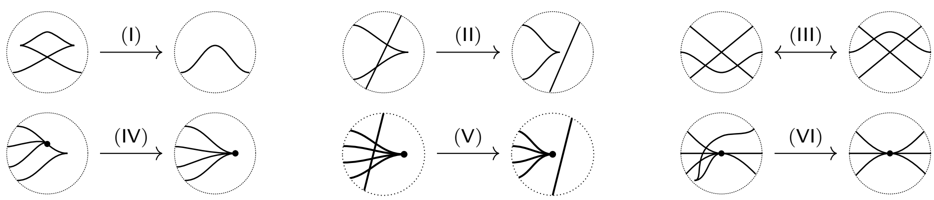
Remark 2.10. It is important to note that during the isotopy between two bordered Legendrian graphs, the ambient manifold $J^1\mathbf{U}_t$ may changes. For example, any translation along the x-axis will give us an equivalence.

Lemma 2.11. *Up to isomorphism, every pair of equivalent front or Lagrangian projections can be connected by a zig-zag sequence of* front *or* Lagrangian Reidemeister moves *depicted in Figures* 2(a) *or* 2(b)*, respectively.*

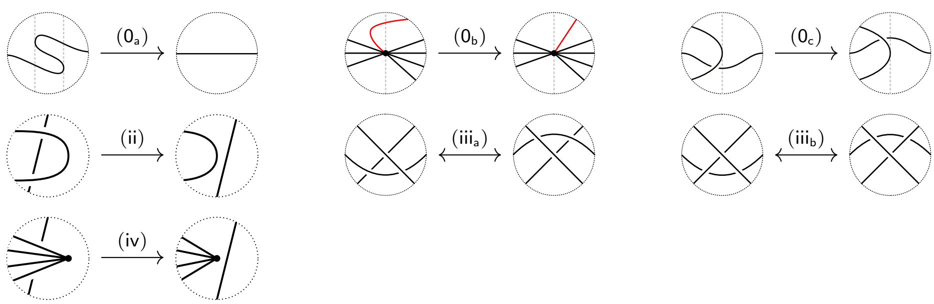
This is well-known and we omit the proof. One may refer [5, Proposition 4.2] or [2, Proposition 2.1]. Notice also that these moves are *not optimal.* Namely, the move (IV) is a special case of (VI).

Notation 2.12. The sets of Lagrangian and front Reidemeister moves will be denoted as follows:

$$\mathrm{RM} \coloneqq \{(\mathsf{I}), (\mathsf{II}), (\mathsf{III}), (\mathsf{V}), (\mathsf{VI})\}\,,$$
$$\mathrm{RM}_{\mathsf{Lag}} \coloneqq \{(\mathsf{0_a}), (\mathsf{0_b}), (\mathsf{0_c}), (\mathsf{ii}), (\mathsf{iii_a}), (\mathsf{iii_b}), (\mathsf{iv})\}\,.$$



(a) Front Reidemeister moves



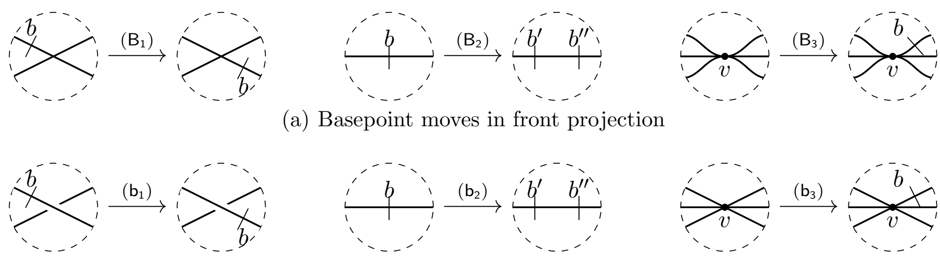
(b) Lagrangian Reidemeister moves

Figure 2: Front and Lagrangian Reidemeister moves: Reflections are possible, the valency of vertex is arbitrary and the vertex can be replaced with a basepoint if it is bivalent.

On the other hand, one can consider the weaker equivalence given by the Legendrian isotopy *up to basepoints.* In other words, two bordered Legendrian graphs are Legendrian isotopic after forgetting basepoints.

Lemma 2.13. *Two regular front or Lagrangian projections of bordered Legendrian graphs are equivalent up to basepoints if and only if they can be connected by a zig-zag sequence of front or Lagrangian Reidemeister moves together with* basepoint *splittings depicted in Figure 3.*

Proof. This is obvious. □



(a) Basepoint moves in front projection

(b) Basepoint moves in Lagrangian projection

Figure 3: Operations on basepoints.

Remark 2.14. Note that the operations (B_1) and (b_1) that move a basepoint through a crossing below or above can be realized as sequences of front and Lagrangian Reidemeister moves, respectively.

Remark 2.15. The operations (B_3) and (b_3) may happen only near vertices. Indeed, these moves are not to forget the base point but to let the vertex absorb the base point.

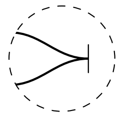
Definition 2.16 (Categories of bordered Legendrian graphs). We regard $\mathcal{BLG}$ and $\mathcal{BLG}_{\mathsf{Lag}}$ of regular front and Lagrangian projections of isomorphic classes of bordered Legendrian graphs as categories, whose morphisms are *freely generated by* front and Lagrangian Reidemeister moves.

Therefore, projections *up to zig-zags of morphisms* are the same as those *up to Reidemeister moves.* Or equivalently, the isomorphism classes in the localized category of $\mathcal{BLG}$ or $\mathcal{BLG}_{\mathsf{Lag}}$ by all Reidemeister moves are the same as the usual equivalent classes of bordered Legendrian graphs.

Example/Definition 2.17. A bordered Legendrian graph $\mathcal{T} \in \mathcal{BLG}$ is said to be in a *normal form* if

1) every vertex is of type $(\mathrm{val}(v), 0)$

2) every vertex is located near J^1U_{R}, and has the larger z-coordinate than any point in the right border T_{R}.
3) every non-vertex x-maximum is a basepoint and *vice versa* so that each x-maximum or a basepoint looks as follows:



See Figure 4 for the example of the normal form.

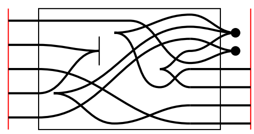
$\mathcal{T} =$  $\in \mathcal{BLG}$

Figure 4: A bordered Legendrian graph in a normal form.

Lemma 2.18. *Let $\mathcal{T} \in \mathcal{BLG}$ be a bordered Legendrian graph. Then there exists a sequence of front Reidemeister moves consisting of* $(\mathsf{V}), (\mathsf{VI})$ *together with* (B_*)*'s which transforms a (non-unique) bordered Legendrian graph in a normal form to $\mathcal{T}$.*

Proof. We first use (B_*) to make each x-maximum to be a basepoint and *vice versa.*

For each vertex $v \in V$ of type (ℓ, r) with $r > 0$, we apply (VI) several times to make v of type $(\ell + r, 0)$ in the reverse direction and we move a small neighborhood U_v of each vertex v to the right upward position by applying only front Reidemeister moves (V) in the reverse direction and we are done. □

2.1.1. Maslov potentials for bordered Legendrian graphs. Let $\mathcal{T} \in \mathcal{BLG}$ and $S = S(T) \subset \mathring{T}$ be the set of x-extreme points in its interior. An $\mathbb{Z}$*-valued Maslov potential* of T is a function $\mu : R \to \mathbb{Z}$ from the set $R :=\pi_0 (T \setminus (V \cup S))$ of connected components of the complement of vertices and cusps such that for all $s \in S \setminus V$,

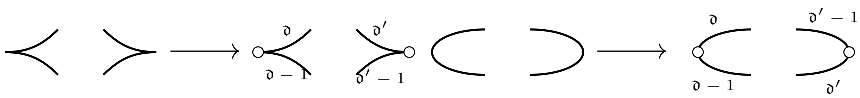
$$\mu(s^+) - \mu(s^-) = 1 \in \mathbb{Z}, \tag{2.1}$$

where s^+ (resp. s^-) is the upper (resp. lower) strand near s.

For $T_{\mathsf{Lag}} \in \mathcal{BLG}_{\mathsf{Lag}}$, let $S_{\mathsf{Lag}} = S(T_{\mathsf{Lag}}) \subset \mathring{T}_{\mathsf{Lag}}$ be the set of x-extreme points. As before we define the set $R_{\mathsf{Lag}} \coloneqq \pi_0(T_{\mathsf{Lag}} \setminus (V_{\mathsf{Lag}} \cup S_{\mathsf{Lag}}))$ of connected components of the complement of vertices and x-extreme points. Then an $\mathbb{Z}$-valued Maslov potential of T_{Lag} is a function $\mu : R_{\mathsf{Lag}} \to \mathbb{Z}$ satisfying the following condition: for each $s \in S_{\mathsf{Lag}} \setminus V_{\mathsf{Lag}}$,

$$\mu(s^+) - \mu(s^-) = \begin{cases} 1 & s \text{ is } x\text{-minimal;} \\ -1 & s \text{ is } x\text{-maximal.} \end{cases} \tag{2.2}$$

Diagrammatically, the above definition is depicted in Figures 5(a) and 5(b).



(a) In front projections (b) In Lagrangian projections

Figure 5: Defining diagrams for Maslov potentials.

Moreover, one can define $\mu_{\mathsf{L}} \coloneqq \mu|_{T_{\mathsf{L}}} : [n_{\mathsf{L}}] \to \mathbb{Z}$ and $\mu_{\mathsf{R}} \coloneqq \mu|_{T_{\mathsf{R}}} : [n_{\mathsf{R}}] \to \mathbb{Z}$ as the restrictions of μ to the connected components containing T_{L} and T_{R}, respectively, under the canonical identifications $T_{\mathsf{L}} \cong [n_{\mathsf{L}}]$ and $T_{\mathsf{R}} \cong [n_{\mathsf{R}}]$.

Definition 2.19 (Maslov potentials for bordered Legendrian graphs). A Maslov potential for a bordered Legendrian graph $\mathcal{T}$ is a triple $(\mu_{\mathsf{L}}, \mu, \mu_{\mathsf{R}})$ denoted simply by $\boldsymbol{\mu}$.

We denote Legendrian graphs with Maslov potentials by using the superscript "μ" such as $\mathcal{BLG}^{\mu}$.

Example 2.20. Recall the bordered Legendrian graphs $\mathcal{T}_n$, $\mathbf{0}_n$ and $\boldsymbol{\infty}_n$ defined in Example/Definition 2.8. Since they have no x-extreme points except for a vertex, there are no conditions for Maslov potentials. That is, any function $\mu : [n] = \{1, \dots, n\} \to \mathbb{Z}$ can be realized as a Maslov potential for $\mathcal{T}_n$, $\mathbf{0}_n$ or $\boldsymbol{\infty}_n$.

Then each Lagrangian Reidemeister move induces a unique isotopy between bordered Legendrian graphs with Maslov potentials.

Lemma 2.21. *Let* $(\mathsf{M}) : \mathcal{T}' \to \mathcal{T}$ *and* $(\mathsf{m}) : \mathcal{T}'_{\mathsf{Lag}} \to \mathcal{T}_{\mathsf{Lag}}$ *be front and Lagrangian Reidemeister moves. Then they lift uniquely to* $(\mathsf{M}) : (\mathcal{T}', \boldsymbol{\mu}') \to$

$(\mathcal{T},\boldsymbol{\mu})$ *and* $(\mathsf{m}):(\mathcal{T}'_{\mathsf{Lag}},\boldsymbol{\mu}')\to(\mathcal{T}_{\mathsf{Lag}},\boldsymbol{\mu})$. *Namely, for given* $\boldsymbol{\mu}'$, *there is a unique Maslov potential* $\boldsymbol{\mu}$ *on each* $\mathcal{T}$ *or* $\mathcal{T}_{\mathsf{Lag}}$ *such that*

$$(\mathsf{M})_*\boldsymbol{\mu}'=\boldsymbol{\mu}\qquad\text{and}\qquad(\mathsf{m})_*\boldsymbol{\mu}'=\boldsymbol{\mu}.$$

Proof. This is an extension of Theorem 2.21 in [1] for Lagrangian Reidemeister moves. The proof is straightforward and we omit the proof. □

Definition 2.22 (Restrictions of Maslov potentials on vertices). For each vertex v of type (ℓ,r) with $\ell+r=n$, the set H_v of half-edges can be identified with $\mathbb{Z}/n\mathbb{Z}$ by Definition 2.7 and we denote the restriction $\mu|_{H_v}:\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}$ of a Maslov potential to H_v by μ_v.

2.1.2. Concatenations of bordered Legendrian graphs. For $i=1,2$, let $\mathcal{T}^i\in\mathcal{BLG}$ be a bordered Legendrian graph of type $\left(n^i_{\mathsf{L}},n^i_{\mathsf{R}}\right)$. Suppose that $n^1_{\mathsf{R}}=n^2_{\mathsf{L}}$. Then we can naturally define the *concatenation* $\mathcal{T}=\mathcal{T}^1\cdot\mathcal{T}^2=(T_{\mathsf{L}}\to T\leftarrow T_{\mathsf{R}})$ such that T is obtained simply by concatenating and identifying T^1_{R} and T^2_{L} up to small isotopy near borders if necessary, and two borders $T_{\mathsf{L}}:=T^1_{\mathsf{L}}$ and $T_{\mathsf{R}}:=T^2_{\mathsf{R}}$ come naturally from $\mathcal{T}^1$ and $\mathcal{T}^2$, respectively.

Remark 2.23. It is important to note that we will not regard the points of concatenations as vertices of T. Therefore T has n-less edges than the disjoint union of T^1 and T^2.

Definition 2.24 (Closure). For $\mathcal{T}\in\mathcal{BLG}$ of type $(n_{\mathsf{L}},n_{\mathsf{R}})$, the *closure* $\widehat{\mathcal{T}}$ is defined by the Legendrian graph obtained by the concatenation

$$\widehat{\mathcal{T}}:=\mathbf{0}_{n_{\mathsf{L}}}\cdot\mathcal{T}\cdot\boldsymbol{\infty}_{n_{\mathsf{R}}}\in\mathcal{LG}$$

as depicted in Figure 6.

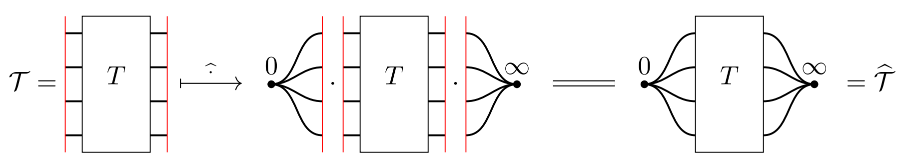


Figure 6: The closure of the front projection $\mathcal{T}$.

Lemma 2.25. *The closure* $\widehat{\cdot}:\mathcal{BLG}^{\mu}\to\mathcal{LG}^{\mu}$ *is a functor.*

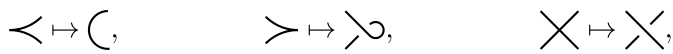
Proof. It is obvious that there is a unique way to extend $\boldsymbol{\mu}$ on $\mathcal{T}$ to $\widehat{\boldsymbol{\mu}}$ on $\widehat{\mathcal{T}}$ by definition of the closure, which is well-defined since any function on $[n]$ can be realized as a Maslov potential of 0_n or ∞_n as seen in Example 2.20.

Each Reidemeister move $(\mathsf{M}) : \mathcal{T}' \to \mathcal{T}$ between bordered Legendrian graphs induces the exactly same move $(\mathsf{M}) : \widehat{\mathcal{T}}' \to \widehat{\mathcal{T}}$ between closures. Therefore it becomes a functor preserving homotopy. □

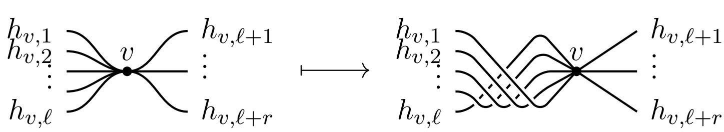
We introduce a combinatorial way, called the *Ng's resolution* to obtain a regular Lagrangian projection $T_{\mathsf{Lag}} \in \mathcal{LG}_{\mathsf{Lag}}$ for given front projection $T \in \mathcal{LG}$ defining the equivalent Legendrian graphs.

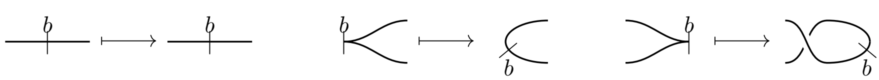
Remark 2.26. This is an extension of the original Ng's resolution for Legendrian knots to Legendrian graphs.

Definition 2.27 (Ng's resolution). [22, Definition 2.1] For $T \in \mathcal{LG}$, the *Ng's resolution* $\mathrm{Res}^{\mathsf{Ng}}(T)$ is a Lagrangian projection obtained by (combinatorially) replacing the local pieces as follows:



and for a vertex v of type (ℓ, r) and a basepoint b, we take a replacement as follows:





Notice that if we have two equivalent front projections, then the Ng's resolution induces equivalences between resolutions. Indeed, for each front Reidemeister move (M), we have a sequence of Lagrangian Reidemeister move(s) $\mathrm{Res}^{\mathsf{Ng}}(\mathsf{M})$.

One candidate for the choice of $\mathrm{Res}^{\mathsf{Ng}}(\mathsf{M})$ for each $(\mathsf{M}) \in \mathrm{RM}$ is given in [3, Figure 6].

Lemma 2.28. *The Ng's resolution* $\mathrm{Res}^{\mathsf{Ng}} : \mathcal{LG}^{\mu} \to \mathcal{LG}^{\mu}_{\mathsf{Lag}}$ *is a functor.*

Proof. The well-definedness is well-known and as observed above, each morphism corresponding to front Reidemeister move (M) will be mapped to the morphism corresponding to the chosen sequence $\mathrm{Res}^{\mathsf{Ng}}(\mathsf{M})$ of Lagrangian Reidemeister moves. Therefore it is functorial. □

2.2. Bordered Chekanov-Eliashberg DGAs

In this section, we recall from [1, 3] some results about Chekanov-Eliashberg algebra for bordered Legendrian graphs which can be seen as a generalization of A *bordered Chekanov-Eliashberg algebra* in [26].

Throughout this paper, we mean by a *differential graded algebra* (DGA) a pair $A = (\mathsf{A}, \partial)$ of a unital associative graded algebra A over a field $\mathbb{K}$ with the unit $\mathbf{1} \in \mathbb{K}$. A DGA $A = (\mathsf{A}, \partial)$ is said to be *semi-free* and generated by S if its underlying algebra A is a tensor algebra of a graded vector space $\mathbb{K}\langle \mathsf{S} \rangle$

$$\mathsf{A} = T(\mathbb{K}\langle \mathsf{S} \rangle) = \bigoplus_{\ell \geq 0} (\mathbb{K}\langle \mathsf{S} \rangle)^{\otimes \ell},$$

for a (possibly infinite) *generating set* S with a grading $|\cdot| : \mathsf{S} \to \mathbb{Z}$.

Remark 2.29. This is the usual notion of the semi-freeness while it is used differently in [21].

We denote the category of semi-free DGAs by $\mathcal{DGA}$.

Assumption 2.30. From now on, we mean by a DGA a semi-free DGA unless mentioned otherwise.

Example/Definition 2.31 (Border DGAs and internal DGAs). There are two important examples of DGAs, called the *border DGA* $A_n(\mu) = (\mathsf{A}_n, \partial_n)$ and the *internal DGA* $I_n(\mu) = (\mathsf{I}_n, \partial_n)$ which are edge-graded over $E_n := [n]$ and defined as follows:

- For a function $\mu : [n] \to \mathbb{Z}$, the algebras A_n and I_n are generated by two sets K_n and $\widetilde{V}_n$

  $$K_n := \{k_{ab} \mid 1 \leq a < b \leq n\}, \qquad \widetilde{V}_n := \{\xi_{a,i} \mid a \in \mathbb{Z}/n\mathbb{Z}, i \geq 1\}.$$

- The edge-gradings on generator k_{ab} and $\xi_{a,i}$ are given as (a, b) and $(a, a+i)$, where all indices are modulo n.

- The homological gradings are

$$|k_{ab}| := \mu(a) - \mu(b) - 1, \qquad |\xi_{a,i}| := \mu(a) - \mu(a+i) + N(n,a,i) - 1,$$

where

$$N(n,a,i) = \sum_{j<i} N(n,a+j,1) \quad \text{and} \quad N(n,a,1) := \begin{cases} 0 & a \neq n; \\ 2 & a = n; \end{cases} \tag{2.3}$$

- The differentials for k_{ab} and $\xi_{a,i}$ are defined as follows:

$$\partial_n(k_{ab}) := \sum_{a<c<b} (-1)^{|k_{ac}|-1} k_{ac} k_{cb}, \tag{2.4}$$

$$\partial_n(\xi_{a,i}) := \delta_{i,n} \mathbf{1}_a + \sum_{i_1+i_2=i} (-1)^{|\xi_{a,i_1}|-1} \xi_{a,i_1} \xi_{a+i_1,i_2}. \tag{2.5}$$

Then it is obvious that there is a canonical inclusion $k_{ab} \mapsto \xi_{a,b-a}$ from $A_n(\mu) \to I_n(\mu)$.

Remark 2.32. We have $A_0(\mu) \cong I_0(\mu) \cong \mathbb{K}$.

Especially, when a vertex v of $n = 2$, of $(1,1)$-type and μ is constant, say m, then the internal DGA $I_2(m) = (\mathsf{I}_2(m), \partial_2)$ is given as

$$\mathsf{I}_2(m) = \mathbb{K}\langle \xi_{1,i}, \xi_{2,i} \mid i \geq 1 \rangle, \tag{2.6}$$

where $|\xi_{1,i}| = |\xi_{2,i}| = i - 1$. Hence, it is independent to the constant m and denoted by I_2 and moreover, we have the DGA morphism

$$I_2 \to (\mathbb{K}[t,t^{-1}], \partial \equiv 0) \quad |t| = |t^{-1}| = 0, \quad \xi_{a,\ell} \mapsto \begin{cases} t & a = 1, \ell = 1; \\ t^{-1} & a = 2, \ell = 1; \\ 0 & \ell > 1. \end{cases} \tag{2.7}$$

Definition 2.33 (Tame isomorphisms). We say that a DGA morphism

$$f : A' = (\mathsf{A}' = T(\mathbb{K}\langle \mathsf{S} \rangle), \partial') \to A = (\mathsf{A} = T(\mathbb{K}\langle \mathsf{S} \rangle), \partial)$$

is called an *elementary isomorphism* if for some $g \in \mathsf{S}$,

$$f(h) = \begin{cases} g + u & \text{if } h = g \\ h & \text{if } h \neq g, \end{cases}$$

where $u \in T(\mathbb{K}\langle \mathsf{S} \setminus \{g\}\rangle)$, and called a *tame isomorphism* if f is a composition of countably many (possibly finite) elementary isomorphisms.

We consider a *stabilization* in the sense of [10, Definition 2.16] and a *generalized stabilization* defined in [1] as follows:

Definition 2.34 (Stabilizations). A *stabilization* of $A = (\mathsf{A} = T(\mathbb{K}\langle \mathsf{S}\rangle), \partial) \in \mathcal{DGA}$ is a DGA which is tame isomorphic to a DGA $SA = (S\mathsf{A}, \overline{\partial})$ obtained from A by adding a countably many (possibly finite) number of canceling pairs of edge-graded generators $\{\widehat{e}^i, e^i \mid i \in I\}$ for some index set I so that $\widehat{e}^i$ and e^i have the same edge-grading and

$$S\mathsf{A} = T\left(\mathbb{K}\left\langle \mathsf{S} \amalg \{\widehat{e}^i, e^i \mid i \in I\}\right\rangle\right), \quad |\widehat{e}^i| = |e^i| + 1, \quad \overline{\partial}(\widehat{e}^i) = e^i, \quad \overline{\partial}(e^i) = 0.$$

Then the canonical forgetful map $\pi : SA \to A$ sending both $\widehat{e}^i$ and e^i to zero induces the isomorphism on homology groups, whose homotopy inverse is precisely the canonical embedding $\iota : A \to SA$.

Definition 2.35 (Generalized stabilizations). For $A = (\mathsf{A} = T(\mathbb{K}\langle \mathsf{S}\rangle), \partial) \in \mathcal{DGA}$ and $\varphi : I_n(\mu) \to A$ for some $\mu : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$, the $\mathfrak{d}$*-th positive* or *negative stabilization* of A with respect to φ is the DGA $S^{\mathfrak{d}\pm}_{\varphi} A = \left(S^{\mathfrak{d}}_{\varphi}\mathsf{A}, \overline{\partial}^{\pm}\right)$, where for $v_{a,i} := \varphi(\xi_{a,i}) \in A$,

- the graded algebra $S^{\mathfrak{d}}_{\varphi}\mathsf{A}$ is given as

$$S^{\mathfrak{d}}_{\varphi}\mathsf{A} := T\left(\mathbb{K}\left\langle \mathsf{S} \amalg \{e^1, \ldots, e^n\}\right\rangle\right), \qquad |e^b| := \mathfrak{d} + \sum_{a<b}(|v_{a,1}| + 1),$$

- the differentials $\overline{\partial}^{\pm}$ for e^b are given as

$$\overline{\partial}^{+} e^b := \sum_{a<b} (-1)^{|e^a|-1} e^a v_{a,b-a}, \qquad \overline{\partial}^{-} e^b := \sum_{a<b} v_{n+1-b,b-a} e^a.$$

As observed in [1, Remark 3.8] and proved in [3, Appendix A], the generalized stabilization is a composition of stabilizations and destabilizations.

Proposition 2.36. *[3, Appendix A] There exists a semi-free edge-graded DGA $\widetilde{S}^{\mathfrak{d}\pm}_{\varphi} A$ which is a common stabilization of both A and $S^{\mathfrak{d}\pm}_{\varphi} A$.*

2.2.1. Bordered DGAs. We consider DGAs together with additional structures, called *bordered DGAs*.

Definition 2.37 (Bordered DGAs). A *bordered* DGA

$$\mathcal{A} = \left(A_{\mathsf{L}} \xrightarrow{\phi_{\mathsf{L}}} A \xleftarrow{\phi_{\mathsf{R}}} A_{\mathsf{R}}\right)$$

of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$ consists of DGAs A, A_{L} and A_{R}, and two DGA morphisms $\phi_{\mathsf{L}} : A_{\mathsf{L}} \to A$ and $\phi_{\mathsf{R}} : A_{\mathsf{R}} \to A$ such that

1) ϕ_{L} is *injective*, and
2) for some $\mu_{\mathsf{L}} : [n_{\mathsf{L}}] \to \mathbb{Z}$ and $\mu_{\mathsf{R}} : [n_{\mathsf{R}}] \to \mathbb{Z}$,

$$A_{\mathsf{L}} \cong A_{n_{\mathsf{L}}}(\mu_{\mathsf{L}}) \quad \text{and} \quad A_{\mathsf{R}} \cong A_{n_{\mathsf{R}}}(\mu_{\mathsf{R}}).$$

A *bordered morphism* $\mathbf{f} : \mathcal{A}' \to \mathcal{A}$ is a triple $(f_{\mathsf{L}}, f, f_{\mathsf{R}})$ of DGA morphisms such that they fit into the following commutative diagram

$$\begin{array}{c} \mathcal{A}' \\ \mathbf{f}\big\downarrow \\ \mathcal{A} \end{array} = \left(\begin{array}{ccccc} A'_{\mathsf{L}} & \xrightarrow{\phi'_{\mathsf{L}}} & A' & \xleftarrow{\phi'_{\mathsf{R}}} & A'_{\mathsf{R}} \\ {\scriptstyle f_{\mathsf{L}}}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_{\mathsf{R}}} \\ A_{\mathsf{L}} & \xrightarrow{\phi_{\mathsf{L}}} & A & \xleftarrow{\phi_{\mathsf{R}}} & A_{\mathsf{R}}. \end{array}\right)$$

We say that $\mathcal{A}$ is a *cofibrant* if ϕ_{R} is also injective.

Definition 2.38 (Stabilizations). We say that $\mathcal{A}' = (A'_{\mathsf{L}} \to A' \leftarrow A'_{\mathsf{R}})$ is a *(weak) stabilization* of $\mathcal{A} = (A_{\mathsf{L}} \to A \leftarrow A_{\mathsf{R}})$ if A' is a stabilization of A and there exists a canonical projection

$$\boldsymbol{\pi} = (\mathrm{Id}_{\mathsf{L}}, \pi, \mathrm{Id}_{\mathsf{R}}) : \mathcal{A}' \to \mathcal{A},$$

and is a *strong stabilization* if it is a weak stabilization and there exists the canonical embedding $\mathbf{i} : \mathcal{A} \to \mathcal{A}'$ as well.

Definition 2.39 (Category of bordered DGAs). The category of bordered DGAs will be denoted by $\mathcal{BDGA}$, and the full subcategory of $\mathcal{BDGA}$ consisting of cofibrants by $\mathcal{BDGA}^c$. Then the category $\mathcal{DGA}$ is the full subcategory of $\mathcal{BDGA}^c$ consisting of bordered DGAs of type $(0, 0)$.

Definition 2.40 (Cofibrant replacements of bordered DGAs). Let $\mathcal{A} = \left(A_{\mathsf{L}} \xrightarrow{\phi_{\mathsf{L}}} A \xleftarrow{\phi_{\mathsf{R}}} A_{\mathsf{R}}\right) \in \mathcal{BDGA}$ be a bordered DGA. The *cofibrant replacement* of $\mathcal{A}$

$$\widehat{\mathcal{A}} \coloneqq \left(A_{\mathsf{L}} \xrightarrow{\widehat{\phi}_{\mathsf{L}}} \widehat{A} \xleftarrow{\widehat{\phi}_{\mathsf{R}}} A_{\mathsf{R}}\right) \in \mathcal{BDGA}^c$$

is defined by the *mapping cylinder construction* as follows: if $A = (\mathsf{A} = T(\mathbb{K}\langle \mathsf{S}\rangle), \partial)$,

- the DGA $\widehat{A} = (\widehat{\mathsf{A}}, \widehat{\partial})$ and its graded algebra $\widehat{\mathsf{A}}$ is defined as

$$\widehat{\mathsf{A}} \coloneqq T\left(\mathbb{K}\left\langle R \amalg K_{\mathsf{R}} \amalg \widehat{K}_{\mathsf{R}}\right\rangle\right), \quad \widehat{K}_{\mathsf{R}} \coloneqq \left\{\widehat{k}_{ab} \;\middle|\; k_{ab} \in K_{\mathsf{R}}\right\}, \quad \left|\widehat{k}_{ab}\right| \coloneqq |k_{ab}| + 1.$$

- The differential $\widehat{\partial}$ for each k_{ab} is the same as $\partial_{\mathsf{R}}(k_{ab})$ and for $\widehat{k}_{ab}$ it is defined as

$$\widehat{\partial}(\widehat{k}_{ab}) \coloneqq k_{ab} - \phi_{\mathsf{R}}(k_{ab}) + \sum_{a<c<b} (-1)^{|\widehat{k}_{ac}|-1} \widehat{k}_{ac}\phi_{\mathsf{R}}(k_{cb}) + k_{ac}\widehat{k}_{cb}. \tag{2.8}$$

- The morphism $\widehat{\phi}_{\mathsf{L}}$ is the composition of ϕ_{L} and the canonical inclusion $A \to \widehat{A}$, and $\widehat{\phi}_{\mathsf{R}}$ is defined by $\widehat{\phi}_{\mathsf{R}}(k_{ab}) = k_{ab} \in \widehat{\mathsf{A}}$.

Let $\widehat{\pi} = (\mathrm{Id}, \widehat{\pi}, \mathrm{Id}) : \widehat{\mathcal{A}} \to \mathcal{A}$ be the bordered morphism which fixes both border DGAs and sends all $\widehat{k}_{ab}$ to zero and each k_{ab} to $\phi_{\mathsf{R}}(k_{ab})$.

Lemma 2.41. *The bordered DGA $\widehat{\mathcal{A}}$ is a weak stabilization.*

Proof. Notice that in each differential $\widehat{\partial}(\widehat{k}_{ab})$ there is one and only one generator k_{ab}. Therefore one may find a tame automorphism Φ on $\widehat{A}$ that sends $\widehat{\partial}(\widehat{k}_{ab})$ to k_{ab} so that the DGA $(\widehat{\mathsf{A}}, \widehat{\partial}_\Phi)$ with the twisted differential $\widehat{\partial}_\Phi \coloneqq \Phi \circ \widehat{\partial} \circ \Phi^{-1}$ becomes a stabilization of A. □

Remark 2.42. The cofibrant replacement $\widehat{\mathcal{A}}$ is *not* a strong stabilization. Indeed, the canonical inclusion $\mathbf{i}$ may involve the DGA homotopy. However it still satisfies the good property, for example, induces an A_∞-quasi-equivalence between augmentation categories. We will see this in Section 4.1.

2.2.2. Bordered Chekanov-Eliashberg DGAs. Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}^\mu$ be a front projection of a bordered Legendrian graph of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$ with a Maslov potential. We consider the Ng's resolution $\widehat{\mathcal{T}}_{\mathsf{Lag}} \coloneqq \mathrm{Res}^{\mathsf{Ng}}(\widehat{\mathcal{T}})$ of the

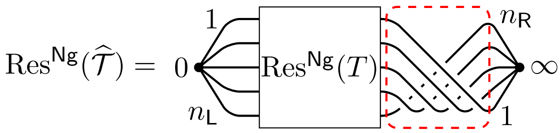


Figure 7: The Ng's resolution of the closure of $\mathcal{T}$.

closure of $\mathcal{T}$, which has two distinguished vertices 0 and ∞ of valency n_{L} and n_{R}, respectively. See Figure 7.

Then it is known that the Chekanov-Eliashberg DGA $A^{\mathsf{CE}}(\widehat{\mathcal{T}}_{\mathsf{Lag}}, \widehat{\boldsymbol{\mu}})$ has the following generating sets:

1) Crossings $C(\widehat{\mathcal{T}}_{\mathsf{Lag}})$;
2) Vertex generators $\widetilde{V}(\widehat{\mathcal{T}}_{\mathsf{Lag}}) = \{v_{a,i} \mid v \in V(\widehat{\mathcal{T}}_{\mathsf{Lag}}), a \in \mathbb{Z}/\operatorname{val}(v)\mathbb{Z}, i \geq 1\}$;
3) Basepoint generators $\widetilde{B}(\widehat{\mathcal{T}}_{\mathsf{Lag}}) = \{b_{a,\ell} \mid b \in B(\widehat{\mathcal{T}}_{\mathsf{Lag}}), a = 1, 2, \ell \geq 1\}$.

Especially, $C(\widehat{\mathcal{T}}_{\mathsf{Lag}})$ has the subset $\widehat{K}_{\mathsf{R}}$ of $\binom{n_{\mathsf{R}}}{2}$ crossings coming from the right border by $\widehat{k}_{ab}$

$$\widehat{K}_{\mathsf{R}} \coloneqq \left\{ \widehat{k}_{ab} \;\middle|\; 1 \leq a < b \leq n_{\mathsf{R}} \right\},$$

which are in the dashed box in Figure 7. Similarly, two distinguished vertices 0 and ∞ have the subsets of vertex generators

$$\begin{aligned} K_{\mathsf{L}} &\coloneqq \{k_{a'b'} = 0_{a',b'-a'} \mid 1 \leq a' < b' \leq n_{\mathsf{L}}\}, \\ K_{\mathsf{R}} &\coloneqq \{k_{ab} = \infty_{a,b-a} \mid 1 \leq a < b \leq n_{\mathsf{R}}\}. \end{aligned}$$

Then it is known that the differential for $\widehat{k}_{ab}$ is given as

$$\widehat{\partial}(\widehat{k}_{ab}) = (-1)^{|\widehat{k}_{ab}|-1} k_{ab} + g_{ab} + \sum_{a<c<b} (-1)^{|\widehat{k}_{ab}|-1} \widehat{k}_{ac} k_{cb} + g_{ac} \widehat{k}_{cb},$$

for some $g_{ab} \in A^{\mathsf{CE}}(T, \mu)$ which is essentially the same as the equation (2.8) by replacing $\widehat{k}_{ab}$ and g_{ab} with $(-1)^{|\widehat{k}_{ab}|-1}\widehat{k}_{ab}$ and $(-1)^{|\widehat{k}_{ab}|} g_{ab}$, respectively. In particular, the assignment $k_{ab} \mapsto (-1)^{|\widehat{k}_{ab}|} g_{ab}$ defines a DGA morphism.

We define a bordered DGA

$$\widehat{A}^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu}) = \left(A^{\mathsf{CE}}(T_{\mathsf{L}}, \mu_{\mathsf{L}}) \xrightarrow{\widehat{\phi}_{\mathsf{L}}} \widehat{A}^{\mathsf{CE}}(T, \mu) \xleftarrow{\widehat{\phi}_{\mathsf{R}}} A^{\mathsf{CE}}(T_{\mathsf{R}}, \mu_{\mathsf{R}}) \right)$$

associated to $(\mathcal{T}, \boldsymbol{\mu})$ as follows: two border DGAs are as before and the DGA $\widehat{A}^{\mathsf{CE}}(T,\mu) = (\widehat{\mathsf{A}}, \widehat{\partial})$ is the DG-subalgebra of $A^{\mathsf{CE}}(\widehat{\mathcal{T}}_{\mathsf{Lag}}, \widehat{\mu})$ generated by all crossings in $\widehat{\mathcal{T}}_{\mathsf{Lag}}$, vertex generators for all but 0 and ∞ and two sets K_{L} and K_{R}.

The *bordered Chekanov-Eliashberg DGA* (CE DGA) or *Legendrian contact homology DGA* (LCH DGA) for $\mathcal{T}^{\mu}$ is given as

$$A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu}) \coloneqq \left(A^{\mathsf{CE}}(T_{\mathsf{L}}, \mu_{\mathsf{L}}) \xrightarrow{\phi_{\mathsf{L}}} A^{\mathsf{CE}}(T, \mu) \xleftarrow{\phi_{\mathsf{R}}} A^{\mathsf{CE}}(T_{\mathsf{R}}, \mu_{\mathsf{R}}) \right),$$

where

1) two DGAs $A^{\mathsf{CE}}(T_{\mathsf{L}}, \mu_{\mathsf{L}})$ and $A^{\mathsf{CE}}(T_{\mathsf{R}}, \mu_{\mathsf{R}})$ are isomorphic to border DGAs $A_{n_{\mathsf{L}}}(\mu_{\mathsf{L}})$ and $A_{n_{\mathsf{R}}}(\mu_{\mathsf{R}})$, respectively, whose generating sets K_{L} and K_{R} are identified with

$$K_{\mathsf{L}} \coloneqq \{k_{a'b'} = 0_{a',b'-a'} \mid 1 \le a' < b' \le n_{\mathsf{L}}\},$$
$$K_{\mathsf{R}} \coloneqq \{k_{ab} = \infty_{a,b-a} \mid 1 \le a < b \le n_{\mathsf{R}}\},$$

2) the DGA $A^{\mathsf{CE}}(T,\mu)$ is the DG-subalgebra generated by crossings, vertex and basepoint generators contained only in $\mathrm{Res}^{\mathsf{Ng}}(T)$ together with K_{L}, and

3) the DGA morphism ϕ_{L} is an obvious inclusion while ϕ_{R} is defined as $k_{ab} \mapsto (-1)^{|\widehat{k}_{ab}|} g_{ab}$ as above.

Theorem 2.43. *[3, Corollary 3.3.7] Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}^{\mu}$. The bordered DGA $A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$ is well-defined and invariant under Legendrian isotopy up to stabilizations.*

Indeed, for each front Reidemeister move $(\mathsf{M}) : (\mathcal{T}', \boldsymbol{\mu}') \to (\mathcal{T}, \boldsymbol{\mu})$, there exists a zig-zag

$$A^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}') \xtwoheadleftarrow{\widehat{\boldsymbol{\pi}}'} \widehat{A}^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}') = \mathcal{A}_0 \underset{\boldsymbol{\pi}_1}{\overset{\mathbf{i}_1}{\rightleftarrows}} \cdots \underset{\boldsymbol{\pi}'_{n-1}}{\overset{\mathbf{i}'_{n-1}}{\leftrightarrows}} \mathcal{A}_n = \widehat{A}^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu}) \xrightarrow{\widehat{\boldsymbol{\pi}}'} A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$$

of stabilizations of bordered DGAs between $A^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}')$ and $A^{\mathsf{CE}}(\mathcal{T}\boldsymbol{\mu})$, where the bordered DGAs $\widehat{A}^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}')$ and $\widehat{A}^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$ are well-defined and the cofibrant replacements of $A^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}')$ and $A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$, respectively, and all but the left- and rightmost are strong stabilizations.

We explain briefly the reason why the DG-subalgebra $A^{\mathsf{CE}}(\mathrm{Res}^{\mathsf{Ng}}(\widehat{\mathcal{T}}, \widehat{\boldsymbol{\mu}}))$ is well-defined. Since two vertex 0 and ∞ face the unbounded region in $\mathbb{R}^2$

especially between two half-edges $h_{0,n_{\mathsf{L}}}$ and $h_{0,1}$, and $h_{\infty,n_{\mathsf{R}}}$ and $h_{\infty,1}$, there are no crossing generators involving vertex generators at 0 or ∞ which pass these unbounded regions in their differentials.

In other words, the bordered DGA $\widehat{A}^{\mathsf{CE}}(\mathcal{T},\boldsymbol{\mu})$ is the same as the LCH DGA of the Legendrian graph $(\widehat{\mathcal{T}},\widehat{\boldsymbol{\mu}})$ together with two DG-subalgebras $A^{\mathsf{CE}}(T_{\mathsf{L}},\mu_{\mathsf{L}})$ and $A^{\mathsf{CE}}(T_{\mathsf{R}},\mu_{\mathsf{R}})$ of the distinguished internal DG-subalgebras I_0 and I_∞. This data is obviously invariant under any front Reidemeister moves fixing two vertices 0 and ∞.

Remark 2.44. The exactly same statement holds for Lagrangian projections. That is, for each $(\mathsf{m}) : (\mathcal{T}'_{\mathsf{Lag}},\boldsymbol{\mu}') \to (\mathcal{T}_{\mathsf{Lag}},\boldsymbol{\mu})$, there is a zig-zag of stabilizations between two bordered LCH DGAs $A^{\mathsf{CE}}(\mathcal{T}'_{\mathsf{Lag}},\boldsymbol{\mu}')$ and $A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}},\boldsymbol{\mu})$.

More precisely, $(\widehat{\mathsf{III}})_*$ is an isomorphism and $(\widehat{\mathsf{M}})_*$ is a canonical projection of a stabilization for $(\mathsf{M}) \in \{(\mathsf{I}),(\mathsf{II}),(\mathsf{VI})\}$. However, we have a zig-zag of stabilization for (VI)

$$
\begin{array}{ccccccc}
 & & & \widetilde{S}A & & & \\
 & & \widetilde{\boldsymbol{\pi}}' \swarrow \nearrow \widetilde{\mathsf{i}}' & & \widetilde{\mathsf{i}} \nwarrow \searrow \widetilde{\boldsymbol{\pi}} & & \\
A^{\mathsf{CE}}(\mathcal{T}',\boldsymbol{\mu}') & \xleftarrow{\ \widehat{\boldsymbol{\pi}}'\ } & \widehat{\mathcal{A}}' & \xrightarrow[(\widehat{\mathsf{VI}})]{} & \widehat{\mathcal{A}} & \xrightarrow{\ \widehat{\boldsymbol{\pi}}\ } & A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T},\boldsymbol{\mu}).
\end{array}
$$

Similarly, for Lagrangian Reidemeister moves (m), we have an isomorphism $(\widehat{\mathsf{m}})_*$ for $(\mathsf{m}) \in \{(\mathsf{0}_*),(\mathsf{iii}_*)\}$, a canonical projection of a stabilization for (ii) and a zig-zag of stabilizations for (iv).

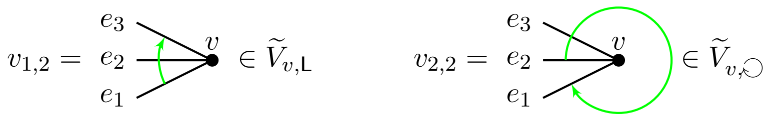
Example 2.45 (DGAs for bordered Legendrian graphs in a normal form). Recall a bordered Legendrian graph $(\mathcal{T},\boldsymbol{\mu}) \in \mathcal{BLG}^{\mu}$ in a normal form in Definition 2.17. Let $(\mathcal{T}_{\mathsf{Lag}},\boldsymbol{\mu})$ be the Ng's resolution of $(\mathcal{T},\boldsymbol{\mu})$. Then the set of vertex generators at each v can be decomposed into two subsets as follows:

$$\widetilde{V}_{v,\mathsf{L}} := \left\{ v_{a,j} \in \widetilde{V}_v \;\middle|\; a+j < \mathrm{val}(v) \right\}, \quad \widetilde{V}_{v,\circlearrowleft} := \left\{ v_{a,j} \in \widetilde{V}_v \;\middle|\; a+j \geq \mathrm{val}(v) \right\}$$

and we denote their unions by $\widetilde{V}_{\mathsf{L}}$ and $\widetilde{V}_{\circlearrowleft}$

$$\widetilde{V}_{\mathsf{L}} := \coprod_{v\in V} \widetilde{V}_{v,\mathsf{L}}, \qquad \widetilde{V}_{\circlearrowleft} := \coprod_{v\in V} \widetilde{V}_{v,\circlearrowleft}.$$

Notice that the elements in $\widetilde{V}_{v,\mathsf{L}}$ correspond to the vertex generators which are lying on the left hand side of the vertex v:

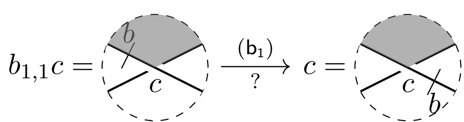


Since all vertices are in the upper right position, there are no immersed polygons for differentials involving generators in $\widetilde{V}_{\circlearrowleft}$ except for infinitesimal ones. That is, due to this geometric reason, no generators in $\widetilde{V}_{\circlearrowleft}$ appears in differentials for any crossing generator and in the image of the right border generators k_{ab}'s under ϕ_{R}.

Before closing this section, we remark the following. The LCH DGA $A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$ has infinitely many generators $b_{a,\ell}$ for each basepoint $b \in B$ while it usually contributes two generator inverses to each other such as t_b and t_b^{-1}. However, as observed in [11, Proposition 14], the internal DGA for each basepoint I_b is quasi-isomorphic to the Laurent polynomial ring $\mathbb{K}[t_b, t_b^{-1}]$. Therefore one can regard I_b as the free resolution of $\mathbb{K}[t_b, t_b^{-1}]$.

Indeed, $\mathbb{K}[t_b, t_b^{-1}]$ can be obtained by taking a quotient by all $b_{a,\ell}$'s with $\ell > 1$, whose degrees are not zero. Due to this observation, we can get one another consequence such that any DGA morphism from $A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$ to a field $K = (\mathbb{K}, \partial = 0)$ factors through the quotient DGA. Therefore there is an isomorphism between sets[2] of DGA morphisms.

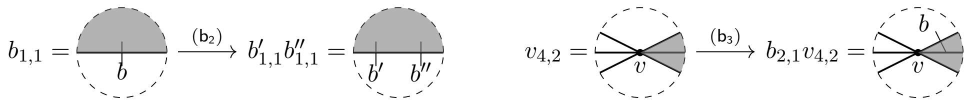
Now let us consider the effects of operations on basepoints to LCH DGAs. As seen earlier, there are three more moves on basepoints. As mentioned earlier, the operation $(\mathsf{b}_1) : (\mathcal{T}'_{\mathsf{Lag}}, \boldsymbol{\mu}') \to (\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu})$ can be viewed as a sequence of front Reidemeister moves. Hence we have a zig-zag of stabilizations which is not easy to describe directly.



On the other hand, two operations (b_2) and (b_3) can be regarded as *tangle replacements* so that they induce DGA morphisms $(\mathsf{b}_2)_*, (\mathsf{b}_3)_* : A^{\mathsf{CE}}(\mathcal{T}', \boldsymbol{\mu}') \to A^{\mathsf{CE}}(\mathcal{T}, \boldsymbol{\mu})$. Indeed, each generator $b_{a,\ell}$ or $v_{a,\ell}$ will be mapped to elements obtained by counting certain polygons in the support of (b_2) or

[2] Indeed, we have an isomorphism between *augmentation varieties*. See [3].

(b_3) in $\mathcal{T}$. For example, $b_{1,1}$ or $v_{4,2}$ will be mapped to $b'_{1,1}b''_{1,1}$ and $b_{2,1}v_{4,2}$. See [1, §6.5] for detail.



However, when we consider the quotient DGA by all generators $b_{a,\ell}$'s with $\ell > 1$, then the move (b_1) induces a DGA isomorphism $(\mathsf{b}_1)_*$ which sends c to $t_b^{-1}c$. The induced map $(\mathsf{b}_2)_*$ sends t_b to $t_{b'}t_{b''}$ and $(\mathsf{b}_3)_*$ sends $v_{a,\ell}$ to either $v_{a,\ell}$, $v_{a,\ell}t_b$ or $t_b^{-1}v_{a,\ell}$ according to where the basepoint b is lying on.

Conversely, one can find DGA morphisms $(\mathsf{b}_\mathsf{i}^{-1})_* : A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu}) \to A^{\mathsf{CE}}(\mathcal{T}'_{\mathsf{Lag}}, \boldsymbol{\mu}')$ for $i = 2, 3$ which are left inverses of $(\mathsf{b}_\mathsf{i})_*$ and defined as

$$(\mathsf{b}_2^{-1})_*(b''_{a,\ell}) := \begin{cases} 1 & \ell = 1; \\ 0 & \ell > 1, \end{cases} \qquad (\mathsf{b}_3^{-1})_*(b_{a,\ell}) := \begin{cases} 1 & \ell = 1; \\ 0 & \ell > 1. \end{cases}$$

In summary, we have the following:

Lemma 2.46. *Let* $(\mathsf{b}_\mathsf{i}) : (\mathcal{T}'_{\mathsf{Lag}}, \boldsymbol{\mu}') \to (\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu})$ *be a basepoint move. Then either*

1) *for* $i = 1$*, there exists a zig-zag of stabilizations between* $A^{\mathsf{CE}}(\mathcal{T}'_{\mathsf{Lag}}, \boldsymbol{\mu}')$ *and* $A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu})$*, or*
2) *for* $i = 2$ *or* 3*, there exists a pair of DGA morphisms*

$$A^{\mathsf{CE}}(\mathcal{T}'_{\mathsf{Lag}}, \boldsymbol{\mu}') \underset{(\mathsf{b}_\mathsf{i}^{-1})_*}{\overset{(\mathsf{b}_\mathsf{i})_*}{\rightleftarrows}} A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu}),$$

where $(\mathsf{b}_\mathsf{i}^{-1})_*$ *is a left inverse of the DGA morphism* $(\mathsf{b}_\mathsf{i})_*$ *induced from the tangle replacement.*

Remark 2.47. For $i = 2$ or 3, there is no issue on the commutativity of $(\mathsf{b}_\mathsf{i}^{-1})_*$with the structure morphisms ϕ_L and ϕ_R.

3. Consistent sequences

In this section, we first briefly review the definition of a consistent sequence of DGAs and the construction of the A_∞-category associated to the consistent sequence of DGAs and we will basically follow the definition and construction described in [21, §3.3] but may use the different notations or conventions.

Let us start with the category Δ_+ of all finite ordinals

$$[1] = \{1\}, \qquad [2] = \{1,2\}, \qquad [3] = \{1,2,3\}, \qquad \cdots,$$

whose morphisms are order-preserving inclusions.

Let $\mathcal{C}$ be a category, together with a sequence of functors $\mathcal{C}^{(m)} \to \mathcal{C}$ for all $m \geq 1$, satisfying the following conditions:

1) For any $h : [m] \to [n]$ in Δ_+, we have a functor $r_h : \mathcal{C}^{(n)} \to \mathcal{C}^{(m)}$.
2) For any other $h' : [\ell] \to [m]$ in Δ_+, we have a natural isomorphism: $r_{h'} \circ r_h \cong r_{h \circ h'}$.

In the above setup, $\mathcal{C}^{(m)}$ will be called the category of m-component objects in $\mathcal{C}$, and an object of $\mathcal{C}^{(m)}$ will be called an m-component object in $\mathcal{C}$. For each $h : [m] \to [n]$ and object $C^{(n)} \in \mathcal{C}^{(n)}$, we denote $r_h(C^{(n)}) = C^{(n)}|_{h([m])}$.

We then define the category of consistent sequences of objects in $\mathcal{C}$ as follows:

Definition 3.1 (Category of consistent sequences). A *consistent sequence* in $\mathcal{C}$ is a sequence of m-component objects $C^{(m)}$ in $\mathcal{C}$ for all $m \geq 1$, denoted by $C^{(\bullet)} = (C^{(m)})_{m \geq 1}$, satisfying the followings: let $h : [m] \to [n]$ and $h' : [\ell] \to [m]$ in Δ_+.

1) We have an isomorphism $C(h) : C^{(n)}|_{h([m])} \cong C^{(m)}$.
2) For $U = h'([\ell])$, we have a composition $C(h)^U$ of isomorphisms

$$C^{(n)}|_{h(U)} \xrightarrow{\cong} (C^{(n)}|_{h([m])})|_U \xrightarrow{r_{h'}(C(h))} C^{(m)}|_U,$$

where the first isomorphism comes from the natural isomorphism $r_{h \circ h'} \cong r_{h'} \circ r_h$, such that $C(h') \circ C(h)^U = C(h \circ h')$.

A *consistent morphism*, denoted by $f^{(\bullet)} = (f^{(m)})_{m \geq 1}$, between two consistent sequences $C^{(\bullet)}$ and $D^{(\bullet)}$ is defined to satisfy the following conditions:

1) For each $m \geq 1$, we have an m-component morphism $f^{(m)} : C^{(m)} \to D^{(m)}$.

2) For any $h : [m] \to [n]$ in Δ_+, we have $r_h(f^{(n)}) : C^{(n)}|_{h([m])} \to D^{(n)}|_{h([m])}$ together with the following commutative diagram:

$$\begin{array}{ccc} C^{(n)}|_{h([m])} & \xrightarrow{r_h(f^{(n)})} & D^{(n)}|_{h([m])} \\ {\scriptstyle C(h)}\downarrow & & \downarrow{\scriptstyle D(h)} \\ C^{(m)} & \xrightarrow{f^{(m)}} & D^{(m)} \end{array}$$

The category of all consistent sequences and consistent morphisms will be denoted by $\mathcal{C}^{(\bullet)}$ and called the *category of consistent sequences* in $\mathcal{C}$.

Notation 3.2. For each $h : [m] \to [n]$, $U \subset [m]$ and $i \in [m]$, we may denote $C^h|_U$, $C^{(m)}|_U$ and $C^{(m)}|_{\{i\}}$ by $C(h)^U$, C^U and C^i, respectively. We further denote $C^{(1)}$ and $f^{(1)}$ simply by C and f.

Lemma 3.3. *Let $C^{(\bullet)} \in \mathcal{C}^{(\bullet)}$ be a consistent sequence in $\mathcal{C}$. Then for each $m \geq 1$ and $i \in [m]$, we have*

$$C^i = C^{(m)}|_i \cong C^{(1)}|_1 = C^1.$$

Proof. This is obvious from the definition by considering $h_i : [1] \to [m]$ with $i = h_i(1)$. □

3.1. Consistent sequences of bordered Legendrian graphs

We define categories of consistent sequences of bordered Legendrian graphs given in terms of both Lagrangian and front projections, which are related via Ng's resolution as usual.

For a bordered graph $\mathbf{\Gamma} = (\Gamma_{\mathsf{L}} \to \Gamma \leftarrow \Gamma_{\mathsf{R}})$, an *$m$-component bordered graph* $\mathbf{\Gamma}^{(m)}$ is a bordered graph defined as

$$\mathbf{\Gamma}^{(m)} := (\Gamma_{\mathsf{L}}^{(m)} \to \Gamma^{(m)} \leftarrow \Gamma_{\mathsf{R}}^{(m)}),$$

where the inclusions preserve the label. Then for each $U \subset [m]$ and $i \in [m]$, let the restriction $\mathbf{\Gamma}^i := (\Gamma_{\mathsf{L}}^i \to \Gamma^i \leftarrow \Gamma_{\mathsf{R}}^i)$ and $\mathbf{\Gamma}^U := \coprod_{i \in U} \mathbf{\Gamma}^i$, where $\Gamma_*^i = \{i\} \times \Gamma_*$ for $* = \mathsf{L}, \mathsf{R}$ or empty.

For each m-component bordered graph $\mathbf{\Gamma}^{(m)}$, its Legendrian embedding defines an m-component bordered Legendrian graph

$$\mathcal{T}^{(m)} = \left(\mathsf{T}_{\mathsf{L}}^{(m)} \to \mathsf{T}^{(m)} \leftarrow \mathsf{T}_{\mathsf{R}}^{(m)}\right) = \left(\mathbf{\Gamma}^{(m)} \to J^1\mathbf{U}\right),$$

whose restriction $\mathcal{T}^U$ on $U \subset [m]$ is obviously defined as $\mathcal{T}^{(m)}|_{\mathbf{\Gamma}^U}$.

We can naturally consider the front and Lagrangian projections of m-component bordered Legendrian graphs which will be denoted by

$$\begin{aligned} &\mathcal{T}^{(m)} = (T_{\mathsf{L}}^{(m)} \to T^{(m)} \leftarrow T_{\mathsf{R}}^{(m)}) := \pi_{\mathsf{fr}}(\mathcal{T}^{(m)}) \\ \text{and} \quad &\mathcal{T}_{\mathsf{Lag}}^{(m)} = (T_{\mathsf{Lag,L}}^{(m)} \to T_{\mathsf{Lag}}^{(m)} \leftarrow T_{\mathsf{Lag,R}}^{(m)}) := \pi_{\mathsf{Lag}}(\mathcal{T}^{(m)}) \end{aligned}$$

and their restrictions on $U \subset [m]$ are denoted by $\mathcal{T}^U$ and $\mathcal{T}_{\mathsf{Lag}}^U$ as before.

One can equip a Maslov potential $\boldsymbol{\mu}^{(m)}$ on $\mathcal{T}^{(m)}$ or $\mathcal{T}_{\mathsf{Lag}}^{(m)}$ so that the restriction $\boldsymbol{\mu}^U$ is the restriction of $\boldsymbol{\mu}^{(m)}$ on $\mathcal{T}^U$ or $\mathcal{T}_{\mathsf{Lag}}^U$.

Notation 3.4. We denote the categories of all regular front and Lagrangian projections of m-component bordered Legendrian graphs with Maslov potentials by $\mathcal{BLG}^{\mu,(m)}$ and $\mathcal{BLG}_{\mathsf{Lag}}^{\mu,(m)}$, whose morphisms are Reidemeister moves preserving m-component structures, respectively.

For a sequence $(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})_{m\geq 1}$ of m-component front projections $(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}) \in \mathcal{BLG}^{\mu,(m)}$, the consistency is as follows: for each $(h : [m] \to [n]) \in \Delta_+$ and $U \subset [m]$, there is an isomorphism between two front projections in the sense of Definition 2.9

$$(\mathcal{T}(h)^U, \boldsymbol{\mu}(h)^U) : (\mathcal{T}^{h(U)}, \boldsymbol{\mu}^{h(U)}) \xrightarrow{\cong} (\mathcal{T}^U, \boldsymbol{\mu}^U).$$

Definition 3.5 (Consistent moves). A *consistent front (or Lagrangian) Reidemeister move* or *basepoint move* $(\mathsf{M})^{(\bullet)} : (\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)}) \to (\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ between two consistent sequences of front (or Lagrangian) projections is a sequence of sequences of front (or Lagrangian) Reidemeister moves or basepoint moves satisfying the following conditions:

1) for each $m \geq 1$, $(\mathsf{M})^{(m)} : (\mathcal{T}'^{(m)}, \boldsymbol{\mu}'^{(m)}) \to (\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})$ is a sequence of front (or Lagrangian) Reidemeister moves or basepoint moves and in particular, $(\mathsf{M})^{(1)} = (\mathsf{M}) \in \mathrm{RM}$ is a (possibly empty) front (or Lagrangian) Reidemeister move or a basepoint move;

2) it is compatible with restrictions, i.e., for each $U \subset [m]$, we have a sequence of front (or Lagrangian) Reidemeister moves or basepoint

moves

$$(\mathsf{M})^U : (\mathcal{T}'^U, \boldsymbol{\mu}'^U) \to (\mathcal{T}^U, \boldsymbol{\mu}^U);$$

3) it satisfies the consistency, i.e., for each $(h : [m] \to [n]) \in \Delta_+$ and $U \subset [m]$, the following diagram is commutative:

$$\begin{array}{ccc} (\mathcal{T}'^{h(U)}, \boldsymbol{\mu}'^{h(U)}) & \xrightarrow{(\mathcal{T}'(h)^U, \boldsymbol{\mu}'(h)^U)} & (\mathcal{T}'^U, \boldsymbol{\mu}'^U) \\ {\scriptstyle (\mathsf{M})^{h(U)}}\downarrow & & \downarrow{\scriptstyle (\mathsf{M})^U} \\ (\mathcal{T}^{h(U)}, \boldsymbol{\mu}'^{h(U)}) & \xrightarrow{(\mathcal{T}(h)^U, \boldsymbol{\mu}(h)^U)} & (\mathcal{T}^U, \boldsymbol{\mu}^U). \end{array}$$

We furthermore say that a consistent move $(\mathsf{M})^{(\bullet)}$ is *elementary* if it consists of Reidemeister moves or basepoint moves of only one type.

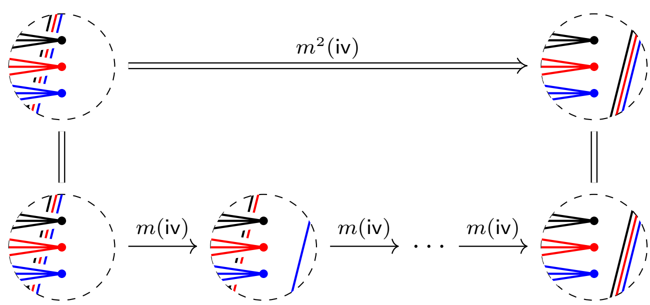
Two consistent sequences of front or Lagrangian projections are said to be (consistently) *equivalent* if and only if they are connected by a zig-zag sequence of consistent front or Lagrangian Reidemeister moves.

Notation 3.6. From now on, we will denote the elementary consistent Reidemeister move or basepoint move by the double arrow "⇒" such as

$$(\mathsf{M})^{(\bullet)} : \mathcal{T}' \Longrightarrow \mathcal{T}.$$

Example 3.7 (Elementary consistent Reidemeister moves). One example for elementary consistent Reidemeister move is a consistent Lagrangian Reidemeister move as follows: for each $m \geq 1$,

(3.1)



Definition 3.8 (Categories of consistent sequences of bordered Legendrian graphs). The categories of consistent sequences and Reidemeister moves of regular front and Lagrangian projections of bordered Legendrian graphs with Maslov potentials will be denoted by $\mathcal{BLG}^{\mu,(\bullet)}$ and $\mathcal{BLG}^{\mu,(\bullet)}_{\mathsf{Lag}}$, and two full subcategories consisting of consistent sequences of non-bordered Legendrian graphs will be denoted by $\mathcal{LG}^{\mu,(\bullet)}$ and $\mathcal{LG}^{\mu,(\bullet)}_{\mathsf{Lag}}$.

Recall the Ng's resolution $\mathrm{Res}^{\mathsf{Ng}} : \mathcal{BLG}^{\mu} \to \mathcal{BLG}^{\mu}_{\mathsf{Lag}}$ which is a functor as seen earlier and therefore it preserves isomorphisms. In other words, each isomorphic pair of front projections will be mapped to an isomorphic pair of Lagrangian projections. It is obvious that $\mathrm{Res}^{\mathsf{Ng}}$ sends each component to the corresponding component, it induces a functor between consistent sequences of front and Lagrangian projections.

Lemma 3.9. *The Ng's resolution induces a functor*

$$\mathrm{Res}^{\mathsf{Ng}^{(\bullet)}} : \mathcal{BLG}^{\mu,(\bullet)} \to \mathcal{BLG}^{\mu,(\bullet)}_{\mathsf{Lag}}$$

which preserves the homotopy relation.

Proof. This is nothing but the Lemma 2.28. □

3.1.1. Canonical front and Lagrangian parallel copies. We introduce a canonical way to obtain a consistent sequence for a bordered Legendrian graph in terms of front projections or Lagrangian projections.

Definition 3.10 (Canonical consistent sequences). Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}^{\mu}$ and $(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu}) \in \mathcal{BLG}^{\mu}_{\mathsf{Lag}}$ be regular front and Lagrangian projections. Then the *canonical consistent sequences* $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ and $(\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}_{\mathsf{Lag}}$ are given by the z- and y-translations as depicted in Figures 8 and 9, respectively.

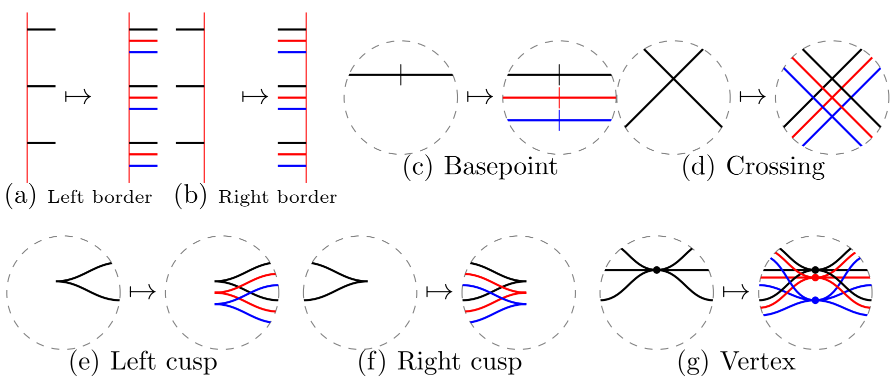


Figure 8: Canonical front parallel copies.

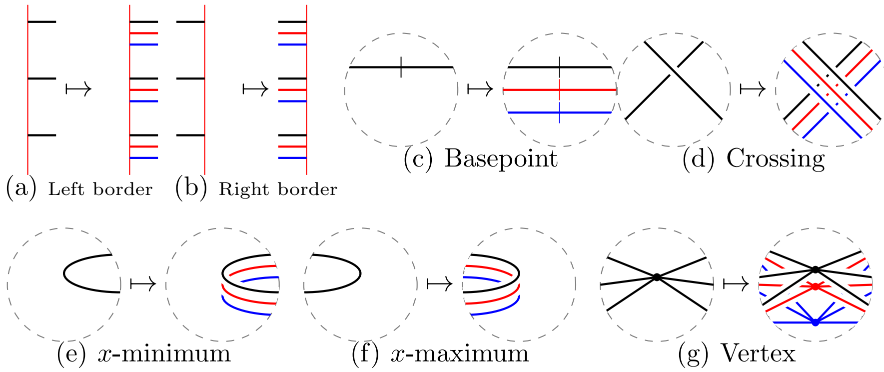


Figure 9: Canonical Lagrangian parallel copies.

Remark 3.11. For Lagrangian projections, the y-translation comes from a strict contactomorphism

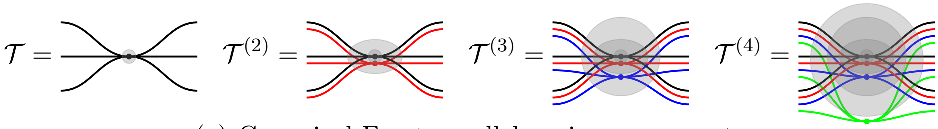
$$(x, y, z) \to (x, y - \epsilon, z + x\epsilon),$$

whose Lagrangian projection is the desired y-translation.

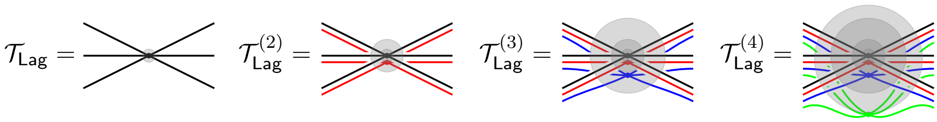
Notice that it is not obvious if $\mathcal{T}^{(m)}$ or $\mathcal{T}^{(m)}_{\mathsf{Lag}}$ is regular again. Indeed, we need to check the regularity holds near each vertex.

There are many ways to achieve the regularity but in this paper, we fix the convention as follows: let U_1 be a neighborhood of the vertex v. If we take a parallel copy and make $\mathcal{T}^{(2)}$ or $\mathcal{T}^{(2)}_{\mathsf{Lag}}$ by the small enough translation, then all additional crossings are contained in a small neighborhood U_2. Now we take the third copy such that the newly appeared crossings *avoid* the region U_2 and are contained in a larger neighborhood U_3 and so on. See Figure 10. The shaded region represents neighborhoods U_i. Then it is obvious that for each $\mathcal{T}$ or $\mathcal{T}^{\mathsf{Lag}}$, the sequence $\mathcal{T}^{(\bullet)}$ or $\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}$ of canonical projections becomes a consistent sequence of m-component Lagrangian projections.

Remark 3.12. In the canonical m-copies near a vertex v of type (ℓ, r), there are exactly $\binom{m}{2}\binom{\ell}{2} + \binom{m}{2}\binom{r}{2}$ many additional crossings.



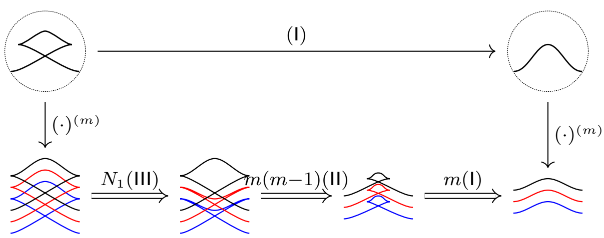
(a) Canonical Front parallel copies near a vertex

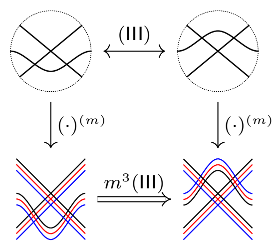


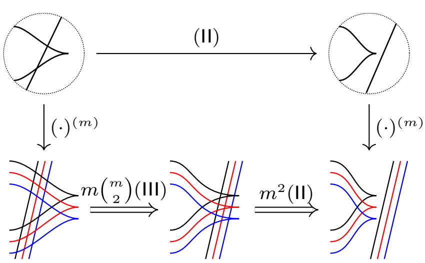
(b) Canonical Lagrangian parallel copies near a vertex

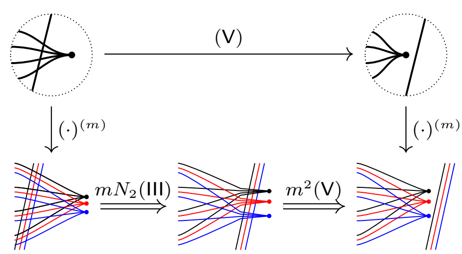
Figure 10: Canonical copies near a vertex.

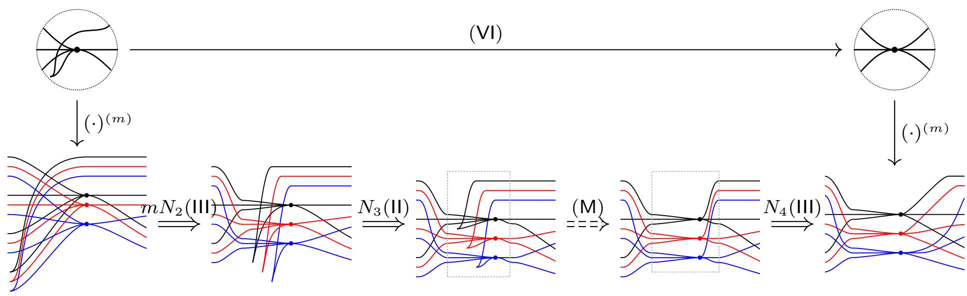
One can choose (zig-zag) sequences of consistent front and Lagrangian Reidemeister moves as follows:

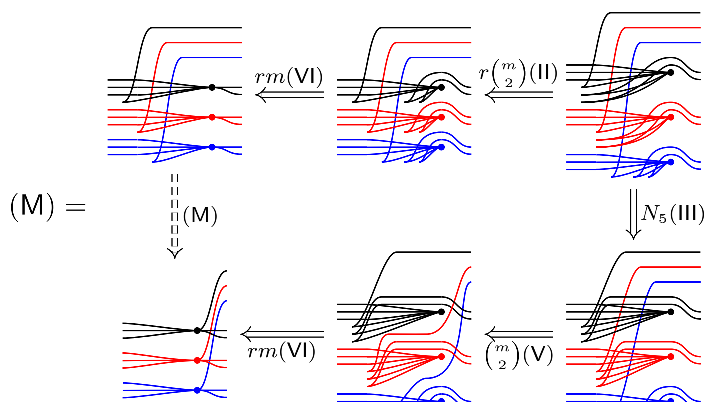


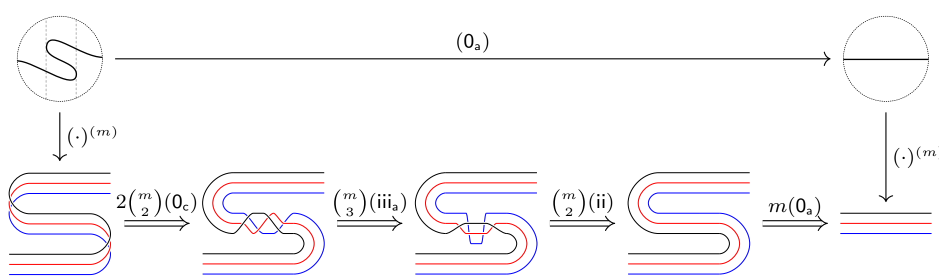


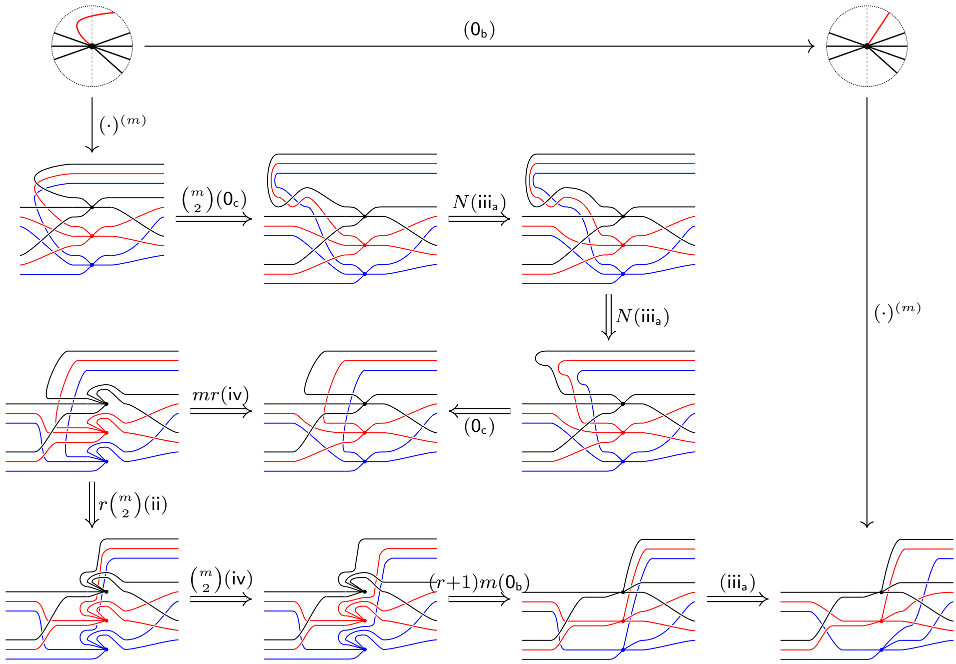




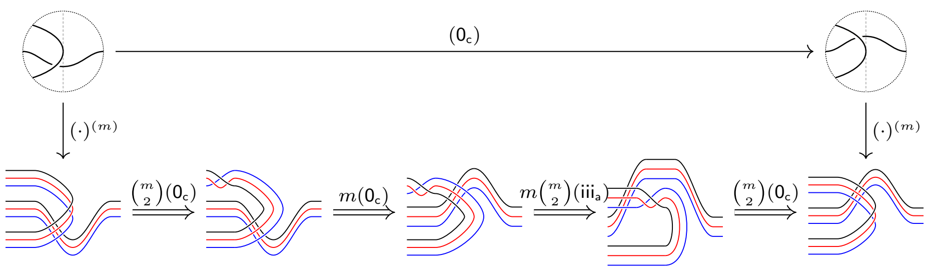

(VI)
$(\cdot)^{(m)}$
$(\cdot)^{(m)}$
mN_2(III)
N_3(II)
(M)
N_4(III)


(M) =
rm(VI)
$r\binom{m}{2}$(II)
(M)
N_5(III)
rm(VI)
$\binom{m}{2}$(V)

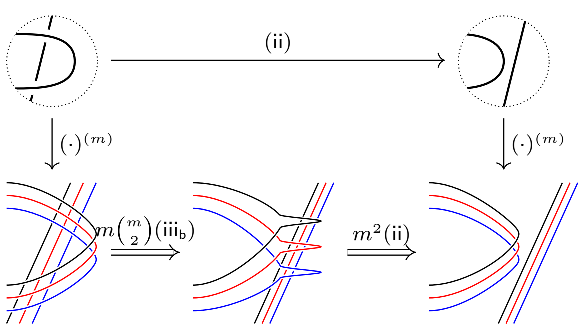

(0_a)
$(\cdot)^{(m)}$
$(\cdot)^{(m)}$
$2\binom{m}{2}$(0_c)
$\binom{m}{3}$(iii_a)
$\binom{m}{2}$(ii)
m(0_a)



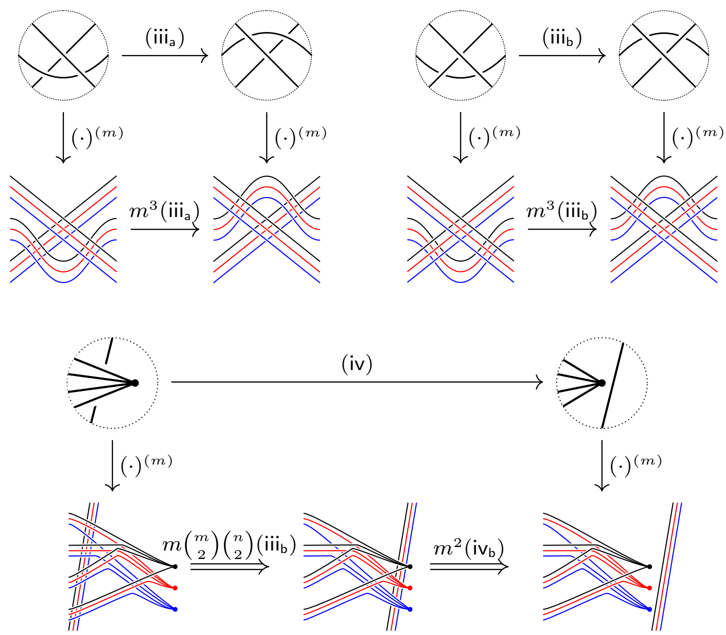
(0_b)
(·)^(m)
(m 2)(0_c)
N(iii_a)
N(iii_a)
mr(iv)
(0_c)
r(m 2)(ii)
(m 2)(iv)
(r+1)m(0_b)
(iii_a)
(·)^(m)



(0_c)
(·)^(m)
(m 2)(0_c)
m(0_c)
m(m 2)(iii_a)
(m 2)(0_c)
(·)^(m)



(ii)
(·)^(m)
m(m 2)(iii_b)
m^2(ii)
(·)^(m)



Here, the numbers N_i are given as

$$N_1 = \frac{m(m-1)(2m-1)}{6}, \quad N_2 = \binom{m}{2}\binom{\ell}{2}, \quad N_3 = \binom{m}{2}(\ell-1),$$
$$N_4 = \binom{m}{3}\binom{r}{2}, \quad N_5 = \binom{m}{2}(2\ell+1),$$

where ℓ and r are the numbers of half-edges on the left and right, respectively.

In addition, one can choose the consistent morphisms corresponding to the operations (B_*) on basepoints as depicted in Figure 11.

The one important observation is that all parallel copies are consistent and all arrows are elementary. Therefore in summary, we have the following proposition.

Proposition 3.13. *The canonical consistent sequences*

$$\mathcal{BLG}^{\mu} \to \mathcal{BLG}^{\mu,(\bullet)} \quad \text{and} \quad \mathcal{BLG}^{\mu}_{\mathsf{Lag}} \to \mathcal{BLG}^{\mu,(\bullet)}_{\mathsf{Lag}}$$

are well-defined and preserves the equivalences.

In particular, each consistent Reidemeister move and basepoint move can be mapped to a zig-zag of elementary consistent Reidemeister moves or basepoint moves.

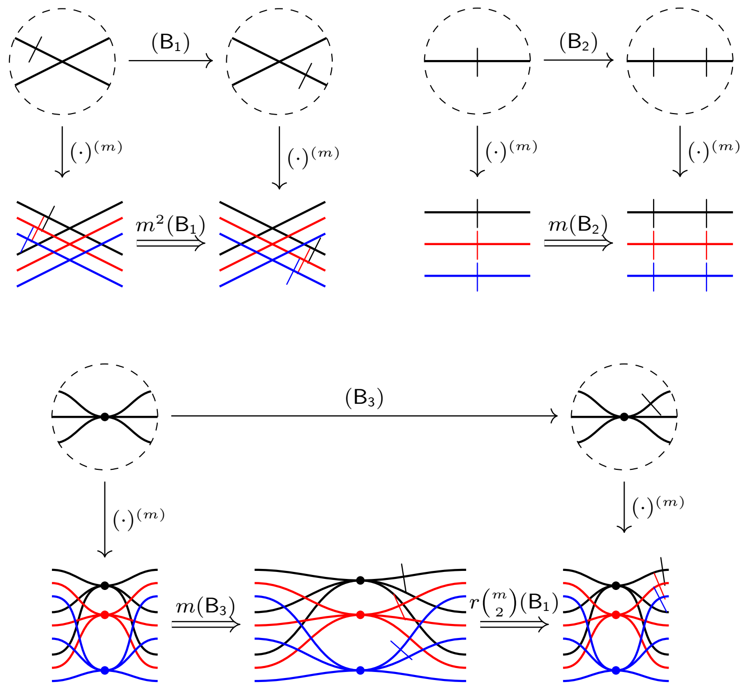


Figure 11: Consistent basepoint moves on the canonical front copies.

Example 3.14. Due to the definition of the bordered Legendrian graph in a normal form, the canonical consistent sequence of a bordered Legendrian graph in a normal form is a sequence of bordered Legendrian graphs in a normal form.

On the other hand, as seen in Lemma 2.18, for each consistent sequence $(\mathcal{T}^{(\bullet)}, \mu^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ of canonical front parallel copies, one can find a representative in a normal form up to consistent front Reidemeister moves. In other words, there is a consistent sequence $(\mathcal{T}'^{(\bullet)}, \mu'^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ of canonical front parallel copies in a normal form and a sequence of consistent front Reidemeister moves between $(\mathcal{T}'^{(\bullet)}, \mu'^{(\bullet)})$ and $(\mathcal{T}^{(\bullet)}, \mu^{(\bullet)})$

$$(\mathcal{T}'^{(\bullet)}, \mu'^{(\bullet)}) \xrightarrow{(\mathsf{M}_1)^{(\bullet)}} \cdots \xrightarrow{(\mathsf{M}_k)^{(\bullet)}} (\mathcal{T}^{(\bullet)}, \mu^{(\bullet)}).$$

Moreover, since the transformation from $\mathcal{T}'^{(1)}$ to $\mathcal{T}^{(1)}$ only needs $(\mathsf{V}), (\mathsf{VI})$ and (B_*), so does the sequence of consistent Reidemeister moves $(\mathsf{M}_\mathsf{i})^{(\bullet)}$.

That is, for each i,

$$(\mathsf{M_i})^{(\bullet)} \in \left\{(\mathsf{V})^{(\bullet)}, (\mathsf{VI})^{(\bullet)}, (\mathsf{B}_1)^{(\bullet)}, (\mathsf{B}_2)^{(\bullet)}, (\mathsf{B}_3)^{(\bullet)}\right\}.$$

Remark 3.15. It is easy to see that the Ng's resolution does not map canonical parallel copies in front projections onto those in Lagrangian projections. Indeed, if $\mathcal{T}$ contains a right cusp or a vertex of type (ℓ, r) with $\ell \geq 2$, then the resolution of the canonical front m-copies is not the same as the canonical Lagrangian m-copies of the resolution.

However, we can always find a consistent Lagrangian Reidemeister move consisting only of (ii)'s and ($\mathsf{iii_a}$)'s between the canonical Lagrangian copy of Ng's resolution and Ng's resolutions of the canonical front copy. See Figure 12.

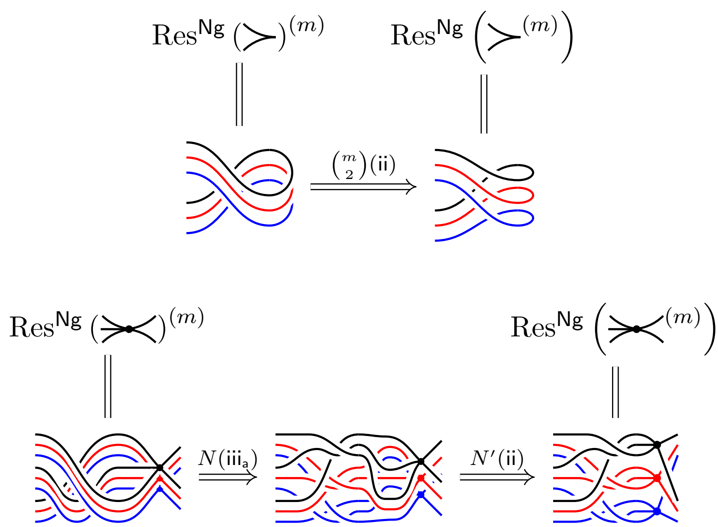


Figure 12: Ng's resolutions of front canonical copies.

3.2. Consistent sequences of bordered Chekanov-Eliashberg DGAs

In this section, we will consider consistent sequences of bordered Chekanov-Eliashberg DGAs.

For each $m \geq 1$, we define the ring $\mathbb{K}^{(m)}$ as

$$\mathbb{K}^{(m)} := \bigoplus_{i\in[m]} \mathbb{K}^i,$$

where $\mathbb{K}^i$ is the copy of $\mathbb{K}$ with the unit $\mathbf{1}^i$. Then $\mathbb{K}^{(m)}$ has the unit $\mathbf{1}^{(m)} := \sum_{i\in[m]} \mathbf{1}^i$ and for each subset $U \subset [m]$, there exists a unique ring homomorphism

$$\mathbb{K}^{(m)} \to \mathbb{K}^U := \bigoplus_{i\in U} \mathbb{K}^i.$$

Therefore $\mathbb{K}^{(m)}$ becomes an m-component object and furthermore it is obviously consistent.

3.2.1. Link-grading and composable DGAs. We first introduce an m-component link-grading on DGAs as described in [21, §3.2].

Definition 3.16 (Link-graded and composable DGAs). Let $A^{(m)} = (\mathsf{A}^{(m)}, \partial^{(m)})$ be a DGA which is not necessarily semi-free. We say that A is *m-component link-graded* for some $m \geq 1$ if there exist functions $r, c : \mathsf{A}^{(m)} \to [m]$ such that for each $g \in \mathsf{A}^{(m)}$ and any word $g_1 \ldots g_k$ in $\partial(g)$ different from $n\mathbf{1}$ for any $n \in \mathbb{Z}$ is *composable* in the sense that

$$r(g) = r(g_1), \qquad c(g) = c(g_k), \qquad r(g_i) = c(g_{i+1})$$

for each $1 \leq i < k$. We say that a composable word $g_1 \ldots g_k$ is said to be *of type* (i, j) if $r(g_1) = i$ and $c(g_k) = j$.

An m-component link-graded DGA $A^{(m)} = (\mathsf{A}^{(m)}, \partial^{(m)})$ is *composable* over $[m]$ if $\mathsf{A}^{(m)}$ is decomposed as

$$\mathsf{A}^{(m)} = \bigoplus_{(i,j)\in[m]^2} \mathsf{A}^{(m)}(i, j)$$

such that it satisfies the following:

- for each $i, j, k, l \in [m]$, the multiplication on $\mathsf{A}^{(m)}(i, j) \otimes \mathsf{A}^{(m)}(k, l)$ does not vanish only if $j = k$. In this case, the result is contained in $\mathsf{A}^{(m)}(i, l)$.
- for each (i, j), the restriction $A^{(m)}(i, j) = (\mathsf{A}^{(m)}(i, j), \partial^{(m)}|_{\mathsf{A}^{(m)}(i,j)})$ is a subchain complex. In particular, $A^{(m)}(i, i)$ becomes the DG-subalgebra.

One equivalent definition of composable DGAs is as follows: a DGA $A^{(m)} = (\mathsf{A}^{(m)}, \partial^{(m)})$ is composable over $[m]$ if and only if it is a subalgebra of the matrix algebra so that

$$\mathsf{A}^{(m)} \cong \mathrm{Mat}_m\left(\mathsf{A}^{(m)}(i,j)\right) := \left\{\left(a^{ij}\right) \;\middle|\; a^{ij} \in \mathsf{A}^{(m)}(i,j)\right\}, \quad \partial^{(m)} \cong (\partial^{(m)}|_{\mathsf{A}(i,j)})$$

and also it is a $\mathbb{K}^{(m)}$-module.

Definition 3.17 (Semi-free composable DGAs). We say that a composable DGA $A^{(m)} = (\mathsf{A}^{(m)}, \partial^{(m)})$ is *semi-free* if it is tensor-algebra-like as follows: there exists a graded set $\mathsf{S}^{(m)}$

$$\mathsf{S}^{(m)} = \coprod_{i,j \in [m]^2} \mathsf{S}^{ij}$$

and the underlying graded algebra $\mathsf{A}^{(m)}$ is the direct sum

$$\mathsf{A}^{(m)} = T_{\mathsf{co}}\mathsf{M}^{(m)} := \bigoplus_{\ell \geq 0} \left(\mathsf{M}^{(m)}\right)^{\ell},$$

where $\mathsf{M}^{(m)} = (\mathsf{M}^{ij}) \subset \mathsf{A}^{(m)}$ is a $\mathbb{K}^{(m)}$-module and each M^{ij} is a free $(\mathbb{K}^i, \mathbb{K}^j)$-bimodule generated by S^{ij}

$$\mathsf{M}^{ij} := \mathbb{K}^i \otimes \mathbb{Z}\langle \mathsf{S}^{ij} \rangle \otimes \mathbb{K}^j,$$

and $\left(\mathsf{M}^{(m)}\right)^{\ell}$ is the ℓ-fold product of $\mathsf{M}^{(m)}$ with respect to the matrix multiplication so that $\left(\mathsf{M}^{(m)}\right)^0 := \mathbb{K}^{(m)}$.

A DGA morphism $f : A'^{(n)} \to A^{(m)}$ between semi-free composable DGAs is said to be an *composable morphism* if there exists $(h : [m] \to [n]) \in \Delta_+$ such that for each $i', j' \in [n]$, $f(A'^{(n)}(i', j'))$ is either contained in $A(i,j)$ for $h(i,j) = (i', j')$ or 0 otherwise.

Let $\mathbf{i} = (i_0, \ldots, i_k)$ be a sequence in $[m]$. We denote $\mathsf{M}^{\mathbf{i}}$ by the submodule defined as

$$\mathsf{M}^{\mathbf{i}} := \mathsf{M}^{i_0 i_1} \mathsf{M}^{i_1 i_2} \ldots \mathsf{M}^{i_{k-1} i_k}.$$

Then it is easy to see that

$$\mathsf{A}^{(m)}(i,j) = \bigoplus_{\mathbf{i}} \mathsf{M}^{\mathbf{i}},$$

where $\mathbf{i}$ ranges over all sequences starting and ending at i and j, respectively.

Notation 3.18. For simplicity, we will use the notation $\mathsf{M}^{(m)} = (\mathbb{K}^{(m)}\langle \mathsf{S}^{(m)}\rangle)$ and so

$$\mathsf{A}^{(m)} = T_{\mathsf{co}}(\mathbb{K}^{(m)}\langle \mathsf{S}^{(m)}\rangle).$$

Definition 3.19 (Category of composable DGAs). We denote the category of all semi-free composable DGAs and morphisms by $\mathcal{DGA}_{\mathsf{co}}$. In particular, the full subcategory of m-components semi-free composable DGAs is denoted by $\mathcal{DGA}_{\mathsf{co}}^{(m)}$.

Remark 3.20. It is obvious that each morphism $f \in \mathcal{DGA}_{\mathsf{co}}^{(m)}$ preserves the link-grading. Especially, the subcategory $\mathcal{DGA}_{\mathsf{co}}^{(1)}$ is equivalent to $\mathcal{DGA}$.

Let $A^{(n)} \in \mathcal{DGA}_{\mathsf{co}}^{(n)}$. Then for each $h : [m] \to [n]$ in Δ_+, there exists a m-component semi-free composable DGA $A^{h([m])} \in \mathcal{DGA}_{\mathsf{co}}^{(m)}$ which is the image of the quotient map

$$\mathrm{res}^h : A^{(n)} \to A^{h([m])}$$

defined by $\mathbf{1}^i \mapsto 0$ for each $i \notin h([m])$. Then the map res^h plays the role of the functor r_h, and moreover, for $h' : [\ell] \to [m]$, we have

$$\mathrm{res}^{h'} \circ \mathrm{res}^h = \mathrm{res}^{h \circ h'}.$$

Example 3.21 (Composable DGAs from link-graded semi-free DGAs). For each m-component link-graded semi-free DGA $A^{(m)} = (\mathsf{A}^{(m)} = T(\mathbb{K}\langle \mathsf{S}^{(m)}\rangle), \partial^{(m)})$, one can associate an m-component composable DGA $A_{\mathsf{co}}^{(m)}$ as described in [21, §3.2]. We will briefly review the construction as follows:

1) Add idempotent elements $\mathbf{1}^i$ for $i \in [m]$ and declare $\mathbf{1}^i\mathbf{1}^j = \delta_{ij}$.
2) For each element g of type (i,j), declare $\mathbf{1}^i g = g = g\mathbf{1}^j$.
3) For each element g of type (i,i) having $n\mathbf{1}$ in $\partial(g)$, replace $n\mathbf{1}$ with $n\mathbf{1}^i$.
4) The unit $\mathbf{1}$ is replaced with $\sum_{i\in[m]} \mathbf{1}^i$.

Then it is easy to check that

$$A_{\mathsf{co}}^{(m)} = (T_{\mathsf{co}}(\mathbb{K}^{(m)}\langle \mathsf{S}^{(m)}\rangle), \partial^{(m)})$$

and therefore it is semi-free composable.

Definition 3.22 (Consistent sequences of composable DGAs). The category of consistent sequences in $\mathcal{DGA}_{\mathsf{co}}$ will be denoted by $\mathcal{DGA}_{\mathsf{co}}^{(\bullet)}$.

More precisely, a sequence $A^{(\bullet)}$ of m-component composable DGAs is consistent if and only if for each $(h : [m] \to [n]) \in \Delta_+$ and $U \subset [m]$, we have an isomorphism

$$A^{h([m])} \to A^{(m)},$$

which identify $\mathbf{1}^{h(i)}$ with $\mathbf{1}^i$ for each i in $[m]$.

Example/Definition 3.23 (Consistent sequences of composable border DGAs). Let $\mu : [n] \to \mathbb{Z}$ be a function. Recall the border DGA $A_n(\mu) \in \mathcal{DGA} = \mathcal{DGA}_{\mathsf{co}}^{(1)}$. For each $m \geq 1$, we will define the *m-component border DGA* $A_n^{(m)}(\mu) \in \mathcal{DGA}_{\mathsf{co}}^{(m)}$ such that $A_n^{(1)}(\mu) = A_n(\mu)$.

Let $A_n^{(m)}(\mu) = (\mathsf{A}_n^{(m)}, \partial_n^{(m)})$ be an m-component link graded DGA defined as follows:

1) The algebra $\mathsf{A}_n^{(m)}$ is the algebra $T_{\mathsf{co}}\mathsf{M}^{(m)}$ over the ring $\mathbb{K}^{(m)}$

$$\mathsf{A}_n^{(m)} := T_{\mathsf{co}}(\mathbb{K}^{(m)}\langle K^{ij} \amalg Y^{ij}\rangle), \quad K_n^{ij} := \{k_{ab}^{ij} \mid 1 \leq a < b \leq n\},$$
$$Y_n^{ij} := \begin{cases} \{y_a^{ij} \mid a \in [n]\} & i < j; \\ \varnothing & i \geq j. \end{cases}$$

2) The link-grading for each k_{ab}^{ij} and y_a^{ij} is defined as (i, j) and the (homological) grading of each is defined as

$$|k_{ab}^{ij}| := |k_{ab}| \quad \text{and} \quad |y_a^{ij}| := -1.$$

3) The differential $\partial_n^{(m)}$ is defined as

$$\partial_n^{(m)}(k_{ab}^{ij}) := \sum_{\substack{a<c<b \\ 1\leq \ell \leq m}} (-1)^{|k_{ac}|-1} k_{ac}^{i\ell} k_{cb}^{\ell j} + \sum_{\ell<j} (-1)^{|k_{ab}|-1} k_{ab}^{i\ell} y_b^{\ell j} + \sum_{i<\ell} y_a^{i\ell} k_{ab}^{\ell j},$$
$$\partial_n^{(m)}(y_a^{ij}) := \sum_{i<\ell<j} y_a^{i\ell} y_a^{\ell j}.$$

Geometrically, generators k_{ab}^{ij} and y_a^{ij} correspond to the Reeb chord from the b-th edge in the j-th copy T_n^j to the a-th edge in the i-th copy T_n^i and the Reeb chord from the a-th edge of T_n^i to a-th edge of T_n^j, respectively.

For each $U \subset [m]$, the restriction $A_n^U(\mu)$ will be defined as the image of the quotient map sending all k_{ab}^{ij}'s and y_a^{ij}'s to zero unless $i, j \in U$. This implies that $A_n^{(m)}(\mu)$ satisfies all axioms for m-component objects.

Moreover, the sequence $(A_n^{(m)}(\mu))_{m\geq 1}$ satisfies the consistency as follows: for each $h : [m] \to [m']$ and $U \subset [m]$, there is an obvious isomorphism $A_n^{h(U)}(\mu) \to A_n^U(\mu)$ sending each $k_{ab}^{i'j'}$ and $y_a^{i'j'}$ to k_{ab}^{ij} and y_a^{ij} if $i' = h(i)$ and $j' = h(j)$. Therefore we have a consistent sequence of composable DGAs $A_n^{(\bullet)}(\mu) \in \mathcal{DGA}_{\mathsf{co}}^{(\bullet)}$.

Corollary 3.24. *The composable DGA $A_n^{(m)}(\mu)$ can be obtained from the semi-free border DGA $A_{mn}(\mu^{(m)})$ via the standard recipe described in Example 3.21, where $\mu^{(m)} : [mn] \cong [m] \times [n] \to \mathbb{Z}$ is defined to be*

$$\mu^{(m)}(i, a) = \mu(a).$$

Proof. This follows directly from the definition and we are done. □

Now we consider consistent sequences of bordered DGAs.

Definition 3.25 (m-component bordered DGAs). An m-component bordered DGA of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$

$$\mathcal{A}^{(m)} = \left(A_{\mathsf{L}}^{(m)} \xrightarrow{\phi_{\mathsf{L}}^{(m)}} A^{(m)} \xleftarrow{\phi_{\mathsf{R}}^{(m)}} A_{\mathsf{R}}^{(m)} \right)$$

is a bordered DGA satisfying the following conditions:

1) All DGAs $A_*^{(m)}$ for $* = \mathsf{L}, \mathsf{R}$ or empty are in $\mathcal{DGA}_{\mathsf{co}}^{(m)}$,
2) two DGAs $A_{\mathsf{L}}^{(m)}$ and $A_{\mathsf{R}}^{(m)}$ are isomorphic to $A_{n_{\mathsf{L}}}^{(m)}(\mu_{\mathsf{L}})$ and $A_{n_{\mathsf{R}}}^{(m)}(\mu_{\mathsf{R}})$ for some μ_{L} and μ_{R},
3) two structure morphisms $\phi_{\mathsf{L}}^{(m)}$ and $\phi_{\mathsf{R}}^{(m)}$ are composable morphisms.

For each $U \subset [m]$, the restriction $\mathcal{A}^U$ for $U \subset [m]$ is defined by the image of the quotient map sending all elements in $A^{(m)}(i, j)$ to zero unless $i, j \in U$

$$\mathcal{A}^U := \left(A_{\mathsf{L}}^U \xrightarrow{\phi_{\mathsf{L}}^U} A^U \xleftarrow{\phi_{\mathsf{R}}^U} A_{\mathsf{R}}^U \right),$$

where ϕ_{L}^U and ϕ_{R}^U are the induced maps by $\phi_{\mathsf{L}}^{(m)}$ and $\phi_{\mathsf{R}}^{(m)}$, respectively.

A bordered composable morphism $\mathcal{A}'^{(n)} \to \mathcal{A}^{(m)}$ between n and m-components bordered composable DGAs is a collection of bordered DGA morphisms $\left\{\mathcal{A}'^{h(U)} \to \mathcal{A}^U \mid U \subset [m]\right\}$ for a uniquely determined $(h : [m] \to [n]) \in \Delta_+$, which is compatible with the restriction map $\mathcal{A}^{(m)} \to \mathcal{A}^U$.

We denote the category of all bordered composable DGAs by $\mathcal{BDGA}_{\mathsf{co}}$ and its full subcategory of m-components by $\mathcal{BDGA}_{\mathsf{co}}^{(m)}$.

By definition, the categories $\mathcal{DGA}_{\mathsf{co}}$ and $\mathcal{DGA}^{(m)}_{\mathsf{co}}$ are full subcategory of $\mathcal{BDGA}_{\mathsf{co}}$ and $\mathcal{BDGA}^{(m)}_{\mathsf{co}}$ of type $(0,0)$, respectively.

Let $\mathcal{A}^{(m)} \in \mathcal{BDGA}^{(m)}_{\mathsf{co}}$ be a sequence of m-components bordered DGAs. Then it is consistent if and only if for each $(h : [m] \to [n]) \in \Delta_+$ and $U \subset [m]$, we have an isomorphism $\mathcal{A}(h)^U : \mathcal{A}^{h(U)} \to \mathcal{A}^U$ between composable bordered DGAs.

Similarly, a sequence $\mathbf{f}^{(\bullet)} : \mathcal{A}'^{(\bullet)} \to \mathcal{A}^{(\bullet)}$ between consistent sequences of bordered DGAs is consistent if and only if for each $m \geq 1$, $\mathbf{f}^{(m)}$ is a bordered morphism between m-component bordered DGAs $\mathcal{A}'^{(m)}$ and $\mathcal{A}^{(m)}$, and it satisfies the consistency, i.e., for each $h : [m] \to [n]$ and $U \subset [m]$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}'^{h(U)} & \xrightarrow{\mathcal{A}'(h)^U} & \mathcal{A}'^{U} \\ {\scriptstyle \mathbf{f}^{h(U)}}\downarrow & & \downarrow{\scriptstyle \mathbf{f}^{U}} \\ \mathcal{A}^{h(U)} & \xrightarrow{\mathcal{A}(h)^U} & \mathcal{A}^{U} \end{array}$$

Definition 3.26 (Consistent sequences of composable bordered DGAs). We denote the category of consistent sequences of composable bordered DGAs by $\mathcal{BDGA}^{(\bullet)}_{\mathsf{co}}$.

In terms of generators, we have the following observation. Let $A^{(m)} = \left(\mathsf{A}^{(m)} = T_{\mathsf{co}}(\mathbb{K}^{(m)}\langle \mathsf{S}^{(m)}\rangle), \partial^{(m)}\right)$ be a composable m-component DGA. Then the restriction on U is now given by the image of the map $\mathbf{1}^i_e \mapsto 0$ for all $e \in E$ unless $i \in U$. Moreover, the consistency implies that for each $(h : [m] \to [n]) \in \Delta_+$, the DGA $A^{(m)}$ is isomorphic to the DGA generated by $\mathsf{S}^{i'j'}$'s for $i', j' \in h([m])$. Therefore we may identify

$$\mathsf{S}^{ii} \cong \mathsf{S}^{11} \quad \text{and} \quad \mathsf{S}^{ij} \cong \mathsf{S}^{12}$$

by considering $h_i : [1] \to [m]$ and $h_{ij} : [2] \to [m]$ with $h_i(1) = i$ and $h_{ij}(1) = i, h_{ij}(2) = j$.

Finally, the differential $\partial^{(m)}$ is the same as the push-forward of $\partial^{(n)}$ on $A(h) : A^{(n)} \to A^{(m)}$. In other words, for each generator $a^{ij} \in \mathsf{S}^{ij}$,

$$\partial^{(m)}\left(a^{ij}\right) = (A(h) \circ \partial^{(n)} \circ A(h)^{-1})(a^{ij}) = A(h)\left(\partial^{(n)}\left(a^{i'j'}\right)\right),$$

where $i' = h(i), j' = h(j)$ and $a^{i'j'}$ is regarded as an element of $\mathsf{S}^{i'j'} \cong \mathsf{S}^{ij}$. In other words, the differential in $A^{(m)}$ can be obtained from the differential in $A^{(n)}$ by removing generators not coming from $A^{(m)}$.

The following is the direct consequence of Lemma 3.3 and summarizes the above discussion.

Corollary 3.27. *[21, Lemma 3.15] Let $\mathcal{A}^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$ be a consistent sequence of bordered DGAs. Then for each $h_i : [1] \to [m]$ with $i = h_i(1)$ and $h_{ij} : [2] \to [m]$ with $i = h_{ij}(1) < j = h_{ij}(2)$, we have isomorphisms between DGAs*

$$\mathcal{A}^{ii} \cong \mathcal{A}^{11} \quad \textit{and} \quad \mathcal{A}^{ij} \cong \mathcal{A}^{12}.$$

Definition 3.28 (Consistent stabilizations). A consistent sequence of bordered DGAs $\mathcal{A}'^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$ is a (*weak*) *consistent stabilization* of $\mathcal{A}^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$ if there exists a consistent morphism $\boldsymbol{\pi}^{(\bullet)} : \mathcal{A}'^{(\bullet)} \to \mathcal{A}^{(\bullet)}$ of canonical projections of stabilization, and is a *strong consistent stabilization* if it is a weak stabilization and there exists a consistent morphism $\mathbf{i}^{(\bullet)} : \mathcal{A}^{(\bullet)} \to \mathcal{A}'^{(\bullet)}$ of canonical embeddings.

Let $A'^{(\bullet)} = (\mathsf{A}'^{(\bullet)} = T_{\mathsf{co}}(\mathbb{K}^{(\bullet)}\langle \mathsf{S}'^{(\bullet)}\rangle), \partial'^{(m)})$ be a consistent stabilization of $A^{(\bullet)} = (\mathsf{A}^{(\bullet)} = T_{\mathsf{co}}(\mathbb{K}^{(\bullet)}\langle \mathsf{S}^{(\bullet)}\rangle), \partial^{(m)})$. Then in terms of generating sets, for each $m \geq 1$, there exists an index set $I^{(m)} = (I^{ij})$ such that

$$\begin{aligned} &\mathsf{S}'^{ij} = \mathsf{S}^{ij} \amalg \{e_k^{ij}, \widehat{e}_k^{ij} \mid k \in I^{ij}\}, \\ &|\widehat{e}_k^{ij}| = |e_k^{ij}| + 1, \quad \partial'^{(m)}(\widehat{e}_k^{ij}) = e_k^{ij}, \quad \partial'^{(m)}(e_k^{ij}) = 0. \end{aligned}$$

Then for each $U \subset \Delta_+$, the restriction A'^U is generated by

$$\mathsf{S}'^U = \mathsf{S}^U \amalg \{e_k^{ij}, \widehat{e}_k^{ij} \mid k \in I^{ij}, i, j \in U\}$$

which is the image of the quotient map sending all e_k^{ij}'s and $\widehat{e}_k^{ij}$'s to zero unless both $i, j \in U$. Finally, for each $(h : [m] \to [n]) \in \Delta_+$, the consistency implies that we have an isomorphism

$$A'^{h(U)} \to A'^U$$

sending each generator $e_k^{i'j'}$ and $\widehat{e}_k^{i'j'}$ to e_k^{ij} and $\widehat{e}_k^{ij}$, respectively, for $h(i,j) = (i', j')$. Therefore it is necessary and sufficient to have the following conditions

$$I^{ij} \cong I^{kl} \quad \text{and} \quad |e_k^{ij}| = |e_k^{kl}|$$

for $(k, l) = (1, 1), (1, 2)$ or $(2, 1)$ according to whether $i = j$, $i < j$ or $i > j$.

As before, the one important example of a consistent stabilization is the cofibrant replacement of a consistent sequence of bordered DGAs defined as follows:

Example/Definition 3.29 (Consistent cofibrant replacement). Let $\mathcal{A}^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$ be a consistent sequence of bordered composable DGAs. For each $m \geq 1$, we consider the cofibrant replacement $\widehat{\mathcal{A}}^{(m)}$ of $\mathcal{A}^{(m)}$ defined in Definition 2.40.

For each $U \subset [m]$, the restriction $\widehat{\mathcal{A}}^{U}$ will be given as the cofibrant replacement of $\mathcal{A}^{U}$ and therefore $\widehat{\mathcal{A}}^{(m)}$ is an m-component bordered DGA as desired.

Finally, the consistency of $\widehat{\mathcal{A}}^{(m)}$ comes easily from the consistency of $\mathcal{A}^{(m)}$ as follows: for each $h : [m] \to [n]$ and $U \subset [m]$, the bordered DGA $\widehat{\mathcal{A}}^{h(U)}$ is the cofibrant replacement of $\mathcal{A}^{h(U)}$ as above which is isomorphic to $\mathcal{A}^{U}$ due to the consistency of $\mathcal{A}^{(\bullet)}$. Since the cofibrant replacement remains the same under isomorphism, we have a desired isomorphism

$$\widehat{\mathcal{A}}^{h(U)} \cong \widehat{\mathcal{A}}^{U}.$$

Lemma 3.30. *For each $\mathcal{A}^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$, its cofibrant replacement $\widehat{\mathcal{A}}^{(\bullet)} \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}$ is well-defined.*

Proof. This is a summary of discussion above and we omit the proof. □

3.2.2. Consistent sequences of LCH DGAs. Let $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ be a consistent sequence of front projections of bordered Legendrian graphs with Maslov potentials. Then for each $m \geq 1$, we have the bordered LCH DGA $A^{\mathsf{CE}}(\mathcal{T}^{(m)}, \mu^{(m)})$

$$\begin{aligned} &A^{\mathsf{CE}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}) \\ &\quad = \left(A^{\mathsf{CE}}(T_{\mathsf{L}}^{(m)}, \mu_{\mathsf{L}}^{(m)}) \xrightarrow{\phi_{\mathsf{L}}^{(m)}} A^{\mathsf{CE}}(T^{(m)}, \mu^{(m)}) \xleftarrow{\phi_{\mathsf{R}}^{(m)}} A^{\mathsf{CE}}(T_{\mathsf{R}}^{(m)}, \mu_{\mathsf{R}}^{(m)}) \right). \end{aligned}$$

It is obvious that the DGA $A^{\mathsf{CE}}(T_{*}^{(m)}, \mu_{*}^{(m)})$ for each $* = \mathsf{L}, \mathsf{R}$ or empty is m-component link-graded but is not composable. However, by using the standard recipe described in Example 3.21, we obtain an m-component composable bordered DGA $A_{\mathsf{co}}^{\mathsf{CE}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})$

$$\begin{aligned} &A_{\mathsf{co}}^{\mathsf{CE}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}) \\ &\quad = \left(A_{\mathsf{co}}^{\mathsf{CE}}(T_{\mathsf{L}}^{(m)}, \mu_{\mathsf{L}}^{(m)}) \xrightarrow{\phi_{\mathsf{L}}^{(m)}} A_{\mathsf{co}}^{\mathsf{CE}}(T^{(m)}, \mu^{(m)}) \xleftarrow{\phi_{\mathsf{R}}^{(m)}} A_{\mathsf{co}}^{\mathsf{CE}}(T_{\mathsf{R}}^{(m)}, \mu_{\mathsf{R}}^{(m)}) \right). \end{aligned}$$

Then it is obvious that two induced morphisms $\phi_{\mathsf{L}}^{(m)}$ and $\phi_{\mathsf{R}}^{(m)}$ are composable morphisms but it is not yet known to satisfy the axiom for m-component objects.

Proposition 3.31. *The bordered composable DGA* $A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}) \in \mathcal{BDGA}^{(m)}_{\mathsf{co}}$ *is an* m*-component bordered composable DGA. Moreover, if* $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ *is a consistent sequence, then the corresponding sequence*

$$A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) := \left(A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})\right)_{m\geq 1}$$

of bordered DGAs is consistent.

Proof. We know that two border DGAs $A^{\mathsf{CE}}_{\mathsf{co}}(T^{(m)}_{\mathsf{L}}, \mu^{(m)}_{\mathsf{L}})$ and $A^{\mathsf{CE}}_{\mathsf{co}}(T^{(m)}_{\mathsf{R}}, \mu^{(m)}_{\mathsf{R}})$ are isomorphic to the m-component border DGAs

$$A^{\mathsf{CE}}_{\mathsf{co}}(T^{(m)}_{\mathsf{L}}, \mu^{(m)}_{\mathsf{L}}) \cong A^{(m)}_{n_{\mathsf{L}}}(\mu_{\mathsf{L}}) \quad \text{and} \quad A^{\mathsf{CE}}_{\mathsf{co}}(T^{(m)}_{\mathsf{R}}, \mu^{(m)}_{\mathsf{R}}) \cong A^{(m)}_{n_{\mathsf{R}}}(\mu_{\mathsf{R}})$$

by Corollary 3.24.

For each $U \subset [m]$, we define the restriction $A^{\mathsf{CE}}_{\mathsf{co}}(T^{(m)}, \mu^{(m)})^U$ as the image of the quotient map sending all generators of type (i,j) to zero unless $i, j \in U$. Hence $A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})$ admits an m-component object structure.

Moreover, we have an isomorphism

$$A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})^U \cong A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^U, \boldsymbol{\mu}^U) \tag{3.2}$$

since we only consider generators and immersed disks lying on T^U in the left hand side.

Now assume that $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ is consistent and let $A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ be the sequence of bordered LCH DGAs of $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$. Then for each $(h : [m] \to [n]) \in \Delta_+$ and $U \subset [m]$, we have isomorphisms between bordered composable DGAs

$$\begin{aligned} A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(n)}, \boldsymbol{\mu}^{(n)})^{h(U)} &\cong A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{h(U)}, \boldsymbol{\mu}^{h(U)}) \\ \text{and} \quad A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})^U &\cong A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^U, \boldsymbol{\mu}^U) \end{aligned}$$

by the observation (3.2). Since two restrictions $(\mathcal{T}^{h(U)}, \boldsymbol{\mu}^{h(U)})$ and $(\mathcal{T}^U, \boldsymbol{\mu}^U)$ are isomorphic due to the consistency of $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$, we have a desired isomorphism

$$A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(n)}, \boldsymbol{\mu}^{(n)})^{h(U)} \cong A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})^U$$

which means the consistency of $A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \mu^{(\bullet)})$. □

Let $(\mathsf{M})^{(\bullet)} : (\mathcal{T}'^{(\bullet)}, \mu'^{(\bullet)}) \to (\mathcal{T}^{(\bullet)}, \mu^{(\bullet)})$ be an elementary consistent front Reidemeister move and let us denote their bordered composable LCH DGAs by $\mathcal{A}'^{(\bullet)}$ and $\mathcal{A}^{(\bullet)}$

$$\mathcal{A}'^{(\bullet)} = \left(A'^{(\bullet)}_{\mathsf{L}} \xrightarrow{\phi'^{(\bullet)}_{\mathsf{L}}} A'^{(\bullet)} \xleftarrow{\phi'^{(\bullet)}_{\mathsf{R}}} A'^{(\bullet)}_{\mathsf{R}}\right) := A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)}),$$
$$\mathcal{A}^{(\bullet)} = \left(A^{(\bullet)}_{\mathsf{L}} \xrightarrow{\phi^{(\bullet)}_{\mathsf{L}}} A^{(\bullet)} \xleftarrow{\phi^{(\bullet)}_{\mathsf{R}}} A^{(\bullet)}_{\mathsf{R}}\right) := A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}).$$

Then as before, we will pass through the cofibrant replacement as follows. By Lemma 3.30, the cofibrant replacements

$$\widehat{\mathcal{A}}'^{(\bullet)} = \left(A'^{(\bullet)}_{\mathsf{L}} \xrightarrow{\widehat{\phi}'^{(\bullet)}_{\mathsf{L}}} \widehat{A}'^{(\bullet)} \xleftarrow{\widehat{\phi}'^{(\bullet)}_{\mathsf{R}}} A'^{(\bullet)}_{\mathsf{R}}\right)$$
$$\text{and} \quad \widehat{\mathcal{A}}^{(\bullet)} = \left(A^{(\bullet)}_{\mathsf{L}} \xrightarrow{\widehat{\phi}^{(\bullet)}_{\mathsf{L}}} \widehat{A}^{(\bullet)} \xleftarrow{\widehat{\phi}^{(\bullet)}_{\mathsf{R}}} A^{(\bullet)}_{\mathsf{R}}\right)$$

are also consistent and furthermore stabilizations of $\mathcal{A}'^{(\bullet)}$ and $\mathcal{A}^{(\bullet)}$, respectively. That is, there are consistent morphisms $\widehat{\pi}'^{(\bullet)}$ and $\widehat{\pi}^{(\bullet)}$ of canonical projections

$$\widehat{\pi}'^{(\bullet)} : \widehat{\mathcal{A}}'^{(\bullet)} \xrightarrow{\simeq} \mathcal{A}'^{(\bullet)} \quad \text{and} \quad \widehat{\pi}^{(\bullet)} : \widehat{\mathcal{A}}^{(\bullet)} \xrightarrow{\simeq} \mathcal{A}^{(\bullet)}.$$

On the other hand, $(\mathsf{M})^{(\bullet)}$ gives us an elementary consistent Reidemeister move

$$(\mathsf{M})^{(\bullet)} : (\widehat{\mathcal{T}}'^{(\bullet)}, \widehat{\boldsymbol{\mu}}'^{(\bullet)}) \to (\widehat{\mathcal{T}}^{(\bullet)}, \widehat{\boldsymbol{\mu}}^{(\bullet)}).$$

Then by the condition (2) of consistent Reidemeister moves in Definition 3.5 and as observed in the discussion after Theorem 2.43, for each $m \geq 1$ it induces a zig-zag of stabilizations

$$A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}'^{(m)}, \boldsymbol{\mu}'^{(\bullet)}) \xleftarrow{\widehat{\boldsymbol{\pi}}'^{(m)}} \widehat{\mathcal{A}}'^{(m)} \xrightarrow[\cong]{(\widehat{\mathsf{III}})^{(m)}} \widehat{\mathcal{A}}^{(m)} \xrightarrow{\widehat{\boldsymbol{\pi}}^{(m)}} A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}),$$

$$A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}'^{(m)}, \boldsymbol{\mu}'^{(\bullet)}) \xleftarrow{\widehat{\boldsymbol{\pi}}'^{(m)}} \widehat{\mathcal{A}}'^{(m)} \underset{\mathbf{i}^{(m)}}{\overset{(\widehat{\mathsf{M}})^{(m)}=\boldsymbol{\pi}^{(m)}}{\rightleftarrows}} \widehat{\mathcal{A}}^{(m)} \xrightarrow{\widehat{\boldsymbol{\pi}}^{(m)}} A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)})$$

for $(\mathsf{M}) \in \{(\mathsf{I}), (\mathsf{II}), (\mathsf{VI})\}$, or

$$\begin{array}{c} \widetilde{S}A^{(m)} \\ \widetilde{\boldsymbol{\pi}}'^{(m)} \swarrow\nearrow \widetilde{\mathbf{i}}'^{(m)} \quad \widetilde{\mathbf{i}}^{(m)} \nwarrow\searrow \widetilde{\boldsymbol{\pi}}^{(m)} \\ A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}'^{(m)}, \boldsymbol{\mu}'^{(m)}) \xleftarrow{\widehat{\boldsymbol{\pi}}'^{(m)}} \widehat{\mathcal{A}}'^{(m)} \xrightarrow[(\widehat{\mathsf{VI}})^{(m)}]{} \widehat{\mathcal{A}}^{(m)} \xrightarrow{\widehat{\boldsymbol{\pi}}^{(m)}} A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(m)}, \boldsymbol{\mu}^{(m)}) \end{array}$$

between m-component bordered composable LCH DGAs. Moreover, the condition (3) in Definition 3.5 implies the consistency of all arrows above and therefore we have a zig-zag of consistent stabilizations corresponding to $(\mathsf{M})^{(\bullet)}$.

Similarly, one can observe the same property for elementary consistent Lagrangian Reidemeister moves as well and we omit the detail.

In summary, we have the following theorem.

Theorem 3.32. *For* $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{\mu,(\bullet)}$ *and* $(\mathcal{T}_{\mathsf{Lag}}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}_{\mathsf{Lag}}^{\mu,(\bullet)}$, *the assignments*

$$
\begin{aligned}
&(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \mapsto A_{\mathsf{co}}^{\mathsf{CE}}\left(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}\right) \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}\\
\text{and}\quad &(\mathcal{T}_{\mathsf{Lag}}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \mapsto A_{\mathsf{co}}^{\mathsf{CE}}\left(\mathcal{T}_{\mathsf{Lag}}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}\right) \in \mathcal{BDGA}_{\mathsf{co}}^{(\bullet)}
\end{aligned}
$$

are well-defined and each elementary consistent front or Lagrangian Reidemeister move induces a zig-zag of consistent stabilizations.

Now let us consider the consistent basepoint move. For an elementary consistent basepoint move $(\mathsf{B}_1)^{(\bullet)}$ or $(\mathsf{b}_1)^{(\bullet)}$, we already know that it induces a zig-zag of consistent stabilizations since it is indeed a sequence of elementary consistent Reidemeister moves.

For the moves $(\mathsf{B}_\mathsf{i})^{(\bullet)}$ or $(\mathsf{b}_\mathsf{i})^{(\bullet)}$ with $i = 2$ or 3, it is obvious that there is a consistent morphisms

$$
A^{\mathsf{CE}}(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}') \underset{(\mathsf{B}_\mathsf{i}^{-1})_*^{(\bullet)}}{\overset{(\mathsf{B}_\mathsf{i})_*^{(\bullet)}}{\rightleftarrows}} A^{\mathsf{CE}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}) \qquad A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}'^{(\bullet)}, \boldsymbol{\mu}') \underset{(\mathsf{b}_\mathsf{i}^{-1})_*^{(\bullet)}}{\overset{(\mathsf{b}_\mathsf{i})_*^{(\bullet)}}{\rightleftarrows}} A^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}^{(\bullet)}, \boldsymbol{\mu}).
$$

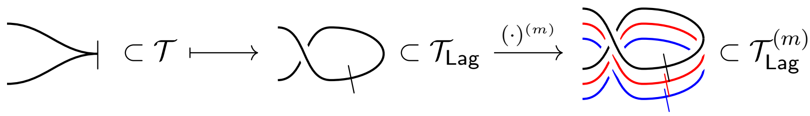
Then as mentioned in Remark 2.47, both bordered consistent morphisms $(\mathsf{B}_\mathsf{i}^{-1})_*^{(\bullet)}$ and $(\mathsf{b}_\mathsf{i}^{-1})_*^{(\bullet)}$ are well-defined and left inverses of consistent morphisms $(\mathsf{B}_\mathsf{i})_*^{(\bullet)}$ and $(\mathsf{b}_\mathsf{i})_*^{(\bullet)}$, respectively.

3.2.3. Legendrian graphs in a normal form. Recall the definition of Legendrian graphs or bordered Legendrian graphs in a normal form defined in Definition 2.17.

Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}_{\mathsf{Lag}}^{\mu}$ be a bordered Legendrian graph of type $(n_\mathsf{L}, n_\mathsf{R})$ in a normal form and $(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu}) \in \mathcal{BLG}_{\mathsf{Lag}}^{\mu}$ be its image of Ng's resolution.

Let us consider the canonical Lagrangian m-copies $\mathcal{T}_{\mathsf{Lag}}^{(m)}$. One of the benefit of being in a normal form is indeed that the DGA $A_{\mathsf{co}}^{\mathsf{CE}}\left(\mathcal{T}_{\mathsf{Lag}}^{(m)}, \boldsymbol{\mu}^{(m)}\right)$ can be easily described in terms of the data of $\mathcal{T}_{\mathsf{Lag}}$ and the DGA $A_{\mathsf{co}}^{\mathsf{CE}}(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu})$.

Recall that the x-maximal points in $\mathcal{T}$ are basepoints and their Ng's resolutions look as depicted in Definition 2.27. Therefore the canonical Lagrangian m-copy near each basepoint $b \in \mathcal{T}_{\mathsf{Lag}}$ (or equivalently, a x-maximum) looks like



Let us consider the set of x-minimal points in $\mathcal{T}_{\mathsf{Lag}}$. Then it has one-to-one correspondence with the set of connected components of the complement of $V \amalg B$, which is the set of edges.

The set of generators for $A^{\mathsf{CE}}(T_{\mathsf{Lag}}, \mu)$ is the union of C, K_{L}, $\widetilde{V}$ and $\widetilde{B}$

$$\begin{aligned} C &= \{c \ \mid c \text{ is a double point}\}\,, \\ K_{\mathsf{L}} &= \{k_{ab} \mid 1 \le a < b \le n_{\mathsf{L}}\}\,, \\ \widetilde{V} &= \{v_{a,\ell} \mid v \in V, 1 \le a \le \mathrm{val}(v), \ell \ge 1\}\,, \\ \widetilde{B} &= \{b_{a,\ell} \mid b \in B, a = 1, 2, \ell \ge 1\}\,. \end{aligned}$$

As seen in Definition 2.17 and Example 2.45, we define the sets $\widetilde{V}_{\mathsf{L}}$ and $\widetilde{V}_{\circlearrowleft}$. Then the generating set for $A^{\mathsf{CE}}\left(T^{(m)}_{\mathsf{Lag}}, \mu^{(m)}\right)$ is the union of the following sets:

$$C^{(m)} \amalg K^{(m)}_{\mathsf{L}} \amalg \widetilde{V}^{(m)}_{\mathsf{L}} \amalg \widetilde{V}^{(m)}_{\circlearrowleft} \amalg \widetilde{B}^{(m)} \amalg X^{(m)} \amalg Y^{(m)},$$

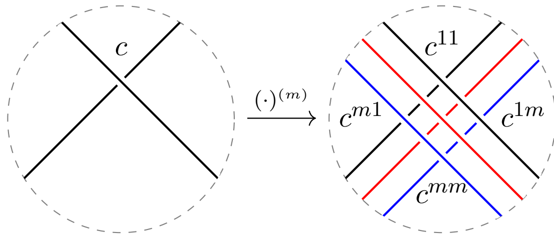
where

$$\begin{aligned} C^{(m)} &:= \left\{c^{ij} \;\middle|\; c \in C, i, j \in [m]\right\}, & \left|c^{ij}\right| &:= |c|, \\ K^{(m)}_{\mathsf{L}} &:= \left\{k^{ij}_{ab} \;\middle|\; 1 \le a < b \le n_{\mathsf{L}}\right\}, & \left|k^{ij}_{ab}\right| &:= |k_{ab}|\,, \\ \widetilde{V}^{(m)}_{\mathsf{L}} &:= \left\{v^{ij}_{a,\ell} \;\middle|\; v_{a,\ell} \in \widetilde{V}_{\mathsf{L}}, i \le j \in [m]\right\}, & \left|v^{ij}_{a,\ell}\right| &:= |v_{a,\ell}|\,, \\ \widetilde{V}^{(m)}_{\circlearrowleft} &:= \left\{v^{ii}_{a,\ell} \;\middle|\; v_{a,\ell} \in \widetilde{V}_{\circlearrowleft}, i \in [m]\right\}, & \left|v^{ii}_{a,\ell}\right| &:= |v_{a,\ell}|\,, \\ \widetilde{B}^{(m)} &:= \left\{b^{ii}_{a,\ell} \;\middle|\; b_{a,\ell} \in \widetilde{B}, i \in [m]\right\}, & |b^{ii}_{a,\ell}| &:= |b_{a,\ell}|, \\ X^{(m)} &:= \left\{x^{ij}_{b} \;\middle|\; b \in B, i < j \in [m]\right\}, & \left|x^{ij}_{b}\right| &:= 0, \\ Y^{(m)} &:= \left\{y^{ij}_{e} \;\middle|\; y_e \in E, i < j \in [m]\right\}, & \left|y^{ij}_{e}\right| &:= -1. \end{aligned}$$

Geometrically, these generating sets corresponding to the crossings and vertex generators of $T^{(m)}$ as follows:

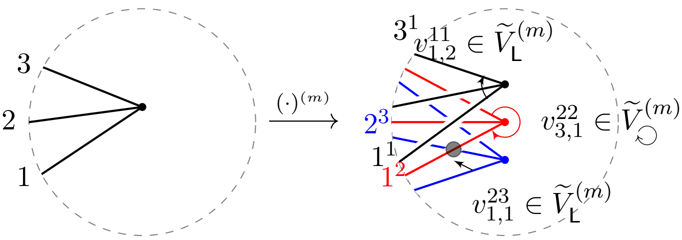
- Crossing: for each $c \in C$, there are m^2-many copies c^{ij} in $C^{(m)}$

(3.3)



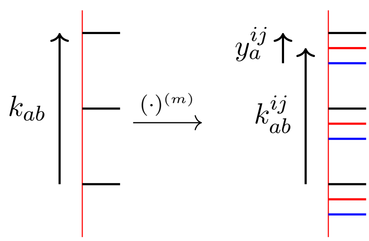
- Vertex: for each $v \in V$, there are two sets of generators $\widetilde{V}^{(m)}_{v,\mathsf{L}}$ and $\widetilde{V}^{(m)}_{v,\circlearrowleft}$

(3.4)



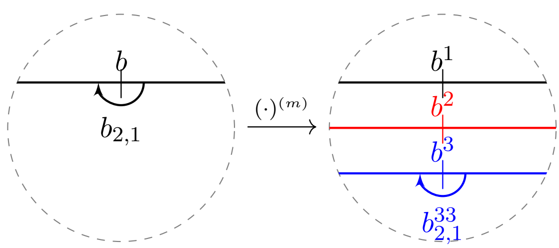
- Left border: for each $k_{ab} \in K_{\mathsf{L}}$ with $a < b$, there is the set $k^{(m)}_{\mathsf{L}}$ of m^2 generators k^{ij}_{ab}, and for each $i < j$ and $1 \le a \le \ell$, there are $\binom{m}{2}$ generators y^{ij}_a,

(3.5)



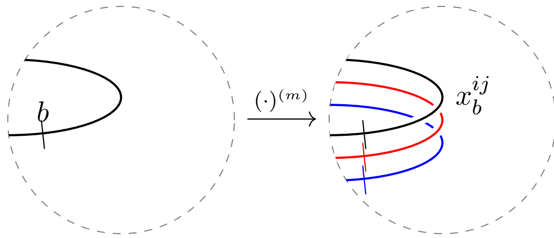
- basepoint: for each $b_{a,\ell} \in \widetilde{B}$, we have the set of m generators $b^{ii}_{a,\ell}$

(3.6)



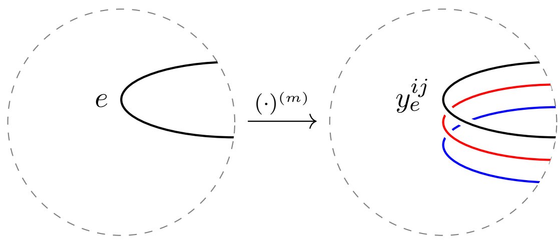
- x-maximum: for each $b \in B$, we have the set of $\binom{m}{2}$ generators x_b^{ij} for $i < j$

(3.7)



- x-minimum: for each $e \in E$, we have the subset of $\binom{m}{2}$ generators y_e^{ij} for $i < j$

(3.8)



Note that for each $v_{a,\ell} \in \widetilde{\mathsf{V}}_{\mathsf{L}}$, the generator $v_{a,\ell}^{ij}$ corresponds to a crossing generator between a-th and $(a+\ell)$-th edges in the i-th and j-th component, respectively, if $i < j$. If $i = j$, then it is the corresponding vertex generator on the i-th component.

On the other hand, for $v_{a,\ell} \in \widetilde{\mathsf{V}}_{\circlearrowleft}$, the generator $v_{a,\ell}^{ii}$ is the corresponding vertex generator as well but there are no generators $v_{a,\ell}^{ij}$ if $i \neq j$.

Let us consider matrices as follows: for each $c \in \mathsf{C}$, $k_{ab} \in K_{\mathsf{L}}^{(m)}$, $v_{a,\ell} \in \widetilde{V}_{\mathsf{L}}$, $w_{a,\ell} \in \widetilde{V}_{\circlearrowleft}$, $b_{a,\ell} \in \widetilde{B}$, $b \in B$ and $e \in E$,

$$\Phi(c) := \begin{pmatrix} c^{11} & c^{12} & \cdots & c^{1m} \\ c^{21} & c^{22} & \cdots & c^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c^{m1} & c^{m2} & \cdots & c^{mm} \end{pmatrix}, \quad \Phi(k_{ab}) := \begin{pmatrix} k_{ab}^{11} & k_{ab}^{12} & \cdots & k_{ab}^{1m} \\ k_{ab}^{21} & k_{ab}^{22} & \cdots & k_{ab}^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ k_{ab}^{m1} & k_{ab}^{m2} & \cdots & k_{ab}^{mm} \end{pmatrix},$$

$$\Phi(v_{a,\ell}) := \begin{pmatrix} v_{a,\ell}^{11} & v_{a,\ell}^{12} & \cdots & v_{a,\ell}^{1m} \\ 0 & v_{a,\ell}^{22} & \cdots & v_{a,\ell}^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_{a,\ell}^{mm} \end{pmatrix}, \quad \Phi(w_{a,\ell}) := \begin{pmatrix} w_{a,\ell}^{11} & 0 & \cdots & 0 \\ 0 & w_{a,\ell}^{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{a,\ell}^{mm} \end{pmatrix},$$

$$\Delta(b_{a,\ell}) \coloneqq \begin{pmatrix} b^{11}_{a,\ell} & 0 & \cdots & 0 \\ 0 & b^{22}_{a,\ell} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^{mm}_{a,\ell} \end{pmatrix}, \qquad X_b \coloneqq \begin{pmatrix} 1 & x^{12}_b & \cdots & x^{1m}_b \\ 0 & 1 & \cdots & x^{2m}_b \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$Y_e \coloneqq \begin{pmatrix} 0 & y^{12}_e & \cdots & y^{1m}_e \\ 0 & 0 & \cdots & y^{2m}_e \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We also define

$$\Phi(b_{1,1}) \coloneqq \Delta(b_{1,1})X_b \quad \text{and} \quad \Phi(b_{2,1}) \coloneqq X_b^{-1}\Delta(b_{2,1}).$$

The differential $\partial^{(m)}$ can be given as follows:

- for each $c^{ij} \in C^{(m)}$ with upper and lower edges $e, e' \in E$,

$$\partial^{(m)}\left(c^{ij}\right) \coloneqq \Phi(\partial(c))_{ij} + \sum_{k<j}(-1)^{|c|-1}c^{ik}y^{kj}_{e'} + \sum_{i<k} y^{ik}_e c^{kj}$$

- for each $k^{ij}_{ab} \in K^{(m)}_{\mathsf{L}}$ with two edges e and e' containing left borders corresponding a and b,

$$\partial^{(m)}\left(k^{ij}_{ab}\right) \coloneqq \Phi(\partial(k_{ab}))_{ij} + \sum_{k<j}(-1)^{|k_{ab}|-1}k^{ik}_{ab}y^{kj}_{e'} + \sum_{i<k} y^{ik}_e k^{kj}_{ab}$$

- for each $v_{a,\ell} \in \widetilde{V}_{\mathsf{L}}$ with two half-edges $h_{v,a} \subset e \in E$ and $h_{v,a+\ell} \subset e' \in E$,

$$\partial^{(m)}\left(v^{ij}_{a,\ell}\right) \coloneqq \Phi(\partial(v_{a,\ell}))_{ij} + \sum_{i<k<j}(-1)^{|v_{a,\ell}|-1}v^{ik}_{a,\ell}y^{kj}_{e'} + y^{ik}_e v^{kj}_{a,\ell}$$

- for each $b \in B$ with two edges (e, e') adjacent to b,

$$\partial^{(m)}\left(x^{ij}_b\right) \coloneqq \sum_{i<k<j} -x^{ik}_b y^{kj}_{e'} + y^{ik}_e x^{kj}_b$$

- for each $e \in E$, $b_{a,\ell} \in \widetilde{B}$ and $w_{a,\ell} \in \widetilde{V}_{\circlearrowleft}$,

$$\partial^{(m)}\left(b^{ii}_{a,\ell}\right) := \Phi(\partial(b_{a,\ell}))_{ii} = \delta_{\ell,2} + \sum_{\ell_1+\ell_2=\ell} (-1)^{|b_{a,\ell_1}|-1} b^{ii}_{a,\ell_1} b^{ii}_{a+\ell_1,\ell_2},$$
$$\partial^{(m)}\left(w^{ii}_{a,\ell}\right) := \Phi(\partial(w_{a,\ell}))_{ii} = \delta_{\ell,\mathrm{val}(w)} + \sum_{\ell_1+\ell_2=\ell} (-1)^{|w_{a,\ell_1}|-1} w^{ii}_{a,\ell_1} w^{ii}_{a+\ell_1,\ell_2},$$
$$\partial^{(m)}\left(y^{ij}_{e}\right) := (Y_e)^2_{ij} = \sum_{i<k<j} y^{ik}_e y^{kj}_e$$

Now consider the structure morphisms $\phi^{(m)}_{\mathsf{L}} : A^{(m)}_{n_{\mathsf{L}}} \to A^{(m)}$ and $\phi^{(m)}_{\mathsf{R}} : A^{(m)}_{n_{\mathsf{R}}} \to A^{(m)}$. Since $\phi^{(m)}_{\mathsf{L}}$ is obvious, we need to consider $\phi^{(m)}_{\mathsf{R}}$. Then due to the geometric definition for $\phi^{(m)}_{\mathsf{R}}$ which counts immersed polygons with the positive end at each generator for $A^{(m)}_{n_{\mathsf{R}}}$, we have the similar description to the differential $\partial^{(m)}$ so that

$$\phi^{(m)}_{\mathsf{R}}(k^{ij}_{a'b'}) = \Phi(\phi_{\mathsf{R}}(k_{a'b'}))_{ij}, \qquad \phi^{(m)}_{\mathsf{R}}(y^{ij}_{c'}) = y^{ij}_e, \tag{3.9}$$

where $1 \le a' < b' \le n_{\mathsf{R}}$, $c' \in [n_{\mathsf{R}}]$ and e is the edge whose one end point is c'.

Here we are using implicitly the fact that any generator in $\widetilde{V}_{\circlearrowleft}$ never appear in a differential of other types of generators or an image of ϕ_{R} as observed in Example 2.45.

4. Augmentation categories

In this section, we review the construction of the augmentation categories which are A_∞-categories obtained from consistent sequences of DGAs.

4.1. Augmentation categories for consistent sequences of bordered DGAs

For the notational simplicity, let $\mathbb{K} = (\mathbb{K}, \partial_{\mathbb{K}} \equiv 0)$ be a DGA consisting of the base field $\mathbb{K}$ with the trivial differential. An *augmentation* for $A = (\mathsf{A} = T\mathsf{M}, \partial) \in \mathcal{DGA}$ is a DGA morphism

$$\epsilon : A \to \mathbb{K}.$$

One can extend naturally the definition of augmentations to m-component DGAs. Let $\mathbb{K}^{(m)} := \mathrm{Mat}_m(\mathbb{K})$. Then it satisfies the axiom of m-component DGAs.

Definition 4.1 (Augmentations for m-component DGAs). For $A^{(m)} = (\mathsf{A}^{(m)} = T\mathsf{M}^{(m)}, \partial^{(m)}) \in \mathcal{DGA}_{\mathsf{co}}^{(m)}$ with $\mathsf{M}^{(m)} = (\mathsf{M}^{ij})$, an augmentation of $A^{(m)}$ is a morphism

$$\epsilon^{(m)} : A^{(m)} \to \mathbb{K}^{(m)},$$

where

$$\epsilon^{(m)} = \left(\epsilon^{(m)}(i,j)\right), \quad \epsilon^{(m)}(i,j) : A^{(m)}(i,j) \to \mathbb{K}, \quad \epsilon^{(m)}(i,i)\left(\mathbf{1}_E^i\right) = \mathbf{1} \in \mathbb{K}.$$

We denote the restriction of $\epsilon^{(m)}$ to the graded submodule M^{ij} by $\epsilon^{ij} := \epsilon^{(m)}(i,j)\big|_{\mathsf{M}^{ij}} : \mathsf{M}^{ij} \to \mathbb{K}$, and we say that $\epsilon^{(\bullet)}$ is *diagonal* if $\epsilon^{ij} \equiv 0$ for $i \neq j$. Conversely, for a sequence $\mathbf{e} = (\epsilon_1, \dots, \epsilon_m)$ of augmentations of A, the diagonal augmentation $\epsilon_{\mathbf{e}}^{(m)} := (\epsilon_{\mathbf{e}}^{ij})$ of $A^{(m)}$ is defined as

$$\epsilon_{\mathbf{e}}^{ii} : \mathsf{M}^{ii} \xrightarrow{\epsilon_i} \mathbb{K}, \qquad\qquad \epsilon_{\mathbf{e}}^{ij} \equiv 0, \quad \forall i \neq j.$$

The DGA $A_{\mathbf{e}}^{(m)} = (\mathsf{A}_{\mathbf{e}}^{(m)}, \partial_{\mathbf{e}}^{(m)})$ is defined as

$$\mathsf{A}_{\mathbf{e}}^{(m)} := \mathsf{A}^{(m)} \quad \text{and} \quad \partial_{\mathbf{e}}^{(m)} := \phi_{\mathbf{e}}^{(m)} \circ \partial^{(m)} \circ \left(\phi_{\mathbf{e}}^{(m)}\right)^{-1},$$

where $\phi_{\mathbf{e}}^{(m)} : \mathsf{A}^{(m)} \to \mathsf{A}_{\mathbf{e}}^{(m)}$ is an algebra tame isomorphism such that for each generator $s \in R^{(m)}$,

$$\phi_{\mathbf{e}}^{(m)}(s) := s + \epsilon_{\mathbf{e}}^{(m)}(s).$$

Lemma 4.2. *The* 0*-th differential* $\partial_{\mathbf{e},0}^{(m)}$ *of the length filtration* $\partial_{\mathbf{e},\ell}^{(m)}$ *vanishes.*

Proof. The proof follows from the direct computation. □

Let $A^{(\bullet)} \in \mathcal{DGA}_{\mathsf{co}}^{(\bullet)}$ and $\mathbf{e} = (\epsilon_m)$ be a sequence of augmentations of A. Then by Lemma 3.3, it is obvious that for any $i < j \in [m]$, there is an isomorphism between $\left(\mathbb{K}_E^i, \mathbb{K}_E^j\right)$-modules

$$\mathsf{M}^{ij} = \mathsf{A}_{\mathbf{e}}^{ij} \cong \mathsf{A}_{(\epsilon_i,\epsilon_j)}^{12}.$$

Let $\mathsf{M}_{ij}^{\vee} := \left(\mathsf{M}^{ij}\right)^*[-1]$ be the dual space of M^{ij} with grading shift by -1. That is, both M^{ij} and $\mathsf{M}_{ij}^{\vee}$ are decomposed into graded pieces

$$\mathsf{M}^{ij} = \bigoplus_{d\in\mathbb{Z}} \left(\mathsf{M}^{ij}\right)^d, \quad \mathsf{M}_{ij}^{\vee} = \bigoplus_{d\in\mathbb{Z}} \mathsf{M}_{ij}^{\vee d}, \quad \mathsf{M}_{ij}^{\vee d+1} := \mathrm{Hom}_{\mathbb{K}}\left(\left(\mathsf{M}^{ij}\right)^d, \mathbb{K}\right).$$

For each increasing sequence $\mathbf{i} := (i_1, \cdots, i_{k+1})$ in $[m]$ and a sequence of integers $\mathbf{d} = (d_1, \cdots, d_k) \in \mathbb{Z}^k$, let us denote the spaces

$$\begin{aligned}
\mathsf{M}^{\mathbf{i}} &:= \mathsf{M}^{i_1 i_2} \otimes \mathsf{M}^{i_2 i_3} \otimes \cdots \otimes \mathsf{M}^{i_k i_{k+1}}, \\
\mathsf{M}^{\vee}_{\mathbf{i}} &:= \mathsf{M}^{\vee}_{i_k i_{k+1}} \otimes \mathsf{M}^{\vee}_{i_{k-1} i_k} \otimes \cdots \otimes \mathsf{M}^{\vee}_{i_1 i_2}, \\
\left(\mathsf{M}^{\mathbf{i}}\right)^{\mathbf{d}} &:= \left(\mathsf{M}^{i_1 i_2}\right)^{d_1} \otimes \left(\mathsf{M}^{i_2 i_3}\right)^{d_2} \otimes \cdots \otimes \left(\mathsf{M}^{i_k i_{k+1}}\right)^{d_k}, \\
\mathsf{M}^{\vee \mathbf{d}}_{\mathbf{i}} &:= \mathsf{M}^{\vee d_k}_{i_k i_{k+1}} \otimes \mathsf{M}^{\vee d_{k-1}}_{i_{k-1} i_k} \otimes \cdots \otimes \mathsf{M}^{\vee d_1}_{i_1 i_2},
\end{aligned}$$

whose elements will be denoted as

$$\begin{aligned}
a^{\mathbf{i}} &= a^{i_1 i_2} \otimes a^{i_2 i_3} \otimes \cdots \otimes a^{i_k i_{k+1}} \in \mathsf{M}^{\mathbf{i}}, & a^{ij} &\in \mathsf{M}^{ij}, \\
a^{\vee}_{\mathbf{i}} &= a^{\vee}_{i_k i_{k+1}} \otimes a^{\vee}_{i_{k-1} i_k} \otimes \cdots \otimes a^{\vee}_{i_1 i_2} \in \mathsf{M}^{\vee}_{\mathbf{i}}, & a^{\vee}_{ij} &\in \mathsf{M}^{\vee}_{ij}.
\end{aligned}$$

We equip a grading on $\mathsf{M}^{\vee}_{\mathbf{i}} \otimes \mathsf{M}^{\mathbf{i}}$ so that for homogeneous elements $a^{\vee}_{\mathbf{i}} \in \mathsf{M}^{\vee}_{\mathbf{i}}$ and $a'^{\mathbf{i}} \in \mathsf{M}^{\mathbf{i}}$

$$|a^{\vee}_{\mathbf{i}} \otimes a'^{\mathbf{i}}| := |a^{\vee}_{\mathbf{i}}| - |a'^{\mathbf{i}}|.$$

Consider the natural pairing $\langle -, - \rangle_{\mathbf{i}} : \mathsf{M}^{\vee}_{\mathbf{i}} \otimes \mathsf{M}^{\mathbf{i}} \to \mathbb{K}$ between $\mathsf{M}^{\vee}_{\mathbf{i}}$ and $\mathsf{M}^{\mathbf{i}}$ which is nonvanishing only on

$$\bigoplus_{\mathbf{d} \in \mathbb{Z}^k} \left(\mathsf{M}^{\vee \mathbf{d}+\mathbf{1}}_{\mathbf{i}} \otimes \left(\mathsf{M}^{\mathbf{i}}\right)^{\mathbf{d}} \right),$$

where $\mathbf{d} + \mathbf{1} := (d_1 + 1, d_2 + 1, \ldots, d_k + 1)$. Since each evaluation $\langle -, - \rangle_{ij} : \mathsf{M}^{\vee d+1}_{ij} \otimes \left(\mathsf{M}^{ij}\right)^{d} \to \mathbb{K}$ is of degree -1, the pairing $\langle -, - \rangle_{\mathbf{i}}$ is of grading $-k$.

Now let us define a composition map $m_{\mathbf{i}} : \mathsf{M}^{\vee}_{\mathbf{i}} \to \mathsf{M}^{\vee}_{i_1 i_{k+1}}$ as follows: for each $a^{\vee}_{\mathbf{i}} \in \mathsf{M}^{\vee}_{\mathbf{i}}$, we require that $m_{\mathbf{i}}\left(a^{\vee}_{\mathbf{i}}\right)$ satisfies that for each $a^{i_1 i_{k+1}} \in \mathsf{M}^{i_1 i_{k+1}}$,

$$\left\langle m_{\mathbf{i}}\left(a^{\vee}_{\mathbf{i}}\right), a^{i_1 i_{k+1}} \right\rangle_{i_1 i_{k+1}} = (-1)^{\sigma} \left\langle a^{\vee}_{\mathbf{i}}, \partial^{(m)}_{\mathbf{e}}\left(a^{i_1 i_{k+1}}\right) \right\rangle_{\mathbf{i}}, \tag{4.1}$$

where $\partial^{(m)}_{\mathbf{e}}$ is the twisted differential on $A^{(m)}_{\mathbf{e}}$.

Definition 4.3. For any $k \geq 1$, and any sequence $(x_1, x_2, \ldots, x_k)$ with grading $|x_i| \in \mathbb{Z}$, define

$$\sigma_k(x_1, x_2, \ldots, x_k) := \frac{k(k-1)}{2} + \sum_{i<j} |x_i||x_j| + \sum_{1 \leq j \leq k} (j-1)|x_j| \tag{4.2}$$

In particular, $\sigma_1 = 0$ and $\sigma_2(x_1, x_2) = 1 + |x_1||x_2| + |x_2|$.

Then σ in (4.1) is given as

$$\sigma = \sigma_k \left(a^\vee_{i_k i_{k+1}}, \ldots, a^\vee_{i_1 i_2}\right)$$

and $m_{\mathbf{i}}\left(a^\vee_{\mathbf{i}}\right)$ is completely determined by this equation.

The degree of $m_{\mathbf{i}}$ can be computed as follows: since the degrees of pairings on the left and right are -1 and $-k$, respectively, we have

$$\left|m_{\mathbf{i}}\left(a^\vee_{\mathbf{i}}\right) \otimes a^{i_1 i_{k+1}}\right| + \left|\langle -, -\rangle_{i_1 i_{k+1}}\right| = 0 = \left|a^\vee_{\mathbf{i}} \otimes \partial^{(m)}_{\mathbf{e}}\left(a^{i_1 i_{k+1}}\right)\right| + \left|\langle -, -\rangle_{\mathbf{i}}\right|.$$

This is equivalent to

$$\left|m_{\mathbf{i}}\left(a^\vee_{\mathbf{i}}\right)\right| - \left|a^\vee_{\mathbf{i}}\right| = 1 - k + \left|a^{i_1 i_{k+1}}\right| - \left|\partial^{(m)}_{\mathbf{e}}\left(a^{i_1 i_{k+1}}\right)\right| = 2 - k,$$

which implies that the composition map $m_{\mathbf{i}}$ is of degree $2-k$.

Definition 4.4. The *augmentation category* $\mathcal{A}\mathsf{ug}_+\left(A^{(\bullet)}; \mathbb{K}\right)$ is an A_∞-category defined as follows:

- The objects are augmentations $\epsilon : A = A^{11} \to \mathbb{K}$;
- The morphisms are graded vector spaces

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) := \left(A^{12}_{(\epsilon_1,\epsilon_2)}\right)^\vee \cong \mathbb{K}\left\langle \mathsf{S}^{12}\right\rangle^\vee;$$

- For $k \geq 1$, the composition map

$$m_k : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_k, \epsilon_{k+1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_{k+1})$$

  is defined as $m_{\mathbf{i}}$ with the sequence $\mathbf{i} = (1, 2, \cdots, k+1)$.

Proposition 4.5 (Functoriality of $\mathcal{A}\mathsf{ug}_+$[21]). *The assignment $A^{(\bullet)} \mapsto \mathcal{A}\mathsf{ug}_+\left(A^{(\bullet)}; \mathbb{K}\right)$ defines a contravariant functor from the category of consistent sequences of DGAs onto the category $\mathcal{A}\mathsf{lg}_\infty$ of A_∞-categories.*

Proof. This is just a combination of Propositions 3.17 and 3.20 in [21] and so we omit the proof. ☐

We briefly review the construction of the A_∞-functor

$$\mathcal{F}^{(\bullet)} = \left(\mathcal{F}^{(k)}\right)_{k\geq 1} : \mathcal{A}\mathsf{ug}_+\left(A^{(\bullet)};\mathbb{K}\right) \to \mathcal{A}\mathsf{ug}_+\left(A'^{(\bullet)};\mathbb{K}\right)$$

induced from the consistent morphism $f^{(\bullet)} : A'^{(\bullet)} \to A^{(\bullet)}$

$$\begin{aligned}
\mathcal{F}^{(0)} &: \mathcal{O}\mathsf{b}\left(\mathcal{A}\mathsf{ug}_+\right) \to \mathcal{O}\mathsf{b}\left(\mathcal{A}\mathsf{ug}'_+\right),\\
\mathcal{F}^{(k)} &: \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\epsilon_k, \epsilon_{k+1}\right) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\epsilon_1, \epsilon_2\right)\\
&\quad \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}'_+}\left(\mathcal{F}^{(0)}\left(\epsilon_1\right), \mathcal{F}^{(0)}\left(\epsilon_{k+1}\right)\right),
\end{aligned}$$

where $\mathbf{e} = (\epsilon_1, \cdots, \epsilon_{k+1})$ is a sequence of augmentations for A, and

$$\mathcal{A}\mathsf{ug}_+ \coloneqq \mathcal{A}\mathsf{ug}_+\left(A^{(\bullet)};\mathbb{K}\right), \qquad \mathcal{A}\mathsf{ug}'_+ \coloneqq \mathcal{A}\mathsf{ug}_+\left(A'^{(\bullet)};\mathbb{K}\right).$$

Let $\mathcal{F}^{(0)}\left(\epsilon_i\right) \coloneqq \epsilon_i^f$ be the pull-back for each i by ϵ_i, which is an augmentation for A.

$$\mathbf{e}^f \coloneqq \left(\epsilon_1^f, \cdots, \epsilon_{k+1}^f\right), \qquad \epsilon_i^f(a) \coloneqq \epsilon_i(f(a)).$$

Recall the DGAs

$$A_{\mathbf{e}}^{(k+1)} = \left(\mathsf{A}_{\mathbf{e}}^{(k+1)}, \partial_{\mathbf{e}}^{(k+1)}\right) \quad \text{and} \quad A_{\mathbf{e}}'^{(k+1)} = \left(\mathsf{A}_{\mathbf{e}^f}'^{(k+1)}, \partial_{\mathbf{e}^f}'^{(k+1)}\right)$$

twisted by the algebra automorphisms $\phi_{\mathbf{e}}^{(k+1)}$ and $\phi_{\mathbf{e}^f}^{(k+1)}$ on $A^{(k+1)}$ and $A'^{(k+1)}$, respectively. Then the composition

$$f_{\mathbf{e}}^{(k+1)} \coloneqq \phi_{\mathbf{e}}^{(k+1)} \circ f^{(k+1)} \circ \left(\phi_{\mathbf{e}^f}^{(k+1)}\right)^{-1} : A_{\mathbf{e}^f}'^{(k+1)} \to A_{\mathbf{e}}^{(k+1)}$$

becomes a DGA morphism since

$$\begin{aligned}
f_{\mathbf{e}}^{(k+1)} \circ \partial_{\mathbf{e}^f}'^{(k+1)} &= \left(\phi_{\mathbf{e}}^{(k+1)} \circ f^{(k+1)} \circ \left(\phi_{\mathbf{e}^f}^{(k+1)}\right)^{-1}\right) \circ \partial_{\mathbf{e}^f}'^{(k+1)}\\
&= \phi_{\mathbf{e}}^{(k+1)} \circ f^{(k+1)} \circ \partial'^{(k+1)} \circ \left(\phi_{\mathbf{e}^f}^{(k+1)}\right)^{-1}\\
&= \phi_{\mathbf{e}}^{(k+1)} \circ \partial^{(k+1)} \circ f^{(k+1)} \circ \left(\phi_{\mathbf{e}^f}^{(k+1)}\right)^{-1}\\
&= \partial_{\mathbf{e}}^{(k+1)} \circ \left(\phi_{\mathbf{e}}^{(k+1)} \circ f^{(k+1)} \circ \left(\phi_{\mathbf{e}^f}^{(k+1)}\right)^{-1}\right)\\
&= \partial_{\mathbf{e}}^{(k+1)} \circ f_{\mathbf{e}}^{(k+1)}.
\end{aligned}$$

Notation 4.6. The ℓ-th length filtration of $f^{(k+1)}_{\mathbf{e}}$ will be denoted by $f^{(k+1)}_{\mathbf{e},\ell}$.

The A_∞-functor $\mathcal{F}^{(\bullet)}$ will be defined by dualizing the composition $f^{(k+1)}_{\mathbf{e}}$. More precisely,

$$\mathcal{F}^{(k)} : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_k, \epsilon_{k+1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\epsilon^f_1, \epsilon^f_{k+1}\right)$$

is defined as follows: for each $a^\vee_{\mathbf{i}} := a^\vee_{k,k+1} \otimes \cdots \otimes a^\vee_{1,2}$ with $a^\vee_{i,i+1} \in \left(A^{12}_{(\epsilon_i,\epsilon_{i+1})}\right)^\vee$,

$$\left\langle \mathcal{F}^{(k)}\left(a^\vee_{\mathbf{i}}\right), a'^{1,k+1} \right\rangle_{1,k+1} = (-1)^\sigma \left\langle a^\vee_{\mathbf{i}}, f^{(k+1)}_{\mathbf{e}}\left(a'^{1,k+1}\right)\right\rangle'_{\mathbf{i}},$$

where σ is the same as (4.2) in Definition 4.3.

Example 4.7 (Augmentation category for border DGAs). Let $\mu : [n] \to \mathbb{Z}$ be a function and $A^{(\bullet)}_n(\mu)$ be the consistent sequence of border DGAs defined in Section 3.2. For simplicity, we denote $\mathcal{A}\mathsf{ug}_+(A^{(\bullet)}_n(\mu); \mathbb{K})$ by $\mathcal{A}\mathsf{ug}_+$.

For each $m \geq 1$, the algebra $\mathsf{A}^{(\bullet)}_n$ has the generating sets

$$K_n := \left\{ k^{ij}_{ab} \;\middle|\; a < b, 1 \leq i, j \leq m \right\}$$
$$\text{and} \quad Y_n := \left\{ y^{ij}_a \;\middle|\; 1 \leq a \leq n, 1 \leq i < j \leq m \right\},$$

where the grading is given as

$$|k^{ij}_{ab}| := \mu(a) - \mu(b) - 1, \qquad |y^{ij}_a| := -1.$$

The differential for each generator is given as follows:

$$\partial^{(m)}_n(k^{ij}_{ab}) := \sum_{\substack{a<c<b\\1\leq\ell\leq m}} (-1)^{|k_{ac}|-1} k^{i\ell}_{ac} k^{\ell j}_{cb} + \sum_{\ell<j} (-1)^{|k_{ab}|-1} k^{i\ell}_{ab} y^{\ell j}_b + \sum_{i<\ell} y^{i\ell}_a k^{\ell j}_{ab},$$
$$\partial^{(m)}_n(y^{ij}_a) := \sum_{i<\ell<j} y^{i\ell}_a y^{\ell j}_a.$$

The set of objects of $\mathcal{A}\mathsf{ug}_+$ is the augmentation variety for $A_n(\mu)$:

$$\mathcal{O}\mathsf{b}\left(\mathcal{A}\mathsf{ug}_+\right) = \mathsf{Aug}(A_n(\mu); \mathbb{K}) := \{\epsilon : A_n(\mu) \to \mathbb{K}\}, \tag{4.3}$$

and for any $\epsilon_1, \epsilon_2 \in \mathcal{A}\mathsf{ug}_+$, the set $\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2)$ of morphisms is

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) = \mathsf{M}^{12\vee} = \mathbb{K}\left\langle k_{ab}^{12\vee}, y_c^{12\vee} \;\middle|\; 1 \le a < b \le n, c \in [n]\right\rangle. \tag{4.4}$$

The composition map m_k is given as follows:

- For $\epsilon_1, \epsilon_2 \in \mathcal{A}\mathsf{ug}_+$, the map

$$m_1 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2)$$

  is defined as

$$\begin{aligned} m_1\left(k_{ab}^{12\vee}\right) &= -\sum_{c<a} \epsilon_1(k_{ca}) k_{cb}^{12\vee} + \sum_{b<d} (-1)^{|k_{ab}^{12\vee}|} k_{ad}^{12\vee} \epsilon_2(k_{bd}), \\ m_1\left(y_c^{12\vee}\right) &= -\sum_{a<c} \epsilon_1(k_{ac}) k_{ac}^{12\vee} + \sum_{c<d} k_{cd}^{12\vee} \epsilon_2(k_{cd}). \end{aligned}$$

- For $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathcal{A}\mathsf{ug}_+$, the map

$$m_2 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_2, \epsilon_3) \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_3)$$

  is defined as

$$\begin{aligned} m_2(k_{cd}^{12\vee} \otimes k_{ab}^{12\vee}) &= \delta_{bc}(-1)^{|k_{ab}||k_{cd}| + |k_{ab}| + |k_{cd}|} k_{ad}^{12\vee}, \\ m_2(y_c^{12\vee} \otimes k_{ab}^{12\vee}) &= -\delta_{bc} k_{ab}^{12\vee}, \\ m_2(k_{cd}^{12\vee} \otimes y_a^{12\vee}) &= -\delta_{ac} k_{cd}^{12\vee}, \\ m_2(y_c^{12\vee} \otimes y_a^{12\vee}) &= -\delta_{ac} y_a^{12\vee}. \end{aligned}$$

- For m_k with $k \ge 3$, the higher composition m_k vanishes.

In particular, the A_∞-category $\mathcal{A}\mathsf{ug}_+(A_n^{(m)}(\mu); \mathbb{K})$ is in fact a DG category.

Notice that for each $\epsilon \in \mathcal{A}\mathsf{ug}_+(A_n^{(\bullet)}(\mu); \mathbb{K})$, the element defined as

$$-y := -\sum_{c \in [n]} y_c^{12\vee} \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon, \epsilon) \tag{4.5}$$

becomes a cocycle since

$$m_1(-y) = \sum_{c \in [n]} \left(\sum_{a<c} \epsilon(k_{ac}) k_{ac}^{12\vee} - \sum_{c<d} k_{cd}^{12\vee} \epsilon(k_{cd}) \right) = 0.$$

Corollary 4.8. *For each* $\epsilon \in \mathcal{A}\mathsf{ug}_+(A_n^{(\bullet)}(\mu);\mathbb{K})$*, the element* $-y \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon,\epsilon)$ *defined in* (4.5) *becomes the unit of the augmentation category* $\mathcal{A}\mathsf{ug}_+(A_n^{(\bullet)}(\mu);\mathbb{K})$*.*

Proof. This is a direct consequence of the above computation. □

Proposition 4.9. *The augmentation category* $\mathcal{A}\mathsf{ug}_+\left(A'^{(\bullet)};\mathbb{K}\right)$ *of a stabilization* $A'^{(\bullet)}$ *of* $A^{(\bullet)}$ *is* A_∞*-quasi-equivalent to the augmentation category* $\mathcal{A}\mathsf{ug}_+\left(A^{(\bullet)};\mathbb{K}\right)$ *of* $A^{(\bullet)}$*.*

Proof. We first identify $A'^{(m)}$ with a stabilization of $A^{(m)}$ for each $m \geq 1$ and denote generators for stabilizations by e's and $\widehat{e}$'s. For simplicity, we will denote augmentation categories of $A^{(\bullet)}$ and $A'^{(\bullet)}$ by $\mathcal{A}\mathsf{ug}_+$ and $\mathcal{A}\mathsf{ug}'_+$, respectively.

Let us consider two consistent morphisms $\iota^{(\bullet)}$ and $\pi^{(\bullet)}$

$$\iota^{(\bullet)} : A^{(\bullet)} \to A'^{(\bullet)}, \qquad \pi^{(\bullet)} : A'^{(\bullet)} \to A^{(\bullet)},$$

where $\iota^{(m)}$ is the canonical inclusion of $A^{(m)} \to A'^{(m)}$ and $\pi^{(m)}$ sends each e and $\widehat{e}$ in $A'^{(m)}$ to zero. Then it is obvious that they are homotopy-inverses to each other so that

$$\pi^{(\bullet)} \circ \iota^{(\bullet)} = \mathrm{Id}_A^{(\bullet)}, \qquad \iota^{(\bullet)} \circ \pi^{(\bullet)} \overset{H^{(\bullet)}}{\simeq} \mathrm{Id}_{A'}^{(\bullet)},$$

where the sequence $H^{(\bullet)}$ of homotopies is given by

$$H^{(m)}(s) := \begin{cases} s & s \in A \subset A'; \\ \widehat{e} & s = e \in A'; \\ 0 & s = \widehat{e} \in A'. \end{cases} \tag{4.6}$$

It is well-known that an A_∞-functor is an A_∞-(quasi-)equivalence if it satisfies two condition, (i) essentially (quasi-)surjective, and (ii) (quasi-)fully faithful. In other words, we need to show the following: let $\mathcal{I}^{(\bullet)}$ be the A_∞-functor induced from $\iota^{(\bullet)}$.

- for each $\epsilon \in \mathcal{A}\mathsf{ug}_+$, there exists $\epsilon' \in \mathcal{A}\mathsf{ug}'_+$ such that $\mathcal{I}^{(0)}(\epsilon') := \iota^*\epsilon'$ and ϵ are isomorphic up to homotopy,

- for $\epsilon_1', \epsilon_2' \in \mathcal{A}\mathsf{ug}_+'$, the induced chain map

$$\mathcal{I}^{(1)} : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+'}(\epsilon_1', \epsilon_2') \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\mathcal{I}^{(0)}(\epsilon_1'), \mathcal{I}^{(0)}(\epsilon_2')\right)$$

  is a quasi-isomorphism, i.e., an isomorphism between their cohomology groups.

The essential quasi-surjectivity is obvious. Indeed, for any $\epsilon \in \mathcal{A}\mathsf{ug}_+$, there is an augmentation $\epsilon' \in \mathcal{A}\mathsf{ug}_+$ extended from ϵ as

$$\epsilon'(s) := \begin{cases} \epsilon(s) & s \in A \subset A'; \\ 0 & \text{otherwise,} \end{cases}$$

such that the pull-back of ϵ' is precisely ϵ

$$\mathcal{I}^{(0)}(\epsilon') = \epsilon' \circ \iota = \epsilon,$$

and therefore it is indeed surjective.

Let $\epsilon_1', \epsilon_2' \in \mathcal{A}\mathsf{ug}_+'$ be two augmentations. Then by the identification of A', we have

(4.7)

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+'}(\epsilon_1', \epsilon_2') \cong \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\mathcal{I}^{(0)}(\epsilon_1'), \mathcal{I}^{(0)}(\epsilon_2')\right) \oplus \left(\bigoplus_{i \in I^{12}} \mathbb{K}_{\mathsf{A}}\left\langle e_{12}^{i\vee}, \widehat{e}_{12}^{i\vee}\right\rangle, m_1'\right),$$

for some index set I^{12}, where

$$m_1'(e_{12}^{i\vee}) = \widehat{e}_{12}^{i\vee} \quad \text{and} \quad m_1'(\widehat{e}_{12}^{i\vee}) = 0. \tag{4.8}$$

Then the induced chain map $\mathcal{I}^{(1)}$ is the projection onto

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}\left(\mathcal{I}^{(0)}(\epsilon_1'), \mathcal{I}^{(0)}(\epsilon_2')\right),$$

which is surjective with the kernel

$$\ker\left(\mathcal{I}^{(1)}\right) \cong \left(\bigoplus_{i \in I^{12}} \mathbb{K}_{\mathsf{A}}\left\langle e_{12}^{i\vee}, \widehat{e}_{12}^{i\vee}\right\rangle, m_1'\right).$$

Since the kernel is acyclic as seen in (4.8), $\mathcal{I}^{(1)}$ is a quasi-isomorphism, which implies the quasi-fully faithfulness, and we are done. □

Corollary 4.10. *If $\mathcal{A}\mathsf{ug}_+'$ is homologically unital, then so is $\mathcal{A}\mathsf{ug}_+$.*

Proof. Since the induced A_∞ functor $\mathcal{I}^{(\bullet)}$ is surjective, $\mathcal{A}\mathsf{ug}_+$ is homologically unital if so is $\mathcal{A}\mathsf{ug}'_+$. □

Remark 4.11. One may expect that the A_∞ functor $\Pi^{(\bullet)} : \mathcal{A}\mathsf{ug}_+ \to \mathcal{A}\mathsf{ug}'_+$ induced from $\pi^{(\bullet)}$ is again an A_∞-quasi-equivalence but the essential quasi-surjectivity is not obvious unless $\mathcal{A}\mathsf{ug}_+$ is unital.

Corollary 4.12. *If $\mathcal{A}\mathsf{ug}_+$ is homologically unital, $\Pi^{(\bullet)}$ is an A_∞-quasi-equivalence and so $\mathcal{A}\mathsf{ug}'_+$ is homologically unital.*

Proof. Let us consider the essential quasi-surjectivity first. For each augmentation $\epsilon' \in \mathcal{A}\mathsf{ug}'_+$, one can find an augmentation $\epsilon \in \mathcal{A}\mathsf{ug}_+$ such that ϵ' and ϵ^π agree with each other on A but may have different values for some $\widehat{e}$.

Similar to (4.7), the chain complex $\mathrm{Hom}_{\mathcal{A}\mathsf{ug}'_+}(\epsilon', \epsilon^\pi)$ is obviously a direct sum

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}'_+}\left(\epsilon', \epsilon^\pi\right) \cong \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon, \epsilon) \oplus \left(\bigoplus_{i \in I^{12}} \mathbb{K}_{\mathsf{A}} \left\langle e_{12}^{i\vee}, \widehat{e}_{12}^{i\vee} \right\rangle, m_1' \right).$$

As before, the second summand is acyclic and therefore the existence of the unit in $\mathcal{A}\mathsf{ug}_+$ implies the existence of an isomorphism in $\mathrm{Hom}_{\mathcal{A}\mathsf{ug}'_+}(\epsilon', \epsilon^\pi)$.

For two augmentations $\epsilon_1, \epsilon_2 \in \mathcal{A}\mathsf{ug}_+$, the induced chain map $\Pi^{(1)}$ sends all $a_{12}^\vee$ to themselves

$$\Pi^{(1)} : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}'_+}\left(\epsilon_1^\pi, \epsilon_2^\pi\right), \qquad \Pi^{(1)}\left(a_{12}^\vee\right) = a_{12}^\vee.$$

Therefore it is injective and its cokernel $\mathrm{Coker}\,\Pi^{(1)}$ is isomorphic to the chain complex

$$\mathrm{Coker}\left(\Pi^{(1)}\right) \cong \left(\bigoplus_{i \in I^{12}} \mathbb{K}_{\mathsf{A}} \left\langle e_{12}^{i\vee}, \widehat{e}_{12}^{i\vee} \right\rangle, m_1' \right)$$

which is acyclic and so it is quasi-fully faithful as desired. □

In summary, we have the following proposition:

Proposition 4.13. *Suppose that there is a zig-zag of stabilizations*

$$A_0^{(\bullet)} \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} A_1^{(\bullet)} \underset{\pi_1'}{\overset{\iota_1'}{\leftrightarrows}} \cdots \underset{\pi_{n-1}'}{\overset{\iota_{n-1}'}{\leftrightarrows}} A_{n-1}^{(\bullet)} \underset{\pi_{n-1}}{\overset{\iota_{n-1}}{\leftrightarrows}} A_n^{(\bullet)}$$

and $\mathcal{A}\mathsf{ug}_+\left(A_0^{(\bullet)}; \mathbb{K}\right)$ is homologically unital. Then $\mathcal{A}\mathsf{ug}_+\left(A_i^{(\bullet)}; \mathbb{K}\right)$ for each i is homologically unital and A_∞-quasi-equivalent to $\mathcal{A}\mathsf{ug}_+\left(A_0^{(\bullet)}; \mathbb{K}\right)$.

Now we consider the bordered version of augmentation categories.

Definition 4.14 (Augmentations for bordered DGAs). An augmentation for $\mathcal{A} = (A_{\mathsf{L}} \to A \leftarrow A_{\mathsf{R}})$ is a bordered DGA morphism

$$\varepsilon = (\epsilon_{\mathsf{L}}, \epsilon, \epsilon_{\mathsf{R}}) : \mathcal{A} \to \mathbb{K} := (\mathbb{K} = \mathbb{K} = \mathbb{K}),$$

which makes the following diagram commutative:

$$\begin{array}{c} \mathcal{A} \\ {\scriptstyle\varepsilon}\downarrow \\ \mathbb{K} \end{array} = \left(\begin{array}{ccccc} A_{\mathsf{L}} & \longrightarrow & A & \longleftarrow & A_{\mathsf{R}} \\ {\scriptstyle\epsilon_{\mathsf{L}}}\downarrow & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon_{\mathsf{R}}} \\ \mathbb{K} & = & \mathbb{K} & = & \mathbb{K} \end{array} \right)$$

Definition 4.15. A *bordered* A_∞-category $(\mathcal{A}_{\mathsf{L}} \leftarrow \mathcal{A} \to \mathcal{A}_{\mathsf{R}})$ of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$ consists of A_∞-categories, $\mathcal{A}_{\mathsf{L}}, \mathcal{A}$ and $\mathcal{A}_{\mathsf{R}}$ and two A_∞-functors $\mathcal{A} \to \mathcal{A}_{\mathsf{L}}$ and $\mathcal{A} \to \mathcal{A}_{\mathsf{R}}$ such that both $\mathcal{A}_{\mathsf{L}}$ and $\mathcal{A}_{\mathsf{R}}$ are A_∞-equivalent to augmentation categories

$$\mathcal{A}_{\mathsf{L}} \cong \mathcal{A}\mathsf{ug}(A^{(\bullet)}_{n_{\mathsf{L}}}(\mu_{\mathsf{L}}); \mathbb{K}) \quad \text{and} \quad \mathcal{A}_{\mathsf{R}} \cong \mathcal{A}\mathsf{ug}(A^{(\bullet)}_{n_{\mathsf{R}}}(\mu_{\mathsf{R}}); \mathbb{K})$$

for some $\mu_{\mathsf{L}} : [n_{\mathsf{L}}] \to \mathbb{Z}$ and $\mu_{\mathsf{R}} : [n_{\mathsf{R}}] \to \mathbb{Z}$, respectively.

The morphism $\mathcal{F}^{(\bullet)}$ between two bordered A_∞-categories is a triple $\left(\mathcal{F}^{(\bullet)}_{\mathsf{L}}, \mathcal{F}^{(\bullet)}, \mathcal{F}^{(\bullet)}_{\mathsf{R}}\right)$ of A_∞-functors making the following diagram commutative:

$$\mathcal{F}^{(\bullet)} = \left(\begin{array}{ccccc} \mathcal{A}_{\mathsf{L}} & \longleftarrow & \mathcal{A} & \longrightarrow & \mathcal{A}_{\mathsf{R}} \\ {\scriptstyle\mathcal{F}^{(\bullet)}_{\mathsf{L}}}\downarrow & & \downarrow{\scriptstyle\mathcal{F}^{(\bullet)}} & & \downarrow{\scriptstyle\mathcal{F}^{(\bullet)}_{\mathsf{R}}} \\ \mathcal{A}'_{\mathsf{L}} & \longleftarrow & \mathcal{A}' & \longrightarrow & \mathcal{A}'_{\mathsf{R}} \end{array} \right).$$

We say that $\mathcal{F}^{(\bullet)}$ is an A_∞-(quasi)-equivalence if so are $\mathcal{F}^{(\bullet)}_{\mathsf{L}}$, $\mathcal{F}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}_{\mathsf{R}}$.

We denote the category of bordered A_∞-categories by $\mathcal{B}\mathcal{A}\mathsf{lg}_\infty$.

Indeed, for each consistent sequence $\mathcal{A}^{(\bullet)} = \left(A^{(\bullet)}_{\mathsf{L}} \xrightarrow{\phi^{(\bullet)}_{\mathsf{L}}} A^{(\bullet)} \xleftarrow{\phi^{(\bullet)}_{\mathsf{R}}} A^{(\bullet)}_{\mathsf{R}} \right)$ of bordered DGAs, we have an associated bordered augmentation A_∞-category

$$\mathcal{A}\mathsf{ug}_+(\mathcal{A}^{(\bullet)}; \mathbb{K}) := \left(\mathcal{A}\mathsf{ug}_+(A^{(\bullet)}_{\mathsf{L}}; \mathbb{K}) \xleftarrow{\mathcal{A}\mathsf{ug}_+(\phi^{(\bullet)}_{\mathsf{L}})} \mathcal{A}\mathsf{ug}_+(A^{(\bullet)}; \mathbb{K}) \xrightarrow{\mathcal{A}\mathsf{ug}_+(\phi^{(\bullet)}_{\mathsf{R}})} \mathcal{A}\mathsf{ug}_+(A^{(\bullet)}_{\mathsf{R}}; \mathbb{K}) \right).$$

Corollary 4.16. *The contravariant functor* $\mathcal{A}\mathsf{ug}_+(-; \mathbb{K}) : \mathcal{BDGA}^{(\bullet)}_{\mathsf{co}} \to \mathcal{B}\mathcal{A}\mathsf{lg}_\infty$ *is well-defined.*

Proof. This is a corollary of Proposition 4.5. □

Proposition 4.17. *Let $\mathcal{A}'^{(\bullet)}$ be a stabilization of $\mathcal{A}^{(\bullet)}$. Then two bordered augmentation categories $\mathcal{A}\mathsf{ug}_+(\mathcal{A}'^{(\bullet)};\mathbb{K})$ and $\mathcal{A}\mathsf{ug}_+(\mathcal{A}^{(\bullet)};\mathbb{K})$ are A_∞-quasi-equivalent.*

Proof. Let $\Pi^{(\bullet)} := \mathcal{A}\mathsf{ug}_+(\boldsymbol{\pi}^{(\bullet)}) : \mathcal{A}\mathsf{ug}_+(\mathcal{A}^{(\bullet)};\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(\mathcal{A}'^{(\bullet)};\mathbb{K})$ be the induced A_∞ functor from the canonical projection $\boldsymbol{\pi} : \mathcal{A}'^{(\bullet)} \to \mathcal{A}^{(\bullet)}$. Since $\mathcal{A}'^{(\bullet)}$ is a stabilization, two induced functors $\mathcal{A}\mathsf{ug}_+(\pi_\mathsf{L})$ and $\mathcal{A}\mathsf{ug}_+(\pi_\mathsf{R})$ on borders are equivalences. Hence it suffices to prove the A_∞-quasi-equivalence for $\mathcal{A}\mathsf{ug}_+(\pi^{(\bullet)})$.

Due to Proposition 4.13, it is obvious that the A_∞ functor $\mathcal{I}^{(\bullet)} : \mathcal{A}\mathsf{ug}_+(A'^{(\bullet)};\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(A^{(\bullet)};\mathbb{K})$ induced from $\iota^{(\bullet)} : A^{(\bullet)} \to A'^{(\bullet)}$ is an A_∞-quasi-equivalence whose quasi-inverse is precisely $\mathcal{A}\mathsf{ug}_+(\pi^{(\bullet)}) : \mathcal{A}\mathsf{ug}_+(A^{(\bullet)};\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(A'^{(\bullet)};\mathbb{K})$ and is also an A_∞-quasi-equivalence as desired. □

4.2. Augmentation categories for bordered Legendrian graphs

Let $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BLG}^{(\bullet)}$ be a consistent sequence of bordered Legendrian graphs. Then by taking $A^{\mathsf{CE}}_{\mathsf{co}}$, we have a consistent sequence of DGAs $A^{\mathsf{CE}}_{\mathsf{co}}\left(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}\right)$ as seen in Theorem 3.32.

Definition 4.18 (Augmentation categories for consistent sequences of bordered Legendrian graphs). Let $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) \in \mathcal{BDGA}^{\mu,(\bullet)}_{\mathsf{co}}$. The *bordered augmentation category* for $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ is the composition

$$\begin{aligned}\mathcal{A}\mathsf{ug}_+\left(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)};\mathbb{K}\right) &= \big(\mathcal{A}\mathsf{ug}_+\left(T^{(\bullet)}_\mathsf{L}, \mu^{(\bullet)}_\mathsf{L};\mathbb{K}\right) \leftarrow \mathcal{A}\mathsf{ug}_+\left(T^{(\bullet)}, \mu^{(\bullet)};\mathbb{K}\right)\\ &\qquad \to \mathcal{A}\mathsf{ug}_+\left(T^{(\bullet)}_\mathsf{R}, \mu^{(\bullet)}_\mathsf{R};\mathbb{K}\right)\big)\\ &:= \mathcal{A}\mathsf{ug}_+(A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)});\mathbb{K}),\end{aligned}$$

where for $* = \mathsf{L}, \mathsf{R}$ or empty,

$$\mathcal{A}\mathsf{ug}_+\left(T^{(\bullet)}_*, \mu^{(\bullet)}_*;\mathbb{K}\right) := \mathcal{A}\mathsf{ug}_+(A^{\mathsf{CE}}_{\mathsf{co}}(T^{(\bullet)}_*, \mu^{(\bullet)}_*);\mathbb{K}).$$

In particular, if $\mathcal{T}^{(\bullet)}$ is the consistent sequence of canonical front copies of $\mathcal{T}$, then we denote simply by

$$\mathcal{A}\mathsf{ug}_+\left(\mathcal{T}, \boldsymbol{\mu};\mathbb{K}\right) := \left(\mathcal{A}\mathsf{ug}_+\left(T_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}\right) \leftarrow \mathcal{A}\mathsf{ug}_+\left(T, \mu;\mathbb{K}\right) \to \mathcal{A}\mathsf{ug}_+\left(T_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}\right)\right).$$

For an example of computing the augmentation category, see Section 6.4.

Theorem 4.19 (Invariance theorem). *The assignment* $(\mathcal{T}, \boldsymbol{\mu}) \to \mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$ *is well-defined and invariant under Legendrian isotopy and basepoint moves up to zig-zags of* A_∞*-quasi-equivalences.*

Proof. The well-definedness is obvious. Indeed, the canonical front copy is well-defined and every morphism is a zig-zag of elementary morphisms by Proposition 3.13, it induces a well-defined consistent sequence of bordered LCH DGAs and elementary morphisms correspond to stabilizations by Theorem 3.32.

Now as always, we pass through the bordered augmentation categories of cofibrant replacements. Then we have a zig-zag of stabilizations of consistent bordered DGAs

$$\mathcal{A}' \xleftarrow{\widehat{\boldsymbol{\pi}}'^{(\bullet)}} \widehat{\mathcal{A}}' = \mathcal{A}_0^{(\bullet)} \underset{\boldsymbol{\pi}_1^{(\bullet)}}{\overset{\mathbf{i}_1^{(\bullet)}}{\rightleftarrows}} \cdots \underset{\boldsymbol{\pi}_{n-1}'^{(\bullet)}}{\overset{\mathbf{i}_{n-1}'^{(\bullet)}}{\leftrightarrows}} \mathcal{A}_n^{(\bullet)} = \widehat{\mathcal{A}} \xrightarrow{\widehat{\boldsymbol{\pi}}'^{(\bullet)}} \mathcal{A},$$

where $\mathcal{A}' = A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)})$ and $\mathcal{A} = A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ and $\widehat{\mathcal{A}}'$ and $\widehat{\mathcal{A}}$ are cofibrant replacements, respectively.

Due to Corollary 4.16, the bordered augmentation categories for these zig-zags are well-defined and by Proposition 4.17, every stabilization gives us an A_∞-quasi-equivalences.

We assume that $(\mathcal{T}', \boldsymbol{\mu}')$ and $(\mathcal{T}, \boldsymbol{\mu})$ are related by a basepoint move $(\mathsf{B_i})$. For $(\mathsf{B_1})$, we are done since $(\mathsf{B_1})^{(\bullet)}$ induces a zig-zag of stabilizations as seen already in Section 3.2.2.

For $(\mathsf{B_i})$ with $i = 2$ or 3, the induced consistent basepoint move on the canonical front copies is a sequence of an elementary basepoint move and a Reidemeister move as seen in Figure 11. Now suppose that two consistent sequences $(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)})$ and $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ are related with an elementary consistent basepoint change move $(\mathsf{B_i})^{(\bullet)}$.

Due to the discussion after Theorem 3.32, we have the induced consistent morphism $(\mathsf{B_i})_*^{(\bullet)}$ and its left inverse $(\mathsf{B_i}^{-1})_*^{(\bullet)}$. Therefore we have a surjective A_∞-functor

$$\mathcal{A}\mathsf{ug}_+((\mathsf{B_i})^{(\bullet)}) : \mathcal{A}\mathsf{ug}_+(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}; \mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)}; \mathbb{K}).$$

For any pair of augmentation $\epsilon_1, \epsilon_2 \in \mathcal{A}\mathsf{ug}_+(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}; \mathbb{K})$, we have a chain map

$$\mathcal{A}\mathsf{ug}_+((\mathsf{B_i})^{(2)}) : \mathrm{Hom}_+(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_+(\epsilon_1', \epsilon_2'),$$

where $\epsilon'_j \in \mathcal{A}\mathsf{ug}_+(\mathcal{T}'^{(\bullet)}, \boldsymbol{\mu}'^{(\bullet)}; \mathbb{K})$ is an induced augmentation via $(\mathsf{B_i})_*$. Notice that the basepoint move $(\mathsf{B_i})$ for $i = 2$ or 3 does not alter the crossings, especially crossings between the 1st and 2nd copies of $\mathcal{T}'^{(2)}$. Therefore the above chain map is an isomorphism between graded vector spaces which implies the fully faithfulness of $\mathcal{A}\mathsf{ug}_+((\mathsf{B_i})^{(\bullet)})$ and we are done. □

The following theorem is a generalization of Proposition 3.25 in [21].

Theorem 4.20. *Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}^\mu$ be a bordered Legendrian graph in a normal form. Then its augmentation category $\mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$ is strictly unital. That is, the A_∞ category $\mathcal{A}\mathsf{ug}_+(T_*, \mu_*; \mathbb{K})$ for each $* = \mathsf{L}, \mathsf{R}$ and empty is strictly unital and two A_∞-functors $\mathcal{A}\mathsf{ug}_+(\phi_{\mathsf{L}}^{(\bullet)}), \mathcal{A}\mathsf{ug}_+(\phi_{\mathsf{R}}^{(\bullet)})$ are unit-preserving.*

Proof. Let $\epsilon \in \mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$. We need to show that there exists an element in $\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon, \epsilon)$ which plays the role of the identity.

Recall the generating sets for $A^{\mathsf{CE}}_{\mathsf{co}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ described in Section 3.2.3. We define an element $-y_{12}^\vee \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon, \epsilon)$ as follows:

$$-y_{12}^\vee := \sum_{e \in \overline{E}} -y_{12}^{e\vee} \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon, \epsilon),$$

where $y_{12}^{e\vee}$ is the dual of y_e^{12}.

Due to the formula of the differential $\partial^{(m)}$ given in Section 3.2.3, the only possibility for s^{13} containing y_e^{12} or $y_e'^{23}$ in its differential is either

$$(-1)^{|s|-1} s^{12} y_{e'}^{23} \quad \text{or} \quad y_e^{12} s^{23},$$

respectively. Note that when $s^{13} = y_e^{13}$ for some e, then these two terms coincide. Moreover, the situation is essentially the same as the augmentation category for border DGAs as computed in Example 4.7.

Therefore, one and only one generator appears in both $m_2\left(-y_{12}^\vee \otimes s^{12}\right)$ and $m_2\left(s^{12} \otimes -y_{12}^\vee\right)$ which is precisely s^{12} itself

$$m_2\left(-y_{12}^\vee \otimes s^{12}\right) = s^{12} = m_2\left(s^{12} \otimes -y_{12}^\vee\right).$$

Finally, the absence of terms of length at least 3 in any differential containing y implies that the higher composition m_k will vanish whenever it contains $-y_{12}^\vee$.

Now we prove that two A_∞ functors $\mathcal{A}\mathsf{ug}_+\left(\phi_*^{(\bullet)}\right) : \mathcal{A}\mathsf{ug}_+(T, \mu; \mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(T_*, \mu_*; \mathbb{K})$ for $* = \mathsf{L}$ and R are unit-preserving. Due to the definition of the induced A_∞-functor briefly reviewed in Section 4 after Proposition 4.5, we have for each augmentation $\epsilon \in \mathcal{A}\mathsf{ug}_+(T,\mu;\mathbb{K})$

$$\mathcal{A}\mathsf{ug}_+\left(\phi_*^{(\bullet)}\right)(-y_{12}^\vee) = \sum_{s^{12}\in R^{12}} (-1)^{\sigma_1}\left\langle -y_{12}^\vee, \phi_{*,\epsilon}^{(2)}(s^{12})\right\rangle s_{12}^\vee,$$

where $\sigma_1 = 0$ as seen in Definition 4.3 and $\phi_{*,\mathbf{e}}^{(2)}$ is a twisted DGA morphism by using the diagonal augmentation $\mathbf{e} = (\epsilon,\epsilon)$.

More precisely, since $\phi_{\mathsf{L}}^{(2)}$ identifies each generators in $A_{\mathsf{co},n_{\mathsf{L}}}^{(2)} \cong A_{\mathsf{co}}^{\mathsf{CE}}(T_{\mathsf{L}}^{(2)},\mu_{\mathsf{L}}^{(2)})$ with the corresponding generator in $A_{\mathsf{co}}^{\mathsf{CE}}(T^{(2)},\mu^{(2)})$,

$$\phi_{\mathsf{L},\mathbf{e}}^{(2)}(k_{ab}^{12}) = k_{ab}^{12} \quad \text{and} \quad \phi_{\mathsf{L},\mathbf{e}}^{(2)}(y_c^{12}) = y_c^{12}$$

and therefore the image of $\phi_{\mathsf{L}}^{(2)}$ has a nontrivial pairing only for y_c^{12}. In other words,

$$\begin{aligned}\mathcal{A}\mathsf{ug}_+\left(\phi_{\mathsf{L}}^{(\bullet)}\right)(-y_{12}^\vee) &= \sum_{c\in[n_{\mathsf{L}}]}\left\langle -y_{12}^\vee, \phi_{\mathsf{L},\epsilon}^{(2)}(y_c^{12})\right\rangle y_{12}^{c\vee}\\ &= \sum_{c\in[n_{\mathsf{L}}]}(-y_{12}^{c\vee}) \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_{+,\mathsf{L}}}(\phi_{\mathsf{L}}^*(\epsilon),\phi_{\mathsf{L}}^*(\epsilon)),\end{aligned}$$

which is the unit as desired.

On the other hand, for the right border, we recall the map $\phi_{\mathsf{R}}^{(m)}$ described in (3.9). That is, for $m = 2$, we have

$$\phi_{\mathsf{R}}^{(2)}(k_{a'b'}^{12}) = \Phi(\phi_{\mathsf{R}}(k_{a'b'}))_{12}, \qquad \phi_{\mathsf{R}}^{(2)}(y_{c'}^{12}) = y_e^{12}. \tag{4.9}$$

Then $\phi_{\mathsf{R}}^{(2)}(k_{a'b'}^{12})$ does not involve any y_e^{12}'s as observed before and therefore the pairing survives only for $y_{c'}^{12}$'s. In that case $\phi_{\mathsf{R},\mathbf{e}}^{(2)}(y_{c'}^{12}) = y_{c'}^{12}$ and we have

$$\begin{aligned}\mathcal{A}\mathsf{ug}_+\left(\phi_{\mathsf{R}}^{(\bullet)}\right)(-y_{12}^\vee) &= \sum_{c'\in[n_{\mathsf{R}}]}\left\langle -y_{12}^\vee, \phi_{\mathsf{R},\mathbf{e}}^{(2)}(y_{c'}^{12})\right\rangle y_{12}^{c'\vee}\\ &= \sum_{c'\in[n_{\mathsf{R}}]}(-y_{12}^{c'\vee}) \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_{+,\mathsf{R}}}(\phi_{\mathsf{R}}^*(\epsilon),\phi_{\mathsf{R}}^*(\epsilon)),\end{aligned}$$

which is the unit as well. This completes the proof. □

Theorem 4.21 (Unitality). *For any bordered Legendrian graph* $(\mathcal{T},\boldsymbol{\mu}) \in \mathcal{BLG}^\mu$, *the augmentation category* $\mathcal{A}\mathsf{ug}_+(\mathcal{T},\boldsymbol{\mu};\mathbb{K})$ *is homologically unital.*

Proof. Due to Lemma 2.18, one can obtain $\mathcal{T}$ from a Legendrian graph $\mathcal{T}'$ in a normal form up to Legendrian isotopy and basepoint moves. Then by Theorem 4.20, the augmentation category $\mathcal{A}\mathsf{ug}_+(\mathcal{T}', \boldsymbol{\mu}'; \mathbb{K})$ is strictly unital.

As seen in the proof of Theorem 4.19, for each Legendrian isotopy, we have a zig-zag of stabilizations between consistent sequences of DGAs for $(\mathcal{T}', \boldsymbol{\mu}')$ and $(\mathcal{T}, \boldsymbol{\mu})$. By Proposition 4.13, we have the homological unitality for $\mathcal{A}\mathsf{ug}_+(T, \mu; \mathbb{K})$ which implies that so is $\mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$ as desired.

For each basepoint move, we have either a zig-zag of stabilizations for (B_1) where the above argument is applicable, or an A_∞-equivalence for (B_i) with $i = 2$ or 3 by the proof of Theorem 4.19 again, which preserves of course the (homological) unitality. □

Proposition 4.22. *Let $(\mathcal{T}, \boldsymbol{\mu}) \in \mathcal{BLG}$ and $(\mathcal{T}_{\mathsf{Lag}}, \boldsymbol{\mu}) := \mathrm{Res}^{\mathsf{Ng}}(\mathcal{T})$. Suppose that $(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)})$ and $(\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}, \boldsymbol{\mu}^{(\bullet)})$ are consistent sequences of canonical front and Lagrangian copies, respectively. Then two augmentation categories $\mathcal{A}\mathsf{ug}_+(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}; \mathbb{K})$ and $\mathcal{A}\mathsf{ug}_+(\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}, \boldsymbol{\mu}^{(\bullet)}; \mathbb{K})$ are A_∞-quasi-equivalent.*

Proof. As seen in Section 3.1.1 and Figure 12, there exists a zig-zag of elementary Lagrangian Reidemeister moves between $\mathrm{Res}^{\mathsf{Ng}}(\mathcal{T}^{(\bullet)})$ and $\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}$. By Theorem 3.32, we have a zig-zag of consistent stabilizations between

$$A^{\mathsf{CE}}(\mathcal{T}^{(\bullet)}, \boldsymbol{\mu}^{(\bullet)}) := A^{\mathsf{CE}}(\mathrm{Res}^{\mathsf{Ng}}(\mathcal{T}^{(\bullet)}), \boldsymbol{\mu}^{(\bullet)}) \quad \text{and} \quad A^{\mathsf{CE}}(\mathcal{T}^{(\bullet)}_{\mathsf{Lag}}, \boldsymbol{\mu}^{(\bullet)}).$$

Finally, Proposition 4.17 completes the proof. □

5. Sheaf categories for Legendrian graphs

In this section we give the preliminaries on the microlocal theory of sheaves, the main reference is [17]. We also establish the necessary combinatorial tools for constructible sheaves, which will be used in the proof of the augmentation-sheaf correspondence in the next section.

5.1. Micro-support and constructible sheaves

For the moment, let M be any smooth manifold, and $\mathbb{K}$ be a base field. We use the same notations as in [25]. Let $\mathcal{S}\mathsf{h}(M; \mathbb{K})$ to be the abelian category of sheaves of $\mathbb{K}$-modules. Let $\mathcal{S}\mathsf{h}_{\mathrm{naive}}(M; \mathbb{K})$ to be the triangulated DG category of complexes of sheaves of $\mathbb{K}$-modules on M whose cohomology sheaves are constructible (that is, locally constant with perfect stalks on each stratum) with respect to some nice stratification (e.g. Whitney stratification). Let

$\mathcal{S}\mathsf{h}(M;\mathbb{K})$ be the DG quotient [9] of the DG category $\mathcal{S}\mathsf{h}_{\mathrm{naive}}(M;\mathbb{K})$ with respect to acyclic complexes. Given a Whitney stratification $\mathcal{S}$ of M, *define* $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$ to be the full subcategory of $\mathcal{S}\mathsf{h}(M;\mathbb{K})$ consisting of objects whose cohomology sheaves are constructible with respect to $\mathcal{S}$.

We firstly recall the notion of micro-support introduced by Kashiwara and Schapira:

Proposition/Definition 5.1 (Micro-support). *[17, Prop.5.1.1, Def.5.1.2] Let $\mathcal{F}$ be a sheaf*[3] *on M, and $p = (x_0, \xi_0) \in T^*M$. We say $p \notin SS(\mathcal{F})$ if one of the following equivalent conditions holds:*

1) *There exists a neighborhood U of p, such that for any $x_1 \in M$ and any C^1-function φ on a neighborhood of x_1 satisfying $(x_1, d\varphi(x_1)) \in U$ and $\varphi(x_1) = 0$, we have*

$$R\Gamma_{\{\varphi(x)\geq 0\}}(\mathcal{F})_{x_1} \simeq 0.$$

Equivalently, by the distinguished triangle

$$R\Gamma_{\{\varphi(x)\geq 0\}}(\mathcal{F}) \to \mathcal{F} \to R\Gamma_{\{\varphi(x)<0\}}(\mathcal{F}) \xrightarrow{+1},$$

we get a quasi-isomorphism

$$\mathcal{F}_{x_1} \xrightarrow{\sim} R\Gamma_{\{\varphi(x)<0\}}(\mathcal{F})_{x_1}.$$

2) *Up to taking an open chart near x_0, we can assume M is an open subset in a vector space E. Then there exists a neighborhood U of x_0, an $\epsilon > 0$, and a proper closed convex cone γ in E with $0 \in \gamma$, satisfying $\gamma \setminus \{0\} \subset \{v \mid \langle v, \xi_0\rangle < 0\}$, such that if we set*

$$H := \{x \mid \langle x - x_0, \xi_0\rangle \geq -\epsilon\} \quad \text{and} \quad L := \{x \mid \langle x - x_0, \xi_0\rangle = -\epsilon\},$$

then $H \cap (U + \gamma) \subset M$ and we have the natural isomorphism:

$$R\Gamma(H \cap (x+\gamma); \mathcal{F}) \xrightarrow{\sim} R\Gamma(L \cap (x+\gamma); \mathcal{F})$$

for all $x \in U$. Recall that a cone is called proper if it contains no lines.

The set $SS(\mathcal{F})$ is called the micro-support *(or* singular support*) of $\mathcal{F}$.*

The micro-support satisfies the following properties:

[3]Whenever we say a sheaf, we mean a complex of sheaves of $\mathbb{K}$-modules.

1) $SS(\mathcal{F}) \cap 0_M = \mathrm{Supp}(\mathcal{F})$ is the support of $\mathcal{F}$, where 0_M the zero section of T^*M.

2) For any sheaf $\mathcal{F}$, $SS(\mathcal{F})$ is a conical (i.e. invariant under the action of $\mathbb{R}_+$ which scales the cotangent fibers) and closed co-isotropic subset of T^*M.

3) If $\mathcal{F}$ is a constructible sheaf with respect to a Whitney stratification $\mathcal{S}$, then $SS(\mathcal{F})$ is a conical Lagrangian (i.e. Lagrangian wherever it is smooth) subset of T^*M with $SS(\mathcal{F}) \subset \bigcup_{i\in\mathcal{S}} T^*_{S_i}M$.

4) (Triangular inequality) If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \xrightarrow{+1}$ is an exact triangle in $\mathcal{S}\mathsf{h}(M;\mathbb{K})$, then $SS(\mathcal{F}_i) \subset SS(\mathcal{F}_j) \cup SS(\mathcal{F}_k)$ for all distinct $i,j,k \in \{1,2,3\}$.

5) (Microlocal Morse lemma) If $f : M \to \mathbb{R}$ is a smooth function such that $(x, df(x)) \notin SS(\mathcal{F})$ for all $x \in f^{-1}([a,b])$, and f is proper on the support of $\mathcal{F}$. Then the restriction map is a quasi-isomorphism:

$$R\Gamma(f^{-1}(-\infty,b);\mathcal{F}) \xrightarrow{\sim} R\Gamma(f^{-1}(-\infty,a);\mathcal{F})$$

From now on, let $M \coloneqq I_x \times \mathbb{R}_z$ be the base manifold, where $I_x = (x_{\mathsf{L}}, x_{\mathsf{R}})$ with $-\infty \le x_{\mathsf{L}} < x_{\mathsf{R}} \le \infty$ is an open interval in $\mathbb{R}_x$. Let $\mathcal{T} = (T_{\mathsf{L}} \leftarrow T \to T_{\mathsf{R}})$ be a bordered Legendrian graph in $J^1 I_x = T^{\infty,-}M$.

Definition 5.2. Given a possibly singular Legendrian $T \subset T^{\infty,-}M$, we define $\mathcal{S}\mathsf{h}(T;\mathbb{K}) = \mathcal{S}\mathsf{h}_T(M;\mathbb{K})$ to be the full subcategory of $\mathcal{S}\mathsf{h}(M;\mathbb{K})$ consisting of those objects $\mathcal{F}$, whose micro-support at infinity is contained in T (i.e. $SS(\mathcal{F}) \subset 0_M \cup \mathbb{R}_{>0}T$). Furthermore, let $\mathcal{S}\mathsf{h}(T;\mathbb{K})_0 = \mathcal{S}\mathsf{h}_T(M:\mathbb{K})_0$ be the full subcategory of $\mathcal{S}\mathsf{h}_T(M;\mathbb{K})$ whose objects are those $\mathcal{F}$ with acyclic stalks for $z \ll 0$.

In particular, given a bordered Legendrian graph $\mathcal{T} = (T_{\mathsf{L}} \to T \leftarrow T_{\mathsf{R}})$ in $J^1 I_x$, by the obvious restriction of sheaves, we obtain a diagram of constructible sheaf categories

$$\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K}) \coloneqq (\mathcal{S}\mathsf{h}(T_{\mathsf{L}};\mathbb{K}) \leftarrow \mathcal{S}\mathsf{h}(T;\mathbb{K}) \to \mathcal{S}\mathsf{h}(T_{\mathsf{R}};\mathbb{K}))$$

and define $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})_0$ similarly.

As in [25, §2.2.1], let $\mathcal{T}^+$ be the extended Legendrian in $T^{\infty,-}M$ as follows: For each crossing c of $\mathcal{T}$, $\mathcal{T}$ meets the semicircle $T^{\infty,-}_c M$ in exactly two points, connected by a unique arc in $T^{\infty,-}_c M$. Then $\mathcal{T}^+$ is the union of $\mathcal{T}$ with all these arcs induced by the crossings. One can similarly define the sheaf categories $\mathcal{S}\mathsf{h}(\mathcal{T}^+;\mathbb{K})$ and $\mathcal{S}\mathsf{h}(\mathcal{T}^+;\mathbb{K})_0$.

Theorem 5.3. *The diagram of DG categories*

$$\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K}) = (\mathcal{S}\mathsf{h}(T_{\mathsf{L}};\mathbb{K}) \leftarrow \mathcal{S}\mathsf{h}(T;\mathbb{K}) \rightarrow \mathcal{S}\mathsf{h}(T_{\mathsf{R}};\mathbb{K}))$$

is a Legendrian isotopy invariant, up to DG equivalence. That is, for any Legendrian isotopy between two bordered Legendrian graphs T, T', there is an equivalence between two diagrams of DG categories taking the form:

$$\begin{array}{ccccc}
\mathcal{S}\mathsf{h}(T_{\mathsf{L}};\mathbb{K}) & \xleftarrow{\mathrm{r}_{\mathsf{L}}} & \mathcal{S}\mathsf{h}(T;\mathbb{K}) & \xrightarrow{\mathrm{r}_{\mathsf{R}}} & \mathcal{S}\mathsf{h}(T_{\mathsf{R}};\mathbb{K}) \\
\Big\Vert\,\mathrm{Id} & \overset{\sim}{\underset{\mathfrak{h}_{\mathsf{L}}}{\diagup\!\!\diagup}} & \mathfrak{K}\Big\downarrow\simeq & \overset{\sim}{\underset{\mathfrak{h}_{\mathsf{R}}}{\diagdown\!\!\diagdown}} & \Big\Vert\,\mathrm{Id} \\
\mathcal{S}\mathsf{h}(T'_{\mathsf{L}};\mathbb{K}) & \xleftarrow[\mathrm{r}'_{\mathsf{L}}]{} & \mathcal{S}\mathsf{h}(T';\mathbb{K}) & \xrightarrow[\mathrm{r}'_{\mathsf{R}}]{} & \mathcal{S}\mathsf{h}(T'_{\mathsf{R}};\mathbb{K})
\end{array}$$

where the arrow $\mathfrak{K}$ in the middle is an equivalence of DG categories which, under restrictions, commutes with the identity functor $\mathrm{Id} : \mathcal{S}\mathsf{h}(T_{\mathsf{L}};\mathbb{K}) \xrightarrow{\sim} \mathcal{S}\mathsf{h}(T'_{\mathsf{L}};\mathbb{K})$ *(resp.* $\mathrm{Id} : \mathcal{S}\mathsf{h}(T_{\mathsf{R}};\mathbb{K}) \xrightarrow{\sim} \mathcal{S}\mathsf{h}(T'_{\mathsf{R}};\mathbb{K})$*) up to specified natural isomorphism $\mathfrak{h}_{\mathsf{L}} : \mathrm{r}_{\mathsf{L}} \overset{\sim}{\Longrightarrow} \mathrm{r}'_{\mathsf{L}} \circ \mathfrak{K}$ (resp. $\mathfrak{h}_{\mathsf{R}} : \mathrm{r}_{\mathsf{R}} \overset{\sim}{\Longrightarrow} \mathrm{r}'_{\mathsf{R}} \circ \mathfrak{K}$). Also, the same holds for $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})_0$.*

Proof. The first proof is a direct consequence of the results of Guillermou-Kashiwara-Schapira [15]. For the more related details, see [25, Thm.4.1, Rmk.4.2,4.3]. □

In the rest of this section, we will give a combinatorial description of the sheaf categories. In particular, we will use this description to give an alternative proof of the invariance theorem. The combinatorial description will also be needed in proving our main result "augmentations are sheaves".

5.2. Combinatorial description for constructible sheaves

As before, let $M = I_x \times \mathbb{R}_z$ be the base manifold, and $\mathcal{T}$ be a bordered Legendrian graph in $J^1 I_x = T^{\infty,-}M$.

We can always assume the front projection T is regular so that the only *singularities* of T are crossings, cusps, and vertices. This induces a *Whitney stratification* $\mathcal{S}_T$ of M whose 0-dimensional strata are the singularities, 1-dimensional strata are the *arcs*—the connected components of $T \setminus \{\text{singularities}\}$, and 2-dimensional strata are the *regions*—connected components of $M \setminus T$. By definition, $\mathcal{S}\mathsf{h}(T;\mathbb{K}) \subset \mathcal{S}\mathsf{h}(T^+;\mathbb{K})$ are full subcategories of $\mathcal{S}\mathsf{h}_{\mathcal{S}_T}(M;\mathbb{K})$.

Given a stratification $\mathcal{S}$, the *star* of a stratum $S \in \mathcal{S}$ is the union of strata whose closure contains S, denoted by $\mathrm{Star}(S)$. Given 2 strata S and S', we denote by $S \leq S'$ and define an arrow $S \to S'$ if $S \subset \overline{S'}$, or equivalently, $\mathrm{Star}(S) \supset \mathrm{Star}(S')$. This defines $\mathcal{S}$ as a poset category. We say that $\mathcal{S}$ is a *regular cell complex* if every stratum is contractible as well as the star of each stratum is contractible. As in [25], we can choose a regular cell complex $\mathcal{S}$ refining the stratification $\mathcal{S}_T$ with the additional 1-dimensional strata away from the crossings of T if necessary.

Assumption 5.4. For the regular cell complex $\mathcal{S}$ refining the stratification $\mathcal{S}_T$ induced by T, we assume that each additional 1-dimensional strata contains no vertical tangent and

1) it is tangent to the existing arcs at the singularity if it has an end at a cusp or a vertex, or
2) it is transverse to the boundary if it has an end at the boundary $\partial M = \{(x, z) \mid x = x_{\mathsf{L}}, x_{\mathsf{R}}\}$.

The assumption can always be satisfied by choosing $\mathcal{S}$ appropriately.

Definition 5.5. For any regular cell complex $\mathcal{S}$ of an oriented manifold M and any stratum $S \in \mathcal{S}$, let us define a *co-standard object* ω_S of $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K})$ as

$$\omega_S := \mathbb{K}_S[\dim S] = R(j_S)_!(\mathbb{K}[\dim S]) \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K}),$$

where $j_S : S \to M$ is the inclusion and $\mathbb{K}$ is regarded as the constant sheaf on S.

Lemma 5.6 ([19, Lem.2.3.2]). *The triangulated DG category $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K})$ is the triangulated envelope of the co-standard objects.*

An immediate corollary is as follows:

Corollary 5.7. *For any $\mathcal{F} \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K})$ and $x \in w \in \mathcal{S}$, we have natural quasi-isomorphisms*

$$\mathcal{F}_x \xrightarrow{\simeq} (R\Gamma(\mathcal{F}))_x \xleftarrow{\simeq} R\Gamma(\mathrm{Star}(w); \mathcal{F}).$$

Here, $\mathcal{F}_x = \varinjlim_{x \in U} \Gamma(U; \mathcal{F})$, $(R\Gamma(\mathcal{F}))_x := \varinjlim_{x \in U} R\Gamma(U; \mathcal{F})$.

To avoid any confusion, let's firstly explain the terminology in the corollary: $\mathcal{F}_x$ means that, we firstly take the ordinary stalks of the cochain complex of sheaves $\mathcal{F}$ degreewise and then form the cochain complex, the point is that this cochain complex only changes by a quasi-isomorphism if we change $\mathcal{F}$ by a quasi-isomorphism. It's in this sense that we can talk about the first quasi-isomorphism in the corollary. As for $(R\Gamma(\mathcal{F}))_x$, we use the notations in [17, §2.6, p.109], hence $R\Gamma(\mathcal{F}) = R\Gamma_M(\mathcal{F})$ is a *cochain complex of sheaves* defined by $U \mapsto R\Gamma(U;\mathcal{F})$. In particular, $R\Gamma(\mathcal{F})$ is not the same as $R\Gamma(M;\mathcal{F})$, the latter being a *cochain complex of* $\mathbb{K}$*-modules*, obtained from the right derived functor of the global sections functor $\mathcal{F} \mapsto \Gamma(M;\mathcal{F})$.

Proof. A direct computation shows that this holds for all the co-standard objects. Moreover, the above property of $\mathcal{F}$ is preserved under taking quasi-isomorphisms, shifts, and cones. Hence by the lemma above, we are done. □

Now we come back to our setting where $M = I_x \times \mathbb{R}_z$. By definition, $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$ contains $\mathcal{S}\mathsf{h}_{\mathcal{S}_T}(M;\mathbb{K})$, hence $\mathcal{S}\mathsf{h}(T^+;\mathbb{K})$ and $\mathcal{S}\mathsf{h}(T;\mathbb{K})$ as full subcategories as well. For an open subinterval J_x of I_x, we denote $M|_{J_x} := J_x \times \mathbb{R}_z$ and define $\mathcal{S}|_{J_x}$ to be the stratification of $M|_{J_x}$ induced by $\mathcal{S}$, whose strata are the connected components of $S \cap (M|_{J_x})$ for all $S \in \mathcal{S}$. Then $\mathcal{S}|_{J_x}$ is a Whitney stratification of $M|_{J_x}$ refining the stratification induced by $T|_{J_x}$ and we obtain a natural dg functor $\mathrm{r} : \mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K}) \to \mathcal{S}\mathsf{h}_{\mathcal{S}|_{J_x}}(M|_{J_x};\mathbb{K})$ coming from the restriction.

Given a regular cell complex $\mathcal{S}$ of M, there is a combinatorial description of $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$ as follows: Denote the induced poset category by $\mathcal{S}$ again and denote the category of cochain complexes of $\mathbb{K}$-modules with cochain maps by $\mathcal{C}\mathsf{h}(\mathbb{K})$.

Definition 5.8. For any poset category $\mathcal{S}$, let $\mathcal{F}\mathsf{un}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ be the DG category of functors from $\mathcal{S}$ to $\mathcal{C}\mathsf{h}(\mathbb{K})$, which are valued on perfect complexes, that is, complexes which are quasi-isomorphic to a bounded complex of finite projective $\mathbb{K}$-modules. We define $\mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})$ to be the DG quotient (see [9]) of $\mathcal{F}\mathsf{un}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ with respect to the thick subcategory of objects taking values in acyclic complexes.

Notation 5.9. We denote the abelian category of functors from $\mathcal{S}$ to the abelian category $\mathbb{K} - \mathrm{Mod}$ of $\mathbb{K}$-modules by $Fun(\mathcal{S},\mathbb{K})$.

We denote by $\mathcal{C}\mathsf{h}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ and $\mathcal{C}\mathsf{h}_{\mathrm{dg}}(\mathcal{S},\mathbb{K})$ the DG category of cochain complexes of objects in the abelian category $Fun(\mathcal{S},\mathbb{K})$ with morphisms the usual complexes of maps between complexes and its DG quotient of $\mathcal{C}\mathsf{h}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ by the full subcategory of acyclic objects, respectively. Then $\mathcal{F}\mathsf{un}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ and $\mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})$ are full subcategories of $\mathcal{C}\mathsf{h}_{\mathrm{naive}}(\mathcal{S},\mathbb{K})$ and $\mathcal{C}\mathsf{h}_{\mathrm{dg}}(\mathcal{S},\mathbb{K})$ respectively.

Observe that we obtain a functor of poset categories $i : \mathcal{S}|_{J_x} \to \mathcal{S}$ by inclusion of strata, which then induces a natural DG functor $i^* : \mathfrak{Fun}(\mathcal{S}, \mathbb{K}) \to \mathfrak{Fun}(\mathcal{S}|_{J_x}; \mathbb{K})$.

Proposition/Definition 5.10. *There is a functor $i_\mathcal{S} : Fun(\mathcal{S}, \mathbb{K}) \to Sh(M; \mathbb{K})$ defined as follows: let $F \in Fun(\mathcal{S}, \mathbb{K})$ and $w \in \mathcal{S}$.*

1) *The stalk $i_\mathcal{S}(F)|_w$ is $F(w)$ viewed as a constant sheaf. In particular, $i_\mathcal{S}(F)$ is constructible with respect to $\mathcal{S}$.*
2) *Let U_w be any contractible open subset of $\mathrm{Star}(w)$ such that $\mathcal{S}|_{U_w}$ is a regular cell complex and the map $\mathcal{S}|_{U_w} \xrightarrow{\sim} \mathcal{S}|_{\mathrm{Star}(w)}$ of partially ordered sets of strata is a bijection. Then $\Gamma(U_w; i_\mathcal{S}(F)) = F(w)$ and the restriction map $\Gamma(\mathrm{Star}(w); i_\mathcal{S}(F)) \to \Gamma(U_w; i_\mathcal{S}(F))$ is the identity. In particular, $(i_\mathcal{S}(F))_x = F(w)$ for all $x \in w$ and it follows that $i_\mathcal{S}$ is exact.*
3) *The restriction map $\Gamma(\mathrm{Star}(w_1); i_\mathcal{S}(F)) \to \Gamma(\mathrm{Star}(w_2); i_\mathcal{S}(F))$ is $F(w_1 \to w_2)$ for all arrows $w_1 \to w_2$ in $\mathcal{S}$.*

Let $\gamma_\mathcal{S} : Sh(M; \mathbb{K}) \to Fun(\mathcal{S}, \mathbb{K})$ be a functor defined as

$$\mathcal{F} \mapsto [S \mapsto \Gamma(\mathrm{Star}(S); \mathcal{F})].$$

Then $\gamma_\mathcal{S} \circ i_\mathcal{S} = \mathrm{Id}$ and $(i_\mathcal{S}, \gamma_\mathcal{S})$ is an adjoint pair and so $\gamma_\mathcal{S}$ is left exact. As a consequence, we obtain an adjoint pair $(i_\mathcal{S}, \Gamma_\mathcal{S})$ in the DG lifting:

$$i_\mathcal{S} : \mathfrak{Fun}(\mathcal{S}, \mathbb{K}) \rightleftarrows \mathsf{Sh}(M; \mathbb{K}) : \Gamma_\mathcal{S} = R\gamma_\mathcal{S} \tag{5.1}$$

and in fact the essential image of $i_\mathcal{S}$ is contained in $\mathsf{Sh}_\mathcal{S}(M; \mathbb{K})$. More explicitly, $\Gamma_\mathcal{S} = R\gamma_\mathcal{S}$ is given by $\mathcal{F}^\bullet \mapsto [w \mapsto R\Gamma(\mathrm{Star}(w); \mathcal{F}^\bullet)]$, and for any $F^\bullet \in \mathfrak{Fun}(\mathcal{S}, \mathbb{K})$ and $\mathcal{G}^\bullet \in \mathsf{Sh}(M; \mathbb{K})$, we have a natural quasi-isomorphism:

$$\mathrm{RHom}^\bullet(i_\mathcal{S}(F^\bullet), \mathcal{G}^\bullet) \simeq \mathrm{RHom}^\bullet(F^\bullet, \Gamma_\mathcal{S}(\mathcal{G}^\bullet)) \tag{5.2}$$

Moreover, we get a natural isomorphism $\beta : \mathrm{Id} \overset{\sim}{\Rightarrow} \Gamma_\mathcal{S} \circ i_\mathcal{S}$.

Proof. Firstly, let us show that for any $F \in Fun(\mathcal{S}, \mathbb{K})$, $i_\mathcal{S}(F)$ indeed defines a sheaf on M. By [28, Thm.2.7.1], it suffices to define $i_\mathcal{S}(F)$ as a sheaf on a base of M.

We take a base $\mathcal{B} = \{B_i\}$ for the topology of M such that each B_i is of the form U_w as in (2). Let $\mathcal{F}^{\mathsf{pre}}$ be a presheaf on $\mathcal{B}$ defined as follows: for each $B_i = U_w$ with $w \in \mathcal{S}$, $\mathcal{F}^{\mathsf{pre}}(B_i) := F(w)$ and for each inclusion $B_j = V_{w_2} \hookrightarrow$

$B_i = U_{w_1}$, we require that $\mathrm{Star}(w_2) \subset \mathrm{Star}(w_1)$. In other words, if $w_1 \to w_2$ is an arrow in $\mathcal{S}$, then the restriction map $\Gamma(U_{w_1}; \mathcal{F}^{\mathsf{pre}}) \to \Gamma(V_{w_2}; \mathcal{F}^{\mathsf{pre}})$ is defined to be $F(w_1 \to w_2)$. Clearly, this is a well-defined presheaf on $\mathcal{B}$. Then $i_{\mathcal{S}}(F) \coloneqq \mathcal{F}$ is defined to be the sheafification of $\mathcal{F}^{\mathsf{pre}}$. Recall that for any open subset U in M, we have

$$\Gamma(U;\mathcal{F}) = \{(f_p \in \mathcal{F}_p^{\mathsf{pre}})_{p\in U} \mid \text{For any } p\text{, there exists a pair } (V_S, s) \text{ such that } V_S \subset U \text{ is a neighborhood of } p \text{ and } V_S \in \mathcal{B} \text{ for some } S \in \mathcal{S}\text{, and } s \in \mathcal{F}^{\mathsf{pre}}(V_S) = F(S) \text{ with } s_q = f_q \text{ on } V_S.\}$$

Also, for all $S \in \mathcal{S}$ with $S \cap U \neq \varnothing$, each $p \in S \cap U$ has a system of neighborhoods in $\mathcal{B}$ of the form U_S. Then by definition, we have $\Gamma(U_S; \mathcal{F}^{\mathsf{pre}}) = F(S) \cong \mathcal{F}_p^{\mathsf{pre}}$. Hence, it follows that any section of $\Gamma(U;\mathcal{F})$ is locally constant on $S \cap U$ with values in $F(S)$. In particular, for any connected component w in $S \cap U$, we have $\Gamma(w;\mathcal{F}) = F(S)$. Thus $\mathcal{F}|_S$ is $F(S)$ viewed as a (locally) constant sheaf. This shows (1), hence $\mathcal{F} \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$.

Moreover, we can rewrite the definition as follows. Let us denote by $\mathcal{S}|_U$ the stratification of U consisting of the connected components of $S \cap U$ for all $S \in \mathcal{S}$ and let $\tau_U : \mathcal{S}|_U \to \mathcal{S}$ be the map of partially ordered sets induced by inclusion of strata. Then the previous definition can be translated into

$$\Gamma(U;\mathcal{F}) = \varprojlim_{w\in\mathcal{S}|_U} (F \circ \tau_U)(w). \tag{5.3}$$

In other words,

$$\Gamma(U;\mathcal{F}) = \left\{ (f(w))_w \in \prod_{w\in\mathcal{S}|_U} (F\circ\tau_U(w)) \;\middle|\; w_1 \leq w_2 \Rightarrow (F\circ\tau_U)(w_1 \to w_2)(f(w_1)) = f(w_2) \right\}.$$

Besides, for any inclusion $V \hookrightarrow U$, there is an induced map of partially ordered sets $\tau_{U,V} : \mathcal{S}|_V \to \mathcal{S}|_U$ via inclusions of strata. Then $\tau_V = \tau_U \circ \tau_{U,V}$, and the restriction map is just $\tau_{U,V}^* : \Gamma(U;\mathcal{F}) \to \Gamma(V;\mathcal{F})$ via the pullback of functions. That is, for any $f \in \Gamma(U;\mathcal{F}) \subset \prod_{w\in\mathcal{S}|_U}(F\circ\tau_U)(w)$, we have $(\tau_{U,V}^* f)(w) = f(\tau_{U,V}(w))$ for all $w \in \mathcal{S}|_V$.

Now, let $U = U_w$ (including $\mathrm{Star}(w)$) as in (2). We obtain $\Gamma(U;\mathcal{F}) \cong F(\tau(w\cap U)) = F(w)$ as $w \cap U$ is the unique minimum in $\mathcal{S}|_U$. For any inclusion $V_{w_2} \hookrightarrow U_{w_1}$ of two open subsets as in (2) with $w_1, w_2 \in \mathcal{S}$, we also know

$w_2 \leq w_1$ and $\tau_{U_{w_1},V_{w_2}}(w_2 \cap V_{w_2}) = w_2 \cap U_{w_1} \leq w_1 \cap U_{w_1}$ in $\mathcal{S}|_{U_{w_1}}$. Then the restriction map $\Gamma(U_{w_1};\mathcal{F}) \cong F(w_1) \to \Gamma(V_{w_2};\mathcal{F}) \cong F(w_2)$ is identified with $F(w_1 \to w_2)$ as follows:

$$
\begin{aligned}
f \cong f(w_1 \cap U) \mapsto \tau^*_{U_{w_1},V_{w_2}} f \cong (\tau^*_{U_{w_1},V_{w_2}} f)(w_2 \cap V) \\
= f(w_2 \cap U) = F(w_1 \to w_2)(f(w_1 \cap U)).
\end{aligned}
$$

This shows (2) and (3) and in fact $\mathcal{F}^{\mathsf{pre}} \cong \mathcal{F}$ is already a sheaf on $\mathcal{B}$.

The functoriality of $i_{\mathcal{S}}$ follows directly from the definition of $\mathcal{F}$ above. By (1), the essential image of $i_{\mathcal{S}}$ is contained in $\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$ and it follows immediately from (2) that $\gamma_{\mathcal{S}} \circ i_{\mathcal{S}} = \mathrm{Id}$.

By [28, Ex.2.7.C], morphisms of sheaves correspond to morphisms of sheaves on a base. By a direct computation, we have an adjunction of $(i_{\mathcal{S}}, \gamma_{\mathcal{S}})$ which yields the following: In (5.2), taken an injective resolution of $\mathcal{G}^\bullet$, say $\mathcal{G}^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$, then by adjunction of $(i_{\mathcal{S}}, \gamma_{\mathcal{S}})$, we have an injective object $\gamma_{\mathcal{S}}(\mathcal{I}^\bullet)$ and it follows that

$$
\begin{aligned}
\mathrm{RHom}^\bullet(i_{\mathcal{S}}(F^\bullet), \mathcal{G}^\bullet) &\simeq \mathrm{Hom}^\bullet(i_{\mathcal{S}}(F^\bullet), \mathcal{I}^\bullet) \\
&\simeq \mathrm{Hom}^\bullet(F^\bullet, \gamma_{\mathcal{S}}(\mathcal{I}^\bullet)) \\
&\simeq \mathrm{RHom}^\bullet(F^\bullet, \Gamma_{\mathcal{S}}(\mathcal{G}^\bullet)).
\end{aligned}
$$

where the last quasi-isomorphism follows from the fact that $\Gamma_{\mathcal{S}}(\mathcal{G}^\bullet) \xrightarrow{\sim} \Gamma_{\mathcal{S}}(\mathcal{I}^\bullet) = \gamma_{\mathcal{S}}(\mathcal{I}^\bullet)$ is a quasi-isomorphism.

Finally, for any $F^\bullet \in \mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})$ and $x \in w \in \mathcal{S}$, by (2) and Corollary 5.7, we have a natural quasi-isomorphism

$$
\begin{aligned}
F^\bullet(w) = \Gamma(\mathrm{Star}(w); i_{\mathcal{S}}(F^\bullet)) &\xrightarrow{\sim} R\Gamma(\mathrm{Star}(w); i_{\mathcal{S}}(F^\bullet)) \\
&= R\gamma_{\mathcal{S}}(i_{\mathcal{S}}(F^\bullet))(w) \simeq (i_{\mathcal{S}}(F^\bullet))_x.
\end{aligned}
$$

Hence, we get a natural isomorphism $\beta : \mathrm{Id} \overset{\sim}{\Rightarrow} \Gamma_{\mathcal{S}} \circ i_{\mathcal{S}}$. This completes the proof. □

We recall the following lemma:

Lemma 5.11. *[25, Prop.3.9], [19, Lem.2.3.2] Let $\mathcal{S}$ be a regular cell complex for M. Then the functor*

$$
\Gamma_{\mathcal{S}} : \mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K}) \to \mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K}), \quad \mathcal{F} \mapsto [S \mapsto R\Gamma(\mathrm{Star}(S);\mathcal{F})]
$$

is a quasi-equivalence with a quasi-inverse $i_{\mathcal{S}}$. Moreover, $i_{\mathcal{S}}$ commutes with the functors induced by restriction to J_x.

Proof. The last statement follows directly from the definition of $i_{\mathcal{S}}$ and it suffices to show the quasi-equivalence.

For any $S \in \mathcal{S}$, we define a functor $\delta_S \in \mathcal{F}\mathsf{un}(\mathcal{S}, \mathbb{K})$ by $\delta_S(w) := 0$ if $w \neq S$, and $\delta_S(S) := \mathbb{K}[\dim S]$, and sends all arrows $w_1 \to w_2$ to zero. Clearly, $\mathcal{F}\mathsf{un}(\mathcal{S}, \mathbb{K})$ is the triangulated envelope of δ_S's. Moreover by Proposition/Definition 5.10, we have $i_{\mathcal{S}}(\delta_S) \simeq \mathbb{K}_S[\dim S] = \omega_S$. Hence, $i_{\mathcal{S}}$ is essentially surjective by Lemma 5.6 and so is $\Gamma_{\mathcal{S}}$ by the natural isomorphism $\beta : \mathrm{Id} \overset{\sim}{\Longrightarrow} \Gamma_{\mathcal{S}} \circ i_{\mathcal{S}}$ in Proposition/Definition 5.10.

It suffices to show that $i_{\mathcal{S}}$ is fully faithful. In fact, for any $F, G \in \mathcal{F}\mathsf{un}(\mathcal{S}, \mathbb{K})$, by the adjunction $(i_{\mathcal{S}}, \Gamma_{\mathcal{S}})$ in Proposition/Definition 5.10, we have

$$\mathrm{RHom}^\bullet(i_{\mathcal{S}}F, i_{\mathcal{S}}G) \simeq \mathrm{RHom}^\bullet(F, \Gamma_{\mathcal{S}}(i_{\mathcal{S}}G)) \simeq \mathrm{RHom}^\bullet(F, G)$$

where the last quasi-isomorphism follows from the natural isomorphism $\beta : \mathrm{Id} \overset{\sim}{\Rightarrow} \Gamma_{\mathcal{S}} \circ i_{\mathcal{S}}$, which then implies the quasi-isomorphism $G \simeq \Gamma_{\mathcal{S}} \circ i_{\mathcal{S}}(G)$. This finishes the proof. $\square$

Remark 5.12. For the last part of the proof above, we can also finish the argument by showing that $\Gamma_{\mathcal{S}}$ is fully faithful in a different way as follows: For all $S, W \in \mathcal{S}$, a direct computation shows

$$\Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S])(W) := R\Gamma(\mathrm{Star}(W); \mathbb{K}_S[\dim S]) \simeq (\mathbb{K}_S)_p[\dim S]$$

for any $p \in W$. That is, $\Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S])(W)$ is acyclic unless $W = S$ when $\Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S])(S) \simeq \mathbb{K}[\dim S]$. In other words, $\Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S]) \simeq \delta_S$. Now, for any $S \in \mathcal{S}$, we take $F^\bullet = \Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S]) \simeq \delta_S$. Then $i_{\mathcal{S}}(F^\bullet) \simeq \mathbb{K}_S[\dim S]$. We apply the adjunction (5.2) to obtain

$$\mathrm{RHom}^\bullet(\mathbb{K}_S[\dim S], \mathcal{G}^\bullet) \simeq \mathrm{RHom}^\bullet(\Gamma_{\mathcal{S}}(\mathbb{K}_S[\dim S]), \Gamma_{\mathcal{S}}(\mathcal{G}^\bullet)).$$

By an argument of taking shifts, exact triangles, and quasi-isomorphisms in the place of $\mathbb{K}_S[\dim S]$, we then obtain

$$\mathrm{RHom}^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{RHom}^\bullet(\Gamma_{\mathcal{S}}(\mathcal{F}^\bullet), \Gamma_{\mathcal{S}}(\mathcal{G}^\bullet))$$

for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K})$. This shows the fully faithfulness of $\Gamma_{\mathcal{S}}$ by a different argument.

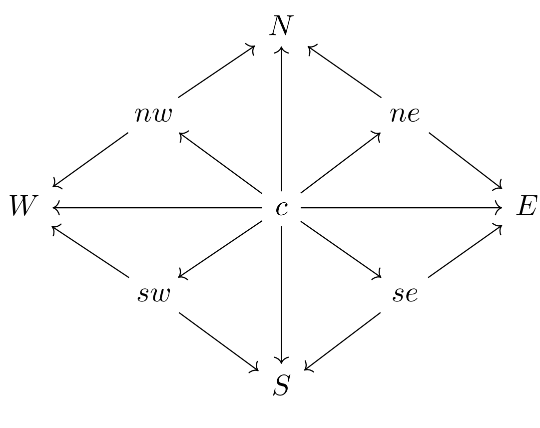
Remark 5.13. By [19, Lem.2.3.2], we also know that for all $S, W \in \mathcal{S}$, the morphism complex is given as

$$\mathrm{RHom}^\bullet_{\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})}(\mathbb{K}_S[\dim S], \mathbb{K}_W[\dim W]) \simeq \begin{cases} \mathbb{K} & S \leq W, \\ 0 & \text{otherwise.} \end{cases}$$

Hence so is $\mathrm{RHom}^\bullet_{\mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})}(\delta_S, \delta_W)$.

Definition 5.14. [25, Def.3.11] Let $\mathcal{T} = (T_{\mathsf{L}} \to T \leftarrow T_{\mathsf{R}}) \in \mathcal{BLG}$ be a bordered Legendrian graph in $J^1 I_x \cong T^{\infty,-}M$ and $\mathcal{S}$ be a regular cell complex refining the stratification $\mathcal{S}_T$ induced by T. Let $\mathcal{F}\mathsf{un}_{T^+}(\mathcal{S},\mathbb{K})$ (resp. $\mathcal{F}\mathsf{un}_{T^+}(\mathcal{S},\mathbb{K})_0$) be the full subcategory of $\mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})$ of objects F satisfying (1) and (2) (resp. (1)–(3)) as follows:

1) Every arrow from a 0-dimensional stratum which is not a crossing, a cusp, or a vertex, or from a 1-dimensional stratum which is not contained in an arc of T, is sent to a quasi-isomorphism. In other words, every arrow from a zero- or one-dimensional stratum contained in $\mathcal{S}$ but not in $\mathcal{S}_T$, is sent to a quasi-isomorphism.
2) If $S, S' \in \mathcal{S}$, with S' bounds S from above, then $S' \to S$ is sent to a quasi-isomorphism. In other words, every downward arrow is sent to a quasi-isomorphism.
3) If $S \in \mathcal{S}$ is contained in the bottom region of T (i.e. the region contains the points with $z \ll 0$), then $F(S)$ is acyclic.
4) (Crossing condition) At each crossing c of $\mathcal{T}$, which is also a 0-dimensional stratum in $\mathcal{S}$, there is an induced subcategory of $\mathcal{S}$ as follows:



All triangles in the diagram are commutative and the total complex of the bicomplex $F(c) \to F(nw) \oplus F(ne) \to F(N)$ is acyclic.

We also define $\mathcal{F}\mathsf{un}_T(\mathcal{S};\mathbb{K})$ and $\mathcal{F}\mathsf{un}_T(\mathcal{S};\mathbb{K})_0$ to be the full subcategories of $\mathcal{F}\mathsf{un}_{T^+}(\mathcal{S};\mathbb{K})$ and $\mathcal{F}\mathsf{un}_{T^+}(\mathcal{S};\mathbb{K})_0$, respectively, consisting of functors F satisfying the extra crossing condition (4).

Then we have the following result similar to [25, Thm.3.12].

Lemma 5.15 (Combinatorial model). *Let $\mathcal{T} \in \mathcal{BLG}$. For a regular cell complex $\mathcal{S}$ for $M = I_x \times \mathbb{R}_z$ obtained by refining the stratification $\mathcal{S}_T$, the functor $\Gamma_{\mathcal{S}}$ induces a quasi-equivalence $\Gamma_{\mathcal{S}} : \mathcal{S}\mathsf{h}(T;\mathbb{K}) \xrightarrow{\sim} \mathcal{F}\mathsf{un}_T(\mathcal{S};\mathbb{K})$ with quasi-inverse $i_{\mathcal{S}}$, and $i_{\mathcal{S}}$ commutes with restriction to J_x. So is*

$$\Gamma_{\mathcal{S}} : \mathcal{S}\mathsf{h}(T;\mathbb{K})_0 \xrightarrow{\sim} \mathcal{F}\mathsf{un}_T(\mathcal{S};\mathbb{K})_0.$$

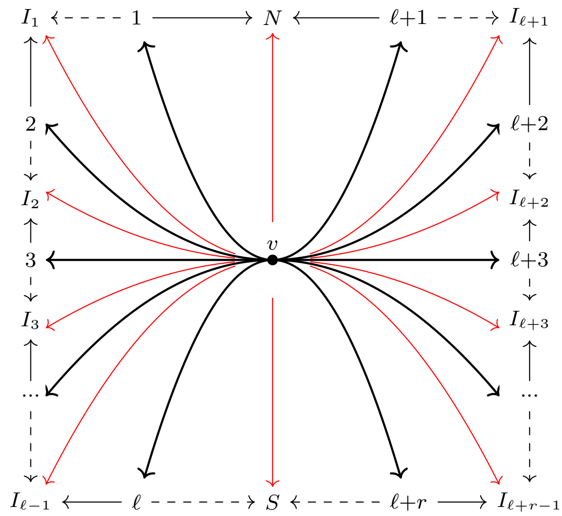
The results also hold when T is replaced by T^+.

Proof. The proof is entirely similar to that of [25, Thm.3.12]. Both the micro-support condition of a sheaf and the properties in Definition 5.14 of functors in $\mathcal{F}\mathsf{un}(\mathcal{S},\mathbb{K})$ can be checked locally. This reduces the proof to match the micro-support condition with the corresponding property in Definition 5.14 near an arc, a cusp, a crossing, and a vertex. The first three cases have been already covered by [25, Thm.3.12] and the only new ingredient is what happens at a vertex.

Local combinatorial model near a vertex. Let $v \in T$ be a vertex of type (ℓ, r). At first we assume that there are no additional 1-dimensional strata in $\mathcal{S}$ ending at v and that $y(v) = 0$ and $T^*_v M \cap (\mathbb{R}_{>0} \cdot T) = \mathbb{R}_{>0} \cdot (-dz)$.

Let $p = \alpha dx + \beta dz \in T^*_v M \setminus \mathbb{R}_{>0} \cdot T$. Then either $\alpha \neq 0$ or $\alpha = 0$ and $\beta > 0$. Near v, we label the region below the arc i by I_i for $1 \leq i \leq \ell + r$, and label the regions above v and below v by N and S, respectively. Then

$I_\ell = S = I_{\ell+r}$ and the poset subcategory of $\mathcal{S}$ near v looks as



Observe that the downward arrow $v \to S$ is the same as the compositions $v \to \ell + r \to S$ and $v \to \ell \to S$. Thus the lemma in this case is equivalent to the following claim.

Claim 5.16. *Let $B_r(v)$ be an r-neighborhood of v for small enough $r \ll 1$. Then for any $\mathcal{F} \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$, the restriction $\mathcal{F}|_{B_r(v)}$ is contained in $\mathcal{S}\mathsf{h}_T(B_r(v);\mathbb{K})$ if and only if the arrows*

$$\{i \to I_i \mid 1 \le i \le \ell + r\} \amalg \{v \to \ell, v \to \ell + r\}$$

which are the dotted arrows in the diagram above, are sent to quasi-isomorphisms under $\Gamma_{\mathcal{S}}(\mathcal{F})$.

Proof. For simplicity, we may assume that v is at the origin $(0,0)$ and up to a local C^1-diffeomorphism near v, we can assume the half-edges $1, 2, \dots, \ell$ are modelled on the graphs of the decreasing functions $z = a_i x^2$ on $(-1, 0]$ with $a_i = 1 - \frac{i}{\ell}$, and the half-edges $\ell + 1, \dots, \ell + r$ are modelled on the graphs of the increasing functions $z = b_i x^2$ on $[0, 1)$ with $b_i = 1 - \frac{\ell+r-i}{r}$.

If $\mathcal{F}|_{B_r(v)} \in \mathcal{S}\mathsf{h}_T(B_r(v);\mathbb{K})$, then by the local model near an arc in the smooth case, we have already known that $\Gamma_{\mathcal{S}}(\mathcal{F})(i \to I_i)$ is a quasi-isomorphism for all $1 \le i \le \ell + r$.

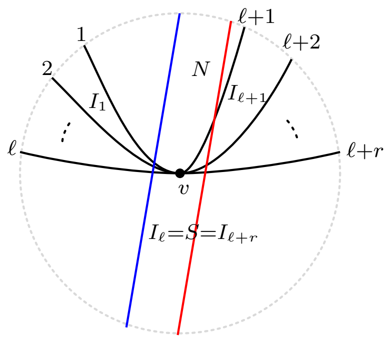
We take a linear Morse function $\varphi = \alpha x + \beta z$ with $\alpha \ne 0$, and then $\varphi(v) = 0, d\varphi(v) = \alpha dx + \beta dz \notin SS(\mathcal{F})$. By definition, $\mathcal{F}_v \xrightarrow{\sim} R\Gamma_{\{\varphi(x)<0\}}(\mathcal{F})_v$

is a quasi-isomorphism, which is equivalent to

$$R\Gamma(B_r(v)\cap\varphi^{-1}(-\infty,\epsilon);\mathcal{F})\xrightarrow{\sim}R\Gamma(B_r(v)\cap\varphi^{-1}(-\infty,-\epsilon);\mathcal{F})$$

for a small enough $0<\epsilon$. We set $Y_\pm := B_r(v)\cap\varphi^{-1}(-\infty,\pm\epsilon)$.

Then Y_+ and Y_- are the regions which are to the left of the red and blue lines or to the right of those lines according to the sign of α:



Hence, we have quasi-isomorphisms

$$R\Gamma(Y_+;\mathcal{F})\simeq\Gamma_{\mathcal{S}}(\mathcal{F})(v)\quad\text{and}$$
$$R\Gamma(Y_-;\mathcal{F})\simeq\operatorname{Cone}\left(\bigoplus_{i=1}^{\ell}\Gamma_{\mathcal{S}}(\mathcal{F})(i)\xrightarrow{d}\bigoplus_{i=1}^{\ell-1}\Gamma_{\mathcal{S}}(\mathcal{F})(I_i)\right)[-1],$$

where the i-th component of d is the difference of the restriction maps induced by $i\to I_i$ and $i+1\to I_i$, for $1\le i\le l-1$. Under this identification, the restriction map $R\Gamma(Y_+;\mathcal{F})\to R\Gamma(Y_-;\mathcal{F})$ is induced by the morphisms $\{v\to i\mid 1\le i\le l\}$ and is a quasi-isomorphism if and only if the total complex of

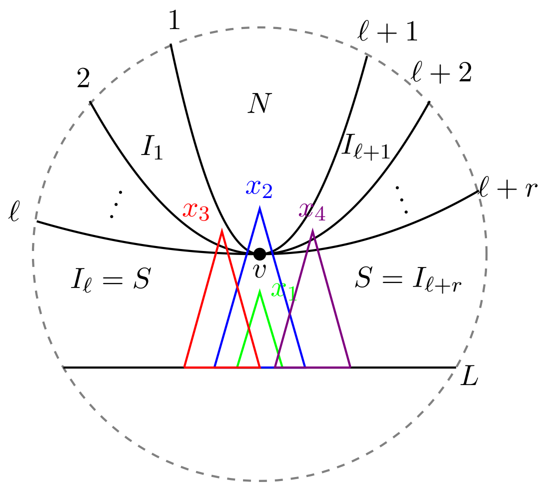
$$\Gamma_{\mathcal{S}}(\mathcal{F})(v)\to\bigoplus_{i=1}^{\ell}\Gamma_{\mathcal{S}}(\mathcal{F})(i)\to\bigoplus_{i=1}^{\ell-1}\Gamma_{\mathcal{S}}(\mathcal{F})$$

is acyclic. As $\Gamma_{\mathcal{S}}(\mathcal{F})(i\to I_i)$ is a quasi-isomorphism for all $1\le i\le\ell-1$, this happens if and only if $\Gamma_{\mathcal{S}}(\mathcal{F})(v\to\ell)$ is a quasi-isomorphism as well.

Conversely, let $\mathcal{F}\in\mathcal{S}\mathsf{h}_{\mathcal{S}}(M;\mathbb{K})$ be a sheaf such that $\Gamma_{\mathcal{S}}(\mathcal{F})(i\to I_i)$ for $1\le i\le\ell$ and $\Gamma_{\mathcal{S}}(\mathcal{F})(v\to\ell),\Gamma_{\mathcal{S}}(\mathcal{F})(v\to\ell+r)$ are all quasi-isomorphisms. In order to prove that $\mathcal{F}|_{B_r(v)}\in\mathcal{S}\mathsf{h}_T(B_r(v);\mathbb{K})$, it suffices to show that $SS(\mathcal{F})\cap(T_v^*M-\{0\})\subset\mathbb{R}_{>0}\cdot(-dz)$ similar to the smooth case.

For any $p=\alpha dx+\beta dz$ in T_v^*M with $\alpha\neq 0$, the same argument as above applies to any C^1-function φ with $d\varphi(v)$ sufficiently close to p, rather than $\alpha x+\beta z$. Hence, $p\notin SS(\mathcal{F})$.

Suppose that $p = (v, \beta dz) \in T^*_v M$ with $\beta > 0$. We use (2) in Proposition/Definition 5.1 to show that $p \notin SS(\mathcal{F})$. As seen in the picture below



we can take a smaller open ball U around v and a small $\epsilon > 0$ defining a line $L \coloneqq \{x \mid \langle x - v, \beta dz\rangle = -\epsilon\}$, and take a proper closed convex cone γ in $E = \mathbb{R}^2_{xz}$ with $0 \in \gamma$ so that

$$\gamma \setminus \{0\} \subset \{w \mid \langle w, \beta dz\rangle < 0\},$$

which is just the lower half-plane. For example, let v_1, v_2 be the two downward unit vectors generating the two green rays as shown above then $\gamma = \{tv_1 + sv_2 : t, s \geq 0\}$. Let $H \coloneqq \{x : \langle x - v, \beta dz\rangle \geq -\epsilon\}$ be the region above L and then we clearly have $H \cap (U + \gamma) \subset B_r(v)$. It suffices to show that for all $x \in U$, we have the natural quasi-isomorphism

$$\text{(5.4)} \qquad \mathrm{r} : R\Gamma(H \cap (x + \gamma); \mathcal{F}) \xrightarrow{\sim} R\Gamma(L \cap (x + \gamma); \mathcal{F})$$

Recall [17, Rmk.2.6.9] that for any compact subset Z of $B_r(v)$, there exist a a quasi-isomorphism

$$\varinjlim_{\mathcal{U} \supset Z} R\Gamma(\mathcal{U}; \mathcal{F}) \xrightarrow{\sim} R\Gamma(Z; \mathcal{F}),$$

where $\mathcal{U}$ runs over the open neighborhoods of Z.

As illustrated above, $H \cap (x + \gamma)$ is the region bounded by a triangle Δ_x whose sides are parallel to 3 lines $\mathbb{R} \cdot v_1, \mathbb{R} \cdot v_2$ and L, where $L \cap (x + \gamma)$ is the bottom edge of Δ_x contained in the region S. Hence, $R\Gamma(L \cap (x + \gamma); \mathcal{F}) \simeq$

$\Gamma_{\mathcal{S}}(\mathcal{F})(S) = R\Gamma(S;\mathcal{F})$. Referring the picture above and according to the choice of x, we have the following four cases:

1) If $x = x_1$ so that $H \cap (x_1 + \gamma)$ has empty intersection with T, then $R\Gamma(H \cap (x + \gamma);\mathcal{F}) \simeq \Gamma_{\mathcal{S}}(\mathcal{F})(S)$ and so we have the natural isomorphism r in (5.4).

2) If $x = x_2$ so that $v \in H \cap (x_2 + \gamma)$, then $R\Gamma(H \cap (x+\gamma);\mathcal{F}) \simeq \Gamma_{\mathcal{S}}(\mathcal{F})(v)$ and the map r is the composition $\Gamma_{\mathcal{S}}(\mathcal{F})(\ell + r \to S) \circ \Gamma_{\mathcal{S}}(\mathcal{F})(v \to \ell + r)$. By the hypothesis of $\mathcal{F}$, each component in the composition is a quasi-isomorphism, hence so is the composition.

3) If $x = x_3$ so that $H \cap (x_3 + \gamma)$ has non-empty intersection with T exactly along half-edges $i, i+1, \ldots, \ell$ for some $1 \le i \le \ell$, then we have

$$R\Gamma(H \cap (x + \gamma);\mathcal{F}) \simeq \mathrm{Cone}\left(\bigoplus_{k=i}^{\ell} \Gamma_{\mathcal{S}}(\mathcal{F})(k) \xrightarrow{d} \bigoplus_{k=i}^{\ell-1} \Gamma_{\mathcal{S}}(\mathcal{F})(I_k)\right)[-1],$$

where the k-th component of d is the difference of the restriction maps induced by $k \to I_k$ and $k+1 \to I_k$ as before. Under this identification, the map (5.4) is induced by the morphism $\ell \to S$. Then by the nine lemma (or the 3×3-lemma) for triangulated categories [18, Lem.2.6], we obtain a diagram in which all squares commute except for the non-displayed one on the bottom right that anti-commutes, and all rows and columns are exact triangles:

$$\begin{array}{ccccccc}
R\Gamma(H \cap (x+\gamma);\mathcal{F}) & \longrightarrow & \bigoplus_{k=i}^{\ell} \Gamma_{\mathcal{S}}(\mathcal{F})(k) & \xrightarrow{d} & \bigoplus_{k=i}^{\ell-1} \Gamma_{\mathcal{S}}(\mathcal{F})(I_k) & \xrightarrow{+1} & \\
\downarrow{\scriptstyle \mathrm{r}} & & \downarrow{\scriptstyle \mathrm{r}'} & & \downarrow & & \\
\Gamma_{\mathcal{S}}(\mathcal{F})(S) & \xrightarrow{\mathrm{Id}} & \Gamma_{\mathcal{S}}(\mathcal{F})(S) & \longrightarrow & 0 & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Cone}(\mathrm{r}) & \longrightarrow & \bigoplus_{k=i}^{\ell-1} \Gamma_{\mathcal{S}}(\mathcal{F})(k)[1] & \xrightarrow{d'} & \bigoplus_{k=i}^{\ell-1} \Gamma_{\mathcal{S}}(\mathcal{F})(I_k)[1] & \xrightarrow{+1} & \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & &
\end{array}$$

where the map r' is induced by $\ell \to S$ and the k-th component of d' is the difference of restriction maps induced by $k \to I_k$ and $k+1 \to I_k$ if $i \le k < \ell - 1$ and is the restriction map induced by $\ell - 1 \to I_{\ell-1}$ if $k = \ell - 1$. By the hypothesis of $\mathcal{F}$, the map $\Gamma_{\mathcal{S}}(\mathcal{F})(k \to I_k)$ is a quasi-isomorphism for $k = \ell-1, \ell-2, \ldots, i$, and so is d'. The exactness of the third row implies that Cone(r) is acyclic and therefore r is a quasi-isomorphism.

4) If $x = x_4$ so that $H \cap (x + \gamma)$ has non-empty intersection with T exactly along half-edges $j, j+1, \ldots, \ell + r$, for some $\ell + 1 \leq j \leq \ell + r$, then the same argument as in the third case holds and we conclude that the map r in (5.4) is a quasi-isomorphism.

This completes the proof of the claim. □

Finally, suppose that there are some additional 1-dimensional strata ending at v. By our assumption on $\mathcal{S}$ at the beginning of Section 5.2, we can regard the extra 1-dimensional strata near v as additional half-edges at v and obtain a new vertex v' with ℓ' left half-edges and r' right half-edges for some $\ell' \geq \ell$ and $r' \geq r$.

We label the half-edges and regions near v' as before. Then exactly the same argument as above holds and proves the lemma for this case. Equivalently, for all $\mathcal{F} \in \mathcal{S}\mathsf{h}_{\mathcal{S}}(M; \mathbb{K})$, we have $\mathcal{F} \in \mathcal{S}\mathsf{h}_T(B_r(v); \mathbb{K})$ if and only if all arrows $\{i \to I_i \mid 1 \leq i \leq \ell' + r'\} \amalg \{v' \to \ell', v' \to \ell' + r'\}$ are sent to quasi-isomorphisms under $\Gamma_{\mathcal{S}}(\mathcal{F})$. □

5.3. A legible model for constructible sheaves

Let $\mathcal{T} = (T_{\mathsf{L}} \to T \leftarrow T_{\mathsf{R}}) \in \mathcal{BLG}$ be a bordered Legendrian graph in $J^1 I_x = T^{\infty,-}M$ as before and $\mathcal{S}$ be a stratification refining $\mathcal{S}_T$. We can simplify the combinatorial model further under the following stronger assumption:

Assumption 5.17. The stratification $\mathcal{S}$ is the induced stratification $\mathcal{S}_{\widetilde{T}}$ of a bordered Legendrian graph $\widetilde{\mathcal{T}} = (T_{\mathsf{L}} \to \widetilde{T} \leftarrow T_{\mathsf{R}})$ which extends $\mathcal{T}$ so that $\widetilde{\mathcal{T}}$ contains no cusps and no vertices with only left or right half-edges.

In particular, this implies that $\mathcal{S}$ is a regular cell complex and satisfies Assumption 5.4.

For a stratification $\mathcal{S}$ satisfying the above assumption, we denote by $G_{\mathcal{S}}$ the finite set of functions f on $(x_{\mathsf{L}}, x_{\mathsf{R}})$ such that f is either $\pm\infty$ or a continuous function whose graph is contained in a union of zero- and one-dimensional strata in $\mathcal{S}$.

We observe the following: suppose that $\mathcal{S}$ satisfies Assumption 5.17. Then any region S in $\mathcal{S}$ is an open disk in M bounded by the graphs of 2 functions

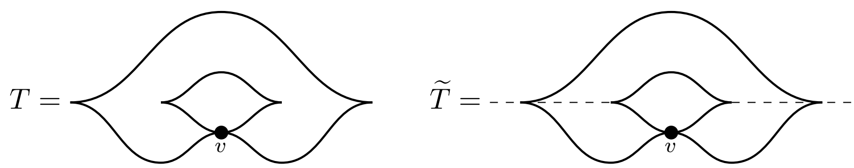


Figure 13: An example of stratifications satisfying Assumption 5.17.

$f_L \le f_U$ in $G_{\mathcal{S}}$. That is, $S = \{(x,z) | x \in (x_{\mathsf{L}}, x_{\mathsf{R}}), f_L(x) < z < f_U(x)\}$.[4] We call (f_L, f_U) a *bounding pair* for S. The choice of the pair (f_L, f_U) might not be unique. However, since S is contractible, there exist $x_{\mathsf{L}} \le x_{1,S} < x_{2,S} \le x_{\mathsf{R}}$ depending only on S such that $f_L < f_U$ on $(x_{1,S}, x_{2,S})$ but $f_L = f_U$ on $(x_{\mathsf{L}}, x_{1,S}] \amalg [x_{2,S}, x_{\mathsf{R}})$. In other words, the graphs of f_L and f_U on $(x_{1,S}, x_{2,S})$ are the lower and upper boundaries of S, which will be denoted by B_S and A_S, respectively.

For any two bounding pairs (f_L, f_U) and (f'_L, f'_U) for S, we denote by $(f_L, f_U) \le (f'_L, f'_U)$ if $f_L \le f'_L$ and $f_U \le f'_U$. In addition, the pair $(\max\{f_L, f'_L\}, \max\{f_U, f'_U\})$ becomes also a bounding pair for S which is a common upper bound. It follows that there is a *unique maximal bounding pair* for S, which will be denoted by (l_S, u_S).

Definition 5.18. Let $\mathcal{S} = \mathcal{S}(\widetilde{T})$ be a regular cell complex as above. A *poset category* or a *finite (acyclic) quiver with relations*, denoted by $\mathcal{R}(\mathcal{S})$, is defined as follows:

- The objects are the 2-dimensional cells of $\mathcal{S}$;
- For any two objects R and S separated by the 1-dimensional stratum s with S below and R above, we assign an arrow $e_s : S \to R$ that impose $S \le R$ and generates the partial order on $\mathcal{R}(\mathcal{S})$.
- For any crossing or vertex v of $\widetilde{T}$, there are unique regions N and S immediately above and below v and exactly two directed *paths* $\gamma_{\mathsf{L}}(v) : S \to N$ and $\gamma_{\mathsf{R}}(v) : S \to N$ which are compositions of arrows, such that $\gamma_{\mathsf{L}}(v)$ and $\gamma_{\mathsf{R}}(v)$ pass through objects corresponding to the regions in $\mathrm{Star}_{\mathcal{S}}(v)$ and go around v from the left and right hand side, respectively. Then we impose the relation $\gamma_{\mathsf{L}}(v) \sim \gamma_{\mathsf{R}}(v)$.

[4]A function f_L or f_U may be constant at $\pm\infty$ and in this case the region is unbounded.

Moreover, we say that an arrow $e : S \to R$ is *simple* if there are no regions between them. We say that a directed path $\gamma : S \to R$ is simple if it is a composition of simple arrows.

We denote by $\mathrm{A}(\mathcal{R}(\mathcal{S}))$ the quotient of the path algebra $\mathbb{K}\langle\mathcal{R}(\mathcal{S})\rangle$ of the acyclic quiver $\mathcal{R}(\mathcal{S})$ by the ideal generated by $(\gamma_{\mathsf{L}}(v) - \gamma_{\mathsf{R}}(v))$'s

$$\mathrm{A}(\mathcal{R}(\mathcal{S})) \coloneqq \mathbb{K}\langle\mathcal{R}(\mathcal{S})\rangle/(\gamma_{\mathsf{L}}(v) - \gamma_{\mathsf{R}}(v) : v \in V(\mathcal{S})).$$

Here, $V(\mathcal{S})$ is the set of 0-dimensional strata of $\mathcal{S}$.

We remark that while the path algebra of an acyclic quiver is of global cohomological dimension 1, our algebra $\mathrm{A}(\mathcal{R}(\mathcal{S}))$ is of global cohomological dimension 2 in general.

Proposition 5.19. *The poset category $\mathcal{R}(\mathcal{S})$ is well-defined.*

Proof. To show that the definition indeed gives a partial order, we need to check the antisymmetric property, that is, for any two 2-dimensional cells R, S, both $R \leq S$ and $S \leq R$ implies that $R = S$. This follows immediately from the last statement in the lemma below. □

Lemma 5.20. *For any two 2-dimensional cells R, S in $\mathcal{S}$, the following are all equivalent:*

1) $u_R \leq l_S$.
2) *S is above the graph of u_R.*
3) $R < S$.

Moreover, $R \leq S$ if and only if $l_R \leq l_S$.

Proof. (1) ⇒ (2) is clear.
(2) ⇒ (1): If S is above the graph of u_R, then $l_S \geq u_R$ on $[x_{1,S}, x_{2,S}]$. Suppose that $l_S(x) < u_R(x)$ for some x, say $x > x_{2,S}$. Then

$$I_W \coloneqq \{x \in (x_{2,S}, x_{\mathsf{R}}) | l_S(x) < u_R(x)\}$$

is non-empty. Let $x_0 \coloneqq \inf(I_W) \geq x_{2,S}$, then $l_S \geq u_R$ on $[x_{1,S}, x_0]$, $u_S(x_0) = l_S(x_0) = u_R(x_0)$, and the graph of u_R is strictly above that of l_S on a small open interval $(x_0, x_0 + \epsilon)$. Define a pair (l'_S, u'_S), which coincides with (l_S, u_S) on $(x_{\mathsf{L}}, x_0]$, and $l'_S = u'_S = \max\{l_S = u_S, u_R\}$ on $[x_0, x_{\mathsf{R}})$. This defines a bounding pair for S, and $(l_S, u_S) < (l'_S, u'_S)$, contradicting to the maximality of (l_S, u_S). Hence, $l_S(s) \geq u_R(x)$ for all $x \in (x_{\mathsf{L}}, x_{\mathsf{R}})$.

$(3) \Rightarrow (2)$: It suffices to show: If R and S are separated by an 1-dimensional stratum s, with R below and S above. Then S is above the graph of u_R.

In fact, as a 1-dimensional stratum, s is the graph of the function $u_R = l_S$ over some open interval $(x_1, x_2) \subset (x_{\mathsf{L}}, x_{\mathsf{R}})$. Then, S contains points which are above the graph of u_R, hence, the whole disk S is above the graph of u_R.

$(2) \Rightarrow (3)$: $S \neq R$ is clear. It suffices to show $R \leq S$. If the lower boundary of $S_0 = S$ is not contained in the graph of u_R, then it contains a 1-dimensional stratum s_1, which is strictly above the graph of u_R. Let S_1 be the 2-dimensional cell below s_1, then S_1 is above the graph of u_R as well. Also by $(3) \Rightarrow (2)$, S_0 is above the graph of u_{S_1}. By $(2) \Rightarrow (1)$, we then have $u_R \leq l_{S_1} \leq u_{S_1} \leq l_{S_0} \leq u_{S_0}$ in $G_{\mathcal{S}}$, with $l_{S_0} \neq u_{S_0}, l_{S_1} \neq u_{S_1}$. If the lower boundary of S_1 is not contained in the graph of u_R, we repeat the procedure above to obtain s_2, S_2. In particular, S_2 is above the graph of u_R, S_1 is above the graph of u_{S_2}, and $u_R \leq l_{S_2} \leq u_{S_2} \leq l_{S_1} \leq u_{S_1} \leq l_{S_0} \leq u_{S_0}$ in $G_{\mathcal{S}}$, with $l_{S_i} \neq u_{S_i}$ for $i = 0, 1, 2$. Since $G_{\mathcal{S}}$ is finite, after repeat the previous procedure finitely many times, we obtain a finite sequence s_i, S_i for $1 \leq i \leq N$ for some $N \geq 0$, such that, s_i separates the 2-dimensional cells S_{i-1}, S_i with S_{i-1} above and S_i below, $u_{S_i} \leq l_{S_{i-1}}$, $u_R \leq l_{S_N}$, and the lower boundary of S_N is contained in the graph of u_R. The last condition just says that $l_{S_N} = u_R$ on $[x_{1,S_N}, x_{2,S_N}]$.

Let us show that there is a 1-dimensional cell s_{N+1} separating R, S_N with R below and S_N above. Otherwise, the open intervals $(x_{1,R}, x_{2,R})$ and (x_{1,S_N}, x_{2,S_N}) have empty intersection. Say, $x_{2,R} \leq x_{1,S_N}$. Then by definition $x_{2,R}$, have $l_R = u_R = l_{S_N}$ on $[x_{1,S_N}, x_{2,S_N}]$. Define a pair (l'_R, u'_R) such that it coincides with (l_R, u_R) outside $[x_{1,S_N}, x_{2,S_N}]$, and $l'_R = u'_R = u_{S_N}$ on $[x_{1,S_N}, x_{2,S_N}]$. This defines a new bounding pair for R with $(l_R, u_R) < (l'_R, u'_R)$, contradiction. Now, we have shown that $S = S_0 \geq S_1 \geq \ldots \geq S_N \geq R$, hence $S \geq R$.

The last statement of the lemma: "$\Rightarrow$" follows from $(3) \Rightarrow (1)$.

$\Leftarrow$: If $R = S$, we are done. Otherwise, assume $l_R \leq l_S$ and $R \neq S$. In particular, S is above the graph of l_R. Then S is above the graph of u_R as well, as $R \neq S$ is the only 2-dimensional cell bounded by the graphs of $l_R \leq u_R$. Now $(2) \Rightarrow (3)$ implies $R < S$.

This finishes the proof of the lemma. □

For a region $R \in \mathcal{R}(\mathcal{S})$, we define an open subset of M to be the upper half-space of $l_R(x)$

$$M_R := \{(x, z) | x_{\mathsf{L}} < x < x_{\mathsf{R}}, z > l_R(x)\}.$$

Then $M_{R'} \subset M_R$ if and only if $R' \geq R$.

Lemma 5.21. *Let $R \in \mathcal{R}(\mathcal{S})$ be fixed. Then for any $R' > R$ in $\mathcal{R}(\mathcal{S})$ with lower boundary $B_{R'}$, the intersection $B_{R'}(R) := B_{R'} \cap M_R$ is non-empty and contractible.*

Proof. $B_{R'}(R)$ *is non-empty:* Since $R' > R$, by definition, there is a 1-dimensional cell s separating R' and another region S, with R' above and S below, such that $S \geq R$. In particular, S is above the graph of l_R. It follows that s is contained in M_R, hence contained in $B_{R'}(R) = B_{R'} \cap M_R$.
$B_{R'}(R)$ *is contractible.* We use the notations at the beginning of Section 5.3. Recall that $B_{R'}$ is simply the graph of $l_{R'}$ over $(x_{1,R'}, x'_{2,R})$. Moreover, $l_R < u_R$ on $(x_{1,R}, x_{2,R})$ and $l_R = u_R$ outside $(x_{1,R}, x_{2,R})$. Also, by Lemma 5.20, we have $u_R \leq l_{R'}$.

If $B_{R'}(R)$ is not contractible, then $B_{R'}$ has a 1-dimensional cell s contained in the graph of l_R, such that, $B_{R'}$ contains points both on the left and right of s. As R' is above the graph of u_R, s has no points in the lower boundary of R. In other words, s is contained in the graph of $l_R = u_R$ over $(x_{\mathsf{L}}, x_{1,R}]$ or $[x_{2,R}, x_{\mathsf{R}})$, say, the latter. Assume s lives over some open interval $(x_1, x_2) \subset (x_{\mathsf{L}}, x_{\mathsf{R}})$, that is, s is the graph of $l_{R'} = u_R = l_R$ over (x_1, x_2). Then $x_{2,R} \leq x_1$. Moreover, $l_{R'} \neq l_R$ as functions on $[x_2, x_{\mathsf{R}})$, as $B_{R'}$ contains points living over (x_2, x_{R}).

Now, can define a pair $(\widetilde{l}_R, \widetilde{u}_R)$ of continuous functions in $G_{\mathcal{S}}$ such that, it coincides with (l_R, u_R) on $(x_{\mathsf{L}}, x_2]$, and $\widetilde{l}_R = \widetilde{u}_R := \max\{l_{R'}, l_R = u_R\} = l_{R'}$ on $[x_2, x_{\mathsf{R}})$. Then $(\widetilde{l}_R, \widetilde{u}_R)$ is a new bounding pair for R, with $(l_R, u_R) < (\widetilde{l}_R, \widetilde{u}_R)$, contradicting to the maximality of (l_R, u_R). This finishes the proof. $\square$

Now there is an induced functor of poset categories $\rho : \mathcal{S} \to \mathcal{R}(\mathcal{S})$, which sends w to $\rho(w)$, the unique 2-dimensional cell *below* w. More precisely, we define $\rho(w)$ to be w if w is a 2-dimensional stratum, or to be the 2-dimensional cell immediately below w otherwise. Then for each region $R \in \mathcal{R}(\mathcal{S})$, the upper half-space M_R is the union of the strata

$$M_R = \bigcup_{\rho(w) \geq R} w$$

by Lemma 5.20.

As usual, we define the abelian category $Fun(\mathcal{R}(\mathcal{S}), \mathbb{K})$, the DG category $\mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}); \mathbb{K})$, and the restriction functors $\mathcal{F}\mathsf{un}(\mathcal{R}; \mathbb{K}) \to \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}|_{J_x}); \mathbb{K})$ for an open sub-interval J_x of I_x. By pre-composition, we get a functor

$$\rho^* : Fun(\mathcal{R}(\mathcal{S}); \mathbb{K}) \to Fun(\mathcal{S}; \mathbb{K}),$$

which is clearly exact. We can then use the same letter ρ^* for its DG lifting

$$\rho^* : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}); \mathbb{K}) \to \mathcal{F}\mathsf{un}(\mathcal{S}; \mathbb{K}).$$

Proposition/Definition 5.22. *Let us define a functor between two abelian categories*

$$\rho_* : Fun(\mathcal{S}, \mathbb{K}) \to Fun(\mathcal{R}(\mathcal{S}), \mathbb{K}), \quad F \mapsto [R \mapsto \Gamma(M_R; i_{\mathcal{S}}(F))].$$

Then $\rho_* \circ \rho^* = \mathrm{Id}$ *and* (ρ^*, ρ_*) *is an adjoint pair*

$$\rho^* : Fun(\mathcal{R}(\mathcal{S}), \mathbb{K}) \rightleftarrows Fun(\mathcal{S}, \mathbb{K}) : \rho_*.$$

As a consequence, we obtain an adjoint pair $(\rho^*, R\rho_*)$ *in the DG liftings*

$$\rho^* : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}), \mathbb{K}) \rightleftarrows \mathcal{F}\mathsf{un}(\mathcal{S}, \mathbb{K}) : R\rho_*$$

where $R\rho_*$ *is given by* $F \mapsto [R \mapsto R\Gamma(M_R; i_{\mathcal{S}}(F))]$*. Moreover, we have a natural isomorphism* $\beta : \mathrm{Id} \overset{\sim}{\Rightarrow} R\rho_* \circ \rho^*$*.*

Proof. We firstly show that $\rho_* \circ \rho^* = \mathrm{Id}$. By definition and formula (5.3), for any $F \in Fun(\mathcal{R}(\mathcal{S}), \mathbb{K})$ and $R \in \mathcal{R}(\mathcal{S})$, we have

$$(\rho_* \circ \rho^*)(F)(R) = \Gamma(M_R; i_{\mathcal{S}}(\rho^* F)) = \varprojlim_{w \in \mathcal{S}|_{M_R}} (\rho^* F)(\tau_{M_R}(w)).$$

Since $\mathcal{S}|_{M_R} = \{w \in \mathcal{S} | \rho(w) \geq R\}$ and τ_{M_R} is just the inclusion map, $(\rho^* F)(\tau_{M_R}(w)) = F(\rho(w))$ for all $w \in \mathcal{S}|_{M_R}$ and it follows that

$$(\rho_* \circ \rho^*)(F)(R) = \varprojlim_{\rho(w) \geq R} F(\rho(w)) \cong F(R).$$

Therefore $\rho_* \circ \rho^* = \mathrm{Id}$.

Next, we show that (ρ^*, ρ_*) is an adjoint pair. For any $F \in Fun(\mathcal{R}(\mathcal{S}), \mathbb{K})$ and $G \in Fun(\mathcal{S}, \mathbb{K})$, a morphism $f \in \mathrm{Hom}(\rho^* F, G)$ is a collection of maps $\{f_w \mid w \in \mathcal{S}\}$ with $f_w : (\rho^* F)(w) = F(\rho(w)) \to G(w)$ such that for any arrow $w_1 \to w_2$, we get a commutative diagram

$$\begin{array}{ccc} F(\rho(w_1)) & \xrightarrow{f_{w_1}} & G(w_1) \\ \downarrow & & \downarrow \\ F(\rho(w_2)) & \xrightarrow{f_{w_2}} & G(w_2) \end{array}$$

Equivalently, we get a collection of maps $\{f_{R,w} \mid \rho(w) \geq R\}$

$$f_{R,w} : F(R) \to F(\rho(w)) = (\rho^* F)(w) \xrightarrow{f_w} G(w)$$

for any fixed region R in $\mathcal{R}(\mathcal{S})$ and $w \in \mathcal{S}|_{M_R}$, i.e., $\rho(w) \geq R$ such that we get a commutative diagram

$$\begin{array}{ccc} F(R_1) & \xrightarrow{f_{R_1,w_1}} & G(w_1) \\ \downarrow & & \downarrow \\ F(R_2) & \xrightarrow{f_{R_2,w_2}} & G(w_2) \end{array}$$

for any $R_1 \leq R_2$ in $\mathcal{R}(\mathcal{S})$ and $w_1 \leq w_2$ in $\mathcal{S}$ with $\rho(w_i) \geq R_i$. This is again equivalent to have a collection of maps $\{\widetilde{f}_R \mid R \in \mathcal{R}(\mathcal{S})\}$ with

$$\widetilde{f}_R = \varprojlim_{\rho(w) \geq R} f_{R,w} : F(R) \to \varprojlim_{\rho(w) \geq R} G(w),$$

such that for any arrow $R_1 \to R_2$ in $\mathcal{R}(\mathcal{S})$, there is a commutative diagram

$$\begin{array}{ccc} F(R_1) & \xrightarrow{\widetilde{f}_{R_1}} & \varprojlim_{\rho(w_1) \geq R_1} G(w_1) \\ \downarrow & & \downarrow \\ F(R_2) & \xrightarrow{\widetilde{f}_{R_2}} & \varprojlim_{\rho(w_2) \geq R_2} G(w_2). \end{array}$$

Since $(\rho_* G)(R) = \Gamma(M_R; i_{\mathcal{S}}(G)) = \varprojlim_{\rho(w) \geq R} G(w)$ by formula (5.3), the above data is identical to a morphism $\widetilde{f}$ in $\mathrm{Hom}(F, \rho_* G)$. That is, we have a natural isomorphism $\mathrm{Hom}(\rho^* F, G) \simeq \mathrm{Hom}(F, \rho_* G)$.

Finally, we show the natural isomorphism $\beta : \mathrm{Id} \stackrel{\sim}{\Rightarrow} R\rho_* \circ \rho^*$. Indeed, for any $F^\bullet \in \mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ and any region R in $\mathcal{R}(\mathcal{S})$, we have a natural morphism

$$\beta_{F^\bullet}(R) : F^\bullet(R) \to (R\rho_* \circ \rho^* F^\bullet)(R) \tag{5.5}$$

defined by

$$F^\bullet(R) = (\rho_* \circ \rho^* F^\bullet)(R) = \Gamma(M_R; i_{\mathcal{S}} \rho^* F^\bullet) \to R\Gamma(M_R; i_{\mathcal{S}} \rho^* F^\bullet).$$

We want to show that $\beta_{F^\bullet}(R)$ is a quasi-isomorphism.

For any region R' in $\mathcal{R}(\mathcal{S})$, let $\delta'_{R'} \in \mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ be a functor defined by $\delta'_{R'}(W) = 0$ if $W \neq R$, and $\delta'_{R'}(R') = \mathbb{K}$ such that $\delta'_{R'}$ sends all the arrows in $\mathcal{R}(\mathcal{S})$ to zero. As seen in the proof of Lemma 5.11, $\mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ is the triangulated envelope of the objects of the form $\delta'_{R'}$. Observe that for fixed R, the property of $\beta_{F^\bullet}(R)$ in (5.5) being a quasi-isomorphism is preserved under taking quasi-isomorphisms, shifts, and cones in the place of $F^\bullet$. Thus, it suffices to show that $\beta_{F^\bullet}(R)$ is a quasi-isomorphism only when $F^\bullet = \delta'_{R'}$ for $R' \in \mathcal{R}(\mathcal{S})$.

By definition, we see that $i_{\mathcal{S}}\rho^*\delta'_{R'} = \mathbb{K}_{E_{R'}}$, where $E_{R'}$ is the union of R' and its upper boundary $A_{R'}$. It suffices to show that

$$R\Gamma(M_R; \mathbb{K}_{E_{R'}}) \simeq \begin{cases} \mathbb{K} & R' = R, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

If R' is not contained in M_R, then neither is $E_{R'}$ ans we have $R\Gamma(M_R; \mathbb{K}_{E_{R'}}) \simeq 0$. Otherwise, if $R' \subset M_R$, then so is $E_{R'}$. If $R' \neq R$, by Lemma 5.21, we have $\overline{R'}(R) := \overline{R'} \cap M_R = E_{R'} \amalg B_{R'}(R)$, where both of $\overline{R'}(R)$ and $B_{R'}(R)$ are non-empty contractible closed subsets of M_R. We then obtain the following exact triangle:

$$R\Gamma(M_R; \mathbb{K}_{E_{R'}}) \to R\Gamma(M_R; \mathbb{K}_{\overline{R'}(R)}) \to R\Gamma(M_R; \mathbb{K}_{B_{R'}(R)}) \xrightarrow{+1},$$

where the second arrow is the same as the quasi-isomorphism $R\Gamma(\overline{R'}(R); \mathbb{K}) \simeq \mathbb{K} \to R\Gamma(B_{R'}(R); \mathbb{K}) \simeq \mathbb{K}$. Therefore, $R\Gamma(M_R; \mathbb{K}_{E_{R'}})$ is acyclic. Finally, if $R' = R$, then E_R is a contractible closed subset of M_R and it follows that $R\Gamma(M_R; \mathbb{K}_{E_R}) \simeq R\Gamma(E_R; \mathbb{K}) \simeq \mathbb{K}$ as desired. □

Definition 5.23. Let $\mathfrak{Fun}_T(\mathcal{R}(\mathcal{S}), \mathbb{K})$ and $\mathfrak{Fun}_T(\mathcal{R}(\mathcal{S}), \mathbb{K})_0$ be full subcategories of $\mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ consisting of functors F satisfying the conditions (1)–(2) and (1)–(3) described below:

1) For any additional 1-dimensional stratum s of $\mathcal{S}$ separating two 2-dimensional cells R_1 below and R_2 above, the morphism $F(e_s) : F(R_1) \xrightarrow{\sim} F(R_2)$ is a quasi-isomorphism.

2) Around any crossing c of T, we label the 4 regions by N, S, W and E as in Definition 5.14 and there exists the following commutative diagram

$$\begin{array}{ccccc} & & F(N) & & \\ & \nearrow & & \nwarrow & \\ F(W) & & & & F(E) \\ & \nwarrow & & \nearrow & \\ & & F(S) & & \end{array}$$

such that the total complex of $F(S) \to F(W) \oplus F(E) \to F(N)$ is acyclic.

3) If R is the bottom region in $\mathcal{S}$, then $F(R)$ is acyclic.

Proposition 5.24 (Legible model). *Let $\mathcal{S}$ be a regular cell complex refining the stratification induced by $\mathcal{T}$ and satisfying Assumption 5.17. Then*

$$\rho^* : \mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K}) \xrightarrow{\sim} \mathfrak{Fun}_{\widetilde{T}^+}(\mathcal{S}, \mathbb{K}) \tag{5.7}$$

is a quasi-equivalence with a quasi-inverse $R\rho_*$, and ρ^* is compatible with restriction to an open sub-interval. Moreover, the similar statement holds for $\rho^* : \mathfrak{Fun}_T(\mathcal{R}(\mathcal{S}), \mathbb{K})_{(0)} \xrightarrow{\sim} \mathfrak{Fun}_T(\mathcal{S}, \mathbb{K})_{(0)}$.

For the proof of the proposition, we need the following lemma.

Lemma 5.25. *For any $\mathcal{F} \in \mathfrak{Sh}_{\widetilde{T}^+}(M; \mathbb{K})$, there exists a filtration*

$$0 = \mathcal{F}_m \to \mathcal{F}_{m-1} \to \cdots \to \mathcal{F}_0 = \mathcal{F}$$

such that each of the associated graded pieces $\mathrm{Gr}_i \mathcal{F}_\bullet$ is contained in $\mathfrak{Sh}_{\widetilde{T}^+}(M; \mathbb{K})$ and supported on a single region R_i of $\mathcal{S}$.

More precisely, for each E_i which is the union of R_i and its upper boundary, we have $\mathrm{Gr}_i \mathcal{F}_\bullet \simeq (A_i)_{E_i}$ for some perfect complex of $\mathbb{K}$-modules A_i regarded as a constant sheaf on M.

Proof. We use the notations at the beginning of Section 5.3. For each function f in $G_{\mathcal{S}}$, we define an open subset of M

$$M_f \coloneqq \{(x, z) \in M \mid x \in (x_{\mathsf{L}}, x_{\mathsf{R}}), z > f(x)\}.$$

Consider the maximal bounding pair (l_{R_1}, u_{R_1}) for the bottom region R_1 in $\mathcal{S}$. Then $f_0 \coloneqq l_{R_1} \equiv -\infty$ and $f_1 \coloneqq u_{R_1}$ is the unique minimum in $G_{\mathcal{S}} \setminus \{f_0\}$. In particular, $M_{f_0} = M$.

If M_{f_1} is non-empty, then $\mathcal{R}(\mathcal{S}|_{M_{f_1}})$ is a finite non-empty partially ordered subset of $\mathcal{R}(\mathcal{S})$ which contains a minimum, say R_2. In other words, R_2 is a minimum among those regions of $\mathcal{S}$ which are above the graph of f_1. Then the lower boundary of R_2 is contained in the graph of f_1. Otherwise, the lower boundary contains a 1-dimensional stratum s which is strictly above the graph of f_1. The region $S = \rho(s)$ below s is less than R_2 and this is a contradiction to the minimality of R_2.

Now we take $f_2 \in G_{\mathcal{S}}$ so that (f_1, f_2) is a bounding pair of R_2. Then by induction, we can repeat the procedure above to obtain a sequence (f_i) in $G_{\mathcal{S}}$ such that for each i, the graphs of f_{i-1} and f_i bound a single region R_i. Since $G_{\mathcal{S}}$ is finite, there exists a $m \in \mathbb{N}$ such that $f_m = \infty$ is the unique maximum in $G_{\mathcal{S}}$. Hence, $M_{f_m} = \varnothing$ and the procedure stops. In fact, m is the number of regions in $\mathcal{S}$.

Let $M_i \coloneqq M_{f_i}$. We obtain a sequence of open inclusions

$$\varnothing = M_m \subset M_{m-1} \subset \cdots \subset M_0 = M$$

and $\mathcal{F}_i \coloneqq \mathcal{F}_{M_i} = R(j_{M_i})_! j_{M_i}^{-1}\mathcal{F}$, where $j_{M_i} : M_i \to M$ is the open inclusion. By [17, Prop.5.4.8.(ii)], we have $\mathcal{F}_i \in \mathcal{S}\mathsf{h}_{T^+(\mathcal{S})}(M;\mathbb{K})$ and obtain a filtration

$$0 = \mathcal{F}_m \to \mathcal{F}_{m-1} \to \cdots \to \mathcal{F}_0 = \mathcal{F},$$

whose associated i-th graded piece is $\mathrm{Gr}_i\,\mathcal{F}_\bullet = \mathcal{F}_{M_{i-1}\setminus M_i}$ and induced by the exact triangle

$$\mathcal{F}_{M_i} \to \mathcal{F}_{M_{i-1}} \to \mathcal{F}_{M_{i-1}\setminus M_i} \xrightarrow{+1}$$

By definition, $\mathrm{Gr}_i\,\mathcal{F}_\bullet$ is supported in a single region R_i. More precisely, it has possibly non-zero stalks only at points in the region R_i and its upper boundary. Moreover, by the triangular inequality for micro-supports, we have $\mathrm{Gr}_i\,\mathcal{F}_\bullet \in \mathcal{S}\mathsf{h}_{\widetilde{\mathcal{T}}^+}(M;\mathbb{K})$. In fact,

$$\mathrm{Gr}_i\,\mathcal{F}_\bullet \simeq \mathcal{F}_{M_{i-1}\setminus M_i} \simeq (F(R_i))_{M_{i-1}\setminus M_i},$$

where the complex $F(R_i) = R\Gamma(R_i;\mathcal{F})$ is regarded as a constant sheaf on M. □

Proof of Proposition 5.24. The proof is similar to that of [25, Prop.3.22], the only nontrivial part is to show the quasi-equivalence (5.7).

At first, let us show that ρ^* is fully faithful. For any $F, G \in \mathsf{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K})$, by the adjunction $(\rho^*, R\rho_*)$ in Proposition/Definition 5.22, we have

$$\mathrm{RHom}^\bullet(\rho^* F, \rho^* G) \simeq \mathrm{RHom}^\bullet(F, R\rho_*\rho^* G) \simeq \mathrm{RHom}^\bullet(F, G),$$

where the last quasi-isomorphism follows from the natural isomorphism $\beta : \mathrm{Id} \overset{\sim}{\Rightarrow} R\rho_* \circ \rho^*$ in Proposition/Definition 5.22, which then implies the quasi-isomorphism $G \simeq R\rho_* \circ \rho^*(G)$. Thus, ρ^* is fully faithful. The natural isomorphism also shows that $R\rho_*$ is essentially surjective.

Now, it suffices to show that ρ^* is essentially surjective. For any functor $F \in \mathsf{Fun}(\mathcal{S}; \mathbb{K})$, let $\mathcal{F} \coloneqq i_{\mathcal{S}}(F)$. Then by Proposition/Definition 5.10 and Corollary 5.7, we have

$$R\Gamma(\mathrm{Star}(w); \mathcal{F}) \simeq \mathcal{F}_x = F(w) = \Gamma(\mathrm{Star}(w); \mathcal{F})$$

for all $x \in w \in \mathcal{S}$. The adjunction $(\rho^*, R\rho_*)$ gives us an adjunction map $\epsilon_F : \rho^* R\rho_* F \to F$, where for any $w \in \mathcal{S}$,

$$\rho^* R\rho_* F(w) = R\rho_* F(\rho(w)) = R\Gamma(M_{\rho(w)}; \mathcal{F}),$$

and by definition of $M_{\rho(w)}$, we have $\mathrm{Star}(w) \subset M_{\rho(w)}$ and therefore

$$\epsilon_F(w) : \rho^* R\rho_* F(w) = R\Gamma(M_{\rho(w)}; \mathcal{F}) \to R\Gamma(\mathrm{Star}(w); \mathcal{F}) \simeq F(w)$$

defined by the restriction of sections.

Hence it suffices to show that $\epsilon_F : \rho^* R\rho_* F \xrightarrow{\sim} F$ is a quasi-isomorphism for all $F \in \mathsf{Fun}_{T^+}(\mathcal{S}; \mathbb{K})$, or equivalently, the restriction

$$R\Gamma(M_{\rho(w)}; \mathcal{F}) \xrightarrow{\sim} R\Gamma(\mathrm{Star}(w); \mathcal{F}) \tag{5.8}$$

is a quasi-isomorphism for any $w \in \mathcal{S}$.

Now we apply Lemma 5.25 to $\mathcal{F}$ to reduce the proof of the quasi-isomorphism (5.8) to the case when $m = 1$, that is, when the sheaf $\mathcal{F}$ is supported on a single region R' and $\mathcal{F} \simeq A_{E_{R'}}$ for some perfect complex of $\mathbb{K}$-modules A with $E_{R'}$ the union of R' and its upper boundary.

However, for $\mathcal{F} \simeq A_{E_{R'}}$, we see that $R\Gamma(\mathrm{Star}(w); \mathcal{F}) \simeq \mathcal{F}_x \simeq \mathcal{F}_y$ for all $w \in \mathcal{S}$ and $x \in w, y \in \rho(w)$ by Corollary 5.7. Therefore

$$R\Gamma(\mathrm{Star}(w); A_{E_{R'}}) \simeq \begin{cases} A & R' = \rho(w), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the same holds for $R\Gamma(M_{\rho(w)}; A_{E_{R'}})$ as seen in (5.6) after replacing $\mathbb{K}$ with A in the same argument. This completes the proof of Proposition 5.24. □

We can further simplify the legible model for sheaf categories by taking resolutions from the knowledge of quiver representations [4]. Indeed, the category $\mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ is nothing but a representation category of the quiver with relations $\mathcal{R}(\mathcal{S})$, i.e., the representation category of $\mathrm{A}(\mathcal{R}(\mathcal{S}))$ with values on perfect complexes (i.e. complexes which are quasi-isomorphic to a bounded complex of finite projective $\mathbb{K}$-modules). Now, for any $F \in \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}), \mathbb{K})$, we can replace F by either a projective or injective resolution.

Let us review a key fact about $\mathrm{A}(\mathcal{R}(\mathcal{S}))$ which we will use later on. Recall some terminology in quiver representation theory. Let us denote by $\mathcal{R}(\mathcal{S})_0$ and $\mathcal{R}(\mathcal{S})_1$ the sets of objects and arrows respectively. For any object $a \in \mathcal{R}(\mathcal{S})_0$, the path of length 0 at a will be denote by λ_a. Then $\mathrm{A}(\mathcal{R}(\mathcal{S}))$ is a free $\mathbb{K}$-module with the basis the set of all paths (of lengths ≥ 0) in $\mathcal{R}(\mathcal{S})$ modulo the relation in Definition 5.18.

Proposition 5.26. *Let $\mathcal{S}$ be a regular cell complex refining the stratification induced by $\mathcal{T}$ and satisfying Assumption 5.17. Then for $\mathrm{A} := \mathrm{A}(\mathcal{R}(\mathcal{S}))$, we have the following characterizations for indecomposable (left) projective and injective modules:*

1) *The indecomposable (left) projective modules of A are $P_a := \mathrm{A}\lambda_a$ with $a \in \mathcal{R}(\mathcal{S})_0$. In particular, $P_a(s)$ is injective for all arrows s in $\mathcal{R}(\mathcal{S})_1$.*
2) *The indecomposable (left) injective modules of A are $I_a := \mathrm{Hom}_{\mathbb{K}}(\lambda_a \mathrm{A}, \mathbb{K})$ with $a \in \mathcal{R}(\mathcal{S})_0$. In particular, $I_a(s)$ is surjective for all arrows s in $\mathcal{R}(\mathcal{S})_1$.*
3) *The (left) simple modules of A are S_a with $a \in \mathcal{R}(\mathcal{S})_0$, where S_a consists of $\mathbb{K}$ at the object a and 0 otherwise, and S_a sends all the arrows in $\mathcal{R}(\mathcal{S})_1$ to zero.*

Proof. This is a standard fact in quiver representation theory. See [4, II.2. Lem.2.4, III.2.Lem.2.1]. □

5.4. A combinatorial proof of invariance of sheaf categories

We can now give an alternative proof of Theorem 5.3 via the combinatorial descriptions of sheaf categories, i.e. Lemmas 5.11 and 5.15, Propositions 5.24 and 5.26.

Proof of Theorem 5.3. It suffices to show the invariance of $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})$ under the 6 types of Legendrian Reidemeister moves in the front projection. First observe that the diagram $\mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})$ is unchanged if we regard each cusp in $\pi_{xz}(T)$ as a 2-valency vertex. Then the Legendrian Reidemeister moves (I) and (IV) are special cases of move (VI), and move (II) is a special case of move (V). So it suffices to show the invariance under moves (III), (V), and (VI). Let T, T' be any pair of bordered Legendrian graphs in $T^{\infty,-}M$, which differ by any one of the above three moves. We can take a pair of regular cell complexes $\mathcal{S}, \mathcal{S}'$ refining the stratifications of M induced by T, T' respectively, such that:

1) $\mathcal{S}, \mathcal{S}'$ satisfy Assumption 5.17 for T, T' respectively.
2) $\mathcal{S}, \mathcal{S}'$ coincide outside the local bordered Legendrian graphs involving the Legendrian Reidemeister move.

Denote by $\mathcal{S}_{\mathsf{L}} = \mathcal{S}'_{\mathsf{L}}$, $\mathcal{S}_{\mathsf{R}} = \mathcal{S}'_{\mathsf{R}}$ the induced stratifications of $\mathcal{S}$ (equivalently, $\mathcal{S}'$) near the left and right boundary of $M = I_x \times \mathbb{R}_z$ respectively. In other words, say T_{L} (resp. T_{R}) lives in over the interval $I_{\mathsf{L}} = (x_{\mathsf{L}}, x_{\mathsf{L}} + \epsilon) \subset I_x$ (resp. $I_{\mathsf{R}} = (x_{\mathsf{R}} - \epsilon, x_{\mathsf{R}}) \subset I_x$), then $\mathcal{S}_{\mathsf{L}} = \mathcal{S}|_{I_{\mathsf{L}}}$ (resp. $\mathcal{S}_{\mathsf{R}} = \mathcal{S}|_{I_{\mathsf{R}}}$) is the stratification with strata the connected components of $S \cap I_{\mathsf{L}} \times \mathbb{R}_z$ (resp. $S \cap I_{\mathsf{R}} \times \mathbb{R}_z$) for all $S \in \mathcal{S}$, as defined at the beginning of Section 5.2.

Apply Lemmas 5.11 and 5.15 and Proposition 5.24, it suffices to show, for each of three moves above, the equivalence between the 2 diagrams of DG categories:

$$\mathcal{F}\mathsf{un}_{\mathcal{T}}(\mathcal{R}(\mathcal{S}),\mathbb{K}) := (\mathcal{F}\mathsf{un}_{T_{\mathsf{L}}}(\mathcal{R}(\mathcal{S}_{\mathsf{L}}),\mathbb{K}) \leftarrow \mathcal{F}\mathsf{un}_{T}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \\ \rightarrow \mathcal{F}\mathsf{un}_{T_{\mathsf{R}}}(\mathcal{R}(\mathcal{S}_{\mathsf{R}}),\mathbb{K})),$$

and

$$\mathcal{F}\mathsf{un}_{\mathcal{T}'}(\mathcal{R}(\mathcal{S}'),\mathbb{K}) = (\mathcal{F}\mathsf{un}_{T'_{\mathsf{L}}}(\mathcal{R}(\mathcal{S}'_{\mathsf{L}}),\mathbb{K}) \leftarrow \mathcal{F}\mathsf{un}_{T'}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \\ \rightarrow \mathcal{F}\mathsf{un}_{T'_{\mathsf{R}}}(\mathcal{R}(\mathcal{S}'_{\mathsf{R}}),\mathbb{K})).$$

This can essentially be shown by the diagrams in Figures 14, 15 and 16.

5.4.1. Move (III). If T, T' differ by a move (III), can take a pair of regular cell complexes $\mathcal{S}, \mathcal{S}'$ as above so that, the local bordered Legendrian graphs involving the move (III) are as in Figures 14(1) and (4), respectively. In the figure, denote by $T_i, \mathcal{S}_i$ the induced bordered Legendrian graphs and regular cell complexes refining $\mathcal{S}_{T_i}$ in (i), for $1 \leq i \leq 4$. In particular, we have $T_1 := T, T_4 := T', T_2 = T_3$, and $\mathcal{S}_1 = \mathcal{S}, \mathcal{S}_4 = \mathcal{S}', \mathcal{S}_2 = \mathcal{S}_3$, and all the $(T_i, \mathcal{S}_i)$'s coincide outside the local bordered Legendrian graphs in the figure. The letters in each picture label the regions.

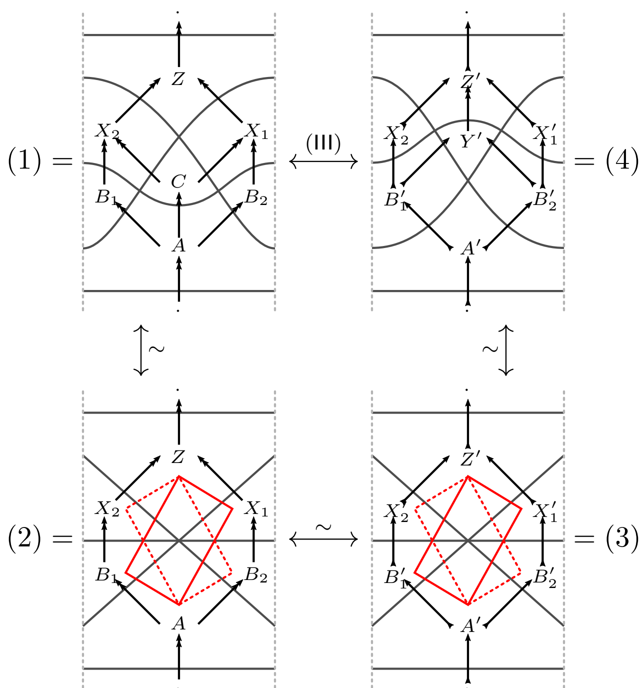


Figure 14: Invariance of sheaf categories: Legendrian Reidemeister move (III).

Proposition/Definition 5.27. *There is an adjunction* (i, π) *of functors between two abelian categories*

$$i : Fun(\mathcal{R}(\mathcal{S}_1), \mathbb{K}) \rightleftarrows Fun(\mathcal{R}(\mathcal{S}_2), \mathbb{K}) : \pi$$

defined as follows:

1) *$i(F_1)$ forgets $F_1(C)$ in (1);*
2) *$\pi(F_2)$ keeps the same data as F_2 on other regions, and $\pi(F_2)(C) \coloneqq F_2(X_1) \times_{F_2(Z)} F_2(X_2) = \ker(F_2(X_1) \oplus F_2(X_2) \xrightarrow{(+,-)} F_2(Z))$. Then the map $\pi(F_2)(A \to C)$ is uniquely determined by the universal property of $\pi(F_2)(C)$ as a kernel.*

Moreover, both of i and π are exact, and $i \circ \pi = id$. As a consequence, we obtain an adjunction (i, π) of DG functors in the DG lifting

$$i : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_1), \mathbb{K}) \rightleftarrows \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_2), \mathbb{K}) : \pi$$

and we get a natural isomorphism $\beta : i \circ \pi \overset{\sim}{\Rightarrow} \mathrm{Id}$, and $\pi : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_2), \mathbb{K}) \to \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_1), \mathbb{K})$ is fully faithful.

Proof. Everything is done by a direct check except the last statement, that is, π is fully faithful, which can be shown as follows. For any $F, G \in \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_2), \mathbb{K})$, by the adjunction (i, π), we have

$$\mathrm{RHom}^\bullet(\pi(F), \pi(G)) \simeq \mathrm{RHom}^\bullet(i \circ \pi(F), G) \simeq \mathrm{RHom}^\bullet(F, G)$$

where the last quasi-isomorphism follows from the natural isomorphism $\beta : i \circ \pi \overset{\sim}{\Rightarrow} \mathrm{Id}$, which gives the quasi-isomorphism $\beta_F : i \circ \pi(F) \xrightarrow{\sim} F$. □

Lemma 5.28. *Let $\mathcal{D}_1 \coloneqq \mathcal{F}\mathsf{un}_{T_1}(\mathcal{R}(\mathcal{S}_1), \mathbb{K})$ be the DG category in (1), and $\mathcal{D}_2$ be the DG full subcategory of $\mathcal{F}\mathsf{un}_{T_2}(\mathcal{R}(\mathcal{S}_2), \mathbb{K})$ whose objects are functors F_2 such that two additional crossing conditions induced by the two red squares in (2) hold, i.e. both of the total complexes $\mathrm{Tot}(F_2(A) \to F_2(B_1) \oplus F_2(X_1) \xrightarrow{(+,-)} F_2(Z))$ and $\mathrm{Tot}(F_2(A) \to F_2(B_2) \oplus F_2(X_2) \xrightarrow{(+,-)} F_2(Z))$ are acyclic. Then the adjunction (i, π) of DG functors in Proposition/Definition 5.27 induces equivalences*

$$i : \mathcal{D}_1 \overset{\sim}{\rightleftarrows} \mathcal{D}_2 : \pi$$

which are quasi-inverses to each other.

Proof. Firstly, the adjunction of DG functors (i, π) in Proposition/Definition 5.27 induces an adjunction of DG functors $i : \mathcal{D}_1 \rightleftarrows \mathcal{D}_2 : \pi$.

To show this, it suffices to show that the essential image of $\mathcal{D}_1$ under i is contained in $\mathcal{D}_2$, and the essential image of $\mathcal{D}_2$ under π is contained in $\mathcal{D}_1$. The former is clear, as for example, for any $F_1 \in \mathcal{D}_1$, the crossing conditions for F_1 at the crossing above and the crossing to the left of C in (1) implies the first crossing condition for $i(F_2)$, i.e. the total complex $\mathrm{Tot}(F_2(A) \to F_2(B_1) \oplus F_2(X_1) \xrightarrow{(+,-)} F_2(Z))$ is acyclic. The latter essentially follows from Proposition 5.26. More precisely, for any $F_2 \in \mathcal{D}_2 \subset \mathfrak{Fun}_{T_2}(\mathcal{R}(\mathcal{S}), \mathbb{K})$, by the proposition, F_2 is quasi-isomorphic to an object in which all the arrows are sent to surjections. So, we can assume F_2 itself has this property. Now, the crossing condition for $\pi(F_2)$ at the crossing above C is automatic, and the crossing conditions for F_2 is equivalent to the natural cochain maps $F_2(A) \xrightarrow{\sim} \ker(F_2(B_1) \oplus F_2(X_1) \xrightarrow{(+,-)} F_2(Z))$, and $F_2(A) \xrightarrow{\sim} \ker(F(B_2) \oplus F_2(X_2) \xrightarrow{(+,-)} F_2(Z))$ being quasi-isomorphisms, which are equivalent to the crossing conditions for $\pi(F_2)$ at the crossings to the left and to the right of C. Hence, $\pi(F_2) \in \mathcal{D}_2$.

Now, the natural isomorphism $\beta : i \circ \pi \overset{\sim}{\Rightarrow} \mathrm{Id} : \mathcal{D}_2 \to \mathcal{D}_2$ implies that $i : \mathcal{D}_1 \to \mathcal{D}_2$ is essentially surjective. By Proposition/Definition 5.27, $\pi : \mathcal{D}_2 \to \mathcal{D}_1$ is fully faithful. It then suffices to show π is essentially surjective.

For any $F_1 \in \mathfrak{Fun}(\mathcal{R}(\mathcal{S}_1), \mathbb{K})$, by Proposition 5.26, F_1 is quasi-isomorphic to an object in which all arrows are sent to surjections. We can then assume F_1 itself has this property. Now, the crossing conditions for F_1 in the local bordered Legendrian graph are equivalent to the natural quasi-isomorphisms $F_1(C) \xrightarrow{\sim} \ker(F_1(X_1) \oplus F_1(X_2) \xrightarrow{(+,-)} F_1(Z))$, $F_1(A) \xrightarrow{\sim} \ker(F_1(B_1) \oplus F_1(C) \xrightarrow{(+,-)} F_1(X_2))$, and $F_1(A) \xrightarrow{\sim} \ker(F_1(B_2) \oplus F_1(C) \xrightarrow{(+,-)} F_1(X_1))$, equivalently, $F_1(C) \xrightarrow{\sim} \pi \circ i(F_1)(C)$, and the two crossing conditions induced by the two red squares in (2) defining $i(F_1) \in \mathcal{D}_2$. Hence, we get a natural quasi-isomorphism $\alpha_{F_1} : F_1 \xrightarrow{\sim} \pi \circ i(F_1)$. In particular, $\pi : \mathcal{D}_2 \to \mathcal{D}_1$ is essentially surjective. □

Similar to $\mathcal{D}_2$, let $\mathcal{D}_3$ be the corresponding DG category defined by (3). Then (2) and (3) are identical, that is, $\mathcal{D}_3 = \mathcal{D}_2$. We just have changed the letters labelling the regions from "X" to "X'", to make it convenient for us to compare with (4).

Let $\mathcal{D}_4 := \mathfrak{Fun}_{T_4}(\mathcal{R}(\mathcal{S}_4), \mathbb{K})$ be the DG category in (4). By a dual argument to that in proving the equivalence between $\mathcal{D}_1$ and $\mathcal{D}_2$, we immediately obtain equivalences

$$p : \mathcal{D}_3 \overset{\sim}{\rightleftarrows} \mathcal{D}_4 : j$$

which are quasi-inverses to each other. Here j is induced from the forgetful functor $j : Fun(\mathcal{R}(\mathcal{S}_4), \mathbb{K}) \to Fun(\mathcal{R}(\mathcal{S}_3), \mathbb{K})$ with $j(F_4)$ forgets $F_4(Y')$, and p is induced from the functor $p : Fun(\mathcal{R}(\mathcal{S}_3), \mathbb{K}) \to Fun(\mathcal{R}(\mathcal{S}_4), \mathbb{K})$ with $p(F_3)(Y') = \mathrm{Coker}(F_3(A') \xrightarrow{(+,-)^t} F_3(B_1') \oplus F_3(B_2'))$.

Now, by composition, we then get an equivalence $\mathcal{F}\mathsf{un}_T(\mathcal{R}(\mathcal{S}),\mathbb{K}) \simeq \mathcal{F}\mathsf{un}_{T'}(\mathcal{R}(\mathcal{S}),\mathbb{K})$. Notice that the composition sends the object $F|_{T_\mathsf{L}}$, F restricted to T_L, to $F'|_{T'_\mathsf{L}}$ which is quasi-isomorphic to $F_{T'_\mathsf{L}=T_\mathsf{L}}$. Hence, the equivalence commutes with $\mathrm{Id} : \mathcal{F}\mathsf{un}_{T_\mathsf{L}}(\mathcal{R}(\mathcal{S}_\mathsf{L}),\mathbb{K}) \xrightarrow{\sim} \mathcal{F}\mathsf{un}_{T'_\mathsf{L}}(\mathcal{R}(\mathcal{S}'_\mathsf{L}),\mathbb{K})$ up to a specified natural isomorphism. The same holds for $T_\mathsf{R} = T'_\mathsf{R}$. Therefore, we get an equivalence of diagrams of DG categories: $\mathcal{F}\mathsf{un}_{\mathcal{T}}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \simeq \mathcal{F}\mathsf{un}_{\mathcal{T}'}(\mathcal{R}(\mathcal{S}),\mathbb{K})$, as desired.

5.4.2. Move (V). The proof proceeds similarly as above. If T, T' differ by a move (V), can take the pair of regular cell complexes $\mathcal{S}, \mathcal{S}'$ so that the local bordered Legendrian graphs involving the move are as in Figures 15(1) and (4), respectively. In the picture, T and T' both have l left and r right half-edges at the vertex, labelled by $1, \dots, l$ and $l+1, \dots, l+r$ from top to bottom respectively. In addition, the bordered Legendrian graph $T(\mathcal{S})$ (resp. $T(\mathcal{S}')$) underlying $\mathcal{S}$ (resp. $\mathcal{S}'$) is T (resp. T') plus the additional dashed arcs. We have added the dashed arcs to ensure that S, S' satisfy Assumption 5.17. As before, let $T_i, \mathcal{S}_i$ be the induced bordered Legendrian graphs and corresponding regular cell complexes in (i), for $1 \le i \le 4$. In particular, $(T_1, \mathcal{S}_1) = (T, \mathcal{S}), (T_4, \mathcal{S}_4) = (T', \mathcal{S}_4)$, and $(T_2, \mathcal{S}_2) = (T_3, \mathcal{S}_3)$.

Similar to Proposition/Definition 5.27, we have the following proposition.

Proposition/Definition 5.29. *There is an adjunction (i, π) of functors between two abelian categories*

$$i : Fun(\mathcal{R}(\mathcal{S}_1),\mathbb{K}) \rightleftarrows Fun(\mathcal{R}(\mathcal{S}_2),\mathbb{K}) : \pi$$

defined as follows:

1) *$i(F_1)$ forgets $F_1(J_k)$'s in (1) for $1 \le k \le l$;*
2) *$\pi(F_2)$ keeps the same data as F_2 on other regions, and $\pi(F_2)(J_k) := F_2(I_k) \times_{F_2(N_1)} F_2(N_2) = \ker(F_2(I_k) \oplus F_2(N_2) \xrightarrow{(+,-)} F_2(N_1))$ for $1 \le k \le l$. Then all the additional maps for $\pi(F_2)$ are uniquely determined by the universal properties of $\pi(F_2)(J_k)$'s as kernels.*

Moreover, both of i and π are exact, and $i \circ \pi = id$. As a consequence, we obtain an adjunction (i, π) of DG functors in the DG lifting

$$i : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_1),\mathbb{K}) \rightleftarrows \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_2),\mathbb{K}) : \pi$$

and we get a natural isomorphism $\beta : i \circ \pi \overset{\sim}{\Rightarrow} \mathrm{Id}$, and $\pi : \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_2),\mathbb{K}) \to \mathcal{F}\mathsf{un}(\mathcal{R}(\mathcal{S}_1),\mathbb{K})$ is fully faithful.

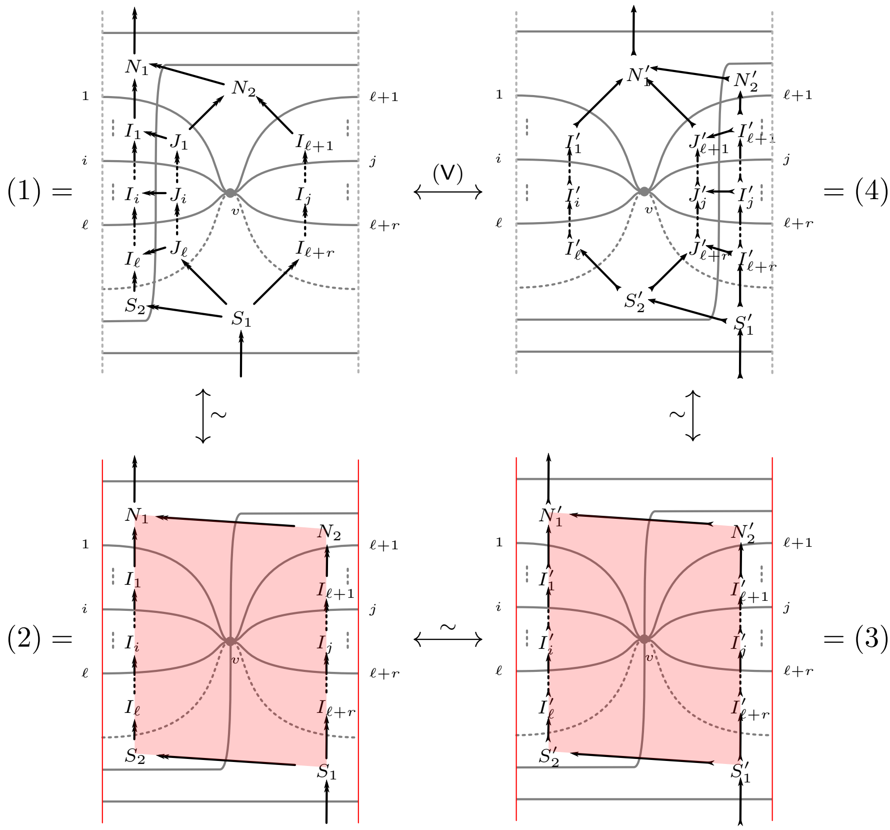


Figure 15: Invariance of sheaf categories: Legendrian Reidemeister move (V).

Proof. The proof is identical to that of Proposition/Definition 5.27. □

Similar to Lemma 5.28, we have the following lemma.

Lemma 5.30. *Let* $\mathcal{D}_1 := \mathfrak{Fun}_{T_1}(\mathcal{R}(\mathcal{S}_1), \mathbb{K})$ *be the DG category in (1), and* $\mathcal{D}_2$ *be the DG full subcategory of* $\mathfrak{Fun}_{T_2}(\mathcal{R}(\mathcal{S}_2), \mathbb{K})$ *whose objects are functors* F_2 *such that the additional crossing condition induced by the red square in (2) holds, i.e. the total complex* $\mathrm{Tot}(F_2(S_1) \to F_2(S_2) \oplus F_2(N_2) \xrightarrow{(+,-)}$

$F_2(N_1)$) is acyclic. Then the adjunction (i,π) of DG functors in Proposition/Definition 5.29 induces equivalences

$$i:\mathcal{D}_1 \overset{\sim}{\rightleftarrows} \mathcal{D}_2:\pi$$

which are quasi-inverses to each other.

Proof. The proof is similar to that of Lemma 5.28. □

Similar to the case of move III, let $\mathcal{D}_3$ be the DG category defined by (3). Then $\mathcal{D}_3=\mathcal{D}_2$. Let $\mathcal{D}_4:=\mathfrak{Fun}_{T_4}(\mathcal{R}(\mathcal{S}_4),\mathbb{K})$ be the DG category in (4). By a dual argument to that in proving the equivalence between $\mathcal{D}_1$ and $\mathcal{D}_2$, we immediately obtain equivalences

$$p:\mathcal{D}_3 \overset{\sim}{\rightleftarrows} \mathcal{D}_4:j$$

which are quasi-inverses to each other. Here j is induced from the forgetful functor $i:Fun(\mathcal{R}(\mathcal{S}_4),\mathbb{K})\to Fun(\mathcal{R}(\mathcal{S}_3),\mathbb{K})$ with $i(F_4)$ forgets $F_4(J'_k)$'s for $l+1\leq k\leq l+r$, and p is induced from the functor $p:Fun(\mathcal{R}(\mathcal{S}_3),\mathbb{K})\to Fun(\mathcal{R}(\mathcal{S}_4),\mathbb{K})$ with $p(F_3)(J'_k)=\mathrm{Coker}(F_3(S'_1)\xrightarrow{(+,-)^t}F_3(I'_k)\oplus F_3(S'_2))$.

Again by composition, we get an equivalence of diagrams of DG categories: $\mathfrak{Fun}_{\mathcal{T}}(\mathcal{R}(\mathcal{S}),\mathbb{K})\simeq\mathfrak{Fun}_{\mathcal{T}'}(\mathcal{R}(\mathcal{S}),\mathbb{K})$, as desired.

5.4.3. Move (VI). If T,T' differ by a move (VI), can take the pair of regular cell complexes $\mathcal{S},\mathcal{S}'$ so that the local bordered Legendrian graphs involving the move are as in Figures 16(1) and (2), respectively. In the picture, T has l left and $r\geq 1$ right half-edges, labelled by $1,\ldots,l$ and $l+1,\ldots,l+r$ from top to bottom respectively. T' has $l+1$ left and $r-1$ right half-edges at the vertex, with the labelling inherited from that of T. In addition, the bordered Legendrian graph $T(\mathcal{S})$ (resp. $T(\mathcal{S}')$) underlying $\mathcal{S}$ (resp. $\mathcal{S}'$) is T (resp. T') plus the additional dashed arcs. We have added the dashed arcs to ensure that S,S' satisfy Assumption 5.17.

Proposition/Definition 5.31. *There is an adjunction (i,π) of functors between two abelian categories*

$$i:Fun(\mathcal{R}(\mathcal{S}'),\mathbb{K})\rightleftarrows Fun(\mathcal{R}(\mathcal{S}),\mathbb{K}):\pi$$

defined as follows:

1) *$i(F')$ forgets $F'(J_k)$'s in (2) for $1\leq k\leq l$;*

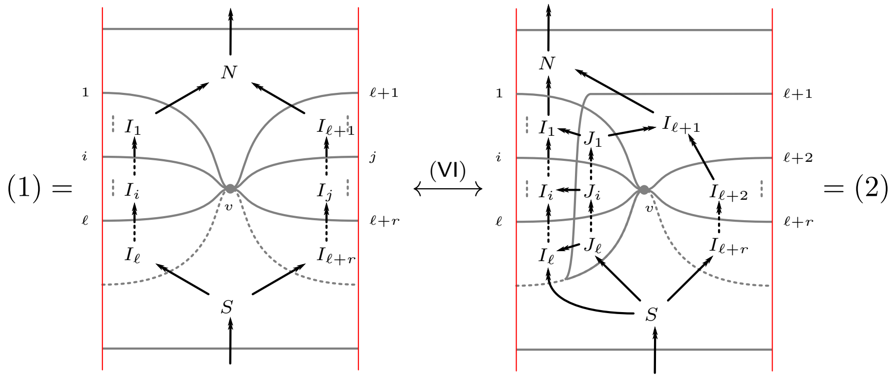


Figure 16: Invariance of sheaf categories: Legendrian Reidemeister move (VI).

2) *$\pi(F)$ keeps the same data as F on other regions, and* $\pi(F)(J_k) := F(I_k) \times_{F(N)} F(I_{l+1}) = \ker(F(I_k) \oplus F(I_{l+1}) \xrightarrow{(+,-)} F(N))$ *for* $1 \leq k \leq l$*. Then all the additional maps for $\pi(F)$ are uniquely determined by the universal properties of $\pi(F)(J_k)$'s as kernels.*

Moreover, both of i and π are exact, and $i \circ \pi = id$. As a consequence, we obtain an adjunction (i, π) of DG functors in the DG lifting

$$i : \mathfrak{Fun}(\mathcal{R}(\mathcal{S}'), \mathbb{K}) \rightleftarrows \mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K}) : \pi$$

and we get a natural isomorphism $\beta : i \circ \pi \overset{\sim}{\Rightarrow} \mathrm{Id}$*, and* $\pi : \mathfrak{Fun}(\mathcal{R}(\mathcal{S}), \mathbb{K}) \to \mathfrak{Fun}(\mathcal{R}(\mathcal{S}'), \mathbb{K})$ *is fully faithful.*

Proof. The proof is identical to that of Proposition/Definition 5.27. □

We also have the following lemma.

Lemma 5.32. *Let* $\mathcal{D} := \mathfrak{Fun}_T(\mathcal{R}(\mathcal{S}), \mathbb{K})$ *and* $\mathcal{D}' := \mathfrak{Fun}_{T'}(\mathcal{R}(\mathcal{S}'), \mathbb{K})$ *be the DG categories in (1) and (2) respectively. Then the adjunction (i, π) of DG functors in Proposition/Definition 5.29 induces equivalences*

$$i : \mathcal{D}' \overset{\sim}{\rightleftarrows} \mathcal{D} : \pi$$

which are quasi-inverses to each other.

Proof. The proof is similar to that of Lemma 5.28. □

Again as before, we get an equivalence of diagrams of DG categories: $\mathcal{F}\mathsf{un}_{\mathcal{T}}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \simeq \mathcal{F}\mathsf{un}_{\mathcal{T}'}(\mathcal{R}(\mathcal{S}),\mathbb{K})$, as desired. Now, We have finished the proof of Theorem 5.3. □

5.5. Microlocal monodromy

Given a bordered Legendrian graph $\mathcal{T} = (T_{\mathsf{L}} \to T \leftarrow T_{\mathsf{R}})$, equipped with a $\mathbb{Z}$-valued Maslov potential $\boldsymbol{\mu}$, let $\mathcal{S}$ be a regular cell complex refining the stratification $\mathcal{S}_{\mathcal{T}}$ induced by $\mathcal{T}$. Denote by $\mathcal{L}\mathsf{oc}(T \setminus V_T)$ the category of local systems of cochain complexes of $\mathbb{K}$-modules on the set of edges, which is the complement $T \setminus V_T$ of vertices.

As in [25, Def.5.4], we define the *microlocal monodromy*.

Definition 5.33 (Microlocal monodromy). There is a natural functor $\mu\mathrm{mon} : \mathcal{S}\mathsf{h}(T;\mathbb{K}) \to \mathcal{L}\mathsf{oc}(T \setminus V_T)$, called *microlocal monodromy*, such that for each edge A of $\mathcal{T}$, we define

$$\mu\mathrm{mon}(\mathcal{F})(A) := \mathrm{Cone}(\mathcal{F}(\mathrm{Star}(a)) \to \mathcal{F}(\mathrm{Star}(N)))[-\mu(a)],$$

where $a \in \mathcal{S}$ is an arc contained in the edge A, N is the region above a so that the arrow $a \to N$ is in $\mathcal{S}$.

Remark 5.34. The above definition is well-defined, i.e., it is independent of the choice of a as observed already in [25, Prop.5.5]. In addition, for any sheaf $F \in \mathcal{S}\mathsf{h}(\mathcal{T};\mathbb{K})$ and any point $p \in \mathcal{T} \setminus \mathcal{V}_{\mathcal{T}}$ which is a smooth Legendrian point of $\mathcal{T}$, by Proposition 7.5.3 in [17], the microlocal monodromy $\mu\mathrm{mon}(\mathcal{F})_p$ at p is nothing but the *microlocal stalk* of $\mathcal{F}$ at p, up to a degree shift. The latter can be used as an alternative (and intrinsic) definition for microlocal monodromy.

Proposition 5.35. *Then the microlocal monodromy* $\mu\mathrm{mon}$ *is invariant under Legendrian isotopy of* $\mathcal{T}$.

Proof. The combinatorial proof is entirely the same as that in [25, §5.1]. In another perspective, one can use an intrinsic characterization of microlocal monodromy: microlocal stalks. See the remark above. □

Definition 5.36 (Subcategory of microlocal rank 1). We define $\mathcal{C}_1(T,\mu;\mathbb{K})$ to be the full DG subcategory of $\mathcal{S}\mathsf{h}(T;\mathbb{K})$ whose objects are $\mathcal{F}$

such that $\mu\mathrm{mon}(\mathcal{F})$ is a local system of rank 1 $\mathbb{K}$-modules in cohomological degree 0, and define the induced diagram of constructible sheaf categories

$$\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K}) \coloneqq (\mathcal{C}_1(T_{\mathsf{L}},\mu_{\mathsf{L}};\mathbb{K}) \leftarrow \mathcal{C}_1(T,\mu;\mathbb{K}) \to \mathcal{C}_1(T_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K})).$$

This DG category $\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K})$ will be the sheaf side of the augmentation-sheaf correspondence in the next section. As a consequence of Theorem 5.3, we obtain

Corollary 5.37. *The category $\mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K})$ is a Legendrian isotopy invariant up to DG equivalence.*

Proof. This follows from Theorem 5.3 and the fact that the notion of the microlocal monodromy is intrinsic as seen in Proposition 5.35. □

Let $\mathcal{S}$ be a regular cell complex which refines $\mathcal{S}_T$ and satisfies Assumption 5.17. We use the notations in Definitions 5.18 and 5.23.

Definition 5.38. We define $\mathfrak{Fun}_{(T,\mu),1}(\mathcal{R}(\mathcal{S}),\mathbb{K})$ to be the full DG subcategory of $\mathfrak{Fun}_T(\mathcal{R}(\mathcal{S}),\mathbb{K})_0$ consisting of functors F such that

$$\mathrm{Cone}(F(e_s))[-\mu(s)] \simeq \mathbb{K}$$

for all arcs s contained in T.

By restriction, we then obtain a diagram of DG categories

$$\begin{aligned}\mathfrak{Fun}_{(\mathcal{T},\boldsymbol{\mu}),1}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \coloneqq (\mathfrak{Fun}_{(T_{\mathsf{L}},\mu_{\mathsf{L}}),1}(\mathcal{R}(\mathcal{S}|_{T_{\mathsf{L}}}),\mathbb{K}) \leftarrow \mathfrak{Fun}_{(T,\mu),1}(\mathcal{R}(\mathcal{S}),\mathbb{K})\\ \to \mathfrak{Fun}_{(T_{\mathsf{R}},\mu_{\mathsf{R}}),1}(\mathcal{R}(\mathcal{S}|_{T_{\mathsf{R}}}),\mathbb{K}))\end{aligned}$$

Corollary 5.39. *There is an A_∞-equivalence:*

$$\mathfrak{Fun}_{(\mathcal{T},\boldsymbol{\mu}),1}(\mathcal{R}(\mathcal{S}),\mathbb{K}) \simeq \mathcal{C}_1(\mathcal{T},\boldsymbol{\mu};\mathbb{K}).$$

Proof. This is a direct corollary of Lemma 5.15 and Proposition 5.24. □

6. Augmentations are sheaves for Legendrian graphs

6.1. Local calculation for augmentation categories

In this section, we will compute the A_∞-structures completely for the trivial bordered Legendrian graphs and the bordered Legendrian graphs containing a vertex of type $(0, n_{\mathsf{R}})$.

6.1.1. Augmentation category for a trivial bordered Legendrian graph. Let $(\mathcal{T}_n, \boldsymbol{\mu}) = (T_n = T_n = T_n, \mu = \mu = \mu)$ be a trivial bordered Legendrian graph of n parallel strands, equipped with a $\mathbb{Z}$-valued Maslov potential μ. We will describe the augmentation category $\mathcal{A}\mathsf{ug}_+(T_n, \mu; \mathbb{K})$, which has been already seen in Corollary 4.8 for the unitality.

Notation 6.1. In this section, we denote $\mathcal{A}\mathsf{ug}_+(T_n, \mu; \mathbb{K})$ by $\mathcal{A}\mathsf{ug}_+$ for simplicity.

As seen in Example 4.7, the Chekanov-Eliashberg DGA $A^{\mathsf{CE}}\left(T_n^{(m)}, \mu^{(m)}\right) \cong A_n^{(m)}(\mu)$ is generated by the set

$$\left\{k_{ab}^{ij} \;\middle|\; a < b, 1 \le i, j \le m\right\} \coprod \left\{y_a^{ij} \;\middle|\; 1 \le a \le n, 1 \le i < j \le m\right\},$$

where the grading is given as

$$|k_{ab}^{ij}| := \mu(a) - \mu(b) - 1, \qquad |y_a^{ij}| := -1.$$

Assumption 6.2. From now on, we denote y_a^{ij} by k_{aa}^{ij}. We regard k_{ab}^{ij} as zero unless it is well-defined.

Then under the above assumption, the differential $\partial^{(m)}$ is simply given as

$$\partial^{(m)} k_{ab}^{ij} = \sum_{\substack{a \le c \le b \\ 1 \le \ell \le m}} (-1)^{|k_{ac}^{i\ell}|-1} k_{ac}^{i\ell} k_{cb}^{\ell j}.$$

Recall from Example 4.7. $\mathcal{A}\mathsf{ug}_+$ is a DG category such that:

1) The objects are the augmentations for $A_n(\mu)$:

$$\mathcal{O}\mathsf{b}\left(\mathcal{A}\mathsf{ug}_+\right) = \mathsf{Aug}(A_n(\mu); \mathbb{K}). \tag{6.1}$$

2) In $A^{(2)}(T_n)$, we have

$$\mathsf{M}^{12} := \mathbb{K}\left\langle k_{ab}^{12} \;\middle|\; 1 \le a \le b \le n\right\rangle.$$

Then, for any two objects ϵ_1, ϵ_2, the set of morphisms is

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) = \mathsf{M}_{12}^{\vee} = \mathbb{K}\left\langle k_{ab}^{12\vee} \;\middle|\; 1 \le a \le b \le n\right\rangle. \tag{6.2}$$

3) For $\epsilon_1, \epsilon_2 \in \mathcal{A}\mathsf{ug}_+$, the map

$$m_1 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2)$$

is defined as

$$m_1\left(k_{ab}^{12\vee}\right) = -\sum_{c<a} \epsilon_1(k_{ca}) k_{cb}^{12\vee} + \sum_{b<d} (-1)^{|k_{ab}^{12\vee}|} k_{ad}^{12\vee} \epsilon_2(k_{bd}) \tag{6.3}$$

4) For $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathcal{A}\mathsf{ug}_+$, the map

$$m_2 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_2, \epsilon_3) \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_3)$$

is defined as

$$\begin{aligned} m_2\left(k_{cd}^{12\vee} \otimes k_{ab}^{12\vee}\right) &= \delta_{bc}(-1)^{\sigma_2}(-1)^{|k_{ab}^{12\vee}|} k_{ad}^{12\vee} \\ &= \delta_{bc}(-1)^{|k_{ab}^{12\vee}||k_{cd}^{12\vee}|+1} k_{ad}^{12\vee}, \end{aligned} \tag{6.4}$$

where

$$\sigma_2 = 1 + |k_{ab}^{12\vee}||k_{cd}^{12\vee}| + |k_{ab}^{12\vee}|.$$

Definition 6.3 (Morse complex). Consider a free graded $\mathbb{K}$-module $C = C(T_n, \mu)$,

$$C := \bigoplus_{1 \le a \le n} \mathbb{K} e_a, \qquad |e_a| := -\mu(a).$$

equipped with a decreasing filtration $F^\bullet$ via

$$F^i C := \bigoplus_{k>i} \mathbb{K} \cdot e_k,$$

for $0 \le i \le n$.

We define the set $\mathrm{MC} = \mathrm{MC}(T_n, \mu; \mathbb{K})$ of *Morse complexes* of $(C, F^\bullet)$, each of which is a complex (C, d) with a $\mathbb{K}$-linear filtration-preserving differential d of degree 1.

Lemma 6.4 ([27, Def.4.4]). *There is a canonical identification*

$$\Xi : \mathsf{Aug}(A_n(\mu); \mathbb{K}) \xrightarrow{\cong} \mathrm{MC}$$
$$\epsilon \mapsto d = d(\epsilon),$$

where

$$de_i \coloneqq \sum_{j>i} (-1)^{\mu(i)} \epsilon(a_{ij}) e_j.$$

The set of Morse complexes MC can be lifted to an obvious DG category $\mathcal{MC} = \mathcal{MC}(T_n, \mu; \mathbb{K})$:

1) The set of objects is MC.
2) The morphism space $(\mathrm{Hom}_{\mathcal{MC}}(d_1, d_2), D)$ is

$$\mathrm{Hom}_{\mathcal{MC}}(d_1, d_2) \coloneqq \mathrm{End}(C, F^\bullet),$$

the complex of endomorphisms of $(C, F^\bullet)$ whose differential is given by

$$Df \coloneqq d_2 \circ f - (-1)^{|f|} f \circ d_1.$$

3) The composition $\cdot$ is the usual composition of endomorphisms of C.

Notation 6.5. We use the identification $\mathrm{End}(C) \cong C \otimes C^*$ and so for $(e \otimes f^*), (e' \otimes f'^*) \in \mathrm{End}(C)$,

$$(e \otimes f^*)(g) \coloneqq \langle f^*, g \rangle e \in C,$$
$$(e \otimes f^*) \cdot (e' \otimes f'^*) \coloneqq \langle f^*, e' \rangle (e \otimes f'^*) \in \mathrm{End}(C).$$

Lemma 6.6 ([21, Theorem 7.25]). *There is a (strict) isomorphism of DG categories*

$$\mathfrak{h} : \mathcal{A}\mathsf{ug}_+(T_n, \mu; \mathbb{K}) \longrightarrow \mathcal{MC}$$

which is given on objects by

$$\epsilon \mapsto d(\epsilon) \coloneqq \sum_{a<b} \left((-1)^{\mu(a)} \epsilon(k_{ab}) \right) (e_b \otimes e_a^*),$$

as in Lemma 6.4, and on morphisms $\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{MC}}(d(\epsilon_1), d(\epsilon_2))$ *by*

$$k_{ab}^{12\vee} \mapsto (-1)^{s(a,b)} (e_b \otimes e_a^*),$$

where the sign $s(a,b)$ is given by

$$s(a,b) := \mu(a)(\mu(b)+1)+1.$$

Remark 6.7. Notice that the sign convention for $s(a,b)$ here is different from the original one, due to the fact that our definition for $d(\epsilon)$ differs by a sign from $\mathfrak{h}(\epsilon)$ in [21, Theorem 7.25].

Proof of Lemma 6.6. For completeness, we give the proof. By Lemma 6.4, it suffices to show that $\mathfrak{h}$ commutes with the differential and composition, that is, $\mathfrak{h}\circ m_1 = D\circ\mathfrak{h}$, and $\mathfrak{h}\circ m_2 = \mathfrak{h}\cdot\mathfrak{h}$.

At first, let us show $\mathfrak{h}\circ m_1 = D\circ\mathfrak{h}$. For two objects $\epsilon_1,\epsilon_2\in\mathcal{A}\mathsf{ug}_+$, let $x\in\mathrm{Hom}_+(\epsilon_1,\epsilon_2)$ be a homogeneous element of the form

$$x = \sum_{1\le a\le b\le n} x_{ab}k_{ab}^{12\vee},$$

whose all nontrivial summand have the same degree. In other words, $x_{ab}\neq 0$ implies that $|x| = |k_{ab}^{12\vee}| = \mu(a)-\mu(b)$ since x is homogeneous.

By definition of $\mathfrak{h}$, we have

$$\mathfrak{h}(x) = \sum_{1\le a\le b\le n}(-1)^{s(a,b)}x_{ab}(e_b\otimes e_a^*).$$

On the other hand, apply the formula (6.3) for m_1, we have

(6.5)

$$\begin{aligned}
\mathfrak{h}\circ m_1(x) &= -\sum_{c<a\le b}\epsilon_1(k_{ca})x_{ab}\mathfrak{h}\left(k_{cb}^{12\vee}\right) + \sum_{a\le b<d}(-1)^{|k_{ab}^{12\vee}|}x_{ab}\epsilon_2(k_{bd})\mathfrak{h}\left(k_{ad}^{12\vee}\right)\\
&= -\sum_{c<a\le b}(-1)^{s(c,b)}\epsilon_1(k_{ca})x_{ab}(e_b\otimes e_c^*)\\
&\quad + \sum_{a\le b<d}(-1)^{\mu(a)-\mu(b)+s(a,d)}x_{ab}\epsilon_2(k_{bd})(e_d\otimes e_a^*)\\
&= -\sum_{c<a\le b}(-1)^{|x|}\left((-1)^{\mu(c)}\epsilon_1(k_{ca})\right)\left((-1)^{s(a,b)}x_{ab}\right)(e_b\otimes e_a^*)\cdot(e_a\otimes e_c^*)\\
&\quad + \sum_{a\le b<d}\left((-1)^{s(a,b)}x_{ab}\right)\left((-1)^{\mu(b)}\epsilon_2(k_{bd})\right)(e_d\otimes e_b^*)\cdot(e_b\otimes e_a^*)\\
&= -(-1)^{|x|}\mathfrak{h}(x)\circ d(\epsilon_1) + d(\epsilon_2)\circ\mathfrak{h}(x)\\
&= D\circ\mathfrak{h}(x)
\end{aligned}$$

as desired. Here, the fourth equality follows directly from the formulas for $\mathfrak{h}(x)$ and $d(\epsilon_i)$ above.

The only nontrivial part is the third equality involving two sign manipulations, which we check as follows: for the first sign manipulation, it suffices to show that, if $\epsilon_1(k_{ca})x_{ab} \neq 0$, then

$$s(c,b) \equiv |x| + \mu(c) + s(a,b) \mod 2.$$

In fact, the condition implies that

$$|k_{ca}| = \mu(c) - \mu(a) - 1 = 0, \quad |x| = |k_{ab}^{12\vee}| = \mu(a) - \mu(b) = \mu(c) - 1 - \mu(b)$$

and therefore

$$\begin{aligned} &s(c,b) - s(a,b) - \mu(c) - |x| \\ &\quad = \mu(b) + 1 - \mu(c) - (\mu(c) - 1 - \mu(b)) \equiv 0 \mod 2 \end{aligned}$$

as desired.

Similarly, for the second one, it suffices to show that

$$x_{ab}\epsilon_2(k_{bd}) \neq 0 \Longrightarrow \mu(a) - \mu(b) + s(a,d) \equiv s(a,b) + \mu(b) \mod 2,$$

or equivalently,

$$s(a,d) - s(a,b) + \mu(a) \equiv 0 \mod 2.$$

Then as before, we have

$$\begin{aligned} &|k_{bd}| = \mu(b) - \mu(d) - 1 = 0 \quad \text{and} \\ &|x| = |k_{ab}^{12\vee}| = \mu(a) - \mu(b) = \mu(a) - \mu(d) - 1 \end{aligned}$$

which implies the desired identity

$$s(a,d) - s(a,b) + \mu(a) = -\mu(a) + \mu(a) \equiv 0 \mod 2.$$

It remains to show $\mathfrak{h} \circ m_2 = \mathfrak{h} \cdot \mathfrak{h}$. For any objects $\epsilon_1, \epsilon_2, \epsilon_3$ in $\mathcal{A}\mathsf{ug}_+(T_n; \mathbb{K})$, let

$$x := \sum_{a \le b} x_{ab} k_{ab}^{12\vee} \in \mathrm{Hom}_+(\epsilon_2, \epsilon_3), \qquad y := \sum_{a \le b} y_{ab} k_{ab}^{12\vee} \in \mathrm{Hom}_+(\epsilon_1, \epsilon_2)$$

be two homogeneous elements. We want to show $\mathfrak{h} \circ m_2(x, y) = \mathfrak{h}(x) \cdot \mathfrak{h}(y)$. By definition of $\mathfrak{h}$, we have

$$\begin{aligned}\mathfrak{h}(x) \cdot \mathfrak{h}(y) &= \sum_{a \le c \le b} \Big((-1)^{s(c,b)} x_{cb}(e_q \otimes e_r^*)\Big) \cdot \Big((-1)^{s(a,c)} y_{ac}(e_c \otimes e_a^*)\Big) \\ &= \sum_{a \le c \le b} (-1)^{s(c,b)+s(a,c)} y_{ac} x_{cb}(e_b \otimes e_a^*).\end{aligned}$$

On the other hand, we have the following by applying the formula (6.4)

$$\begin{aligned}\mathfrak{h} \circ m_2(x, y) &= \mathfrak{h}\left(\sum_{a \le c \le b} x_{cb} y_{ac} m_2\left(k_{cb}^{12\vee} \otimes k_{ac}^{12\vee}\right)\right) \\ &= \sum_{a \le c \le b} y_{ac} x_{cb} (-1)^{1+|k_{ac}^{12\vee}||k_{cb}^{12\vee}|} (-1)^{s(a,b)} (e_b \otimes e_a^*) \\ &= \mathfrak{h}(x) \cdot \mathfrak{h}(y)\end{aligned}$$

as desired. Here, the last equality follows from the previous formula for $\mathfrak{h}(x) \cdot \mathfrak{h}(y)$, and the following sign manipulation:

$$\begin{aligned}&1 + |k_{ac}^{12\vee}||k_{cb}^{12\vee}| + s(a, b) \\ &= 1 + (\mu(a) - \mu(c))(\mu(c) - \mu(b)) + \mu(a)(\mu(b) + 1) + 1 \\ &\equiv \mu(a)\mu(c) + \mu(c)\mu(b) + \mu(c) + \mu(a) + 2 \\ &\equiv s(a, c) + s(c, b) \mod 2\end{aligned}$$

This finishes the proof of the lemma. □

6.1.2. Augmentation category for a vertex. Let $(\mathcal{V}, \boldsymbol{\mu}) \in \mathcal{BLG}$ be a bordered Legendrian graph of type $(n_{\mathsf{L}}, n_{\mathsf{R}})$ in $J^1\mathbf{U}$, which looks as the left picture in Figure 17. In particular, $\mathcal{V}$ contains a vertex v of type $(0, r)$ right, whose half-edges are labelled from top to bottom by $1, 2, \ldots, r$. Then $n_{\mathsf{R}} = n_{\mathsf{L}} + r$.

As usual, we label the left (resp. right) ends from top to bottom by $1, 2, \ldots, n_{\mathsf{L}}$ (resp. $1, 2, \ldots, n_{\mathsf{R}}$). We assume the half-edges $\{h_{v,1}, \ldots, h_{v,r}\}$ of v connect the right ends $\{a_v, a_v + 1, \ldots, a_v + r - 1\}$ from top to bottom. In other words, the half-edge $h_{v,i}$ is connected to the right end $a_1 + i - 1$. For simplicity, we denote

$$a' := a \text{ for } 1 \le a < a_v, \text{ and } b' := b + r \text{ for } a_v \le b \le n_{\mathsf{L}}.$$

We would like to compute the augmentation category

$$\mathcal{A}\mathsf{ug}_+(\mathcal{V}, \boldsymbol{\mu}; \mathbb{K}) = \left(\mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L}; \mathbb{K}) \leftarrow \mathcal{A}\mathsf{ug}_+(V, \mu; \mathbb{K}) \rightarrow \mathcal{A}\mathsf{ug}_+(V_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K})\right).$$

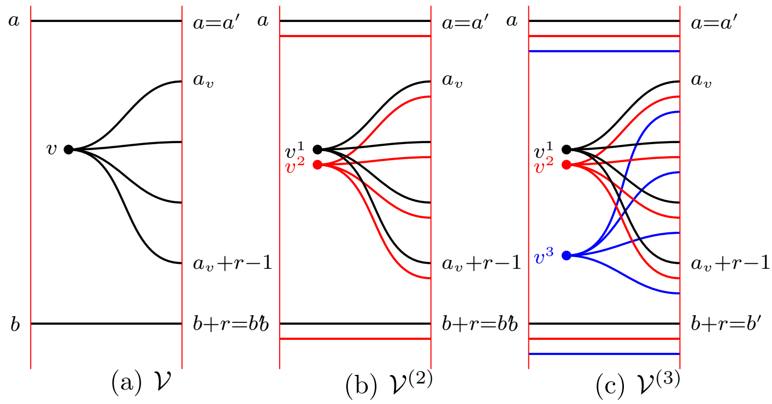


Figure 17: Front projection m-copy near a vertex.

For $m \geq 1$, let $\mathcal{V}^{(m)} = \left(V_\mathsf{L}^{(m)} \rightarrow V^{(m)} \leftarrow V_\mathsf{R}^{(m)}\right) \in \mathcal{BLG}^{(m)}$ be the canonical front projection m-copy of $\mathcal{V}$. For examples, $\mathcal{V}^{(m)}$'s for $m = 1, 2, 3$ are shown as in Figure 17 from left to right.

Remark 6.8. Here we are using the convention of the canonical front parallel copies described in Section 3.1.1.

Notation 6.9. For simplicity, let us denote

$$A_*^{(m)} := A^{\mathsf{CE}}\left(V_*^{(m)}, \mu_*^{(m)}\right) \quad \text{and} \quad \mathcal{A}\mathsf{ug}_{+,*} := \mathcal{A}\mathsf{ug}_+(V_*, \mu_*; \mathbb{K}),$$

for $* = \mathsf{L}, \mathsf{R}$ or empty.

As usual, for each copy $\mathcal{V}^i$ in $\mathcal{V}^{(m)}$, we label the left ends and right ends from top to bottom by $1, 2, \ldots, n_\mathsf{L}$ and $1, 2, \ldots, n_\mathsf{R}$ respectively, label the vertex by v^i and the right half-edges from top to bottom by $1, 2, \ldots, r \in \mathbb{Z}/r\mathbb{Z}$.

In the bordered Legendrian graph $\mathcal{V}^{(m)}$, label the Reeb chords of $V_{\mathsf{L}}^{(m)}$ and $V_{\mathsf{R}}^{(m)}$ corresponding to the pairs of strands by

$$K_{\mathsf{L}} := \left\{ k_{ab}^{ij} \,\middle|\, 1 \le a < b \le n_{\mathsf{L}}, 1 \le i, j \le m \text{ or } 1 \le a = b \le n_{\mathsf{L}}, 1 \le i < j \le m \right\}$$

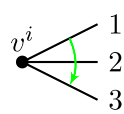
and

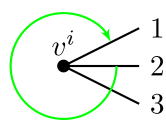
$$K_{\mathsf{R}} := \left\{ k_{ab}^{\prime ij} \,\middle|\, 1 \le a < b \le n_{\mathsf{R}}, 1 \le i, j \le m \text{ or } 1 \le a = b \le n_{\mathsf{R}}, 1 \le i < j \le m \right\},$$

each of which is a Reeb chord (or line segment) of $V_{\mathsf{L}}^{(m)}$ and $V_{\mathsf{R}}^{(m)}$ going from the b-th strand of the jth copy to the a-th strand of the ith copy.

Label the Reeb chords of V^i at the vertex v^i by either c_{ab}^{ii} if $1 \le a < b \le r$ or $v_{a,\ell}^{ii}$ otherwise. In other words, c_{ab}^{ii} or $v_{a,\ell}^{ii}$ is the "Reeb chord" starting from the initial half-edge $a + \ell \in \mathbb{Z}/r\mathbb{Z}$, traveling around the vertex v^i counterclockwise and covering ℓ minimal sectors, hence ending at the half-edge $a \in \mathbb{Z}/r\mathbb{Z}$ of v^i.

Remark 6.10. This is the same as the construction for Legendrian graphs in a normal form described in Section 3.2.3 but the reflected manner. In other words, the generators c_{ab}^{ii} correspond to vertex generators lying on the *right side* of v.

$c_{13}^{ii} =$ 

$v_{2,2}^{ii} =$ 

There are additional Reeb chords of $V^{(m)}$ corresponding to the crossings of the right half-edges of the vertices. Label the Reeb chord going from the b-th right half-edge of v^j in V^j to the a-th right half-edge of v^i in V^i by c_{ab}^{ij} for $1 \le b < a \le r$ and $1 \le i < j \le m$. See Figure 18 for an example when $r = 3$ and $m = 4$.

We denote the set of Reeb chords by $R^{ij} := \left\{ k_{pq}^{ij}, v_{a,\ell}^{ij}, c_{ab}^{ij} \right\}$ starting at the j-th copy V^j and ending at the i-th copy V^i. Then the LCH DGA $A^{(m)} = (\mathsf{A}^{(m)}, \partial^{(m)}) := A^{\mathsf{CE}}(V^{(m)}, \mu^{(m)})$ is the free associative algebra generated by the Reeb chord in $V^{(m)}$

$$\mathsf{A}^{(m)} := \mathbb{Z}\left\langle R^{ij} \;\middle|\; 1 \le i \le j \le m \right\rangle.$$

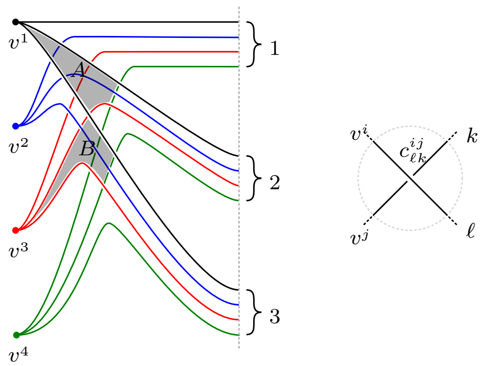


Figure 18: The crossings in the front projection m-copy near a vertex when $r = 3$ and $m = 4$.

The grading for each generator is defined as follows: let $\mu_{\mathsf{L}} \coloneqq \mu|_{V_{\mathsf{L}}}$. Then

$$|k^{ij}_{ab}| = \mu_{\mathsf{L}}(a) - \mu_{\mathsf{L}}(b) - 1,$$
$$|v^{ii}_{a,\ell}| = |v_{a,\ell}| = \mu(a) - \mu(a+\ell) + N(n,a,\ell) - 1,$$
$$|c^{ij}_{ab}| = \begin{cases} \mu(a) - \mu(b) - 1 & i = j; \\ \mu(a) - \mu(b) & i < j, \end{cases}$$

where $N(n, a, \ell)$ is the same as described in (2.3).

The differentials of k^{ij}_{ab} and $v^{ii}_{a,\ell}$ are given by the differentials of border DGAs and internal DGAs defined in Example/Definition 2.31. Indeed, under Assumption 6.2, we have

$$\partial^{(m)} k^{ij}_{ab} = \sum_{\substack{a \le c \le b \\ 1 \le \ell \le m}} (-1)^{|k^{i\ell}_{ac}|+1} k^{i\ell}_{ac} k^{\ell j}_{cb}, \tag{6.6}$$

$$\partial^{(m)} v^{ii}_{a,\ell} = \delta_{\ell,r} + \sum_{\ell_1+\ell_2=\ell} (-1)^{|v^{ii}_{a,\ell_1}|-1} v^{ii}_{a,\ell_1} v^{ii}_{a+\ell_1,\ell_2}. \tag{6.7}$$

The differential of c^{ij}_{ab} is obtained by counting certain admissible disks of degree 1 in $(J^1U, V^{(m)})$.

For simplicity, we denote $\widetilde{a} \coloneqq (-1)^{|a|-1} a$ for any Reeb chord a. Then for $i < j$,

$$\partial^{(m)} c^{ij}_{ab} = \sum_{\mathbf{a},\mathbf{i}} (-1)^{\eta} c^{\mathbf{i}}_{\mathbf{a}}, \tag{6.8}$$

where for sequences $\mathbf{a} = (a_1, \ldots, a_{\ell+1})$ and $\mathbf{i} = (i_1, \ldots, i_{\ell+1})$ with $\ell \geq 1$

$$c_{\mathbf{a}}^{\mathbf{i}} := c_{a_1 a_2}^{i_1 i_2} c_{a_2 a_3}^{i_2 i_3} \cdots c_{a_\ell a_{\ell+1}}^{i_\ell i_{\ell+1}}$$

and the summation is over all possible sequences $\mathbf{a}$ and $\mathbf{i}$ in $[r]$ and $[m]$ such that

$$a_1 = a, \qquad a_{\ell+1} = b, \qquad i_1 = i, \qquad i_{\ell+1} = j,$$

and either

$$a_1 < a_2 > \cdots > a_{\ell+1}, \qquad i_1 = i_2 < \cdots < i_{\ell+1}, \qquad \eta = 0, \tag{6.9}$$

or

$$a_1 > a_2 < a_3 > \cdots > a_{\ell+1}, \quad i_1 < i_2 = i_3 < \cdots < i_{\ell+1}, \quad \eta = |c_{a_1 a_2}^{i_1 i_2}| - 1. \tag{6.10}$$

In Figure 18, two shaded regions labelled by A and B correspond to differentials of two types in (6.9) and (6.10), respectively.

In fact, we have a bordered DGA

$$\mathcal{A}^{(m)} = \left(A_{\mathsf{L}}^{(m)} \xrightarrow{\phi_{\mathsf{L}}^{(m)}} A^{(m)} \xleftarrow{\phi_{\mathsf{R}}^{(m)}} A_{\mathsf{R}}^{(m)} \right),$$

where $\phi_{\mathsf{L}}^{(m)}$ is the inclusion of the sub-DGA $A_{\mathsf{L}}^{(m)}$ generated by K_{L}. The algebra $A_{\mathsf{R}}^{(m)}$ is generated by K_{R} with the differential similar to $\partial_{\mathsf{L}}^{(m)}$ as in (6.6), that is,

$$\partial^{(m)} k'^{ij}_{ab} = \sum_{\substack{a \leq c \leq b \\ 1 \leq \ell \leq m}} (-1)^{|k'^{i\ell}_{ac}|+1} k'^{i\ell}_{ac} k'^{\ell j}_{cb}.$$

Notice that in the case when $m = 1$, we have the identification

$$\phi_{\mathsf{L}}^{(1)} = \phi_{\mathsf{L}} : A_{\mathsf{L}}^{(1)} = A_{\mathsf{L}} \to A^{(1)} = A \leftarrow A_{\mathsf{R}}^{(1)} = A_{\mathsf{R}} : \phi_{\mathsf{R}}^{(1)} = \phi_{\mathsf{R}},$$

with $k_{ab} := k_{ab}^{11}, v_{pq} := v_{pq}^{11}$ and $k'_{ab} := k'^{11}_{ab}$.

In addition, the DGA map $\phi_{\mathsf{R}}^{(m)}$ is defined via counting certain admissible disks of index 0 in $(J^1U, V^{(m)})$. More precisely, we have the following:

1) If $1 \le a \le b \le r$ and $1 \le i, j \le m$ such that $k'^{ij}_{a_v+a-1,a_v+b-1}$ is well-defined, then

$$\begin{aligned}(6.11)\quad \phi^{(m)}_{\mathsf{R}}(k'^{ij}_{a_v+a-1,a_1+b-1}) \\ = \sum v^{ii}_{a,a_1-a} c^{ii_2}_{a_1a_2}\cdots c^{i_\ell j}_{a_\ell b} + \sum \widetilde{c^{ii_1}_{aa_0}} v^{i_1i_1}_{a_0,a_1-a_0} c^{i_1i_2}_{a_1a_2}\cdots c^{i_\ell j}_{a_\ell b},\end{aligned}$$

where the summation is over all possible such composable words so that the summand is well-defined. In particular, this implies that

$$\phi^{(m)}_{\mathsf{R}}(k'^{ij}_{a_v+a-1,a_v+b-1}) = \begin{cases} v^{ii}_{a,b-a} & i = j; \\ 0 & i \ne j. \end{cases}$$

2) If $1 \le a \le b \le n_{\mathsf{L}}, 1 \le i, j \le m$ such that $k'^{ij}_{a'b'}$ is well-defined, in particular, $1 \le a' \le b' \le n_{\mathsf{R}}$ and $a', b' \notin \{a_v, a_v+1, \ldots, a_v+r-1\}$, then

$$\phi^{(m)}_{\mathsf{R}}(k'^{ij}_{a'b'}) = k^{ij}_{ab}.$$

3) Otherwise, that is, if $1 \le a \le b \le n_{\mathsf{R}}, 1 \le i, j \le m$ such that exactly one of a and b belongs to $\{a_v, a_v+1, \ldots, a_v+r-1\}$ and k'^{ij}_{ab} is well-defined, then

$$\phi^{(m)}_{\mathsf{R}}(k'^{ij}_{ab}) = 0.$$

For example, in the case when $r = 3$ and $m = 4$ as in Figure 18, we have

$$\iota^{(4)}_{\mathsf{R}}(k'^{14}_{a_v+1,a_v+1}) = v^{11}_{2,1}c^{14}_{32} + \widetilde{c^{12}_{21}}v^{22}_{1,2}c^{24}_{32} + \widetilde{c^{13}_{21}}v^{33}_{1,2}c^{34}_{32} + \widetilde{c^{14}_{21}}v^{44}_{1,1}.$$

Notice that when $m = 1$, we have

$$(6.12)\quad \begin{cases} \phi_{\mathsf{R}}(k'_{a_v+a-1,a_v+b-1}) = v_{a,b-a} & 1 \le a < b \le r; \\ \phi_{\mathsf{R}}(k'_{a'b'}) = k_{ab} & 1 \le a < b \le n_{\mathsf{L}}; \\ \phi_{\mathsf{R}}(k'_{ab}) = 0 & \text{otherwise.} \end{cases}$$

To describe the augmentation category $\mathcal{A}\mathsf{ug}_+(V, \mu; \mathbb{K})$, firstly observe that $A^{(m)}$ is the push-out of $A^{(m)}_{\mathsf{L}}$ and $A^{(m)}_v$, where $A^{(m)}_v$ is the sub-DGA of $A^{(m)}$ generated by $\{v^{ii}_{a,\ell}, c^{ij}_{ab}\}$. That is, for $m \ge 1$, we have the following push-out diagram of DGAs:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & A_v^{(m)} \\ \downarrow & & \downarrow \\ A_{\mathsf{L}}^{(m)} & \xrightarrow{\phi_{\mathsf{L}}^{(m)}} & A^{(m)} \end{array}$$

Also, the DGA $A_v^{(m)}$ is nothing but the LCH DGA $A^{\mathsf{CE}}(v^{(m)}, \mu_v^{(m)})$ for the front m-copy of v, viewed as a bordered Legendrian graph, obtained from V by removing the extra parallel strands. Here, μ_v is the restriction of the Maslov potential μ to the set of half-edges of v. It follows that

$$\mathcal{A}\mathsf{ug}_+(V, \mu; \mathbb{K}) \cong \mathcal{A}\mathsf{ug}_+(V_{\mathsf{L}}, \mu_{\mathsf{L}}) \times \mathcal{A}\mathsf{ug}_+(v; \mathbb{K})$$

is a strict product of two A_∞-categories. By Section 6.1.1, we have already seen the identification $\mathfrak{h}_{\mathsf{L}} : \mathcal{A}\mathsf{ug}_+(V_{\mathsf{L}}; \mathbb{K}) \cong \mathcal{MC}(V_{\mathsf{L}}; \mathbb{K})$. It suffices to describe $\mathcal{A}\mathsf{ug}_+(v; \mathbb{K})$.

At first, we introduce a Morse complex for a vertex as follows:

Definition 6.11. [3, Def.4.3.1] Let $\mathbb{K}[Z]$ be the graded polynomial ring in one variable Z with $|Z| = 1$. We define a free graded left $\mathbb{K}[Z]$-module

$$C(v) := \mathbb{K}[Z]\langle e_1, \ldots, e_r\rangle, \qquad |e_a| := -\mu_v(a)$$

and a decreasing filtration

$$C(v) \supset F^1C(v) \supset \cdots \supset F^rC(v) \supset Z^2 \cdot F^{r+1}C(v) = Z^2 \cdot F^1C(v)$$

of $C(v)$ by free graded left $\mathbb{K}$-submodules such that for each $a \in \mathbb{Z}/r\mathbb{Z}$,

$$F^aC(v) := \mathbb{K}\langle e_a\rangle \oplus \bigoplus_{i>0} \mathbb{K}\langle Z^{N(v,a,i)} e_{a+i}\rangle.$$

We define $\mathrm{MC}(v; \mathbb{K})$ to be the set of all $\mathbb{K}[Z]$-superlinear endomorphisms d of $C(v)$ of degree 1 which preserves $F^\bullet$ and satisfies $d^2 + Z^2 = 0$.

Lemma 6.12. *[3, Lem.4.3.3] There is a canonical identification*

$$\begin{aligned} \Xi_v : \mathsf{Aug}(v; \mathbb{K}) &\overset{\cong}{\to} \mathrm{MC}(v; \mathbb{K}); \\ \epsilon &\mapsto d = d(\epsilon), \end{aligned}$$

where

$$(-1)^{\mu(i)} de_i = \sum_{j>0} \epsilon(v_{i,j}) Z^{N(v,i,j)} e_{i+j}.$$

From now on, we will always use the identification above.

Lemma 6.13. *[3, Lem. 4.3.6] Let v be a vertex as above. There are decompositions of $\mathsf{Aug}(v;\mathbb{K})$ and $\mathrm{MC}(v;\mathbb{K})$ over the finite set $\mathrm{NR}(v)$*

$$\mathrm{MC}(v;\mathbb{K}) = \coprod_{\rho\in\mathrm{NR}(v)} B(v;\mathbb{K})\cdot d_\rho$$

In particular, $\mathrm{MC}(v;\mathbb{K}) = \varnothing$ if $\mathrm{val}(v)$ is odd.

Now we can describe $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$.

Objects. Notice that $A_v^{(1)} \cong I_v$ with $v_{a,\ell} = v_{a,\ell}^{11}$, where $I_v = I_r(\mu_v)$ is the internal DGA of v defined in Example/Definition 2.31. Hence, the set of objects of $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$ is the variety of augmentations for I_v

$$\mathcal{O}\mathsf{b}\ \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) = \mathsf{Aug}(v;\mathbb{K}).$$

Notice also that by Lemma 6.13, $\mathsf{Aug}(v;\mathbb{K}) = \varnothing$ if r is odd.

Assumption 6.14. From now on, we will assume r is even.

Morphisms. As in Section 6.1.1, define M^{12} to be the free $\mathbb{K}$-module generated by $\{c_{ab}^{12}\}$ in $I_v^{(2)}$. Then, for any two augmentations ϵ_1, ϵ_2, the set of morphisms in $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$ is

$$\mathrm{Hom}_+(\epsilon_1,\epsilon_2) = \mathsf{M}_{12}^\vee = \mathbb{K}\langle c_{ab}^+, 1\le b < a \le r\rangle$$

as a free $\mathbb{K}$-module, where $\mathsf{M}_{12}^\vee := (\mathsf{M}^{12})^*[-1]$, and $x^+ := (x^{12})^\vee = (x^{12})^*[-1]$. In particular, $|c_{ab}^+| = |c_{ab}^{12}| + 1$.

Compositions $\boldsymbol{m_K}$. For any $K\ge 1$, and objects $\epsilon_1,\epsilon_2,\ldots,\epsilon_{K+1}$, the composition map

$$\begin{aligned} m_K : \mathrm{Hom}_+(\epsilon_K,\epsilon_{K+1}) \otimes \ldots \otimes \mathrm{Hom}_+(\epsilon_2,\epsilon_3) \otimes \mathrm{Hom}_+(\epsilon_1,\epsilon_2) \\ \to \mathrm{Hom}_+(\epsilon_1,\epsilon_{K+1}) \end{aligned}$$

is defined as before.

- Let $K = 1$ and $\mathbf{e} = (\epsilon_1, \epsilon_2)$. Then by (6.8), we compute

$$\partial_{\mathbf{e}}^{(2)} c_{ab}^{12} = \sum_{A>a} \phi_{\mathbf{e}}(c_{aA}^{11}) c_{Ab}^{12} + \sum_{B<b} \widetilde{c_{aB}^{12}} \phi_{\mathbf{e}}(c_{Bb}^{22})$$

where $1 \leq a < b \leq r$. It follows that

$$m_1(c_{AB}^+) = \sum_{B<a<A} \epsilon_1(v_{a,A-a}) c_{aB}^+ + (-1)^{|c_{AB}^+|} \sum_{B<b<A} \epsilon_2(v_{B,b-B}) c_{Ab}^+ \tag{6.13}$$

where $1 \leq B < A \leq r$.

- Let $K \geq 2$ and $\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{K+1})$ be given. We need to compute $m_K(c_{i_K}^+, \ldots, c_{i_2}^+, c_{i_1}^+)$, where $c_{i_j}^+ \in \{c_{ab}^+\}$ is a generator for $\mathrm{Hom}_+(\epsilon_j, \epsilon_{j+1})$. To do that, we need to calculate $\partial_{\mathbf{e}}^{(K+1)} c_{ab}^{1\,K+1}$ and look at the the $\mathbb{K}$-coefficient of the monomial $c_{i_1}^{12} c_{i_2}^{23} \ldots c_{i_k}^{K\,K+1}$ in the result. By the formula (6.8) for $\partial^{(K+1)} c_{ab}^{1\,K+1}$, with $1 \leq b < a \leq r$, we have

$$\begin{aligned} \partial_{\mathbf{e}}^{(K+1)}(c_{ab}^{1K+1}) = & \sum \phi_{\mathbf{e}}(c_{aa_1}^{11}) c_{a_1a_2}^{12} \ldots c_{a_Kb}^{KK+1} \\ & + \sum \widetilde{c_{aa_1}^{12}} \phi_{\mathbf{e}}(c_{a_1a_2}^{22}) c_{a_2a_3}^{23} \ldots c_{a_Kb}^{KK+1} + \cdots \end{aligned} \tag{6.14}$$

in which only the first two terms contribute to m_K. In addition, the first summation is over all $1 \leq a_1, \ldots, a_K \leq r$ such that $a < a_1 > a_2 > \ldots > a_K > b$, and the second summation is over all $1 \leq a_1, \ldots, a_K \leq r$ such that $a > a_1 < a_2 > a_3 > \ldots > a_K > b$.

We can then divide the computation of $m_K(c_{i_K}^+ \otimes \cdots \otimes c_{i_2}^+ \otimes c_{i_1}^+)$ into 4 cases:

1) If $K \geq r+1$, then both of the two summations above are empty. Hence, we have

$$m_K \equiv 0 \quad \forall K \geq r+1.$$

2) For $2 \leq K \leq r$ and any $1 \leq a = a_0, a_1, a_2, \ldots, a_K, a_{K+1} = b \leq r$ such that $a_0 > a_1 < a_2 > a_3 > \cdots > a_K > a_{K+1}$ and $a_0 > a_{K+1}$, we have

$$m_K(c_{a_Ka_{K+1}}^+ \otimes c_{a_{K-1}a_K}^+ \otimes \cdots \otimes c_{a_2a_3}^+ \otimes c_{a_0a_1}^+) = (-1)^{\sigma_K + |c_{a_0a_1}^+|} \epsilon_2(v_{a_1,a_2-a_1}) c_{ab}^+.$$

3) For $2 \leq K \leq r-1$ and any $1 \leq a < a_1, a_2, \ldots, a_{K+1} = b \leq r$ such that $a_1 > a_2 > \ldots > a_{K+1}$, we have

$$m_K(c_{a_Ka_{K+1}}^+ \otimes \cdots \otimes c_{a_2a_3}^+ \otimes c_{a_1a_2}^+) = (-1)^{\sigma_K} \sum_{a_{K+1}<a<a_1} \epsilon_1(v_{a,a_1-a}) c_{aa_{K+1}}^+.$$

4) For $2 \leq K \leq r$ and $(c^+_{i_1}, c^+_{i_2}, \dots, c^+_{i_K})$ which is not of the form $(c^+_{a_0a_1}, c^+_{a_2a_3}, \dots, c^+_{a_Ka_{K+1}})$ in (2) or $(c^+_{a_1a_2}, c^+_{a_2a_3}, \dots, c^+_{a_Ka_{K+1}})$ in (3) above, then

$$m_K(c^+_{i_K} \otimes \cdots \otimes c^+_{i_2} \otimes c^+_{i_1}) = 0.$$

Here, we use $\sigma_K = \sigma_K(c^+_{a_0a_1}, c^+_{a_2a_3}, \dots, c^+_{a_Ka_{K+1}})$ as in Definition 4.3, and this gives the description of $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$.

Now we want to compute the whole diagram of augmentation categories

$$\mathcal{A}\mathsf{ug}_+(\mathcal{V},\boldsymbol{\mu};\mathbb{K}) := (\mathcal{A}\mathsf{ug}_+(V_\mathsf{L},\mu_\mathsf{L};\mathbb{K}) \xleftarrow{\mathfrak{r}_\mathsf{L}} \mathcal{A}\mathsf{ug}_+(V,\mu;\mathbb{K}) \xrightarrow{\mathfrak{r}_\mathsf{R}} \mathcal{A}\mathsf{ug}_+(V_\mathsf{R},\mu_\mathsf{R};\mathbb{K})).$$

The functor $\mathfrak{r}_\mathsf{L}$ is clear under the identification

$$\mathcal{A}\mathsf{ug}_+(V,\mu;\mathbb{K}) \cong \mathcal{A}\mathsf{ug}_+(V_\mathsf{L},\mu_\mathsf{L};\mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}),$$

which is the obvious restriction functor

$$\mathcal{A}\mathsf{ug}_+(V_\mathsf{L},\mu_\mathsf{L};\mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(V_\mathsf{L},\mu_\mathsf{L};\mathbb{K}).$$

It suffices to describe $\mathfrak{r}_\mathsf{R}$.

$\mathfrak{r}_\mathsf{R}$ on objects. Given any object ϵ in $\mathcal{A}\mathsf{ug}_+(V;\mathbb{K})$, denote $\epsilon_\mathsf{L} := \epsilon \circ \iota_\mathsf{L}$, $\epsilon_\mathsf{R} := \epsilon \circ \iota_\mathsf{R}$, and $\epsilon_v := \epsilon \circ \iota_v$, where $\iota_v : I_v \hookrightarrow A$ is the natural inclusion of DGAs. Then by definition, $\mathfrak{r}_\mathsf{R}(\epsilon) = \epsilon_\mathsf{R}$.

$\mathfrak{r}_\mathsf{R}$ on objects also admits an alternative description. We will use the identification *of objects*

$$\mathfrak{h}_* : \mathsf{Aug}_+(V_*,\mu_*;\mathbb{K}) \cong \mathrm{MC}(V_*,\mu_*;\mathbb{K}) \quad \text{for} \quad * = \mathsf{L},\mathsf{R}$$

and

$$\mathfrak{h}_v : \mathsf{Aug}_+(v;\mathbb{K}) \cong \mathrm{MC}(v;\mathbb{K})$$

by Lemmas 6.6 and 6.12. Then $\epsilon = (d_\mathsf{L}, d_v)$, where $(C_* := C(V_*,\mu_*), d_* := d(\epsilon_*))$ for $* = \mathsf{L},\mathsf{R}$ and $(C_v := C(v), d_v := d(\epsilon_v))$ are Morse complexes defined as in Lemmas 6.4 and 6.12, respectively.

Now, under the identification, $\mathfrak{r}_{\mathsf{R}}$ is in fact a quotient of cochain complexes

$$\mathfrak{r}_{\mathsf{R}} : (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_v, d_v) \twoheadrightarrow (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}}) \cong (C_{\mathsf{R}}, d_{\mathsf{R}})$$

as follows: By definition, we have

$$C_{\mathsf{L}} = \bigoplus_{a=1}^{n_{\mathsf{L}}} \mathbb{K} e_{a,\mathsf{L}}, \qquad C_{\mathsf{R}} = \bigoplus_{a=1}^{n_{\mathsf{R}}} \mathbb{K} e_{a,\mathsf{R}} \quad \text{and} \quad C_v = \bigoplus_{a=1}^{r} \mathbb{K}[Z] e_{a,v}$$

with the basis elements $e_{a,\mathsf{L}}$, $e_{a,\mathsf{R}}$ and $e_{a,v}$ corresponding to the a-th strand of V_{L} and V_{R}, and the a-th right half-edge $h_{v,a}$ of v, respectively.

We define $\varphi_V : C_{\mathsf{L}} \oplus C_v \twoheadrightarrow C_{\mathsf{R}}$ by

$$\varphi_V(e_{a,\mathsf{L}}) = e_{a',\mathsf{R}}, \quad \varphi_V(e_{a,v}) = e_{a_v+a-1,\mathsf{R}}, \quad \text{and} \quad \varphi_V(Z^n e_{a,v}) = 0 \quad \forall n > 0.$$

Then by the formula (6.12) for ϕ_{R},

$$\mathfrak{r}_{\mathsf{R}} = \varphi_V : (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_v, d_v) \twoheadrightarrow (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}}) \cong (C_{\mathsf{R}}, d_{\mathsf{R}})$$

is indeed a quotient of complexes, where $(C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}})$ is the subcomplex of $(C_{\mathsf{R}}, d_{\mathsf{R}})$ with $C_{v_{\mathsf{R}}} = \bigoplus_{1\le a\le r} \mathbb{K} e_{a_v+a-1,\mathsf{R}}$.

Remark 6.15. Similar to the definition of $\mathcal{MC}(T_n, \mu; \mathbb{K})$, one can define the DG-category $\mathcal{MC}(v; \mathbb{K})$ as the canonical lift of $\mathrm{MC}(v; \mathbb{K})$. Observe that φ_V in fact defines naturally a DG functor

$$\begin{aligned}\mathcal{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) &\times \mathcal{MC}(v; \mathbb{K}) \\ &\to \mathcal{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \times \mathcal{MC}(v_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K}) \hookrightarrow \mathcal{MC}(V_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K})\end{aligned}$$

of DG categories. The second (but not the first) functor will be used below.

$\mathfrak{r}_{\mathsf{R}}$ on morphisms. By definition, the A_∞-functor $\mathfrak{r}_{\mathsf{R}}$ on morphisms are given by a collection of assignments $(\mathfrak{r}_{\mathsf{R}}^{(K)})_{K\ge 1}$. That is, for any $K \ge 1$ and $K+1$ objects $\epsilon_1, \epsilon_2, \dots, \epsilon_{K+1}$ in $\mathcal{A}\mathsf{ug}_+(V, \mu; \mathbb{K})$, we have a map of degree $1-K$

$$\begin{aligned}\mathfrak{r}_{\mathsf{R}}^{(K)} : \mathrm{Hom}_+(\epsilon_K, \epsilon_{K+1}) &\otimes \cdots \otimes \mathrm{Hom}_+(\epsilon_2, \epsilon_3) \otimes \mathrm{Hom}_+(\epsilon_1, \epsilon_2) \\ &\to \mathrm{Hom}_+(\epsilon_{1,\mathsf{R}}, \epsilon_{K+1,\mathsf{R}}),\end{aligned}$$

where $\epsilon_{i,\mathsf{R}} := \mathfrak{r}_{\mathsf{R}}(\epsilon_i)$. It suffices to determine $\mathfrak{r}_{\mathsf{R}}^{(K)}(c_{i_K}^+ \otimes \cdots \otimes c_{i_2}^+ \otimes c_{i_1}^+)$ for any collection of generators $c_{i_j}^+$ for a free $\mathbb{K}$-module $\mathrm{Hom}_+(\epsilon_j, \epsilon_{j+1}) = \mathbb{K}\langle k_{ab}^+ \mid 1 \le a \le b \le n_{\mathsf{L}}\rangle \oplus \mathbb{K}\langle c_{ab}^+ \mid 1 \le b < a \le r\rangle$.

As in Section 6.1.1, the sequences $\mathbf{e} \coloneqq (\epsilon_1, \epsilon_2, \dots, \epsilon_{K+1})$ and $\mathbf{e}_\mathsf{R} \coloneqq (\epsilon_{1,\mathsf{R}}, \epsilon_{2,\mathsf{R}}, \dots, \epsilon_{K+1,\mathsf{R}})$ define diagonal augmentations for $A^{(K+1)}$ and $A_\mathsf{R}^{(K+1)}$, respectively. We define

$$\phi_{\mathbf{e},\mathsf{R}}^{(K+1)} \coloneqq \phi_\mathbf{e} \circ \phi_\mathsf{R}^{(K+1)} \circ \phi_{\mathbf{e}_\mathsf{R}}^{-1} : A_\mathsf{R}^{(K+1)} \to A^{(K+1)}.$$

Recall that we have

$$\begin{aligned}(6.15)\quad \mathfrak{r}_\mathsf{R}^{(K)}(c_{i_K}^+ \otimes \cdots \otimes c_{i_2}^+ \otimes c_{i_1}^+) \\ = (-1)^{\sigma_K} \sum k_{ab}'^+ \langle \phi_{\mathbf{e},\mathsf{R}}^{(K+1)} k_{ab}'^{1,K+1}, c_{i_1}^{12} c_{i_2}^{23} \dots c_{i_K}^{K,K+1} \rangle,\end{aligned}$$

where $\langle \phi_{\mathbf{e},\mathsf{R}}^{(K+1)} k_{ab}'^{1,K+1}, c_{i_1}^{1,2} c_{i_2}^{2,3} \dots c_{i_K}^{K,K+1} \rangle$ denotes the $\mathbb{K}$-coefficient of the monomial $c_{i_1}^{12} c_{i_2}^{23} \dots c_{i_K}^{K,K+1}$ in $\phi_{\mathbf{e},\mathsf{R}}^{(K+1)}(k_{ab}'^{1,K+1})$, and $\sigma_K \coloneqq \sigma_K(c_{i_1}^+, c_{i_2}^+, \dots, c_{i_K}^+)$ as in Definition 4.3. Observe that, by the formula for $\phi_\mathsf{R}^{(K+1)}$, the A_∞-functor $\mathfrak{r}_\mathsf{R}$ is simply the composition

$$\begin{aligned}\mathfrak{r}_\mathsf{R} : \mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L}; \mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v; \mathbb{K}) \\ \xrightarrow{\mathrm{Id} \times \mathfrak{r}_{v,\mathsf{R}}} \mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L}; \mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K}) \hookrightarrow \mathcal{A}\mathsf{ug}_+(V_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K})\end{aligned}$$

Here, the functor $\mathfrak{r}_{v,\mathsf{R}} = \mathfrak{r}_\mathsf{R}|_{\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})} : \mathcal{A}\mathsf{ug}_+(v; \mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K})$ is the right restriction functor for the bordered Legendrian graph v obtained by removing the extra parallel strands of V as before. The second arrow in the composition is the natural inclusion of DG categories

$$\mathcal{MC}(V_\mathsf{L}, \mu_\mathsf{L}; \mathbb{K}) \times \mathcal{MC}(v_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K}) \hookrightarrow \mathcal{MC}(V_\mathsf{R}, \mu_\mathsf{R}; \mathbb{K})$$

in Remark 6.15 under the obvious identification. Therefore it suffices to describe $\mathfrak{r}_{v,\mathsf{R}}$.

Let $K = 1$ and $\mathbf{e} = (\epsilon_1, \epsilon_2)$. Apply the formula (6.11) for $1 \le a \le b \le r$ to compute

$$\phi_{\mathbf{e},\mathsf{R}}^{(2)}(k_{a_v+a-1,a_v+b-1}'^{12}) = \sum_{A>b} \phi_\mathbf{e}(v_{a,A-a}^{11}) c_{Ab}^{12} + \sum_{B<a} \widetilde{c_{aB}^{12}} \phi_\mathbf{e}(v_{B,b-B}^{22}).$$

Then by (6.15), for $1 \le B < A \le r$, we have

$$\begin{aligned}(6.16)\qquad \mathfrak{r}_{v,\mathsf{R}}^{(1)}(c_{AB}^+) = \sum_{a \le B} \epsilon_1(v_{a,A-a}) k_{a_v+a-1,a_v+B-1}'^+ \\ + (-1)^{|c_{AB}^+|} \sum_{b \ge A} \epsilon_2(v_{B,b-B}) k_{a_v+A-1,a_v+b-1}'^+.\end{aligned}$$

In general, let $K \geq 2$ and $\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{K+1})$. We apply the formula (6.11) again for $1 \leq a \leq b \leq r$ to obtain

$$\begin{aligned}\phi^{(K+1)}_{\mathbf{e},\mathsf{R}}(k'^{1,K+1}_{a_v+a-1,a_v+b-1}) = &\sum \phi_{\mathbf{e}}(v^{11}_{a,a_1-a})c^{12}_{a_1a_2}\cdots c^{KK+1}_{a_Kb}\\ &+\sum \widetilde{c^{12}_{aa_1}}\phi_{\mathbf{e}}(v^{22}_{a_1,a_2-a_1})c^{23}_{a_2a_3}\cdots c^{KK+1}_{a_Kb}+\cdots\end{aligned}$$

in which only the first two terms contribute to $(\mathfrak{r}_{v,R})_K$. In addition, the first summation is over all $1 \leq a_1, \ldots, a_K \leq r$ such that $a < a_1 > a_2 > \ldots > a_K > b$, and the second summation is over all $1 \leq a_1, \ldots, a_K \leq r$ such that $a > a_1 < a_2 > a_3 > \ldots > a_K > b$.

Remark 6.16. Notice that this formula is exactly the same as (6.14) for the computation of m_K.

Similar to m_K, we can divide the computation of $\mathfrak{r}^{(K)}_{v,\mathsf{R}}(c^+_{i_K} \otimes \cdots \otimes c^+_{i_2} \otimes c^+_{i_1})$ into 4 cases:

1) If $K \geq r+1$, then both of the two summations above are empty. Hence, we have

$$\mathfrak{r}^{(K)}_{v,\mathsf{R}} \equiv 0 \quad \forall K \geq r+1. \tag{6.17}$$

2) For $2 \leq K \leq r$ and any $1 \leq a = a_0, a_1, a_2, \ldots, a_K, a_{K+1} = b \leq r$ such that $a_0 > a_1 < a_2 > a_3 > \cdots > a_K > a_{K+1}$ and $a_0 \leq a_{K+1}$, we have

$$\begin{aligned}&\mathfrak{r}^{(K)}_{v,\mathsf{R}}(c^+_{a_Ka_{K+1}} \otimes \cdots \otimes c^+_{a_2a_3} \otimes c^+_{a_0a_1})\\ &\qquad = (-1)^{\sigma_K+|c^+_{a_0a_1}|}\epsilon_2(v_{a_1,a_2-a_1})k'^{+}_{a_v+a-1,a_v+b-1}.\end{aligned} \tag{6.18}$$

3) For $2 \leq K \leq r-1$ and any $1 \leq a < a_1, a_2, \ldots, a_{K+1} = b \leq r$ such that $a_1 > a_2 > \cdots > a_{K+1}$, we have

$$\begin{aligned}&\mathfrak{r}^{(K)}_{v,\mathsf{R}}(c^+_{a_Ka_{K+1}} \otimes \cdots \otimes c^+_{a_1a_2})\\ &\qquad = (-1)^{\sigma_K}\sum_{a \leq a_{K+1}} \epsilon_1(v_{a,a_1-a})k'^{+}_{a_v+a-1,a_v+a_{K+1}-1}.\end{aligned} \tag{6.19}$$

4) For $2 \leq K \leq r$ and $(c^+_{i_1}, c^+_{i_2}, \ldots, c^+_{i_K})$ which is not of the form $(c^+_{a_0a_1}, c^+_{a_2a_3}, \ldots, c^+_{a_Ka_{K+1}})$ in (2) or $(c^+_{a_1a_2}, c^+_{a_2a_3}, \ldots, c^+_{a_Ka_{K+1}})$ in (3) above, then

$$\mathfrak{r}^{(K)}_{v,\mathsf{R}}(c^+_{i_K} \otimes \cdots \otimes c^+_{i_2} \otimes c^+_{i_1}) = 0. \tag{6.20}$$

Here we use $\sigma_K = \sigma_K(c^+_{a_0a_1}, c^+_{a_2a_3}, \ldots, c^+_{a_K a_{K+1}})$ as in Definition 4.3 as always, and this finishes the description of $\mathfrak{r}_{v,R} : \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R};\mathbb{K})$, hence that of $\mathfrak{r}_\mathsf{R}$.

Furthermore, the following gives a simpler description of $\mathcal{A}\mathsf{ug}_+(\mathcal{V}, \boldsymbol{\mu};\mathbb{K})$:

Lemma 6.17. *The A_∞-functor*

$$\mathfrak{r}_{v,\mathsf{R}} : \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R};\mathbb{K})$$

is an A_∞-equivalence. As a consequence, we obtain a commutative diagram of A_∞-categories:

$$\begin{array}{ccccc}
\mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}) & \xleftarrow{\ \mathfrak{r}_\mathsf{L}\ } & \mathcal{A}\mathsf{ug}_+(V, \mu;\mathbb{K}) & \xrightarrow{\ \mathfrak{r}_\mathsf{R}\ } & \mathcal{A}\mathsf{ug}_+(V_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}) \\
\| & & \simeq \big\downarrow \mathrm{Id}\times\mathfrak{r}_{v,\mathsf{R}} & & \| \\
\mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}) & \xleftarrow{\ \mathfrak{p}_1\ } & \mathcal{A}\mathsf{ug}_+(V_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}) & \xhookrightarrow{\ \mathfrak{i}\ } & \mathcal{A}\mathsf{ug}_+(V_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}) \\
\big\downarrow \cong & & \cong \big\downarrow & & \cong \big\downarrow \\
\mathcal{MC}(V_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}) & \xleftarrow{\ \mathfrak{p}_1\ } & \mathcal{MC}(V_\mathsf{L}, \mu_\mathsf{L};\mathbb{K}) \times \mathcal{MC}(v_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}) & \xhookrightarrow{\ \mathfrak{i}\ } & \mathcal{MC}(V_\mathsf{R}, \mu_\mathsf{R};\mathbb{K})
\end{array}$$

Here $\mathfrak{p}_1$'s are the projections to the first factors, $\mathfrak{i}$'s are the natural inclusions and the functor $\mathrm{Id}\times\mathfrak{r}_\mathsf{R}$ *is defined via*

$$\begin{aligned}
\mathcal{A}\mathsf{ug}_+(V;\mathbb{K}) &\cong \mathcal{A}\mathsf{ug}_+(V_\mathsf{L};\mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) \\
&\xrightarrow{\mathrm{Id}\times\mathfrak{r}_{v,\mathsf{R}}} \mathcal{A}\mathsf{ug}_+(V_\mathsf{L};\mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v_\mathsf{R}, \mu_\mathsf{R};\mathbb{K}).
\end{aligned}$$

Therefore we have equivalences between all the three rows with each row viewed as bordered A_∞-categories. In particular, the result implies that $\mathcal{A}\mathsf{ug}_+(\mathcal{V}, \boldsymbol{\mu};\mathbb{K})$ is equivalent to the third row of Morse complex categories, which involves only DG categories and DG functors. This will be one key step in showing our main result "augmentations are sheave".

An algebraic proof of Lemma 6.17. Notice that the equivalence between the second and third row is just a direct consequence of Lemma 6.6. The only nontrivial part is to show $\mathfrak{r}_{v,\mathsf{R}}$ is an A_∞-equivalence. For simplicity, we assume that $V = v$, that is, there are no parallel strands and so $(n_\mathsf{L}, n_\mathsf{R}) = (0, r)$, $a_v = 1$, and $\mathfrak{r}_{v,\mathsf{R}} = \mathfrak{r}_\mathsf{R}$.

Remember that on objects $\mathfrak{r}_\mathsf{R}$ is just the map of augmentation varieties $r_\mathsf{R} : \mathcal{A}\mathsf{ug}(V, \mu;\mathbb{K}) \to \mathcal{A}\mathsf{ug}(V_\mathsf{R}, \mu_\mathsf{R};\mathbb{K})$, which is surjective. Therefore it suffices to show that $\mathfrak{r}_\mathsf{R}$ is fully faithful, i.e., for any two augmentations ϵ_1, ϵ_2 in

$\mathcal{A}\mathsf{ug}_+(V,\mu;\mathbb{K})$, the cochain map

$$\mathfrak{r}_{\mathsf{R}}^{(1)} : \mathrm{Hom}_V(\epsilon_1,\epsilon_2) \to \mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}}) \tag{6.21}$$

is a quasi-isomorphism. Here $\mathrm{Hom}_T(-,-) \coloneqq \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(T;\mathbb{K})}(-,-)$ for any bordered Legendrian graph T.

Equivalently, it suffices to show that the cone of $-\mathfrak{r}_{\mathsf{R}}^{(1)}$ is acyclic. By definition, the cone of $-\mathfrak{r}_{\mathsf{R}}^{(1)}$ is the cochain complex

$$\mathrm{Cone}(-\mathfrak{r}_{\mathsf{R}}^{(1)}) \coloneqq (\mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}}) \oplus \mathrm{Hom}_V(\epsilon_1,\epsilon_2)[1], \delta)$$

with differential

$$\delta \coloneqq \begin{pmatrix} m_{1,\mathsf{R}} & -\mathfrak{r}_{\mathsf{R}}^{(1)} \\ 0 & -m_1 \end{pmatrix}$$

Let $\mathrm{End}^*(C_{\mathsf{R}})$ be the free $\mathbb{Z}$-graded $\mathbb{K}$-module of $\mathbb{K}$-linear endomorphisms of $C_{\mathsf{R}} = \bigoplus_{1\le a\le r} \mathbb{K}e_p$ without filtration. For $i=1,2$, remember that $d(\epsilon_{i,\mathsf{R}})$ is the image of $\epsilon_{i,\mathsf{R}}$ under the DG equivalence $\mathfrak{h}_{\mathsf{R}} : \mathcal{A}\mathsf{ug}_+(V_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K}) \cong \mathcal{MC}(V_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K})$ in Lemma 6.6 for V_{R}. Then we define a cochain complex $(\mathcal{E}, D) \coloneqq (\mathrm{End}^*(C_{\mathsf{R}}), D)$ with differential

$$D(f) \coloneqq d(\epsilon_{2,\mathsf{R}}) \circ f - (-1)^{|f|} f \circ d(\epsilon_{1,\mathsf{R}}).$$

Notice that we have

$$\mathrm{Hom}_V(\epsilon_1,\epsilon_2)[1] = \bigoplus_{1\le b<a\le r} \mathbb{K}c_{ab}^+[1], \qquad \mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}}) = \bigoplus_{1\le a\le b\le r} \mathbb{K}k_{ab}'^+.$$

In particular, $|c_{ab}^+[1]| = |c_{ab}^+| - 1 = \mu(a) - \mu(b)$ as before. Let $\alpha : \mathrm{Cone}(-\mathfrak{r}_{R,1}) \to \mathcal{E}$ be a $\mathbb{K}$-linear map defined as

$$\begin{aligned} \alpha(k_{ab}'^+) &\coloneqq (-1)^{s(a,b)}(e_b \otimes e_a^*), && 1 \le a \le b \le r; \\ \alpha(c_{ab}^+[1]) &\coloneqq (-1)^{s(a,b)}(e_b \otimes e_a^*), && 1 \le b < a \le r, \end{aligned}$$

where $s(a,b) = \mu(a)(\mu(b)+1)+1$.

Claim 6.18. *The map* $\alpha : \mathrm{Cone}(-\mathfrak{r}_{\mathsf{R}}^{(1)}) \to \mathcal{E}$ *is an isomorphism of cochain complexes.*

Proof. Clearly, the map α is a $\mathbb{K}$-linear isomorphism of degree 0 and it suffices to show that α is a cochain map. Firstly observe that $\alpha|_{\mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}})}$

is just the composition

$$\mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}}) \xrightarrow{\mathfrak{h}_{\mathsf{R}}^{(1)}} \mathrm{Hom}_{\mathcal{MC}(V_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K})}(d(\epsilon_{1,\mathsf{R}}),d(\epsilon_{2,\mathsf{R}})) \hookrightarrow \mathcal{E},$$

where $\mathfrak{h}_{\mathsf{R}}^{(1)}$ is induced from the DG equivalence $\mathfrak{h}_{\mathsf{R}}:\mathcal{A}\mathsf{ug}_+(V_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K})\cong\mathcal{MC}(V_{\mathsf{R}},\mu_{\mathsf{R}};\mathbb{K})$ and the second arrow is the inclusion of complexes. Hence, it follows by Lemma 6.6 that $\alpha|_{\mathrm{Hom}_{V_{\mathsf{R}}}(\epsilon_{1,\mathsf{R}},\epsilon_{2,\mathsf{R}})}$ is a cochain map and it remains to show that $\alpha\circ\delta(x)=D\circ\alpha(x)$ for any homogeneous element $x\in\mathrm{Hom}_V(\epsilon_1,\epsilon_2)[1]$.

Let $x=\sum_{1\le B<A\le r}x_{AB}c_{AB}^+[1]$ and denote $x^+:=x[-1]$. In particular, $x_{AB}\neq 0$ implies that

$$|x|=|c_{AB}^+[1]|=\mu(A)-\mu(B).$$

Hence we have

$$\begin{aligned}
\alpha(x) &= \sum_{B<A}(-1)^{s(A,B)}x_{AB}(e_B\otimes e_A^*)\\
d(\epsilon_{1,\mathsf{R}}) &= \sum_{a<A}(-1)^{\mu(a)}\epsilon_{1,\mathsf{R}}(k'_{a,A})(e_A\otimes e_a^*)\\
d(\epsilon_{2,\mathsf{R}}) &= \sum_{B<b}(-1)^{\mu(B)}\epsilon_{2,\mathsf{R}}(k'_{B,b})(e_b\otimes e_B^*).
\end{aligned}$$

On the other hand, by definition, $\delta(x)=-\mathfrak{r}_{\mathsf{R}}^{(1)}(x^+)-m_1(x^+)[1]$. We apply the formula (6.13) to obtain

$$\begin{aligned}
&\alpha(m_1(x^+)[1])\\
&=\sum_{B<a<A}\epsilon_1(v_{a,A-a})x_{AB}\alpha(c_{aB}^+[1])+\sum_{B<b<A}(-1)^{|c_{AB}^+|}x_{AB}\epsilon_2(v_{B,b-B})\alpha(c_{Ab}^+[1])\\
&=\sum_{b<a<A}\epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}\alpha(c_{ab}^+[1])+\sum_{B<b<a}(-1)^{|c_{aB}^+|}x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})\alpha(c_{ab}^+[1])\\
&=\sum_{b<a<A}(-1)^{s(a,b)}\epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}(e_b\otimes e_a^*)\\
&\quad+\sum_{B<b<a}(-1)^{|c_{aB}^+|+s(a,b)}x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})(e_b\otimes e_a^*).
\end{aligned}$$

Similarly, by the formula (6.16), we have the following.

$$\begin{aligned}
&\alpha(\mathfrak{r}_{\mathsf{R}}^{(1)}(x^+)) \\
&= \sum_{a\le B<A} \epsilon_1(v_{a,A-a})x_{AB}\alpha(k'^{+}_{aB}) + \sum_{B<A\le b} (-1)^{|c^+_{AB}|} x_{AB}\epsilon_2(v_{B,b-B})\alpha(k'^{+}_{Ab}) \\
&= \sum_{a\le b<A} \epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}\alpha(k'^{+}_{ab}) + \sum_{B<a\le b} (-1)^{|c^+_{aB}|} x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})\alpha(k'^{+}_{ab}) \\
&= \sum_{a\le b<A} (-1)^{s(a,b)}\epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}(e_b\otimes e_a^*) \\
&\quad + \sum_{B<a\le b} (-1)^{|c^+_{aB}|+s(a,b)} x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})(e_b\otimes e_a^*).
\end{aligned}$$

By combining the two computations above, we then obtain

$$\begin{aligned}
&\alpha\circ\delta(x) \\
&= -\sum_{a,b<A} (-1)^{s(a,b)}\epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}(e_b\otimes e_a^*) \\
&\quad - \sum_{B<a,b} (-1)^{|c^+_{aB}|+s(a,b)} x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})(e_b\otimes e_a^*) \\
&= -\sum_{a,b<A} (-1)^{s(a,b)}\epsilon_{1,\mathsf{R}}(k'_{aA})x_{Ab}(e_b\otimes e_a^*) \\
&\quad + \sum_{B<a,b} (-1)^{\mu(a)-\mu(B)+s(a,b)} x_{aB}\epsilon_{2,\mathsf{R}}(k'_{Bb})(e_b\otimes e_a^*) \\
&= -\sum_{a,b<A} (-1)^{|x|}((-1)^{\mu(a)}\epsilon_{1,\mathsf{R}}(k'_{aA}))((-1)^{s(A,b)}x_{Ab})(e_b\otimes e_A^*)\circ(e_A\otimes e_a^*) \\
&\quad + \sum_{B<a,b} ((-1)^{s(a,B)}x_{aB})((-1)^{\mu(B)}\epsilon_{2,\mathsf{R}}(k'_{Bb}))(e_b\otimes e_B^*)\circ(e_B\otimes e_a^*) \\
&= -(-1)^{|x|}\alpha(x)\circ d(\epsilon_{1,\mathsf{R}}) + d(\epsilon_{2,\mathsf{R}})\circ\alpha(x) \\
&= D\circ\alpha(x)
\end{aligned}$$

as desired.

In the computation above, the third equality follows directly from the formulas for $\alpha(x)$ and $d(\epsilon_{i,\mathsf{R}})$ above. The only nontrivial part is the second equality involving two sign manipulations, which are of the same form as those in (6.5). Hence it follows by exactly the same argument and finishes the proof. □

Now, by the claim, it suffices to show that $(\mathcal{E}, D)$ is acyclic, which is purely a homological algebra problem. In fact, we can prove a stronger statement as follows, which finishes the proof of the lemma. □

Claim 6.19. *Let* $C \coloneqq \bigoplus_{a=1}^{r} \mathbb{K}e_a$ *be a free* $\mathbb{Z}$*-graded* $\mathbb{K}$*-module and* $(C, d_1), (C, d_2) \in \mathrm{MC}(C;\mathbb{K})$ *be two Morse complexes as in Definition 6.3 such that either* (C, d_1) *or* (C, d_2) *is acyclic. Let* $(\mathcal{E}, D) \coloneqq (\mathrm{End}^\bullet(C), D)$ *be the complex of* $\mathbb{K}$*-linear endomorphisms of* C *with differential given by* $D(f) \coloneqq d_2 \circ f - (-1)^{|f|} f \circ d_1$*. Then* $(\mathcal{E}, D)$ *is chain homotopic to zero.*

To prove the claim, we firstly make some preparation.

Definition 6.20. Let $\mathrm{GNR}(C)$ be the set of all involutions ρ *possibly with fixed points* on $I(C) \coloneqq \{1, 2, \ldots, n\}$ such that

$$|e_i| = |e_j| - 1$$

for all $i < j = \rho(i)$, and let $\mathrm{NR}(C)$ be the subset of $\mathrm{GNR}(C)$ consisting of fixed-point-free involutions.

Let $B(C;\mathbb{K})$ be the automorphism group of $(C, F^\bullet)$, where $F^\bullet$ is a filtration described in Definition 6.3. Then $B(C;\mathbb{K})$ acts on $\mathrm{MC}(C;\mathbb{K})$ via conjugation such that for any $g \in B(C;\mathbb{K})$ and $d \in \mathrm{MC}(C;\mathbb{K})$,

$$g \cdot d = g \circ d \circ g^{-1}.$$

Definition 6.21 (Canonical differentials). For each involution $\rho \in \mathrm{GNR}(C)$, a *canonical differential* $d_\rho \in \mathrm{MC}(C;\mathbb{K})$ is defined as

$$(-1)^{\mu(i)} d_\rho e_i = \begin{cases} e_j & i < j = \rho(i); \\ 0 & \text{otherwise.} \end{cases}$$

Recall the following lemma.

Lemma 6.22. *[27, Lem.4.5] The group action of* $B(C;\mathbb{K})$ *induces a decomposition of* $\mathrm{MC}(C;\mathbb{K})$ *into finitely many orbits*

$$\mathrm{MC}(C;\mathbb{K}) = \coprod_{\rho \in \mathrm{GNR}(C)} B(C;\mathbb{K}) \cdot d_\rho.$$

Now, we can prove Claim 6.19.

Proof of Claim 6.19. By Lemma 6.22, if (C, d_i) is acyclic, then $d_i = g_i \cdot d_{\rho_i} = g_i \circ d_{\rho_i} \circ g_i^{-1}$ for some $g_i \in B(C; \mathbb{K})$ and some involution $\rho_i \in \mathrm{NR}(C)$. Here, d_{ρ_i} is the canonical differential associated to ρ_i as in Definition 6.21. By definition, $d_{\rho_i} e_a = e_b$ if $a < b = \rho_i(a)$ and $d_{\rho_i} e_a = 0$ otherwise.

Let us define $\delta_{\rho_i} \in \mathcal{E}^{-1}$ as $\delta_{\rho_i} e_b = e_a$ if $a < b = \rho_i(a)$ and $\delta_{\rho_i} e_b = 0$ otherwise. Then clearly we have $\delta_{\rho_i}^2 = 0$ and

$$d_{\rho_i} \circ \delta_{\rho_i} + \delta_{\rho_i} \circ d_{\rho_i} = \mathrm{Id}\,.$$

Define $\delta_i \in \mathcal{E}^{-1}$ as $\delta_i \coloneqq g_i \cdot \delta_{\rho_i} = g_i \circ \delta_{\rho_i} \circ g_i^{-1}$. It follows immediately that $\delta_i^2 = 0$ and $d_i \circ \delta_i + \delta_i \circ d_i = \mathrm{Id}$.

Now, if (C, d_2) is acyclic, we define a map $H_2 : \mathcal{E} \to \mathcal{E}$ of degree -1 by $H_2(f) \coloneqq \delta_2 \circ f$. Then we have the following computation.

$$\begin{aligned}
[D, H_2](f) &= D \circ H_2(f) + H_2 \circ D(f) \\
&= D(\delta_2 \circ f) + \delta_2 \circ D(f) \\
&= d_2 \circ (\delta_2 \circ f) - (-1)^{|f-1|}(\delta_2 \circ f) \circ d_1 \\
&\quad + \delta_2 \circ (d_2 \circ f - (-1)^{|f|} f \circ d_1) \\
&= f
\end{aligned}$$

That is, $\mathrm{Id} = [D, H_2] : \mathcal{E} \to \mathcal{E}$ is chain homotopic to zero as desired. If (C, d_1) is acyclic, by a similar argument, one can show that $\mathrm{Id} = [D, H_1]$ with $H_1(f) = (-1)^{|f|} f \circ \delta_1$ as desired. □

Now we give an alternative geometric proof of Lemma 6.17.

A geometric proof of Lemma 6.17. Let us start by introducing an alternative parallel copies, say $v'^{(\bullet)}$, of the vertex v of type $(0, r)$ as depicted in Figure 19.

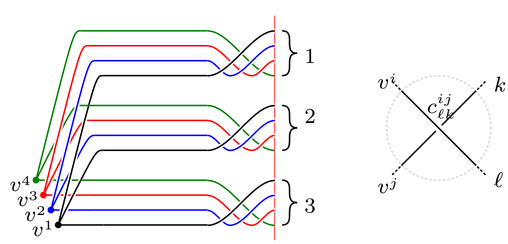


Figure 19: An alternative parallel m-copy near a vertex for $r = 3$ and $m = 4$.

Clearly $v'^{(\bullet)}$ is in $\mathcal{BLG}^{(\bullet)}_{\mathsf{Lag}}$ and its consistent LCH DGA will be denoted by $A^{(\bullet)}(v')$. By using the same labeling convention of the crossing as in Figure 18, the DGA $A^{(\bullet)}(v')$ is generated by the following sets of generators:

$$C_1 := \{c^{ij}_{aa} \mid 1 \leq i < j \leq m,\ 1 \leq a \leq r\};$$
$$C_2 := \{c^{ij}_{ab} \mid 1 \leq i < j \leq m,\ 1 \leq a < b \leq r\};$$
$$\widetilde{V} := \{v^i_{a,\ell} \mid 1 \leq i \leq m,\ 1 \leq a \leq r,\ \ell \in \mathbb{N}\}.$$

Note that the generators in C_1 are the crossings near the right border while the ones in C_2 are crossings near the vertices v^i in Figure 19. The grading is given by

$$|c^{ij}_{ab}| = \mu(a) - \mu(b) - 1, \qquad |v^i_{a,\ell}| = \mu(a) - \mu(a+\ell) + (N-1),$$

where $N = N(n, a, \ell)$ is the same as in (2.3).

The differential for the vertex generators $\widetilde{V}$ is the same as before, see (2.5) in Example/Definition 2.31. For the generators in C_1 and C_2, the differential is given by

$$\begin{aligned}
\partial'^{(m)} c^{ij}_{aa} &= \sum_{i<k<j} c^{ik}_{aa} c^{kj}_{aa};\\
\partial'^{(m)} c^{ij}_{ab} &= (-1)^{|v^i_{a,b-a}|+1} v^i_{a,b-a} c^{ij}_{bb} + \sum_{a \leq c < b} (-1)^{|c^{ij}_{ac}|+1} c^{ij}_{ac} v^j_{c,b-c}\\
&\quad + \sum_{\substack{i<k<j\\ a<c<b}} (-1)^{|c^{ik}_{ac}|+1} c^{ik}_{ac} c^{kj}_{cb}.
\end{aligned}$$

Now describe the corresponding augmentation category $\mathcal{A}\mathsf{ug}_+(v'; \mathbb{K})$. The objects of $\mathcal{A}\mathsf{ug}_+(v')$ is the augmentations of $A(v)$:

$$\mathcal{O}\mathsf{b}(\mathcal{A}\mathsf{ug}_+(v')) = \mathsf{Aug}(v; \mathbb{K}).$$

For any two augmentations ϵ_1, ϵ_2, the set of morphism becomes

$$\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2) = \mathbb{K}\left\langle c^{12\vee}_{ab} \mid 1 \leq a \leq b \leq r \right\rangle.$$

By dualizing the differential $\partial'^{(m)}$, we have the following A_∞ structure $\{m'_K\}_{K\geq 1}$ as follows:

- For $\epsilon_1, \epsilon_2 \in \mathsf{Aug}(v; \mathbb{K})$, the map

$$m'_1 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2)$$

  is defined as

$$m'_1\left(c_{ab}^{12\vee}\right) = -\sum_{c<a} \epsilon_1(v_{c,a-c}) c_{cb}^{12\vee} + \sum_{b<d} (-1)^{|c_{ab}^{12\vee}|} c_{ad}^{12\vee} \epsilon_2(v_{b,d-b}) \tag{6.22}$$

- For $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathsf{Aug}(v; \mathbb{K})$, the map

$$m'_2 : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_2, \epsilon_3) \otimes \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_3)$$

  is defined as

$$m'_2\left(c_{cd}^{12\vee} \otimes c_{ab}^{12\vee}\right) = \delta_{bc}(-1)^{|c_{ab}^{12\vee}|} c_{ad}^{12\vee}.$$

- For m'_K with $K \geq 3$, the higher composition m'_K vanishes.

Note that

$$\mathbf{1}' := \sum_{1 \leq a \leq r} -c_{a,a}^{12\vee} \in \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon, \epsilon)$$

is the unit for all $\epsilon \in \mathsf{Aug}(v')$, and hence $\mathcal{A}\mathsf{ug}_+(v')$ is a (strictly) unital A_∞-category.

We show that $\mathfrak{r}'_{\mathsf{R}} : \mathcal{A}\mathsf{ug}_+(v') \to \mathcal{A}\mathsf{ug}_+(v'_{\mathsf{R}})$ is an A_∞-equivalence. Similar as before, it suffices to check that the A_∞-functor $\mathfrak{r}'_{\mathsf{R}}$ is essentially surjective and fully-faithful. Indeed, $\mathfrak{r}'_{\mathsf{R}}$ is surjective (and hence essentially surjective). The main issue again is to show the following lemma.

Lemma 6.23. *As in the above setup,*

$$\mathfrak{r}'^{(1)}_{\mathsf{R}} : \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+(v'_{\mathsf{R}})}(\epsilon_{1,\mathsf{R}}, \epsilon_{2,\mathsf{R}})$$

is a quasi-isomorphism.

Proof. The DGA $A^{(m)}(v'_{\mathsf{R}})$ is nothing but a trivial bordered Legendrian described in Section 6.1.1. The generators are

$$\left\{k_{aa}^{ij} \mid 1 \leq a \leq r,\ 1 \leq i < j \leq m\right\} \coprod \left\{k_{ab}^{ij} \,\middle|\, 1 \leq a < b \leq r,\ 1 \leq i, j \leq m\right\},$$

with grading $|k^{ij}_{ab}| := \mu(a) - \mu(b) - 1$, and the differential

$$\partial^{(m)}_{\mathsf{R}} k^{ij}_{ab} = \sum_{\substack{a\le c\le b\\ 1\le k\le m}} (-1)^{|k^{ik}_{ac}|-1} k^{ik}_{ac} k^{kj}_{cb}.$$

Here we use the convention on generators in Assumption 6.2.

Now consider the DGA map

$$\phi^{(m)}_{\mathsf{R}} : A^{(m)}(v'_{\mathsf{R}}) \to A^{(m)}(v').$$

By a direct counting of polygons in Figure 19, we have

$$\phi^{(m)}_{\mathsf{R}}(k^{ii}_{ab}) = v^{i}_{a,b-a} \quad \text{and} \quad \phi^{(m)}_{\mathsf{R}}(k^{ij}_{ab}) = c^{ij}_{ab}.$$

Then by the construction below Proposition 4.5, we obtain

$$\mathfrak{r}'^{(1)}_{\mathsf{R}} : \operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+(v')}(\epsilon_1, \epsilon_2) \to \operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+(v'_{\mathsf{R}})}(\epsilon_{1,R}, \epsilon_{2,R});$$
$$c^{12\vee}_{ab} \mapsto k^{12\vee}_{ab}$$

for $1 \le a \le b \le r$. By comparing (6.3) and (6.22), we conclude that $\mathfrak{r}'^{(1)}_{\mathsf{R}}$ is a (strict) isomorphism between two chain complexes, in particular, a quasi-isomorphism. □

As a second step, we will show the following lemma.

Lemma 6.24. *There is a A_∞ quasi equivalence between $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$ and $\mathcal{A}\mathsf{ug}_+(v';\mathbb{K})$.*

Proof. Let us consider a zig-zag sequence of elementary consistent Lagrangian Reidemeister moves between $v^{(m)}$ and $v'^{(m)}$ as shown in Figure 20. Even though the illustration is given when $r = 3$ and $m = 4$, the similar works for arbitrary r and m. Note that each of them is in $\mathcal{BLG}^{(\bullet)}_{\mathsf{Lag}}$, so it makes sense to consider the corresponding consistent DGAs.

Since each arrow is elementary and consistent, there is a zig-zag of stabilizations between $A^{(\bullet)}(v)$ and $A^{(\bullet)}(v')$. Moreover, we already argued that $\mathcal{A}\mathsf{ug}_+(v')$ is strictly unital. Then by Proposition 4.13, we conclude that $\mathcal{A}\mathsf{ug}_+(v;\mathbb{K})$ and $\mathcal{A}\mathsf{ug}_+(v';\mathbb{K})$ are A_∞ quasi-equivalent to each other which proves Lemma 6.24. □

Note that the consistent Reidemeister moves in Figure 20 fix the right border. So Lemma 6.24 and the fact that $\mathfrak{r}'_{\mathsf{R}} : \mathcal{A}\mathsf{ug}_+(v';\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(v'_{\mathsf{R}};\mathbb{K})$

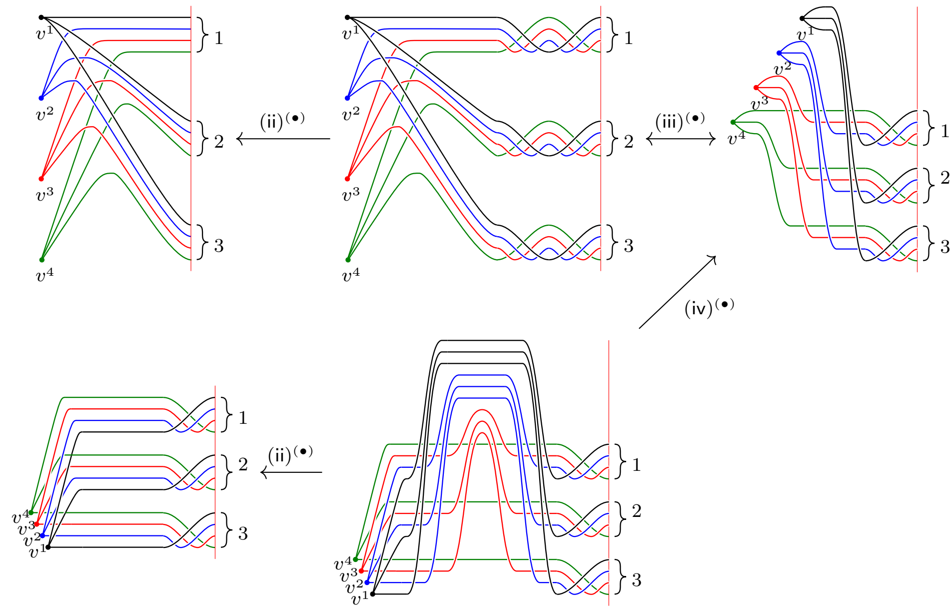


Figure 20: A sequence of consistent Reidemeister moves between $v^{(m)}$ and $v'^{(m)}$.

is an A_∞ equivalence imply that

$$\mathfrak{r}_{\mathsf{R}} : \mathcal{A}\mathsf{ug}_+(v;\mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(v_{\mathsf{R}}, \mu_{\mathsf{R}};\mathbb{K})$$

is also a A_∞ quasi equivalence. This gives a geometric proof of Lemma 6.17. □

6.2. Sheaf property of augmentation categories

Similar to [21], for the purpose of proving that "augmentations are sheaves" for Legendrian graphs, or more generally, bordered Legendrian graphs, we will need a sheaf property of augmentation categories, which we now explain.

Let $\mathcal{C}$ be a constructible sheaf of (homotopically) unital A_∞-categories on an interval $(x_{\mathsf{L}}, x_{\mathsf{R}})$, with respect to a stratification $\mathcal{X}$ consisting of a zero dimensional stratum $\{x_i\}$ and one-dimensional stratum $\{u_{i,i+1} = (x_i, x_{i+1})\}$. Equivalently, the restriction maps near 0-dimensional stratum x_i induce a diagram of unital A_∞-categories and (homotopically) unital A_∞-functors:

$$\mathcal{C}(u_{i-1,i}) \xleftarrow{\mathfrak{g}_L} \mathcal{C}_{x_i} \xrightarrow{\mathfrak{g}_R} \mathcal{C}(u_{i,i+1}).$$

It then follows from the sheaf axiom that if $x_i < x_{i+1} < \cdots < x_j$ are the zero dimensional strata in the interval $(a, b) \subset (x_{\mathsf{L}}, x_{\mathsf{R}})$, then

$$\mathcal{C}((a,b)) \cong \mathcal{C}_{x_i} \times_{\mathcal{C}(u_{i,i+1})} \mathcal{C}_{x_{i+1}} \times \cdots \times \mathcal{C}_{x_j},$$

is a fiber product. Here, the objects are tuples $(\xi_i, \xi_{i+1}, \ldots, \xi_j; f_{i,i+1}, \ldots, f_{j-1,j})$, where $\xi_k \in \mathcal{C}_{x_k}$ and $f_{k,k+1} : \mathfrak{g}_{\mathsf{R}}(\xi_k) \to \mathfrak{g}_{\mathsf{L}}(\xi_{k+1})$ is an isomorphism in $\mathcal{C}$, i.e. a closed degree 0 morphism in $\mathrm{Hom}^*_{\mathcal{C}(u_{k,k+1})}(\mathfrak{g}_{\mathsf{R}}(\xi_k), \mathfrak{g}_{\mathsf{L}}(\xi_{k+1}))$, whose cohomology is invertible by passing to the cohomological category of $\mathcal{C}(u_{k,k+1})$.

On the other hand, there is a full subcategory of this fiber product, called a *strict fiber product*

$$(\mathcal{C}_{x_i} \times_{\mathcal{C}(u_{i,i+1})} \mathcal{C}_{x_{i+1}} \times \cdots \times \mathcal{C}_{x_j})_{\mathsf{strict}},$$

in which all the $f_{k,k+1}$'s are identity morphisms, that is, $\mathfrak{g}_{\mathsf{R}}(\xi_k) = \mathfrak{g}_{\mathsf{L}}(\xi_{k+1})$.

Lemma 6.25. *[21, Lemma 7.4] Suppose that all $\mathfrak{g}_{\mathsf{L}}$'s satisfy the* isomorphism lifting property *such that any isomorphism $\phi : \mathfrak{g}_{\mathsf{L}}(\xi) \cong \eta'$ is the image under $\mathfrak{g}_{\mathsf{L}}$ of some isomorphism $\psi : \xi \cong \xi'$. Then the inclusion of the strict fiber product in the actual fiber product is an A_∞-equivalence.*

In our case for augmentation categories, we have the following lemma.

Lemma 6.26. *In the front projection picture, let $(\mathcal{T}, \boldsymbol{\mu})$ be any of the following elementary bordered Legendrian graph: (i) n parallel strands, (ii) a single crossing, (iii) a single left cusp, (iv) a single right cusp with a basepoint, or (v) a single vertex as in Example 6.1.2. Then $\mathcal{A}\mathsf{ug}_+(T_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \xleftarrow{\mathfrak{r}_{\mathsf{L}}} \mathcal{A}\mathsf{ug}_+(T, \mu; \mathbb{K})$ satisfies the isomorphism lifting property as in Lemma 6.25.*

Proof. The smooth cases are covered in [21, §7.2]. The only remaining case is when $\mathcal{T} = \mathcal{V}$ is a single vertex as in Example 6.1.2. By Lemma 6.17, the restriction functor $\mathfrak{r}_{\mathsf{L}}$ is just the obvious projection $\mathfrak{p}_1 : \mathcal{A}\mathsf{ug}_+(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \times \mathcal{A}\mathsf{ug}_+(v; \mathbb{K}) \to \mathcal{A}\mathsf{ug}_+(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K})$, which clearly satisfies the isomorphism lifting property. □

Then the following proposition is a consequence of the previous lemma.

Proposition 6.27. *Let $(\mathcal{T}, \boldsymbol{\mu})$ be any bordered Legendrian graph in $J^1\mathbf{U}$ with $\mathbf{U} = [x_{\mathsf{L}}, x_{\mathsf{R}}]$ such that the x-coordinates of the singularities in its front projection are all distinct, denoted by $x_{\mathsf{L}} = x_0 < x_1 < x_2 < \ldots < x_N < x_{N+1} = x_{\mathsf{R}}$. Assume that each right cusp has a basepoint and each vertex has no left half-edges.*

Let $\mathcal{C}$ be the constructible sheaf of unital A_∞-categories on $(x_{\mathsf{L}}, x_{\mathsf{R}})$ defined by

$$\begin{aligned}&\Big(\mathcal{C}(u_{x_{k-1,k}}) \xleftarrow{\mathfrak{g}_{\mathsf{L}}} \mathcal{C}_{x_k} \xrightarrow{\mathfrak{g}_{\mathsf{R}}} \mathcal{C}(u_{k,k+1})\Big)\\ &\coloneqq \Big(\mathcal{A}\mathsf{ug}_+(T|_{u_{k-1,k}}, \mu|_{u_{k-1,k}}; \mathbb{K}) \xleftarrow{\mathfrak{r}_{\mathsf{L}}} \mathcal{A}\mathsf{ug}_+(T|_{(x_{k-1},x_{k+1})}, \mu|_{(x_{k-1},x_{k+1})}; \mathbb{K})\\ &\qquad\qquad \xrightarrow{\mathfrak{g}_{\mathsf{R}}} \mathcal{A}\mathsf{ug}_+(T|_{u_{k,k+1}}, \mu|_{u_{k,k+1}}; \mathbb{K})\Big),\end{aligned}$$

where $u_{i,i+1} = (x_i, x_{i+1})$. Then we have the following equivalences between the rows of diagrams of unital A_∞-categories:

$$\begin{array}{ccccc}\mathcal{A}\mathsf{ug}_+(T_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) & \xleftarrow{\mathfrak{r}_{\mathsf{L}}} & \mathcal{A}\mathsf{ug}_+(T, \mu; \mathbb{K}) & \xrightarrow{\mathfrak{r}_{\mathsf{R}}} & \mathcal{A}\mathsf{ug}_+(T_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K})\\ \| & & \| & & \|\\ \mathcal{C}(u_{0,1}) & \xleftarrow{\mathfrak{g}_{\mathsf{L}}} & (\mathcal{C}_{x_1} \times_{\mathcal{C}(u_{1,2})} \mathcal{C}_{x_2} \times \ldots \times \mathcal{C}_{x_N})_{\mathsf{strict}} & \xrightarrow{\mathfrak{g}_{\mathsf{R}}} & \mathcal{C}(u_{N,N+1})\\ \| & & \big\downarrow \simeq & & \|\\ \mathcal{C}(u_{0,1}) & \xleftarrow{\mathfrak{g}_{\mathsf{L}}} & \mathcal{C}_{x_1} \times_{\mathcal{C}(u_{1,2})} \mathcal{C}_{x_2} \times \ldots \times \mathcal{C}_{x_N} & \xrightarrow{\mathfrak{g}_{\mathsf{R}}} & \mathcal{C}(u_{N,N+1})\end{array}$$

In particular, the augmentation categories $\{\mathcal{A}\mathsf{ug}_+(T|_{(a,b)}, \mu|_{(a,b)}; \mathbb{K})\}_{(a,b)\subset\mathbf{U}}$ form a sheaf.

Proof. Similar to [21, §7.2], the identification between the first two rows follows directly from the definition of diagrams of augmentation categories, and the co-sheaf property of LCH DGAs associated to bordered Legendrian graphs. It remains to show the middle-lower inclusion is an equivalence, which is done by Lemmas 6.26 and 6.25 above. □

6.3. Augmentations are sheaves

Let us start by recalling the result and strategy from [21].

Theorem 6.28. *[21] Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian knot. Then there is an A_∞ equivalence between two categories*

$$\mathcal{A}\mathsf{ug}_+(\Lambda; \mathbb{K}) \xrightarrow{\cong} \mathcal{C}_1(\Lambda; \mathbb{K}).$$

The first step of the proof is to cut the front $\pi_{\mathsf{fr}}(\Lambda) \subset \mathbb{R}_{xz}$ diagram into elementary pieces, each of which is a bordered Legendrian in $(0,1) \times \mathbb{R}_z$ having only one of a cusp, a crossing, or a basepoint. The next step is to show the equivalence for each pieces and then argue the sheaf property in both sides to conclude the global equivalence.

Among many ingredients in the proof, let us recall the equivalence when a bordered Legendrian is the bordered Legendrian graph T_n of trivial n-strands.

Proposition 6.29. *[21] Let $(\mathcal{T}_n, \boldsymbol{\mu})$ be the trivial bordered Legendrian graph with n-strands. Then there are DG equivalences:*

$$\mathcal{A}\mathsf{ug}_+(T_n, \mu; \mathbb{K}) \xrightarrow{\cong} \mathcal{MC}(T_n, \mu; \mathbb{K}) \quad \textit{and} \quad \mathcal{MC}(T_n, \mu; \mathbb{K}) \xrightarrow{\simeq} \mathcal{C}_1(T_n, \mu; \mathbb{K}).$$

The first equivalence has been described in Lemma 6.6. The second equivalence can be described as follows: Recall that

$$C = C(T_n) \coloneqq \bigoplus_{i=1}^{n} \mathbb{K} \cdot e_i$$

with $|e_i| = -\mu(i)$ and C is equipped with a filtration

$$F^\bullet : F^i = F^i C \coloneqq \oplus_{j>i} \mathbb{K} \cdot e_j.$$

In particular, $F^0 = C$, $F^n = 0$ and $\mathcal{S}_{T_n}$ is the regular cell complex induced by T_n. Then by Definition 5.18, the poset category $\mathcal{R}(\mathcal{S}_{T_n})$ is the A_{n+1}-quiver:

$$n \ \to \ n-1 \ \to \ \ldots \ \to \ 0,$$

where the node i corresponds to the region immediately below the strand i for $1 \leq i \leq n$ and 0 corresponds to the upper region.

Recall the indecomposable projective representation P_i of the quiver A_{n+1}

$$P_i \coloneqq (0 \to \cdots \to \mathbb{K} \xrightarrow{\mathrm{Id}} \mathbb{K} \xrightarrow{\mathrm{Id}} \ldots \mathbb{K} \xrightarrow{\mathrm{Id}} \mathbb{K}),$$

which consists of a copy of $\mathbb{K}$ at all nodes $k \leq i$. In general, any indecomposable representation of A_{n+1} is one of $S_{i,j}$

$$S_{ij} \coloneqq P_j / P_{i-1} = (0 \to \cdots \to 0 \to \mathbb{K} \xrightarrow{\mathrm{Id}} \mathbb{K} \xrightarrow{\mathrm{Id}} \ldots \xrightarrow{\mathrm{Id}} \mathbb{K} \to 0 \to \cdots \to 0).$$

In other words, S_{ij} consists of a copy of $\mathbb{K}$ at all nodes k for $i \leq k \leq j$, which admits a projective resolution

$$S'_{ij} \coloneqq (P_{i-1} \hookrightarrow P_j) \twoheadrightarrow S_{ij}.$$

Moreover, the path algebra $\mathbb{K}\langle A_{n+1}\rangle$ of A_{n+1} is of global cohomological dimension one, hence every submodule of a projective for $\mathbb{K}\langle A_{n+1}\rangle$ is still projective. It then follows that each object in the DG category $\mathcal{F}\mathsf{un}(A_{n+1}, \mathbb{K})$ (see Definition 5.8) splits and quasi-isomorphic to a direct sum of $S_{ij}[s]$'s.

The lemma below is an immediate consequence.

Lemma 6.30. *[21, Lem.7.38] Every representation R in $\mathcal{F}\mathsf{un}(A_{n+1}, \mathbb{K})$ is quasi-isomorphic to a representation R' defined as*

$$R'_n \hookrightarrow R'_{n-1} \hookrightarrow \cdots \hookrightarrow R'_0 \quad \text{and} \quad R'_{n+1} := 0$$

such that

1) *The maps $R'_{i+1} \hookrightarrow R'_i$ are injective;*
2) *The quiver representation R' in each cohomological degree is projective;*
3) *The differential on each R'_i/R'_{i+1} is zero.*

There is a natural DG equivalence

$$\mathcal{R}_{T_n,\mu} : \mathrm{MC}(T_n, \mu; \mathbb{K}) \xrightarrow{\sim} \mathcal{F}\mathsf{un}_{(T_n,\mu),1}(\mathcal{R}(\mathcal{S}) = A_{n+1}, \mathbb{K})$$

such that

$$\mathcal{R}_{T_n,\mu}(C, d) := ((F^n, d|_{F^n}) \hookrightarrow (F^{n-1}, d|_{F^{n-1}}) \hookrightarrow \ldots \hookrightarrow (F^0, d)),$$

and $\mathcal{R}_{T_n,\mu}$ sends Hom-complexes in $\mathrm{MC}(T_n, \mu; \mathbb{K})$ literally to the corresponding identical Hom-complexes in $\mathcal{F}\mathsf{un}_{(T_n,\mu),1}(A_{n+1}, \mathbb{K})$. It follows immediately that $\mathcal{R}_{T_n,\mu}$ is fully faithful. The essential surjectivity of $\mathcal{R}_{T_n,\mu}$ is a direct corollary of Lemma 6.30 above. Now, the second equivalence in Proposition 6.29 is just the composition of the equivalence in Corollary 5.39 with $\mathcal{R}_{T_n,\mu}$.

Finally, we come to our main theorem.

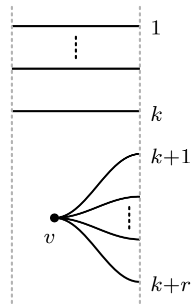
Theorem 6.31 (Augmentations are sheaves). *Let $(\mathcal{T}, \boldsymbol{\mu})$ be a bordered Legendrian graph. Then there is an A_∞-equivalence:*

$$\mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K}) \xrightarrow{\sim} \mathcal{C}_1(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$$

Proof. Let us recall the invariance property Theorem 4.19, Corollary 5.37 of the both sides, and sheaf property Proposition 6.27 of $\mathcal{A}\mathsf{ug}_+(\mathcal{T}, \boldsymbol{\mu}; \mathbb{K})$. Thus it suffices to check the equivalence for a new type of an elementary piece

$$\mathcal{V} = (V_{\mathsf{L}} \xrightarrow{\iota_{\mathsf{L}}} V \xleftarrow{\iota_{\mathsf{R}}} V_{\mathsf{R}}) \subset J^1\mathbf{U},$$

which is a bordered Legendrian graph of type $(k, k+r)$ consisting of one vertex v of type $(0, r)$ and trivial k-strands as follows:



As in Section 5.3, let $\mathcal{S}_V$ be the Whitney stratification of $\mathbf{U} \times \mathbb{R}_z$ induced by V and $\mathcal{S}$ be the regular cell complex for $\mathbf{U} \times \mathbb{R}_z$ refining $\mathcal{S}_V$ by adding one left half-edge at v. Clearly, $\mathcal{S}$ satisfies Assumption 5.17. Then by Lemma 6.17 and Corollary 5.39, it suffices to show the equivalence

$$\mathcal{R}_{\mathcal{V},\boldsymbol{\mu}} : \mathcal{MC}(\mathcal{V}, \boldsymbol{\mu}; \mathbb{K}) \to \mathfrak{Fun}_{(\mathcal{V},\boldsymbol{\mu}),1}(\mathcal{R}(\mathcal{S}), \mathbb{K})$$

between the diagram of Morse complex categories

$$\mathcal{MC}(\mathcal{V}, \boldsymbol{\mu}; \mathbb{K}) := \Big(\mathcal{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \xleftarrow{\mathfrak{p}_1} \mathcal{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \times \mathcal{MC}(v_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K}) \\ \xhookrightarrow{\mathfrak{i}} \mathcal{MC}(V_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K})\Big)$$

and $\mathfrak{Fun}_{(\mathcal{V},\boldsymbol{\mu}),1}(\mathcal{R}(\mathcal{S}), \mathbb{K})$.

Notice that the poset category $\mathcal{R}(\mathcal{S}|_{V_{\mathsf{R}}}) = A_{n+1}$ with $n = k + r$ and

$$\mathfrak{Fun}_{(V,\mu),1}(\mathcal{R}(\mathcal{S}), \mathbb{K}) \hookrightarrow \mathfrak{Fun}_{(V_{\mathsf{R}},\mu_{\mathsf{R}}),1}(\mathcal{R}(\mathcal{S}|_{V_{\mathsf{R}}}), \mathbb{K})$$

is just the fully faithful embedding into the full DG subcategory of $\mathfrak{Fun}_{(V_{\mathsf{R}},\mu_{\mathsf{R}}),1}(A_{n+1}, \mathbb{K})$ which consists of functors F such that $F(k + r \to k)$ is a quasi-isomorphism.

On the other hand, by the discussion above, we have the DG equivalence

$$\mathcal{R}_{V_{\mathsf{R}},\mu_{\mathsf{R}}} : \mathrm{MC}(V_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K}) \xrightarrow{\sim} \mathfrak{Fun}_{(V_{\mathsf{R}},\mu_{\mathsf{R}}),1}(A_{n+1}, \mathbb{K}).$$

Notice that the image of $\mathfrak{i}$ of $\mathrm{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \times \mathrm{MC}(v_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K})$ consists of Morse complexes $(C_{\mathsf{R}}, d_{\mathsf{R}}) = (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}})$, such that each summand is acyclic. Then, the composition $\mathcal{R}_{V_{\mathsf{R}},\mu_{\mathsf{R}}} \circ \mathfrak{i}$ induces a fully faithful DG functor

$$\mathcal{R}_{V,\mu} : \mathrm{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \times \mathrm{MC}(v_{\mathsf{R}}, \mu_{\mathsf{R}}; \mathbb{K}) \hookrightarrow \mathfrak{Fun}_{(V,\mu),1}(\mathcal{R}(\mathcal{S}), \mathbb{K}),$$

which is also essentially surjective.

In fact, the essential surjectivity of $\mathcal{R}_{V,\mu}$ implies that any functor $F \in \mathfrak{Fun}_{(V,\mu),1}(\mathcal{R}(\mathcal{S}), \mathbb{K})$ is quasi-isomorphic to $\mathcal{R}_{V,\mu}((C_{\mathsf{R}}, d_{\mathsf{R}}))$ for some Morse complex $(C_{\mathsf{R}}, d_{\mathsf{R}})$, which we can assume to be canonical (or Barannikov's normal form). Moreover, the summand $(C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}})$ of $(C_{\mathsf{R}}, d_{\mathsf{R}})$ is acyclic since $F(k + r \to k)$ is a quasi-isomorphism. Therefore it follows that $(C_{\mathsf{R}}, d_{\mathsf{R}}) = (C_{\mathsf{L}}, d_{\mathsf{L}}) \oplus (C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}})$ for some canonical Morse complexes $(C_{\mathsf{L}}, d_{\mathsf{L}})$ and $(C_{v_{\mathsf{R}}}, d_{v_{\mathsf{R}}})$. We are done.

Finally, it is direct to check that $\mathcal{R}_{V,\mu}$ commutes with the DG equivalence

$$\mathcal{R}_{V_{\mathsf{L}},\mu_{\mathsf{L}}} : \mathrm{MC}(V_{\mathsf{L}}, \mu_{\mathsf{L}}; \mathbb{K}) \xrightarrow{\sim} \mathfrak{Fun}_{(V_{\mathsf{L}},\mu_{\mathsf{L}}),1}(\mathcal{R}(\mathcal{S}|_{V_{\mathsf{L}}}), \mathbb{K})$$

defined as in the discussion above up to a specified natural isomorphism. Thus, we obtain a DG equivalence

$$\mathcal{R}_{\mathcal{V},\boldsymbol{\mu}} : \mathcal{MC}(\mathcal{V}, \boldsymbol{\mu}; \mathbb{K}) \xrightarrow{\sim} \mathfrak{Fun}_{(\mathcal{V},\boldsymbol{\mu}),1}(\mathcal{R}(\mathcal{S}), \mathbb{K})$$

as desired. This finishes the proof of Theorem 6.31. □

6.4. Example

Let us consider a Legendrian graph (Λ, μ) having the following front with Maslov potential and Lagrangian projection. Note that Λ has one vertex v and two (oriented) edges e_1, e_2:

6.4.1. Augmentation category. Fix the base field $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$. Firstly, let us consider the augmentation category $\mathcal{A}\mathsf{ug}_+ = \mathcal{A}\mathsf{ug}_+(\Lambda, \mu; \mathbb{K})$. We consider CE DGA $A = A^{\mathsf{CE}}(\Lambda, \mu)$ with $\mathbb{Z}/2\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$-coefficient. Here, t_i corresponds to the basepoint on the edge e_i for $i = 1, 2$ in Figure 21. Then the

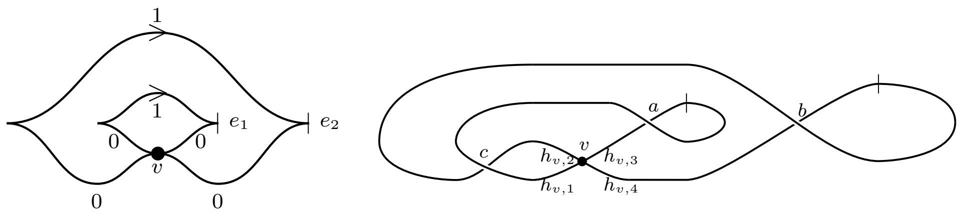


Figure 21: The front and Lagrangian projection of Λ.

DGA $(\mathsf{A}, |\cdot|, \partial)$ is

$$\mathsf{A} = \mathbb{Z}/2\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]\langle \mathsf{G}\rangle;$$
$$\mathsf{G} = \{a, b, c\} \amalg \{v_{i,\ell} \mid i \in \mathbb{Z}/4\mathbb{Z}, \ell \in \mathbb{N}_{>0}\},$$

and the grading is given by

$$|a| = |b| = 1, \qquad |c| = 0, \qquad |v_{i,\ell}| = \mu(i) - \mu(i+\ell) + N(v, i, \ell) - 1.$$

The differential on the crossing generators are as follows:

$$\partial a = t_1 + v_{1,2} + cv_{2,1};$$
$$\partial b = t_2 + v_{2,2} + v_{2,1}t_1^{-1}(v_{1,3} + cv_{2,2} + av_{3,1});$$
$$\partial c = v_{1,1}.$$

The following is the list of degree zero generators:

$$c, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{3,2}, v_{3,3}, v_{4,1}, v_{4,2}.$$

For any augmentation ϵ, denote $\overline{g} := \epsilon(g)$ for any generator g. Among infinitely many equations of differential on vertex generators, let us list relations coming from the differential of degree one generators that the augmentation should satisfy:

$$\overline{v_{1,2}v_{3,2}} + \overline{v_{1,3}v_{4,1}} = 1, \quad \overline{v_{2,1}v_{3,3}} + \overline{v_{2,2}v_{4,2}} = 1,$$
$$\overline{v_{3,2}v_{1,2}} + \overline{v_{3,3}v_{2,1}} = 1, \quad \overline{v_{4,1}v_{1,3}} + \overline{v_{4,2}v_{2,2}} = 1;$$
$$\overline{v_{1,2}v_{3,3}} + \overline{v_{1,3}v_{4,2}} = 0, \quad \overline{v_{3,2}v_{1,3}} + \overline{v_{3,3}v_{2,2}} = 0,$$
$$\overline{v_{2,1}v_{3,2}} + \overline{v_{2,2}v_{4,1}} = 0, \quad \overline{v_{4,1}v_{1,2}} + \overline{v_{4,2}v_{2,1}} = 0.$$

Then by direct computation, we can check that there are eight possible ($\mathbb{Z}/2\mathbb{Z}$-valued) augmentations:

i	$\epsilon_i(c)$	$\epsilon_i(v_{1,2})$	$\epsilon_i(v_{1,3})$	$\epsilon_i(v_{2,1})$	$\epsilon_i(v_{2,2})$	$\epsilon_i(v_{3,2})$	$\epsilon_i(v_{3,3})$	$\epsilon_i(v_{4,1})$	$\epsilon_i(v_{4,2})$
1	1	0	1	1	0	0	1	1	0
2	0	1	0	0	1	1	0	0	1
3	0	1	1	1	0	0	1	1	1
4	1	0	1	1	1	1	1	1	0
5	0	1	1	0	1	1	1	0	1
6	1	1	0	0	1	1	0	0	1
7	1	1	1	0	1	1	1	0	1
8	0	1	0	1	1	1	0	1	1

Obviously $\epsilon(t_1) = \epsilon(t_2) = 1$.

Remark 6.32. Note that there are only two equivalence classes of augmentations up to isomorphisms. One can check that

$$\epsilon_1 \sim \epsilon_3 \sim \epsilon_4 \sim \epsilon_8, \qquad \epsilon_2 \sim \epsilon_5 \sim \epsilon_6 \sim \epsilon_7.$$

In the rest of example, we mainly consider two non-equivalent augmentations ϵ_1 and ϵ_2. Moreover, two (equivalence classes of) augmentation corresponds to two possible resolutions at vertex v

$$\epsilon_1 \in \mathsf{Aug}^{\rho_v^1}(\Lambda, \mu; \mathbb{K}), \qquad \epsilon_2 \in \mathsf{Aug}^{\rho_v^2}(\Lambda, \mu; \mathbb{K}),$$

where $\rho_v^1 = \{\{1,4\},\{2,3\}\}$, and $\rho_v^2 = \{\{1,3\},\{2,4\}\}$. See [3, §5] for the details.

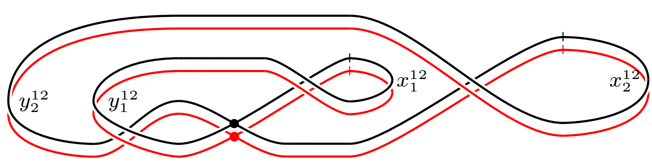


Figure 22: Two copies of Λ in the Lagrangian projection.

Now consider two copies $\Lambda^{(2)}$. The labeling of the crossing is given as in Section 3.2.3. Then the corresponding algebra $\mathsf{A}^{(2)}$ is generated by vertex generators

$$\{v^i_{k,\ell} \mid i = 1, 2, k \in \mathbb{Z}/4\mathbb{Z}, \ell \in \mathbb{N}_{>0}\},$$

pure Reeb chords

$$\{a^{ii}, b^{ii}, c^{ii} \mid i = 1, 2\},$$

and mixed Reeb chords

$$\{a^{12}, b^{12}, c^{12}, v^{12}_{1,1}, v^{12}_{4,-1}, x^{12}_1, x^{12}_2, y^{12}_1, y^{12}_2\} \amalg \{a^{21}, b^{21}, c^{21}\}.$$

Here $v^{12}_{1,1}, v^{12}_{4,-1}$ are the crossings near two-copies of vertex, see (3.4) for the label convention near m-copy of the vertex, and the crossings x^{12}_i, y^{12}_i arise from the right- and left cusps, respectively. The grading for the generator is straight forward from the DGA $A(\Lambda)$ except the following

$$\begin{aligned} |x^{12}_i| &= 0, & |y^{12}_i| &= -1 \quad \text{for } i = 1, 2; \\ |v^{12}_{1,1}| &= -1, & |v^{12}_{4,-1}| &= 0. \end{aligned}$$

Now let us consider the mixed chords in $\Lambda^{(2)}$, especially from the second to the first copy. Note that there are such Reeb chords a^{12}, b^{12} of degree 1, $c^{12}, v^{12}_{4,-1}, x^{12}_1, x^{12}_2$ of degree 0, and $y^{12}_1, y^{12}_2, v^{12}_{1,1}$ of degree -1.

For convenience of the computation, let us restrict the differential to the following submodule

$$\mathbf{M}^{(2)}_0 := (\mathsf{M}^{11}_0)^{\otimes n_1} \otimes \mathsf{M}^{12} \otimes (\mathsf{M}^{22}_0)^{\otimes n_2},$$

where n_i is a nonnegative integer, and M^{ii}_0 denotes the submodule spanned by degree 0 elements for $i = 1, 2$. Then the restriction of the differential becomes

$$\begin{aligned}
\partial a^{12}|_{\mathbf{M}^{(2)}_0} &= x^{12}_1 t^2_1 + c^{11} v^{11}_{2,2} v^{12}_{4,-1} + c^{12} v^{22}_{2,1} + v^{11}_{1,3} v^{12}_{4,-1}; \\
\partial b^{12}|_{\mathbf{M}^{(2)}_0} &= x^{12}_2 t^2_2 + v^{11}_{2,1} (t^1_1)^{-1} c^{12} v^{22}_{2,2} \\
&\quad + \left(v^{11}_{2,1} (t^1_1)^{-1} x^{12}_1 + v^{11}_{2,2} v^{12}_{4,-1} (t^2_1)^{-1}\right) \left(c^{22} v^{22}_{2,2} + v^{22}_{1,3}\right) \\
\partial c^{12}|_{\mathbf{M}^{(2)}_0} &= y^{12}_1 c^{22} + c^{11} y^{12}_2 + v^{12}_{1,1}; \\
\partial v^{12}_{4,-1}|_{\mathbf{M}^{(2)}_0} &= v^{11}_{4,1} v^{12}_{1,1} v^{22}_{2,1} + v^{11}_{4,2} y^{12}_2 v^{22}_{2,1} + v^{11}_{4,1} y^{12}_1 v^{22}_{1,2}; \\
\partial x^{12}_1|_{\mathbf{M}^{(2)}_0} &= t^1_1 \left(v^{11}_{3,2} v^{12}_{1,1} v^{22}_{2,1} + v^{11}_{3,2} y^{12}_1 v^{22}_{1,2} + v^{11}_{3,3} y^{12}_2 v^{22}_{2,1}\right) (t^2_1)^{-1} + y^{12}_1; \\
\partial x^{12}_2|_{\mathbf{M}^{(2)}_0} &= t^1_2 \left(v^{11}_{4,1} v^{12}_{1,1} v^{22}_{2,2} + v^{11}_{4,2} y^{12}_2 v^{22}_{2,2} + v^{11}_{4,1} y^{12}_1 v^{22}_{1,3}\right) (t^2_2)^{-1} + y^{12}_2; \\
\partial v^{12}_{1,1}|_{\mathbf{M}^{(2)}_0} &= 0; \\
\partial y^{12}_1|_{\mathbf{M}^{(2)}_0} &= 0; \\
\partial y^{12}_2|_{\mathbf{M}^{(2)}_0} &= 0.
\end{aligned}$$

Note that the above restriction only contribute the bi-linearized differentials. For example, we have the following bi-linearized differential with respect to a pair (ϵ_1, ϵ_1):

$$\begin{aligned} &\partial_{\epsilon_1,\epsilon_1} a^{12} = x_1^{12} + c^{12} + v_{4,-1}^{12}, && \partial_{\epsilon_1,\epsilon_1} b^{12} = x_1^{12} + x_2^{12}, && \partial_{\epsilon_1,\epsilon_1} c^{12} = y_1^{12} + y_2^{12} + v_{1,1}^{12};\\ &\partial_{\epsilon_1,\epsilon_1} v_{4,-1}^{12} = v_{1,1}^{12}, && \partial_{\epsilon_1,\epsilon_1} x_1^{12} = y_1^{12} + y_2^{12}, && \partial_{\epsilon_1,\epsilon_1} x_2^{12} = y_1^{12} + y_2^{12};\\ &\partial_{\epsilon_1,\epsilon_1} y_1^{12} = 0, && \partial_{\epsilon_1,\epsilon_1} y_2^{12} = 0, && \partial_{\epsilon_1,\epsilon_1} v_{1,1}^{12} = 0. \end{aligned}$$

For a pair (ϵ_1, ϵ_2), we have

$$\begin{aligned} &\partial_{\epsilon_1,\epsilon_2} a^{12} = x_1^{12} + v_{4,-1}^{12}, && \partial_{\epsilon_1,\epsilon_2} b^{12} = x_2^{12} + c^{12}, && \partial_{\epsilon_1,\epsilon_2} c^{12} = y_2^{12} + v_{1,1}^{12};\\ &\partial_{\epsilon_1,\epsilon_2} v_{4,-1}^{12} = y_1^{12}, && \partial_{\epsilon_1,\epsilon_2} x_1^{12} = y_1^{12}, && \partial_{\epsilon_1,\epsilon_2} x_2^{12} = v_{1,1}^{12} + y_2^{12};\\ &\partial_{\epsilon_1,\epsilon_2} y_1^{12} = 0, && \partial_{\epsilon_1,\epsilon_2} y_2^{12} = 0, && \partial_{\epsilon_1,\epsilon_2} v_{1,1}^{12} = 0. \end{aligned}$$

In a case of (ϵ_2, ϵ_1), we deduce

$$\begin{aligned} &\partial_{\epsilon_2,\epsilon_1} a^{12} = x_1^{12} + c^{12}, && \partial_{\epsilon_2,\epsilon_1} b^{12} = x_2^{12} + v_{4,-1}^{12}, && \partial_{\epsilon_2,\epsilon_1} c^{12} = y_1^{12} + v_{1,1}^{12};\\ &\partial_{\epsilon_2,\epsilon_1} v_{4,-1}^{12} = y_2^{12}, && \partial_{\epsilon_2,\epsilon_1} x_1^{12} = v_{1,1}^{12} + y_1^{12}, && \partial_{\epsilon_2,\epsilon_1} x_2^{12} = y_2^{12};\\ &\partial_{\epsilon_2,\epsilon_1} y_1^{12} = 0, && \partial_{\epsilon_2,\epsilon_1} y_2^{12} = 0, && \partial_{\epsilon_2,\epsilon_1} v_{1,1}^{12} = 0. \end{aligned}$$

When a pair is (ϵ_2, ϵ_2), then

$$\partial_{\epsilon_2,\epsilon_2} a^{12} = x_1^{12}, \qquad \partial_{\epsilon_2,\epsilon_2} b^{12} = x_2^{12}, \qquad \partial_{\epsilon_2,\epsilon_2} c^{12} = v_{1,1}^{12},$$

with all other differentials are trivial.

For any pair of augmentation (ϵ_i, ϵ_j), we have

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) = \mathrm{Hom}^0_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) \oplus \mathrm{Hom}^1_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) \oplus \mathrm{Hom}^2_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j);\\ &\mathrm{Hom}^0_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) = \mathbb{Z}/2\mathbb{Z}\langle y_1^{12}, y_2^{12}, v_{1,1}^{12}\rangle^\vee;\\ &\mathrm{Hom}^1_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) = \mathbb{Z}/2\mathbb{Z}\langle c^{12}, v_{4,-1}^{12}, x_1^{12}, x_2^{12}\rangle^\vee;\\ &\mathrm{Hom}^2_{\mathcal{A}\mathsf{ug}_+}(\epsilon_i, \epsilon_j) = \mathbb{Z}/2\mathbb{Z}\langle a^{12}, b^{12}\rangle^\vee. \end{aligned}$$

By dualizing the bi-linearized differential $\partial_{\epsilon_1,\epsilon_1}$, we have

$$\begin{aligned} &m_1^{\epsilon_1,\epsilon_1}(y_1^{12})^\vee = (c^{12})^\vee + (x_1^{12})^\vee + (x_2^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(x_1^{12})^\vee = (a^{12})^\vee + (b^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(a^{12})^\vee = 0;\\ &m_1^{\epsilon_1,\epsilon_1}(y_2^{12})^\vee = (c^{12})^\vee + (x_1^{12})^\vee + (x_2^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(x_2^{12})^\vee = (b^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(b^{12})^\vee = 0;\\ &m_1^{\epsilon_1,\epsilon_1}(v_{1,1}^{12})^\vee = (c^{12})^\vee + (v_{4,-1}^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(v_{4,-1}^{12})^\vee = (a^{12})^\vee, && m_1^{\epsilon_1,\epsilon_1}(c^{12})^\vee = (a^{12})^\vee. \end{aligned}$$

The only non-trivial cohomology class with respect to $m_1^{\epsilon_1,\epsilon_1}$ is $\alpha := [(y_1^{12} + y_2^{12})^\vee] \in H^0\,\mathrm{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_1)$.

By the similar computation for (ϵ_1,ϵ_2), (ϵ_2,ϵ_1), and (ϵ_2,ϵ_2), we have the following non-trivial cohomology classes

$$\beta \coloneqq [(y_2^{12}+v_{1,1}^{12})^\vee] \in H^0\operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1,\epsilon_2), \tag{6.23}$$
$$\gamma \coloneqq [(y_1^{12}+v_{1,1}^{12})^\vee] \in H^0\operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_2,\epsilon_1);$$
$$\delta_1 \coloneqq [(y_1^{12})^\vee], \delta_2 \coloneqq [(y_2^{12})^\vee] \in H^0\operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_2,\epsilon_2), \tag{6.24}$$
$$\delta_3 \coloneqq [(v_{4,-1}^{12})^\vee] \in H^1\operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_2,\epsilon_2),$$

respectively.

For the A_∞ structure, especially m_2, we consider three copy of Λ as in Figure 6.4.1. Again the labeling and grading convention for generators are as

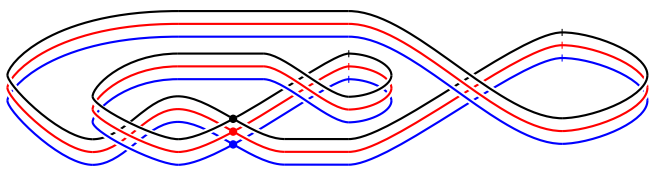


Figure 23: Three copies of Λ.

in Section 3.2.3. Since the DGA $A(\Lambda^{(3)},\mu^{(3)})$ is complicated, for simplicity, let us list the terms in the differential which are relevant to our construction of A_∞ structure. Especially, we only list term of type $\mathbf{M}_0^{(3)}$, where

$$\mathbf{M}_0^{(k+1)} \coloneqq \mathbf{M}_0^{(k)} \otimes \mathsf{M}^{k\,k+1} \otimes (\mathsf{M}_0^{k+1\,k+1})^{\otimes n_{k+1}}.$$

Here n_{k+1} is a nonnegative integer, and $\mathsf{M}_0^{k+1\,k+1}$ denotes the submodule spanned by degree 0 elements.

Then the restriction of the differential becomes

$$\begin{aligned}
\partial a^{13}|_{\mathbf{M}_0^{(3)}} &= c^{12}v_{2,2}^{22}v_{4,-1}^{23};\\
\partial b^{13}|_{\mathbf{M}_0^{(3)}} &= \left(v_{2,2}^{11}v_{4,-1}^{12}(t_1^2)^{-1} + v_{2,1}^{11}(t_1^1)^{-1}x_1^{12}\right)\left(x_1^{23}c^{33}v_{2,2}^{33} + c^{23}v_{2,2}^{33} + x_1^{23}v_{1,3}^{33}\right);\\
\partial c^{13}|_{\mathbf{M}_0^{(3)}} &= y_1^{12}c^{23} + c^{12}y_2^{23};\\
\partial v_{4,-1}^{13}|_{\mathbf{M}_0^{(3)}} &= v_{4,2}^{11}y_2^{12}v_{2,2}^{22}v_{4,-1}^{23} + v_{4,1}^{11}\left(y_1^{12}v_{1,3}^{22} + v_{1,1}^{12}v_{2,2}^{22}\right)v_{4,-1}^{23} + v_{4,-1}^{12}v_{3,2}^{22}y_1^{23}v_{1,2}^{33}\\
&\quad + v_{4,-1}^{12}\left(v_{3,3}^{22}y_2^{23} + v_{3,2}^{22}v_{1,1}^{23}\right)v_{2,1}^{33};
\end{aligned}$$

$$\begin{aligned}
\partial x_1^{13}|_{\mathbf{M}_0^{(3)}} &= y_1^{12}x_1^{23} + x_1^{12}t_1^2\left(v_{3,2}^{22}y_1^{23}v_{1,2}^{33} + v_{3,2}^{22}v_{1,1}^{23}v_{2,1}^{33} + v_{3,3}^{22}y_2^{23}v_{2,1}^{33}\right)(t_1^3)^{-1}\\
&\quad + t_1^1\left(v_{3,2}^{11}y_1^{12}v_{1,3}^{22} + v_{3,2}^{11}v_{1,1}^{12}v_{2,2}^{22} + v_{3,3}^{11}y_2^{12}v_{2,2}^{22}\right)v_{4,-1}^{23}(t_1^3)^{-1}\\
\partial x_2^{13}|_{\mathbf{M}_0^{(3)}} &= y_2^{12}x_2^{23} + x_2^{12}t_2^2\left(v_{4,1}^{22}y_1^{23}v_{1,3}^{33} + v_{4,1}^{22}v_{1,1}^{23}v_{2,2}^{33} + v_{4,2}^{22}y_2^{23}v_{2,2}^{33}\right)(t_2^3)^{-1}\\
\partial v_{1,1}^{13}|_{\mathbf{M}_0^{(3)}} &= v_{1,1}^{12}y_2^{23} + y_1^{12}v_{1,1}^{23};\\
\partial y_1^{13}|_{\mathbf{M}_0^{(3)}} &= y_1^{12}y_1^{23};\\
\partial y_2^{13}|_{\mathbf{M}_0^{(3)}} &= y_2^{12}y_2^{23}.
\end{aligned}$$

If we choose the triple $(\epsilon_1, \epsilon_1, \epsilon_1)$, then $\partial_{(\epsilon_1,\epsilon_1,\epsilon_1)}|_{\mathbf{M}_0^{(3)}}$ induces a multiplication on cohomology classes

$$\overline{m}_2^{(\epsilon_1,\epsilon_1,\epsilon_1)} : H^* \operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_1) \otimes H^* \operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_1) \to H^* \operatorname{Hom}_{\mathcal{A}\mathsf{ug}_+}(\epsilon_1, \epsilon_1)$$

is given by

$$\overline{m}_2^{(\epsilon_1,\epsilon_1,\epsilon_1)}(\alpha \otimes \alpha) = \alpha.$$

By the similar computation,

$$\begin{aligned}
&\overline{m}_2^{(\epsilon_1,\epsilon_1,\epsilon_2)}(\beta \otimes \alpha) = \beta, && \overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_1)}(\gamma \otimes \beta) = 0,\\
&\overline{m}_2^{(\epsilon_2,\epsilon_1,\epsilon_1)}(\alpha \otimes \gamma) = \gamma, && \overline{m}_2^{(\epsilon_2,\epsilon_1,\epsilon_2)}(\beta \otimes \gamma) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_2)}(\delta_1 \otimes \beta) = 0, \quad \overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_2)}(\delta_2 \otimes \beta) = \beta, \quad \overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_2)}(\delta_3 \otimes \beta) = 0;\\
&\overline{m}_2^{(\epsilon_2,\epsilon_2,\epsilon_1)}(\gamma \otimes \delta_1) = \gamma, \quad \overline{m}_2^{(\epsilon_2,\epsilon_2,\epsilon_1)}(\gamma \otimes \delta_2) = 0, \quad \overline{m}_2^{(\epsilon_2,\epsilon_2,\epsilon_1)}(\gamma \otimes \delta_3) = 0.
\end{aligned}$$

For the triple $(\epsilon_2, \epsilon_2, \epsilon_2)$, we deduce the following table of multiplication:

$\overline{m}_2$	δ_1	δ_2	δ_3
δ_1	δ_1	0	δ_3
δ_2	0	δ_2	0
δ_3	0	δ_3	0

For the higher multiplication m_k, $k \geq 3$, let us consider $(k+1)$-copy of Λ, the induced differential, and its restriction to $\mathbf{M}_0^{(k+1)}$. The followings are possible terms which may contribute to m_k, $k \geq 3$:

$$\begin{aligned}
\partial b^{14}|_{\mathbf{M}_0^{(4)}} &= \left(v_{2,2}^{11} v_{4,-1}^{12} (t_1^2)^{-1} + v_{2,1}^{11} (t_1^1)^{-1} x_1^{12}\right) x_1^{23} \\
&\quad \left(c^{34} v_{2,2}^{44} + x_1^{34} v_{1,3}^{44} + x_1^{34} c^{44} v_{2,2}^{44}\right); \\
\partial v_{4,-1}^{14}|_{\mathbf{M}_0^{(4)}} &= v_{4,-1}^{12} \left(v_{3,2}^{22} y_1^{23} v_{1,3}^{33} + v_{3,2}^{22} v_{1,1}^{23} v_{2,2}^{33} + v_{3,3}^{22} y_2^{23} v_{2,2}^{33}\right) v_{4,-1}^{34}; \\
\partial x_1^{14}|_{\mathbf{M}_0^{(4)}} &= x_1^{12} t_1^2 \left(v_{3,2}^{22} y_1^{23} v_{1,3}^{33} + v_{3,2}^{22} v_{1,1}^{23} v_{2,2}^{33} + v_{3,3}^{22} y_2^{23} v_{2,2}^{33}\right) v_{4,-1}^{34} (t_1^4)^{-1}; \\
\partial b^{1m}|_{\mathbf{M}_0^{(m)}} &= \left(v_{2,2}^{11} v_{4,-1}^{12} (t_1^2)^{-1} + v_{2,1}^{11} (t_1^1)^{-1} x_1^{12}\right) x_1^{23} x_1^{34} \cdots x_1^{m-2\,m-1} \\
&\quad \left(c^{m-1\,m} v_{2,2}^{m\,m} + x_1^{m-1\,m} v_{1,3}^{m\,m} + x_1^{m-1\,m} c^{m\,m} v_{2,2}^{m\,m}\right)
\end{aligned}$$

For example, a term in $\partial b^{14}|_{\mathbf{M}_0^{(4)}}$ has a chain level contribution

$$\langle m_3^{(\epsilon_2,\epsilon_1,\epsilon_1,\epsilon_2)}(c^{12} \otimes x_1^{12} \otimes v_{4,-1}^{12}), b^{12} \rangle = 1.$$

6.4.2. Sheaf category. On the other hand, let us consider the sheaf category $\mathcal{C}_1(\Lambda, \mu; \mathbb{K})$. By the discussion in Sections 5.2 and 5.3, a sheaf $\mathcal{F} \in \mathcal{C}_1(\Lambda, \mu; \mathbb{K}) \subset \mathcal{S}\mathsf{h}_\Lambda(\mathbb{R}^2; \mathbb{K})$ of microlocal rank 1 whose micro-support lies in Λ can be identified with a representation of the quiver diagram in Figure 24.

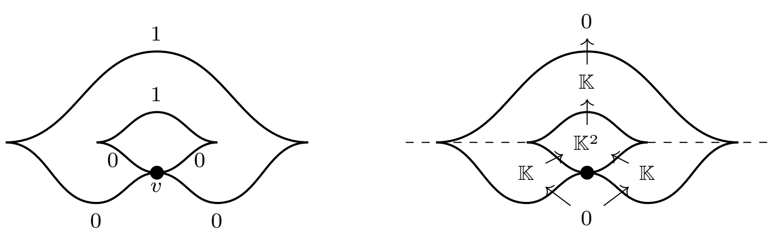
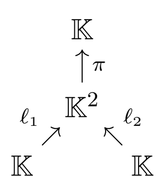


Figure 24: The front diagram of Λ, and the induced legible model.

More precisely, the representation of the above quiver diagram is determined by three lines



$$[\ell_1] \coloneqq \operatorname{Image} \ell_1 \hookrightarrow \mathbb{K}^2, \quad [\ell_2] \coloneqq \operatorname{Image} \ell_2 \hookrightarrow \mathbb{K}^2, \quad [\ell_0] \coloneqq \ker \pi \hookrightarrow \mathbb{K}^2,$$

satisfying that $\pi \circ \ell_i$ is quasi-isomorphism for $i = 1, 2$, in other words,

$$[\ell_0] \neq [\ell_1], \qquad\qquad [\ell_0] \neq [\ell_2].$$

in $\mathbb{P}^1(\mathbb{K})$. There are two possible cases, $[\ell_1] \neq [\ell_2]$ and $[\ell_1] = [\ell_2]$, and let us denote the corresponding representation of quiver by Q_1 and Q_2, respectively. Note that $Q_2 \cong Q_2^1 \oplus Q_2^2$, where

$$Q_2^1 \coloneqq \begin{array}{ccc} & \mathbb{K} & \\ & \uparrow & \\ & \mathbb{K} & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & \mathbb{K} \end{array} \qquad\qquad Q_2^2 \coloneqq \begin{array}{ccc} & 0 & \\ & \uparrow & \\ & \mathbb{K} & \\ \nearrow & & \nwarrow \\ 0 & & 0 \end{array}$$

Let us list all projective indecomposable representations of the above quiver of type D_4 as follows:

$$P_1 \coloneqq \begin{array}{ccc} & \mathbb{K} & \\ & \uparrow & \\ & 0 & \\ \nearrow & & \nwarrow \\ 0 & & 0 \end{array} \quad P_2 \coloneqq \begin{array}{ccc} & \mathbb{K} & \\ & \uparrow & \\ & \mathbb{K} & \\ \nearrow & & \nwarrow \\ 0 & & 0 \end{array} \quad P_3 \coloneqq \begin{array}{ccc} & \mathbb{K} & \\ & \uparrow & \\ & \mathbb{K} & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & 0 \end{array} \quad P_4 \coloneqq \begin{array}{ccc} & \mathbb{K} & \\ & \uparrow & \\ & \mathbb{K} & \\ \nearrow & & \nwarrow \\ 0 & & \mathbb{K} \end{array}$$

6.4.3. Computation of differential. In the rest of example, we assume that our base field is $\mathbb{Z}/2\mathbb{Z}$. Recall for a vector space V that $V[i]$ means the degree shift by $-i$.[5]

Note that

$$\operatorname{Hom}(P_i[k], P_j[\ell]) \cong \begin{cases} \mathbb{K}[\ell - k] & \text{if } i = j \text{ or } (i,j) = (1,2), (1,3), (1,4), (2,3), (2,4); \\ 0 & \text{otherwise.} \end{cases}$$

We sometimes use $\mathbb{K}_{AB}$ instead of $\operatorname{Hom}(A, B)$ when its cohomology is $\mathbb{K}[0]$.

[5]We omit the degree shift notation $[\ell]$ when it is clear.

The quivers Q_1, Q_2^1 and Q_2^2 admit the following projective resolutions which will be denoted by $\widetilde{Q}_1$, $\widetilde{Q}_2^1$ and $\widetilde{Q}_2^2$, respectively:

$$0 \longrightarrow P_1 \xrightarrow{f} P_3 \oplus P_4 \longrightarrow Q_1 \longrightarrow 0;$$
$$0 \longrightarrow P_2 \xrightarrow{g} P_3 \oplus P_4 \longrightarrow Q_2^1 \longrightarrow 0;$$
$$0 \longrightarrow P_1 \xrightarrow{h} P_2 \longrightarrow Q_2^2 \longrightarrow 0.$$

Here a matrix form of f, g and h are given by $\binom{1}{1}$, $\binom{1}{1}$, and (1) with respect to a canonical choice of basis of each ordered summand. Then we compute that

$$R\operatorname{Hom}(Q_1, Q_1) = \operatorname{Hom}(\widetilde{Q}_1, Q_1) \cong \left(\mathbb{K}_{P_3Q_1} \oplus \mathbb{K}_{P_4Q_1} \xrightarrow{(1\ 1)} \mathbb{K}_{P_1Q_1}[-1] \right) \cong \mathbb{K}.$$

Let b_1 and b_2 be the non-zero element of $\mathbb{K}_{P_3Q_1}$ and $\mathbb{K}_{P_4Q_1}$, then the only non-trivial cohomology class in $H^* \operatorname{Hom}(\widetilde{Q}_1, Q_1)$ is $[b_1 + b_2]$.

Similarly we verify that

$$\begin{aligned} &\operatorname{Hom}(\widetilde{Q}_2^1, Q_2^1) \cong \left(\mathbb{K}^2 \twoheadrightarrow \mathbb{K}[-1]\right) \cong \mathbb{K}, \\ &\operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2) \cong (0 \to \mathbb{K}[-1]) \cong \mathbb{K}[-1], \\ &\operatorname{Hom}(\widetilde{Q}_2^2, Q_2^1) \cong \left(\mathbb{K} \xrightarrow{\cong} \mathbb{K}[-1]\right) \cong 0, \\ &\operatorname{Hom}(\widetilde{Q}_2^2, Q_2^2) \cong (\mathbb{K} \to 0[-1]) \cong \mathbb{K}, \end{aligned} \tag{6.25}$$

which conclude

$$R\operatorname{Hom}(Q_2, Q_2) = \operatorname{Hom}(\widetilde{Q}_2, Q_2) \cong \bigoplus_{i,j=1,2} \operatorname{Hom}(\widetilde{Q}_2^i, Q_2^j) \cong \mathbb{K}^2 \oplus \mathbb{K}[-1].$$

Also compute

$$\begin{aligned} &\operatorname{Hom}(\widetilde{Q}_1, Q_2^1) \cong \left(\mathbb{K}_{P_3Q_2^1} \oplus \mathbb{K}_{P_4Q_2^1} \xrightarrow{(1\ 1)} \mathbb{K}_{P_1Q_2^1}[-1]\right) \cong \mathbb{K}; \\ &\operatorname{Hom}(\widetilde{Q}_1, Q_2^2) \cong (0 \to 0[-1]) \cong 0; \\ &\operatorname{Hom}(\widetilde{Q}_2^1, Q_1) \cong \left(\mathbb{K}_{P_3Q_1} \oplus \mathbb{K}_{P_4Q_1} \xrightarrow{\cong} \operatorname{Hom}(P_2, Q_1)[-1]\right) \cong 0; \\ &\operatorname{Hom}(\widetilde{Q}_2^2, Q_1) \cong (\operatorname{Hom}(P_2, Q_1) \twoheadrightarrow \mathbb{K}_{P_1Q_1}[-1]) \cong \mathbb{K}, \end{aligned}$$

which imply

$$
\begin{aligned}
R\operatorname{Hom}(Q_1, Q_2) &\cong \operatorname{Hom}(\widetilde{Q}_1, Q_2^1) \oplus \operatorname{Hom}(\widetilde{Q}_1, Q_2^2) \cong \mathbb{K};\\
R\operatorname{Hom}(Q_2, Q_1) &\cong \operatorname{Hom}(\widetilde{Q}_2^1, Q_1) \oplus \operatorname{Hom}(\widetilde{Q}_2^2, Q_1) \cong \mathbb{K}.
\end{aligned}
$$

Note that the above computation coincide with the one in $\mathcal{A}\mathsf{ug}_+(\Lambda)$, see (6.23).

6.4.4. Computation of multiplication. Now consider the compositions between Hom spaces. Especially look at

$$
m : R\operatorname{Hom}(Q_1, Q_1) \otimes R\operatorname{Hom}(Q_1, Q_1) \to R\operatorname{Hom}(Q_1, Q_1).
$$

The domain is isomorphic to $\operatorname{Hom}(\widetilde{Q}_1, Q_1) \otimes \operatorname{Hom}(\widetilde{Q}_1, \widetilde{Q}_1)$ and hence becomes the following:

$$
\left(\mathbb{K}_{P_3Q_1} \oplus \mathbb{K}_{P_4Q_1} \xrightarrow{(1\ 1)} \mathbb{K}_{P_1Q_1}[-1]\right)
\otimes \left(\mathbb{K}_{P_1P_1} \oplus \mathbb{K}_{P_3P_3} \oplus \mathbb{K}_{P_4P_4} \xrightarrow{\left(\begin{smallmatrix}1&1&0\\1&0&1\end{smallmatrix}\right)} \mathbb{K}_{P_1P_3}[-1] \oplus \mathbb{K}_{P_1P_4}[-1]\right).
$$

Let c_1, c_2, and c_3 be the non-zero element in $\mathbb{K}_{P_1P_1}$, $\mathbb{K}_{P_3P_3}$, and $\mathbb{K}_{P_4P_4}$, respectively, then $[c_1 + c_2 + c_3]$ is the unique non-trivial cohomology class in $H^* \operatorname{Hom}(\widetilde{Q}_1, \widetilde{Q}_1)$.

Let us denote the domain of m_2 by $D^\bullet$, then the i-th degree part D^i becomes as follows for $i = 0, 1, 2$:

$$
\begin{aligned}
D^0 \cong{}& (\mathbb{K}_{P_3Q_1} \otimes \mathbb{K}_{P_1P_1}) \oplus (\mathbb{K}_{P_3Q_1} \otimes \mathbb{K}_{P_3P_3}) \oplus (\mathbb{K}_{P_3Q_1} \otimes \mathbb{K}_{P_4P_4})\\
&\oplus (\mathbb{K}_{P_4Q_1} \otimes \mathbb{K}_{P_1P_1}) \oplus (\mathbb{K}_{P_4Q_1} \otimes \mathbb{K}_{P_3P_3}) \oplus (\mathbb{K}_{P_4Q_1} \otimes \mathbb{K}_{P_4P_4});\\
D^1 \cong{}& (\mathbb{K}_{P_3Q_1} \otimes \mathbb{K}_{P_1P_3}) \oplus (\mathbb{K}_{P_3Q_1} \otimes \mathbb{K}_{P_1P_4}) \oplus (\mathbb{K}_{P_4Q_1} \otimes \mathbb{K}_{P_1P_3})\\
&\oplus (\mathbb{K}_{P_4Q_1} \otimes \mathbb{K}_{P_1P_4}) \oplus (\mathbb{K}_{P_1Q_1} \otimes \mathbb{K}_{P_1P_1})\\
&\oplus (\mathbb{K}_{P_1Q_1} \otimes \mathbb{K}_{P_3P_3}) \oplus (\mathbb{K}_{P_1Q_1} \otimes \mathbb{K}_{P_4P_4});\\
D^2 \cong{}& (\mathbb{K}_{P_1Q_1} \otimes \mathbb{K}_{P_1P_3}) \oplus (\mathbb{K}_{P_1Q_1} \otimes \mathbb{K}_{P_1P_4})
\end{aligned}
$$

The differential $d_i : D^i \to D^{i+1}$ for $i = 0, 1$ can be expressed by the following matrix form

$$d_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad d_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

with respect to basis $\{e^0_i\}_{i=1,\dots,6}$, $\{e^1_j\}_{j=1,\dots,7}$, $\{e^2_k\}_{k=1,2}$ for D^0, D^1, D^2. These are chosen by a canonical element in each ordered summand. Note that the only non-trivial cohomology class in the domain $D^\bullet$ is

$$[e^0_1 + \cdots + e^0_6] \cong [b_1 + b_2] \otimes [c_1 + c_2 + c_3].$$

In the codomain of m, $\mathbb{K}_{P_3Q_1} \oplus \mathbb{K}_{P_4Q_1} \xrightarrow{(1\ 1)} \mathbb{K}_{P_1Q_1}[-1]$, let f_1 and f_2 be the non-zero element in $\mathbb{K}_{P_3Q_1}$ and $\mathbb{K}_{P_4Q_1}$, respectively. Then $[f_1 + f_2]$ is the only non-trivial cohomology class.

The multiplication m for each summand is defined by

$$m\left(\mathbb{K}_{WR} \otimes \mathbb{K}_{ST}\right) = \begin{cases} \mathbb{K}_{SR} & \text{if } T = W; \\ 0 & \text{otherwise.} \end{cases} \tag{6.26}$$

and the induced multiplication

$$\overline{m} : H^* \operatorname{Hom}(\widetilde{Q}_1, Q_1) \otimes H^* \operatorname{Hom}(\widetilde{Q}_1, \widetilde{Q}_1) \to H^* \operatorname{Hom}(\widetilde{Q}_1, Q_1)$$

sends $[e^0_1 + \cdots + e^0_6]$ to $[f_1 + f_2]$ which coincides with

$$\overline{m}_2 : H^* \operatorname{Hom}(\epsilon_1, \epsilon_1) \otimes H^* \operatorname{Hom}(\epsilon_1, \epsilon_1) \to H^* \operatorname{Hom}(\epsilon_1, \epsilon_1).$$

Now move onto the multiplication

$$m : R\operatorname{Hom}(Q_2, Q_2) \otimes R\operatorname{Hom}(Q_2, Q_2) \to R\operatorname{Hom}(Q_2, Q_2).$$

which can be decomposed into the following four parts

$$m^{ij} : \Big(R\operatorname{Hom}(Q^1_2, Q^j_2) \otimes R\operatorname{Hom}(Q^i_2, Q^1_2)\Big) \oplus \Big(R\operatorname{Hom}(Q^2_2, Q^j_2) \otimes R\operatorname{Hom}(Q^i_2, Q^2_2)\Big) \to R\operatorname{Hom}(Q^i_2, Q^j_2)$$

for $(i, j) = (1,1), (1,2), (2,1)$ and $(2,2)$.

Note that the cochain complex computation for $(i, j) = (1,1)$ is identical to the above case. Let $[a]$ be the non-trivial cohomology class of $R\operatorname{Hom}(Q_2^1, Q_2^1)$, then the induced multiplication $\overline{m}$ on the cohomology gives $\overline{m}([a],[a]) = [a]$.

When $(i, j) = (1,2)$, the multiplication m^{ij} becomes

$$m^{12} : \left(\operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2) \otimes \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^1)\right) \\ \oplus \left(\operatorname{Hom}(\widetilde{Q}_2^2, Q_2^2) \otimes \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^2)\right) \to \operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2).$$

By the computation in (6.25), we already know that each summand of the domain is non-trivial. More precisely,

$$\begin{aligned} H^* \operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2) &\cong H^* \operatorname{Hom}\left(P_2 \xrightarrow{\binom{1}{1}} P_3 \oplus P_4, Q_2^2\right) \cong \langle [b] \rangle; \\ H^* \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^1) &\cong H^* \operatorname{Hom}\left(P_2 \xrightarrow{\binom{1}{1}} P_3 \oplus P_4, P_2 \xrightarrow{\binom{1}{1}} P_3 \oplus P_4\right) \\ &\cong H^* \left(\mathbb{K}_{P_2P_2} \oplus \mathbb{K}_{P_3P_3} \oplus \mathbb{K}_{P_4P_4} \xrightarrow{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)} \mathbb{K}_{P_2P_3}[-1] \oplus \mathbb{K}_{P_2P_4}[-1]\right) \\ &\cong \langle [c_2 + c_3 + c_4] \rangle, \end{aligned}$$

where b and c_i are the nonzero element in $\mathbb{K}_{P_2Q_2^2}[-1]$ and $\mathbb{K}_{P_iP_i}$, $i = 2,3,4$, respectively. Then we have $m^{12}(b \otimes (c_2 + c_3 + c_4)) = b$, see (6.26). Similarly,

$$\operatorname{Hom}(\widetilde{Q}_2^2, Q_2^2) \cong \mathbb{K}_{P_2Q_2^2} \cong \langle e \rangle, \qquad \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^2) \cong \mathbb{K}_{P_2P_2}[-1] \cong \langle f \rangle,$$

where e and f are nonzero elements of the corresponding vector spaces. Then the induced multiplication

$$\overline{m}^{12} : \left(H^* \operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2) \otimes H^* \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^1)\right) \\ \oplus \left(H^* \operatorname{Hom}(\widetilde{Q}_2^2, Q_2^2) \otimes H^* \operatorname{Hom}(\widetilde{Q}_2^1, \widetilde{Q}_2^2)\right) \to H^* \operatorname{Hom}(\widetilde{Q}_2^1, Q_2^2).$$

satisfies

$$\overline{m}^{12}([b] \otimes [c_2 + c_3 + c_4]) = [b], \qquad \overline{m}^{12}([e] \otimes [f]) = [b].$$

When $(i, j) = (2,1)$, the domain and the codomain of the multiplication become acyclic, hence the multiplication is trivial.

Let $(i,j)=(2,2)$, then by (6.25), we have

$$m^{22}:\operatorname{Hom}(\widetilde{Q}_2^2,Q_2^2)\otimes\operatorname{Hom}(\widetilde{Q}_2^2,\widetilde{Q}_2^2)\to\operatorname{Hom}(\widetilde{Q}_2^2,Q_2^2).$$

Note that

$$\begin{aligned}H^*\operatorname{Hom}(\widetilde{Q}_2^2,\widetilde{Q}_2^2)&\cong H^*\operatorname{Hom}(P_1\to P_2,P_1\to P_2)\\&\cong H^*\left(\mathbb{K}_{P_1P_1}\oplus\mathbb{K}_{P_2P_2}\xrightarrow{(1\ 1)}\mathbb{K}_{P_1P_2}[-1]\right)\\&\cong\langle[g_1+g_2]\rangle,\end{aligned}$$

where g_i is the nonzero element of $\mathbb{K}_{P_iP_i}$ for $i=1,2$. Then the multiplication satisfies $m^{22}(e\otimes(g_1+g_2))=e$, and hence induces

$$\begin{aligned}\overline{m}^{22}:H^*\operatorname{Hom}(\widetilde{Q}_2^2,Q_2^2)\otimes H^*\operatorname{Hom}(\widetilde{Q}_2^2,\widetilde{Q}_2^2)&\to H^*\operatorname{Hom}(\widetilde{Q}_2^2,Q_2^2);\\ [e]\otimes[g_1+g_2]&\mapsto[e].\end{aligned}$$

Let δ_1, δ_2, and δ_3 be the cohomology class of $R\operatorname{Hom}(Q_2^2,Q_2^2)$, $R\operatorname{Hom}(Q_2^1,Q_2^1)$, and $R\operatorname{Hom}(Q_2^1,Q_2^2)$, respectively, then the above multiplications recover the table of $\overline{m}_2$ with respect to the triple $(\epsilon_2,\epsilon_2,\epsilon_2)$.

Now consider the multiplication

$$m:R\operatorname{Hom}(Q_1,Q_2)\otimes R\operatorname{Hom}(Q_2,Q_1)\to R\operatorname{Hom}(Q_2,Q_2)$$

which can be reduced into

$$m:\operatorname{Hom}(\widetilde{Q}_1,Q_2^1)\otimes\operatorname{Hom}(\widetilde{Q}_2^2,Q_1)\to\operatorname{Hom}(\widetilde{Q}_2^2,Q_2^1)\cong 0.$$

Obviously, the above multiplication vanishes. In a similar manner, the following also vanishes:

$$m:R\operatorname{Hom}(Q_2,Q_1)\otimes R\operatorname{Hom}(Q_1,Q_2)\to R\operatorname{Hom}(Q_1,Q_1).$$

These coincide with the computation

$$\overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_1)}(\gamma\otimes\beta)=0,\qquad\qquad \overline{m}_2^{(\epsilon_2,\epsilon_1,\epsilon_2)}(\beta\otimes\gamma)=0.$$

The remaining cases are

$$\begin{aligned}
m &: R\operatorname{Hom}(Q_1,Q_2)\otimes R\operatorname{Hom}(Q_1,Q_1)\to R\operatorname{Hom}(Q_1,Q_2);\\
m &: R\operatorname{Hom}(Q_2,Q_2)\otimes R\operatorname{Hom}(Q_1,Q_2)\to R\operatorname{Hom}(Q_1,Q_2);\\
m &: R\operatorname{Hom}(Q_2,Q_1)\otimes R\operatorname{Hom}(Q_2,Q_2)\to R\operatorname{Hom}(Q_2,Q_1);\\
m &: R\operatorname{Hom}(Q_1,Q_1)\otimes R\operatorname{Hom}(Q_2,Q_1)\to R\operatorname{Hom}(Q_2,Q_1),
\end{aligned}$$

whose non-trivial parts boil down to

$$\begin{aligned}
m &: \operatorname{Hom}(\widetilde{Q}_1,Q_2^1)\otimes \operatorname{Hom}(\widetilde{Q}_1,\widetilde{Q}_1)\to \operatorname{Hom}(\widetilde{Q}_1,Q_2^1);\\
m &: \operatorname{Hom}(\widetilde{Q}_2^1,Q_2^1)\otimes \operatorname{Hom}(\widetilde{Q}_1,\widetilde{Q}_2^1)\to \operatorname{Hom}(\widetilde{Q}_1,Q_2^1);\\
m &: \operatorname{Hom}(\widetilde{Q}_2^2,Q_1)\otimes \operatorname{Hom}(\widetilde{Q}_2^2,\widetilde{Q}_2^2)\to \operatorname{Hom}(\widetilde{Q}_2^2,Q_1);\\
m &: \operatorname{Hom}(\widetilde{Q}_1,Q_1)\otimes \operatorname{Hom}(\widetilde{Q}_2^2,\widetilde{Q}_1)\to \operatorname{Hom}(\widetilde{Q}_2^2,Q_1),
\end{aligned}$$

respectively. These give non-trivial contribution to the multiplication and match with the following non-trivial $\overline{m}_2$ computation in the augmentation side:

$$\begin{aligned}
\overline{m}_2^{(\epsilon_1,\epsilon_1,\epsilon_2)}(\beta\otimes\alpha) &= \beta, & \overline{m}_2^{(\epsilon_1,\epsilon_2,\epsilon_2)}(\delta_2\otimes\beta) &= \beta,\\
\overline{m}_2^{(\epsilon_2,\epsilon_2,\epsilon_1)}(\gamma\otimes\delta_1) &= \gamma, & \overline{m}_2^{(\epsilon_2,\epsilon_1,\epsilon_1)}(\alpha\otimes\gamma) &= \gamma,
\end{aligned}$$

respectively.

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