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Iso-contact embeddings of manifolds in co-dimension 2

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The purpose of this article is to study co-dimension 2 iso-contact embeddings of closed contact manifolds. We first show that a closed contact manifold (M^{2n-1}, ξ_M) iso-contact embeds in a contact manifold (N^{2n+1}, ξ_N), provided M contact embeds in (N, ξ_N) with trivial normal bundle and the contact structure induced on M via this embedding is overtwisted and homotopic as an almost-contact structure to ξ_M. We apply this result to show that a closed contact 3–manifold having no 2–torsion in its second integral cohomology iso-contact embeds in the standard contact 5–sphere if and only if the first Chern class of the contact structure is zero. Finally, we discuss iso-contact embeddings of closed simply connected contact 5–manifolds.

1. Introduction

The study of embeddings of manifolds in Euclidean spaces has been a classical and well studied topic, which has lead to developments of many important tools in geometric topology. H.Whitney in [Wh] established that every smooth n–manifold admits an embedding in $\mathbb{R}^{2n}$. He also demonstrated that $\mathbb{R}P^2$ does not admit an embedding in $\mathbb{R}^3$, thereby showing that this result is optimal in general. However, M. Hirsch generalized the result for odd dimensional closed orientable manifolds to establish that every $(2n+1)$– dimensional manifold admits an embedding in $\mathbb{R}^{4n-1}$. This, in particular, implies that every closed orientable 3–manifold admits an embedding in $\mathbb{R}^5$. On the other hand, J. Nash in [Na] established that every closed Riemannian n–manifold admits a C^∞–isometric embedding in $\frac{n}{2}(3n+11)$–dimensional flat Euclidean space and also proved that the problem of finding a C^1–isometric embedding is completely unobstructed [Na2].

In this article, we study *iso-contact* embeddings of contact manifolds. Recall that by a contact structure on a manifold M, we mean a maximally nowhere integrable hyperplane field ξ on M. The contact structure is said to be co-orientable, provided ξ is the kernel of a 1–form defined on M. A contact manifold M with a contact structure ξ is denoted by the pair (M, ξ). When ξ is co-oriented and ξ is the kernel of a 1–form α defined on M, then we also denote the contact manifold (M, ξ) by the pair $(M, Ker\{\alpha\})$. In this article, we will always work with co-orientable contact structures defined on orientable manifolds.

Let $(M_1, Ker\{\alpha_1\})$ and $(M_2, Ker\{\alpha_2\})$ be two contact manifolds. We say that $(M_1, Ker\{\alpha_1\})$ admits an *iso-contact embedding* in $(M_2, Ker\{\alpha_2\})$, provided there exists a smooth embedding $f : M_1 \hookrightarrow M_2$ such that $f^*\alpha_2 = g\alpha_1$, for some everywhere positive function $g : M_1 \to \mathbb{R}$. In case a manifold M_1 admits an embedding into a contact manifold (M_2, ξ_2) such that the restriction of ξ_2 to M_1 is a contact structure on M_1, we say M_1 admits a *contact embedding* in (M_2, ξ_2).

For a contact manifold $(M_1, Ker\{\alpha_1\})$ to admit an iso-contact embedding in the manifold $(M_2, Ker\{\alpha_2\})$, there must exist a smooth embedding f of M_1 in M_2 and a monomorphism $F : TM_1 \to TM_2$ which covers the map f and satisfies the property that the bundle $(F_*\xi_1, F_*d\alpha_1)$ is a conformal symplectic sub-bundle of $(M_2, Ker\{\alpha_2\})$. If such a pair (f, F) exists, then we say that we have a formal iso-contact embedding of $(M_1, Ker\{\alpha_1\})$ in $(M_2, Ker\{\alpha_2\})$. We refer to [EM, Chpt-12] for more on formal iso-contact embeddings.

Questions related to iso-contact embeddings of closed contact manifolds in an arbitrary contact manifold (N, ξ) abide by the h-principle provided that the co-dimension of an embedding is greater or equal to 4. This was proved by M. Gromov [Gr, Chapter 2.4] using convex integration techniques. Gromov also established the h-principle for iso-contact embeddings for the category of open contact manifolds provided that the co-dimension of an embedding is greater than or equal to 2 [EM, Chapter 12]. Gromov [Gr, Chapter 3.4] also proved the h-principle for co-dimension 2 immersions.

Our main focus is on understanding iso-contact embeddings of closed contact manifolds in co-dimension 2. In this co-dimension, the techniques developed by Gromov in [Gr] are generally not sufficient for a complete answer. Henceforth, unless mentioned explicitly co-dimensions of embeddings will be assumed to be 2.

The systematic study of co-dimension 2 iso-contact embeddings of closed contact manifolds was initiated by J. Etnyre and R. Fukuwara in [EF].[1] Iso-contact embeddings of contact 3–manifolds in $M \times S^2$ were also constructed in [NP] using the fact that the co-tangent bundle of any closed

[1] In [EF] the term contact embedding means iso-contact embedding. We on the other hand follow the conventions from [EM].

orientable 3–manifold is trivial. In [EL], it is shown that every closed contact 3–manifold admits an iso-contact embedding in an overtwisted contact $\mathbb{S}^2 \times \mathbb{S}^3$. We would also like to point out that A. Mori in [Mr] also produced iso-contact embeddings of all contact 3–manifolds in the contact manifold $\left(\mathbb{R}^7, Ker\{dz + \sum_{i=1}^{3} x_i dy_i\}\right)$ using open books and D. Martinez-Torres in [Ma] produced an iso-contact embedding of any contact manifold M^{2n+1} in $\left(\mathbb{R}^{4n+3}, Ker\{dz + \sum_{i=1}^{2n+1} x_i dy_i\}\right)$.

In order to state iso-contact embedding results of this article, we need the notion of *overtwisted* contact manifolds due to M. Borman, Y. Eliashberg and E. Murphy discussed in [BEM].

Recall that a contact manifold (M, ξ) is said to be overtwisted, provided it admits an iso-contact embedding of an overtwisted ball. For a precise definition of an overtwisted ball, refer [BEM]. For the purpose of this article what is important is the following fact proved in [BEM]:

In every homotopy class of almost contact structures, there exists a unique overtwisted contact structure up to isotopy. Here, by an almost-contact structure on a manifold M, we mean a hyperplane-field ξ together with a conformal class of a symplectic structure on it. We would like to remark that a contact structure $Ker\{\alpha\}$ can be naturally regarded as an almost-contact structure. This is because $d\alpha$ restricted to $Ker\{\alpha\}$ provides the conformal class of a symplectic structure on the hyperplanes $Ker\{\alpha\}$. Now, we state our main result of this article.

Theorem 1. *Let (M, ξ_M) be a closed contact manifold of dimension $2n-1$. Let ξ_M^{ot} denote the unique overtwisted contact structure in the almost contact class of ξ_M. If (M, ξ_M^{ot}) admits an iso-contact embedding in a contact manifold (N, ξ_N) of dimension $2n+1$ with trivial normal bundle, then so does (M, ξ_M).*

Our proof of Theorem 1 relies on certain *flexibility* discovered in iso-contact embeddings of contact manifolds in neighborhoods of a special class of closed contact overtwisted manifolds which are assumed to be embedded in a given contact manifold. See Proposition 18 for a precise statement.

After this article was announced, there has been significant developments in the study of iso-contact embeddings. O. Lazarev in [La] provided a new proof of Theorem 1, the article [CPP] establishes existence h-principle for iso-contact embeddings, and the article [CE] by R. Casals and J. Etnyre provides first examples of formally isotopic but not contact isotopic contact submanifolds in higher dimensions.

Let us now discuss some applications of Theorem 1. We first discuss co-dimension 2 iso-contact embeddings of contact manifolds in the standard contact spheres. Recall that by the standard contact structure ξ_{std} on the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$, we mean the kernel of the 1-form $\sum_{i=1}^{n} x_i dy_i - y_i dx_i$ restricted to $\mathbb{S}^{2n-1}$.

The techniques developed to prove Theorem 1 also establishes the following proposition.

Proposition 2. *A closed contact manifold (M^{2n-1}, ξ_M) admits an iso-contact embedding in the standard contact sphere $(\mathbb{S}^{2n+1}, \xi_{std})$ if and only if M admits a contact embedding in $(\mathbb{S}^{2n+1}, \xi_{std})$ and the induced contact structure on M by the embedding is homotopic to ξ_M as an almost-contact structure.*

There are many interesting classes of smooth manifolds which admit smooth co-dimension 2 embeddings in the standard spheres. For example, as mentioned earlier, M. Hirsch in [Hi] showed that every closed smooth 3–manifold admits a smooth embedding in $\mathbb{S}^5$. There are now many proofs of this result. See for example, [HLM] for what is now known as braided embedding and [PPS] for embeddings using open books.

N. Kasuya in [Ka] first observed that not all contact 3–manifolds admit iso-contact embeddings in the standard contact $\mathbb{S}^5$. He showed that the necessary condition for the existence of such an embedding is that the first Chern class of the contact structure must be zero. In [Ka], Kasuya also showed that every closed contact 3–manifold (M, ξ) admits an iso-contact embedding in some contact $\mathbb{R}^5$.

In [EF], Etnyre and Fukuwara obtained various iso-contact embedding results. One of the most striking results which they established states that every overtwisted contact 3–manifold (M, ξ_{ot}) with no 2–torsion in the second integral cohomology iso-contact embeds in the standard contact $\mathbb{S}^5$ if and only if the first Chern class of the overtwisted contact structure ξ_{ot} is zero.

Applying Proposition 2 about iso-contact embeddings in the spheres and the result about iso-contact embeddings of overtwisted 3–manifolds in $\mathbb{S}^5$ proved in [EF], we prove the following:

Theorem 3. *Let M be a closed orientable 3–manifold. Then, we have the following:*

1) *In case, M has no 2–torsion in $H^2(M, \mathbb{Z})$, then M together with any contact structure ξ on it admits an iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$ if and only if the first Chern class $c_1(\xi)$ is zero.*
2) *In case, M has a 2–torsion in $H^2(M, \mathbb{Z})$, then there exits a homotopy class $[\xi]$ of plane fields on M such that M together with any contact structure homotopic to a plane field belonging to the class $[\xi]$ over a 2–skeleton of M admits an iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$.*

Finally, we discuss iso-contact embeddings of simply-connected contact 5–manifolds in $(\mathbb{S}^7, \xi_{std})$. In particular, we establish:

Theorem 4. *Let (M, ξ) be a closed simply connected contact 5–manifold with the second Steifel-Whitney class $w_2(M) = 0$ in $H^2(M, \mathbb{Z}/2\mathbb{Z})$. Then, (M, ξ) admits an iso-contact embedding in $(\mathbb{S}^7, \xi_{std})$ if and only if the first Chern class $c_1(\xi) = 0$.*

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2. Preliminaries

In this section, we briefly review notions necessary for the article pertaining to open books, contact structures and relationships between them. Some good references for these are [Gi], [Et], and [Ko1].

2.1. Open books

Let us review few results related to open book decompositions of manifolds. We first recall the following:

Definition 5 (Open book decomposition). *An open book decomposition of a closed oriented manifold M consists of a co-dimension 2 oriented submanifold B with a trivial normal bundle in M and a locally trivial fibration $\pi : M \setminus B \to \mathbb{S}^1$ such that $\pi^{-1}(\theta)$ is an interior of a co-dimension 1 submanifold N_θ satisfying $\partial N_\theta = B$ for all $\theta \in \mathbb{S}^1$. Furthermore, the normal bundle $\mathcal{N}(B)$ of the submanifold B is trivialized such that π restricted to $\mathcal{N}(B) \setminus B \to \mathbb{S}^1$ is given by the angular co-ordinate in $\mathbb{D}^2$–factor.*

The submanifold B is called the *binding*, and N_θ is called a *page* of the open book. We denote the open book decomposition of M by $(M, \mathcal{O}b(B, \pi))$, or sometimes simply by $\mathcal{O}b(B, \pi)$.

Next, we discuss the notion of an *abstract open book decomposition*. To begin with, let us recall that the *mapping class group* of a manifold $(\Sigma, \partial\Sigma)$ is the group of isotopy classes of orientation preserving diffeomorphisms of $(\Sigma, \partial\Sigma)$ which are the identity near the boundary $\partial\Sigma$.

Definition 6 (Mapping torus). *Let Σ be a manifold with non-empty boundary $\partial\Sigma$. Let ϕ be an element of the mapping class group of Σ. By the mapping torus $\mathcal{MT}((\Sigma, \partial\Sigma), \phi)$, we mean*

$$\Sigma \times [0, 1]/ \sim,$$

where $\sim$ is the equivalence relation identifying $(x, 0)$ with $(\phi(x), 1)$.

Observe that by the definition of $\mathcal{MT}((\Sigma, \partial\Sigma), \phi)$, there exists a collar of the boundary $\partial\mathcal{MT}((\Sigma, \partial\Sigma), \phi)$ in $\mathcal{MT}((\Sigma, \partial\Sigma), \phi)$ which can be identified with $(-\epsilon, 0] \times \partial\Sigma \times \mathbb{S}^1$. This is because the diffeomorphism ϕ is the identity in a collar $(-\epsilon, 0] \times \partial\Sigma$ of the boundary of Σ. We will sometimes denote the mapping torus $\mathcal{MT}((\Sigma, \partial\Sigma), \phi)$ just by $\mathcal{M}(\Sigma, \phi)$. We are now in a position to define an abstract open book decomposition.

Definition 7 (Abstract open book). *Let Σ and ϕ as in the previous definition. An abstract open book decomposition of M is pair (Σ, ϕ) such that M is diffeomorphic to*

$$\mathcal{MT}(\Sigma, \phi) \cup_{id} \partial\Sigma \times \mathbb{D}^2,$$

where id denotes the identity mapping of $\partial\Sigma \times \mathbb{S}^1$

The map ϕ is called the *monodromy* of the open book. We will denote an abstract open book decomposition by $\mathcal{A}ob(\Sigma, \phi)$. Note that the mapping class ϕ uniquely determines $M = \mathcal{A}ob(\Sigma, \phi)$ up to diffeomorphism.

One can easily see that an abstract open book decomposition of M gives an open book decomposition of M up to diffeomorphism and vice versa. Hence, sometimes we will not distinguish between open books and abstract open books. In particular, we will continue to use the notation $\mathcal{A}ob(\Sigma, \phi)$ to denote the open book decomposition associated to the abstract open book $\mathcal{A}ob(\Sigma, \phi)$.

Examples 8. 1) *Notice that* $\mathbb{S}^n$ *admits an open book decomposition with pages* $\mathbb{D}^{n-1}$ *and monodromy the identity map of* $\mathbb{D}^{n-1}$. *We call this open book the* trivial open book *of* $\mathbb{S}^n$. *For more details regarding open books, refer the lecture notes [Et] and [Gi, Chapter 4.4.2].*

2) *The manifold* $\mathbb{S}^3 \times \mathbb{S}^2$ *admits an open book decomposition with pages disk co-tangent bundle* $\mathcal{D}T^*\mathbb{S}^2$ *and monodromy the identity.*

3) *In [Al], it was shown that every closed orientable* 3*–manifold admits an open book decomposition. This result was further generalized to all odd dimensional closed orientable manifold of dimension bigger than* 5 *by Quinn [Qu].*

Given two abstract open books $M_1^n = \mathcal{A}ob(\Sigma_1, \phi_1)$ and $M_2^n = \mathcal{A}ob(\Sigma_2, \phi_2)$, if we make the boundary connected sum of the pages of $\mathcal{A}ob(\Sigma_1, \phi_1)$ and $\mathcal{A}ob(\Sigma_1, \phi_2)$, then we get an abstract open book decomposition $\mathcal{A}ob(\Sigma_1 \#_\partial \Sigma_2, \phi_1 \# \phi_2)$ of $M_1 \# M_2$. This was first demonstrated in [Ga] by D. Gabai for 3–manifolds.

There exists an intimate connection between the open books and the contact structures on the manifolds. This was discovered first by W. Thurston and H. Winkelnkemper [TW] and was later strengthened by E. Giroux in [Gi]. In order to understand this correspondence, we first recall the notion of a contact structure supported by an open book.

2.2. Contact manifolds and supporting open books

Definition 9 (Open book supporting a contact form). *Let $(M^{2n+1}, Ker\{\alpha\})$ be a contact manifold. We say that an open book decomposition $\mathcal{O}b(B,\pi)$ supports the contact form α provided:*

1) *The binding B is a contact submanifold of M.*
2) *The 2–form $d\alpha$ is a symplectic form on each page of the open book.*
3) *The boundary orientation on B coming from the orientation of the pages induced by $(d\alpha)^n$ is the same as the orientation given by $\alpha|_B \wedge (d\alpha|_B)^{n-1}$.*

We would like to remark that if α_1 and α_2 are two contact forms on a contact manifold M which are supported by the same open book $\mathcal{O}b(B,\pi)$, then they are isotopic as contact structures. See, for details, [Ko1].

Let (M,ξ) be a contact manifold. We say that ξ is supported by an open book decomposition $\mathcal{O}b(B,\pi)$ of M provided that there exists a contact 1–form α inducing the contact structure ξ on M such that α is supported by $\mathcal{O}b(B,\pi)$.

Giroux in [Gi] provided a one to one correspondence between the open books up to *positive stabilizations* and the supported contact structures up to isotopy for closed orientable 3–manifolds. See the notes [Ko1] by O. van Koert and [Et] by Etnyre for more on this. The purpose of the next subsection is to recall a few notions and the results associated to the Giroux's correspondence.

2.3. Contact abstract open book and the Giroux's correspondence

We begin this subsection by recalling the notion of the *Generalized Dehn twist*. This notion is necessary to understand the notion of positive stabilization. This notion was first introduced by Arnold in the Floer memorial volume [Ar]. Here, we are following the explicit formula due to P. Seidel given in [Se1]. See also [Se2].

Definition 10 (Generalized Dehn twist). *Consider,*

$$T^*\mathbb{S}^n = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} |\ x.y = 0, ||y|| = 1\}.$$

Define a diffeomorphism τ *of* $T^*\mathbb{S}^n$ *as follows:*

$$\tau(x,y) = \begin{pmatrix} \cos g(y) & |y|^{-1}\sin g(y) \\ -|y|\sin g(y) & \cos g(y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where, g is a function of y which is the identity near 0 *and is zero outside a compact set containing* 0*. The diffeomorphism* τ *is called the generalized Dehn twist while* τ^{-1} *is called the negative generalized Dehn twist.*

It is relatively easy to check that τ is a compactly supported symplectomorphism of $T^*\mathbb{S}^n$. Furthermore, τ can be isotoped to a symplectomorphism which is compactly supported in an arbitrary small neighborhood of the zero section of $T^*\mathbb{S}^n$. This, in particular, implies that τ and τ^{-1} can be regarded as diffeomorphisms of the disk co-tangent bundle $\mathcal{D}T^*\mathbb{S}^n$. We refer to [MS, page-186] and the notes [Ko1] for more details. We recommend [KN] for a nice exposition on how to produce a compactly supported generalized Dehn twist.

Next, we discuss the notion of a *contact abstract open book.* We refer to [Ko, Section–2] for a more detailed description of this.

Let $(\Sigma, d\lambda)$ be a Weinstein manifold, and let ϕ be an exact symplectomorphism of Σ which is the identity near the boundary of Σ. Giroux generalized the construction of Thurston and Winkelnkemper given in [TW] to produce a contact form on the manifold with open book $\mathcal{A}ob(\Sigma,\phi)$ such that the contact form is supported by the open book $\mathcal{A}ob(\Sigma,\phi)$ in the sense explained in Subsection 2.2. We will generally denote this contact form by $\alpha_{(\Sigma,\phi)}$. See lecture notes by O. van Koert [Ko] for the details of this construction. The article [GM] is also a good reference for this. We call this contact manifold a *contact abstract open book.*

In this article, unless stated otherwise, whenever we talk of a contact structure ξ supported by an abstract open book $\mathcal{A}ob(\Sigma,\phi)$, we will always mean that Σ is a Weinstein manifold, ϕ an exact symplectomorphism of Σ which when restricted to a collar of its boundary is the identity, and the contact structure ξ is contactomorphic to $Ker\{\alpha_{(\Sigma,\phi)}\}$ described earlier.

Examples 11. 1) *Let* $\mathbb{D}^{2n}$ *denote the unit* $2n$*–disk in* $\mathbb{R}^{2n}$*. Let* λ_{std} *denote the canonical* 1*–form on* $\mathbb{D}^{2n}$ *given by* $\sum_{i=1}^{n} x_i dy_i - y_i dx_i$ *which induces the standard symplectic structure on* $\mathbb{D}^{2n}$*. The standard contact sphere* $(\mathbb{S}^{2n+1}, \xi_{std})$ *is contactomorphic to the contact abstract open book* $(\mathcal{A}ob(\mathbb{D}^{2n}, id), Ker\{\alpha_{(\mathbb{D}^{2n},id)}\})$.

2) *Consider $\mathcal{D}T^*\mathbb{S}^n$, the unit disk bundle associated to the co-tangent bundle of $\mathbb{S}^n$ and the Generalized Dehn twist τ on $\mathcal{D}T^*\mathbb{S}^n$. It is well known that the contact abstract open book $\mathcal{A}ob(\mathcal{D}T^*\mathbb{S}^n, \tau)$ is contactomorphic to the standard contact sphere $(\mathbb{S}^{2n+1}, \xi_{std})$.*

3) *The contact abstract open book $\mathcal{A}ob(\mathcal{D}T^*\mathbb{S}^n, \tau^{-1})$ induces an overtwisted contact structure on $\mathbb{S}^{2n+1}$. This is clearly discussed for 3–manifolds in [KN1]. In general, this follows from [CMP]. We will denote this overtwisted contact structure by ξ_{stot}. We will denote a contact 1–form inducing the contact structure ξ_{stot} by α_{stot}.*

We now define the notion of a *generalized contact abstract open book*:

Definition 12 (Generalized contact abstract open book).
Let $((W, \partial W), d\lambda)$ be a Weinstein cobordism with a connected convex boundary M. Let ϕ be a symplectomorphism of $(W, d\lambda)$ which is the identity in a small collar of the boundary of W. Consider the quotient manifold N defined as:

$$N = \mathcal{MT}(W, \phi) \cup_{id} M \times \mathbb{D}^2.$$

Notice that N admits a contact structure analogous to the one discussed earlier for the contact abstract open book. We call N a generalized contact abstract open book with the binding M, the page W, and the monodromy ϕ.

By a slight abuse of notation, we will use the same notation $\mathcal{A}ob(W, \phi)$ for the generalized contact abstract open book as well. By a Weinstein manifold or Weinstein domain, we will always mean a Weinstein cobordism with an empty concave boundary and a connected convex boundary. Note that whenever the Weinstein cobordism associated to a generalized contact abstract open book is a Weinstein manifold, we get usual contact abstract open book.

We would like to think of $M \times \mathbb{D}^2_\varepsilon$ as an open book with pages $[0, \epsilon) \times M$ and the binding M. This abstract open book is a special case of a generalized abstract open book. Since we will need it time and again, we introduce a special terminology for it.

Definition 13 (ε–partial open book). *Consider the contact manifold $(M, Ker\{\alpha\})$. The contact manifold $M \times \mathbb{D}^2_\epsilon$ with the contact form $e^{-r}\alpha + r^2 d\theta$ is called an ε–partial open book associated to $(M, Ker\{\alpha\})$.*

Definition 14 (Generalized contact abstract connected sum). *Let $(\mathcal{A}ob(W_1, \phi_1), \alpha_{(W_1,\phi_1)})$ and $(\mathcal{A}ob(W_2, \phi_2), \alpha_{(W_2,\phi_2)})$ be two generalized contact abstract open books. Observe that we can perform the boundary connected sum $W_1 \#_\partial W_2$ of W_1 and W_2 along their connected convex boundaries to produce a new Weinstein cobordism $W_1 \#_\partial W_2$ with connected convex boundary $\partial W_1 \# \partial W_2$. Let $\mathcal{A}ob(W_1, \phi_1) \# \mathcal{A}ob(W_2, \phi_2)$ be the generalized abstract open book obtained by performing the boundary connected sum of their pages along convex boundaries together with the monodromy $\phi \# \phi_2$ which corresponds to the monodromy ϕ_1 on W_1 extended by the identity along the Weinstein 1–handle to get a symplectomorphism of $W_1 \#_\partial W_2$ followed by ϕ_2 extended to $W_1 \#_\partial W_2$ in exactly the same fashion.*

Since the page of the generalized abstract open book are Weinstein cobordism $W_1 \#_\partial W_2$ with connected convex boundary $M_1 \# M_2$ – where M_i is the convex boundary of W_i for each $i = 1, 2$ – it is clear that this generalized abstract open book carries a natural contact structure supported by the generalized open book having pages $W_1 \#_\partial W_2$ and the monodromy $\phi_1 \# \phi_2$. This contact structure will be denoted by $Ker\{\alpha_{(W_1 \#_\partial W_2, \phi_1 \# \phi_2)}\}$. We call this contact manifold the generalized contact abstract connected sum.

Remark 15. 1) *Observe that the binding of a generalized contact abstract boundary connected sum is the connected sum of the bindings of the generalized contact abstract open books.*

2) *When W_1 and W_2 are Weinstein manifolds, then the generalized contact abstract connected sum $\mathcal{A}ob(W_1, \phi_1) \#_\partial \mathcal{A}ob(W_2, \phi_2)$ is the contact connected sum of $\mathcal{A}ob(W_1, \phi_1)$ and $\mathcal{A}ob(W_2, \phi_2)$. The contact structure $Ker\{\alpha_{(W_1 \#_\partial W_2, \phi_1 \# \phi_2)} = \alpha\}$ is supported by the open book with pages $W_1 \#_\partial W_2$ and the monodromy $\phi_1 \# \phi_2$.*

3) *We will sometime use the notation $\mathcal{A}ob(W_1 \#_\partial W_2, \phi_1 \# \phi_2)$ to denote the generalized abstract connected sum $\mathcal{A}ob(W_1, \phi_1) \# \mathcal{A}ob(W_2, \phi_2)$. This notation will be used to emphasis the abstract open book decomposition of $\mathcal{A}ob(W_1, \phi_1) \# \mathcal{A}ob(W_2, \phi_2)$. More importantly to emphasis that the monodromy $\phi_1 \# \phi_2$ is the monodromy which restricts to ϕ_1 on W_1, ϕ_2 on W_2, and is the identity when restricted to the Weinstein 1–handle connecting W_1 and W_2.*

2.4. Iso-contact open book embeddings

In this subsection, we discuss the notion of *iso-contact open book embeddings.* For more on open book embeddings, refer [EL] and [PPS].

Definition 16. *Let $M = \mathcal{A}ob(\Sigma, \phi)$ and $N = \mathcal{A}ob(W, \Psi)$ be two generalized contact abstract open books. Let $F : M \to N$ be a proper iso-contact embedding of M in N. We say that this embedding is a contact abstract open book embedding, provided the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}T(\Sigma,\phi) & \overset{F}{\hookrightarrow} & \mathcal{M}T(W,\Psi) \\ & \searrow^{\pi_1} & \downarrow^{\pi_2} \\ & & \mathbb{S}^1. \end{array}$$

Here, $\pi_1 : \mathcal{M}T(\Sigma, \phi) \to \mathbb{S}^1$ and $\pi_2 : \mathcal{M}T(W, \Psi) \to \mathbb{S}^1$ are the natural projections associated to the mapping tori.

We end this section by proving a proposition. This proposition, in particular, establishes that if (M_1, ξ_{M_1}) iso-contact embeds in $(N_1, Ker\{\alpha_1\})$ and (M_2, ξ_{M_2}) iso-contact embeds in $(N_2, Ker\{\alpha_2\})$, then the *contact connected sum* $(M_1\#M_2, \xi_{M_1}\#\xi_{M_2})$ iso-contact embeds in the contact connected sum of N_1 and N_2 given by

$$(N_1\#N_2, Ker\{\alpha_1\}\#Ker\{\alpha_2\}) = (N_1\#N_2, Ker\{\alpha_1\#\alpha_2\}).$$

This was already proved by Etnyre and Fukuwara in [EF].

Proposition 17. *If a contact abstract open book $(\mathcal{A}ob(\Sigma_i^{2n-2}, \phi_i), \eta_i)$ iso-contact open book embeds in a generalized contact abstract open book $(\mathcal{A}ob(W_i^{2n}, \Psi_i), \xi_i)$, for $i = 1, 2$, then the contact abstract connected sum $(\mathcal{A}ob(\Sigma_1, \phi_1)\#\mathcal{A}ob(\Sigma_2, \phi_2), \eta_1\#\eta_2)$ iso-contact open book embeds in the generalized contact abstract connected sum $(\mathcal{A}ob(W_1, \Psi_1)\#\mathcal{A}ob(W_2, \Psi_2), \xi_1\#\xi_2)$.*

Furthermore, if Σ_i is contained in an arbitrary small collar of the convex (connected) boundary component M_i of ∂W_i of W_i, then we can ensure that the page $\Sigma_1\#_\partial\Sigma_2$ of $\mathcal{A}ob(\Sigma_1\#_\partial\Sigma_2, \phi_1\#\phi_2)$ is contained in an arbitrary small collar of the convex boundary of the page of $\mathcal{A}ob(W_1\#_\partial W_2, \Psi_1\#\Psi_2)$.

Proof. First of all notice that since the boundary connected sum of W_1 with W_2 can be regarded as adding a 1–handle to $W_1 \sqcup W_2$, we can perform the boundary connected sum of W_1 with W_2 along their convex boundaries in such way that the boundary connected sum of Σ_1 with Σ_2 properly symplectically embeds in $W_1\#_\partial W_2$. To achieve this, notice that in order to perform the boundary connected sum, we need to fix a small Darboux ball U_1 around a point p_1 in $M_1 \subset \partial W_1$ and a small Darboux ball U_2 around a point p_2 in $M_2 \subset \partial W_2$. We fix these balls in such a way that they restrict to Darboux balls $\widetilde{U}_i$ containing the point p_i in $\partial\Sigma_i$, for each $i = 1, 2$. Now, if we perform the boundary connected sum of W_1 with W_2, we get an induced boundary connected sum of Σ_1 with Σ_2 which is contained in $W_1\#_\partial W_2$.

Observe that we have not yet achieved the second property. In order to achieve this, we first observe that $\Sigma_1\#_\partial\Sigma_2 \subset W_1\#_\partial W_2$ can be made disjoint from the core of the 1–handle B associated to $W_1\#_\partial W_2$ by a sufficiently small C^∞ perturbation whose support is contained in a small tubular neighborhood of $B \cap \Sigma_1\#_\partial\Sigma_2 \subset W_1\#_\partial W_2$. See Figure 1 for a pictorial description.

Let ϵ_1 be such that Σ_1 is contained in the symplectic collar $([0, \epsilon_1] \times M_1, d(e^t\alpha_1))$ of M_1 in W_1, where $e^t\alpha_1$ is the Liouville 1–form on the symplectic collar of the convex boundary M_1.

Let ϵ_2 be such that Σ_2 is contained in the symplectic collar $([0, \epsilon_2] \times \partial M_2, d(e^t\alpha_2))$ of M_2 in W_2, where $e^t\alpha_2$ is the Liouville 1–form on the symplectic collar of the convex boundary of M_2.

Let us denote by $B = \mathbb{D}^{2n-1}(\delta) \times \mathbb{D}(1)$ the band of length 1 and radius δ used in the boundary connected sum $W_1\#_\partial W_2$. Clearly, by the construction $\widetilde{B} = \mathbb{D}^{2n-3}(\delta) \times \mathbb{D}(1)$ is the band associated to the induced boundary connected sum $\Sigma_1\#_\partial\Sigma_2$.

Let A_δ denote the annulus $[\frac{\delta}{10}, \delta] \times \mathbb{S}^{2n-2} \times \mathbb{D}^1$. Notice that the part of the boundary of the band B corresponding to $\mathbb{D}^{2n-1} \times \partial\mathbb{D}(1)$ can be assumed to have the symplectic collar A_δ. Hence, if the C^∞–perturbation that we perform in order to make $\Sigma_1\#_\partial\Sigma_2$ disjoint from the core of 1– handle is done such that perturbed $\Sigma_1\#_\partial\Sigma_2$ is contained in A_δ and the support of the perturbation is contained in the complement of the annulus $[\frac{9\delta}{10}, \delta] \times \mathbb{S}^{2n-1} \times \mathbb{D}^1$, then the perturbed band $\widetilde{B}$ associated to $\Sigma_1\#_\partial\Sigma_2$ is contained in A_δ and its intersection with the boundary of the annulus A_δ is the same as the intersection of unperturbed $\widetilde{B}$. Observe that such a perturbation is always possible.

Next, choose δ such that $\delta < min\ \{\epsilon_1, \epsilon_2\}$. Observe that for this choice of δ the perturbed $\Sigma_1\#_\partial\Sigma_2$ lies in a small symplectic neighborhood of $\partial W_1\#\partial W_2$ as claimed. See Figure 2.

Finally, observe that since the symplectomorphisms Ψ_1 and Ψ_2 are the identity in suitable collars of the boundaries of W_1 and W_2 respectively, the symplectomorphism $\Psi_1\#\Psi_2$ naturally induces the symplectomorphism $\phi_1\#\phi_2$ on the symplectically embedded $\Sigma_1\#_\partial\Sigma_2 \subset W_1\#_\partial W_2$ that we just described. This proves the proposition. $\square$

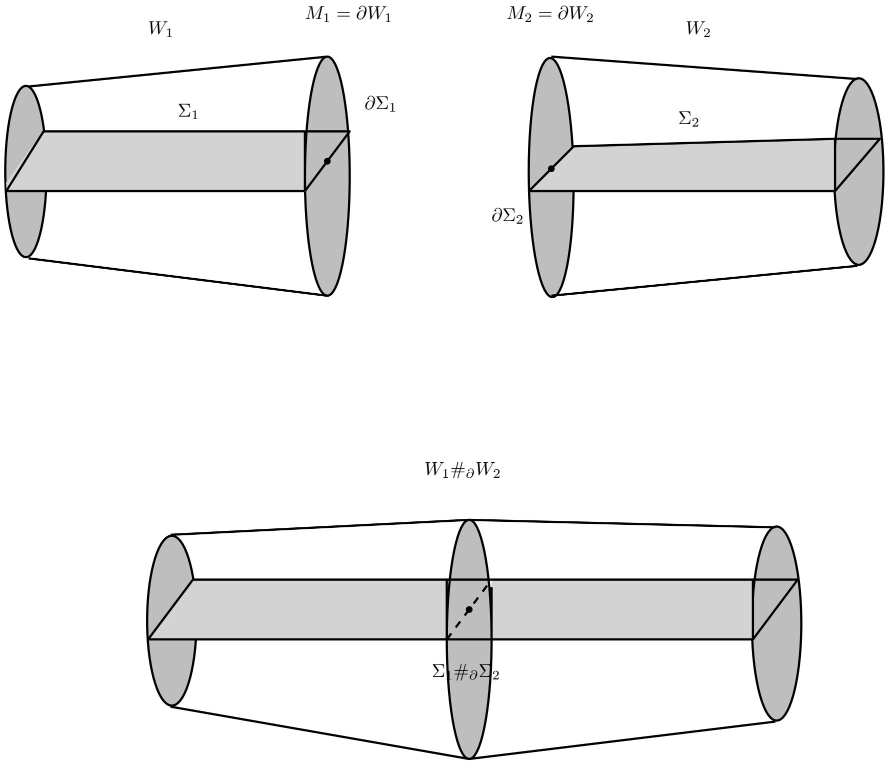

Figure 1: The figure on the top depicts small Darboux neighborhoods of the attaching spheres p_i contained in $\partial\Sigma_i \subset \partial W_i$ together with a small collar inside W_i used in performing the boundary connected sum $W_1\#_\partial W_2$. The picture of on the bottom depicts the embedding of the Darboux ball of $\Sigma_1\#_\partial\Sigma_2 \subset W_1\#_\partial W_2$. The embedded $\Sigma_1\#_\partial\Sigma_2$ is then perturbed to miss the point p.

3. Proof of Theorem 1

The purpose of this section is to prove Theorem 1. There are two main steps in establishing Theorem 1. We state these steps in the form of Proposition 18 and Proposition 19. Proposition 18 is the deepest one and it can be claimed to contain the most original argument of the article. We give a proof of this proposition in Section 4.

In order to state Proposition 18, we need to introduce the following notation. Let (M^{2n+1}, ξ) be a contact manifold. The contact structure obtained

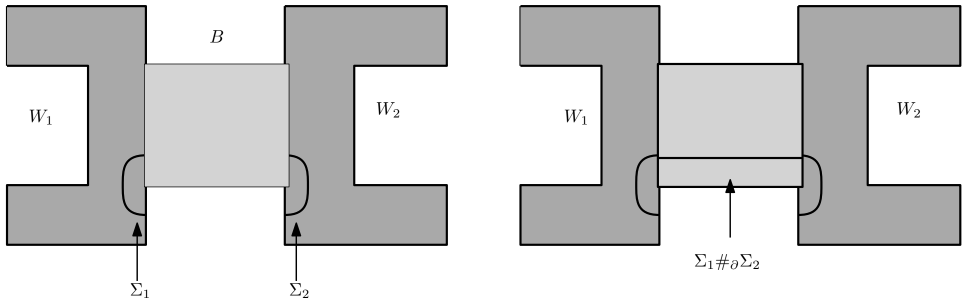

Figure 2: The figure on the left depicts a collar of $\partial W_i = M_i$ containing Σ_i for each i together with the band B. The red line at the center of the band B is the core of the attaching band. The figure on the right depicts $\Sigma_1 \#_\partial \Sigma_2$ embedded in $W_1 \#_\partial W_2$ close to the boundary $\partial(W_1 \#_\partial W_2)$ and disjoint from the core of the 1-handle B.

by the contact connected sum of (M, ξ) with the standard overtwisted sphere $(\mathbb{S}^{2n+1}, \xi_{stot})$ will be denoted by ξ^{stot}. Notice that if ξ is supported by an open book decomposition $\mathcal{A}ob(\Sigma, \phi)$, then ξ^{stot} is supported by the open book $\mathcal{A}ob(\Sigma \#_\partial \mathcal{D}T^*\mathbb{S}^n, \phi \# \tau^{-1})$. This follows from [CMP]. From now on, let us call ξ^{stot} the *standard overtwisted structure* on M associated to the given contact structure ξ on M. Notice that it is false in general that ξ and ξ^{stot} are homotopic as almost contact structures.

Proposition 18. *Let M^{2n-1} be a closed smooth manifold. Let ξ be a contact structure on M. Suppose that (M, ξ^{stot}) admits an iso-contact embedding in a contact manifold (N^{2n+1}, ξ_N) with trivial normal bundle, then (M, ξ) also admits an iso-contact embedding in (N, ξ_N).*

So, we just claim that if a manifold M with the standard overtwisted contact structure associated to a contact structure ξ on M admits an iso-contact embedding in (N, ξ_N) with trivial normal bundle then so does the contact manifold (M, ξ).

This is the key step and its proof is divided into several smaller steps. As mentioned earlier, we will prove each step in Section 4. Let us assume it by now and let us realize that the main result of the article, Theorem 1, is just a straight-forward consequence of Proposition 18.

In fact, everything is reduced to proving the following result, whose proof we provide by completeness since it was known to the experts. Actually we just follow [NP, Example: 1.b].

Proposition 19. *There exists an iso-contact open book embedding of* $(\mathbb{S}^{2n-1}, \xi_{stot})$ *in the standard contact sphere* $(\mathbb{S}^{2n+1}, \xi_{std})$.

Proof. Consider the function $f : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}$ given by $f(z_1, \cdots, z_n) \to \overline{z_1}^2 + \cdots + \overline{z_n}^2$. Define an embedding Φ of $\mathbb{S}^{2n-1}$ in $\mathbb{C}^{n+1}$ by $\Phi((z_1, \cdots, z_n)) = (z_1, \cdots, z_n, f(z_1, \cdots, z_n))$. It has been shown in [NP, Example: 1.b] that this embedding is contained in a star shaped standard contact sphere in $\mathbb{C}^{n+1}$. Moreover, there is an obvious open book structure given by the argument of the function $\overline{z_1}^2 + \cdots + \overline{z_n}^2$. □

Let us now discuss how the two previous statements readily imply Theorem 1.

Proof of Theorem 1. In a nutshell, we just start by fixing the iso-contact open book embedding that we have obtained in Proposition 19, and the iso-contact embedding of (M, ξ_M^{ot}) given in the hypothesis of the theorem. We now apply Proposition 17 to this pair of embeddings. This creates an iso-contact embedding of the contact manifold (M, ξ_M^{stot}) in the same target manifold because of [BEM] as ξ_M^{ot} and ξ_M are homotopic as almost-contact structures. But now, we are in the hypothesis of Proposition 18, so we conclude that the contact manifold (M, ξ_M) admits an iso-contact embedding as claimed. □

Let us now discuss how Proposition 2 follows from Theorem 1.

Proof of Proposition 2. Since the Euler class of the normal bundle of any embedded closed orientable manifold M in $\mathbb{S}^k$ has to be zero, we get that the manifold M^{2n-1} admits an embedding in $\mathbb{S}^{2n+1}$ with trivial normal bundle.

Next, assume that M admits an embedding in $(\mathbb{S}^{2n+1}, \xi_{std})$ such that the induced contact structure ξ is homotopic to ξ_M. Proposition 19 implies that there exits an iso-contact embedding of $(\mathbb{S}^{2n-1}, \xi_{stot})$ in $(\mathbb{S}^{2n+1}, \xi_{std})$. Hence, it follows from Proposition 17 that there exists an iso-contact embedding of $(M, \xi \# \xi_{stot})$ in $(\mathbb{S}^{2n+1}, \xi_{std})$. Now, since the overtwisted contact structures $\xi \# \xi_{stot}$ and $\xi_M \# \xi_{stot}$ are homotopic as almost contact structures, by the uniqueness of an overtwisted contact structure in a given homotopy class of almost contact structures, we get that there is an iso-contact embedding of $(M, \xi_M \# \xi_{stot}) = (M, \xi_M^{stot})$ in the standard contact sphere.

Proposition 18 now implies that there is an iso-contact embedding of (M, ξ_M) in the standard contact sphere as claimed. This completes our argument. □

The next section is devoted to the proof of Proposition 18.

4. Proof of Proposition 18

We employ this whole Section proving Proposition 18. The proof has three steps.

The first step is just to recall a standard fact. It is just writing the normal model for contact structures on neighborhoods of contact sub-manifolds (see for instance [Ge, Theorem:2.5.15]). It can be stated as follows:

Lemma 20. *[Ge, Theorem:2.5.15] Let (N, ξ_N) be a contact manifold. Let (M, ξ_M) be a contact submanifold of (N, ξ_N) with trivial normal bundle. If $Ker\{\alpha\}$ is contactomorphic to ξ_M on M, then there exists an ε_0–positive such that there is an iso-contact embedding of an ε–partial open book associated to $(M, Ker\{\alpha\})$ in (N, ξ_N) for every ε smaller than ε_0.*

The second step establishes the following:

Lemma 21. *Let $(M, Ker\{\alpha\}) = (\mathcal{A}ob(\Sigma, \phi), Ker\{\alpha_{(\Sigma,\phi)}\})$. Let $\varepsilon_0 > 0$ be given. There exists a contact abstract open book embedding F of $(M, Ker\{\alpha\})$ in ε_0–partial open book $M \times \mathbb{D}^2_{\varepsilon_0}$.*

Proof. Given $(M, Ker\{\alpha\})$ as in the hypothesis, we know that α is supported by the open book decomposition with the page Σ and the monodromy ϕ. Let B be the binding of this open book and let $\pi : M \setminus B \to \mathbb{S}^1$, where $\mathbb{S}^1$ is a circle of radius ϵ in $\mathbb{C}$ for some $0 < \epsilon < \epsilon_0$, be the fibration inducing the open book decomposition. Let $\psi : M \to \mathbb{D}^2_\epsilon$ be a Bourgeios function [Bo, page: 1573] such that $\pi = \frac{\psi}{|\psi|}$ for $\psi \neq 0$.

Observe that the graphical embedding of M given by $x \to (x, \psi(x))$ is the required contact open book embedding of M for a small ϵ as claimed. This is because the embedding is clearly an open book embedding and any page embeds symplectically in the symplectization of M corresponding to the symplectic page $(M \times [0, \epsilon), d(e^{-r}\alpha))$ of the ϵ_0–partial open book. □

We would like to point out that the way in which we state and prove this result was suggested to us by Patrick Massot. This makes the original argument more transparent.

The third step shows:

Lemma 22. *For every $\varepsilon > 0$, there exists a contact open book embedding of the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_{std})$ in the ε–partial open book associated to $(\mathbb{S}^{2n-1}, Ker\{\alpha_{stot}\})$, where the standard contact sphere is regarded as an abstract open book with pages the standard symplectic $(2n-2)$–disk and monodromy the identity.*

Proof. First of all observe that in the standard contact manifold $\mathbb{R}^{2n-1} \times \mathbb{D}^2_\epsilon$ with the contact structure $e^{-r}(dz + x_1dy_1 + \cdots + x_{n-1}dy_{n-1}) + r^2d\theta$, there exist a contact open book embedding of $\mathbb{S}^{2n-1}$, where the contact open book of $\mathbb{R}^{2n-1} \times \mathbb{D}^2_\epsilon$ is the ϵ–partial open book of $\mathbb{R}^{2n-1} \times \mathbb{D}^2_\epsilon$ with the binding $\mathbb{R}^{2n-1} \times \{0\}$ and the pages θ equal to constant. By choosing a Darboux ball in $(\mathbb{S}^{2n-1}, \alpha_{stot})$, we get that there exists an open book embedding of $(\mathbb{S}^{2n-1}, Ker\{\alpha_{std}\})$ in the ϵ–partial open book associated to $(\mathbb{S}^{2n-1}, Ker\{\alpha_{stot}\})$ as claimed. □

Now that we have proved all three steps needed for the proof of the Proposition 18 in the form of Lemmas 20, 21, and 22, we proceed to complete the proof.

Proof of Proposition 18. We can assume that (M, ξ) is an abstract open book $(\mathcal{A}ob(\Sigma, \phi), Ker\{\alpha_{(\Sigma,\phi)}\})$ by Giroux theorem [Gi] on the existence of adapted open books (for a detailed proof see [Pr2]). We first notice that Lemma 20 implies that given an $\varepsilon > 0$, it is sufficient to iso-contact open book embed (M, ξ) in the ε–partial open book associated to the contact manifold $(M\#\mathbb{S}^{2n-1}, \xi\#\xi_{stot}) = (M, \xi^{stot})$.

Let $\Sigma_\theta = \pi_1^{-1}(\theta)$, where $\pi_1 : \mathcal{MT}(\Sigma, \phi) \to \mathbb{S}^1$ is the fibration associated to $\mathcal{A}ob(\Sigma, \phi)$.

By Lemma 21, there exists a contact abstract open book embedding of the contact manifold $(M, Ker\{\alpha\})$ in the ε–partial open book associated to $(M, Ker\{\alpha\})$. Next, by Lemma 22, there exists an iso-contact abstract open book embedding of the standard contact $(2n-1)$–sphere having open book with pages $\mathbb{D}^{2n-2}$ and monodromy that identity in the ε–partial open book associated to the standard overtwisted sphere $(\mathbb{S}^{2n-1}, Ker\{\alpha_{stot}\})$.

It now follows from Proposition 17 that there exists an iso-contact abstract open book embedding of $(\mathcal{A}ob(\Sigma\#_\partial\mathbb{D}^{2n-2}, \phi\#id), Ker\{\alpha\#\alpha_{std}\})$ in the ε–partial open book associated to $(M\#\mathbb{S}^{2n-1}, \xi\#\xi_{stot})$.

Since the contact abstract open book

$$\mathcal{A}ob(\Sigma\#_\partial\mathbb{D}^{2n-2}, Ker\{\alpha_{(\Sigma\#_\partial\mathbb{D}^{2n-2},\phi\#id)}\})$$

is contactomorphic to (M, ξ), and since $(M\#\mathbb{S}^{2n-1}, \xi\#\xi_{stot})$ – by definition – is (M, ξ^{stot}), the Proposition follows. □

In the next couple of sections, we will give applications of the Theorem 1.

5. Contact embedding of 3-manifolds in the standard contact $\mathbb{S}^5$

The purpose of this section is to show that every contact 3–manifold (M, ξ) contact embeds in $(\mathbb{S}^5, \xi_{std})$, provided the first Chern class of ξ is zero and M has no 2-torsion in $H^2(M, \mathbb{Z})$. The result essentially follows from [EF, Theorem:1.20] and Proposition 18. However, for the sake of completeness, we provide a slightly more detailed argument. We begin this section by reviewing a few facts about the homotopy classes of plane fields on an orientable 3–manifold.

5.1. Homotopy classes of oriented plane fields on orientable 3-manifolds

Let ξ be an oriented 2–plane field on a closed oriented 3–manifold M. Recall that any two such plane fields are homotopic over the 1–skeleton of a triangulation of M. R. Gompf in [Go] established that when M has no two torsion in $H^2(M, \mathbb{Z})$, the first Chern class $c_1(\xi)$ completely determines homotopy of plane fields over the 2–skeleton. See [Go, Theorem:4.5].

It also follows from [Go, Theorem:4.5] that if $c_1(\xi) = 0$, then homotopy over the 3–skeleton is completely determined by the 3–dimensional invariant $d_3(\xi)$, which is defined as follows:

It was shown in [Go] that it is possible to choose an almost complex manifold (X, J) with $\partial X = M$ and whose complex tangencies are (M, ξ). More precisely, $\xi = TM \cap J(TM)$. Given this one defines $d_3(\xi)$ as:

$$d_3(\xi) = C_1^2(X, J) - 3\sigma(X) - 2(\chi(X) - 1).$$

We would like to point out that this formula is slightly different from the one given in [Go], as we are subtracting 1 from the Euler characteristic of X in the formula. This is just to ensure that the formula for d_3 is additive when one considers the connected sums. More precisely,

Let (M_1, ξ_1) be a contact manifold with $c_1(\xi_1) = 0$ and (M_2, ξ_2) be another contact manifold with $c_1(\xi_2) = 0$, then for the contact connected sum $(M_1 \# M_2, \xi_1 \# \xi_2)$, we have

$$d_3(\xi_1 \# \xi_2) = d_3(\xi_1) + d_3(\xi_2).$$

To begin with, we need the following result of Etnyre and Fukuwara from [EF]. For the sake of completeness, we will provide a short sketch of the proof of this result here.

Theorem 23 (Etnyre and Fukuwara). *Let M be a closed 3–manifold. If M is orientable, then there exists an embedding of M in $\mathbb{S}^5$ such that the contact structure ξ_{std} on $\mathbb{S}^5$ induces a contact structure on M.*

Proof. To begin with, we observe that if there exists an embedding $F : M \to \mathbb{S}^3 \times D^2$ given by $F(x) = (f_1(x), f_2(x))$ which satisfies the following properties:

1) the map $f_1 : M \to \mathbb{S}^3$ is a branch covering,
2) the branch locus L in $\mathbb{S}^3$ for the branch cover $f_1 : M \to \mathbb{S}^3$ is transversal to the standard contact structure on $\mathbb{S}^3$,

then there exists an $\varepsilon_0 > 0$ such that for all ε less than ε_0, the embedding $F_\varepsilon : M^3 \to \mathbb{S}^3 \times D^2$ given by $F_\varepsilon(x) = (f_1(x), \varepsilon f_2(x))$ is a contact embedding of M in $(\mathbb{S}^3 \times D^2, Ker\{\alpha_{std} + r^2 d\theta\})$. See [EF] for the computation establishing that the pulled back form $F_\varepsilon^*(\alpha + r^2 d\theta)$, in fact, induces a contact structure on M.

Now, Remark 3 on the page 375 of [HLM] and the fact that transversality is a generic property implies that there exists an embedding of M in $\mathbb{S}^3 \times D^2$ satisfying the two properties mentioned above. This clearly implies that in an arbitrarily small neighborhood of contact $(\mathbb{S}^3, \xi_{std})$ inside $(\mathbb{S}^5, \xi_{std})$ admits an embedding of M such that it is a contact embedding. □

We are now in a position to prove Theorem 3. Recall that Theorem 3 states that a necessary and sufficient condition for an iso-contact embedding of a contact 3–manifold (M, ξ) in $(\mathbb{S}^5, \xi_{std})$ is that $c_1(\xi) = 0$ provided M has no 2–torsion in $H^2(M, \mathbb{Z})$. In case, M has a 2-torsion in $H^2(M, \mathbb{Z})$, the statement claims that there is a homotopy class $[\xi]$ of the plane fields such that M with every contact structure homotopic to a plane field in the class $[\xi]$ over the 2–skeleton of M admits an iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$.

Proof of Theorem 3. We know from [Ka] that $c_1(\xi) = 0$ is a necessary condition for having an iso-contact embedding of any contact (M, ξ) in $(\mathbb{S}^5, \xi_{std})$. We know from Theorem 23 that there exist a contact structure η on every 3–manifold M with $c_1(\eta) = 0$ such that (M, η) admits an iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$.

In case, M has no 2–torsion in $H^2(M, \mathbb{Z})$, it follows from [Go, Theorem:4.5] that every overtwisted contact structure η_2^{ot} on M which is homotopic to η over a 2–skeleton of M can be obtained by making a contact connected sum of M with a suitably chosen overtwisted $\mathbb{S}^3$. We already know from [EF, Theorem:1:20] that every contact $\mathbb{S}^3$ embeds in $(\mathbb{S}^5, \xi_{std})$.

Hence, we conclude that if (M, η) iso-contact embeds in $(\mathbb{S}^5, \xi_{std})$, then so does (M, η_2^{ot}) provided η^{ot} is an overtwisted contact structure on M which is homotopic to η over the 2–skeleton of M. But, this implies that every (M, η^{ot}) iso-contact embeds in $(\mathbb{S}^5, \xi_{std})$, provided the first Chern class of η^{ot} is zero.

The case of no 2–torsion in $H^2(M, \mathbb{Z})$ is now a straightforward consequence of Corollary 1.

In case, M has a 2–torsion in $H^2(M, \mathbb{Z})$ – by an argument similar to the one discussed above – it is clear that every overtwisted contact structure ξ^{ot} on M such that ξ^{ot} is homotopic to η as an almost contact plane field over a 2–skeleton on M admits iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$. Again, applying Theorem 1, we conclude that every contact structure homotopic as a plane field over 2–skeleton to η admits an iso-contact embedding in $(\mathbb{S}^5, \xi_{std})$. This completes our argument. □

6. Embeddings of simply connected 5–manifolds in $(\mathbb{S}^7, \xi_{std})$

We begin this section by observing the following:

Proposition 24. *Let ξ be a contact structure on $\mathbb{S}^{2n-1}$. If ξ is co-orientable and homotopic as an almost-contact structure to the standard contact structure on $\mathbb{S}^{2n-1}$, then $(\mathbb{S}^{2n-1}, \xi)$ admits an iso-contact embedding in $(\mathbb{S}^{2n+1}, \xi_{std})$. In particular, every contact $(\mathbb{S}^5, \xi)$ iso-contact embeds in $(\mathbb{S}^7, \xi_{std})$.*

Proof. The first part of the proposition is an immediate consequence of Proposition 2. In order to prove the second part, recall that there exists a unique almost-contact class on $\mathbb{S}^5$. This was established in [Ge1]. See also [Ha]. But this implies ξ_{stot} is homotopic as an almost-contact plane field to ξ. Hence the proposition follows. □

Next, we show that any contact structure on $\mathbb{S}^2 \times \mathbb{S}^3$ with trivial first Chern class iso-contact embeds in $(\mathbb{S}^7, \xi_{std})$. More precisely, we establish:

Lemma 25. *Let ξ be a co-orientable contact structure on $\mathbb{S}^2 \times \mathbb{S}^3$. The contact manifold $(\mathbb{S}^2 \times \mathbb{S}^3, \xi)$ iso-contact embeds in $(\mathbb{S}^7, \xi_{std})$ if and only if the first Chern class $c_1(\xi)$ of the contact structure is zero.*

Proof. Recall that in [Ge1, Ha] it is established that two almost-contact plane fields ξ_1 and ξ_2 are homotopic as almost-contact structures if and only if their first Chern classes coincide.

Next, Kasuya in [Ka] showed that a necessary condition for a contact manifold (M^{2n+1}, ξ) to admit an iso-contact embedding in $(\mathbb{S}^{2n+3}, \xi_{std})$ is that $c_1(\xi) = 0$.

Hence, from Corollary 2, we can see that if there exist a contact embedding of $\mathbb{S}^2 \times \mathbb{S}^3$ in $(\mathbb{S}^7, \xi_{std})$, then the lemma follows. So, we now show that there is a contact embedding of $\mathbb{S}^2 \times \mathbb{S}^3$ in $(\mathbb{S}^7, \xi_{std})$.

Notice that the contact abstract open book $\mathcal{A}ob(\mathcal{D}T^*\mathbb{S}^2, id)$ is contact manifold diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$. Clearly, $\mathcal{A}ob(\mathcal{D}T^*\mathbb{S}^2, id)$ iso-contact open book embeds in the contact abstract open book $\mathcal{A}ob(\mathbb{D}^6, id)$ as the disk cotangent bundle $\mathcal{D}T^*\mathbb{S}^2$ can be found as Liouville hypersurface in the standard contact $\mathbb{S}^5$. Since contact abstract open book $\mathcal{A}ob(D^6, id)$ is contactomorphic to $(\mathbb{S}^7, \xi_{std})$, the lemma follows. □

It was established by H. Geiges in [Ge, Chapter–8] that a necessary condition to produce a contact structure on any 5–manifold is that the third integral Steifel-Whitney class W_3 is zero. D. Barden in [Ba] had given a complete classification of simply connected 5–manifolds. Using this classification, it is easy to list all the simply connected prime 5-manifolds with vanishing W_3. We now proceed to describe this list. First of all, recall that for each $2 \leq k < \infty$, there exists a unique prime simply connected manifold M_k characterized by the property that $H_2(M_k, \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$. Next, recall that there exists a unique non-trivial orientable real rank 4 vector-bundle over $\mathbb{S}^2$. By $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^3$, we denote the unit sphere bundle associated to this vector bundle.

We are now in a position to state Barden's theorem that we will need to establish the Theorem 4.

Theorem 26 (Barden). *Every closed simply connected almost contact 5–manifold can be uniquely decomposed into a connected sum of prime manifolds $M_k, 2 \leq k < \infty$, $\mathbb{S}^2 \times \mathbb{S}^3$ and $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^3$. Furthermore, the decomposition has no copy of $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^3$ provided the second Steifel-Whitney class is zero.*

Proof of Theorem 4. Notice that it is sufficient to establish that any (M, ξ) satisfying the hypothesis with $c_1(\xi) = 0$ admits an iso-contact embedding in $(\mathbb{S}^7, \xi_{std})$.

Let M be a closed simply connected 5–manifold with $w_2(M) = 0$ and let ξ be a contact structure on it with $c_1(\xi) = 0$. In order to establish Theorem 4, we first show that M admits a contact embedding in $(\mathbb{S}^7, \xi_{std})$ such that the induced contact structure has its first Chern class 0.

Notice that if M is as in the hypothesis, then it follows from Theorem 26 of Barden stated above that in its connected sum decomposition, there is no $\mathbb{S}^2\widetilde{\times}\mathbb{S}^3$ factor. See also, [Ge, Theorem:8.2.9] for a proof of this. Next, we have already observed that if $M = N_1\#N_2\#\cdots\#N_l$ and each N_i contact embeds in $(\mathbb{S}^7, \xi_{std})$, then there exist a contact embedding of M in $(\mathbb{S}^7, \xi_{std})$.

We have shown in Lemma 25 that $\mathbb{S}^2 \times \mathbb{S}^3$ contact embeds in $(\mathbb{S}^7, \xi_{std})$. Hence, in order to show that M contact embeds in $(\mathbb{S}^7, \xi_{std})$, we just need to show that each prime manifold M_k described in Theorem 26 above must contact embed in $(\mathbb{S}^7, \xi_{std})$. It is well known that each M_k is a Brieskorn 5–sphere. Hence, they admit contact embedding in $(\mathbb{S}^7, \xi_{std})$. See, for example, [Ko, Remark 4.2].

Thus, we have shown that every simply connected 5–manifold satisfying the hypothesis admits a contact embedding in $(\mathbb{S}^7, \xi_{std})$. Next, recall that if a 5–manifold admits a formal contact embedding in $(\mathbb{S}^7, \xi_{std})$, then it was shown in [Ka] that the first Chern class of the induced contact structure has to be trivial.

Finally, observe that it was established in [Ge, Ge1, Chpt–8] (also see [Ha, chpt-VII] for a precise formulation) that any two contact structures on a closed simply connected 5–manifold having their first Chern classes trivial are homotopic as almost-contact structures. It now follows from the Proposition 2 that (M, ξ) admits an iso-contact embedding in $(\mathbb{S}^7, \xi_{std})$. □

References

[Al] J. W. Alexander, A lemma on systems of knotted curves, *Proc. Nat. Acad. Sci. USA*, **9** (1923), 93–95.

[Ar] V. I. Arnold, Some remarks on symplectic monodromy of Milnor fibrations *The Floer Memorial Volume*, Progress in Mathematics book series (PM, volume 133), pp. 99–103.

[Ba] D. Barden, Simply connected five-manifolds, *Ann. of Math* **82** (1965), no. 2, 365–385.

[Bo] F. Bourgeios, Odd dimensional tori are contact manifolds, *Int. Math. Res. Not.* **2002** (2002), no. 3,, 1571–1574.

[BEM] M. Borman, Y. Elaishberg and E. Murphy, Existence and classification of overtwisted contact structures in all dimensions, *Acta Mathematica* **215** (2015), no. 2, 281–361.

[CE] R. Casals and J. Etnyre, Non-simplicity of isocontact embeddings in all higher dimensions. Geometric and Functional Analysis **30** (2020), 1–33.

[CMP] R. Casals, E. Murphy and F. Presas, Geometric criteria for overtwistedness, *J. Amer. Math. Soc.* **32** (2019), 563–604.

[CPP] R. Casals, D. M. Pancholi and F. Presas, The Legendrian Whitney trick, `arXiv:1908.04828v1 [math.SG]` (To appear Geometry and Topology).

[Et] J. Etnyre, Lectures on open book decompositions and contact structures, *Floer Homology, Gauge Theory, and Low-Dimensional Topology* **5**, 103–141.

[EM] Y. Eliashberg and N. Mishachev, Introduction to the h–principle, *Graduate Studies in Mathematics, AMS,* Vol. 48, (2002).

[EF] J. Etnyre and R. Fukuwara, Braided embeddings of contact 3–manifolds in standard contact 5–spheres, *Journal of Topology* **10** (2017), no. 2, 412–446.

[EL] J. Etnyre, J and Y. Lekili, Embedding all contact 3–manifolds in a fixed contact 5–manifold, *Journal of The LMS* **99** (2019), no. 1, 52–68.

[Ga] D. Gabai, The Murasugi sum is a natural geometric operation, *Low-dimensional topology (San Francisco, Calif., 1981)*, 131 –143, (1983).

[Ge] H. Geiges, An Introduction to Contact Topology, *Cambridge Studies in Advanced Mathematics* **109**.

[Ge1] H. Geiges, Contact structures on 1–connected 5–manifolds, *Mathematika* **38** (1991), no. 2, 303–311.

[Gi] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, *Proceedings of the ICM*, Beijing 2002, Vol. 2, 405–414.

[Gi1] E. Giroux, Structures de contact en dimension trois et bifurcations des feuilletages de surfaces. *Invent. Math.* **141** (2000), no. 3, 615–689.

[Go] R. Gomph, Handle body constructions on Stein surfaces *Annals of Math* **148** (1998), no. 2 619–693.

[Gr] M. Gromov, Partial Differential Relations, *Ergeb. Math. Grenzgeb.* **9** (1986), Springer–Verlag.

[GM] E. Giroux and J-P. Mohsen, Contact structures and symplectic fibrations over the circle, Lecture Notes.

[Ha] M. Hamilton, On symplectic 4–manifolds and contact 5–manifolds, Ph.D Thesis.

[Hi] M. Hirsch, On imbedding differentiable manifolds in euclidean space. *Ann. of Math.* **73** (1961), no. 2, 566–571.

[Ho] K. Honda, On the classification of tight contact structures. I. *Geom. Topol.* **4** (2000), 309–368 (electronic).

[HLM] H. M. Hilden, M. T. Lozano and J. M. Montesinos, All three-manifolds are pullbacks of a branched covering S^3 to S^3, *Trans. Amer. Math. Soc.* **279** (1983), no. 2, 729–735.

[Ka] N. Kasuya, An obstruction for codimension two contact embeddings in the odd dimensional Euclidean spaces, *J. Math. Soc. Japan* **68** (2016), no. 2, 737–743.

[Ko] O. van Koert, Open books on contact five-manifolds, *Annales de l'Institut Fourier* **58** (2008), 139–157.

[Ko1] O. van Koert, Lectures on stablization of open books, *Münster J. Math.* **10** (2017), no. 2, 425–455.

[KN] O. van Koert and K. Niederkrüger, Open book decompositions for contact structures on Brieskorn manifolds, *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3679–3686.

[KN1] O. van Koert and K. Niederkrüger, Every contact manifolds can be given a nonfillable contact structure, *Int. Math. Res. Not. IMRN* **23** (2007), Art. ID rnm115.

[La] O. Lazarev, Maximal contact and symplectic structure, *Journal of Topology* **13** (2020), no. 3.

[Ma] D. Martinez-Torres, Contact embeddings in standard contact spheres via approximately holomorphic geometry, *J. Math. Sci. Univ. Tokyo* **18** (2011), no. 2, 139–154.

[Mr] A. Mori, Global models of contact forms, *J. Math. Sci. Univ. Tokyo* **11** (2004), no. 4, 447–454.

[Mr1] The Reeb foliation arises as a family of Legendrian submanifolds at the end of a deformation of the standard S^3 in S^5. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 1-2, 67–70.

[MS] D. McDuff, D. Salamon, Introduction to Symplectic Topology, *Oxford Math. Monogr.* (1998), Oxford.

[Na2] J. Nash, C^1–isometric imbeddings, *Annals of Mathematics* **60** (1954), no. 2, 383–396.

[Na] J. Nash, The imbedding problem for Riemannian manifolds, *Annals of Mathematics* **63** (1956), no. 1, 20–63.

[NP] K. Niederkruger and F. Presas, Some remarks on the size of tubular neighborhoods in contact topology and fillability, *Geom. Topol* **14** (2010), no. 2, 719–754.

[Pr] F. Presas, Lefschetz type pencils on contact manifolds, *Asian J. Math.* **6** (2002), 277–302.

[Pr2] F. Presas, Geometric decompositions of almost contact manifolds, Contact and Symplectic Topology, pp. 137–172, Bolyai Soc. Math. Stud. **26**, János Bolyai Math. Soc., Budapest, (2014).

[PPS] D. M. Pancholi, S. Pandit and K. Saha, Open books and embeddings of manifolds, `arXiv:1806.09784v2 [math.GT]`.

[Qu] F. Quinn, Open book decompositions, and the bordism of automorphisms, *Topology*, **18** (1979), no. 1, 55–73.

[Se1] P. Seidel, Floer homology and the symplectic isotopy problem, Ph.D. thesis, Oxford.

[Se2] P. Seidel, Lectures on four-dimensional Dehn twists, *Symplectic 4-Manifolds and Algebraic Surfaces*, pp. 231–267, Lecture Notes in Math. **1938**, Springer, Berlin, (2008).

[TW] W. Thurston, H. Winkelnkemper, On the existence of contact forms, *Proceedings of AMS* Vol. 52, (October 1975), 345–347.

[Wh] H. Whitney, The self-intersections of a smooth n–manifold in $2n$–space, *Ann. of Math.* **45** (1944), no. 2, 220–246.

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