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Splitting formulas for the local real Gromov-Witten invariants

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Motivated by the real version of the Gopakumar-Vafa conjecture for 3-folds, the authors introduced in [GI] the notion of local real Gromov-Witten invariants associated to local 3-folds over Real curves. This article is devoted to the proof of a splitting formula for these invariants under target degenerations. It is used in [GI] to show that the invariants give rise to a 2-dimensional Klein TQFT and to prove the local version of the real Gopakumar-Vafa conjecture.

1. Introduction

A central problem in Gromov-Witten theory is understanding the structure and properties of the Gromov-Witten invariants. Motivated by the real version of the Gopakumar-Vafa conjecture and the work of Bryan and Pandharipande [BP2], the authors introduced and studied in [GI] the notion of local real Gromov-Witten invariants. In this article we prove the splitting formula for these invariants, as outlined in [GI, §4] and used to establish the structural results of [GI].

A symmetric (or Real[1]) Riemann surface is a (possibly disconnected and marked) Riemann surface Σ together with an anti-holomorphic involution $c : \Sigma \to \Sigma$ (also referred to as a real structure). Throughout this paper we restrict attention to the case in which none of the marked points are real, and we denote by V the collection of marked points of Σ. Consider the real relative moduli space

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$$

of degree d real maps $f : C \to \Sigma$ from (possibly disconnected) domains of Euler characteristic χ and having ramification profile $\vec{\mu}$ over the divisor V, reviewed in §3.3. Here $\vec{\mu}$ is a collection of partitions of d, one for each pair of conjugate points in V. These moduli spaces are orientable, but not *a priori* canonically oriented; their orientation depends on a choice of (twisted) orientation data $\mathfrak{o}$ on Σ, cf. §8.

When $L \to \Sigma$ is a holomorphic line bundle, the local RGW invariants are defined as the pairing

$$RGW^{c,\mathfrak{o}}_{d,\chi}(\Sigma, L)_{\vec{\mu}} = \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \int_{[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma,V)]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(-\mathrm{ind}\,\overline{\partial}_L) \tag{1.1}$$

between the appropriate Chern class of the index bundle $\mathrm{Ind}\,\overline{\partial}_L$ (regarded as an element in the usual complex K-theory) and the virtual fundamental class of the real relative moduli space. Here b is the (virtual) dimension[2] of the moduli space and the presence of the factor $|\mathrm{Aut}(\vec{\mu})|$ is explained in Remark 3.2. In particular, (1.1) are the coefficients of the generating series [GI, (2.24)].

[1]We use Real with capital R for spaces or bundles with anti-holomorphic involutions, following Atiyah.

[2]Unless otherwise noted, all dimensions appearing in this paper are real dimensions rather than complex dimensions.

The pairing (1.1) is invariant under smooth deformations of the target Σ (as a symmetric marked curve) together with L and $\mathfrak{o}$; in particular it is independent of the holomorphic structure on L. By considering a family $\cup_s \Sigma_s$ of smooth symmetric marked curves degenerating to a symmetric nodal curve Σ_0 (with a conjugate pair of nodes), the splitting formula proved in this paper relates the local RGW invariants of a smooth fiber Σ_s to those of the normalization of the singular fiber Σ_0.

Throughout this paper, Σ_0 denotes a symmetric nodal curve with a single pair of conjugate nodes and r pairs of conjugate marked points; in particular, we assume that Σ_0 has no real special points. We always denote by

$$\mathcal{F}_{/\Delta} = \bigcup_{s\in\Delta} \Sigma_s \tag{1.2}$$

a (flat) family of deformations of Σ_0 parametrized by the unit disk Δ as described in §3.1. The fibers over $s \neq 0$ are smooth (marked, symmetric) curves Σ_s. We also denote by

$$\widetilde{\Sigma} \to \Sigma_0$$

the normalization (or resolution) of the singular fiber Σ_0 (as a symmetric marked curve). We denote by c_s, $\widetilde{c}$ and V_s, $\widetilde{V}$ the corresponding real structures and markings.

If $L \to \mathcal{F}$ is a complex line bundle over the family, we denote by L_s and $\widetilde{L}$ its pullback to Σ_s and respectively $\widetilde{\Sigma}$. There is also a notion of twisted orientation data $\mathfrak{o}_{\mathcal{F}}$ on the family $\mathcal{F}$, which pulls-back to give twisted orientation data $\mathfrak{o}_s$ on Σ_s and $\widetilde{\mathfrak{o}}$ on $\widetilde{\Sigma}$, cf. §8.1.

With these preliminaries, the main result of this paper is the proof of the following theorem.

Theorem 1.1 (RGW Splitting Theorem). *Let $\mathcal{F} = \cup_s \Sigma_s$ be a family of deformations of Σ_0 and $\widetilde{\Sigma}$ be its normalization as described above. Fix $L \to \mathcal{F}$ a complex line bundle and $\mathfrak{o}_{\mathcal{F}}$ a twisted orientation on $\mathcal{F}$. Then the local RGW invariants* (1.1) *associated to the smoothing Σ_s are related to those associated to the normalization $\widetilde{\Sigma}$ as follows:*

$$RGW_{d,\chi}^{c_s,\mathfrak{o}_s}(\Sigma_s, L_s)_{\vec{\mu}} = \sum_{\lambda\vdash d} \zeta(\lambda) RGW_{d,\chi+4\ell(\lambda)}^{\widetilde{c},\widetilde{\mathfrak{o}}}(\widetilde{\Sigma}, \widetilde{L})_{\vec{\mu},\lambda,\lambda} \tag{1.3}$$

for all $s \neq 0$ and all d, χ, and $\vec{\mu}$. Here λ is a partition of d, $\ell(\lambda)$ is its length and the coefficient $\zeta(\lambda)$ is given by (2.2).

Theorem 1.1 is an extension to the real setting of the splitting formula of Bryan-Pandharipande [BP2, Theorem 3.2] proved in [BP1, Appendix A]. A direct consequence of Theorem 1.1 is [GI, Theorem 4.1] cf. Corollary 10.2 and the discussion after it. This is used in [GI] to show that the local RGW invariants give rise to a 2-dimensional Klein TQFT and to prove the local version of the real Gopakumar-Vafa conjecture.

The proof of Theorem 1.1 is a consequence of the splitting properties of the total Chern class of the index bundle established by Bryan and Pandharipande, combined with a splitting formula for the virtual fundamental class of the real relative moduli space. The proof of the latter – Theorem 1.2 below – occupies the majority of this paper. The basic idea is an adaptation to the real setting of the classical proof of the splitting theorem [LR, Li, IP2].

We consider a family of real moduli spaces

$$\overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}) \;=\; \underset{s\in\Delta}{\cup}\, \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s) \tag{1.4}$$

associated to the family $\mathcal{F}$ of targets, cf. (3.14). For every partition λ of d, we also consider the real relative moduli space

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})$$

associated to the normalization $\widetilde{\Sigma}$ of Σ_0; note that $\widetilde{\Sigma}$ has two additional pairs of conjugate marked points (corresponding to the pair of conjugate nodes of Σ_0) and we restrict to the case when the ramification profile is λ over all these additional points. There is a natural map

$$\Phi : \bigsqcup_{\lambda\vdash d} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V}) \to \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_0, V_0) \tag{1.5}$$

which attaches pairs of marked points of both the domain and target to produce nodes, cf. §3.5.

Theorem 1.2 (VFC Splitting Theorem). *With the notation above, for every d, χ, and $\vec{\mu}$ and all $s \in \Delta \setminus 0$ the equality*

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)]^{\mathrm{vir},\mathfrak{o}_s} = \sum_{\lambda\vdash d} \frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2} \Phi_* [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})]^{\mathrm{vir},\widetilde{\mathfrak{o}}} \tag{1.6}$$

holds in the rational Cech homology of the family (1.4) *of moduli spaces.*

We now outline the key steps involved in the proof of the splitting formula (1.6). In this paper we use Ruan-Tian perturbations adapted to our setting in combination with the thin compactification method of [IP5] as summarized in §2.3. This approach allows us to use standard arguments after adapting them to the real setting.

The notion of Ruan-Tian perturbations extends to the family $\mathcal{F} = \cup_s \Sigma_s$ of targets in a way that is compatible with the real structures and the divisors, cf. §4.1. In particular, every RT perturbation ν on the family is real and pulls back to a RT perturbation on Σ_s and $\widetilde{\Sigma}$, compatible with their divisors. Denote by $\mathcal{P}$ the space of such RT perturbations ν on the family $\mathcal{F}$.

As ν varies over the parameter space $\mathcal{P}$, we get various families of moduli spaces over $\mathcal{P}$, cf. §4.2. Denote by $\overline{m}(\Sigma_s)$ and $\overline{m}(\widetilde{\Sigma})$ the family (over $\mathcal{P}$) of *real relative* moduli spaces associated to the fiber Σ_s and respectively the normalization $\widetilde{\Sigma}$ of the nodal fiber Σ_0. Fix a segment $I = [0, s_0] \subset \Delta$, where $s_0 \neq 0$. As in the usual proof of the splitting formula, we consider a family of moduli spaces

$$\overline{m}(\mathcal{F}_{/I}) = \bigcup_{s \in I} \overline{m}(\Sigma_s).$$

The attaching map (1.5) extends to a proper map

$$\overline{m}(\widetilde{\Sigma}) \xrightarrow{\Phi} \overline{m}(\Sigma_0).$$

These are all families over the parameter space $\mathcal{P}$ of RT perturbations ν and we denote by $\overline{m}(-)_\nu$ the corresponding fiber of $\overline{m}(-)$.

In section 5.2, we first show that

$$\overline{m}(\Sigma_{s_0}), \quad \overline{m}(\widetilde{\Sigma}), \quad \text{and} \quad \overline{m}(\Sigma_0) \tag{1.7}$$

are thinly compactified families over the parameter space $\mathcal{P}$ (in the sense of [IP5]) and therefore carry a VFC, cf. Theorem 5.3. This involves proving that for generic ν all the strata of the moduli spaces $\overline{m}(-)_\nu$ are cut transversally. There is one subtlety: when the real locus of the target is nonempty, it also involves gluing across the codimension 1 strata to show that for generic ν, the corresponding subsets

$$\widetilde{m}(\Sigma_{s_0})_\nu, \quad \widetilde{m}(\widetilde{\Sigma})_\nu, \quad \text{and} \quad \widetilde{m}(\Sigma_0)_\nu,$$

(defined as the union of codimension at most 1 strata) are orientable topological manifolds (without boundary). As a consequence of transversality, generically the union of the codimension at least 2 strata of $\overline{m}(-)_\nu$ have homological codimension at least 2 and thus the thin compactification method applies to define the VFC.

To establish the relation (1.6) it suffices to compare the fundamental classes

$$[\widetilde{m}(\Sigma_{s_0})_\nu], \quad \Phi_*[\widetilde{m}(\widetilde{\Sigma})_\nu], \quad \text{and} \quad [\widetilde{m}(\Sigma_0)_\nu] \quad \text{in} \quad \check{\mathrm{H}}_*(\widetilde{m}(\mathcal{F}_{/I})_\nu;\mathbb{Q})$$

for generic $\nu \in \mathcal{P}$. These classes depend on the choice of orientation of the moduli spaces, which in turn is determined by a choice of twisted orientation data $\mathfrak{o}_{\mathcal{F}}$ on $\mathcal{F}$, cf. §8. There are two natural perspectives on the moduli space associated to the nodal target Σ_0, one coming from the normalization and the other from the deformation, cf. §6. Each perspective gives rise to a natural orientation (after fixing $\mathfrak{o}_{\mathcal{F}}$) and a key step is showing that these two orientations agree, proved in §8.3.

We prove in §6 that for generic $\nu \in \mathcal{P}$ the attaching map Φ restricts to a proper, finite degree map between the two oriented topological manifolds $\widetilde{m}(\widetilde{\Sigma})_\nu$ and $\widetilde{m}(\Sigma_0)_\nu$ and thus

$$\Phi_*[\widetilde{m}(\widetilde{\Sigma})_\nu] = \deg\Phi \cdot [\widetilde{m}(\Sigma_0)_\nu] \quad \text{in} \quad \check{\mathrm{H}}_*(\widetilde{m}(\Sigma_0)_\nu;\mathbb{Q}). \tag{1.8}$$

Lemma 6.1 expresses the degree of Φ in terms of specific combinatorial factors.

Theorem 1.2 would follow if $\widetilde{m}(\mathcal{F}_{/I})_\nu = \underset{s\in I}{\cup}\widetilde{m}(\Sigma_s)_\nu$ was generically a topological cobordism; however, as in the usual splitting formula, in general it is branched along $s=0$. After passing to a cover $\widehat{m}(\Sigma_0)$ of $\widetilde{m}(\Sigma_0)$, we construct in §7 an auxiliary space

$$\widehat{m}(\mathcal{F}_{/I}) = \widehat{m}(\Sigma_0) \cup \underset{s\neq 0}{\cup}\widetilde{m}(\Sigma_s).$$

This space comes with a proper continuous projection to $\widetilde{m}(\mathcal{F}_{/I})$ which restricts to a map $q_0 : \widehat{m}(\Sigma_0) \to \widetilde{m}(\Sigma_0)$. We prove in §7 that for generic $\nu \in \mathcal{P}$,

(i) q_0 restricts to a proper map between two oriented manifolds $\widehat{m}(\Sigma_0)_\nu$ and $\widetilde{m}(\Sigma_0)_\nu$. Thus

$$(q_0)_*[\widehat{m}(\Sigma_0)_\nu] = (\deg q_0)\cdot[\widetilde{m}(\Sigma_0)_\nu] \quad \text{in} \quad \check{\mathrm{H}}_*(\widetilde{m}(\Sigma_0)_\nu;\mathbb{Q});$$

(ii) $\widehat{\mathfrak{M}}(\mathcal{F}_{/I})_\nu$ is an oriented topological cobordism between $\widetilde{\mathfrak{M}}(\Sigma_{s_0})_\nu$ and $\widehat{\mathfrak{M}}(\Sigma_0)_\nu$. Thus

$$[\widehat{\mathfrak{M}}(\Sigma_0)_\nu] = [\widetilde{\mathfrak{M}}(\Sigma_{s_0})_\nu] \quad \text{in} \quad \check{\mathrm{H}}_*(\widehat{\mathfrak{M}}(\mathcal{F}_{/I})_\nu; \mathbb{Q}).$$

This allows us to relate $[\widetilde{\mathfrak{M}}(\Sigma_{s_0})_\nu]$ to $[\widetilde{\mathfrak{M}}(\Sigma_0)_\nu]$ up to specific combinatorial factors. Combined with (1.8), the precise formulas for the degrees of Φ and q_0, and the properties of thin compactifications, this implies (1.6), completing the proof of Theorem 1.2, cf. §9.

Outline of paper. In Section 2 we review notation and background; a summary of the thin compactification method is included in §2.3 and at the beginning of §5. In Section 3 we review the construction of the real relative moduli space and extend it to a family of targets. In Sections 4 and 5 we construct the relevant VFCs by turning on Ruan-Tian perturbations to obtain transversality strata-wise and applying the thin compactification method. The moduli space of maps to a nodal target is analyzed in Section 6, including several equivalent descriptions of the linearization and its orientation sheaf. The auxiliary space $\widehat{\mathfrak{M}}(\mathcal{F}_{/I})$ is constructed and analyzed in Section 7. The orientations of the various moduli spaces involved are discussed in Section 8. The main result of Section 8 is Proposition 8.3 which compares the two natural orientations on the moduli space associated to the nodal target. Theorems 1.2 and 1.1 are proved in Sections 9 and 10. The appendices provide more details on the various linearizations considered and the relations between them.

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2. Background and notation

2.1. Notations for partitions

Let $d \in \mathbb{Z}$. A partition λ of d, denoted $\lambda \vdash d$, is a finite sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ such that the sum of its parts, denoted $|\lambda|$, is equal to d. The number of its parts, called length of the partition, is denoted $\ell(\lambda)$. We can also write a partition in the form $\lambda = (1^{m_1} 2^{m_2} \dots)$ where m_k is the number of parts of λ equal to k. Then

$$d = |\lambda| = \sum_{i=1}^{\ell} \lambda_i = \sum_{k=1}^{\infty} k m_k \quad \text{and} \quad \ell(\lambda) = \ell = \sum_{k=1}^{\infty} m_k.$$

Let $\mathrm{Aut}(\lambda)$ be the automorphism group of λ; its order is

$$|\mathrm{Aut}(\lambda)| = \prod m_k!. \tag{2.1}$$

We also consider the following combinatorial factor

$$\zeta(\lambda) = \prod m_k! k^{m_k}. \tag{2.2}$$

2.2. The local RGW invariants

A symmetric (or Real) curve (C, σ) is a closed, oriented, possibly nodal, possibly disconnected, possibly marked complex curve C together with an anti-holomorphic involution σ, called the real structure. In this paper we only consider the case when all marked points of C come in conjugate pairs.

Let (X, ω) be a symplectic manifold and ϕ an anti-symplectic involution on X. A real map

$$f : (C, \sigma) \longrightarrow (X, \phi) \tag{2.3}$$

is a map $f : C \to X$ such that $f \circ \sigma = \phi \circ f$. In this paper we restrict to almost complex structures J on X which are real i.e. satisfy $\phi^* J = -J$ and are tamed by ω. Denote by

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\ell}(X) \tag{2.4}$$

the (absolute) real moduli space consisting of equivalence classes (up to reparametrization of the domain) of stable degree d J-holomorphic real maps

(2.3) from symmetric curves of Euler characteristic χ and ℓ pairs of conjugate marked points.

Throughout this paper we restrict ourselves to target manifolds X which are (families of) symmetric curves; in this case we denote the target curve by (Σ, c) and the family of such by $\mathcal{F}$, see §3.1.

Assume (Σ, c) is a (smooth) symmetric curve with r (ordered) pairs of conjugate marked points

$$V = \{(x_1^+, x_1^-), \dots, (x_r^+, x_r^-)\}, \quad \text{where} \quad x_i^- = c(x_i^+), \tag{2.5}$$

and no other marked points. For a collection $\vec{\mu} = (\mu^1, \dots, \mu^r)$ of r partitions of d, denote by

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \tag{2.6}$$

the real relative moduli space reviewed in §3.3. Note that when Σ has no marked points $V = \emptyset$ and the real relative moduli space (2.6) becomes an absolute real moduli space.

The virtual dimension of the moduli space (2.6) is equal to

$$b = d\chi(\Sigma) - \chi - 2\delta(\vec{\mu}), \quad \text{where} \quad \delta(\vec{\mu}) = \sum_{i=1}^{r} (d - \ell(\mu^i)), \tag{2.7}$$

cf. (A.46).

In general, the (absolute) real moduli spaces (2.4) are not always orientable, but there are some criteria that ensure orientability. In [GZ1, Definitions 1.2 and 5.1] a notion of real orientation was introduced. When the target X is odd complex dimensional, the existence of a real orientation on TX ensures that the real moduli spaces $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\ell}(X)$ are orientable, and a choice determines a canonical orientation of the moduli space cf. [GZ1, Theorem 1.3]. This was extended in [GI, Definitions 2.1 and A.1] to the notion of a twisted orientation, reviewed in Definition 8.1 below, and used to orient the real relative moduli spaces. While a real orientation in the sense of [GZ1, Definition 1.2] does not exist on a symmetric curve with even genus and fixed-point free involution, a twisted orientation exists on every symmetric marked curve (with no real special points), cf. §8.1.

For a smooth marked symmetric curve Σ as in (2.5), its relative tangent bundle $\mathcal{T}_\Sigma$ is defined by

$$\mathcal{T}_\Sigma = T\Sigma \otimes \mathcal{O}\big(-\sum_i x_i^+ - \sum_i x_i^-\big). \tag{2.8}$$

It is a holomorphic line bundle over Σ and has a canonical real structure $c_{\mathcal{T}}$ induced by c, see also (B.6).

As described after [GI, (A.13)] and reviewed in §8, the moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ is orientable, and can be canonically oriented by a choice of twisted orientation data $\mathfrak{o}$ on the relative tangent bundle $\mathcal{T}_\Sigma$. This gives rise to a virtual fundamental class denoted $[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)]^{\mathrm{vir},\mathfrak{o}}$, which depends on $\mathfrak{o}$. For any holomorphic line bundle $L \to \Sigma$, we can consider its index bundle $\mathrm{Ind}\, \overline{\partial}_L$ as an element in K-theory, see (10.1). Denote

$$RGW^{c,\mathfrak{o}}_{d,\chi}(\Sigma, L)_{\vec{\mu}} \stackrel{\mathrm{def}}{=} \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \int\limits_{[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma,V)]^{\mathrm{vir},\mathfrak{o}}} c_{b/2}(-\mathrm{Ind}\, \overline{\partial}_L). \tag{2.9}$$

Here b is the (virtual) dimension (2.7) of the moduli space and $c_k(E)$ denotes the k'th Chern class of E. The local RGW invariant defined by [GI, (2.24)] is then equal to

$$RGW^{c,\mathfrak{o}}_{d}(\Sigma, L)_{\vec{\mu}} = \sum_{\chi} RGW^{c,\mathfrak{o}}_{d,\chi}(\Sigma, L)_{\vec{\mu}}\, t^{-\chi/2}(u/t)^{b/2+dk} \tag{2.10}$$

cf. Remark 3.2. Here $k = c_1(L)[\Sigma]$ is the degree of the line bundle.

While there may be different ways of defining the VFC, in this paper we describe a specific construction of the VFC

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)]^{\mathrm{vir},\mathfrak{o}} \in \check{\mathrm{H}}_b(\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V); \mathbb{Q}) \tag{2.11}$$

as an element of rational Cech homology, using the thin compactification method as introduced in [IP5, §2] and briefly reviewed in §2.3. This has the advantage that it is very concrete and does not use sophisticated virtual techniques. In the case when the target is a Riemann surface, turning on certain geometric perturbations ν of the J-holomorphic map equation as introduced by Ruan-Tian [RT] suffices to obtain transversality strata-wise for the real relative moduli space (after passing to a cover of the Deligne-Mumford moduli space; see Remark 4.1). This ensures that the moduli space is generically a thin compactification, and thus carries a VFC for all Ruan-Tian perturbations, including for $\nu = 0$. We present the details of this construction in §4 and §5.

2.3. Thin compactifications

Here we briefly summarize the method of thin compactifications as introduced in [IP5] and how it applies to construct the virtual fundamental class (VFC) of the real relative moduli spaces considered in this paper; the details of this construction are presented in §5.

Throughout this paper, by a d-dimensional manifold we mean a Hausdorff space M locally modeled on $\mathbb{R}^d$. If M is also oriented (but not necessarily compact), it has a fundamental class in Steenrod homology and we denote by $[M] \in \check{\mathrm{H}}_d(M;\mathbb{Q})$ its image in rational Cech homology. There are some subtleties involved when working with these homology theories; in particular, to pushforward such classes one needs a *proper* continuous map (between locally compact Hausdorff spaces), cf. [IP5, §1]. However, rational Cech homology satisfies a continuity axiom, a relative homeomorphism axiom and is also exact; in particular, there is a natural long exact sequence

$$\ldots \to \check{\mathrm{H}}_*(A;\mathbb{Q}) \to \check{\mathrm{H}}_*(X;\mathbb{Q}) \xrightarrow{\rho} \check{\mathrm{H}}_*(X \setminus A;\mathbb{Q}) \to \ldots \tag{2.12}$$

associated to a closed pair (X, A), where ρ is the "restriction" to an open set, cf. [M] and [IP5, §1].

If M is an oriented d-dimensional manifold, a thin compactification of M (in the sense of [IP5, §2]) is a compact Hausdorff space $\overline{M}$ containing M such that $S = \overline{M} \setminus M$ is closed and (homologically) codimension 2, i.e. $\check{\mathrm{H}}_*(S;\mathbb{Q}) = 0$ for all $* > d-2$. Then $\rho : \check{\mathrm{H}}_*(\overline{M};\mathbb{Q}) \to \check{\mathrm{H}}_*(M;\mathbb{Q})$ is an isomorphism for $* = d$, cf. (2.12), thus the fundamental class $[M] \in \check{\mathrm{H}}_*(M;\mathbb{Q})$ uniquely lifts to a class on $\overline{M}$ denoted $[\overline{M}] \in \check{\mathrm{H}}_*(\overline{M};\mathbb{Q})$. Thinly compactified cobordisms are defined similarly, cf. [IP5, §2.3].

Roughly speaking, the thin compactifications method of [IP5] applies to a moduli problem whenever (i) for generic perturbation the moduli space is a thinly compactified manifold, i.e. is an oriented manifold away from (homologically) codimension 2 strata and (ii) the moduli space over a generic 1-parameter family of perturbations is a thinly compactified cobordism, i.e. is an oriented cobordism away from (homologically) codimension 2 strata. Since the fundamental class (in rational Cech homology) of a manifold uniquely extends to any thin compactification, the first condition defines the fundamental class of the moduli space for generic parameter. For non-generic parameter, the virtual fundamental class is obtained as the limit (in rational Cech homology) of nearby classes, which is well defined by condition (ii). For more details, see [IP5, Theorem 4.2].

The ideal situation in which the thin compactification method applies is when all the strata of the moduli space are cut transversally, and the boundary strata have (virtual) codimension at least 2, cf. [IP5, Lemmas 2.2 and 5.3]. More generally, it suffices to cover the boundary by images of codimension 2 manifolds, as in [IP6, Lemma 2.2].

Because the targets considered in this paper are holomorphic curves, we are in a very special situation where geometric Ruan-Tian perturbations work particularly well, and are sufficient to obtain transversality strata-wise on all strata of the real relative moduli spaces, with one exception. When the target is a sphere without any marked points, the (absolute) moduli space of maps from a genus 1 domain with no marked points has a few strata which are not cut transversally, cf. Remark 4.7. In all the other cases, the RT perturbations suffice to ensure that all the strata are cut transversally, as we outline below. For this reason, we work under the mild technical assumption that the target Σ has a marked point on each spherical component, which suffices for our considerations, cf. Remark 9.2; for precise details, see Lemma 4.5.

Classically, it is known that one can use Ruan-Tian perturbations ν to get transversality on all stable components of the domain (after first passing to a cover of the Deligne-Mumford moduli space as in Remark 4.1). However, Ruan-Tian perturbations identically vanish on unstable domain components, so these components must be handled by other methods. When the domain is a genus g curve with at least $3-2g$ marked points, these are the spherical components collapsed to points under the map to the Deligne-Mumford moduli space. Otherwise, the entire domain is unstable (when it has genus 1 and no marked points or genus 0 with fewer than 3 marked points).

However, when the target is aspherical (e.g. a smooth higher genus curve) unstable spherical domain components cannot occur for maps in the moduli space. When the target is a sphere, there could be unstable spherical domain components, but these are well understood classically, and in particular are cut transversally (even though they are multiple covers!). This is also true for all the strata of the moduli space of (holomorphic) maps from a genus 1 curve without marked points, as long as the target has genus one (or higher).

For the relative moduli spaces, there are also rubber components involved, but in our case these project to constant maps into the divisor which satisfy an additional condition that depends on $\nabla\nu$, cf. (4.16). This condition is automatically satisfied on genus 0 components and is again cut transversally on the other rubber components because we can use $\nabla\nu$

to get transversality. Moreover, any stratum with rubber components has codimension at least 2. For more details, see §4.2.

The moduli spaces of real maps considered in this paper may have codimension 1 strata. In this case, we first argue that for generic parameter ν the union of the codimension at most one strata is an oriented topological manifold (without boundary), and thus carries a fundamental class (in rational Cech homology); this involves only standard gluing techniques at an ordinary real node of the domain (and no rubber components). All the other boundary strata have codimension at least two, thus the method of thin compactifications applies. For details, see §5.

3. Family moduli spaces

A key step in proving the splitting formula for the RGW invariants is to consider a family of moduli spaces associated to a family of symmetric marked curves degenerating to a nodal symmetric curve (with a conjugate pair of nodes). This family moduli space, denoted $\overline{\mathcal{M}}(\mathcal{F}_{/\Delta})$, serves as the ambient space where we can compare the VFCs and integrands used in defining the RGW invariants. In this and the following two sections we set up the necessary notation, review the constructions and show that the moduli spaces involved in defining the RGW invariants (2.10) extend over families of symmetric marked curves, including across the singular fibers.

3.1. Families of symmetric curves

Recall that if Σ_0 is a complex nodal marked curve, then we can consider

(a) a smooth normalization $\widetilde{\Sigma}$ of Σ_0 that replaces each node by a pair of marked points;

(b) a (flat) family $\mathcal{F}$ of deformations Σ_s of Σ_0 smoothing out the nodes.

Let (Σ_0, c_0) be a nodal symmetric curve with r pairs of conjugate marked points V_0 and a single pair of conjugate nodes $x^{\pm}$. Let

$$\pi : (\mathcal{F}, c_{\mathcal{F}}) \longrightarrow (\Delta^2, c_\Delta), \tag{3.1}$$

denote a (flat) family of deformations of (Σ_0, c_0) smoothing the nodes as defined in [GZ2, §4.2], cf. §B.1. Here $\Delta \subset \mathbb{C}$ is the unit disk and $\Delta^2 = \Delta \times \Delta$. The total space $\mathcal{F}$ is a smooth Kähler manifold with complex structure J and the projection is holomorphic. Moreover, $c_{\mathcal{F}}$ is a real structure on $\mathcal{F}$ which

is anti-holomorphic and restricts to the real structure c_0 on the central fiber Σ_0, while $c_\Delta : \Delta^2 \to \Delta^2$ is defined by $(s_1, s_2) \mapsto (\bar{s}_2, \bar{s}_1)$. Finally, marked points give rise to sections of $\mathcal{F} \to \Delta^2$; their images define a smooth divisor $V \subset \mathcal{F}$.

The fiber of (3.1) over $(s, \bar{s})$ is a symmetric marked curve denoted Σ_s; its real structure c_s is the restriction of $c_\mathcal{F}$ to Σ_s while its marked points correspond to the restriction V_s of V to Σ_s. This determines a family

$$\mathcal{F}_{/\Delta} \stackrel{\text{def}}{=} \bigcup_{s\in\Delta} \Sigma_s \tag{3.2}$$

of symmetric marked curves over Δ (the pullback of (3.1) via the map $\Delta \to \Delta^2$, $s \mapsto (s, \bar{s})$ for all $s \in \Delta$). We will later consider restrictions of this family (3.2) to a line, path, etc.

Consider also

$$(\widetilde{\Sigma}, \widetilde{c}) \to (\Sigma_0, c_0) \tag{3.3}$$

the normalization of the singular fiber Σ_0 (as a marked symmetric curve). Here $\widetilde{c}$ denotes the real structure of $\widetilde{\Sigma}$ and we denote by $\widetilde{V}$ the collection of marked points of $\widetilde{\Sigma}$ consisting of one point over each marked point of Σ_0, and a pair of points over each node of Σ_0. The curves Σ_s and $\widetilde{\Sigma}$ come with natural maps

$$\iota_s : (\Sigma_s, c_s) \hookrightarrow (\mathcal{F}, c_\mathcal{F}) \quad \text{and} \quad \phi : (\widetilde{\Sigma}, \widetilde{c}) \to (\mathcal{F}, c_\mathcal{F}). \tag{3.4}$$

into the total space $\mathcal{F}$ of the family; the second map factors through the nodal fiber.

Finally, let

$$(\mathcal{T}, c_\mathcal{T}) \to (\mathcal{F}, c_\mathcal{F}) \tag{3.5}$$

denote the relative tangent bundle to the family (3.1); here $\mathcal{T}$ is a holomorphic line bundle (locally free sheaf) over $\mathcal{F}$ which comes with an induced real structure $c_\mathcal{T}$, cf. [GZ2, Lemma 4.8] and (B.6). The pullback of $(\mathcal{T}, c_\mathcal{T})$ to both Σ_s and $\widetilde{\Sigma}$ under (3.4) gives their corresponding relative tangent bundle (2.8). The relative tangent bundle $\mathcal{T}_{\Sigma_0}$ to the nodal curve Σ_0 is defined as the restriction of $\mathcal{T}$ to Σ_0, and it fits in the normalization short exact sequence

of holomorphic sheaves

$$0 \to \mathcal{T}_{\Sigma_0} \to \mathcal{T}_{\widetilde{\Sigma}} \to \mathcal{T}_{|nodes} \to 0 \tag{3.6}$$

compatible with the real structures.

3.2. Family moduli space

The real relative moduli spaces of maps into the smooth fibers Σ_s naturally form a family which can be compactified by adding a fiber over $s = 0$. In parallel with the proof of the usual splitting formula e.g [BP1, (14)], we consider the family

$$\overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}) = \bigcup_{s\in\Delta} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s) \longrightarrow \Delta$$

of moduli spaces whose fiber at $s \in \Delta \setminus 0$ is the real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma_s, V_s)$, while the fiber $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma_0, V_0)$ over $s = 0$ includes maps with rubber components over both the nodes and the marked points of Σ_0. We describe these spaces in more detail below.

3.3. Relative real moduli spaces

Even in the complex category, there are several versions of the relative moduli space of holomorphic maps to a complex curve Σ relative to a divisor V. The version used by Bryan-Pandharipande in [BP2, Definition 3.1] is more convenient for computational purposes, and is a finite quotient of the standard one defined by Jun Li in [Li]. The latter has the property that all the contact points are ordered, and is more convenient for analytical considerations, including for constructing the VFC and describing its behavior under target degenerations. Of course, the virtual fundamental classes of these two versions of the moduli space are essentially the same up to a combinatorial factor.

In this section we outline the construction of the relative moduli space, in which all the contact points are ordered, adapted to the real setting; see also Remark 3.2. We include some of the standard arguments for ease of reference when we extend these arguments to families of degenerating targets.

Let (Σ, c) be a (smooth) symmetric marked curve with r pairs of conjugate marked points V as in (2.5). Fix d, χ and a collection $\vec{\mu} = (\mu^1, \ldots, \mu^r)$

of r partitions of d. Consider first the "top stratum"

$$\mathcal{M}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \tag{3.7}$$

consisting of equivalence classes (up to reparametrizations of the domain) of J-holomorphic real maps $f : (C, \sigma) \to (\Sigma, c)$ such that

(i) the domain C is a smooth, marked, possibly disconnected symmetric Riemann surface of Euler characteristic χ;

(ii) f has ramification pattern μ^i over x_i^+ (and thus also over its conjugate x_i^-), for all $i = 1, \ldots, r$;

(iii) f is nontrivial on each connected component of C;

(iv) f is a degree d map, i.e. the image of f represents $d[\Sigma]$ in $H_2(\Sigma; \mathbb{Z})$;

The points in $f^{-1}(V)$ are called contact points of f to V, and here all the contact points are *marked* points of the domain (and the domain has no other marked points); see also Remark 3.2 below. Specifically, condition (ii) means that

- $f^{-1}(x_i^{\pm}) = \{y_{ij}^{\pm}\}_{j=1,\ldots,\ell(\mu^i)}$ for every $i = 1, \ldots, r$; thus $f(y_{ij}^+) = x_i^+$.
- the ramification order of f at $y_{ij}^{\pm}$ is μ_j^i and $y_{ij}^{\pm}$ are conjugate points;
- $\{y_{ij}^{\pm} \mid j = 1, \ldots, \ell(\mu^i),\ i = 1, \ldots, r\}$ are marked points of the domain.

A map f which satisfies these conditions is said to have its contact to V prescribed by $\vec{\mu}$.

The moduli space (3.7) has a compactification $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ in which both the domain and target of the maps is allowed to degenerate. We start by briefly describing the targets, denoted $\Sigma[m]$, and called buildings obtained by rescaling Σ around to V; for the general rescaling construction normal to a divisor see for example [I, §4]. In our case $\Sigma[m]$ is a nodal symmetric marked curve, obtained from Σ by attaching chains of spheres at its marked points in the following manner. Let N_V be the normal bundle of V in Σ, and consider the projectivization $\mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C})$. Then

$$\mathbb{P}_V = \mathbb{P}^1 \times V = \bigsqcup_{i=1}^{r} \mathbb{P}^1 \times \{x_i^{\pm}\} = \bigsqcup_{i=1}^{r} \mathbb{P}_{x_i^{\pm}} \tag{3.8}$$

is a $\mathbb{P}^1$ bundle over $V = \{x_1^{\pm}, \ldots, x_r^{\pm}\}$ with

(i) a zero and infinity section V_0 and V_∞,

(ii) a real structure induced by the one on N_V, thus covering $c(x_i^+) = x_i^-$ in the base, and

(iii) a $\mathbb{C}^*$ action on each $\mathbb{P}_{x_i^\pm}$, fixing (i) pointwise and compatible with (ii) i.e. satisfying $c(\lambda \cdot z) = \overline{\lambda} \cdot c(z)$.

Assume first for simplicity that V consists of a single pair of marked points, i.e. $r = 1$. Starting with Σ and inductively rescaling it m times around V gives rise to the building

$$\Sigma[m] = \Sigma \underset{V=V_\infty}{\cup} \mathbb{P}_V \underset{V_0=V_\infty}{\cup} \cdots \underset{V_0=V_\infty}{\cup} \mathbb{P}_V \longrightarrow \Sigma,$$

with a divisor $V[m] \subset \Sigma[m]$ corresponding to the zero divisor of the last copy of $\mathbb{P}_V$. For the general case, we allow Σ to be *independently* rescaled m_i times around each pair $\{x_i^\pm\}$ of conjugate points in $V = \{x_1^\pm, \ldots, x_r^\pm\}$. This similarly gives rise to a building $\Sigma[m]$ with a divisor $V[m] = \{x_i^\pm\}_{i=1}^r$. The $\mathbb{C}^*$ action on each $\mathbb{P}_{x_i^\pm}$ induces a $(\mathbb{C}^*)^{|m|}$ action on the building $\Sigma[m]$, called the rescaling action, and denoted

$$t \mapsto R_t \in \mathrm{Aut}(\Sigma[m]);$$

here $m = (m_1, \ldots, m_r)$ and $|m| = m_1 + \cdots + m_r$. Finally, let

$$p : \Sigma[m] \longrightarrow \Sigma \tag{3.9}$$

denote the projection induced by collapsing all the m_i copies of $\mathbb{P}_{x_i^\pm}$ for $i = 1, \ldots, r$ down to V.

Then the compactification $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ of the top stratum (3.7) is defined as follows.

Definition 3.1. *An element of the moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ is an equivalence class, up to reparametrizations of the domain and rescaling the target, of real J-holomorphic maps*

$$f : C \to \Sigma[m] \tag{3.10}$$

from some possibly nodal, possibly disconnected symmetric Riemann surface C to some symmetric building $\Sigma[m]$ such that

(i) f has prescribed contact to the marked points of the target, i.e. the preimage of $\{x_i^\pm\}$ consists only of marked points of the domain, denoted

$\{y_{ij}^{\pm}\}$, *and* f *has order of contact* μ_j^i *at* $y_{ij}^{\pm}$ *to* $x_i^{\pm}$*; thus* $f(y_{ij}^{+}) = x_i^{+}$ *and* $y_{ij}^{\pm}$ *are conjugate marked points.*

(ii) f *satisfies the following matching condition: the preimage of the nodes of the target consists only of nodes of the domain with the same order of contact on the two local branches.*

(iii) the restriction of f *to every connected component of the domain is nontrivial.*

(iv) f *has degree* d *and its domain has (virtual) Euler characteristic* χ*.*

(v) f *is* relatively stable*, i.e.* $\mathrm{Aut}f$ *is finite; here an automorphism of* $f : C \to \Sigma[m]$ *is a pair* $(\varphi, t) \in \mathrm{Aut}(C) \times (\mathbb{C}^*)^{|m|}$ *such that* $R_t \circ f \circ \varphi = f$*.*

The real relative moduli space

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \tag{3.11}$$

comes with natural maps induced by forgetting some of the data such as the real structures, the order of the contact points, the divisor V, etc. Forgetting the real structure defines a map to the (usual) relative moduli space of holomorphic maps to Σ relative the divisor V.

Remark 3.2. Unlike in [GI, Definition 2.5] (or [BP2, Definition 3.1]), throughout this paper we are using the standard definition of the relative moduli space (cf. [Li] or [IP1]) in which all the contact points are marked. The moduli space (3.11) comes with a group action permuting the contact points with same image and multiplicity; the quotient by this $\mathrm{Aut}(\vec{\mu})$ action is the moduli space in [GI, Definition 2.5]. In particular, the former moduli space is the degree $|\mathrm{Aut}(\vec{\mu})| = \prod_i |\mathrm{Aut}(\mu^i)|$ cover of the latter moduli space, obtained by ordering the contact points.

3.4. Rubber components and stratification

There is an equivalent description of the elements of the relative moduli space that is more convenient for transversality purposes. Decompose any real map (3.10) satisfying conditions (i)-(v) of Definition 3.1 into components, obtained by restricting f to the irreducible components of its domain. These components can be grouped according to their image in $\Sigma[m]$; those that are mapped to Σ are called level zero components and the rest are called rubber components. In turn, the rubber components can be grouped according to which copy of $\mathbb{P}_{x_i^{\pm}}$ in $\Sigma[m]$ their image lands in, and the vertices of

the dual graph of $f : C \to \Sigma[m]$ are decorated by this discrete data (along with the genus and the homology class represented by each irreducible component).

By Definition 3.1(ii), each node q of the domain has an associated order of contact $\lambda(q)$, where $\lambda(q) = 0$ if q is an ordinary node i.e. mapped away from the nodes of $\Sigma[m]$. Each marked point of the domain also comes with an associated order of contact cf. Definition 3.1(i). Altogether, this associates to every special point q of the domain a multiplicity $\lambda(q)$, and the dual graph of f is decorated by this data as well.

Remark 3.3. Note that every rubber component f_k of f, regarded as a map $f_k : C_k \to \mathbb{P}_V$ either represents 0 in homology (and its image is disjoint from the 0 and ∞ divisors) or else its domain has at least two special points (the inverse image of 0 and ∞), cf. Definition 3.1(i)-(ii).

For each decorated dual graph τ, let $\mathcal{M}_\tau$ denote the corresponding stratum of the real relative moduli space. Elements of this stratum can also be described in terms of their projection under the collapsing map (3.9), and the lifts of this projection to meromorphic sections of a line bundle as follows.

Since V is 0 dimensional, each rubber component projects to a constant map to V under (3.9), and is a holomorphic map to one of the $\mathbb{P}_V$'s, with prescribed contact to 0 and ∞ (determined by τ), cf. Definition 3.1(i)-(ii). Equivalently, it is a meromorphic section $\xi \neq 0$ of the pullback normal bundle of V with zeros and poles at the special points of prescribed order (and no other zeros/poles). Note that any two such sections, if they exist, must be constant multiples of each other. For more details, see §A.8.

Some rubber components may be multiple covers $\mathbb{P}^1 \to \mathbb{P}^1$ totally ramified over 0 and ∞; these are called trivial components, and their domain is a sphere with precisely two special points. All the other (nontrivial) rubber components project to a stable map to V cf. Remark 3.3.

In particular, the projection (3.9) induces a forgetful map

$$\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V) \to \overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, \emptyset) \tag{3.12}$$

to the (absolute) real moduli space; it takes $f : C \to \Sigma[m]$ to the map obtained from $p \circ f : C \to \Sigma$ after collapsing all the chains of trivial components in the domain to obtain a stable map to Σ.

Remark 3.4. (Stability vs relative stability) Let $f : C \to \Sigma[m]$ be a real map which satisfies conditions (i)-(iii) of Definition 3.1. Then f is relatively stable if and only if (a) f is stable as a map to $\Sigma[m]$ and (b) every copy of $\mathbb{P}_{x_i^{\pm}}$ in $\Sigma[m]$ contains at least one nontrivial rubber component of f.

Remark 3.5. By definition, the image of the forgetful map (3.12) consists of stable, real holomorphic maps $f : C \to \Sigma$ that have a real holomorphic lift $\widehat{f} : \widehat{C} \to \Sigma[m]$ which satisfies the matching conditions over the singular locus of $\Sigma[m]$ and has prescribed contact with $V[m]$, cf. Definition 3.1. In particular, the components of f that are mapped to V must have a lift to a rubber component.

3.5. Nodal targets

Definition 3.1 extends verbatim to nodal targets, as long as we include maps with rubber components over both the marked points and the nodes of the target. Assume Σ_0 is a nodal symmetric marked curve with one ordered pair $\{x^{\pm}\}$ of conjugate nodes, r pairs of conjugate marked points V_0, and no other special points. Then the real relative moduli space

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_0, V_0)$$

is defined as in Definition 3.1 except that $\Sigma[m]$ is replaced by a building $\Sigma_0[m]$ obtained by rescaling Σ_0 at both the marked points and the nodes; in particular, chains of spheres are also inserted at the nodes of Σ_0, in addition to those inserted at the marked points. Therefore an element of this moduli space is an equivalence class of real J-holomorphic maps

$$f : C_0 \to \Sigma_0[m]$$

satisfying conditions (i)-(v) of Definition 3.1. Now the domain C_0 must be nodal and the top stratum $\mathcal{M}^{\mathbb{R}}(\Sigma_0, V_0)$ is the subset consisting of maps to Σ_0 whose domain has no other nodes besides those in the inverse image of the nodes of Σ_0.

Recall that (3.3) attaches pairs of marked points of $\widetilde{\Sigma}$ to produce the nodes of Σ_0; it extends to a map between any building associated to $\widetilde{\Sigma}$ and the corresponding building associated to Σ_0. The attaching map

$$\bigsqcup_{\lambda \vdash d} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V}) \xrightarrow{\Phi} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_0, V_0) \tag{3.13}$$

is then induced by attaching pairs of marked points of both the domain and target to produce nodes (then forgetting the order of these nodes). Note that in the domain of (3.13) the contact points are *ordered*; however Φ factors through the quotient by the diagonal $\mathrm{Aut}(\lambda)$ action to produce *unordered* nodes in its image.

3.6. Maps to the family

Consider the family

$$\overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}) = \bigcup_{s\in\Delta} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s) \longrightarrow \Delta \tag{3.14}$$

moduli spaces associated to the family (3.2) of targets; its fiber over $s \in \Delta$ is the real relative moduli space associated to the fiber of $\mathcal{F}$ over s defined above. In particular, the fiber $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma_0, V_0)$ over $s = 0$ includes maps with rubber components over both the nodes and the marked points of Σ_0.

The inclusion of Σ_s into $\mathcal{F}$ is holomorphic, and induces a proper map

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s) \longhookrightarrow \overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}) \tag{3.15}$$

for every $s \in \Delta$. The map (3.3) is also holomorphic and induces the proper map (3.13) at the level of moduli spaces; the composition of the latter with the map (3.15) for $s = 0$ is a proper map

$$\bigsqcup_{\lambda\vdash d} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V}) \longrightarrow \overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}). \tag{3.16}$$

Remark 3.6. The topology of these real relative moduli spaces is a refinement of the usual Gromov topology, defined by a process of rescaling the target normal to the divisors cf. [IP1] and [IP2]; in particular, the topology on the subset (3.7) consisting of maps without rubber components is the usual Gromov topology.

4. Perturbations and transversality

In this section we introduce spaces of Ruan-Tian perturbations adapted to our setting and show that the strata of the real relative moduli spaces described above are cut transversally over such parameter spaces.

4.1. Ruan-Tian perturbations

To obtain transversality stata-wise, we fix the (integrable) complex structure J on the target, but turn on Ruan-Tian perturbations ν adapted to the situation. In particular, for the real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$, we restrict to the space $\mathcal{JV}^{\mathbb{R}}(\Sigma, V)$ of RT-perturbations on Σ compatible with both the real structure and the divisor V, as described in this section.

In general, recall that if J is an (almost) complex structure on X and ν is a Ruan-Tian perturbation [RT], then a (J, ν)-holomorphic map to X is a solution $f : C \to X$ of the equation

$$\overline{\partial}_J f = \nu|_f \tag{4.1}$$

or equivalently the graph F of f is J_ν-holomorphic, as reviewed for example in [IP4, §3.1]. If the domain C has trivial automorphism group, one can use the variation in ν to show that the linearization is cut transversally, essentially because the graph of f is an embedding. This extends to the case the domain C is stable after passing to a regular cover of the Deligne-Mumford moduli space as in Remark 4.1 below.

Remark 4.1. In general, for higher genus domains, passing to covers of the moduli spaces is needed to kill the automorphism groups and turn on Ruan-Tian perturbations, as reviewed for example in [IP4, §3.1]. This is classically achieved by working on a regular cover of the Deligne-Mumford moduli space, defined by considering curves with extra decorations such as level structures or twisted bundles eg. as constructed by Abramovich-Corti-Vistoli in [ACV], cf. [ACG, Chapter XVI, Theorem 7.1]; see also Chapter XVI, §10 of [ACG]. Such regular cover comes with an universal curve $\mathcal{U}$ whose fiber at a (decorated stable) curve C is C.

When (X, c) is a manifold with a real structure c, denote by $\mathcal{JV}^{\mathbb{R}}(X)$ the space of real Ruan-Tian perturbations on X, as defined for example in [Z, §2 and §3.1]. These are constructed as follows. Using the forgetful map from the real Deligne-Mumford moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell}$ to the complex one, and a regular cover of the latter, one constructs a cover of $\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell}$ and a universal curve $(\mathcal{U}, c_{\mathcal{U}})$. The fiber of $(\mathcal{U}, c_{\mathcal{U}})$ at a (decorated stable) symmetric curve (C, σ) is (C, σ).

A real RT perturbation on X is an element $\nu \in \mathrm{Hom}^{01}(\mathcal{T}_{\mathcal{U}}, TX)$ defined on $\mathcal{U} \times X$, such that

(4.2) ν is real and is supported away from the special points of the domain.

Here $\mathcal{T}_{\mathcal{U}} \to \mathcal{U}$ is the relative tangent bundle to the fibers of $\mathcal{U}$, cf. (B.6), and Hom^{01} denotes the space of anti-complex linear homomorphisms i.e. such that $J\nu + \nu j = 0$, where J is the (almost) complex structure on (X, c), and j is the family of complex structures on the fibers of $\mathcal{U}$. Such a homomorphism ν can be regarded as section of the bundle $(\mathcal{T}_{\mathcal{U}})^\vee \boxtimes_{\mathbb{C}} TX$ which has an induced real structure; ν is called *real* if this section is invariant with respect to the real structures. For details, see [Z, §2 and §3.1].

When $X = \Sigma$ is a symmetric curve with conjugate marked points V, consider the subset

$$\mathcal{JV}^{\mathbb{R}}(\Sigma, V) \subseteq \mathcal{JV}^{\mathbb{R}}(\Sigma) \tag{4.3}$$

of real RT perturbations (J, ν) which are compatible with V in the sense of [IP1, Definition 3.2]. Since here J is integrable and V is 0 dimensional, these conditions reduce to the requirement that ν vanishes along $\mathcal{U} \times V$ and $\nabla\nu$ is complex linear along $\mathcal{U} \times V$. Specifically, the conditions are

(4.4)

$$\nu|_{\mathcal{U}\times V} = 0 \quad \text{and} \quad (\nabla_{Jw}\nu)(v) = J(\nabla_w \nu)(v) \text{ for all } w \in TX|_V \text{ and } v \in \mathcal{T}_{\mathcal{U}}.$$

Note that the last condition is equivalent to

$$\nabla_w \nu = \alpha_V \otimes_{\mathbb{C}} w \quad \text{for all } w \in N_V \tag{4.5}$$

where $\alpha_V \in \mathrm{Hom}^{01}(\mathcal{T}_{\mathcal{U}}; \mathbb{C})$ is defined on $\mathcal{U} \times V$.

Thus, as in [IP1, §3], the compatibility condition with the divisor V ensures that

(a) ν restricts to a RT perturbation on V (i.e. ν takes values in TV along V);

(b) for maps with image in V, the normal operator $L^N = \overline{\partial} - \nabla\nu$ is complex linear;

(c) the 1-jet of ν determines a RT perturbation on $\mathbb{P}_V$ and on (the normalization of) the building $X[m]$ obtained by rescaling X normal to V.

Property (c) follows using the fact that

$$\mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C}) = \mathbb{P}(\mathbb{C} \oplus (N_V)^\vee) \tag{4.6}$$

has a $\mathbb{C}^*$ action and an involution swapping the zero and infinity divisors (induced by $z \mapsto z^{-1}$). Let ζ denote the canonical $\mathbb{C}^*$-equivariant vector field on $\mathbb{P}_V$. This vanishes along the zero and infinity divisors $V_0 \sqcup V_\infty$, while its restriction to $N_V \subset \mathbb{P}_V$ under the inclusion $w \mapsto [w, 1]$ is $\zeta_w = (w; w) \in T_w N_V$

for all $w \in N_V$. Then the restriction (4.5) of $\nabla\nu$ induces a RT perturbation on $N_V \subset \mathbb{P}_V$, defined as the tensor product of the pullbacks of α_V and ζ. This tensor product $\alpha_V \boxtimes \zeta$ can be regarded as a $\mathbb{C}^*$-equivariant RT perturbation on $\mathbb{P}_V$, compatible with its zero and infinity divisors as in (4.4). Thus ν induces a RT perturbation on $X[m]$, whose restriction to $X \subset X[m]$ is equal to ν, and the restriction to each copy $\mathbb{P}_V$ of $X[m]$ is equal to $\alpha_V \boxtimes \zeta$.

Remark 4.2. Note that a map $f : C \to \Sigma[m]$ to the building is (J, ν)-holomorphic (with respect to this lift of ν to $\Sigma[m]$) if and only if

(i) the projection $\rho := p(f) : C \to \Sigma$ is (J, ν)-holomorphic

(ii) each rubber part $f_r := f|_{C_r} : C_r \to \mathbb{P}_V$ of f, regarded as a section ξ of the pullback normal bundle $\rho_r^* N_V$, satisfies $\overline{\partial}\xi = \nabla_\xi \nu$ i.e. $L^N \xi = 0$; for details, see §A.8.

In fact, since L^N is complex linear, then ξ is a meromorphic section; thus up to scale is determined by its zero and infinity divisor. Note that if $\xi \neq 0$ then $\zeta = \xi^{-1}$ is a section of the dual bundle $\rho_r^*(N_V)^\vee$ which solves $\overline{\partial}\zeta = -\nabla_\zeta \nu$; the poles of ζ correspond to the zeros of ξ and viceversa.

This perspective neatly encodes the conditions describing the relative stable map compactitication constructed in [IP1], cf. [IP1, Remark 7.7 and Definition 7.2].

The space of RT perturbations (4.3) easily extends to the family of targets. As before, we start with the space of real Ruan-Tian perturbations ν on the total space $(\mathcal{F}, c_{\mathcal{F}})$ of the family (3.1). These satisfy condition (4.2) for X equal to the total space of the family $\mathcal{F}$. Then we restrict to the subspace

$$\mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta}) \tag{4.7}$$

of such perturbations which additionally satisfy the following conditions:

(a) ν is compatible with the fibration i.e. $\pi_* \nu = 0$;

(b) ν is compatible with the divisor V of $\mathcal{F}$ i.e. the 1-jet condition (4.4) holds for X equal to the total space of the family $\mathcal{F}$ and $V \subset X$ the union of the marked points of the family.

(c) ν is compatible with the nodal locus i.e. the pullback $\widetilde{\nu} = \phi^* \nu$ to the normalization $\widetilde{\Sigma}$ takes values in $T\widetilde{\Sigma} \subset \phi^* T\mathcal{F}$ and satisfies the 1-jet condition (4.4) for X equal to $\widetilde{\Sigma}$ and V equal to the preimage of the nodes of Σ_0.

Condition (a) implies that any (J,ν) holomorphic map to $\mathcal{F}$ projects to a holomorphic, thus (locally) constant map to Δ, and therefore its image is contained in a fiber of $\mathcal{F}$ (if the domain is connected). Moreover, conditions (a)-(b) imply that the pullback ν_s of ν to Σ_s is a real RT perturbation on Σ_s, which is compatible with the divisor $V_s = V \cap \Sigma_s$. Similarly, the pullback $\widetilde{\nu}$ of ν to the normalization $\widetilde{\Sigma}$ is a real RT perturbation on $\widetilde{\Sigma}$ compatible with the divisor $\widetilde{V}$ (the inverse image of the special points of Σ_0). This means that the maps ι_s and ϕ in (3.4) induce maps

$$\begin{array}{ccc} \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta}) \to \mathcal{JV}^{\mathbb{R}}(\Sigma_s, V_s) & & \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta}) \to \mathcal{JV}^{\mathbb{R}}(\widetilde{\Sigma}, \widetilde{V}) \\ \nu \mapsto \nu_s & & \nu \mapsto \widetilde{\nu} \end{array} \tag{4.8}$$

at the level of parameter spaces.

Note that the family $\mathcal{F}$ has a specific local model (B.4) in a neighborhood of the nodes of Σ_0, and is a product away from a smaller neighborhood. Thus in (4.7), condition (a) implies (c) away from $\mathcal{U} \times \{nodes\}$. A standard calculation shows that (a) and (c) imply that for every node q of Σ_0,

$$\nu|_{\mathcal{U}\times q} = 0 \quad \text{and} \quad \alpha_{q_1} + \alpha_{q_2} = 0. \tag{4.9}$$

Here q_i are the lifts of q to $\widetilde{\Sigma}$, and $\alpha_{q_i} \in \mathrm{Hom}^{01}(\mathcal{T}_{\mathcal{U}}; \mathbb{C})$ over $\mathcal{U} \times q$ is determined by $\widetilde{\nu}$ via (4.5).

Remark 4.3. Property (4.9) implies that the lift of ν to the building $\Sigma_0[m]$ is well defined, i.e. its value on $\mathbb{P}_q$ is independent of the choice of lift $q_i \in \widetilde{\Sigma}$ of the node q of Σ_0. Note that for each node q of Σ_0 we get two copies $\mathbb{P}_{q_1}$ and $\mathbb{P}_{q_2}$ in $\mathbb{P}_{\widetilde{V}}$ and there is an identification $\mathbb{P}_{q_1} \cong \mathbb{P}_{q_2}$ swapping the zero and infinity divisors, induced by $(q_1, z) \mapsto (q_2, z^{-1})$. For a map $\rho : C \to \{nodes\} \subset \Sigma_0$, each lift ρ_i to $\widetilde{\Sigma}$ has an associated normal operator L^N. Property (4.9) implies that the condition $\ker L^N \neq 0$ is well defined, independent of the choice of lift of ρ to the normalization $\widetilde{\Sigma}$, cf. Remark 4.2.

4.2. Transversality for each stratum

Since our targets are curves, a generic real RT perturbation ν compatible with the divisor ensures that all strata are of expected dimension and therefore as in [IP5], the real relative moduli spaces carry a VFC for all parameters, including $\nu = 0$. For completeness, we provide the details of this argument here.

Assume first Σ is a smooth symmetric curve with conjugate marked points V as in (2.5) and a fixed integrable complex structure J. As ν varies in the space of RT perturbations $\mathcal{JV}^{\mathbb{R}}(\Sigma, V)$ defined in (4.3), we get a family

$$\overline{m}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \xrightarrow{\pi} \mathcal{JV}^{\mathbb{R}}(\Sigma, V) \tag{4.10}$$

of real relative moduli spaces, one for each choice of topological data d, χ, $\vec{\mu}$ and we denote by $\overline{m}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)_\nu$ its fiber at ν; it can be regarded as a deformation of the moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ in (3.11), which corresponds to $\nu = 0$. Specifically, an element of the moduli space (4.10) is a pair $([f], \nu)$ where

$$f : C \to \Sigma[m] \tag{4.11}$$

is a (relatively stable) real (J, ν)-holomorphic map satisfying the properties (i)-(v) listed in Definition 3.1.

As in §3.4, the moduli space (4.10) is stratified with strata $\mathcal{M}_\tau$ indexed by the decorated dual graph τ of the maps (4.11); therefore, along a stratum, the topological type of the domain and target is fixed, as is the ramification pattern of f over the special points of the target. Note that the preimage of the nodes/marked points of $\Sigma[m]$ consists only of nodes/marked points of C. Denote by $\mathbf{x}$ the collection of special points of the target $\Sigma[m]$, and by $\mathbf{y} = f^{-1}(\mathbf{x})$ its preimage. An element y of $\mathbf{y}$ is called a contact point, and comes decorated by the contact multiplicity $\lambda(y)$ of f at y, cf. (A.39). The other points of the domain are called ordinary points; they are mapped away from the special points of the target, and their contact multiplicity is 0. Note that some of the nodes could be ordinary ones, but all the marked points are contact points.

Fix a stratum $\mathcal{M}_\tau$ of the moduli space. Restrict first to the case $m = 0$, i.e. the target $\Sigma[m] = \Sigma$ is smooth; here the domain could be nodal, but all its nodes are ordinary (mapped away from V). When $f : C \to \Sigma$ has contact order λ at the points $\mathbf{y} = f^{-1}(V)$, the linearization (to this moduli problem) is

$$\begin{aligned} &L_f : \Gamma_{\lambda;\mathbf{y}}(f^*T\Sigma)^{\mathbb{R}} \oplus T_C\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell} \to \Gamma_{\lambda-1;\mathbf{y}}(\Omega^{01}_C \otimes_{\mathbb{C}} f^*T\Sigma)^{\mathbb{R}} \\ &L_f(\xi, h) = \overline{\partial}\xi - [\nabla_\xi \nu + \tfrac{1}{2} J df h]^{01}, \end{aligned} \tag{4.12}$$

cf. (A.42) and §A.6. Here $\Gamma_{\lambda;\mathbf{y}}(E)^{\mathbb{R}}$ denotes the subspace of invariant sections of E which vanish to order $\lambda(y)$ at y, for all y in $\mathbf{y}$, and $\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell}$ denotes the real

Deligne-Mumford moduli space containing C. Here for simplicity we assume C is stable, but as in [IP3, §4.2], one can always use a "local slice" (B.5) to parametrize the variations in the domains, after first locally stabilizing the domains if necessary. Those considerations extend to our case, as long as we are working with symmetric choices throughout, by adding pairs of conjugate marked points in the domain, mapped to fixed pairs of non-special conjugate points of the target. For the strata-wise linearization, the second term in (4.12) is replaced by the tangent space $T_C\mathcal{S}$ to the stratum $\mathcal{S}$ containing C.

As in [MS, §3.1-§3.2], there are several choices of norms/completions one can use to locally describe a stratum of the family (4.10) as the zero locus of a Fredholm map

$$\Psi(f,\nu) = \overline{\partial}_J f - \nu|_f \tag{4.13}$$

between Banach manifolds[3]. When Ψ is transverse to 0 at f (i.e. $\delta\Psi$ is onto) then a neighborhood of f in its stratum is a Banach manifold modeled on the kernel of the strata-wise linearization. For more details on the standard set-up, we refer the reader to [MS, §3.1-§3.2]; see also [IP3, §4.2] for a summary of some analytic preliminaries.

After completing (4.12) in the λ-weighted Sobolev norms (A.34), the strata-wise linearization (to this moduli problem) at $f : C \to \Sigma$ becomes the Fredholm operator

$$\begin{aligned} &\mathcal{L}_f : \mathcal{E}_f^{k,p} \oplus T_C\mathcal{S} \to \mathcal{F}_f^{k-1,p} \\ &\mathcal{L}_f(\xi,h) = \overline{\partial}\xi - [\nabla_\xi \nu + \tfrac{1}{2} J df h]^{01} \end{aligned} \tag{4.14}$$

cf. (A.44). While the weighted completions $\mathcal{E}_f^{k,p}$ and $\mathcal{F}_f^{k-1,p}$ depend on the choice of $k, p \geq 1$ (where $kp \geq 2$) the kernel and cokernel of $\mathcal{L}_f$ are independent of these choices, and consist of smooth elements. The full linearization $\delta\Psi_f$, which also includes the variation $\mu = \delta\nu$ in the parameter ν, is given by

$$(\xi, h, \mu) \mapsto \mathcal{L}_f(\xi, h) - \mu|_f. \tag{4.15}$$

When $m \neq 0$, (J,ν)-holomorphic maps $f : C \to \Sigma[m]$ satisfying the properties listed in Definition 3.1 can be analyzed as in Remark 4.2 by considering their projection $p(f) : C \to \Sigma$ under the collapsing map $\Sigma[m] \to \Sigma$ in the

[3]the Banach manifolds used here are Hausdorff, separable, and paracompact, so that Sard-Smale theorem applies.

target. The rubber components of f continue to project to constant maps to V (since V is 0 dimensional), and satisfy the condition

$$\ker L^N \neq 0 \tag{4.16}$$

along this projection, cf. [IP1, (6.3) and (7.1)]. Here L^N is the normal operator, given by

$$L^N \xi = \overline{\partial}\xi - \nabla_\xi \nu, \tag{4.17}$$

see (A.67)-(A.70). When the index of L^N is negative, condition (4.16) imposes additional restrictions on the rubber components as in [IP1, Lemma 6.4].

Remark 4.4. Assume Σ is connected and $V \neq \emptyset$. If $f : C \to \Sigma[m]$ satisfies conditions (i)-(iii) of Definition 3.1, then every connected component of its domain has at least one marked point (the preimage of $V[m]$). Thus the unstable domain components of a map in the moduli space (4.10) have genus 0; see also Remarks 3.3 and 3.4. One can also show that when (Σ, V) is a *stable* curve, the trivial rubber components are the only unstable domain components of the maps in the moduli space (4.10).

Below we assume for simplicity that Σ is connected and $V \neq \emptyset$, or more generally that every rational component of Σ has at least one marked point; otherwise, see Remark 4.7 below.

Standard arguments (cf. [MS, §3.2]) imply the following result whose proof we sketch for completeness.

Lemma 4.5. *Assume that every rational component of Σ has at least one marked point. Then over the parameter space $\mathcal{JV}^{\mathbb{R}}(\Sigma, V)$ of real RT-perturbations on Σ compatible with V, every stratum of the real relative moduli space* (4.10) *is cut transversally.*

Proof. As in the proof of [MS, Proposition 3.2.1], transversality follows provided we have enough variations in the parameters ν to ensure that the linearization of the equations cutting out each stratum of the moduli space has trivial cokernel.

Assume $f : C \to \Sigma[m]$ is a real (J, ν)-holomorphic map as in (4.11) for some $\nu \in \mathcal{JV}^{\mathbb{R}}(\Sigma, V)$. Decompose its domain C into stable and unstable components. When the divisor V is non-empty, also decompose f into rubber components (collapsed to points in V under the projection $p : \Sigma[m] \to \Sigma$)

and non-rubber ones (level zero components), and consider the projection of f to Σ; see also (A.63)-(A.66).

By definition, the Ruan-Tian perturbations ν used here are pulled back from $\mathcal{U} \times \Sigma$ and ν must vanish along $\mathcal{U} \times V$. Therefore such perturbations identically vanish on both unstable domain components and on the projection to V of the rubber components, so these must be treated separately. To obtain transversality on the stable components, we use the fact that the restriction to these components defines an embedding into $\mathcal{U} \times \Sigma$. Transversality then follows by using variations either in ν (on the non-rubber components) or else in $\nabla\nu$ (on the rubber components) which have prescribed values at a suitable collection of points; we just need to ensure that the variations can be chosen so that they are tangent to the parameter space $\mathcal{JV}^{\mathbb{R}}(\Sigma, V)$, i.e. satisfy conditions (4.2) and (4.4).

Specifically, decompose f as in (A.64) into a non-rubber part $f_0 : C_0 \to \Sigma$ and rubber parts $f_r : C_r \to \mathbb{P}_V$ and consider the projection

$$\rho := p(f) : C \to \Sigma \tag{4.18}$$

of f. Then the restriction of ρ to C_0 is f_0 while its restriction ρ_r to C_r is a map to V. Moreover,

1) $\rho : C \to \Sigma$ is (J, ν)-holomorphic
2) the restriction $\rho_r : C_r \to V$ is a real map which satisfies the condition (4.16), cf. (A.67)-(A.70).

We will first consider the conditions on the non-rubber part and then on each rubber part, and show they are cut transversally.

Step 1. (Non-rubber components) Consider first the conditions on the non-rubber part of f, i.e. that $f_0 : C_0 \to \Sigma$ is (J, ν) holomorphic (here $C_0 \subseteq C$ may be nodal). As in the proof of [MS, Proposition 3.2.1], the surjectivity of the linearization $\delta\Psi_{f_0}$ fails only if we can find a nonzero $\eta \in (\mathcal{F}^{0,p}_{f_0})^\vee$ such that

$$\int_{C_0} \langle \mathcal{L}_{f_0}\xi, \eta\rangle = 0 \quad \text{and} \quad \int_{C_0} \langle \mu, \eta\rangle = 0 \tag{4.19}$$

for all $\xi \in \mathcal{E}^{1,p}_{f_0}$ and all variations μ in the parameter ν. But then $\eta \in \operatorname{coker} \mathcal{L}_{f_0}$, thus by elliptic regularity η is continuous on each component of C_0. We next show that the restriction of η must vanish on every component of C_0.

Step 1a. (Non-rubber, stable components) Consider first the restriction of f to a stable, non-rubber component. Assume that the restriction

of η to this component is nonzero. Then we can find a point x on it where $\eta(x) \neq 0$. By the continuity of η, we can assume that

(i) $x \in \mathcal{U}$ is not real and not special

(ii) $f(x)$ is not in V (because the image of a non-rubber component cannot lie entirely in V).

Regard $\eta(x)$ as an element of $\mathrm{Hom}^{01}(T_xC, T_{f(x)}\Sigma)$. Then as in the standard transversality proof [MS, Proposition 3.2.1] (but after symmetrizing), we can find a symmetric variation μ in ν supported sufficiently close to the image of x (and its conjugate) such that the values of μ and η agree at x (and therefore also at $c(x)$), i.e.

$$\mu|_{(x,f(x))} = \eta(x). \tag{4.20}$$

By construction, such variation μ satisfies condition (4.2). Since $f(x) \in X \setminus V$, it also satisfies (4.4) whenever μ is supported sufficiently close to $(x, f(x))$ and its conjugate. Furthermore, if β_ε is a symmetric bump function supported near $(x, f(x))$ and its conjugate, then $\beta_\varepsilon\mu$ is also a variation in ν satisfying (4.20) and whose support is arbitrarily close to x (and its conjugate). But η is continuous and $\eta(x) \neq 0$, thus for sufficiently small ε,

$$\int_{C_0} \langle \beta_\varepsilon\mu, \eta\rangle \neq 0$$

contradicting (4.19). Therefore η vanishes on all the stable, non-rubber components.

Step 1b. (Non-rubber, unstable components) Consider next the restriction of $f : C \to \Sigma[m]$ to the union $C_{0u} \subseteq C$ of all the non-rubber but unstable domain components. This is a J-holomorphic map $f_{0u} : C_{0u} \to \Sigma$ since the perturbation ν vanishes on these components. Consider the restriction f_i of f to a connected component C_i of C_{0u} (here $C_i \subseteq C$ may be nodal). Then the preimage of $V \subset \Sigma$ consists of finitely many points $\mathbf{y}_i$ of C_i, all of them special points of C. Moreover, f_i is a stable map (since f is relatively stable), thus has positive degree. Assume for simplicity that Σ is connected. Then either (i) C_i has genus 0, and $\mathbf{y}_i$ and V have at most 2 points or else (ii) C_i has genus 1 and $V = \emptyset$; see also Remark 4.4.

Case (i). This situation cannot occur when the target Σ satisfies $g(\Sigma) \geq 1$ since such Σ is aspherical. So the only possibility left is when $V \subset \Sigma = S^2$ consists of at most one pair of conjugate points. Then the restriction of η is in the cokernel of a holomorphic $\overline{\partial}$-operator on this component C_i (since

$\nabla\nu$ vanishes on unstable domain components). But C_i is a genus zero curve thus $H^1(C_i, f_i^*\mathcal{T}_\Sigma)$ vanishes by Serre duality since $c_1(\mathcal{T}_\Sigma) = \chi(\Sigma \setminus V) \geq 0$ in this case.

Case (ii). This situation cannot occur when $g(\Sigma) \geq 2$, since there are no positive degree holomorphic maps from a genus 1 curve to a higher genus curve. Since we assumed that Σ has no genus 0 components without marked points, the only possibility left is that $g(\Sigma) = 1$ and f_i is an unramified cover of a torus by a torus. But then the cokernel of the restriction of the linearization (4.14) to C_i also vanishes.

Therefore η also vanishes on all the unstable, non-rubber components.

Thus $\eta = 0$ on all non-rubber components. This implies that $\delta\Psi_{f_0}$ is surjective, which means that the conditions on the non-rubber part f_0 of f are cut transversally.

Step 2. (Rubber components) Finally, consider the conditions on rubber parts of f described in the paragraph containing (4.18), which we also want to show are cut transversally.

Since $V = V^+ \sqcup c(V^+)$ cf. (2.5), the rubber parts of f come in conjugate pairs, so it suffices to restrict attention to each connected component C_i^+ which is mapped to V^+ under the projection $\Sigma[m] \to \Sigma$.

Consider the restriction of f to such connected component C_i^+. Here $C_i^+ \subset C$ may be nodal, but note that it is either a trivial rubber component or else it is stable cf. Remark 3.3. Either way, the pullback of ν identically vanishes along this component since it projects to a point in V. The restriction ρ_i of $\rho = p(f)$ to C_i^+ is a constant map to V^+ which satisfies the additional condition

$$\ker L^N_{\rho_i} \neq 0. \tag{4.21}$$

Here $L^N_{\rho_i} = \overline{\partial} - \nabla\nu$ is a holomorphic $\overline{\partial}$-operator on the trivial complex line bundle $\rho_i^* N_V$ over C_i^+; its complex index is equal to $1 - g(C_i^+)$ cf. (A.69). In particular, (4.21) is automatic on the genus zero components C_i^+, so we may assume that C_i^+ is stable, has genus at least 1, and $\operatorname{ind} L^N_{\rho_i} \leq 0$.

To check that the subset of such pairs (ρ, ν) which satisfy the condition (4.21) is cut transversally, we similarly adapt the proof of [IP1, Lemma 6.4] to the real setting; see also the proof of [IP3, Proposition 5.3]. Regard the normal operator as a section $(\rho, \nu) \mapsto L^N_\rho$ of a bundle $\mathrm{Fred} \to \mathcal{B}$ of *complex* Fredholm operators. Here $\mathcal{B}$ is the space of pairs (ρ, ν) where $\nu \in \mathcal{JV}^{\mathbb{R}}(\Sigma, V)$, and $\rho : C_i^+ \to V^+$ is a constant map on $C_i^+ \subseteq C$. Since C_i^+ is connected, then by the last sentence of Remark A.12 it suffices to prove that this section is

transverse to

$$\mathrm{Fred}^1 = \{\mathrm{D} \in \mathrm{Fred} \mid \dim_{\mathbb{C}} \ker \mathrm{D} = 1\}.$$

The fiber at D of the normal bundle to Fred^1 is $\mathrm{Hom}(\ker D; \mathrm{coker}\, D)$.

Let $0 \neq \kappa \in \ker L^N$ and $\eta \neq 0$ an element of the cokernel of L^N. By unique continuation, we can find a point x on the domain so that

(i) both $\kappa(x)$ and $\eta(x)$ are nonzero

(ii) $x \in \mathcal{U}$ is not real and not special, but of course x must be mapped by ρ to V.

It therefore suffices to construct a variation μ in ν such that

$$\int_{C_i^+} \langle (\delta_\mu L^N)(\kappa),\ \eta\rangle = -\int_{C_i^+} \langle \nabla_\kappa \mu,\ \eta\rangle \neq 0. \tag{4.22}$$

Here $\delta_\mu L^N = -\nabla\mu$ is the variation in the normal operator as we vary ν but fix the map $\rho : C_i^+ \to V$. Since C_i^+ is stable, then as in the proof of [IP1, Lemma 6.4], after symmetrizing, we can find a variation μ in ν compatible with V, supported near the image of x (and its conjugate) such that the values of $\nabla_\kappa \mu$ and η agree at x (and therefore also at its conjugate), i.e.

$$(\nabla_\kappa \mu)|_{(x,p(f)(x))} = \eta(x). \tag{4.23}$$

This variation μ then satisfies both conditions (4.2) and (4.4). The result then follows as before: multiplying μ by (symmetric) bump functions β_ε supported around the image of x (and its conjugate) in $\mathcal{U} \times \Sigma$ gives a sequence of variations $\beta_\varepsilon \mu$, still satisfying (4.2), (4.4) and (4.23), but whose support is arbitrarily close to x (and its conjugate). This shows that (4.22) is satisfied. □

Remark 4.6. The virtual dimension of a stratum is the index of the strata-wise linearization. For the real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ or more generally for (4.10), the virtual (co)dimensions of its strata can be calculated as in [IP1, §7], see also Remark A.8. In particular, a stratum consisting of maps $f : C \to \Sigma$ (i.e. without rubber components) has virtual codimension l, where l is the number of (ordinary) nodes of the domain. A stratum consisting of maps $f : C \to \Sigma[m]$ has virtual codimension (at least) $l + 2|m|$.

Remark 4.7. In the proof above, the assumption that every rational component of Σ has at least one marked point is only used in Step 1B case (ii).

When $V = \emptyset$ and Σ has genus 0 components, there are a few strata of the (absolute) moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi}(\Sigma, \emptyset)$ in (4.10) which are not cut transversally. These strata consist of stable maps $f : C \to \Sigma$ whose domain has a connected component C_i which is a nodal genus 1 curve without any marked points. The restriction of f to C_i is holomorphic since RT perturbations vanish on such components. If C_i contains a torus on which f is constant, then the cokernel of the linearization (4.14) on C_i is 1-dimensional. All the other strata are cut transversally.

4.3. Transversality for a family of targets

Consider next a family $\mathcal{F} = \cup_s \Sigma_s$ of targets as in (3.1), and let $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ be the space of real RT perturbations defined in (4.7). For the rest of the paper, we fix the topological data $d, \chi, \vec{\mu}$ and denote by

$$\overline{\mathcal{M}}(\Sigma_s) = \bigcup_{\nu} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\mu}(\Sigma_s, V_s)_\nu \to \mathcal{P} \tag{4.24}$$

the family of real relative moduli spaces defined as in (4.10) for $\Sigma = \Sigma_s$ but for parameters $\nu \in \mathcal{P}$. Note that the fiber over $\nu \in \mathcal{P}$ depends only on the pullback ν_s to Σ_s, cf. (4.8). We also fix $s_0 \neq 0$ in Δ and denote by I the segment $[0, s_0] \subset \Delta$.

The families in (3.14) and (3.13) similarly extend to families

$$\overline{\mathcal{M}}(\mathcal{F}_{/I}) = \bigcup_{s \in I} \overline{\mathcal{M}}(\Sigma_s) \quad \text{and} \quad \overline{\mathcal{M}}(\widetilde{\Sigma}) = \bigsqcup_{\lambda \vdash d} \overline{\mathcal{M}}_\lambda(\widetilde{\Sigma}), \tag{4.25}$$

over $\mathcal{P}$. Here $\widetilde{\Sigma}$ is the normalization of the nodal fiber Σ_0 and the fiber of $\overline{\mathcal{M}}_\lambda(\widetilde{\Sigma})$ at $\nu \in \mathcal{P}$ is by definition

$$\overline{\mathcal{M}}_\lambda(\widetilde{\Sigma})_\nu = \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})_\nu. \tag{4.26}$$

Remark 4.8. Recall that elements f in $\overline{\mathcal{M}}(\Sigma_0)$ have the property that the preimage of the nodes of the target is a subset of the nodes of the domain, cf. §3.5 and Definition 3.1(ii). When f has no rubber components, let $\widetilde{C}$ be the (partial) normalization of its domain at these contact nodes, and let $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ be the lift of f to the normalizations. If $f : C_0 \to \Sigma_0$ is a real map satisfying the conditions (i)-(v) in Definition 3.1, then its lift $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ is also a real map satisfying essentially the same conditions, the only difference being that each contact node y has been replaced by two points y_1, y_2 at which $\widetilde{f}$ has the same contact multiplicity $\lambda(y)$. Thus *locally* there is no difference between the conditions describing a stratum of $\overline{\mathcal{M}}(\Sigma_0)$

near $f : C_0 \to \Sigma_0$ and a stratum of $\overline{\mathcal{M}}(\widetilde{\Sigma})$ near $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$. The same is true when f has rubber components.

Below we assume that Σ_0 has no connected component which has genus 0 but has no marked points on it. Then the normalization $\widetilde{\Sigma}$ of Σ_0 and the smooth deformations Σ_s of Σ_0 also have this property.

The proof of Lemma 4.5 extends essentially verbatim, after replacing Σ by the family $\mathcal{F}$, to give the following result.

Lemma 4.9. *Assume that every rational connected component of Σ_0 has at least one marked point. Then over the space $\mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ of real RT perturbations on the family, every stratum of the following families of real relative moduli spaces*

$$\bigcup_{s\in I} \overline{\mathcal{M}}(\Sigma_s), \quad \overline{\mathcal{M}}(\Sigma_{s_0}), \quad \overline{\mathcal{M}}(\widetilde{\Sigma}), \quad \text{and} \quad \overline{\mathcal{M}}(\Sigma_0) \tag{4.27}$$

are cut transversally.

Proof. A stratum of each one of these families consists of maps f whose domain and target are symmetric surfaces with fixed topological type. It suffices to check that the variations in ν constructed in the proof of Lemma 4.5 can be chosen tangent to the space of perturbations $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ on the family. This means they can be defined on $\mathcal{U} \times \mathcal{F}$ and chosen so that they satisfy (a) $\pi_*\nu = 0$, (b) condition (4.2) for $X = \mathcal{F}$, and (c) condition (4.4) for $X = \mathcal{F}$ and $V \subset X$ the union of the marked divisor of $\mathcal{F}$ and the nodes of Σ_0.

Let f be an element of any one of the moduli spaces (4.27). Its domain is a real (possibly nodal) marked curve C and its target is a building $\Sigma[m]$ as in (4.11), where Σ is either some fiber Σ_s of $\mathcal{F}$ for $s \in I$ or else is the normalization $\widetilde{\Sigma}$ of the nodal fiber Σ_0. In turn, both Σ_s and $\widetilde{\Sigma}$ map to $\mathcal{F}$ cf. (3.4). In particular, f descends to a map $p(f)$ to $\mathcal{F}$, such that its rubber components are mapped to the marked points of Σ_s or else to the marked points and nodes of Σ_0.

Moreover, locally every stratum of $\overline{\mathcal{M}}(\Sigma_0)$ can be described by the same local defining equations as those of a stratum of $\overline{\mathcal{M}}(\widetilde{\Sigma})$ cf. Remark 4.8.

Thus it suffices to show that the strata of the moduli spaces $\overline{\mathcal{M}}(\widetilde{\Sigma})$ and $\overline{\mathcal{M}}(\Sigma_{s\neq 0})$ are cut transversally over the parameter space $\mathcal{P}$; for these moduli spaces, the rest of the proof proceeds as in that of Lemma 4.5.

We again separately consider the condition that $p(f) : C \to \Sigma_s \subset \mathcal{F}$ be (J, ν)-holomorphic, and the additional condition $\ker L^N \neq 0$ on the rubber

components. Unstable domain components continue to be holomorphic, and must be treated separately exactly as in the proof of Lemma 4.5.

Step 1 (Non-rubber components). Here it suffices to construct a variation μ tangent to $\mathcal{P}$ satisfying (4.20) at a point x on the domain such that

(i) $x \in \mathcal{U}$ is not real and not special

(ii) the image $f(x) \in \Sigma_s \subset \mathcal{F}$ is not a marked point nor a node.

For that, we use the fact that both $\mathcal{U}$ and $\mathcal{F}$ are locally trivial fibrations in a sufficiently small neighborhood of such a point $(x, f(x))$ (and its conjugate), thus the variation μ constructed on a single fiber of $\mathcal{F}$ as in Lemma 4.5 extends to the family.

Step 2 (Rubber components) For each rubber component C' of the domain, denote by ρ the restriction of $p(f)$ to C'. Then ρ maps C' to a single point of $\mathcal{F}$; denote it $q \in \Sigma_s \subset \mathcal{F}$. Note that q must be a special point of Σ_s. We want to show that the condition on (ρ, ν) that $\ker L^N \neq 0$ is cut transversally. Here we separate the case when

(a) q is a marked point of the family; then L^N_ρ is an operator on $\rho^* N_q = \rho^* T_q \Sigma_s$.

(b) q is a node of Σ_0; then q has two lifts q_1, q_2 to the normalization. For each one, we get a normal operator $L^N_{\rho_i}$ on $\rho^* N_{q_i} = \rho^* T_{q_i} \widetilde{\Sigma}$. However, the condition $\ker L^N_{\rho_i} \neq 0$ is independent of the choice of lift, cf. Remark 4.3.

Step 2a. Assume q is a marked point of Σ_s. As in the paragraph containing (4.22), it suffices to ensure that there exists a variation μ tangent to $\mathcal{P}$ satisfying (4.23) at a non-real, non-special point $x \in C'$ where $\kappa(x) \neq 0$ and $\eta(x) \neq 0$. But at a marked point q of the family $\mathcal{F}$, π is a local fibration in a sufficiently small neighborhood of the marked point, thus the variation μ constructed on a single fiber of $\mathcal{F}$ as in Lemma 4.5 extends to the family.

Step 2b. Assume q is a node of Σ_0, and chose a lift $q_1 \in \widetilde{\Sigma}$ to the normalization. As in the proof of Lemma 4.5, we have $\kappa \in \ker L^N$ and $\eta \in \operatorname{coker} L^N$ such that both $\kappa(x)$ and $\eta(x)$ are nonzero, cf. (i)-(ii) above (4.22). Here x is a fixed nonspecial, nonreal point of C, $\kappa(x) \in T_{q_1}\widetilde{\Sigma} \hookrightarrow T_q \mathcal{F}$ while $\eta(x) \in \operatorname{Hom}^{01}(T_x C, T_{q_1}\widetilde{\Sigma})$. It suffices to check that we can construct a variation μ tangent to $\mathcal{P}$ such that (4.23) holds, i.e. $(\nabla_\kappa \mu)|_{(x,q)} = \eta(x)$.

Using the local model (B.4) of $\mathcal{F}$ around the node q, we can construct a variation μ defined in a neighborhood of the point (x, q) in $\mathcal{U} \times \mathcal{F}$ as

follows. Identify a neighborhood of q in $\mathcal{F}$ with a neighborhood of 0 in $\mathbb{C}^2$ as in the paragraph containing (B.4). Let $\mu \in \mathrm{Hom}^{01}(\mathcal{T}_{\mathcal{U}}, T\mathcal{F})$ be given by $\mu_{(w,z)} = (z_1\alpha_1(w), z_2\alpha_2(w))$ for all $w \in \mathcal{U}$ close to x and $z = (z_1, z_2) \in \mathbb{C}^2$ small. Here $\alpha_i \in \mathrm{Hom}^{01}(\mathcal{T}_{\mathcal{U}}, \mathbb{C})$ and $\alpha_1 + \alpha_2 = 0$. This ensures that μ satisfies conditions (a)-(c) below (4.7); see also (4.9). Moreover, since $\kappa(x), \eta(x) \neq 0$ we can arrange that $\alpha_1(x) \otimes \kappa(x) = \eta(x)$ at the point x, thus $(\nabla_\kappa \mu)|_{(x,q)} = \eta(x)$. Multiplying by a bump function and symmetrizing gives the desired variation tangent to $\mathcal{P}$. □

Combining the Sard-Smale Theorem with Lemma 4.9 gives the following result.

Corollary 4.10. *There exists a Baire subset $\mathcal{P}^* \subset \mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ such that for all $\nu \in \mathcal{P}^*$, all the strata of the fibers over ν of all the moduli spaces* (4.27) *are cut transversally, thus are smooth manifolds of the expected dimension.*

By a generic parameter we mean an element of a Baire subset of the parameter space (a Baire subset is a countable intersection of open and dense sets).

5. VFCs for real relative moduli spaces

In this section we use the transversality results in §4.2 and §4.3 to show that the moduli spaces considered there are thin compactifications (as briefly reviewed in §2.3) or more generally thinly compactified families in the sense of [IP5, §3]. This implies that the moduli spaces carry a virtual fundamental class (VFC) in rational Cech homology, cf. [IP5, §4].

5.1. Thinly compactified families and their VFC

First we recall the general results of [IP5, §4]. Roughly speaking, a proper map

$$\overline{m} \to \mathcal{P} \tag{5.1}$$

is a thinly compactified family over $\mathcal{P}$ provided (i) for generic $\nu \in \mathcal{P}$ the fiber $\overline{m}_\nu$ is a thin compactification, i.e. an oriented topological manifold away from a (homologically) codimension 2 strata and (ii) over a generic path γ in $\mathcal{P}$, $\overline{m}_\gamma$ is a thinly compactified cobordism, i.e. an oriented cobordism

away from a (homologically) codimension 2 strata. See [IP5, Defn 2.1 and 3.1] and [IP5, §2.4] for the precise details.

Since the fundamental class (in rational Cech homology) of a manifold uniquely extends to any thin compactification by [IP5, Thm 2.4], condition (i) above defines the fundamental class of the moduli space for generic parameter ν. For non-generic parameters, the virtual fundamental class (VFC) is obtained as the limit (in rational Cech homology) of nearby classes, which is well defined by condition (ii). In particular, as in [IP5, Thm 4.2] any thinly compactified family $\overline{\mathcal{M}} \to \mathcal{P}$ carries a unique VFC[4]

$$[\overline{\mathcal{M}}_\nu]^{\mathrm{vir}} \in \check{\mathrm{H}}_*(\overline{\mathcal{M}}_\nu; \mathbb{Q}) \tag{5.2}$$

defined for all $\nu \in \mathcal{P}$, and satisfying a consistency condition over all paths γ in $\mathcal{P}$, cf. [IP5, Defn 4.1].

Finally, thin compactifications behave well under passing to covers, cf. [IP5, §2.1] or enlarging the parameter spaces cf. [IP5, §6].

5.2. The VFC for real relative moduli spaces

We next apply these consideration to the families of moduli spaces considered in section §4. Before we proceed, note that being a finite dimensional manifold is a topological condition, which can be verified by providing a local model; on the other hand, its orientability and a choice of orientation are global questions. This section summarizes why the transversality results proved so far suffice to address the first question for the generic fiber of these moduli spaces (away from virtual codimension 2 strata), and to identify their (actual) orientation sheaf with a certain determinant line bundle (away from virtual codimension 2 strata). Orientations are deferred to §8.

We start with the case of a fixed, smooth target, and then move to a family of targets. Let Σ be a (smooth) symmetric curve with r pairs of conjugate marked points V, and assume that Σ has no rational components without any marked points. For every d, χ, $\vec{\mu}$ fixed, consider the family

$$\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \longrightarrow \mathcal{JV}^{\mathbb{R}}(\Sigma, V) \tag{5.3}$$

of moduli spaces defined by (4.10). Its virtual orientation line at f is the determinant of the linearization. Recall that when $f : C \to \Sigma$ (i.e. f has no rubber components) the linearization L_f is given by (4.12) and $\det L_f$ is

[4]called the relative fundamental class in [IP5].

defined using its Fredholm completion (A.44). However, it is more convenient here to use an equivalent description of the linearization, as an operator of the form

$$\begin{aligned} &D_f : \Gamma(f^*\mathcal{T})^{\mathbb{R}} \oplus T_C\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell} \to \Lambda^{01}(f^*\mathcal{T})^{\mathbb{R}} \\ &D_f(\xi, h) = \overline{\partial}_{f^*(\mathcal{T},c_\mathcal{T})}\xi + A_f(\xi) + b_f(h) \end{aligned} \tag{5.4}$$

cf. (A.47). Here $\mathcal{T} = \mathcal{T}_\Sigma$ is the relative tangent bundle (2.8) of the marked curve Σ. The first term of D_f is the pullback of the $\overline{\partial}_{(\mathcal{T},c_\mathcal{T})}$ operator on the target, while A_f and b_f are 0'th order terms. By construction, the image of both A_f and b_f are (smooth) $(0,1)$-forms supported away from a neighborhood of the special points of the domain. Operators of the form (5.4) are completed to Fredholm operators as in (A.1)-(A.2); we refer the reader to Lemma A.9 and rest of §A.6 for the identification between the two linearizations – L_f in (4.12) and D_f in (5.4) – as well as between the kernels and cokernels of their Fredholm completions.

Up to 0'th order terms, the operator D_f in (5.4) is the same as $\overline{\partial}_{f^*(\mathcal{T},c_\mathcal{T})} \oplus 0_W$, where $W = T_C\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell}$ parametrizes the variations in the domain of f and $0_W : W \to 0$. In particular, the virtual dimension of the moduli space is the index of D_f, which is equal to (2.7). Moreover, since the virtual relative orientation sheaf $\mathfrak{o}_\mathcal{M}$ of the family (5.3) is the determinant line bundle of the family of linearizations D_f, we get canonical (up to homotopy) identifications

$$\mathfrak{o}_\mathcal{M} \cong \det D \cong \det(\overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \oplus 0) = \det \overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}, \tag{5.5}$$

cf. [GI, (A.13)]. Here $\det \overline{\partial}_{(\mathcal{T},c_\mathcal{T})}$ denotes the determinant of the family of pullback operators $\overline{\partial}_{f^*(\mathcal{T},c_\mathcal{T})}$ as f varies in $\mathcal{M}$, and

$$\mathfrak{f} : \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V) \to \overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}$$

denotes the forgetful morphism to the real Deligne-Mumford moduli space parametrizing the domains.

The following theorem defines the VFC of the real relative moduli spaces.

Theorem 5.1. *The family of real relative moduli spaces* (5.3) *is a thinly compactified family. It therefore carries a VFC*

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)_\nu]^{\mathrm{vir}} \in \check{\mathrm{H}}_b(\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)_\nu; \mathbb{Q}) \tag{5.6}$$

for all $\nu \in \mathcal{JV}^{\mathbb{R}}(\Sigma, V)$*, including* $\nu = 0$*, with* b *given by* (2.7)*. The VFC depends on the choice of (twisted) orientation data on the relative tangent bundle* $\mathcal{T}$ *of* Σ *as described in §8.*

Proof. Fix d, χ and $\vec{\mu}$. For simplicity, denote the family (5.3) by $\overline{\mathcal{M}} \to \mathcal{P}$ as in (5.1) and by $\overline{\mathcal{M}}_\nu$ its fiber over $\nu \in \mathcal{P}$. Consider the open subset

$$\widetilde{\mathcal{M}} \subseteq \overline{\mathcal{M}} \tag{5.7}$$

consisting of the union of strata of virtual codimension at most one. The elements of these strata are maps $f : C \to \Sigma$ whose domains have at most 1 real node and this node must be an ordinary real node, i.e. its image is away from V; in particular these elements cannot have any rubber components and they are all maps into a fixed, smooth target Σ. All the other strata have virtual codimension at least 2 by the formula in Remark 4.6.

By Lemma 4.5 and Sard-Smale Theorem, for generic parameter $\nu \in \mathcal{P}$, all the strata of the real relative moduli space are cut transversally; in particular, the cokernel of the linearization along the top stratum and along the codimension one stratum of $\widetilde{\mathcal{M}}_\nu$ vanishes. Furthermore all the maps in $\widetilde{\mathcal{M}}_\nu$ have the same fixed, smooth target $X = \Sigma$ and at most one node, which is an ordinary node. Therefore the proof of the usual gluing theorem in [P, §B] or [LT] applies verbatim to the moduli space $\widetilde{\mathcal{M}}_\nu$ after making the following inconsequential changes:

(a) working throughout with symmetric choices (of metrics, bump functions etc) as in [GZ1].

(b) the linearization is completed in norms which are weighted around the ordinary node as in [P, §B.4], and have different weights around the contact points. But these contact points are marked points of the domain so they stay *away* from the node where the gluing is performed.

Therefore as in the usual gluing theorem, the generic fiber $\widetilde{\mathcal{M}}_\nu$ of (5.7) is a b-dimensional topological manifold (without boundary), locally modeled on the kernel of the linearization, cf. [P, Theorem B.1.1]. Moreover, the same arguments show that the determinant of the family of linearizations is locally trivial (thus forms a line bundle) as $f : C \to \Sigma$ varies in $\widetilde{\mathcal{M}}$, exactly as in [GZ1, (3.3)] but with $E = TX$ replaced in our case by $E = \mathcal{T}_\Sigma$.

Let $\det D$ be the determinant line bundle of the family of linearizations D_f as f varies in $\widetilde{\mathcal{M}}$, cf. (5.4). As in [GZ1], there is a canonical homotopy

class of isomorphisms

$$\det D \cong \det(\overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \oplus 0_W) = \det \overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \tag{5.8}$$

as $f : C \to \Sigma$ varies over $\widetilde{m}$, even as the domains become nodal. This identifies the virtual (relative) orientation sheaf $\mathfrak{o}_{\widetilde{m}}$ of $\widetilde{m}$ with

$$\mathfrak{o}_{\widetilde{m}} \cong \det D \cong \det \overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}. \tag{5.9}$$

In §8 below we recall how a choice of (twisted) orientation data trivializes (5.9). Note that a trivialization of $\det D$ orients the generic fibers $\widetilde{m}_\nu$ of the family $\widetilde{m} \to \mathcal{P}$.

Consider next the "singular locus"

$$\mathcal{S} = \overline{m} \setminus \widetilde{m}. \tag{5.10}$$

By Lemma 4.5 and the Sard-Smale theorem, for generic parameter $\nu \in \mathcal{P}$, its fiber $\mathcal{S}_\nu = \overline{m}_\nu \setminus \widetilde{m}_\nu$ is stratified by smooth manifolds of the expected dimension, which is at most $\dim \widetilde{m}_\nu - 2$ as noted above. Then by [IP5, Lemma 2.2] the singular locus $\mathcal{S}_\nu$ is homologically codimension 2. Thus for generic ν, $\overline{m}_\nu$ is a thin compactification of the oriented manifold $\widetilde{m}_\nu$ and therefore carries a fundamental class $[\overline{m}_\nu]$.

Similarly, the moduli space $\overline{m}_\gamma$ over a generic path γ is a thinly compactified cobordism in the sense of [IP5, §2.4]. This means that the corresponding subset $\widetilde{m}_\gamma$ is an oriented topological manifold with boundary $\widetilde{m}_{\partial\gamma}$, and its complement in $\overline{m}_\gamma$ is stratified by codimension 2 manifolds, as is the complement of $\widetilde{m}_{\partial\gamma}$ in $\overline{m}_{\partial\gamma}$.

Therefore by [IP5, Lemma 5.4], $\overline{m} \to \mathcal{P}$ is a thinly compactified family in the sense of [IP5, Definition 3.2]. Consequently, by [IP5, Theorem 4.2], it carries a VFC as in (5.2) for all $\nu \in \mathcal{P}$, including $\nu = 0$. The VFC is defined by $[\overline{m}_\nu]$ for generic parameter ν, and extended uniquely to all parameters using the continuity property of rational Cech holomogy. □

Remark 5.2. In particular, the VFC (2.11) of the (unperturbed) real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ can be defined by turning on RT perturbations ν in $\mathcal{JV}^{\mathbb{R}}(\Sigma, V)$ to obtain transversality strata-wise and then turning them off by taking $\nu \to 0$.

We next show that the proof of Theorem 5.1 extends to the moduli spaces

$$\overline{m}(\Sigma_{s\neq 0}), \quad \overline{m}(\widetilde{\Sigma}) = \bigsqcup_{\lambda\vdash d} \overline{m}_\lambda(\widetilde{\Sigma}), \quad \text{and} \quad \overline{m}(\Sigma_0) \tag{5.11}$$

described in the paragraph containing (4.24), for fixed $d, \chi, \vec{\mu}$ as in §4.3. As in §4.3, we continue to assume that the nodal curve Σ_0 has no real special points, and no rational connected components without any marked points. Note that (5.11) are families of real relative moduli spaces over the parameter space $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$.

Theorem 5.3. *Assume Σ_0 has no rational connected components without any marked points. Then the moduli spaces* (5.11) *are thinly compactified families over $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$. In particular, each one carries a VFC*

$$[\overline{m}(-)_\nu]^{\mathrm{vir}} \in \check{\mathrm{H}}_b(\overline{m}(-)_\nu;\, \mathbb{Q})$$

for all $\nu \in \mathcal{P}$. The VFC depends on the choice of (twisted) orientation data $\mathfrak{o}_{\mathcal{F}}$ on the relative tangent bundle $\mathcal{T} \to \mathcal{F}$ to the family $\mathcal{F}$; the virtual dimension b is given by (2.7).

Proof. The strata of the moduli spaces (5.11) are cut transversally by Lemma 4.9 and every stratum with at least one rubber component has codimension at least 2, cf. Remark 4.6. Denote by

$$\widetilde{m}(\Sigma_{s\neq 0}), \quad \widetilde{m}(\widetilde{\Sigma}) = \bigsqcup_{\lambda} \widetilde{m}_\lambda(\widetilde{\Sigma}), \quad \text{and} \quad \widetilde{m}(\Sigma_0) \tag{5.12}$$

the union of their codimension at most 1 strata. Note that besides the nodes in the preimage of the nodes of the target, the domains of maps in $\widetilde{m}(\Sigma_0)$ can have at most one additional node, which must be a real ordinary node. Moreover, these maps have no rubber components.

Recall that by definition the fibers of the families $\overline{m}(\Sigma_s)$ and $\overline{m}_\lambda(\widetilde{\Sigma})$ at $\nu \in \mathcal{P}$ are the real relative moduli spaces $\overline{m}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)_\nu$ and respectively $\overline{m}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})_\nu$, and that Σ_s and $\widetilde{\Sigma}$ are smooth for $s \neq 0$. These moduli spaces have the same virtual dimension

$$b = d\chi(\Sigma_s \setminus V_s) - \chi + 2\ell(\vec{\mu}) = d\chi(\widetilde{\Sigma} \setminus \widetilde{V}) - (\chi + 4\ell(\lambda)) + 2\ell(\vec{\mu}) + 4\ell(\lambda)$$

cf. (2.7); see also (A.46). Moreover, the relative orientation sheaf of the family $\widetilde{m}(\Sigma_{s\neq 0})$ and respectively $\widetilde{m}_\lambda(\widetilde{\Sigma})$ over the parameter space $\mathcal{P}$ continues

to be given by the corresponding formula (5.9) and is similarly orientable, as reviewed in §8 below.

Case 1. When $\Sigma = \Sigma_{s\neq 0}$ or $\widetilde{\Sigma}$, the rest of the proof of Theorem 5.1 applies verbatim to $\overline{m}(\Sigma) \to \mathcal{P}$.

Case 2. When $\Sigma = \Sigma_0$, recall that every stratum of $\overline{m}(\Sigma_0)$ can be described by the same local defining equations as those of a stratum of $\overline{m}(\widetilde{\Sigma})$, cf. Remark 4.8. In particular, since for generic ν every stratum of $\overline{m}(\widetilde{\Sigma})_\nu$ is a manifold, then so is every stratum of $\overline{m}(\Sigma_0)_\nu$; the same is true over a generic path.

Consider next $\widetilde{m}(\Sigma_0)$, the subset consisting of maps $f_0 : C_0 \to \Sigma_0$ whose domain has at most one *ordinary* node, in addition to the contact nodes (recall that ordinary nodes are mapped away from the special points of Σ_0, while contact nodes are the preimage of the nodes of Σ_0). At the beginning of §6.1 below we show that a finite (global) cover of $\widetilde{m}(\Sigma_0)$ (obtained by ordering the contact nodes) can be identified with $\widetilde{m}(\widetilde{\Sigma})$. Since the generic fiber of $\widetilde{m}(\widetilde{\Sigma})$ is a finite dimensional manifold, then so is the generic fiber of $\widetilde{m}(\Sigma_0)$; the same is true over a generic path.

The relative orientation sheaf of $\widetilde{m}(\Sigma_0)$ is analyzed in §6 below, and is orientable as well, cf. paragraph above Lemma 6.1. The rest of the proof proceeds as in that of Theorem 5.1. □

The same proof also provides the following specific result about the generic fibers of these families. Let $\mathcal{P}^*$ be the Baire subset of $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ appearing in Corollary 4.10.

Proposition 5.4. *Assume Σ_0 has no rational connected components without any marked points. Then for all $\nu \in \mathcal{P}^*$, we have the following properties:*

(i) the spaces

$$\bigcup_{s\in(0,s_0)} \widetilde{m}(\Sigma_s)_\nu, \quad \widetilde{m}(\Sigma_{s_0})_\nu, \quad \widetilde{m}(\Sigma_0)_\nu, \quad \text{and} \quad \widetilde{m}(\widetilde{\Sigma})_\nu \tag{5.13}$$

are orientable topological manifolds;

(ii) $\overline{m}(\Sigma)_\nu$ is a thin compactification of $\widetilde{m}(\Sigma)_\nu$ for Σ equal to Σ_{s_0}, Σ_0, and respectively $\widetilde{\Sigma}$;

(iii) $\bigcup_{s\in(0,s_0]} \widetilde{m}(\Sigma_s)_\nu$ is a topological manifold with boundary $\widetilde{m}(\Sigma_{s_0})_\nu$;

(iv) the complement of $\bigcup_{s\in[0,s_0]} \widetilde{m}(\Sigma_s)_\nu$ in $\bigcup_{s\in[0,s_0]} \overline{m}(\Sigma_s)_\nu$ is homologically codimension 2.

In general, $\bigcup_{s\in[0,s_0]} \widetilde{\overline{m}}(\Sigma_s)_\nu$ is branched over $s=0$, but a cover of it is a topological cobordism cf. §7.

A priori, the VFC as defined in [IP5] may depend on the parameter space of perturbations used to achieve transversality. Using the results of [IP5, §6] we end this section by comparing the VFCs of the (unperturbed) real moduli spaces $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ constructed using perturbations defined on $\Sigma = \Sigma_{s\neq 0}$ or $\widetilde{\Sigma}$ with those constructed using perturbations defined on the corresponding family $\mathcal{F}$.

Lemma 5.5. *For the first two families in* (5.11)*, the VFC associated by Theorem 5.3 agrees under* (4.8) *with the VFC associated by Theorem 5.1.*

Proof. Assume for simplicity Σ is equal to $\Sigma_{s\neq 0}$ (the proof for $\widetilde{\Sigma}$ is similar). Denote by $\mathcal{P} \to \mathcal{P}'$ the map $\mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta}) \to \mathcal{JV}^{\mathbb{R}}(\Sigma_s, V_s)$ defined as in (4.8). We can then consider the family of moduli spaces

$$\bigcup_\nu \overline{m}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)_\nu$$

as ν varies over either $\mathcal{P}$ or $\mathcal{P}'$; denote these two families by $\overline{m}$ and respectively $\overline{m}'$. By [IP5, Lemma 6.2] it suffices to check that the conditions in [IP5, Definition 6.1] are satisfied. But the fiber $\overline{m}_\nu$ of $\overline{m} \to \mathcal{P}$ is the same as the fiber of $\overline{m}' \to \mathcal{P}'$ over the image of ν in $\mathcal{P}'$; moreover the former fiber is cut transverally strata-wise if and only if the latter one is (because the linearization is the same for both problems). Therefore the Baire subsets of regular parameters (where the strata are cut transversally) satisfy the required conditions of [IP5, Definition 6.1], completing the proof. □

6. Moduli spaces for nodal targets

In this section we describe in more detail the family of real relative moduli spaces associated to the nodal curve Σ_0 and analyze its orientation sheaf. Recall that Σ_0 is a nodal curve with a single pair $x^\pm$ of conjugate nodes and r pairs of conjugate marked points, and $\widetilde{\Sigma}$ is its normalization (3.3). We fix the topological data $d, \chi, \vec{\mu}$ and consider the moduli spaces $\overline{m}(\Sigma_0)$ and $\overline{m}(\widetilde{\Sigma})$ introduced in (4.24)-(4.27). We continue working over the parameter space $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ of RT perturbations on the family $\mathcal{F}$. Then the attaching map (3.13) extends to give a map

$$\Phi : \overline{m}(\widetilde{\Sigma}) \to \overline{m}(\Sigma_0) \tag{6.1}$$

of moduli spaces over the parameter space $\mathcal{P}$. Its restriction $\widetilde{\overline{\mathcal{M}}}(\widetilde{\Sigma}) \to \widetilde{\overline{\mathcal{M}}}(\Sigma_0)$ to the codimension at most 1 strata (5.12) can be described as follows. Consider the cover of $\widetilde{\overline{\mathcal{M}}}(\Sigma_0)$ obtained by ordering the nodes in the preimage of the node x^+ of Σ_0. It comes with a group action reordering those nodes that have the same contact multiplicity. This cover is in fact homeomorphic to $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$ and $\widetilde{\mathcal{M}}(\Sigma_0)$ is the quotient by the corresponding group action, as we next show.

6.1. The degree of the attaching map

We start with a useful result about the attaching map (6.1); to state it precisely we need to introduce some extra notation. Recall that by definition $\overline{\mathcal{M}}(\widetilde{\Sigma})$ is a disjoint union of spaces $\overline{\mathcal{M}}_\lambda(\widetilde{\Sigma})$ indexed by λ, cf. (4.25), thus so is its open subset $\widetilde{\overline{\mathcal{M}}}(\widetilde{\Sigma}) = \bigsqcup_{\lambda \vdash d} \widetilde{\overline{\mathcal{M}}}_\lambda(\widetilde{\Sigma})$.

Moreover the elements $f_0 : C_0 \to \Sigma_0$ in $\widetilde{\overline{\mathcal{M}}}(\Sigma_0)$ have nodal domains and the preimage $\mathbf{y}^+ = f_0^{-1}(x^+)$ of the target node x^+ consists only of nodes with matching contact multiplicity. We can similarly decompose the moduli space

$$\widetilde{\overline{\mathcal{M}}}(\Sigma_0) = \bigsqcup_{\lambda \vdash d} \widetilde{\overline{\mathcal{M}}}_\lambda(\Sigma_0) \tag{6.2}$$

as a disjoint union of (open and closed) subsets indexed by the contact order λ at the nodes $\mathbf{y}^+$. Denote by x_1^+, x_2^+ the two marked points of $\widetilde{\Sigma}$ that get attached to produce the node x^+ of Σ_0. Order the $\ell = \ell(\lambda)$ nodes $y_1^+, \dots, y_\ell^+$ in $\mathbf{y}^+$ so that the contact multiplicity of f_0 at y_i^+ is equal to λ_i, the i'th part of the partition λ.

Then C_0 is a nodal curve with ℓ pairs of conjugate nodes $\mathbf{y} = \{y_1^\pm, \dots, y_\ell^\pm\}$ in the preimage of the nodes of Σ_0 and at most one other node. Denote by $\widetilde{C}$ the (partial) normalization of C_0 resolving *only* the nodes in $\mathbf{y}$, and by $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ the unique lift of f_0 which satisfies the following conditions:

(i) the lift to $\widetilde{C}$ of the node y_i^+ consists of two marked points denoted y_{i1}^+ and y_{i2}^+.

(ii) $\widetilde{f}(y_{i1}^+) = x_1^+$ for all $i = 1, \dots, \ell$.

These conditions imply that the preimage $\widetilde{f}^{-1}(x_2^+)$ consists of the marked points y_{i2}^+, for $i = 1, \dots, \ell$ and the contact multiplicity of $\widetilde{f}$ at both y_{i1}^+ and y_{i2}^+ is equal to λ_i, exactly as is the case for an element of $\widetilde{\overline{\mathcal{M}}}_\lambda(\widetilde{\Sigma})$.

Ordering the contact nodes $\mathbf{y}^+$ as above therefore gives rise to a cover of $\widetilde{\mathcal{M}}_\lambda(\Sigma_0)$, and this cover is canonically identified with $\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma})$; the elements of $\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma})$ are precisely the lifts $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ of the elements $f_0 : C_0 \to \Sigma_0$ of $\widetilde{\mathcal{M}}_\lambda(\Sigma_0)$. Moreover, $\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma})$ comes with two natural $\mathrm{Aut}(\lambda)$ actions permuting the marked points in the preimage of x_1^+ and respectively x_2^+ (and therefore also permuting the corresponding conjugate points); the restriction

$$\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma}) \xrightarrow{\Phi|} \widetilde{\mathcal{M}}_\lambda(\Sigma_0). \tag{6.3}$$

of the attaching map is the quotient by the diagonal $\mathrm{Aut}(\lambda)$ action.

Both moduli spaces in (6.3) are families over the parameter space $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$. The first one is relatively orientable over $\mathcal{P}$ and an orientation is determined by a choice of twisted orientation data $\widetilde{\mathfrak{o}}$ on $\widetilde{\Sigma}$, as reviewed in §8 below. Moreover, the group action above is orientation preserving, thus the quotient is oriented. In particular, the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ is (relatively) orientable over $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$.

Lemma 6.1. *The attaching map* (6.1) *restricts to a proper map* (6.3) *of degree* $|\mathrm{Aut}(\lambda)|$ *with respect to the orientations described above. Therefore, if* Σ_0 *has no rational connected components without any marked points, then*

$$\Phi_*[\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma})_\nu] = |\mathrm{Aut}\ \lambda| \cdot [\widetilde{\mathcal{M}}_\lambda(\Sigma_0)_\nu] \tag{6.4}$$

for generic ν *(i.e. for all* $\nu \in \mathcal{P}^*$*).*

Proof. Since the restriction of Φ is the quotient map by the $\mathrm{Aut}(\lambda)$ action, it is a proper map and all its fibers are finite; the generic fiber consists of $|\mathrm{Aut}\ \lambda|$ points. Moreover, for generic ν, the fibers of both families of moduli spaces in (6.3) are topological manifolds by Proposition 5.4. They are oriented by the procedure described above and the group action is orientation preserving. Therefore their fundamental classes in rational Cech homology are related by (6.4), cf. [IP5, §1 and §2.1]. □

6.2. Linearizations and the orientation sheaf

This section provides several equivalent descriptions of the linearization to the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ and its orientation sheaf. The first one is induced via (6.3) from that of $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$; another one naturally extends over the family of moduli spaces $\cup_s \widetilde{\mathcal{M}}(\Sigma_s)$. For the proof of the splitting formula we must keep

track of the identifications between these two perspectives, as they affect the induced orientations and therefore the signs of the coefficients in (1.6).

Recall that the elements of $\widetilde{\mathcal{M}}(\Sigma_0)$ are maps $f : C_0 \to \Sigma_0 \subset \mathcal{F}$ without any rubber components, and that the total space $X = \mathcal{F}$ of the family (3.1) is smooth. When

$$(E, c_E) \to (\mathcal{F}, c_{\mathcal{F}}) \tag{6.5}$$

is a Real bundle, and W is a finite dimensional real vector bundle over $\widetilde{\mathcal{M}}(\Sigma_0)$, then as f varies in $\widetilde{\mathcal{M}}(\Sigma_0)$ we get a family of (pullback) operators $\overline{\partial}_{f^*(E,c_E)} \oplus 0_{W_f}$. The families of linearizations to $\widetilde{\mathcal{M}}(\Sigma_0)$ described in this section are of this form up to 0'th order terms. In that case, the results of [GZ1], applied to families of real maps to the smooth target $(\mathcal{F}, c_{\mathcal{F}})$, imply that the determinant of such family of operators is identified with

$$\det(\overline{\partial}_{(E,c_E)} \oplus 0_W) = \det \overline{\partial}_{(E,c_E)} \otimes \det W \tag{6.6}$$

(canonically up to homotopy).

The linearization at $f_0 : C_0 \to \Sigma_0$ to the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ is induced by the linearization

$$D_{\widetilde{f}} : \Gamma(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \oplus T_{\widetilde{C}}\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \tag{6.7}$$

at a lift $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ to the (partial) normalizations cf. §6.1. Because $\widetilde{\Sigma}$ is smooth, $D_{\widetilde{f}}$ is given by the formula (5.4) with f replaced by $\widetilde{f}$. In particular, as in (5.9), this identifies the orientation sheaf of $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$ with

$$\mathfrak{o}_{\widetilde{\mathcal{M}}(\widetilde{\Sigma})} \cong \det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}}, c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}} \tag{6.8}$$

(uniquely up to homotopy). Above $\mathcal{T}_{\widetilde{\Sigma}}$ is the relative tangent bundle of $\widetilde{\Sigma}$, while $T\overline{\mathcal{M}}^{\mathbb{R}}$ denotes the tangent bundle of the Real Deligne-Mumford moduli space parametrizing the variations in $\widetilde{C}$ (We assume for simplicity $\widetilde{C}$ is stable).

Remark 6.2. For any complex bundle $E \to C_0$, denote by $\widetilde{E} \to \widetilde{C}$ its lift to the normalization $\widetilde{C}$ of C_0. Recall that $\Gamma(C_0; E) \subset \Gamma(\widetilde{C}; \widetilde{E})$ is the subspace of sections that match at the nodes of C_0 while $\Lambda^{01}(C_0; E) := \Lambda^{01}(\widetilde{C}; \widetilde{E})$. Moreover, note that the pullback $\widetilde{E}$ of $E = f_0^*\mathcal{T}_{\Sigma_0}$ to the normalization $\widetilde{C}$ is equal to $\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}}$.

Using (3.6), together with the normalization short exact sequence (A.22) and (A.23) for the bundle $E = f_0^*\mathcal{T}_{\Sigma_0}$ allows us to rewrite the linearization (6.7) as

$$\widetilde{D}_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus (f_0^*\mathcal{T}_{\Sigma_0})_{|\mathbf{y}^+} \oplus T_{\widetilde{C}}\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}}. \tag{6.9}$$

Here an element $\zeta \in \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}}$ gives rise to a variation in f_0 with fixed domain, target, and also fixed product of the leading coefficients (A.39) at each one of the contact nodes, while the middle term records the variation in the product of leading coefficients, cf. §A.7. The operator (6.9) has the form

$$\widetilde{D}_{f_0}(\zeta, \alpha, h) = \overline{\partial}_{f_0^*(\mathcal{T}_{\Sigma_0}, c)}\zeta + A_{f_0}(\zeta) + \gamma_{f_0}(\alpha) + b_{f_0}(h). \tag{6.10}$$

where A_{f_0} and b_{f_0} are as in (5.4), while γ_{f_0} is induced by (A.23) after a choice of splitting of (A.22) for $E = f_0^*\mathcal{T}_{\Sigma_0}$, as reviewed in §A.7. Moreover, we can arrange that γ_f be induced by pullback from a splitting of the normalization short exact sequence for the bundle $E = \mathcal{T}_{\Sigma_0}$.

As f_0 varies in $\widetilde{\mathcal{M}}(\Sigma_0)$, the middle term in the domain of (6.9) is intrinsically the pullback $\mathrm{ev}_{\mathbf{y}^+}^*(\mathcal{T}_{|x^+})$ of $\mathcal{T}_{|x^+}$ under the evaluation map at the ℓ nodes $\mathbf{y}^+$ in the preimage of the node x^+, while the last term is $T_{C_0}\overline{\mathcal{N}}_\ell^{\mathbb{R}}$. Here ℓ is the number of nodes of C_0 in the inverse image of x^+, and $\overline{\mathcal{N}}_\ell^{\mathbb{R}}$ denotes the nodal stratum of the real Deligne-Mumford moduli space consisting of Real curves with ℓ pairs of conjugate nodes. Up to 0'th order terms, the family of operators $\widetilde{D}_{f_0}$ given by (6.9) is the same as $\overline{\partial}_{(\mathcal{T}_{\Sigma_0}, c_\mathcal{T})} \oplus 0 \oplus 0$. This describes a family $\widetilde{D}$ of linearizations to $\widetilde{\mathcal{M}}(\Sigma_0)$, and identifies the virtual orientation sheaf of $\widetilde{\mathcal{M}}(\Sigma_0)$ with

$$\mathfrak{o}_{\widetilde{\mathcal{M}}(\Sigma_0)} \cong \det \widetilde{D} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0}, c_\mathcal{T})} \otimes \det \mathrm{ev}_{\mathbf{y}^+}^*(\mathcal{T}_{|x_+}) \otimes \mathfrak{f}^* \det T\overline{\mathcal{N}}_\ell^{\mathbb{R}} \tag{6.11}$$

(uniquely up to homotopy).

Remark 6.3. When f is in $\widetilde{\mathcal{M}}(\Sigma_0)$, for every node y in the preimage $\mathbf{y}^+$ of the node x^+ of Σ_0, we can consider the product $a_y(f) := a(f; y_1)a(f; y_2)$ of the two leading coefficients (A.39)-(A.41) of f at y. Then after passing to the cover of the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ obtained by ordering the nodes as in §6.1, this product is a nowhere vanishing section of a complex line bundle.

This induces a canonical complex linear isomorphism

$$\mathrm{ev}_y^*(\mathcal{T}_{|x^+}) \cong (\mathcal{T}_{|y})^{\lambda(y)} \tag{6.12}$$

for each y in $\mathbf{y}^+$, where $\lambda(y)$ is the contact multiplicity at y. Here $\mathcal{T}_{|y} = \mathcal{T}_{|y_1} \otimes \mathcal{T}_{|y_2}$ denotes the relative tangent bundle at the node y, which is the pullback of the relative tangent bundle $\mathcal{T}$ on the universal curve by the section $f \mapsto (f, y)$. Note that x^+ is fixed, so the first term in (6.12) is trivial (and canonically trivialized by a choice of a complex linear isomorphism $\mathcal{T}_{|x^+} \cong \mathbb{C}$).

For a map $f_0 : C_0 \to \Sigma_0$ to a nodal target, there is another description of the linearization (6.9) that is better suited for smoothing the target and is used in the next two sections. It corresponds to the linearization defined in [IP2, §7]. For that, we consider instead the operator

$$D_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}}, \tag{6.13}$$

defined by the same formula (5.4) as $D_{\widetilde{f}}$, but with $f = f_0$ instead of $f = \widetilde{f}$. Note that the first term of the domain of D_{f_0} is a subspace of the first term of the domain of $D_{\widetilde{f}}$, while the second term $T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}}$ involves all variations in C_0, not just the ones tangent to the nodal stratum (the latter correspond to $T_{\widetilde{C}}\overline{\mathcal{M}}^{\mathbb{R}}$). Note that the operators D_{f_0} and $D_{\widetilde{f}}$ have the same target, cf. Remark 6.2.

Decompose $T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}}$ into tangent and normal directions to the nodal stratum $\mathcal{N}_\ell$ as in (A.31), where the normal direction is identified with $\mathcal{T}_{|\mathbf{y}^+}$. As reviewed in §A.7, we can use the normalization short exact sequence for the bundle $E = \mathcal{T}_{C_0}$ to rewrite (6.13) as an operator

$$\widehat{D}_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus \mathcal{T}_{|\mathbf{y}^+} \oplus T_{\widetilde{C}}\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \tag{6.14}$$

$$\widehat{D}_{f_0}(\zeta, v, h) = \overline{\partial}_{f_0^*(\mathcal{T}_{\Sigma_0}, c)}\zeta + A_{f_0}(\zeta) + \widehat{\gamma}_{f_0}(v) + b_{f_0}(h).$$

Here A_{f_0} and b_{f_0} are exactly as in the formula (6.10) for the operator $\widetilde{D}_{f_0}$, while $\widehat{\gamma}_{f_0}$ depends on the splitting of the normalization sequence associated to $E = \mathcal{T}_{C_0}$. The middle term in (6.10) keeps track of the variation in the product of the leading coefficients at the nodes, while the middle term in (6.14) keeps track instead of the variation normal to the nodal stratum in the real Deligne-Mumford moduli space.

Remark 6.4. The discussion above describes the linearization along $\widetilde{m}(\Sigma_0)$ in several equivalent ways: the linearization $\widetilde{D}_{f_0}$ is obtained from $D_{\tilde{f}}$ via a splitting of the normalization short exact sequence for $E = \mathcal{T}_{C_0}$, while the linearization $\widehat{D}_{f_0}$ is obtained from D_{f_0} via a splitting of the normalization short exact sequence for $E = \mathcal{T}_{\Sigma_0}$. The intermediate operators $\widetilde{D}_{f_0}$ and $\widehat{D}_{f_0}$ defined by (6.9) and (6.14) are equivalent via a complex linear isomorphism between the middle terms of their domain induced by the linearization of (6.12) cf. §A.7. In particular, these linearizations are all equivalent (just written using different coordinates); either one could be used to obtain a local model for $\widetilde{m}(\Sigma_0)$ or to (globally) identify its virtual orientation sheaf.

In particular, we get the following result.

Lemma 6.5. *The (relative) orientation sheaf $\mathfrak{o}_{\widetilde{m}(\Sigma_0)}$ of $\widetilde{m}(\Sigma_0)$ is canonically (up to homotopy) isomorphic to:*

$$\mathfrak{o}_{\widetilde{m}(\Sigma_0)} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0}, c_\mathcal{T})} \otimes \det \mathrm{ev}^*_{\mathbf{y}^+}(\mathcal{T}_{|x_+}) \otimes \mathfrak{f}^* \det T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell} \tag{6.15}$$

$$\mathfrak{o}_{\widetilde{m}(\Sigma_0)} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0}, c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}. \tag{6.16}$$

Proof. The first isomorphism is (6.11), obtained by using the family of linearizations (6.10). The second one is obtained the same way as (6.11), but using instead the family of linearizations (6.13). In (6.16), the last term $T\overline{\mathcal{M}}^{\mathbb{R}}$ denotes the tangent bundle of the real Deligne-Mumford moduli space parametrizing all the variations in the nodal domain C_0, including those that smooth out the nodes. □

7. The construction of the cobordism

In this section we continue working with the moduli spaces (5.13) over the parameter space $\mathcal{P} = \mathcal{JV}^{\mathbb{R}}(\mathcal{F}_{/\Delta})$ of RT perturbations on the family $\mathcal{F}$. We define an auxiliary space denoted $\widehat{m}(\mathcal{F}_{/I})$ and prove that generically its fiber $\widehat{m}(\mathcal{F}_{/I})_\nu$ is a cobordism between $\widetilde{m}(\Sigma_{s_0})_\nu$ and a cover $\widehat{m}(\Sigma_0)$ of $\widetilde{m}(\Sigma_0)_\nu$. This will allow us to compare their fundamental classes (in rational Cech homology).

7.1. The cover $\widehat{m}(\Sigma_0)$ of $\widetilde{m}(\Sigma_0)$

We start by defining a cover

$$q_0 : \widehat{m}(\Sigma_0) \to \widetilde{m}(\Sigma_0) \tag{7.1}$$

of the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$. Decompose $\widetilde{\mathcal{M}}(\Sigma_0)$ as in (6.2) according to the contact multiplicity λ at the nodes in the preimage of the node x^+ of the target Σ_0. Fix local coordinates on Σ_0 around x^+; this induces a trivialization $\mathcal{T}_{x^+} \cong \mathbb{C}$. The elements of $\widetilde{\mathcal{M}}_\lambda(\Sigma_0)$ are (J,ν)-holomorphic maps $f : C_0 \to \Sigma_0$, without any rubber components, and such that

(i) the inverse image $f^{-1}(x^+) = \mathbf{y}^+$ consists only of nodes y_i^+ of the domain with matching contact multiplicities λ_i, for $i = 1, \ldots, \ell(\lambda)$ and

(ii) the domain has at most 1 other node (which must be real).

The cover

$$\widehat{\mathcal{M}}(\Sigma_0) = \bigsqcup_{\lambda \vdash d} \widehat{\mathcal{M}}_\lambda(\Sigma_0) \tag{7.2}$$

consists of essentially the same objects, except that

(iii) for each contact node y_i^+ with contact multiplicity λ_i, we also choose a λ_i-root $\alpha_i \in \mathcal{T}^\vee_{|y_i^+}$ of the product of the leading coefficients of f at y_i^+, for every $i = 1, \ldots, \ell(\lambda)$, cf. (A.39)-(A.41) and [IP2, (6.1)-(6.2)].

Denote the objects of $\widehat{\mathcal{M}}(\Sigma_0)$ by

$$(f, \alpha), \quad \text{where } f \in \widetilde{\mathcal{M}}(\Sigma_0) \quad \text{and} \quad \alpha = (\alpha_i)_i \in (\mathcal{T}_{|\mathbf{y}^+})^\vee \tag{7.3}$$

satisfies

$$\alpha_i^{\lambda_i} = a(f; y_{i1}^+) a(f; y_{i2}^+) \quad \text{for all} \quad i = 1, \ldots, \ell(\lambda). \tag{7.4}$$

Here $a(f; y) = a_y(f)$ denotes the leading coefficient of f at the point y cf. (A.39)–(A.41), while λ_i is the contact multiplicity at $y = y_{i1}^+$ and y_{i2}^+.

With this definition, the map (7.1) is given by $q_0(f, \alpha) = f$.

Recall that the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ over the parameter space $\mathcal{P}$ is relatively orientable as discussed above Lemma 6.1. Fix any orientation on it and pull it back to $\widehat{\mathcal{M}}(\Sigma_0)$.

Lemma 7.1. *The map $q_0 : \widehat{\mathcal{M}}_\lambda(\Sigma_0) \to \widetilde{\mathcal{M}}_\lambda(\Sigma_0)$ is proper and has degree $\zeta(\lambda)/|\mathrm{Aut}(\lambda)|$ with respect to these orientations. Therefore if Σ_0 has no rational connected components without any marked points,*

$$(q_0)_* [\widehat{\mathcal{M}}_\lambda(\Sigma_0)_\nu] = \frac{\zeta(\lambda)}{|\mathrm{Aut}\, \lambda|} \cdot [\widetilde{\mathcal{M}}_\lambda(\Sigma_0)_\nu] \tag{7.5}$$

for generic ν (i.e. for all $\nu \in \mathcal{P}^*$).

Proof. By construction $\widehat{m}(\Sigma_0)$ comes with a group action on the choice of root α_i^+ by a λ_i-root of 1. So the quotient map q_0 is proper and the order of the group is

$$\prod_i \lambda_i = \frac{\zeta(\lambda)}{|\mathrm{Aut}\,\lambda|}$$

cf. (2.1)-(2.2). Since here the orientation of $\widehat{m}(\Sigma_0)$ is pulled back from that of $\widetilde{m}(\Sigma_0)$, then the action is orientation preserving. Moreover for generic ν, the fiber of $\widetilde{m}(\Sigma_0)_\nu$ is an oriented topological manifold by Proposition 5.4, and therefore so is its cover (with the pullback orientation), giving (7.5). □

7.2. The cobordism moduli space and its topology

Start with the subset $\widetilde{m}(\mathcal{F}_{/I}) = \bigcup_{s\in I} \widetilde{m}(\Sigma_s)$ of the family (4.25). Consider next the *set*

$$\widehat{m}(\mathcal{F}_{/I}) = \widehat{m}(\Sigma_0) \sqcup \bigcup_{s\neq 0} \widetilde{m}(\Sigma_s) \tag{7.6}$$

with the topology defined below. Extend q_0 as the identity on $\bigcup_{s\neq 0} \widetilde{m}(\Sigma_s)$ to get a map

$$q : \widehat{m}(\mathcal{F}_{/I}) \to \widetilde{m}(\mathcal{F}_{/I}) = \bigcup_{s\in I} \widetilde{m}(\Sigma_s). \tag{7.7}$$

The RHS is a subset of (J,ν)-holomorphic maps $f : C \to \Sigma_s \subset \mathcal{F}$ without any rubber components, for some $s \in I$. Therefore it comes with the topology induced by the usual Gromov topology on the moduli space of real (J,ν)-holomorphic maps to the total space $X = \mathcal{F}$ of the family of targets, cf. [GZ1, §4.2].

Below we define a Hausdorff topology on the LHS of (7.7) with the following properties: (a) q is proper and continuous (b) q restricts to the covering map q_0 for $s = 0$ and to the identity for $s \neq 0$. In particular, since q will be continuous, the topology on the LHS will be a refinement of the Gromov topology as considered in [IP2, §4-5].

By definition, (7.6) is the disjoint union of two sets $\widehat{m}(\Sigma_0)$ and $\bigcup_{s\neq 0} \widetilde{m}(\Sigma_s)$ and condition (b) canonically determines the topology on each one of these two subsets, where $\widehat{m}(\Sigma_0) = q^{-1}(\widetilde{m}(\Sigma_0))$ must be a closed subset by (a). Therefore it suffices to describe what it means for a sequence of maps f_n :

$C_n \to \Sigma_{s_n}$ with $s_n \neq 0$ to converge to an element $(f_0, \alpha) \in \widehat{\mathcal{M}}(\Sigma_0)$, beyond the fact that it must converge to f_0 in the usual Gromov topology. In short, as in [IP2], the domains of these maps must converge to the domain C_0 of f_0 along a fixed direction normal to the nodal stratum of domains containing C_0. To state this precisely, we use the set-up of [IP2, §3-5], adapted to our setting. We start with a choice of local coordinates as follows; see also [IP2, §4].

Fix an element (f_0, α) of $\widehat{\mathcal{M}}(\Sigma_0)$ and let C_0 denote the domain of f_0. Let w_1, w_2 be the fixed local coordinates around the node x^+ of the target; then in a neighborhood of x^+ the curve $\Sigma_s \subset \mathcal{F}$ is described by $w_1 w_2 = s$, where $s \in I = [0, s_0] \subset \Delta$ is the gluing parameter. After rotating the disk Δ parametrizing the family $\mathcal{F}_{/\Delta}$ of targets, we may assume that $I \subset \mathbb{R}_{\geq 0}$. Fix local coordinates z_1, z_2 around each node y of C_0 as in (B.3) and construct a local slice $\mathcal{S}$ as in (B.5) parametrizing local deformations $C_{\tau,u}$ of C_0. Here u are local coordinates along the nodal stratum $\mathcal{N}_\ell$ of the real Deligne-Mumford moduli space while the gluing parameters

$$\tau = (\tau_i)_i \in \mathcal{T}_{|\mathbf{y}^+}, \quad \text{where } \tau_i \in \mathcal{T}_{|y_i^+} \text{ for all } i = 1, \ldots, \ell(\lambda) \tag{7.8}$$

cf. (B.4) provide normal coordinates to this stratum cf. (A.30)-(A.31). Note that a choice of local coordinates on C_0 at y_i^+ induces a local trivialization of the relative tangent bundle $\mathcal{T}$ around y_i^+.

Below we also use the notion and properties of δ-flat maps in the sense of [IP2, Definition 3.1] to describe a neighborhood of $f_0 \in \widetilde{\mathcal{M}}(\Sigma_0) \subset \cup_s \widetilde{\mathcal{M}}(\Sigma_s)$ in the Gromov topology on the target of (7.7); for precise details see [IP2, (3.1)-(3.5)] and [IP2, Lemma 3.2]. Roughly speaking, δ-flat maps $f : C \to \Sigma_s \subset \mathcal{F}$ do not have enough energy in the pre-image of a δ-neighborhood $U_\delta \subset \mathcal{F}$ of the nodes $\mathbf{x}$ of $\Sigma_0 \subset \mathcal{F}$ for a rubber component to start forming in there, cf. [IP2, (3.1)-(3.5)], thus giving rise to the uniform estimates of [IP2, §5]. As $\delta \to 0$, δ-flat maps exhaust the complement of the strata with at least one rubber component, cf. [IP2, Lemma 3.2].

Consider now a sequence of (J, ν)-holomorphic maps $f_n : C_n \to \Sigma_{s_n}$ with $s_n \neq 0$ and which converges in the Gromov topology to the map $f_0 \in \widetilde{\mathcal{M}}(\Sigma_0)$. Since f_0 has no components mapped to the singular locus $\mathbf{x}$ of Σ_0, it is δ-flat in the sense of [IP2, Definition 3.1] for all sufficiently small $\delta > 0$, cf. [IP2, (3.1)-(3.4)]. Gromov convergence then implies that after decreasing δ, f_n is δ-flat for all sufficiently large n. Moreover, up to reparametrizations of the domains and for large n, we may identify

$$C_n = C_{\tau_n, u_n}$$

as in the paragraph containing (7.8), where $(\tau_n, u_n) \to 0$ as $n \to \infty$. Therefore by [IP2, Lemma 5.3]

$$\lim_{n\to\infty} \frac{s_n}{\tau_{i,n}^{\lambda_i}} = a(f_0; y_{i1}^+) a(f_0; y_{i2}^+) \tag{7.9}$$

for each contact node y_i^+ of C_0. Here $\tau_n = (\tau_{i,n})_i$ are the gluing parameters (7.8) corresponding to $C_n = C_{\tau_n, u_n}$ (for n large), while as before $a(f; y) \neq 0$ is the leading coefficient of f at y (in these local coordinates). In particular, since $s_n \neq 0$, this implies that $\tau_{i,n} \neq 0$ for all $i = 1, \ldots, \ell$ and n large.

With these preliminaries understood, a sequence $f_n : C_n \to \Sigma_{s_n}$ with $s_n \neq 0$ is said to converge in $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ to (f_0, α) if and only if f_n converges to f_0 in the Gromov topology and moreover

$$\lim_{n\to\infty} \frac{(s_n)^{1/\lambda_i}}{\tau_{i,n}} = \alpha_i \tag{7.10}$$

for all $i = 1, \ldots, \ell$. Here $s_n \in \mathbb{R}_{>0}$ (by the simplifying assumption $I \subset \mathbb{R}_{\geq 0}$) thus it has a unique λ_i-root $(s_n)^{1/\lambda_i} \in \mathbb{R}_{>0}$. The limit (f_0, α) if it exists is unique and is independent of the choice of local coordinates around C_0, cf. (7.3) and (7.8). This completes the definition of the topology on the domain of (7.7), which by construction satisfies all the properties listed in the paragraph after (7.7).

Remark 7.2. In light of (7.4) and (7.9), condition (7.10) can be equivalently replaced by

$$\lim_{n\to\infty} \arg \tau_{i,n} = -\arg \alpha_i,$$

where $\arg w$ denotes the argument of w; in particular, this means that the domains $C_n = C_{\tau_n, u_n}$ of f_n converge to the domain C_0 of f_0 along a fixed direction normal to the nodal stratum $\mathcal{N}_\ell$; see also [IP1, Lemma 4.2 and (4.10)]. By [IP2, Lemma 5.4], this also implies that f_n converge to f_0 in the λ-weighted Sobolev norms of §A.5 used in the proof of the splitting formula; see also (A.44).

This motivates considering the following local model, regarded as a section of the (pullback) bundle $\mathcal{T}_{\mathbf{y}_+}$ over $\widehat{\mathcal{M}}(\Sigma_0) \times \mathbb{R}_{\geq 0}$. It consists of tuples

$$(f, \alpha, \tau, s) \quad \text{such that} \quad \alpha_i \tau_i = s^{1/\lambda_i} \quad \text{for all} \quad i = 1, \ldots, \ell. \tag{7.11}$$

Here (f,α) is an element of $\widehat{\mathcal{M}}(\Sigma_0)$, while $\tau=(\tau_i)_i\in\mathcal{T}_{\mathbf{y}_+}$ and $s\in\mathbb{R}_+$. Note that in cylindrical coordinates the conditions defining (7.11) become linear (i.e. $\log\alpha_i+\log\tau_i=1/\lambda_i\log s$).

7.3. The cobordism

Consider the family

$$\widehat{\mathcal{M}}(\mathcal{F}_{/I})=\widehat{\mathcal{M}}(\Sigma_0)\cup\bigcup_{s\neq 0}\widetilde{\mathcal{M}}(\Sigma_s)\to\mathcal{P} \tag{7.12}$$

with the topology defined above; it is a family over the parameter space $\mathcal{P}=\mathcal{JV}(\mathcal{F}_{/\Delta})$ of Ruan-Tian perturbations, as well as a family over the interval $I=[0,s_0]$. We next verify that for generic $\nu\in\mathcal{P}$, the fiber $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ of (7.12) is a topological cobordism. This is the main step in the proof of Theorem 1.2.

Proposition 7.3. *Assume Σ_0 has no rational connected components without any marked points. Then for generic RT perturbation $\nu\in\mathcal{P}$, the fiber $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ of (7.12) is a (possibly non-compact) orientable topological cobordism between $\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu$ and the cover $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ of $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ defined by (7.1).*

Proof. As in Proposition 5.4, for generic ν,

(i) $\bigcup_{s\in(0,s_0]}\widetilde{\mathcal{M}}(\Sigma_s)_\nu$ is an orientable topological manifold with boundary $\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu$ and

(ii) $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ is an orientable manifold and therefore so is its cover $\widehat{\mathcal{M}}(\Sigma_0)_\nu$.

Moreover, as in (4.27), for generic ν all the strata of $\overline{\mathcal{M}}(\Sigma_0)$ are cut transversally, i.e. the cokernel of the linearization D_f is trivial at all elements f in the fiber over ν.

Fix such a generic parameter ν. We next describe the local model of a neighborhood in $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ of a fixed point $f_0:C_0\to\Sigma_0$ in $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$. Recall that by definition $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})=\cup_{s\in I}\widetilde{\mathcal{M}}(\Sigma_s)$ is an *open* subset of the family $\overline{\mathcal{M}}(\mathcal{F}_{/I})=\cup_{s\in I}\overline{\mathcal{M}}(\Sigma_s)$ of moduli spaces and consists of maps without any rubber components. Moreover, since f_0 has no rubber components, a sequence f_n converges to f_0 in the topology of the compactified family $\overline{\mathcal{M}}(\mathcal{F}_{/I})=\cup_{s\in I}\overline{\mathcal{M}}(\Sigma_s)$ if and only if $f_n:C_n\to\Sigma_{s_n}\subset\mathcal{F}$ converge to f_0 in the Gromov topology on the space of real (J,ν)-holomorphic maps $f:C\to\mathcal{F}$. Therefore a sufficiently small neighborhood U_{f_0} of f_0 in the family $\overline{\mathcal{M}}(\mathcal{F}_{/I})_\nu=\cup_s\overline{\mathcal{M}}(\Sigma_s)_\nu$ consists only of δ-flat maps for some $\delta>0$. Finally,

the cokernel of the linearization at f_0 is onto, and shrinking the neighborhood U_{f_0} we may assume that it is a subset the open subset $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ of $\overline{\mathcal{M}}(\mathcal{F}_{/I})_\nu$, thus disjoint from any of the codimension at least two strata.

Therefore, the gluing argument of [IP2] applies to describe a sufficiently small neighborhood U_{f_0} of f_0 in terms of the space of approximate maps, which in turn are constructed starting from the model space (7.11), cf. [IP2, Definitions 6.1-6.2] (noting the change in notation).

For generic ν, the local model of the neighborhood U_{f_0} of f_0 in $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ is therefore described by tuples

$$(f,\tau,s) \text{ such that } a(f;y_{i1}^+)a(f;y_{i2}^+)\tau_i^{\lambda_i}=s \text{ for all } \quad i=1,\dots,\ell. \tag{7.13}$$

Here $f : C \to \Sigma_0$ belongs to a neighborhood $\mathcal{O}_{f_0}$ of $f_0 : C_0 \to \Sigma_0$ in the moduli space $\overline{\mathcal{M}}(\Sigma_0)_\nu$ and $a(f;y)$ is the leading coefficient of f at y. As before, $\tau=(\tau_i)_i$ are the gluing parameters of the domain and $s \in \mathbb{R}_{\geq 0}$ is the gluing parameter of the target. Moreover $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ is an open subset of $\overline{\mathcal{M}}(\Sigma_0)_\nu$, therefore we may assume that $\mathcal{O}_{f_0}$ is a neighborhood of f_0 in $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ after possibly shrinking it.

By the definition (7.7) of the space $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ and of its topology, the local model of $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ around the point $(f_0,\alpha_0) \in \widehat{\mathcal{M}}(\Sigma_0)_\nu$ is therefore described by tuples (f,α,τ,s) as in (7.11) where (f,α) is in the neighborhood of (f_0,α_0) in $\widehat{\mathcal{M}}(\Sigma_0)_\nu$, while $\tau=(\tau_i)_i$ and s are the gluing parameters as above. But $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ is a topological manifold and we can uniquely solve the equations defining (7.11) for the variable τ in terms of (f,α) and $s \in \mathbb{R}_{\geq 0}$, thus the fiber of the local model is a topological manifold with boundary at $s=0$, and therefore so is the fiber of $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ in a neighborhood of $s=0$. □

We next consider the orientation sheaves of these spaces. When $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})=\cup_{s\in I}\widetilde{\mathcal{M}}(\Sigma_s)$ is regarded as a family over the parameter space $\mathcal{P}\times I$, its fiber at (ν,s) is $\widetilde{\mathcal{M}}(\Sigma_s)_\nu$, and the (fiber-wise) linearization D_f is given by the formula (5.4), including when $s=0$ cf. (6.13). The kernel of this linearization encodes the (formal) tangent space to the fiber and its cokernel is the obstruction. When $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})=\cup_{s\in I}\widetilde{\mathcal{M}}(\Sigma_s)$ is regarded as a family over $\mathcal{P}$, we can similarly consider the fiberwise linearization $\mathbb{D}_f$, as s is now allowed to vary in the interval I; the operator $\mathbb{D}_f$ is an extension of D_f and its domain has an extra term keeping track of the variation $\delta s \in T_sI \cong \mathbb{R}$ in s. As $f : C \to \Sigma_s \subset \mathcal{F}$ varies in $\widetilde{\mathcal{M}}(\mathcal{F}_{/I})$, we get a family of real maps to a *smooth* target $X=\mathcal{F}$ and up to 0'th order terms, the family of operators $\mathbb{D}_f$ is the same as a (pullback) family of the form $\overline{\partial}_{f^*(\mathcal{T},c_\mathcal{T})} \oplus 0_{W_f} \oplus 0_{\mathbb{R}}$; see also paragraphs containing (5.8) and (6.5). Therefore as before, we get an

identification

$$\det \mathbb{D} \cong \det(\overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \oplus 0_W \oplus 0_{\mathbb{R}})$$

over $\widetilde{m}(\mathcal{F}_{/I})$.

By definition, the (virtual) relative orientation sheaves of the moduli spaces $\widehat{m}(\Sigma_0)$ and $\widehat{m}(\mathcal{F}_{/I})$ over the parameter space $\mathcal{P}$ are the (pullback) of the determinant bundle of the family of operators D_f and respectively $\mathbb{D}_f$ as f varies in $\widetilde{m}(\mathcal{F}_{/I})$. The proof above also implies that for generic ν, the fiber $\widehat{m}(\mathcal{F}_{/I})_\nu$ is a topological cobordism whose orientation sheaf is canonically identified with $\det \mathbb{D}_f$ (up to homotopy), including along the codimension 1 strata, while the orientation sheaf of its boundary $\widehat{m}(\Sigma_0)_\nu \sqcup \widetilde{m}(\Sigma_{s_0})_\nu$ is similarly given by $\det D_f$. Both statements follow from the proof of the gluing theorem, which shows that the corresponding moduli space is a small deformation of the space of approximate maps, and that the tangent space to the space of approximate maps is a small deformation of the kernel of the linearization.

This gives the following result.

Corollary 7.4. *The (relative) orientation sheaf of the family $\widehat{m}(\mathcal{F}_{/I})$ over $\mathcal{P}$ can be canonically identified (up to homotopy) with the pullback under* (7.7) *of the bundle*

$$\det \overline{\partial}_{(\mathcal{T},c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}} \otimes \pi^* \det TI. \tag{7.14}$$

Here $\mathcal{T} \to \mathcal{F}$ is the relative tangent bundle (3.5).

The bundle (7.14) is orientable over $\widetilde{m}(\mathcal{F}_{/I})$ and canonically oriented after choosing a twisted orientation data on $\mathcal{T} \to \mathcal{F}$, as described after (8.17). In particular, any choice of orientation on (7.14) determines an orientation of the cobodism $\widehat{m}(\mathcal{F}_{/I})_\nu$ as well as on its two boundaries. With respect to this choice, Proposition 7.3 gives the equality of the fundamental classes

$$[\widetilde{m}(\Sigma_0)_\nu] = [\widehat{m}(\Sigma_{s_0})_\nu] \quad \text{in} \quad \check{\mathrm{H}}_*(\widehat{m}(\mathcal{F}_{/I})_\nu; \mathbb{Q}) \tag{7.15}$$

for generic ν in $\mathcal{P}$.

Remark 7.5. Here we are using the convention that if M_0 and M_1 are oriented topological manifolds, and W is an oriented cobordism from M_0 to M_1 then W is a manifold whose boundary is $\partial W = -M_0 \sqcup M_1$ as oriented manifolds.

8. Orientations

In this section we review the orientation procedure for all the moduli spaces considered in §5-§7 and show that a choice of (twisted) orientation data on Σ or on the family $\mathcal{F}$ canonically orients these moduli spaces (over the space of Ruan-Tian perturbations on Σ or $\mathcal{F}$).

First, recall that if X is a space with an involution c, then a Real[5] bundle over (X, c) is a pair (E, ϕ) where $E \to X$ is a complex vector bundle and ϕ is an anti-complex linear involution on E covering c; then ϕ is called the real structure on E. An isomorphism between two Real bundles (E_i, ϕ_i) is a complex linear isomorphism ψ compatible with the real structures i.e. such that $\psi^*\phi_2 = \phi_1$. We denote by $\overline{E}$ is the same real bundle as E, but with opposite complex structure $J \mapsto -J$.

Definition 8.1 ([GI, Appendix]). *Let (X, c) be a Real symplectic manifold and (W, ϕ) a Real bundle over it. A* twisted orientation *$\mathfrak{o} = (L, \psi, \mathfrak{s})$ for the bundle (W, ϕ) consists of*

(i) a complex line bundle $L \to X$ such that the Real vector bundle $(E, c_E) = (L \oplus c^*\overline{L}, c_{tw})$ satisfies:

$$w_2(W^\phi) = w_2(E^{c_E}) \quad \text{and} \quad \Lambda^{\text{top}}(W, \phi) \cong \Lambda^{\text{top}}(E, c_E) \tag{8.1}$$

(ii) a homotopy class $[\psi]$ of isomorphisms satisfying (8.1).

(iii) a spin structure $\mathfrak{s}$ on the real vector bundle $W^\phi \oplus (E^\vee)^{c_E^\vee}$ over the real locus, compatible with the orientation induced by ψ.

Here $E^\vee$ denotes the dual of E and the real structure c_{tw} on $L \oplus c^*\overline{L}$ is defined by

$$c_{tw}(x; v, w) = (c(x); w, v) \tag{8.2}$$

for all $x \in X$, $v \in L_x$ and $w \in \overline{L}_{c(x)}$. Note that given any complex line bundle L satisfying condition (i), one can then *chose* (not uniquely) a homotopy class $[\psi]$ as in (ii) and a spin structure $\mathfrak{s}$ as in (iii) such that together they give a choice of twisted orientation on (W, ϕ).

[5] called a real bundle pair in [GZ1, §1.1]

As in [GZ1, Proposition 5.2], the conditions in Definition 8.1 determine a canonical homotopy class of isomorphisms

$$f^*(W \oplus E^\vee, \phi \oplus c_E^\vee) \cong (C \times \mathbb{C}^{n+2}, c_{std}) \tag{8.3}$$

which vary continuously as f varies in a space of real maps $f : C \to X$. (Here n is the complex rank of W). In particular, as in the proof of [GZ1, Theorem 1.3] they determine a canonical homotopy class of isomorphisms

$$\det \overline{\partial}_{(W,\phi)} \otimes \det \overline{\partial}_{(E,c_E)^\vee} \cong \det \overline{\partial}_{(\mathbb{C}^{n+2},c_{std})}. \tag{8.4}$$

But since $(E, c_E)^\vee = (L \oplus c^*\overline{L}, c_{tw})^\vee$ there is also a canonical isomorphism

$$\det \overline{\partial}_{(E,c_E)^\vee} \cong p_1^* \det \overline{\partial}_{L^\vee} \tag{8.5}$$

obtained as in [GI, (2.9)] by projecting onto the first factor. The RHS of (8.5) is the determinant bundle of a family of complex operators thus has a canonical (complex) orientation. Together with the canonical orientation on the square of a bundle, (8.4) induces a trivialization

$$\det \overline{\partial}_{(W,\phi)} \otimes \det \overline{\partial}_{(\mathbb{C}^n,c_{std})} \xrightarrow[\cong]{\mathfrak{o}} \underline{\mathbb{R}} \tag{8.6}$$

which depends on the choice of twisted orientation data $\mathfrak{o}$. Here $\underline{\mathbb{R}}$ denotes the trivial real line bundle (canonically trivialized). Note that the existence of a complex line bundle L satisfying condition (i) of Definition 8.1 ensures that the real line bundle on the right-hand side of (8.6) is *orientable*, while conditions (ii)-(iii) of Definition 8.1 fix an orientation on it.

8.1. Existence of twisted orientations

We next show that, for symmetric curves *without any real special points*, there exits a choice of twisted orientation data on their relative tangent bundle. We also show that the same is true for the family of targets (3.1).

First, when (Σ, c) is a (possibly nodal) symmetric curve, then up to isomorphism, Real line bundles (W, ϕ) over X are classified by $c_1(W)$ and $w_1(W^\phi)$, cf. [GZ3, Theorem 1.1]. Thus such a bundle has a twisted orientation data if and only if $w_1(W^\phi) = 0$ and there exits a complex line bundle

$L \to \Sigma$ such that

$$c_1(L \otimes c^*\overline{L}) = c_1(W) \tag{8.7}$$

cf. (8.1); the first condition in (8.1) is automatic in this case.

Let (Σ, c) be a (possibly nodal) symmetric marked curve with marked points V and without any real special points. Let $W = \mathcal{T}_\Sigma$ be its relative tangent bundle, regarded here as a Real line bundle, i.e. a complex line bundle with a real structure $c_{\mathcal{T}}$. Recall that $\mathcal{T}_\Sigma$ is given by (2.8) when Σ is smooth and by (3.6) when Σ is nodal. Since none of the special points are real, the real locus of $\mathcal{T}_\Sigma$ is orientable. Thus a twisted orientation data $\mathfrak{o} = (\Theta, \psi, \mathfrak{s})$ on $\mathcal{T}_\Sigma$ exists provided we can find a complex line bundle $\Theta \to \Sigma$ such that

$$c_1(\Theta \otimes c^*\overline{\Theta}) = c_1(\mathcal{T}_\Sigma). \tag{8.8}$$

We next show that such a bundle Θ exists for any such (Σ, c). When Σ is smooth and connected, then the bundle Θ must be a complex line bundle whose Chern number is equal to $\frac{1}{2}c_1(\mathcal{T}_\Sigma)[\Sigma] = \frac{1}{2}\chi(\Sigma \setminus V) \in \mathbb{Z}$; such a bundle exists and is unique up to complex linear isomorphism.

When Σ is a smooth doublet (i.e. has two connected components interchanged by c) then Θ can be any complex line bundle on Σ whose Chern numbers on the two halves sum to $\frac{1}{2}\chi(\Sigma \setminus V) \in \mathbb{Z}$; see [GI, Example 2.2]. Therefore a bundle satisfying condition (8.8) exists on every *smooth* symmetric curve Σ without real marked points. Finally, when Σ_0 is a nodal symmetric curve which has no real special points, the bundle Θ_0 on Σ_0 satisfying condition (8.8) can be obtained from one on the normalization after choosing an identification at the marked points corresponding to the nodes, compatible with the real structure; such a choice is unique up to homotopy.

Consider next the family of targets (3.1), and let $\mathcal{T} \to \mathcal{F}$ be its relative tangent bundle (3.5), regarded as a Real line bundle. Note that $(\mathcal{F}, c_{\mathcal{F}})$ deformation retracts to (Σ_0, c_0) through real maps, so $\iota_0 \circ r$ is real homotopic to the identity, where $r : \mathcal{F} \to \Sigma_0$ denotes the retraction. Moreover, isomorphism classes of Real bundles on a Real space X are in 1-1 correspondence with homotopy classes of real maps from X to the classifying space, cf. [E, Proposition 2.1]. In particular, $r^*\mathcal{T}_{\Sigma_0} \cong \mathcal{T}$ as Real bundles. Since $\mathcal{F}$ deformation retracts to Σ_0, every bundle on Σ_0 can be extended over the family $\mathcal{F}$. Thus the complex line bundle $\Theta_0 \to \Sigma_0$ described above extends to a complex line bundle $\Theta_{\mathcal{F}}$ over $\mathcal{F}$ with the property that there

exists an isomorphism of Real bundles

$$\psi_{\mathcal{F}} : (\mathcal{T}, c_{\mathcal{T}}) \longrightarrow (\Theta_{\mathcal{F}} \otimes c_{\mathcal{F}}^* \overline{\Theta}_{\mathcal{F}}, c_{tw}).$$

This, together with the fact that both the real part of $\mathcal{T}$ and the restriction of $\Theta_{\mathcal{F}}$ to $\mathcal{F}^{c_{\mathcal{F}}}$ are spin (i.e. have vanishing first and second Stiefel-Whitney classes) implies that a twisted orientation $\mathfrak{o}_{\mathcal{F}} = (\Theta_{\mathcal{F}}, \psi_{\mathcal{F}}, \mathfrak{s}_{\mathcal{F}})$ exists on the Real bundle $\mathcal{T} \to \mathcal{F}$ for any family $\mathcal{F}$ as in (3.1).

Remark 8.2. When Σ_0 has no real special points, one can also show that every choice of twisted orientation data on the relative tangent bundle of the normalization $\widetilde{\Sigma}$ descends to one on the nodal curve Σ_0 and extends to one on the relative tangent bundle $\mathcal{T} \to \mathcal{F}$ of the family of deformations of Σ_0.

8.2. Orienting the moduli spaces

The considerations above are used to orient the real relative moduli spaces as described after [GI, (A.13)], and in fact to simultaneously orient all the moduli spaces considered in §5-§7. For that it suffices to construct a homotopy class of (global) trivializations of the determinant bundle of the linearization over the union of virtual codimension at most 1 strata for each one of these moduli spaces over their parameter space.

Specifically, assume $X = \Sigma$ is a (smooth) marked symmetric curve, and consider the union $\widetilde{\mathcal{M}}(\Sigma)$ of the virtual codimension at most 1 strata of the real relative moduli space associated to Σ. Over the space of Ruan-Tian perturbations, identify the (relative) orientation sheaf of $\widetilde{\mathcal{M}}(\Sigma)$ with

$$\mathfrak{o}_{\widetilde{\mathcal{M}}(\Sigma)} \cong \det \overline{\partial}_{(\mathcal{T}_\Sigma, c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi, \ell(\vec{\mu})} \tag{8.9}$$

cf. (5.9). It suffices to describe a homotopy class of trivializations of the real line bundle (8.9). A choice of twisted orientation data $\mathfrak{o} = (\Theta, \psi, \mathfrak{s})$ on the relative tangent bundle $\mathcal{T}_\Sigma$ of Σ gives rise to a canonical homotopy class of trivializations

$$\det \overline{\partial}_{(\mathcal{T}_\Sigma, c_{\mathcal{T}})} \otimes \det \overline{\partial}_{(\mathbb{C}, c_{std})} \xrightarrow[\cong]{\mathfrak{o}} \underline{\mathbb{R}} \tag{8.10}$$

cf. (8.6) with $W = \mathcal{T}_\Sigma$. Combining this with the canonical homotopy class of isomorphisms [GZ1, (3.1)]

$$\mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi, \ell} = \det \overline{\partial}_{(\mathbb{C}, c_{\mathrm{std}})} \tag{8.11}$$

induces a trivialization of (8.9), canonically up to homotopy.

When $\Sigma = \Sigma_0$ has a pair of conjugate nodes the procedure above also applies to the moduli space $\widetilde{m}(\Sigma_0)$ by trivializing the line bundle

$$\mathfrak{o}_{\widetilde{m}(\Sigma_0)} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0}, c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}, \tag{8.12}$$

cf. (6.16). In this case a choice of twisted orientation data $\mathfrak{o}_0 = (\Theta_0, \psi_0, \mathfrak{s}_0)$ on $\mathcal{T}_{\Sigma_0}$ as in §8.1 determines by [GZ2, Proposition 4.3] a canonical homotopy class of isomorphisms as in (8.3) and consequently as in (8.6) with $W = \mathcal{T}_{\Sigma_0}$ and $L = \Theta_0$.

This procedure also extends to *simultaneously* induce trivializations of the determinant line bundles of the moduli spaces in (5.12), (7.1) and (7.7) as follows.

Fix a choice of twisted orientation data $\mathfrak{o}_{\mathcal{F}} = (\Theta_{\mathcal{F}}, \psi_{\mathcal{F}}, \mathfrak{s}_{\mathcal{F}})$ for the bundle $(\mathcal{T}, c_{\mathcal{T}})$ over the family $(\mathcal{F}, c_{\mathcal{F}})$ as in §8.1. The pullback under (3.4) of $\mathcal{T}$ to Σ_s and $\widetilde{\Sigma}$ gives their corresponding relative tangent bundles. Furthermore $\mathfrak{o}_{\mathcal{F}}$ pulls back to a twisted orientation data

$$\begin{aligned} &\mathfrak{o}_s = \iota_s^* \mathfrak{o}_{\mathcal{F}} \quad \text{on} \quad \mathcal{T}_{\Sigma_s} \longrightarrow \Sigma_s \\ &\text{and respectively} \quad \widetilde{\mathfrak{o}} = \phi^* \mathfrak{o}_{\mathcal{F}} \quad \text{on} \quad \widetilde{\mathcal{T}} \longrightarrow \widetilde{\Sigma}. \end{aligned} \tag{8.13}$$

Applying the procedure in (8.9)-(8.11) for $\mathfrak{o}_s$ and $\widetilde{\mathfrak{o}}$ to induce a canonical (relative) orientation on each of the families of moduli spaces

$$\widetilde{m}(\Sigma_{s\neq 0}), \quad \widetilde{m}(\Sigma_0), \quad \text{and} \quad \widetilde{m}(\widetilde{\Sigma}) \tag{8.14}$$

over $\mathcal{P}$ defined in (5.12); note that for $s = 0$ this procedure is applied to orient (8.12), and it is the same as the one described in the paragraph containing it. In particular, a choice of $\mathfrak{o}_{\mathcal{F}}$ determines a canonical orientation on the generic fibers $\widetilde{m}(-)_\nu$ of the moduli spaces in (8.14).

Consider next the families

$$\widehat{m}(\mathcal{F}_{/I}) \to \widetilde{m}(\mathcal{F}_{/I}) \tag{8.15}$$

over $\mathcal{P}$ defined in (7.7), and recall that by construction they restrict to a covering

$$q_0 : \widehat{m}(\Sigma_0) \to \widetilde{m}(\Sigma_0) \tag{8.16}$$

over $s = 0$. By Corollary 7.4 the relative orientation sheaf of $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ is canonically isomorphic to the pullback by (8.15) of

$$\det \overline{\partial}_{(\mathcal{T},c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \otimes \pi^* \det TI. \tag{8.17}$$

Moreover, after orienting the segment I from 0 to s_0, the line bundle (8.17) is canonically oriented by a choice of twisted orientation data $\mathfrak{o}_{\mathcal{F}}$ using the same procedure of combining (8.11) with (8.6) for $W = \mathcal{T}$ over the moduli space $\widetilde{\mathcal{M}}(\mathcal{F}_{/I}) = \cup_{s\in I}\widetilde{\mathcal{M}}(\Sigma_s)$. Thus such a choice canonically orients the cobordism $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ for generic ν, and therefore also its boundary, in such a way that

(i) $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ is an oriented cobordism from the cover $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ of $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ to $\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu$, and

(ii) the orientation on $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ is the pullback of one on $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$.

Furthermore, these orientations on $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$ and $\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu$ agree with those described in the paragraph containing (8.14) for the same choice of $\mathfrak{o}_{\mathcal{F}}$. With respect to these orientations, the degree of the covering map q_0 is given by Lemma 7.1.

However, there is *another* natural procedure of fixing a relative orientation of the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ over the parameter space $\mathcal{P}$, which may *a priori* lead to a different orientation on its generic fibers than the one described in the paragraph above. As described above Lemma 6.1, the relative orientation of $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$ induced by $\widetilde{\mathfrak{o}}$ descends to a relative orientation on $\widetilde{\mathcal{M}}(\Sigma_0)$ under the attaching map

$$\Phi : \widetilde{\mathcal{M}}(\widetilde{\Sigma}) \to \widetilde{\mathcal{M}}(\Sigma_0). \tag{8.18}$$

The degree of this cover with respect to *these* orientations is given by Lemma 6.1.

8.3. Comparing orientations

The purpose of this section is to show that the two natural orientations on $\widetilde{\mathcal{M}}(\Sigma_0)$ described in §8.2 (both induced by $\mathfrak{o}_{\mathcal{F}}$) agree, cf. Proposition 8.3 below. This is the crucial sign computation in this paper, which fixes the signs that might otherwise enter in the VFC splitting formula (1.6). Its proof is a rather tedious but straightforward combination of many diagram chases, while keeping careful track at each step of the identifications made

in the process of trivializing the determinant bundles of various families of operators.

Recall that as described in §6.2, the relative orientation sheaf of the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ has several equivalent descriptions obtained by writing the linearization in different coordinates, cf. Remark 6.4. Each of these can be trivialized in more than one way, depending on the choice of the orientation *procedure*, leading to *possibly different* orientations on the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$. However, the difference between any two such resulting orientations can be, at least in principle, specifically determined, and our goal here is to compare two such specific orientations. In practice, one way to do so is by describing a natural family of identifications between the two families of linearizations (*de facto* describing the linearization of the identity in two types of charts on the moduli space) and then calculating the sign of the resulting isomorphism at the level of determinant bundles with respect to an *a priori* specified procedure of orienting each one of them (independent of the identification between the two linearizations).

We start with some preliminary considerations comparing the pullback under Φ of the determinant bundle of a family of operators to the determinant bundle of the pullback family over the normalization.

Assume (E, c_E) is Real vector bundle over Σ_0 and let $(\widetilde{E}, c_{\widetilde{E}})$ denote its pullback to the normalization $\widetilde{\Sigma}$ of Σ_0. Pulling back $\overline{\partial}_{(E,c_E)}$ and $\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})}$ to $\widetilde{\mathcal{M}}(\Sigma_0)$ and respectively $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$ gives rise to two families of operators over these moduli spaces; let $\det \overline{\partial}_{(E,c_E)}$ and $\det \overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})}$ or more precisely

$$\det\,[\overline{\partial}_{(E,c_E)} \to \widetilde{\mathcal{M}}(\Sigma_0)] \quad \text{and} \quad \det\,[\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})} \to \widetilde{\mathcal{M}}(\widetilde{\Sigma})] \tag{8.19}$$

denote their corresponding determinant line bundles. Pulling back the normalization short exact sequence (A.21) over the moduli space (and using (A.25) for the pullback operators) gives an exact sequence

$$0 \to \det \overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})} \to \Phi^* \det \overline{\partial}_{(E,c_E)} \to \det(\mathrm{ev}_{\mathbf{y}}^* E)^{\mathbb{R}} \to 0, \tag{8.20}$$

inducing a canonical isomorphism

$$\Phi^* \det\, \overline{\partial}_{(E,c_E)} = \det\, \overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})} \otimes \det(\mathrm{ev}_{\mathbf{y}}^* E)^{\mathbb{R}}. \tag{8.21}$$

Here $\mathbf{y}$ denotes the collection of contact nodes of the domains, i.e. those in the preimage of the nodes $\mathbf{x}$ of Σ_0. But recall that Σ_0 has a pair of conjugate nodes denoted $x^{\pm}$ (and no other nodes). The restriction to x^+

induces a canonical identification

$$(E_{\mathbf{x}}, c_E)^{\mathbb{R}} \cong E_{x^+}. \tag{8.22}$$

Furthermore, over the moduli space $\widetilde{\mathcal{M}}(\Sigma_0)$ this induces a canonical identification

$$(\mathrm{ev}_{\mathbf{y}}^* E)^{\mathbb{R}} \cong \mathrm{ev}_{\mathbf{y}^+}^*(E) = \mathrm{ev}_{\mathbf{y}^+}^*(E_{x^+}), \tag{8.23}$$

where $\mathbf{y}^+$ denotes the collection of nodes in the preimage of x^+. But x^+ is fixed, and E_{x^+} is a complex vector space. Thus the isomorphism (8.21), after using the complex orientation induced by (8.23) on its last term, induces the isomorphism

$$\det\,[\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})} \to \widetilde{\mathcal{M}}(\widetilde{\Sigma})] \cong \Phi^* \det\,[\overline{\partial}_{(E,c_E)} \to \widetilde{\mathcal{M}}(\Sigma_0)]. \tag{8.24}$$

Next recall the relation described in §6.2 between the family of linearizations D_{f_0} given by (6.13) as f_0 varies in $\widetilde{\mathcal{M}}(\Sigma_0)$, and the family of linearizations $D_{\widetilde{f}}$ given by (6.7), associated to the lifts $\widetilde{f}$, as $\widetilde{f}$ varies in $\widetilde{\mathcal{M}}(\widetilde{\Sigma})$. The relation between these two families of linearizations, described in Remark 6.4, is obtained via intermediate identifications with the linearizations $\widehat{D}_{f_0}$ given by (6.14) and $\widetilde{D}_{f_0}$ given by (6.9). Note that up to 0'th order terms these are pullback families of operators of the form $\overline{\partial}_{(E,c_E)} \oplus 0_W$, as described in paragraph containing (6.6). The natural identifications between their domains, described in Remark 6.4, induce the following isomorphisms at the level of the determinant bundles

$$\Phi^* \det D_{f_0} \cong \Phi^* \det \widehat{D}_{f_0} \cong \Phi^* \det \widetilde{D}_{f_0} \cong \det D_{\widetilde{f}}, \tag{8.25}$$

where

$$\det D_{f_0} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det \mathfrak{f}^* T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}, \tag{8.26}$$

$$\det \widehat{D}_{f_0} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det(\mathcal{T}_{\mathbf{y}^+}) \otimes \det \mathfrak{f}^* T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell} \tag{8.27}$$

$$\det \widetilde{D}_{f_0} \cong \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det \mathrm{ev}_{\mathbf{y}^+}^*(\mathcal{T}_{|x^+}) \otimes \det \mathfrak{f}^* T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell}. \tag{8.28}$$

$$\det D_{\widetilde{f}} \cong \det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}. \tag{8.29}$$

Specifically, as in Remark 6.4, the first identification in (8.25), between the pullbacks of (8.26) and (8.27), is induced by the decomposition (A.31) into tangent and normal directions to the nodal stratum $\mathcal{N}_\ell$ (with $\ell = \ell(\lambda)$

pairs of conjugate nodes), cf. (A.26) and (A.30), which induces the isomorphism

$$\mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \cong \det(\mathcal{T}_{\mathbf{y}^+}) \otimes \mathfrak{f}^* \det T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell}. \tag{8.30}$$

Here $T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell}$ parametrizes those variations in C_0 that do not smooth the nodes. Note that its pullback under the attaching map is $T\overline{\mathcal{M}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}$ which parametrizes the variations in the normalization $\widetilde{C}$ of C_0, so

$$\Phi^*\mathfrak{f}^* \det T\overline{\mathcal{N}}^{\mathbb{R}}_{\ell} \cong \mathfrak{f}^* \det \overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}. \tag{8.31}$$

The pullback of (8.30) combines with (8.31) to give

$$\Phi^*\mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \cong \det \mathcal{T}_{\mathbf{y}^+} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}. \tag{8.32}$$

The second identification in (8.25), between the pullbacks of (8.27) and (8.28) is induced by a complex linear isomorphism – the linearization of (6.12) – cf. Remark 6.4. Finally, the last identification in (8.25), between (8.29) and the pullback of (8.28), comes from the isomorphism (8.31) and the isomorphism

$$\det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})} \underset{(8.21)}{\cong} \Phi^* \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det \mathrm{ev}^*_{\mathbf{y}^+}(\mathcal{T}_{|x_+}) \tag{8.33}$$

induced by the normalization short exact sequence for $\mathcal{T}_{\Sigma_0}$, cf. Remark 6.4.

With these preliminary considerations understood, the main result in this section is the following.

Proposition 8.3. *Fix a choice of orientation data $\mathfrak{o}_{\mathcal{F}}$ on the family $\mathcal{F}$, and orient $\widetilde{m}(\widetilde{\Sigma})$ and $\widetilde{m}(\Sigma_0)$ by the procedure outlined in the paragraph containing* (8.14)*. Then the attaching map Φ in* (8.18) *is orientation preserving with respect to these orientations.*

Proof. Orient $\widetilde{m}(\Sigma_0)$ by identifying its orientation sheaf with (8.26) and trivializing the latter by combining

$$\begin{aligned} &\det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \underset{(8.10)}{\overset{\mathfrak{o}_0}{\cong}} \det \overline{\partial}_{(\mathbb{C},c_{std})} \\ &\text{and} \quad \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \underset{(8.11)}{\cong} \det \overline{\partial}_{(\mathbb{C},c_{std})}. \end{aligned} \tag{8.34}$$

We must compare the pullback under Φ of this orientation with the orientation of $\widetilde{m}(\widetilde{\Sigma})$ obtained by trivializing (8.29) using

$$\begin{aligned} &\det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})} \underset{(8.10)}{\overset{\widetilde{\mathfrak{d}}}{\cong}} \det \overline{\partial}_{(\mathbb{C},c_{std})} \\ &\text{and} \quad \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell} \underset{(8.11)}{\cong} \det \overline{\partial}_{(\mathbb{C},c_{std})}. \end{aligned} \tag{8.35}$$

This means calculating the sign, with respect to these orientations, of the isomorphism

$$\begin{aligned} \Phi^*(\det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}) \\ \cong \det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell} \end{aligned} \tag{8.36}$$

obtained from the composition (8.25) described above. The discussion after (8.25) implies that (8.36) is the composition

$$\begin{aligned} &\Phi^* \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \Phi^*\mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} \\ &\quad \underset{(8.32)}{\cong} \Phi^* \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det(\mathcal{T}_{\mathbf{y}^+}) \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell} \\ &\quad \underset{(6.12)}{\cong} \Phi^* \det \overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})} \otimes \det \mathrm{ev}^*_{\mathbf{y}^+}(\mathcal{T}_{|x_+}) \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell} \\ &\quad \underset{(8.33)}{\cong} \det \overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}. \end{aligned}$$

The second isomorphism comes from the complex identification induced by the linearization of (6.12).

We are now ready to calculate to sign of the isomorphism (8.36); we do this in several steps. In addition to (8.32) and (8.33) we will also use the identification

$$\Phi^* \det \overline{\partial}_{(\mathbb{C},c_{std})} \otimes \det \mathrm{ev}^*_{\mathbf{y}_+}(\mathbb{C}) \underset{(8.21)}{\cong} \det \overline{\partial}_{(\mathbb{C},c_{std})} \tag{8.37}$$

induced by the normalization short exact sequence for $E = \mathbb{C}$.

Step 1. Consider first the tensor product of the isomorphisms (8.32) and (8.37); it corresponds to [GZ2, (4.41)] (taking into account the change of notation). After using the complex orientation on $\mathcal{T}_{\mathbf{y}^+}$ and $\mathrm{ev}^*_{\mathbf{y}_+}(\mathbb{C})$ it

induces the isomorphism

$$\Phi^*\big(\det\overline{\partial}_{(\mathbb{C},c_{std})}\otimes\mathfrak{f}^*\det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})}\big)\cong\det\overline{\partial}_{(\mathbb{C},c_{std})}\otimes\mathfrak{f}^*\det T\overline{\mathcal{M}}^{\mathbb{R}}_{\chi+4\ell,\ell(\vec{\mu})+2\ell}. \tag{8.38}$$

Both sides are oriented via the canonical orientation on the tensor product of the terms in (8.11), i.e. by the second isomorphism in (8.34) and respectively (8.35). By [GZ2, Proposition 4.18], the sign of the isomorphism (8.38), with respect to these orientations, is given by the number mod 2 of pairs of conjugate nodes of the domain i.e. is equal to $(-1)^\ell$.

Step 2. Consider next the tensor product of the isomorphisms (8.33) and (8.37) corresponding to the normalization short exact sequences for $E=\mathcal{T}_{\Sigma_0}$ and respectively $E=\mathbb{C}$. After using the complex orientation on $\mathrm{ev}^*_{\mathbf{y}^+}(\mathcal{T}_{|x_+})$ and $\mathrm{ev}^*_{\mathbf{y}_+}(\mathbb{C})$ it induces the isomorphism

$$\Phi^*\big(\det\overline{\partial}_{(\mathcal{T}_{\Sigma_0},c_{\mathcal{T}})}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})}\big)\underset{(8.24)}{\cong}\det\overline{\partial}_{(\mathcal{T}_{\widetilde{\Sigma}},c_{\mathcal{T}})}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})}. \tag{8.39}$$

Both sides are oriented via the canonical trivialization (8.10) for $\mathfrak{o}_0$ on $\mathcal{T}_{\Sigma_0}$ and $\widetilde{\mathfrak{o}}$ on $\mathcal{T}_{\widetilde{\Sigma}}$, i.e. by the first isomorphisms in (8.34) and respectively (8.35). By Lemma 8.4 below, the sign of the isomorphism (8.39), with respect to these orientations is also $(-1)^\ell$.

Step 3. Finally, consider the tensor product of two copies of the isomorphism (8.37); after using the complex orientation on $\mathrm{ev}^*_{\mathbf{y}_+}(\mathbb{C})$ twice, it induces the isomorphism

$$\Phi^*\big(\det\overline{\partial}_{(\mathbb{C},c_{std})}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})}\big)\underset{(8.24)}{\cong}\det\overline{\partial}_{(\mathbb{C},c_{std})}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})}. \tag{8.40}$$

Both sides are oriented via the canonical trivialization of twice of a bundle; with respect to these orientations, (8.40) is orientation preserving since the normalization short exact sequence is natural and the complex orientation on $\mathrm{ev}^*_{\mathbf{y}_+}(\mathbb{C}\oplus\mathbb{C})$ agrees with the one induced by regarding it as twice of a bundle.

The proof is completed by observing that the tensor product of (8.38) and (8.39) agrees (up to homotopy) with the tensor product of (8.40) and (8.36) and the orientations discussed above also agree. Thus the sign of (8.36) with respect to the orientations induced by (8.34) and respectively (8.35) is $(-1)^\ell\cdot(-1)^\ell=+1$. □

We conclude this section with two results on the behavior of the trivialization (8.6) under pullback to the normalization. A special case of the following result was used in Step 2 above.

Lemma 8.4. *Assume $\mathfrak{o}_0$ is a twisted orientation on a Real line bundle (W,ϕ) over the nodal symmetric marked curve Σ_0, and let $\widetilde{\mathfrak{o}}$ denote its pullback, which is a twisted orientation on the pullback bundle $(\widetilde{W},\widetilde{\phi})$ over the normalization $\widetilde{\Sigma}$. Consider the isomorphism*

$$\Phi^*(\det\overline{\partial}_{(W,\phi)}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})})\underset{(8.24)}{\cong}\det\overline{\partial}_{(\widetilde{W},\widetilde{\phi})}\otimes\det\overline{\partial}_{(\mathbb{C},c_{std})}$$

where the determinant line bundles are as in (8.19)*. Orient both sides by* (8.6) *for $\mathfrak{o}_0$ and respectively $\widetilde{\mathfrak{o}}$. Then, with respect to these orientations, this isomorphism has sign $(-1)^\ell$, where ℓ is the number of conjugate pairs of nodes of the domain.*

Proof. The isomorphism (8.6) is obtained by combining the isomorphisms (8.3)-(8.5) with the canonical orientation on the square of a bundle. Therefore it suffices to understand how each one of these isomorphisms behaves under pullback by Φ.

First of all, when $(E,c_E)=(2L,2c_L)$ is twice a Real bundle, the isomorphism

$$\Phi^*\det\overline{\partial}_{(E,c_E)}\underset{(8.24)}{\cong}\det\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})} \tag{8.41}$$

is orientation preserving with respect to orienting both $\det\overline{\partial}_{(E,c_E)}$ and $\det\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})}$ as a square of a real bundle. This follows by the naturality of the normalization short exact sequence and the fact that both the pullback of E and the restriction of E to $\mathbf{y}^+$ is twice a bundle. Moreover, the complex orientation on twice of a complex bundle agrees with the canonical orientation on twice a bundle.

We next focus on the isomorphism (8.4), which is induced by the canonical homotopy class of isomophisms (8.3). But (8.3) pulls back to the normalization to give (up to homotopy) the isomorphism (8.3) for the pullback structure; it also restricts to the nodes $\mathbf{y}^+$ as a complex linear isomorphism

(unique up to homotopy). Therefore the diagram

$$\begin{array}{ccc} \Phi^*\det\overline{\partial}_{(W,\phi)\oplus(E,c_E)^\vee} & \xrightarrow[(8.4)]{\mathfrak{o}_0} & \Phi^*\det\overline{\partial}_{(\mathbb{C}^{n+2},c_{std})} \\ \Big\downarrow{\scriptstyle (8.24)} & & \Big\downarrow{\scriptstyle (8.24)} \\ \det\overline{\partial}_{(\widetilde{W},\widetilde{\phi})\oplus(\widetilde{E},c_{\widetilde{E}})^\vee} & \xrightarrow[(8.4)]{\widetilde{\mathfrak{o}}} & \det\overline{\partial}_{(\mathbb{C}^{n+2},c_{std})} \end{array}$$

commutes up to homotopy. Equivalently, consider the isomorphism

$$\begin{aligned} (8.42)\qquad & \Phi^*(\det\overline{\partial}_{(W,\phi)}\otimes\det\overline{\partial}_{(E,c_E)^\vee}\otimes\det\overline{\partial}_{(\mathbb{C}^{n+2},c_{std})}) \\ & \underset{(8.24)}{=}\det\overline{\partial}_{(\widetilde{W},\widetilde{\phi})}\otimes\det\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})^\vee}\otimes\det\overline{\partial}_{(\mathbb{C}^{n+2},c_{std})} \end{aligned}$$

induced by the normalization short exact sequence and the complex orientation on the restriction of the bundles W, E and $\mathbb{C}^{n+2}$ to the nodes $\mathbf{y}^+$. Orient both sides of (8.42) by the canonical isomorphism induced by the tensor product of the two sides of (8.4) for $\mathfrak{o}_0$ and respectively $\widetilde{\mathfrak{o}}$. Then with respect to these orientations, the isomorphism (8.42) is orientation preserving.

It remains to understand the behavior of the isomorphism (8.5) under pullback. For that, consider the isomorphism (8.24) for the bundle $(E,c_E)=(L\oplus c^*\overline{L},c_{tw})$. By Lemma 8.5 below the sign of this isomorphism, with respect to the complex orientations induced by (8.5), is the number mod 2 of pairs of conjugate nodes of the domain i.e. is equal to $(-1)^\ell$. This completes the proof. □

Lemma 8.5. *Assume L is a complex line bundle over Σ_0. Let $(E,c_E)=(L\oplus c^*\overline{L},c_{tw})$ and denote by $\widetilde{E}$ its pullback to the normalization $\widetilde{\Sigma}$. Consider the isomorphism*

$$\Phi^*\det\overline{\partial}_{(E,c_E)}\underset{(8.24)}{\cong}\det\overline{\partial}_{(\widetilde{E},c_{\widetilde{E}})}$$

where the determinant line bundles are as in (8.19)*. Orient both sides by the complex orientation induced by the isomorphism* (8.5) *for E and respectively $\widetilde{E}$. Then with respect to these orientations, this isomorphism has sign $(-1)^\ell$, where ℓ is the number of pairs of conjugate nodes of the domain.*

Proof. The isomorphism (8.24) is induced by the normalization short exact sequence (A.21) together with the identification (8.23). Consider also the

normalization short exact sequence

$$0 \longrightarrow L \longrightarrow \widetilde{L} \longrightarrow L_{\mathbf{x}} \longrightarrow 0.$$

for the complex line bundle L. Compare the pullback of the short exact sequences for E and L with the isomorphisms (8.5) for E and $\widetilde{E}$ obtained by projecting onto the first factor. We would get a commutative diagram if we used the corresponding identification

$$(\mathrm{ev}_{\mathbf{y}}^* E)^{\mathbb{R}} \underset{p_1^*}{\cong} \mathrm{ev}_{\mathbf{y}}^* L = \mathrm{ev}_{\mathbf{y}^+}^* L \oplus \mathrm{ev}_{\mathbf{y}^-}^* L.$$

However, we are using instead the identification (8.23) induced by the restriction to $\mathbf{y}^+$, i.e.

$$(\mathrm{ev}_{\mathbf{y}}^* E)^{\mathbb{R}} \cong \mathrm{ev}_{\mathbf{y}^+}^* E = \mathrm{ev}_{\mathbf{y}^+}^* (L \oplus c^* \overline{L}) = \mathrm{ev}_{\mathbf{y}^+}^* L \oplus \mathrm{ev}_{\mathbf{y}^-}^* \overline{L}.$$

The difference between the complex orientations of the last two displayed equations is $(-1)^\ell$, coming from the rank ℓ of the complex bundle $\mathrm{ev}_{\mathbf{y}^-}^* L$. □

9. Proof of the VFC Splitting Theorem 1.2

Let Σ_0 be a nodal symmetric curve with r pairs of conjugate marked points V_0 as in (2.5) and a pair of conjugate nodes $x^\pm$ (and no other special points). We first assume that Σ_0 has no rational connected components without marked points; this assumption is removed at the end of this section, cf. Remark 9.2.

Consider the family $\mathcal{F} = \cup_s \Sigma_s$ of targets from §3.1. Fix a parameter $s_0 \neq 0$ and let I denote the segment $[0, s_0] \subset \Delta$. Fix also the degree d, the Euler characteristic χ, and the ramification profile $\vec{\mu} = (\mu^1, \ldots, \mu^r)$ and consider the families

$$\overline{m}(\mathcal{F}_{/I}) = \underset{s\in I}{\cup} \overline{m}(\Sigma_s) \quad \text{and} \quad \overline{m}(\widetilde{\Sigma})$$

of real relative moduli spaces defined in (4.25). Recall that these are families over the parameter space $\mathcal{P} = \mathcal{JV}(\mathcal{F}_{/\Delta})$ of RT-perturbations ν, whose fibers $\overline{m}(-)_\nu$ at ν are respectively

$$\begin{aligned} &\overline{m}(\mathcal{F}_{/I})_\nu = \underset{s\in I}{\cup} \overline{m}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)_\nu \\ &\text{and} \quad \overline{m}(\widetilde{\Sigma})_\nu = \underset{\lambda\vdash d}{\sqcup} \overline{m}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})_\nu. \end{aligned} \tag{9.1}$$

By construction we have proper maps

$$\overline{m}(\widetilde{\Sigma}) \xrightarrow{\Phi} \overline{m}(\Sigma_0) \qquad \text{and} \qquad \overline{m}(\Sigma_s) \hookrightarrow \overline{m}(\mathcal{F}_{/I}) = \underset{s\in I}{\cup} \overline{m}(\Sigma_s)$$

for all $s \in I$, as in (3.13) and (3.15) (which correspond to $\nu = 0$). Moreover, restricting to the open subsets $\widetilde{m}(-) \subseteq \overline{m}(-)$, cf. (5.12), gives rise to proper maps

$$\widetilde{m}(\widetilde{\Sigma}) \xrightarrow{\Phi|} \widetilde{m}(\Sigma_0) \qquad \text{and} \qquad \widetilde{m}(\Sigma_s) \hookrightarrow \widetilde{m}(\mathcal{F}_{/I}) \stackrel{\text{def}}{=} \underset{s\in I}{\cup} \widetilde{m}(\Sigma_s).$$

Notice that throughout this argument we must work with proper maps in order to push forward classes in rational Cech homology (e.g. maps that extend continuously to some compactification).

Fix an orientation data $\mathfrak{o}_{\mathcal{F}}$ on the family $\mathcal{F}$ and let $\mathfrak{o}_s$ and respectively $\widetilde{\mathfrak{o}}$ be its pullback to Σ_s and $\widetilde{\Sigma}$.

Proof of the VFC Splitting Theorem 1.2. Assume that Σ_0 has no rational connected components without any marked points. Then by Theorem 5.3, $\overline{m}(\Sigma_{s_0})$, $\overline{m}(\Sigma_0)$ and $\overline{m}(\widetilde{\Sigma})$ are thinly compactified families over the parameter space $\mathcal{P} = \mathcal{JV}(\mathcal{F}_{/\Delta})$. In particular, for generic ν, the fibers of these families are thin compactifications of topological manifolds, cf. Proposition 5.4. The latter are canonically oriented by the choice of $\mathfrak{o}_{\mathcal{F}}$ as described in the paragraph containing (8.14). Our goal is to compare the images of the fundamental classes

$$[\widetilde{m}(\Sigma_{s_0})_\nu], \quad \Phi_*[\widetilde{m}(\widetilde{\Sigma})_\nu], \quad \text{and} \quad [\widetilde{m}(\Sigma_0)_\nu] \qquad \text{in} \quad \check{\mathrm{H}}_b(\widetilde{m}(\mathcal{F}_{/I})_\nu; \mathbb{Q})$$

of these manifolds (for generic ν). Here b is the dimension of these moduli spaces, cf. (2.7).

For generic ν, by Lemma 6.1 and Proposition 8.3, the attaching map Φ restricts to a finite degree orientation preserving proper map between the oriented topological manifolds $\widetilde{m}(\widetilde{\Sigma})_\nu$ and $\widetilde{m}(\Sigma_0)_\nu$; each of these spaces decomposes as a disjoint union of open and closed subsets indexed by λ. Moreover by (6.4)

$$[\widetilde{m}_\lambda(\Sigma_0)_\nu] = \frac{1}{|\mathrm{Aut}\ \lambda|} \cdot \Phi_*[\widetilde{m}_\lambda(\widetilde{\Sigma})_\nu] \tag{9.2}$$

in $\check{\mathrm{H}}_b(\widetilde{m}_\lambda(\Sigma_0)_\nu; \mathbb{Q})$ and therefore in $\check{\mathrm{H}}_b(\widetilde{m}(\Sigma_0)_\nu; \mathbb{Q})$.

Consider next the auxiliary spaces $\widehat{\mathcal{M}}(\Sigma_0)$ and $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ constructed in §7. They are also families over the parameter space $\mathcal{P}$ and come with proper maps

$$\begin{array}{ccc} \widehat{\mathcal{M}}(\Sigma_0) & \longhookrightarrow & \widehat{\mathcal{M}}(\mathcal{F}_{/I}) \\ {\scriptstyle q_0}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \widetilde{\mathcal{M}}(\Sigma_0) & \longhookrightarrow & \widetilde{\mathcal{M}}(\mathcal{F}_{/I}) \end{array}$$

cf. (7.1) and (7.7). But for generic ν, by Proposition 7.3 the fiber $\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ of $\widehat{\mathcal{M}}(\mathcal{F}_{/I})$ is a cobordism between the manifold $\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu$ and the cover $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ of the manifold $\widetilde{\mathcal{M}}(\Sigma_0)_\nu$. This cobordism is also canonically oriented by the choice $\mathfrak{o}_{\mathcal{F}}$ and the restriction of this orientation to the boundary agrees with the one above, cf. the paragraph containing (8.17). Thus

$$[\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu] = [\widehat{\mathcal{M}}(\Sigma_0)_\nu] = \sum_\lambda [\widehat{\mathcal{M}}_\lambda(\Sigma_0)_\nu] \tag{9.3}$$

in $\check{\mathrm{H}}_b(\widehat{\mathcal{M}}(\mathcal{F}_{/I})_\nu;\mathbb{Q})$, where the first equality is (7.15), and the second equality holds because, as in (7.2), the manifold $\widehat{\mathcal{M}}(\Sigma_0)_\nu$ is a disjoint union of open and closed subsets indexed by λ.

Furthermore, by Lemma 7.1

$$(q_0)_*[\widehat{\mathcal{M}}_\lambda(\Sigma_0)]_\nu = \frac{\zeta(\lambda)}{|\mathrm{Aut}\ \lambda|} \cdot [\widetilde{\mathcal{M}}_\lambda(\Sigma_0)_\nu] \tag{9.4}$$

in $\check{\mathrm{H}}_b(\widetilde{\mathcal{M}}(\Sigma_0)_\nu;\mathbb{Q})$.

Pushing forward (9.3) by the proper map q and combining it with the pushforward of (9.4) and (9.2) by the proper inclusion $\widetilde{\mathcal{M}}(\Sigma_0)_\nu \hookrightarrow \widetilde{\mathcal{M}}(\mathcal{F}_{/I})_\nu$ gives the equality

$$[\widetilde{\mathcal{M}}(\Sigma_{s_0})_\nu] = \sum_\lambda \frac{\zeta(\lambda)}{|\mathrm{Aut}\ \lambda|} \cdot [\widetilde{\mathcal{M}}_\lambda(\Sigma_0)_\nu] = \sum_\lambda \frac{\zeta(\lambda)}{|\mathrm{Aut}\ \lambda|^2} \cdot \Phi_*[\widetilde{\mathcal{M}}_\lambda(\widetilde{\Sigma})_\nu] \tag{9.5}$$

in $\check{\mathrm{H}}_b(\widetilde{\mathcal{M}}(\mathcal{F}_{/I})_\nu;\mathbb{Q})$; this holds for generic ν in $\mathcal{P}$.

It remains to show that the equality (9.5) lifts to an equality between the fundamental classes of the thin compactifications (for generic ν) and thus uniquely extends to an equality between their VFCs for all ν.

For that, as in (5.10), consider the difference

$$\mathcal{S}_\nu = \overline{m}(\mathcal{F}_{/I})_\nu \setminus \widetilde{m}(\mathcal{F}_{/I})_\nu$$

which is a closed subset of $\overline{m}(\mathcal{F}_{/I})_\nu$. For generic ν, by Proposition 5.4, it has homological dimension at most $(b+1)-2 = b-1$, i.e. $\check{\mathrm{H}}_*(\mathcal{S}_\nu;\mathbb{Q})=0$ for all $* > b-1$. As in the proof of [IP5, Lemma 2.10], the long exact sequence (2.12) in rational Cech homology

$$\ldots \to \check{\mathrm{H}}_*(\mathcal{S}_\nu;\mathbb{Q}) \to \check{\mathrm{H}}_*(\overline{m}(\mathcal{F}_{/I})_\nu;\mathbb{Q}) \xrightarrow{\rho} \check{\mathrm{H}}_*(\widetilde{m}(\mathcal{F}_{/I})_\nu;\mathbb{Q}) \to \ldots$$

associated to the closed pair $(\overline{m}(\mathcal{F}_{/I})_\nu, \mathcal{S}_\nu)$ then implies that ρ is *injective* in dimension $* = b$. Therefore (9.5) implies that for generic ν

$$[\overline{m}(\Sigma_{s_0})_\nu] = \sum_\lambda \frac{\zeta(\lambda)}{|\mathrm{Aut}\ \lambda|^2} \cdot \Phi_*[\overline{m}_\lambda(\widetilde{\Sigma})_\nu] \tag{9.6}$$

in $\check{\mathrm{H}}_b(\overline{m}(\mathcal{F}_{/I})_\nu;\mathbb{Q})$ as follows. It suffices to check that ρ maps (9.6) to (9.5). This follows from the naturality of the long exact sequence (2.12), combined with the facts that by construction (i) the intersection of $\widetilde{m}(\mathcal{F}_{/I})_\nu$ with $\overline{m}(\Sigma_s)_\nu$ is equal to $\widetilde{m}(\Sigma_s)_\nu$, (ii) the inverse image of $\widetilde{m}(\Sigma_0)_\nu$ under Φ is equal to $\widetilde{m}(\widetilde{\Sigma})_\nu$, and (iii) $\overline{m}(\Sigma_{s_0})_\nu$ and $\overline{m}(\widetilde{\Sigma})_\nu$ are thin compactifications of the manifolds $\widetilde{m}(\Sigma_{s_0})_\nu$ and respectively $\widetilde{m}(\widetilde{\Sigma})_\nu$ (for generic ν in $\mathcal{P}$) by Proposition 5.4.

Finally, recall that by Theorem 5.3, $\overline{m}(\Sigma_{s_0})$ and $\overline{m}(\widetilde{\Sigma})$ are thinly compactified families over $\mathcal{P}$, thus each fiber carries a VFC as in (5.2). By [IP5, Lemma 3.4] the relation (9.6) for generic ν extends uniquely to the corresponding relation between the VFCs for all parameters $\nu \in \mathcal{P}$, including for $\nu = 0$. Applying Lemma 5.5 and switching back to the original notation (9.1) for the fibers at $\nu = 0$ gives (1.6). This completes the proof of Theorem 1.2 under the assumption that all rational connected components of Σ_0 have at least one marked point. The remaining case is treated in Remark 9.2 below. □

9.1. A relation between the absolute and relative VFCs

We end this section with a consequence of the VFC Splitting Theorem, describing how the VFC behaves under adding a pair of conjugate marked points of the target. To state it, we need to first introduce some notation.

Recall that if Σ is a smooth complex curve, there is a flat family $\mathcal{C} \to \mathbb{C}$ of deformations of Σ whose central fiber is the nodal curve $\Sigma \cup \mathbb{P}^1$ with one node. It can be obtained by fixing a point $x \in \Sigma$ and blowing up the constant family $\Sigma \times \mathbb{C} \to \mathbb{C}$ at the point $x \times 0$. The exceptional divisor is $\mathbb{P}(T_x\Sigma \oplus \mathbb{C}) = \mathbb{P}^1$. So the restriction of $\mathcal{C}$ over $\mathbb{C} \setminus 0$ is the product family, but $\mathcal{C}$ has a nodal fiber $\Sigma \cup \mathbb{P}^1$ over $s = 0$.

Assume next that (Σ, V) is a smooth symmetric curve with r pairs of conjugate marked points as in (2.5), and choose $\mathbf{x} = \{x^{\pm}\}$ an additional pair of conjugate points on Σ, disjoint from V. Blowing up the product family $\Sigma \times \mathbb{C}^2$ along the submanifold $(x^+ \times 0 \times \mathbb{C}) \sqcup (x^- \times \mathbb{C} \times 0)$ similarly yields a flat family $\mathcal{F}$ of deformations as in (3.1)-(3.2). Each smooth fiber of $\mathcal{F}_{/\Delta} = \bigcup_{s \in \Delta} \Sigma_s$ is $\Sigma_s = \Sigma$ for all $s \neq 0$, while the central fiber is a nodal curve

$$\Sigma_0 = \Sigma \cup \mathbb{P}_{\mathbf{x}} \tag{9.7}$$

with a pair of conjugate nodes and an additional pair $\mathbb{P}_{\mathbf{x}}$ of conjugate spherical components; see also (3.8). The marked points V_0 of Σ_0 correspond to the marked points V of Σ under the inclusion $\Sigma \subset \Sigma_0$.

Let $p : \Sigma_0 \to \Sigma$ denote the map which collapses $\mathbb{P}_{\mathbf{x}}$ to $\mathbf{x}$ and restricts to the identity on Σ. This induces a collapsing map from any building associated to $\Sigma_0 = \Sigma \cup \mathbb{P}_{\mathbf{x}}$ to one associated to Σ (collapsing the entire chain $\mathbb{P}_{\mathbf{x}} \underset{\mathbf{x_0}=\mathbf{x}_\infty}{\cup} \mathbb{P}_{\mathbf{x}} \underset{\mathbf{x_0}=\mathbf{x}_\infty}{\cup} \dots \underset{\mathbf{x_0}=\mathbf{x}_\infty}{\cup} \mathbb{P}_{\mathbf{x}}$ attached at $\mathbf{x}$, but keeping the chains attached at V).

Therefore p induces a proper continuous map

$$\mathfrak{p} : \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_0, V_0) \to \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$$

at the level of (unperturbed) moduli spaces.

Consider the family of targets described above (9.7), and let $\mathfrak{o}_{\mathcal{F}}$ be a choice of twisted orientation data on it. Assume that (Σ, V) has no rational components without any special points; then the same is true for $(\widetilde{\Sigma}, \widetilde{V})$ and (Σ_0, V_0). Therefore the moduli spaces $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$, $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma_0, V_0)$ and $\overline{\mathcal{M}}^{\mathbb{R}}(\widetilde{\Sigma}, \widetilde{V})$ have a well defined VFC, as defined in §5, and the VFC Splitting Theorem holds, as proved at the beginning of §9. This has the following consequence.

Corollary 9.1. *For every d, χ, and $\vec{\mu}$, the equality*

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)]^{\mathrm{vir},\mathfrak{o}} = \sum_{\lambda \vdash d} \frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2}\, \mathfrak{p}_* \Phi_* [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})]^{\mathrm{vir},\widetilde{\mathfrak{o}}} \tag{9.8}$$

holds in the rational Cech homology of $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$.

Proof. By the VFC Splitting Theorem 1.2, for every $s \in \Delta \setminus 0$, we get the equality

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)]^{\mathrm{vir},\mathfrak{o}_s} = \sum_{\lambda \vdash d} \frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2}\, \Phi_*[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})]^{\mathrm{vir},\widetilde{\mathfrak{o}}} \tag{9.9}$$

in the rational Cech homology of the family $\bigcup\limits_{s\in\Delta} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)$. It suffices to show that (9.9) pushes forward to (9.8) under a suitable proper continuous map.

The collapsing map $p : \Sigma_0 \to \Sigma$ is the restriction of the composition

$$\mathcal{F} \to \Sigma \times \Delta^2 \xrightarrow{\mathrm{pr}_\Sigma} \Sigma \tag{9.10}$$

where the first arrow collapses the exceptional divisor and the second arrow is the projection onto Σ; note that both of these are holomorphic maps, compatible with the real structures. Then the composition (9.10) similarly induces a proper continuous map

$$\mathfrak{p} : \bigcup_{s\in\Delta} \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s) \to \overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$$

at the level of (unperturbed) moduli spaces, and therefore a map between their rational Cech homology groups. Consider next the pushforward of (9.9) under this map. Note that for $s \neq 0$, the map (9.10) restricts to the identity $\Sigma_s = \Sigma$, therefore

$$\mathfrak{p}_*[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)]^{\mathrm{vir},\mathfrak{o}_s} = [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)]^{\mathrm{vir},\mathfrak{o}}$$

for all $s \neq 0$. Thus the pushforward of (9.9) is (9.8). □

Remark 9.2. So far both the construction of the VFC and the proof the VFC Splitting Theorem were done under the technical assumption that the target curves have no spherical components without any special points. In general, we can always first deform Σ to a nodal curve Σ_0 whose normalization $\widetilde{\Sigma} = \Sigma \sqcup \mathbb{P}_{\mathbf{x}}$ has at least one marked point on each spherical component cf. (9.7), and then *define* the VFC of $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ by the formula (9.8). This is well defined, and the VFC Splitting Theorem 1.2 *automatically* holds in

this case. In particular, when $V = \emptyset$, (9.8) becomes

$$[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi}(\Sigma)]^{\mathrm{vir},\mathfrak{o}} = \sum_{\lambda \vdash d} \frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2}\, \mathfrak{p}_* \Phi_* [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\lambda,\lambda}(\Sigma \sqcup \mathbb{P}_{\mathbf{x}}, \mathbf{x} \sqcup \mathbf{x}_\infty)]^{\mathrm{vir},\widetilde{\mathfrak{o}}}.$$

10. Proof of the RGW Splitting Theorem 1.1

We begin with a brief review of the definition of the integrand in (2.9) and its properties.

10.1. Index bundles

Let E be a holomorphic vector bundle over a complex curve Σ. The complex operator $\overline{\partial}_E$ determines by pullback a family of complex operators over the moduli spaces of holomorphic maps to Σ; the fiber at $f : C \to \Sigma$ is the pullback operator $\overline{\partial}_{f^*E}$. Denote by

$$\mathrm{Ind}\, \overline{\partial}_E = R^\bullet \pi_* \mathrm{ev}^*(E) \tag{10.1}$$

the index bundle associated to this family of operators, regarded as an element in K-theory. Here $\pi : \mathcal{C} \to \overline{\mathcal{M}}$ is the universal curve over the moduli space and $\mathrm{ev} : \mathcal{C} \to \Sigma$ is the evaluation map. When Σ is a marked curve, Bryan and Pandharipande considered the index bundle $\mathrm{Ind}\, \overline{\partial}_E$ over the relative moduli space associated to Σ, cf. [BP2, §2.2] (they denote the evaluation map ev by f). This index bundle is defined by the same formula (10.1), but now the domain of π is the universal curve over the relative moduli space. Let

$$c_k(-\mathrm{Ind}\, \overline{\partial}_E) \tag{10.2}$$

denote the corresponding k'th Chern class, regarded as an element in the Cech cohomology of the relative moduli space.

10.2. The integrand

Let (Σ, c) be a symmetric marked curve with marked points V as in (2.5), and let $L \to \Sigma$ be a holomorphic line bundle over the underlying complex curve Σ. Fix the topological data d, χ and $\vec{\mu}$ and let b be as in (2.7).

Consider the real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$. It comes with a forgetful map to the (usual) relative moduli space associated to the complex

marked curve Σ. Denote by

$$I(\Sigma; L) = c_{b/2}(-\mathrm{Ind}\,\overline{\partial}_L) \in \check{\mathrm{H}}^b(\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V); \mathbb{Q}) \tag{10.3}$$

the pullback of the corresponding Chern class (10.2) on the usual (complex) relative moduli space. This is the integrand that appears in (2.9), regarded as a cohomology class on the real relative moduli space. The RGW invariant (2.9) is then the pairing

$$RGW^{c,\mathfrak{o}}_{d,\chi}(\Sigma, L)_{\vec{\mu}} = \langle I(\Sigma; L),\ \frac{1}{|\mathrm{Aut}(\vec{\mu})|}[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)]^{\mathrm{vir},\mathfrak{o}}\rangle \tag{10.4}$$

with the corresponding VFC of the moduli space.

10.3. Splitting the integrand

Assume next that $L \to \mathcal{F}$ is a holomorphic line bundle over the holomorphic family $\mathcal{F} = \cup_s \Sigma_s$ where $\mathcal{F}$ is as in (3.1). Denote by L_s and respectively $\widetilde{L}$ the pullback of L to Σ_s and respectively the normalization $\widetilde{\Sigma}$ of Σ_0. As s varies in Δ, consider the class

$$I(\Sigma_s, L_s) = c_{b/2}(-\mathrm{ind}\,\overline{\partial}_{L_s}) \in \check{\mathrm{H}}^b(\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s); \mathbb{Q})$$

associated to Σ_s as in (10.3). Denote by

$$I(\mathcal{F}, L) = c_{b/2}(-\mathrm{ind}\,\overline{\partial}_L) \in \check{\mathrm{H}}^b(\overline{\mathcal{M}}_{d,\chi,\vec{\mu}}(\mathcal{F}_{/\Delta}); \mathbb{Q}) \tag{10.5}$$

the corresponding class on the family (3.14) of real relative moduli spaces; as in (10.3), this is the pullback of the class $c_{b/2}(-R^\bullet\pi_*\mathrm{ev}^*L)$ on the family of (complex) relative moduli spaces. We also consider the class $I(\widetilde{\Sigma}, \widetilde{L})$ associated by (10.3) to $\widetilde{\Sigma}$. These classes are related as follows.

Lemma 10.1. *The pullback of $I(\mathcal{F}, L)$ under the inclusion* (3.15) *is equal to the class $I(\Sigma_s, L_s)$. Moreover, the pullback of $I(\mathcal{F}, L)$ under the attaching map* (3.16) *is the corresponding class $I(\widetilde{\Sigma}, \widetilde{L})$ associated to $\widetilde{\Sigma}$, i.e.*

$$\Phi^* I(\mathcal{F}, L) = I(\widetilde{\Sigma}, \widetilde{L}). \tag{10.6}$$

Proof. Forgetting the real structure commutes with the maps at level of moduli spaces induced by $\Sigma_s \hookrightarrow \mathcal{F}$ and $\widetilde{\Sigma} \to \Sigma_0 \hookrightarrow \mathcal{F}$. The lemma then follows from the corresponding results for the usual (complex) family of relative moduli spaces proved by Bryan-Pandharipande, as given in the proofs of [BP2, Theorem 3.2] and [BP1, Proposition A.1].

Specifically, the first statement is an immediate consequence of the fact that the universal curve of the relative moduli space associated to Σ_s is the pullback of the universal curve on the family of moduli spaces.

The second statement follows from the normalization exact sequence

$$0 \longrightarrow L_{|\Sigma_0} \longrightarrow \widetilde{L} \longrightarrow L_{|x^+} \oplus L_{|x^-} \longrightarrow 0,$$

for the holomorphic line bundle L, where $x^{\pm}$ is the pair of (conjugate) nodes of Σ_0. Pulling this back over the (complex) moduli space gives

$$\Phi^* c(-\mathrm{Ind}\,\overline{\partial}_{L_{|\Sigma_0}}) = c(-\mathrm{Ind}\,\overline{\partial}_{\widetilde{L}})$$

for the total Chern classes as in [BP2], since the pullback of the last term is a trivial rank $2\ell(\lambda)$ bundle. □

Proof of the RGW Splitting Theorem 1.1. Consider the invariant (2.9) associated to Σ_s, written in the form (10.4). *A priori*, it is a pairing between a homology and a cohomology class defined on the moduli space associated to Σ_s. However, by Lemma 10.1 the integrand is pulled back from the class (10.5) on the family. Combining this with the splitting of the VFC formula (1.6) gives

$$\begin{aligned}
RGW^{c_s,\mathfrak{o}_s}_{d,\chi}(\Sigma_s, L_s)_{\vec{\mu}} &= \frac{1}{|\mathrm{Aut}(\vec{\mu})|}\langle I(\mathcal{F}, L),\ [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma_s, V_s)]^{\mathrm{vir},\mathfrak{o}_s}\rangle \\
&= \frac{1}{|\mathrm{Aut}(\vec{\mu})|}\sum_{\lambda\vdash d}\frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2}\langle I(\mathcal{F}, L),\ \Phi_*[\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})]^{\mathrm{vir},\widetilde{\mathfrak{o}}}\rangle \\
&= \frac{1}{|\mathrm{Aut}(\vec{\mu})|}\sum_{\lambda\vdash d}\frac{\zeta(\lambda)}{|\mathrm{Aut}(\lambda)|^2}\langle \Phi^* I(\mathcal{F}, L),\ [\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi+4\ell(\lambda),\vec{\mu},\lambda,\lambda}(\widetilde{\Sigma}, \widetilde{V})]^{\mathrm{vir},\widetilde{\mathfrak{o}}}\rangle \\
&= \sum_{\lambda\vdash d}\zeta(\lambda) RGW^{\widetilde{c},\widetilde{\mathfrak{o}}}_{d,\chi+4\ell(\lambda)}(\widetilde{\Sigma}, \widetilde{L})_{\vec{\mu},\lambda,\lambda},
\end{aligned}$$

where the last equality uses (10.6). □

Using the fact that the local RGW invariants are constant under smooth deformations of the target and passing to the (shifted) generating functions (2.10) we also obtain the following consequence.

Corollary 10.2. *Let (Σ_0, c_0) be a symmetric marked curve with a pair of conjugate nodes (and no real marked points), L_0 a complex line bundle on Σ_0, and $\mathfrak{o}_0$ a choice of twisted orientation data on Σ_0. Then the local*

RGW invariants of any smooth deformation $(\Sigma, c, L, \mathfrak{o})$ *of* $(\Sigma_0, c_0, L_0, \mathfrak{o}_0)$ *are related to those of the normalization* $\widetilde{\Sigma}$ *of* Σ_0 *by*

$$RGW_d^{c,\mathfrak{o}}(\Sigma, L)_{\vec{\mu}} = \sum_{\lambda \vdash d} \zeta(\lambda) t^{2\ell(\lambda)} RGW_d^{\widetilde{c},\widetilde{\mathfrak{o}}}(\widetilde{\Sigma}, \widetilde{L})_{\vec{\mu},\lambda,\lambda}.$$

As discussed in [GI, §4], every smooth symmetric curve Σ (without any real marked points) can be deformed into such a nodal symmetric curve Σ_0 by pinching a pair of *conjugate splitting circles*. Furthermore, every complex line bundle L over Σ and choice $\mathfrak{o}$ of twisted orientation data for Σ can also be deformed to the nodal curve Σ_0 and then lifted to the normalization $\widetilde{\Sigma}$ of Σ_0. Conversely, every complex line bundle $\widetilde{L}$ and twisted orientation data $\widetilde{\mathfrak{o}}$ on $\widetilde{\Sigma}$ descend to Σ_0 and can be deformed to a complex line bundle L and orientation data $\mathfrak{o}$ on Σ. Therefore Corollary 10.2 implies [GI, Theorem 4.1].

Appendix A. Linearizations

In this appendix we include more details about the various linearizations considered and the precise relation between them. For transversality and gluing arguments, we use Fredholm completions in the type of (weighted) Sobolev norms defined in [IP1, p. 80], [IP2, p. 971] and [IP2, (1.1) of Corrigendum], reviewed in §A.5 and used in [IP1] and [IP2] to analyze the relative moduli spaces (relative a smooth divisor) appearing in the symplectic sum construction. For the absolute moduli spaces, similar norms are defined in [P, §B.4], and used by Pardon for a proof of the usual gluing theorem. For orientability and the construction of determinant line bundles of real CR-operators we use [GZ1, Appendix A.2], where the authors take Fredholm completions in a different variant of these norms, those originally used by Li-Tian in the proof of the usual gluing theorem, cf. [LT, §3]. The monograph [MS] uses both usual Sobolev norms reviewed in [MS, §B.1] for transversality [MS, §3.2] and another version of weighted Sobolev norms [MS, §10.3] for the proof of the usual gluing theorem [MS, §10.1].

A.1. General considerations

Let (C, c) be a (smooth) symmetric marked curve, and $\mathbf{x}$ denote a symmetric subset of the marked points of C. Let $(E, c_E) \to (C, c)$ be a Real bundle, i.e. a complex vector bundle with a real structure c_E covering c (called a real bundle pair in [GZ1]). Denote by $\Gamma(E)^{\mathbb{R}}$ the space of smooth sections of E

which are real, i.e. invariant under the involution $\xi \mapsto c_E \circ \xi \circ c$. Let

$$D : \Gamma(E)^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}} = \Gamma(\Omega^{01}_C \otimes_{\mathbb{C}} E)^{\mathbb{R}}, \quad D\xi = \overline{\partial}\xi + A\xi \tag{A.1}$$

be a real CR-operator on the Real bundle (E, c_E). Here $\overline{\partial}$ is a holomorphic $\overline{\partial}$-operator on E compatible with the real structure (i.e. such that c_E is anti-holomorphic), and

$$A \in \Gamma(C; \mathrm{Hom}_{\mathbb{R}}(E, \Omega^{01}_C \otimes_{\mathbb{C}} E))^{\mathbb{R}}$$

is a 0'th order term; see [GZ1, (4.4)]. After completing in (weighted) Sobolev norms, D becomes a Fredholm operator. Examples include

(i) Sobolev completions $\mathcal{D} : W^{k,p}\Gamma(E)^{\mathbb{R}} \to W^{k-1,p}\Lambda^{01}(E)^{\mathbb{R}}$ for $p > 1$ and $kp > 2$ as in [MS, §B.1].

(ii) completions as in [LT, §3].

(iii) completions $\mathcal{D} : W^{k,p,\delta}\Gamma(E)^{\mathbb{R}} \to W^{k-1,p,\delta}\Lambda^{01}(E)^{\mathbb{R}}$ as in [P, §B.4], weighted at the points in $\mathbf{x}$, for $\delta \in (0,1)$ and k, p so that $W^{k,p,\delta} \hookrightarrow C^0$ eg. $k \geq 1$, $p \geq 4$ (Pardon uses $p = 2$ and $k \geq 6$).

By elliptic regularity, the kernels and cokernels of all these Fredholm completions are the same, and consist of smooth elements, see eg. [MS, §C.1] and [P, Lemma B.5.2]. Therefore we let

$$\ker D := \ker \mathcal{D}, \quad \mathrm{coker}\, D := \mathrm{coker}\, \mathcal{D} \tag{A.2}$$

where $\mathcal{D} : \mathcal{E} \to \mathcal{F}$ is *any* of the completions (i)-(iii) of the real CR operator $D : \Gamma(E)^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}}$. Note that subspace $\ker D \subset \Gamma(E)^{\mathbb{R}}$ is *independent* of these choices. Moreover, $\mathrm{coker}\, D = \ker D^*$, where D^* is the formal adjoint of (A.1), regarded as a real CR operator on the bundle $\overline{F}$, where $F = \Omega^{01}_C \otimes E$. When $(E, c_E) \to (C, c)$ is a Real line bundle, the index of (A.1) is equal to

$$\mathrm{ind}\, D = \mathrm{ind}\, \overline{\partial}_{(E,c_E)} = c_1(E) + \tfrac{1}{2}\chi(C). \tag{A.3}$$

Consider the short exact sequence of vector spaces

$$0 \to \Gamma_{\mathbf{x}}(E)^{\mathbb{R}} \to \Gamma(E)^{\mathbb{R}} \xrightarrow{\mathrm{ev}_{\mathbf{x}}} (E_{\mathbf{x}}, c_E)^{\mathbb{R}} \to 0. \tag{A.4}$$

Here $\Gamma_{\mathbf{x}}(E)^{\mathbb{R}}$ denotes the subspace of sections which vanish at all the points in $\mathbf{x}$, and $(E_{\mathbf{x}}, c_E)$ denotes the restriction of (E, c_E) to $\mathbf{x}$. A choice of splitting

of (A.4) induces an isomorphism

$$\Gamma(E)^{\mathbb{R}} \cong \Gamma_{\mathbf{x}}(E)^{\mathbb{R}} \oplus (E_{\mathbf{x}})^{\mathbb{R}}, \tag{A.5}$$

where $(E_{\mathbf{x}})^{\mathbb{R}} := (E_{\mathbf{x}}, c_E)^{\mathbb{R}}$ denotes the real locus of $(E_{\mathbf{x}}, c_E)$.

Remark A.1. A splitting of (A.4) can be obtained as follows. Fix a trivialization of E in a neighborhood of $\mathbf{x}$ and bump functions β_x at each point $x \in \mathbf{x}$. Starting with $\alpha = (\alpha_x)_{x\in\mathbf{x}} \in (E_{\mathbf{x}})^{\mathbb{R}}$, extend the values $\alpha_x \in E_x$ to a constant section of E in a neighborhood of $x \in \mathbf{x}$ and multiply the result by the bump function to get a section $\sum_{x\in\mathbf{x}} \beta_x\alpha_x$ of E with value α_x at $x \in \mathbf{x}$. Finally symmetrize it to obtain a real section.

Consider next the restriction

$$D' : \Gamma_{\mathbf{x}}(E)^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}} \tag{A.6}$$

of the real CR operator (A.1) to the subspace in (A.4). Regard D as defined on the right hand side of (A.5), where it becomes the operator

$$\Gamma_{\mathbf{x}}(E)^{\mathbb{R}} \oplus (E_{\mathbf{x}})^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}}, \quad (\zeta, \alpha) \mapsto D'\zeta + \gamma(\alpha). \tag{A.7}$$

Here $\gamma : (E_{\mathbf{x}})^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}}$ depends on the choice of splitting in (A.5). Note that when the 0'th order term A is supported away from $\mathbf{x}$, then the splitting γ can be chosen so that its image consists of forms supported away from $\mathbf{x}$. After completing in any of the (weighted) Sobolev norms (i)-(iii), (A.4) becomes an exact sequence of Banach spaces and D and therefore D' become Fredholm operators. This induces a long exact sequence

$$0 \to \ker D' \to \ker D \to (E_{\mathbf{x}})^{\mathbb{R}} \to \operatorname{coker} D' \to \operatorname{coker} D \to 0. \tag{A.8}$$

The map $(E_{\mathbf{x}})^{\mathbb{R}} \to \operatorname{coker} D'$ is the projection of γ to the cokernel.

Remark A.2. In fact, any short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{D}'} & & \downarrow{\scriptstyle \mathcal{D}} & & \downarrow{\scriptstyle \mathcal{D}''} & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array} \tag{A.9}$$

of Fredholm operators induces (by the Snake Lemma) a long exact sequence

$$\begin{aligned} 0 \to \ker \mathcal{D}' \to \ker \mathcal{D} \to \ker \mathcal{D}'' \\ \to \operatorname{coker} \mathcal{D}' \to \operatorname{coker} \mathcal{D} \to \operatorname{coker} \mathcal{D}'' \to 0. \end{aligned} \tag{A.10}$$

Examples include the restriction $\mathcal{D}'$ of a Fredholm operator $\mathcal{D}$ to a closed, finite codimension subspace, e.g. as in (A.8).

A.2. Conjugating by a section

Consider next the holomorphic line bundle $\mathcal{O}(\mathbf{x})$ over (C, c) with its natural real structure induced by the action of c on $\mathbf{x}$. This is defined as follows. Fix local holomorphic coordinates on C at the points x in $\mathbf{x}$, compatible with the real structure as in (B.2). Let

$$s(z) = z \quad \text{for all } z \in U_x \text{ and } x \in \mathbf{x}, \tag{A.11}$$

and $s(z) = 1$ outside a smaller neighborhood U' of $\mathbf{x}$. Then s is a holomorphic section of $\mathcal{O}(\mathbf{x})$ whose divisor is $\mathbf{x}$. The real structure on C induces a real structure on $(\bigsqcup_{x\in\mathbf{x}} U_x) \times \mathbb{C}$ and $U' \times \mathbb{C}$, and thus on the bundle $\mathcal{O}(\mathbf{x})$, and the section s is equivariant with respect to the real structures.

When $E \to C$ is a holomorphic bundle (with a compatible real structure c_E), multiplication by s gives rise to a short exact sequence of holomorphic sheaves

$$0 \to E \otimes \mathcal{O}(-\mathbf{x}) \xrightarrow{\cdot s} E \to E_{\mathbf{x}} \to 0 \tag{A.12}$$

relating the two $\overline{\partial}$-operators on $E \otimes \mathcal{O}(-\mathbf{x})$ and E; this exact sequence is compatible with the real structures. Therefore it induces a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(E \otimes \mathcal{O}(-\mathbf{x}))^{\mathbb{R}} \to H^0(E)^{\mathbb{R}} \to (E_{\mathbf{x}}, c_E)^{\mathbb{R}} \\ \to H^1(E \otimes \mathcal{O}(-\mathbf{x}))^{\mathbb{R}} \to H^1(E)^{\mathbb{R}} \to 0. \end{aligned} \tag{A.13}$$

Consider next a real CR operator (A.1), and assume the 0'th order term A vanishes in a neighborhood of the special points of C. Conjugate

$D = \overline{\partial} + A$ by s to obtain the operator

$$D^s = s^{-1} D s = \overline{\partial} + s^{-1} A s. \tag{A.14}$$

This can be regarded as a real CR-operator on the Real bundle $E \otimes \mathcal{O}(-\mathbf{x})$ (well defined at least when the 0'th order term A vanishes on a neighborhood of $\mathbf{x}$). The following standard result for $\overline{\partial}$-operators extends to real CR operators.

Lemma A.3. *Multiplication by s induces canonical identifications*

$$\ker D^s \cong \ker D' \quad \text{and} \quad \operatorname{coker} D^s \cong \operatorname{coker} D', \tag{A.15}$$

where D^s and D' are given by (A.14) *and* (A.6).

Proof. Multiplication by s maps sections of $E \otimes \mathcal{O}(-\mathbf{x})$ to sections of E which vanish at $\mathbf{x}$ so we get a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Gamma(E \otimes \mathcal{O}(-\mathbf{x}))^{\mathbb{R}} & \xhookrightarrow{\cdot s} & \Gamma_{\mathbf{x}}(E)^{\mathbb{R}} \\ & & \downarrow{\scriptstyle D^s} & & \downarrow{\scriptstyle D'} \\ 0 & \longrightarrow & \Lambda^{01}(E \otimes \mathcal{O}(-\mathbf{x})) & \xhookrightarrow{\cdot s} & \Lambda^{01}(E)^{\mathbb{R}}. \end{array} \tag{A.16}$$

After completing in the usual Sobolev norms as above (A.2) and passing to the quotient, we get a short exact sequence $0 \to D^s \to D' \to D'' \to 0$ of Fredholm operators, cf. (A.9)-(A.10). Note that $\operatorname{ind} D^s = \operatorname{ind} D'$. When E is a Real line bundle, then $c_1(E \otimes \mathcal{O}(-\mathbf{x})) = c_1(E) - \sum_{x \in \mathbf{x}} 1$ thus $\operatorname{ind} D^s = \operatorname{ind} D - \sum_{x \in \mathbf{x}} 1 = \operatorname{ind} D'$, cf. (A.3) and (A.8).

As in the proof of [P, Lemma B.5.2], this implies (A.15) as follows. It suffices to check that the induced map (i) $\ker D^s \to \ker D'$ is surjective (it is injective by (A.10)), and (ii) $\operatorname{coker} D^s \to \operatorname{coker} D'$ is injective (surjectivity then follows from (i) and the fact that $\operatorname{ind} D^s = \operatorname{ind} D'$).

This can be verified directly using (A.8) and elliptic regularity. Assume $\xi \in \ker D'$. Thus $\xi \in \ker D$ and ξ vanishes at all the points in $\mathbf{x}$. Then by elliptic regularity it has an expansion $\xi(z) = \alpha z(1 + O(|z|))$ around a point in $\mathbf{x}$, thus $s^{-1}\xi \in \ker D^s$.

It remains to show that $\operatorname{coker} D^s \to \operatorname{coker} D'$ is injective. Let $\eta \in \operatorname{coker} D^s$. Since D^s is a real CR operator, then $\eta \in \ker(D^s)^*$ is smooth by elliptic regularity. It suffices to show that if $s\eta = D\xi$ where $\xi \in W^{1,p}$ and $\xi(x) = 0$ for all x in $\mathbf{x}$ then $s^{-1}\xi \in W^{1,p}$. By elliptic regularity ξ is smooth and has an expansion $\xi(z) = \alpha_1 \overline{z} + az(1 + O(|z|))$ around each point in $\mathbf{x}$. But

then $D\xi = \alpha_1 + O(|z|)$ while $s\eta = bz(1 + O(|z|))$ thus $\alpha_1 = 0$. This implies $s^{-1}\xi \in W^{1,p}$ completing the proof. □

One can similarly obtain the exact sequence

$$0 \to \ker D^s \to \ker D \to (E_{\mathbf{x}}, c_E)^{\mathbb{R}} \to \operatorname{coker} D^s \to \operatorname{coker} D \to 0 \tag{A.17}$$

generalizing (A.13), which corresponds to $D = \overline{\partial}_{(E,c_E)}$ (i.e. $A = 0$).

More generally, fix $\lambda = (\lambda_x)_{x\in\mathbf{x}}$ a symmetric sequence of positive integer multiplicities. Let $s = s_\lambda$ be a symmetric section of the Real line bundle $\mathcal{O}(\lambda \cdot \mathbf{x}) = \mathcal{O}(\sum_{x\in\mathbf{x}} \lambda_x x)$ such that

$$s(z) = z^{\lambda_x} \quad \text{for all} \quad z \in U_x \text{ and } x \text{ in } \mathbf{x}, \tag{A.18}$$

and s is smooth and nonzero outside $U = \sqcup_{x\in\mathbf{x}} U_x$. Then the conjugate operator (A.14) is a real CR operator D^s on $E \otimes_{\mathbb{C}} \mathcal{O}(-\lambda \cdot \mathbf{x})$, and multiplication by s induces a canonical isomorphism

$$\ker D^s \cong \ker D^\lambda, \quad \operatorname{coker} D^s \cong \operatorname{coker} D^\lambda \tag{A.19}$$

as in Lemma A.3. . Here

$$D^\lambda : \Gamma_{\lambda;\mathbf{x}}(E)^{\mathbb{R}} \to \Lambda^{01}_{\lambda-\mathbf{1};\mathbf{x}}(E)^{\mathbb{R}} \tag{A.20}$$

is the restriction of $D = \overline{\partial} + A$ to the subspace $\Gamma_{\lambda;\mathbf{x}}(E)^{\mathbb{R}}$ of sections which vanish to order λ at $\mathbf{x}$ and whose target is the subspace $\Lambda^{01}_{\lambda-\mathbf{1};\mathbf{x}}(E)^{\mathbb{R}}$ of (0,1)-forms which vanish to order $\lambda - \mathbf{1}$ at $\mathbf{x}$; it becomes a Fredholm operator $\mathcal{D}^\lambda$ after completing in the usual Sobolev norms for k sufficiently large (so that $W^{k,p} \hookrightarrow C^{\lambda_x}$ for all $x \in \mathbf{x}$). As in (A.2), these completions $\mathcal{D}^\lambda$ have the same kernel and cokernel. The $\lambda = \mathbf{1}$ case of (A.19) corresponds to (A.15); for $\lambda > \mathbf{1}$, (A.19) follows by induction from (A.16) with D replaced by $D^{s_{\lambda-\mathbf{1}}}$.

Remark A.4. The kernel and cokernel of the restriction D^λ of D consists of those elements of $\ker D$ and $\operatorname{coker} D = \ker D^*$ which vanish to order λ and respectively $\lambda - \mathbf{1}$ at $\mathbf{x}$. There are other Fredholm completions of (A.20) that have the same kernel and cokernel, cf. §A.5.

A.3. The normalization short exact sequence

The situation is similar in the case C_0 is a nodal symmetric curve. Let $\mathbf{y} = \{y_1, \ldots, y_\ell\}$ denote the collection of nodes of C_0. Let $\widetilde{C}$ denote the

normalization of C_0, and let y_{i1}, y_{i2} be the two points of $\widetilde{C}$ which are the preimage of the node y_i. Then $\widetilde{C}$ has a symmetric subset $\widetilde{\mathbf{y}}$ of marked points, where

$$\widetilde{\mathbf{y}} = \mathbf{y}_1 \sqcup \mathbf{y}_2, \quad \text{and} \quad \mathbf{y}_k = \{y_{1k}, \ldots, y_{\ell k}\}, \quad \text{for } k = 1, 2.$$

If $(E, c_E) \to (C_0, c)$ is a Real bundle, let $(\widetilde{E}, c_{\widetilde{E}})$ denote the pullback bundle. Then we have a normalization short exact sequence

$$0 \longrightarrow (E, c_E) \longrightarrow (\widetilde{E}, c_{\widetilde{E}}) \longrightarrow (E_{\mathbf{y}}, c_E) \longrightarrow 0. \tag{A.21}$$

It induces a short exact sequence

$$0 \to \Gamma(C_0; E)^{\mathbb{R}} \to \Gamma(\widetilde{C}, \widetilde{E})^{\mathbb{R}} \to (E_{\mathbf{y}}, c_E)^{\mathbb{R}} \to 0, \tag{A.22}$$

recording the fact that a continuous section of E is a section of $\widetilde{E}$ that has the same value on the pair of points of $\widetilde{C}$ that correspond to a node of C_0. Choosing a splitting of (A.22) provides an isomorphism

$$\Gamma(\widetilde{C}, \widetilde{E})^{\mathbb{R}} \cong \Gamma(C_0; E)^{\mathbb{R}} \oplus (E_{\mathbf{y}})^{\mathbb{R}} \tag{A.23}$$

as before. Then similarly one can relate a real CR operator

$$D_0 : \Gamma(C_0; E)^{\mathbb{R}} \to \Lambda^{01}(C_0; E)^{\mathbb{R}}, \quad D_0\xi = \overline{\partial}\xi + A_0(\xi) \tag{A.24}$$

on E to the pullback operator $\widetilde{D}$ on $\widetilde{E}$. In particular, we get a long exact sequence

$$0 \to \ker D_0 \to \ker \widetilde{D} \to (E_{\mathbf{y}})^{\mathbb{R}} \to \operatorname{coker} D_0 \to \operatorname{coker} \widetilde{D} \to 0. \tag{A.25}$$

Remark A.5. A splitting of (A.23) can be obtained by first lifting $v \in E_{\mathbf{y}}$ to the normalization as $(v, -v) \in \widetilde{E}_{\mathbf{y}_1} \oplus \widetilde{E}_{\mathbf{y}_2}$ and then proceeding as in Remark A.1 to obtain the section of $\Gamma(E)^{\mathbb{R}}$ supported in a neighborhood of the marked points $\mathbf{y}_1 \sqcup \mathbf{y}_2$ of $\widetilde{C}$ and taking opposite values on any pair of marked points that correspond to a node of C_0.

A.4. Variations in the domain

Recall from [GZ1, p. 721] that the Kodaira-Spencer deformation theory canonically identifies the tangent space of the real Deligne-Mumford moduli

space at a stable symmetric curve C with

$$T_C\overline{\mathcal{M}}^{\mathbb{R}} \cong H^1(\mathcal{T}_C)^{\mathbb{R}} \tag{A.26}$$

(after passing to covers in the presence of automorphisms); see also [L, §3]. Here $\mathcal{T}_C$ is the relative tangent bundle to C, cf. §B.2. In fact, as in [IP3, §4.1], the variation in C as a marked (symmetric) curve can be regarded as a (symmetric) $(0,1)$-form h with values in $\mathcal{T}_C$ and supported away from the special points of C (here h is the variation δj in the complex structure j). This gives a diagram

$$\begin{array}{ccc} T_C\overline{\mathcal{M}}^{\mathbb{R}} & \hookrightarrow & \Lambda^{01}(\mathcal{T}_C)^{\mathbb{R}} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \pi_{\mathrm{coker}}} \\ & & H^1(\mathcal{T}_C)^{\mathbb{R}} \end{array} \tag{A.27}$$

where the vertical arrow is the projection onto the cokernel $H^1(\mathcal{T}_C)^{\mathbb{R}}$ of the $\overline{\partial}$ operator on $(\mathcal{T}_C, c_{\mathcal{T}})$.

Remark A.6. Assume C is a stable symmetric curve with marked points $\mathbf{x} = \{x_1, \ldots, x_m\}$ and let $(\mathcal{T}_C)|_{x_i}$ denote the restriction of $\mathcal{T}_C$ to x_i. As C varies in the real Deligne-Mumford moduli space, $(\mathcal{T}_C)|_{x_i}$ defines a complex orbibundle denoted $\mathcal{T}_{|x_i}$ over the moduli space, equal to the pullback of the relative tangent bundle $\mathcal{T}$ on the universal curve via the section $C \mapsto (C, x_i)$; see §B.2. An element $(C, v_i) \in \mathcal{T}_{|x_i} \setminus 0$ can be regarded as a curve C together with a germ of a local holomorphic coordinate z_i at x_i. Moreover, the exact sequence (A.12) for $E = \mathcal{T}_C$ induces the exact sequence

$$0 \longrightarrow (\mathcal{T}_{|\mathbf{x}})^{\mathbb{R}} \longrightarrow H^1(\mathcal{T}_C \otimes \mathcal{O}(-\mathbf{x}))^{\mathbb{R}} \longrightarrow H^1(\mathcal{T}_C)^{\mathbb{R}} \longrightarrow 0, \tag{A.28}$$

cf. (A.13). The last term parametrizes the variation in the marked symmetric curve C, cf. (A.26). The middle term in (A.28) encodes the variation in (C, v), where $v = (v_x)_{x \in \mathbf{x}}$ consists of germs of holomorphic coordinates at the marked points compatible with the real structure as in (B.2). The first term encodes just the variation in v (with C fixed).

Assume next C_0 is a nodal curve, let $\mathbf{y}$ denote its collection of nodes and $\widetilde{C}$ its normalization. Then the normalization short exact sequence for

the holomorphic bundle $E = \mathcal{T}_{C_0}$ induces the long exact sequence

$$0 \to H^0(\mathcal{T}_{C_0})^{\mathbb{R}} \to H^0(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}} \to (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}} \to H^1(\mathcal{T}_{C_0})^{\mathbb{R}} \to H^1(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}} \to 0. \tag{A.29}$$

When C_0 is stable so is $\widetilde{C}$. Therefore both H^0 terms vanish, and the H^1 terms model the tangent spaces to the real Deligne-Mumford spaces containing C_0 and respectively $\widetilde{C}$, so (A.29) reduces to

$$0 \longrightarrow (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}} \longrightarrow H^1(\mathcal{T}_{C_0})^{\mathbb{R}} \longrightarrow H^1(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}} \longrightarrow 0. \tag{A.30}$$

The last term parametrizes variations in C_0 that do not smooth its nodes, while the middle one parametrizes all variations in C_0, including those that smooth the nodes. A choice of splitting of the short exact sequence (A.30) and the isomorphism (A.26) for $C = C_0$ and $\widetilde{C}$ determines a decomposition

$$T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}} = (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}} \oplus T_{C_0}\overline{\mathcal{N}}^{\mathbb{R}}. \tag{A.31}$$

Here $\overline{\mathcal{N}}^{\mathbb{R}}$ denotes the stratum containing C_0, and $(\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}}$ can be regarded as its normal bundle. Moreover, those variations in C_0 that map to 0 in $H^1(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}}$ via (A.30) can be regarded as variations in the gluing parameters for a family (B.5) of versal deformations of C_0 smoothing the nodes. In local coordinates, an element of $(\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}}$ can therefore be regarded as a variation $\delta\tau$ in the gluing parameter τ (or in cylindrical coordinates as $\frac{\delta\tau}{\tau} = \delta(\log\tau)$), for example via similar considerations as those in Remark A.6.

Remark A.7. In (A.30), $H^1(\mathcal{T}_{C_0})^{\mathbb{R}}$ and $H^1(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}}$ are the cokernels of the $\overline{\partial}$-operator on the relative tangent bundle to C_0 and respectively $\widetilde{C}$; the former operator is the restriction of the latter to a subspace. As in (A.25), the second arrow in (A.30) is the projection onto the cokernel of $\overline{\partial}_{(\mathcal{T}_{C_0},c)}$ of

$$h : (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}} \to \Lambda^{01}(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}} = \Lambda^{01}(\mathcal{T}_{C_0})^{\mathbb{R}}, \tag{A.32}$$

where $h(v) = \overline{\partial}_{(\mathcal{T}_{\widetilde{C}},c)} r(v)$ for some choice of splitting $r : (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}} \to \Gamma(\mathcal{T}_{\widetilde{C}})^{\mathbb{R}}$ of (A.23) for $E = \mathcal{T}_{C_0}$ as in Remark A.5.

A.5. Weighted Sobolev norms and real CR operators

Let C be a smooth curve. Fix a subset $\mathbf{y}$ of marked points of C, and $\lambda = (\lambda_y)_{y\in\mathbf{y}}$ a collection of multiplicities associated to $\mathbf{y}$, where $\lambda_y \in \mathbb{Z} \setminus 0$ for all

y in $\mathbf{y}$. Fix $z = e^{-(t+i\theta)}$ a local holomorphic coordinate at each point y in $\mathbf{y}$. Let

$$C' = C \setminus \mathbf{y}$$

regarded as a manifold with infinite cylindrical ends parametrized by $(t, \theta) \in [0, \infty) \times S^1$, one for each point $y \in \mathbf{y}$. Fix a weight function ρ and a cylindrical metric g' on C' such that

$$\rho = |z| = e^{-t}, \quad g' = dt^2 + d\theta^2 \tag{A.33}$$

in the neighborhood U_y of each point $y \in \mathbf{y}$ and $\rho > 0$ elsewhere; denote by $U_y' \subset U_y$ a slightly smaller neighborhood of y. When E is a bundle over C, let

$$\begin{aligned} \|\zeta\|^p_{k,p,\delta;\lambda} = &\int_{C\setminus U_y'} \left(\sum_{|m|\leq k} |\nabla^m \zeta|^p \right) dvol_{g'} \\ &+ \sum_{y\in\mathbf{y}} \int_{C'\cap U_y} \left(\sum_{|m|\leq k} |\rho^{1-\lambda_y-\delta} \nabla^m \zeta|^p \right) dvol_{g'} \end{aligned} \tag{A.34}$$

denote the λ-weighted Sobolev norm on $\Gamma(E)$, associated to $k \geq 1$, $p \geq 2$, weight $\delta \in \mathbb{R} \setminus \mathbb{Z}$ and the fixed multiplicities $\lambda = (\lambda_y)_{y\in\mathbf{y}}$. The λ-norms on $\Lambda^{01}(E) = \Gamma(\Omega^{01}_C \otimes_{\mathbb{C}} E)$ are defined by the same formula. These norms are defined with respect to a fixed metric and compatible connection on E, and the cylindrical metric g' and its Levi-Civita connection on $TC' = TC|_{C'}$.

When (C, c) is a symmetric curve and (E, c_E) is a Real bundle over it, the norms on $\Gamma(E)^{\mathbb{R}}$ and $\Lambda^{01}(E)^{\mathbb{R}}$ are defined by the same formulas by using throughout choices compatible with the real structure, i.e. (i) symmetric weight functions ρ and multiplicities $\lambda = (\lambda_y)_{y\in\mathbf{y}}$ associated to a symmetric collection of points $\mathbf{y}$ and (ii) symmetric hermitian metrics and connections on E and TC', with symmetric cylindrical ends.

Let $D : \Gamma(E)^{\mathbb{R}} \to \Lambda^{01}(E)^{\mathbb{R}}$, $D = \overline{\partial} + A$ be a real CR operator on (E, c_E). Completing in the λ-norms (A.34) gives rise to an operator

$$\mathcal{D}^\lambda : W^{k,p,\delta}_{\lambda;\mathbf{y}} \Gamma(C'; E)^{\mathbb{R}} \to W^{k-1,p,\delta}_{\lambda;\mathbf{y}} \Lambda^{01}(C'; E)^{\mathbb{R}}, \qquad \mathcal{D} = \overline{\partial} + A \tag{A.35}$$

on the punctured curve $C' = C \setminus \mathbf{y}$, which is Fredholm as in Lockhart and McOwen [LM, Theorem 6.2] since $\delta \notin \mathbb{Z}$ is not an eigenvalue of the asymptotic linearized operator on any end (the eigenvalues are $\mathbb{Z}$, corresponding

to the powers z^n on the punctured disk). When (E, c_E) is a line bundle and $\delta \in (0,1)$, the index of (A.35) is equal to

$$\text{ind}\, \mathcal{D}^\lambda = c_1(E)[C] + \tfrac{1}{2}\chi(C) - \sum_{y \in \mathbf{y}} \lambda_y. \tag{A.36}$$

Assume next that the multiplicities λ are positive, and fix $\delta \in (0,1)$. Then (A.35) can be regarded as an operator on the closed curve C as follows. Restrict to $k \geq 1$, $p \geq 4$ (or else $k \geq 2$ and $p \geq 2$) so that $W^{k,p}_{loc} \hookrightarrow C^0_{loc}$ by the Sobolev embedding theorem. Then sections of $E|_{C'}$ with finite λ-norm are continuous on C' and exponentially decay on the ends, thus extend continuously to sections of E on C which vanish to order λ at $\mathbf{y}$. Let

$$D^\lambda : \Gamma_{\lambda;\mathbf{y}}(E)^{\mathbb{R}} \to \Lambda^{01}_{\lambda-\mathbf{1};\mathbf{y}}(E)^{\mathbb{R}} \tag{A.37}$$

denote the restriction of $D = \overline{\partial} + A$ defined as in (A.20). Note that on C, smooth sections of E have finite λ-norm if and only if they vanish to order λ at $\mathbf{y}$; however, since $|dz| = |z|$ in cylindrical coordinates, (0,1)-forms with finite λ-norm only vanish to order $\lambda - \mathbf{1}$ at $\mathbf{y}$. Fixing $\delta \in (0,1)$ and completing (A.37) in the same λ-norms as (A.35) gives rise to a Fredholm operator

$$D^\lambda : \mathcal{E}^{k,p} \to \mathcal{F}^{k-1,p} \tag{A.38}$$

on the closed curve C, canonically identified with (A.35). By elliptic regularity its kernel and cokernel are independent of $\delta \in (0,1)$ and k, p in the range above, and consist of those smooth elements in the kernel and respectively cokernel of D with finite λ-norm. In particular, the kernel and cokernel of the Fredholm completion of (A.37) in these λ-norms is the same as that of the Fredholm completion in the usual Sobolev norms (for k sufficiently large) considered in §A.2.

A.6. Linearizations for smooth targets

The considerations above allow us to regard the linearization to the relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ in several equivalent ways. Assume $f : C \to \Sigma$ is an element of the relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$, and that f has no rubber components. Then by definition it is an element of the absolute moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, \emptyset)$ which has a certain branching over V, where $f^{-1}(V)$ consists only of marked points of the domain. This means that in local coordinates on the

target around $x \in V$ and on the domain in the neighborhood of each marked point $y \in f^{-1}(x)$, the map f has an expansion

$$f(z) = a_y z^{\lambda(y)} \widehat{f}(z)\,, \quad \text{where} \quad \widehat{f}(z) = 1 + O(|z|), \tag{A.39}$$

$a_y \neq 0$ is the leading coefficient of f at y, and $\lambda(y) > 0$ is the contact multiplicity. Intrinsically, the leading coefficient $a_y(f)$ of f at y is an element

$$a_y(f) \in (T_y C)^{-\lambda(y)} \otimes_{\mathbb{C}} T_{f(y)}\Sigma. \tag{A.40}$$

As $f : C \to \Sigma$ varies in $\overline{\mathcal{M}}(\Sigma, V)^{\mathbb{R}}$, this determines a section a_y of the complex line bundle

$$(\mathcal{T}_y)^{-\lambda(y)} \otimes_{\mathbb{C}} \mathrm{ev}_y^* T\Sigma \tag{A.41}$$

cf. [IP2, Lemma 7.1]; note that $f(y) = x \in V$, thus the second factor in (A.41) is trivial.

Conditions (i)-(iv) of Definition 3.1 imply that a local variation in the map $f : C \to \Sigma$ with fixed domain and target (including fixed marked points) also fixes the collection of contact points $\mathbf{y}$ and their contact multiplicity λ. Variations δf in f along $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ will vanish to order $\lambda(y)$ at y, and in fact will continue to have an expansion of the form (A.39). The leading coefficient of the expansion of δf at y is the variation δa_y in the leading coefficient of f at y (assuming the coordinate systems are fixed). In particular, variations δf which vanish to next order at all the contact points fix the leading coefficients of f. Of course, if a smooth function f vanishes of order $\lambda(y)$ at the point y, then $\overline{\partial} f$ vanishes to order $\lambda(y) - 1$.

Fix a map f with contact multiplicity λ and contact points $\mathbf{y}$. Denote by $\Gamma_{\lambda;\mathbf{y}}(f^*T\Sigma)^{\mathbb{R}}$ the subspace of sections that vanish to order $\lambda(y)$ at y for $y \in \mathbf{y}$, as in (A.20). Then the linearization L_f to $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ is the restriction

$$L_f : \Gamma_{\lambda;\mathbf{y}}(f^*T\Sigma)^{\mathbb{R}} \oplus T_C \overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}_{\lambda-\mathbf{1};\mathbf{y}}(f^*T\Sigma)^{\mathbb{R}} \tag{A.42}$$

of the "usual" linearization

$$\begin{aligned} &\mathbb{L}_f : \Gamma(f^*T\Sigma)^{\mathbb{R}} \oplus T_C \overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(f^*T\Sigma)^{\mathbb{R}} \\ &\mathbb{L}_f(\xi, h) = \overline{\partial}\xi + [\nabla_\xi \nu + \tfrac{1}{2} J df h]^{01}. \end{aligned} \tag{A.43}$$

This follows as in the proof of [IP1, Lemma 4.2], which also applies in our case by working with symmetric choices throughout. Specifically, fix

$\delta \in (0,1)$ and restrict to $k \geq 1,\ p \geq 4$ (or else $k \geq 2$ and $p \geq 2$). After passing to completions as in (A.38), the linearization (A.42) to $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ at a map f with contact multiplicity λ becomes the Fredholm operator

$$\begin{aligned} &\mathcal{L}_f : \mathcal{E}_f^{k,p} \oplus T_C\overline{\mathcal{M}}^{\mathbb{R}} \to \mathcal{F}_f^{k-1,p} \\ &\mathcal{L}_f(\xi, h) = \overline{\partial}\xi + [\nabla_\xi \nu + \tfrac{1}{2} J df h]^{01} \ = \ \overline{\partial}_{f^*(T\Sigma, dc)}\xi + \hat{A}_f(\xi) + \hat{b}_f(h) \end{aligned} \tag{A.44}$$

obtained by completing (A.42) using the λ-weighted Sobolev norms of §A.5, cf. [IP2, p. 971], as corrected in [IP2, eq. (1.1) of Corrigendum]. Here

$$\hat{A}_f \in \mathrm{Hom}_{\mathbb{R}}(f^*T\Sigma, \Lambda^{01}_C \otimes_{\mathbb{C}} f^*T\Sigma)^{\mathbb{R}}$$

is a (symmetric) 0'th order term given by $\hat{A}_f(\xi) = [\nabla_\xi \nu]^{01}$, while the term $\hat{b}_f : T_C\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(f^*T\Sigma)^{\mathbb{R}}$ is induced by the map $h \mapsto \frac{1}{2} J df h$. Here h is a $(0,1)$-form with values in the relative tangent bundle to C which vanishes in a sufficiently small neighborhood of the special points of C (a variation in C can be regarded as such a form h via (A.27), as reviewed in §A.4).

For a map f with contact multiplicity λ, consider (a) the completion (A.44) of L_f for k, p in the range above and fixed $\delta \in (0,1)$ or (b) the completion $\mathcal{L}_f$ of L_f in the usual Sobolev norms as above (A.2), but with $k \gg 0$. Then as in (A.37)-(A.38), $\ker \mathcal{L}_f$ and $\mathrm{coker}\, \mathcal{L}_f$ are independent of these choices, and are denoted

$$\ker L_f = \ker \mathcal{L}_f \quad \text{and} \quad \mathrm{coker}\, L_f = \mathrm{coker}\, \mathcal{L}_f. \tag{A.45}$$

Remark A.8. The linearization (A.42) to the real relative moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ is precisely the restriction to the invariant part of the domain and target of the linearization to the (usual) relative moduli space $\overline{\mathcal{M}}(\Sigma, V)$ defined in [IP1]. In particular, the index of the former is half the index of the latter. Thus the virtual dimension counts of [IP1, Lemmas 7.5-7.6] directly translate into counts for the moduli spaces $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ considered in this paper. In particular, the virtual dimension of the moduli space $\overline{\mathcal{M}}^{\mathbb{R}}_{d,\chi,\vec{\mu}}(\Sigma, V)$ is

$$\begin{aligned} b &= c_1(T\Sigma)A + \tfrac{1}{2}\chi - V \cdot A + \dim \overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} = d\chi(\Sigma \setminus V) - \chi + 2\ell(\vec{\mu}) \\ &= d\chi(\Sigma) - \chi + 2\delta(\vec{\mu}). \end{aligned} \tag{A.46}$$

Here $A = f_*[C] = d[\Sigma] \in H_2(\Sigma)$ and V has $2r$ points, so $V \cdot A = 2dr$, while $\dim \overline{\mathcal{M}}^{\mathbb{R}}_{\chi,\ell(\vec{\mu})} = -\frac{3}{2}\chi + 2\ell(\vec{\mu})$.

An equivalent description of the linearization at f is obtained as follows. Consider an invariant section σ of $\mathcal{O}(\mathbf{x})$ on the target Σ whose divisor is $V = \mathbf{x}$ and such that σ is holomorphic in a neighborhood of $\mathbf{x}$, and smooth and nonzero elsewhere. Pull it back to get a section $s = f^*\sigma$ of $f^*\mathcal{O}(\mathbf{x})$ which vanishes to order λ at the points $\mathbf{y}$ and nowhere else. Conjugating (A.43) by s as in §A.2 and §A.5 allows us to regard the linearization to $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ as

$$D_f : \Gamma(f^*\mathcal{T}_\Sigma)^{\mathbb{R}} \oplus T_C\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(f^*\mathcal{T}_\Sigma)^{\mathbb{R}} \tag{A.47}$$

given by the formula

$$\begin{aligned} D_f(\xi, h) &= \overline{\partial}_{f^*(\mathcal{T}, c_\mathcal{T})}\xi + s^{-1}[\nabla_{s\xi}\nu + \tfrac{1}{2}Jdfh]^{01} \\ &= \overline{\partial}_{f^*(\mathcal{T}, c_\mathcal{T})}\xi + A_f(\xi) + b_f(h). \end{aligned} \tag{A.48}$$

By construction both $\nabla\nu$ and h vanish in a sufficiently small neighborhood U of the special points of the domain, so we can chose the section σ (depending on f) so that $s = f^*\sigma$ is identically 1 outside this neighborhood. In that case (A.47) is given by exactly the same formula as L_f, except that $T\Sigma$ is replaced by $\mathcal{T}_\Sigma$, cf. (2.8).

Note that the linearization D_f is a 0'th order perturbation of $\overline{\partial}_{(E,c_E)} \oplus 0$, where $\overline{\partial}_{(E,c_E)}$ is a real CR operator (A.1) on the Real bundle $(E, c_E) = f^*(\mathcal{T}_\Sigma, c_\mathcal{T})$. Let $\mathcal{D}_f$ denote the Fredholm operator obtained by completing D_f in the usual Sobolev norms or any other choice from those listed above (A.2). Then as in (A.2), $\ker \mathcal{D}_f$ and $\operatorname{coker} \mathcal{D}_f$ are independent of these choices, and are denoted $\ker D_f$ and $\operatorname{coker} D_f$.

Lemma A.9. *Dividing by s induces an identification*

$$\ker L_f \cong \ker D_f \quad \text{and} \quad \operatorname{coker} L_f \cong \operatorname{coker} D_f, \tag{A.49}$$

canonical up to homotopy in s. Thus either one of these linearizations locally describes the moduli space $\overline{\mathcal{M}}^{\mathbb{R}}(X, V)$ around f. This induces canonical (up to homotopy) identifications between the determinant lines

$$\begin{aligned} \det L_f &\underset{(A.49)}{\cong} \det D_f \underset{(A.48)}{\cong} \det(\overline{\partial}_{f^*(\mathcal{T}_\Sigma, c_\mathcal{T})} \oplus 0) \\ &= \det \overline{\partial}_{f^*(\mathcal{T}_\Sigma, c_\mathcal{T})} \otimes \mathfrak{f}^* \det T\overline{\mathcal{M}}^{\mathbb{R}}. \end{aligned} \tag{A.50}$$

Proof. Consider the operator $\mathbb{L}_f$ given by (A.43). Up to 0'th order terms, $\mathbb{L}_f$ is the same as $\overline{\partial}_{(E,c_E)} \oplus 0$, where $\overline{\partial}_{(E,c_E)}$ is a real CR operator (A.1) on

the bundle $(E, c_E) = f^*(T\Sigma, dc)$. The linearization L_f is the restriction of $\mathbb{L}_f$ to subspaces of sections which vanish to order λ cf. (A.42) while D_f is the conjugate operator $s^{-1}\mathbb{L}_f s$ cf. (A.48). Therefore (A.49) follows the same way as (A.19).

Since up to 0'th order terms D_f is the same as $\overline{\partial}_{f^*(\mathcal{T}_\Sigma, c_\mathcal{T})} \oplus 0$ cf. (A.48), the second identification in (A.50) follows as in [GZ1, (3.3) and §4.3] with $E = TX$ replaced by $E = \mathcal{T}_\Sigma$. □

Remark A.10. Under the transformation $\xi \mapsto \frac{1}{f^*\sigma}\xi$, a variation δf in f (regarded as a section of $f^*T\Sigma$) is mapped to a section $\xi = \frac{\delta f}{f^*\sigma}$ of $f^*\mathcal{T}_\Sigma$; the leading coefficient of δf at y is mapped to the value

$$\xi(y) = \frac{\delta a_y}{a_y} = \delta(\log a_y) \tag{A.51}$$

where δa_y denotes the variation in the leading coefficient a_y of f at y, cf. (A.39).

We can also separate the leading coefficient of the variation in f as in [IP2]. For the linearization (A.47), this corresponds to separating the value $\xi(y) \in (f^*\mathcal{T})_y$ at each contact point y, which is given by (A.51). Specifically, using a decomposition such as in (A.7) for $E = f^*\mathcal{T}_\Sigma$ over C and $\mathbf{x}^+$ replaced by the contact points $\mathbf{y}^+$, the linearization (A.47) rewrites as the operator

$$\begin{aligned} &\mathbf{D}_f : \Gamma_{\mathbf{y}}(f^*\mathcal{T}_\Sigma)^{\mathbb{R}} \oplus (f^*\mathcal{T}_\Sigma)_{\mathbf{y}^+} \oplus T_C\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{0,1}(f^*\mathcal{T}_\Sigma)^{\mathbb{R}} \\ &\mathbf{D}_f(\zeta, \alpha, h) = D_f(\zeta, h) + \gamma(\alpha). \end{aligned} \tag{A.52}$$

Here $\zeta \in \Gamma(f^*\mathcal{T}_\Sigma)^{\mathbb{R}}$ gives rise to a variation in f with fixed domain and target and also fixed leading coefficient (A.39) at each one of the marked points, while the middle term records the variation in the leading coefficients, cf. (A.51). The term $\gamma : (f^*\mathcal{T}_\Sigma)_{\mathbf{y}^+} \to \Lambda^{01}(f^*\mathcal{T}_\Sigma)^{\mathbb{R}}$ depends on the choice of splitting, see Remark A.1.

A.7. Linearizations for nodal targets

The discussion above extends to the case where the target is a nodal curve Σ_0 with complex conjugate nodes; here we are discussing the case when the target has a single pair of complex conjugate nodes $x^\pm$. Consider a continuous real map $f_0 : C_0 \to \Sigma_0$ such that the preimage of x^+ consists only of nodes of the domain. Denote by $\mathbf{y}^+ = \{y_1^+, \dots y_\ell^+\}$ the preimage of the node x^+ of the target, and let $\mathbf{y} = \mathbf{y}^+ \sqcup \mathbf{y}^-$ where $y_i^- = c(y_i^+)$. Then

$\mathbf{y}$ is a subset of nodes of C, and let $\widetilde{C}$ be the (partial) normalization of C resolving the nodes in $\mathbf{y}$; let $\widetilde{\Sigma}$ be the normalization of Σ_0. Then $f_0 : C_0 \to \Sigma_0$ lifts to a continuous real map

$$\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma} \tag{A.53}$$

with smooth target (but possibly nodal domain). When $f_0 : C_0 \to \Sigma_0$ is an element of the real relative moduli space $\overline{\mathcal{M}}(\Sigma_0)$ of (4.24), then (local) variations in f_0 can be described in terms of those of its lift (A.53).

We can thus consider the linearization at $f_0 : C_0 \to \Sigma_0$ to $\overline{\mathcal{M}}(\Sigma_0)$ (with fixed target and fixed ν) induced by the linearization

$$D_{\widetilde{f}} : \Gamma(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \oplus T_{\widetilde{C}}\mathcal{M}^{\mathbb{R}} \to \Lambda^{0,1}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \tag{A.54}$$

at a lift $\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma}$ of f_0 to the normalizations. This has the form

$$(\xi, h) \mapsto \overline{\partial}\xi + A_{\widetilde{f}}(\xi) + b_{\widetilde{f}}(h) \tag{A.55}$$

cf. (A.48) with f replaced by $\widetilde{f}$.

Using (3.6) and the normalization short exact sequence (A.21) for $E = f_0^*\mathcal{T}_{\Sigma_0}$ we can rewrite (A.54) as a map

$$\widetilde{D}_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus (f_0^*\mathcal{T}_{\Sigma_0})_{|\mathbf{y}^+} \oplus T_{\widetilde{C}}\mathcal{M}^{\mathbb{R}} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \tag{A.56}$$

An element $\zeta \in \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}}$ corresponds to a variation in f_0 with fixed domain, target, and also fixed product of the leading order terms at each one of the nodes in the inverse image of x^+, while the middle term records the variation in the product of leading coefficients (or in cylindrical coordinates the variation $\frac{\delta(a_1a_2)}{a_1a_2} = \frac{\delta a_1}{a_1} + \frac{\delta a_2}{a_2}$). In particular, the linearization (A.56) has the form

$$\widetilde{D}_{f_0}(\zeta, \alpha, h) = \overline{\partial}\zeta + A_{f_0}(\zeta) + \gamma(\alpha) + b_{f_0}(h) \tag{A.57}$$

where A_{f_0} is the restriction of $A_{\widetilde{f}}$ to $\Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}}$, and b_{f_0} is equal to $b_{\widetilde{f}}$ cf. (A.55). Here γ depends on the choice of splitting of (A.21) for $E = f_0^*\mathcal{T}_{\Sigma_0}$; such a splitting can be obtained for example by pulling back a splitting for $E = \mathcal{T}_{\Sigma_0}$. The last term $b_{f_0} = b_{\widetilde{f}}$ is induced by the map

$$\Lambda^{01}(\widetilde{C}; \mathcal{T}_{\widetilde{C}})^{\mathbb{R}} \to \Lambda^{01}(\widetilde{C}; \widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \tag{A.58}$$

obtained from $h \mapsto \frac{1}{2}Jdfh$ after dividing by the section $\widetilde{s} = \widetilde{f}^*\sigma$ and projecting to the (0,1)-part.

There is another perspective for the linearization at f_0 that is better suited when deforming the nodal target by smoothing its nodes. For that consider instead the operator

$$\begin{aligned} &D_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \\ &D_{f_0}(\zeta, h) = \overline{\partial}\zeta + A_{f_0}(\zeta) + b_{f_0}(h) \end{aligned} \tag{A.59}$$

where A_{f_0} is as in (A.57) and b_{f_0} is induced by the same formula (A.58). However here, unlike in (A.57), the variation $h \in \Lambda^{01}(\mathcal{T}_{C_0})^{\mathbb{R}}$ is tangent to the entire Real Deligne-Mumford moduli space, not just to the nodal stratum containing C_0. Using the exact sequence (A.30) and a decomposition of (A.31), this operator can therefore be identified with

$$\begin{aligned} &\widehat{D}_{f_0} : \Gamma(f_0^*\mathcal{T}_{\Sigma_0})^{\mathbb{R}} \oplus \mathcal{T}_{|\mathbf{y}^+} \oplus T_{\widetilde{C}}\mathcal{M}^{\mathbb{R}} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}} \\ &\widehat{D}_{f_0}(\zeta, v, h) = \overline{\partial}\zeta + A_{f_0}(\zeta) + \widehat{\gamma}(v) + b_{f_0}(h). \end{aligned} \tag{A.60}$$

Here A_{f_0} and b_{f_0} are exactly as in (A.57), but $\widehat{\gamma} : \mathcal{T}_{|\mathbf{y}^+} \to \Lambda^{01}(\widetilde{f}^*\mathcal{T}_{\widetilde{\Sigma}})^{\mathbb{R}}$ depends on the choice of splitting of (A.22) for $E = \mathcal{T}_{C_0}$, while (A.57) depends on one for $E = \mathcal{T}_{\Sigma_0}$. Specifically, $\widehat{\gamma}$ is the composition of (A.58) and (A.32).

The two forms (A.57) and (A.60) of the linearization at f_0 are in fact equivalent: using the formulas for γ and $\widehat{\gamma}$ in terms of the two splittings one sees that the linearization (A.60) is the same as the linearization (A.56). Their middle terms are identified via the complex linear isomorphism induced by the linearization of (6.12), given by

$$\frac{\delta a_y}{a_y} \mapsto -\lambda(y)\frac{\delta \tau_y}{\tau_y}. \tag{A.61}$$

Here a_y denotes the product of the leading coefficients of f_0 at y, while τ_y can be regarded as the gluing parameter of the domain at the node y.

Remark A.11. The leading coefficients depend on the germs of holomorphic coordinates used, as do the gluing parameters. The product a_y induces the isomorphism (6.12), relating the gluing parameter in the target with a power of that of the domain; its linearization gives (A.61).

A.8. The normal operator

The discussion above extends to the strata of the real relative moduli space $\overline{m}^{\mathbb{R}}(\Sigma, V)$ containing maps $f : C \to \Sigma[m]$ with rubber components (i.e. with

$m \neq 0$); the target $\Sigma[m]$ is nodal but comes with an extra rescaling action, cf. §3.3. To describe such strata, we adapt [IP1, Definition 7.2 and Remark 7.7] to our setting. It is based on the following simple observation: away from the infinity divisor, $\mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C})$ is equal to N_V, via the canonical embedding $w \mapsto [w, 1]$ in homogenous coordinates. Thus for a map $\rho : C \to V$, sections ξ of $\rho^* N_V$ (with finitely many zeros and poles) can be equivalently regarded as lifts

$$f = (\rho, \xi) \tag{A.62}$$

of ρ to $\mathbb{P}_V$ (with finitely many points in the preimage of the zero and infinity divisor).

Let $f : C \to \Sigma[m]$ represent an equivalence class in $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$, cf. (4.10). Let $\mathbf{y} \subset C$ denote the special points of the domain which are in the inverse image of the special points $\mathbf{x}$ of the target $\Sigma[m]$. Let

$$\widetilde{f} : \widetilde{C} \to \widetilde{\Sigma[m]} \tag{A.63}$$

denote the lift of f defined as in (A.53), where $\widetilde{\Sigma[m]}$ is the normalization of the building $\Sigma[m]$, thus a disjoint union of $\widetilde{\Sigma}$ and m_i copies of $\mathbb{P}_{x_i^{\pm}} \subset \mathbb{P}_V$, cf. (3.8). We can then decompose the lift (A.63) as a disjoint union of pieces

$$f_0 : C_0 \to \Sigma \subset \Sigma[m] \quad \text{and} \quad f_r : C_r \to \mathbb{P}_V, \ \text{ for } r \geq 1, \tag{A.64}$$

where C_0 denotes the union of all level 0 (i.e. non-rubber) components, while the other pieces C_r, for $r \geq 1$, are symmetric curves consisting of (rubber) components mapped to one of the m_i copies of $\mathbb{P}_{x_i^{\pm}} \subset \mathbb{P}_V$ of the building $\Sigma[m]$.

In particular, for $r \geq 1$, each rubber piece $f_r : C_r \to \mathbb{P}_V$ of the decomposition (A.64) is a real map with prescribed multiplicity $|\lambda_r| > 0$ of contact to the 0 and ∞-divisor of $\mathbb{P}^1_V$ at the special points $\mathbf{y}_r$ of C_r; equivalently f_r is a nontrivial section ξ_r of the pullback normal bundle $N_V = T\Sigma_{|V}$ to V with zeros and poles of prescribed *order* $\lambda_r : \mathbf{y}_r \to \mathbb{Z}$ at the special points and no other zeros/poles (recall that the order of a meromorphic section is *negative* at a pole and positive at a zero). Therefore

$$\sum_{y \in \mathbf{y_r}} \lambda_r(y) = 0 \quad \text{for all } r \geq 1, \tag{A.65}$$

since the intersection numbers of f_r with the zero and the infinity divisor of $\mathbb{P}_V$ are equal. Note that the $\mathbb{C}^*$ action on $\mathbb{P}_V$ induces a natural $\mathbb{C}^*$ action both on maps $f_r : C_r \to \mathbb{P}_V$ and on sections ξ_r of the pullback normal bundle.

Moreover, along a stratum of $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ the topological type of the maps $f : C \to \Sigma[m]$ and of their lifts (A.63)-(A.64) is fixed, including the topological type of the domain, target and the contact multiplicities. Therefore the conditions describing a stratum of $\overline{\mathcal{M}}^{\mathbb{R}}(\Sigma, V)$ represented by maps

$$f : C \to \Sigma[m] \quad \text{with } m \neq 0$$

can be expressed in terms of conditions on their lifts (A.63)-(A.64), or equivalently on their projections

$$f_0 : C_0 \to \Sigma \subset \Sigma[m] \quad \text{and} \quad \rho_r : C_r \to V, \ \text{ for } r \geq 1 \tag{A.66}$$

under the collapsing map to Σ, as in [IP1, §7]. If we denote by $\rho = p(f)$ the projection of f under the collapsing map $\Sigma[m] \to \Sigma$, then these conditions become (a) ρ is a real (J, ν)-holomorphic map and (b) its restriction $\rho_r : C_r \to V$ to each rubber piece C_r is a real map which satisfies the additional condition that there exits an element

$$\xi_r \in \ker L^N_{\rho_r} \tag{A.67}$$

such that ξ_r is non-trivial on each irreducible component of C_r. Here

$$\begin{aligned} &L^N_{\rho_r} : W^{k,p,\delta}_{\lambda_r;\mathbf{y_r}} \Gamma(C_r'; \rho_r^* N_V)^{\mathbb{R}} \to W^{k-1,p,\delta}_{\lambda_r;\mathbf{y_r}} \Lambda^{01}(C_r'; \rho_r^* N_V)^{\mathbb{R}} \\ &L^N_{\rho_r} \xi = \overline{\partial} \xi - \nabla_\xi \nu \end{aligned} \tag{A.68}$$

denotes the normal operator associated to any real map $\rho_r : C_r \to V$, completed to a Fredholm operator on the punctured curve $C_r' = C_r \setminus \mathbf{y_r}$ in the λ_r-norms as in (A.34)-(A.35), with $\delta \in (0, 1)$. By (A.36) and (A.65) the index of this operator is

$$\operatorname{ind} L^N_{\rho_r} = c_1(\rho_r^* N_V) + \tfrac{1}{2}\chi(C_r) - \sum_{y \in \mathbf{y_r}} \lambda_r(y) = \tfrac{1}{2}\chi(C_r). \tag{A.69}$$

Note that V is 0-dimensional, so $E = \rho_r^* N_V$ is topologically trivial, thus $c_1(E) = 0$.

Since $V = V^+ \sqcup V^-$ consists only of pairs of complex conjugate points, then after further decomposing $\widetilde{C}$ into its connected components and reindexing the rubber ones, we may assume without loss of generality that we get a decomposition (A.64) where (a) C_0 is the union of all level 0 (i.e. non-rubber) components and (b) the other pieces C_r, for $r \geq 1$, are symmetric

curves

$$C_r = C_r^+ \sqcup C_r^-$$

consisting of rubber components, where $C_r^\pm$ are *connected* and mapped to a point $p_r^\pm$ in $V^\pm$. Then the restriction of (A.68) to C_r^+ is a *complex* linear Fredholm operator by the second condition in (4.4), and therefore is a holomorphic $\overline{\partial}$-operator on C_r^+ with values in $(\rho_r^+)^* N_V$. Every nontrivial element ξ in the kernel of $L^N_{\rho_r^+}$ is meromorphic and has finite λ_r-norm. Therefore ξ must have zeros/poles of *order* at least $\lambda_r(y)$ at the points $y \in \mathbf{y}_r^+ := \mathbf{y}_r \cap C_r^+$ and no other poles (otherwise its λ_r-norm would not be finite). But the first Chern class of the bundle $(\rho_r^+)^* N_V$ is 0, and it is also equal to the sum of the orders of the zeros and poles of any nontrivial meromorphic section of it. Therefore ξ must have order exactly $\lambda_r(y)$ at each $y \in \mathbf{y}_r^+$ and no other zeros nor poles. Note that the ratio of any two such sections is constant (on connected components).

Remark A.12. An element ξ_r satisfying (A.67) can be equivalently regarded as a real map

$$\xi_r : C_r \to \mathbb{P}_V \quad \text{which is a solution of} \quad \overline{\partial}\xi = \nabla_\xi \nu \tag{A.70}$$

and which has prescribed multiplicity of contact to 0 and ∞ at the special points $\mathbf{y}_r$ of C_r. The $\mathbb{C}^*$ action rescaling $\mathbb{P}_V$ corresponds to the $\mathbb{C}^*$-action on $\ker L^N$. Moreover as explained above, whenever C_r^+ is *connected* then either $\ker L^N = 0$ or else $\ker L^N \cong \mathbb{C}$.

Appendix B. Families

B.1. Smoothings of symmetric curves

Let (C_0, c_0) be a (possibly nodal) symmetric marked curve, and denote by $\mathbf{x}$ and $\mathbf{y}$ the collection of marked points and respectively nodes of C_0. In this section C_0 is allowed to have real nodes or real marked points. Flat families

$$(\mathcal{C}, c) \to (B, c_B) \tag{B.1}$$

of deformations of C_0 are defined at the beginning of [GZ1, §4.2]; their restriction to $b \in B^{\mathbb{R}}$ is a symmetric marked curve C_b with the induced real structure c_b.

Such families (B.1) can be obtained by smoothing the nodes of C_0 with gluing parameters $\tau \in B$, using standard local models in a neighborhood U

of the nodes. This construction is described in [GZ1, §6.2] in the case C_0 has a single node (which must be real), and in [GZ2, §4.2] in the case C_0 has a pair of conjugate nodes. The general case when C_0 has more nodes is defined similarly as we briefly outline below.

Choose holomorphic coordinates $z = z_x$ on C_0 around each marked point $x \in \mathbf{x}$ identifying a neighborhood of x in C_0 with the unit disk $|z| \leq 1$. We can furthermore arrange that in this neighborhood of $\mathbf{x}$ the real structure c_0 has a standard form in these coordinates, i.e. is given by

$$z_x \mapsto \overline{z}_{c_0(x)} \tag{B.2}$$

as a map from the neighborhood of x to the neighborhood of $c_0(x)$. Similarly, we can fix holomorphic coordinates z_1, z_2 on C_0 at each node $y \in \mathbf{y}$ identifying a neighborhood of y in C_0 with the locus

$$z_1 z_2 = 0, \quad |z_1|, |z_2| \leq 1 \tag{B.3}$$

and in which the real structure on C_0 has a standard form (depending on the type of node). Note that the coordinates z_1, z_2 on C_0 at the node y correspond to local holomorphic coordinates on the normalization $\widetilde{C}$ at the two points y_1, y_2 in the pre-image of $y \in \mathbf{y}$.

Then in a neighborhood $U_y \subset \mathcal{C}$ of the node y, the fiber C_τ of $\mathcal{C}$ over $\tau = (\tau_y)_{y \in \mathbf{y}} \in B$ is given by

$$z_1 z_2 = \tau_y, \qquad |z_1|, |z_2| \leq 1, \tag{B.4}$$

where τ_y is called the gluing parameter at the node y, for $y \in \mathbf{y}$. Here $B \subset \mathbb{C}^{\mathbf{y}}$ is a sufficiently small neighborhood of 0 and its real structure c_B is defined by $\tau_y \mapsto \overline{\tau}_y$ if y is a real node and by $\tau_{y^\pm} \mapsto \overline{\tau}_{y^\mp}$ for a pair $y^\pm$ of conjugate nodes. The real structure on $\mathcal{C}$ also has a standard form in these coordinates, given by [GZ1, (6.11)] in a neighborhood of a real node, and by [GZ2, (4.9)] in the neighborhood of a pair of conjugate nodes. Finally, outside a smaller neighborhood U' of the nodes the family (B.1) is identified with the product family $(C_0 \setminus U', c_0) \times (B, c_B)$.

Intrinsically, the gluing parameters τ encode variations normal to the nodal stratum, cf. (A.30)-(A.31); for every node y of C_0, the gluing parameter can be regarded as an element $\tau_y \in \mathcal{T}_{|y}$. Here $\mathcal{T}_{|y} = \mathcal{T}_{|y_1} \otimes \mathcal{T}_{|y_2}$ denotes the relative tangent bundle at the node y, and $\tau = (\tau_y)_{y \in \mathbf{y}} \in \mathcal{T}_{|\mathbf{y}} = \oplus_{y \in \mathbf{y}} \mathcal{T}_{|y}$.

One can also include local deformations C_{0u} of C_0 along the nodal stratum $\mathcal{N}$ containing C_0 e.g. obtained by varying j away from the nodes and marked points. In that case, one gets flat families (B.1) parametrized by pairs $(\tau, u) \in B = B_N \times B_T$, where $u \in B_T$ parametrizes local deformations C_{0u} of C_0 preserving its nodes, and $C_{\tau u}$ is obtained from C_{0u} by smoothing its nodes with gluing parameter τ. In particular, when C_0 is stable and (τ, u) vary in a sufficiently small neighborhood $\mathcal{S}$ of 0 in $\mathcal{T}_{|\mathbf{y}} \oplus T_{C_0}\mathcal{N} = T_{C_0}\overline{\mathcal{M}}$, we get versal families

$$(\mathcal{C}, c) \longrightarrow (\mathcal{S}, c_{\mathcal{S}}) \tag{B.5}$$

called "local slices" at C_0. Their restriction to $\mathcal{S}^{\mathbb{R}} \subset T_{C_0}\overline{\mathcal{M}}^{\mathbb{R}}$ is a family $C_{\tau u}$ of symmetric marked curves, where $u \in T_{C_0}\mathcal{N}^{\mathbb{R}}$ is a variation along the nodal stratum $\mathcal{N}^{\mathbb{R}}$ containing C_0 and $\tau \in (\mathcal{T}_{|\mathbf{y}})^{\mathbb{R}}$ is a normal variation to it, cf. §A.4.

B.2. The relative tangent bundle

Let $(\mathcal{C}, c) \to (B, c_B)$ be a flat family of deformations of a fixed marked symmetric curve as in §B.1. Denote by

$$(\mathcal{T}, c_{\mathcal{T}}) \to (\mathcal{C}, c) \tag{B.6}$$

the relative tangent bundle to the fibers of this family. It is a holomorphic line bundle (locally free sheaf) over $\mathcal{C}$. Its dual $\mathcal{T}^\vee$ is the relative dualizing sheaf of the family:

(i) in local coordinates (B.4) near a node y, it is generated by the sections $\frac{dz_1}{z_1}$ and $\frac{dz_2}{z_2}$ modulo the relation $0 = \frac{d(z_1 z_2)}{z_1 z_2} = \frac{dz_1}{z_1} + \frac{dz_2}{z_2}$.

(ii) in local coordinates (B.2) near a marked point x, it is generated by the section $\frac{dz}{z}$, where the marked point corresponds to $z = 0$.

Both $\mathcal{T}$ and its dual $\mathcal{T}^\vee$ come with a natural real structure induced by c; see [GZ1, Lemma 6.8] for the case C_0 has one real node and [GZ2, Lemma 4.8] for the case C_0 has a pair conjugate nodes (the general case follows similarly). The restriction of (B.6) to a fiber C_b of $\mathcal{C}$ with $b \in B^{\mathbb{R}}$ is the relative tangent bundle (2.8) of (C_b, c_b). Moreover, if $(\widetilde{C}, \widetilde{c})$ denotes the normalization of (C_0, c_0) then the pullback of $\mathcal{T}$ to $\widetilde{C}$ is $\mathcal{T}_{\widetilde{C}}$.

By the Kodaira-Spencer deformation theory, infinitesimal automorphisms of the marked symmetric curve C_0 correspond to $H^0(\mathcal{T}_{C_0})^{\mathbb{R}}$ while infinitesimal variations correspond to $H^1(\mathcal{T}_{C_0})^{\mathbb{R}}$, cf. [GZ1, (5.22) and p. 736].

B.3. Families of real CR operators

Let (E, c_E) be a Real bundle over a flat family $(\mathcal{C}, c)$ of deformations of a fixed symmetric curve as in §B.1. When ∇ is a c_E-compatible (complex linear) connection on E, and A is a smooth 0'th order term, they determine a (pull-back) family of CR operators

$$D_b : \Gamma(C_b; \iota_b^* E)^{\mathbb{R}} \to \Lambda^{01}(C_b; \iota_b^* E)^{\mathbb{R}} \tag{B.7}$$

as in [GZ1, (4.7)-(4.8)], for $b \in B^{\mathbb{R}}$; here $\iota_b : C_b \to \mathcal{C}$ is the inclusion. Note that (0,1)-forms on C_b with values in $\iota_b^* E$ can be regarded as sections of $\iota_b^* F = \Omega^{0,1}_{C_b} \otimes_{\mathbb{C}} \iota_b^* E$ where $F = \Omega^{0,1}_{\mathcal{C}/B} \otimes_{\mathbb{C}} E$. Let

$$\det D = \cup_b \det D_b$$

denote the determinant line bundle of the family (B.7) of CR-operators, defined as in [GZ1, (4.9) and Appendix A]. As explained there, the fact that $\det D$ is a line bundle (i.e. is locally trivial over $B_{\mathbb{R}}$) follows from the proof of the gluing theorem (for linear CR operators).

Families of real CR-operators often arise by pulling back data associated to Real bundles over smooth targets as in [GZ1, §4.3, p 711]. Specifically, assume (X, J, c_X) is a a smooth almost complex manifold with an anti-complex involution, and $(E, c_E) \to (X, c_X)$ is Real bundle over it. Fix a c_E-compatible (complex linear) connection ∇ on E. Given a family of real maps $f : C \to X$, we can consider a family of CR operators

$$\begin{aligned} &D_f : \Gamma(C; f^* E)^{\mathbb{R}} \to \Lambda^{01}(C, f^* E)^{\mathbb{R}}, \\ &D_f(\xi) = \overline{\partial}^{\nabla}_{f^*(E, c_E)} \xi + A_f \xi \end{aligned} \tag{B.8}$$

defined as in [GZ1, §4.3, p 711], where A is a 0'th order term.

When Σ is smooth, $\mathcal{T}_\Sigma \to \Sigma$ is a Real bundle over a smooth target $X = \Sigma$, and we can similarly consider the family of linearizations

$$D_f : \Gamma(C; f^* \mathcal{T}_\Sigma)^{\mathbb{R}} \oplus T_C \overline{\mathcal{M}}^{\mathbb{R}} \to \Lambda^{01}(C; f^* \mathcal{T}_\Sigma)^{\mathbb{R}} \tag{B.9}$$

cf. (A.47) associated to a family of real maps $f : C \to \Sigma$. Note that up to 0'th order terms, D_f is equal to $\overline{\partial}_{f^*(\mathcal{T}_\Sigma, c_\mathcal{T})} \oplus 0_{W_f}$, where $W_f = T_C \overline{\mathcal{M}}^{\mathbb{R}}$ and the considerations in [GZ1, Appendix A] continue to apply in this case.

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