Contact categories of disks

KO HONDA AND YIN TIAN

In the first part of the paper we associate a pre-additive category $\mathcal{C}(\Sigma)$ to a closed oriented surface Σ, called the *contact category* and constructed from contact structures on $\Sigma \times [0, 1]$. There are also $\mathcal{C}(\Sigma, F)$, where Σ is a compact oriented surface with boundary and $F \subset \partial\Sigma$ is a finite oriented set of points which bounds a submanifold of $\partial\Sigma$, and universal covers $\widetilde{\mathcal{C}}(\Sigma)$ and $\widetilde{\mathcal{C}}(\Sigma, F)$ of $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma, F)$. In the second part of the paper we prove that the universal cover of the contact category of a disk admits an embedding into its "triangulated envelope."

1. Introduction

The goal of this paper is twofold. The first goal is to associate a pre-additive category $\mathcal{C}(\Sigma)$ to a closed oriented surface Σ, called the *contact category* and constructed from contact structures on $\Sigma \times [0,1]$.[1] There are also $\mathcal{C}(\Sigma, F)$, where Σ is a compact oriented surface with boundary and $F \subset \partial\Sigma$ is a finite oriented set of points which bounds a submanifold of $\partial\Sigma$, and universal covers $\widetilde{\mathcal{C}}(\Sigma)$ and $\widetilde{\mathcal{C}}(\Sigma, F)$ of $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma, F)$. The contact category $\mathcal{C}(\Sigma)$ admits a decomposition

$$\mathcal{C}(\Sigma) = \coprod_{i \in \mathbb{Z}} \mathcal{C}(\Sigma, i)$$

into connected components, where i is the Euler class (= first Chern class) of the contact structure evaluated on Σ.

The contact categories, a priori, have no reason to satisfy any nontrivial axioms of a triangulated category. In spite of such an inauspicious start, the contact categories partially satisfy the axioms of a triangulated category, and, in particular, have distinguished triangles that we call the *bypass exact triangles*.

The second goal of this paper is to study the universal covers of contact categories of a disk in more detail. When $\Sigma = D^2$, $\#F = 2n + 2$, and we are in the component where the Euler class is $n - 2e$, we abbreviate

$$\widetilde{\mathcal{C}}_{n,e} := \widetilde{\mathcal{C}}(D^2, F; n - 2e).$$

We prove that $\widetilde{\mathcal{C}}_{n,e}$ admits an embedding into its "triangulated envelope"; more precisely, we have:

Theorem 1.1. *There exist a family of triangulated categories $\widetilde{\mathcal{D}}_{n,e}$ and additive functors $\widetilde{\mathcal{F}}_{n,e} : \widetilde{\mathcal{C}}_{n,e} \to \widetilde{\mathcal{D}}_{n,e}$ such that $\widetilde{\mathcal{F}}_{n,e}$ are fully faithful and images of $\widetilde{\mathcal{F}}_{n,e}$ generate $\widetilde{\mathcal{D}}_{n,e}$ under taking iterated cones. Moreover, $\widetilde{\mathcal{F}}_{n,e}$ is exact, i.e., takes bypass exact triangles to distinguished triangles.*

In this paper we take $\widetilde{\mathcal{C}}_{n,e}$ and $\widetilde{\mathcal{D}}_{n,e}$ to be $\mathbb{F}_2$-linear, where $\mathbb{F}_2$ is the field of two elements. We believe that the analogue of Theorem 1.1 holds for any ground field, but technically difficult to keep track of signs.

---

[1]The contact category was discovered by the first author around 2007, but never written up systematically. We hope that this is the first in a series of papers which develops the theory of contact categories. The idea of constructing a contact category was also pursued by Kevin Walker (unpublished) at around the same time.

The category $\widetilde{\mathcal{D}}_{n,e}$ is the homotopy category of bounded chain complexes of finitely generated left projective $R_{n,e}$-modules, where $R_{n,e}$ is isomorphic to the homology of a strands algebra over a disk (cf. [LOT] and [Za]). The contact categories and their relation to Heegaard Floer homology have been extensively studied by Mathews in a series of papers [M1, M2, M3, M4].

In [Co], Cooper defines the *formal contact category* $\mathcal{K}o_+(\Sigma, F)$, which can be interpreted as an abstractly constructed "triangulated envelope" of $\widetilde{\mathcal{C}}(\Sigma, F)$. (Strictly speaking, the version in [Co] is ungraded, but it is expected that his construction works in the graded case; we are referring to the graded version as $\mathcal{K}o_+(\Sigma, F)$.) He also proves the equivalence of $\mathcal{K}o_+(D^2, F)$ and $\widetilde{\mathcal{D}}_{n,e}$: the functor $\mathcal{K}o_+(D^2, F) \to \widetilde{\mathcal{D}}_{n,e}$ is defined using the universal property of $\mathcal{K}o_+(D^2, F)$ and the functor $\widetilde{\mathcal{D}}_{n,e} \to \mathcal{K}o_+(D^2, F)$ is constructed using the work of Zarev [Za]. Combining Theorem 1.1 and (the extension of) Cooper's work gives:

Theorem 1.2. $\widetilde{\mathcal{C}}_{n,e}$ *embeds in the triangulated envelope* $\mathcal{K}o_+(D^2, F)$.

It is a very interesting problem to understand whether the contact category for a general surface embeds in its triangulated envelope. For algebraic applications, the contact categories of rectangles and annuli were used by the second author to give categorifications of the quantized Lie superalgebra $\mathfrak{sl}(1|1)$) and the Clifford algebras [T1, T2, T3, T4].

Index of notation. We have provided an index of notation at the end of the paper, which we hope the reader will find useful starting with Section 4.

Organization of the paper. In Section 2 we define the contact categories and their universal covers for general surfaces Σ and (Σ, F); from Section 3 we restrict to contact categories $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$ over disks. In Section 3 we introduce the Serre functors of $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$ which provide essential simplifications in the proof of Theorem 1.1. In Section 4 we introduce notation to algebraically describe $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$ and in Section 5 we define a family of triangulated categories $\widetilde{\mathcal{D}}_{n,e}$. In Section 6 we construct a family of functors $\mathcal{F}_{n,e} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ of additive categories and in Section 7 we extend $\mathcal{F}_{n,e}$ to $\widetilde{\mathcal{F}}_{n,e} : \widetilde{\mathcal{C}}_{n,e} \to \widetilde{\mathcal{D}}_{n,e}$ and show that the $\widetilde{\mathcal{F}}_{n,e}$ preserve the shift functors and distinguished triangles. Finally in Section 8 we show that the $\widetilde{\mathcal{F}}_{n,e}$ are fully faithful and the images of $\widetilde{\mathcal{F}}_{n,e}$ generate $\widetilde{\mathcal{D}}_{n,e}$ under taking iterated cones.

2. The contact category

The goal of this section is to define the contact categories $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma, F)$. We first recall some properties of bypasses and contact structures.

2.1. Contact structures and bypasses

For more details, the reader is referred to [H1].

2.1.1. Convex surfaces. Let Σ be a compact oriented surface and $\Gamma \subset \Sigma$ be an oriented, properly embedded 1-manifold (i.e., a multicurve) which divides $\Sigma - \Gamma$ into alternating positive and negative regions in the sense that the sign changes every time Γ is crossed once transversely. The positive region (resp. negative region) will be denoted $R_+(\Gamma)$ (resp. $R_-(\Gamma)$), and the orientation of Γ and the boundary orientation of $R_+(\Gamma)$ agree. Such a 1-manifold Γ is called a *dividing set* of Σ.

Recall that an oriented embedded surface Σ in a contact 3-manifold (M, ξ) is ξ-*convex* (or simply *convex*) if there is a contact vector field X which is positively transverse to Σ. The *dividing set of Σ with respect to* (ξ, X) is the locus

$$\Gamma = \{x \in \Sigma \mid X(x) \in \xi(x)\}.$$

In this paper, our convex surfaces are either closed or compact with Legendrian boundary. In such cases, Γ is an oriented, properly embedded 1-manifold and its isotopy class is independent of the choice of X. The positive (resp. negative) region is the set of points $x \in \Sigma$ for which the orientation induced by $X(x)$ on $\xi(x)$ coincides with (resp. is opposite of) the orientation on $\xi(x)$. A dividing set Γ on Σ will usually be viewed as the dividing set Γ_{Σ_0} with respect to a $[-\varepsilon, \varepsilon]$-invariant contact structure ξ_Γ on $\Sigma \times [-\varepsilon, \varepsilon]$, where we write $\Sigma_t = \Sigma \times \{t\}$; in other words, we are locally taking $X = \partial_t$.

Let Σ be a convex surface with dividing set Γ. According to a criterion of Giroux [Gi1], Σ has a tight neighborhood if and only if either (i) $\Sigma = S^2$ and $\Gamma = S^1$, or (ii) $\Sigma \neq S^2$ and Γ has no homotopically trivial closed component.

2.1.2. Bypasses. An embedded Legendrian arc $\delta \subset \Sigma$ is an *arc of attachment of a bypass* if δ is transverse to Γ and has exactly three intersections with Γ, two of which are points of $\partial\delta$. We also write $\delta = \delta_+ \cup \delta_-$, where $\delta_\pm$ is the closure of $\delta \cap R_\pm(\Gamma)$. Let $U \subset \Sigma$ be a disk neighborhood of δ, which transversely intersects Γ along three arcs and whose boundary is Legendrian. Consider an overtwisted disk $(\{(r,\theta) \mid r \leq 1\}, \zeta)$, where (r,θ) are polar coordinates and ζ is the germ of a contact structure such that its characteristic foliation is of "wheel-and-spokes" type with leaves $r = 1$ and $\theta = const$. Then a *bypass D* is $(\{r \leq 1, 0 \leq \theta \leq \pi\}, \zeta|_{0\leq\theta\leq\pi})$, i.e., one-half of an overtwisted disk.

We now attach D to the invariant contact structure $(\Sigma \times [0,\varepsilon], \xi_\Gamma)$ along Σ_ε (resp. $(\Sigma \times [-\varepsilon, 0], \xi_\Gamma)$ along $\Sigma_{-\varepsilon})$ so that the diameter of D is glued to $\delta \times \{\varepsilon\}$ (resp. $\delta \times \{-\varepsilon\}$). If D is attached to Σ_ε (resp. $\Sigma_{-\varepsilon}$), then we say the bypass is attached *from the front* (resp. *from the back*). When D is attached from the front, a small one-sided neighborhood of $(\Sigma \times [0,\varepsilon]) \cup D$ can be viewed as $(\Sigma \times [0,1], \xi)$, where the dividing set Γ_{Σ_0} is Γ and Γ_{Σ_1} is obtained from Γ by performing the local operation on U as in Figure 1. More specifically, in the rest of this paper, we:

(B1) fix a model contact structure $(D^2 \times [0,1], \zeta)$ such that $D^2 \times \{0,1\}$ is convex with Legendrian boundary, ζ is ∂_t-invariant on $D^2 \times [0,\varepsilon]$ and a neighborhood of $(D^2 \times \{1\}) \cup (\partial D^2 \times [0,1])$, the dividing sets $\Gamma_{D^2 \times \{0\}}$, $\Gamma_{D^2 \times \{1\}}$ and the Legendrian arc δ_0 are as given in Figure 1, and ζ is contactomorphic to a small one-sided neighborhood of $(D^2 \times [0,\varepsilon]) \cup_{\delta_0} D$, where the contactomorphism is the identity on $D^2 \times [0,\varepsilon]$;

(B2) choose an identification $\phi : U \xrightarrow{\sim} D^2$ such that $\phi(\Gamma_{U \times \{0\}}) = \Gamma_{D^2 \times \{0\}}$ and $\phi(\delta) = \delta_0$; and

(B3) let $\xi|_{(\Sigma-U)\times[0,1]}$ be t-invariant with dividing sets $\Gamma_{(\Sigma-U)\times\{t\}} = (\Gamma - U) \times \{t\}$ and $\xi|_{U\times[0,1]} = (\phi \times id_{[0,1]})^*\zeta$.

In other words, a bypass attachment depends on the choices of $U \supset \delta$ and $\phi : U \xrightarrow{\sim} D^2$. We remark that topologically Γ_{Σ_1} is obtained from $\Gamma = \Gamma_{\Sigma_0}$ by applying two band sums in succession.

Suppose that $(\Sigma \times [0,1], \xi)$ is obtained by attaching a bypass along δ to (Σ_0, Γ_0) from the front. If $U_0 \subset \Sigma_0$ is the disk neighborhood of δ and U_1 is the corresponding disk on Σ_1 (in particular, $\Gamma_0|_{\Sigma_0-U_0} = \Gamma_1|_{\Sigma_1-U_1}$), then there is a bypass arc of attachment $\delta' \subset U_1$ which gives (a contact manifold isotopic to) $(\Sigma \times [0,1], \xi)$ when attached to (Σ_1, Γ_1) from the back. We will call δ' the *anti-bypass* arc of the bypass arc δ.
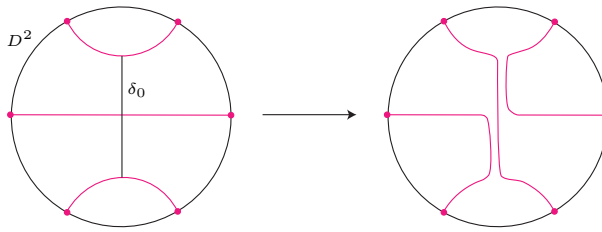
Figure 1: Effect of a bypass attachment along δ_0 from the front. The left-hand side is $D^2 \times \{0\}$ and the right-hand side is $D^2 \times \{1\}$. The red arcs are the dividing curves.

Convention 2.1.2.1. If we do not explicitly mention from which side the bypass is attached, we always assume the bypass is attached from the front.

A bypass is *overtwisted* (resp. *trivial*) if there exists a disk neighborhood $U \subset \Sigma$ of the arc of attachment δ such that:

1) $\Gamma \pitchfork U$ and $\Gamma|_U$ consists of two arcs;

2) if Γ' is the result of attaching the bypass, then $\Gamma'|_U$ has a homotopically trivial component (resp. $\Gamma'|_U$ is homotopic to $\Gamma|_U$).

See Figure 2.

Figure 2: Overtwisted bypass (top) and trivial bypass (bottom).
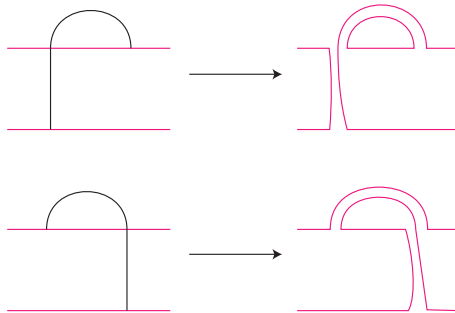
2.1.3. Bypass rotation. We will now discuss *bypass rotation*, which was introduced in [HKM]. Let Σ be a convex surface with dividing set Γ. The ambient contact manifold for Σ is the $[-\varepsilon, \varepsilon]$-invariant contact neighborhood of $\Sigma = \Sigma_0$. Let δ_0 and δ_1 be arcs of attachment as given in Figure 3. In

particular, δ_0 is obtained from δ_1 by rotating one endpoint in the counter-clockwise direction. The bypasses are to be attached "from the front". We will call such an operation *left rotation.*
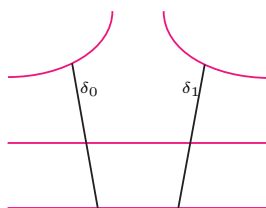

Figure 3: The arc δ_0 is to the left of δ_1.

Lemma 2.1.3.1 (Bypass Rotation). *Let* $(\Sigma \times [0,1], \xi_{\delta_1})$ *be the contact manifold obtained from* (Σ, Γ) *by attaching a bypass from the front along* δ_1. *If* δ_0 *is obtained from* δ_1 *by left rotation, then there exists a bypass along* δ_0 *inside* $(\Sigma \times [0,1], \xi_{\delta_1})$.

The lemma is completely local, i.e., it is valid when $\Sigma = D^2$ and Γ consists of the four arcs given in Figure 3.

2.2. The contact category $\mathcal{C}(\Sigma)$

Let Σ be a closed, oriented surface. In this subsection we assign to each Σ a category $\mathcal{C}(\Sigma)$, called the *contact category.*

Definition 2.2.1. A surface Σ is *collared* if it is equipped with auxiliary data $(\Sigma \times [-\varepsilon, \varepsilon], X)$, where:

(i) $\Sigma \times [-\varepsilon, \varepsilon]$ is a thickening of Σ with coordinates (x, t) so that $\Sigma = \Sigma \times \{0\}$; and

(ii) X is the nonsingular vector field ∂_t on $\Sigma \times [-\varepsilon, \varepsilon]$, i.e., the pullback of ∂_t under the projection $\Sigma \times [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$.

The manifold $\Sigma \times [-\varepsilon, \varepsilon]$ is a *collar* or *collar neighborhood* of Σ.

We often write $\Sigma_t = \Sigma \times \{t\}$.

2.2.1. The category Cont(Σ). We first define the category **Cont**(Σ), where Σ is a collared surface with $\varepsilon > 0$ small.

The objects of $\mathbf{Cont}(\Sigma)$ are dividing sets Γ on Σ, where a *dividing set* Γ is an oriented embedded 1-manifold which is the oriented boundary of an open 2-dimensional submanifold $R_+(\Gamma)$ of Σ. The submanifold $R_+(\Gamma)$ has the same orientation as Σ and $R_-(\Gamma) = \Sigma - \overline{R_+(\Gamma)}$ has the opposite orientation as Σ. The collection of objects of $\mathbf{Cont}(\Sigma)$ will be denoted by $\mathfrak{ob}(\mathbf{Cont}(\Sigma))$.

Remark 2.2.1.1. We take the objects to be 1-manifolds, not *isotopy classes* of 1-manifolds. See Section 2.2.2 for more details.

Next we define $\mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$ to be the set of homotopy classes of contact structures ξ on $\Sigma \times [0, 1]$ such that:

1) the boundary $\partial(\Sigma \times [0, 1]) = \Sigma_1 - \Sigma_0$ is ξ-convex;

2) there exists an extension of ξ to $\Sigma \times [-\varepsilon, 1 + \varepsilon]$ so that $(\Sigma \times [-\varepsilon, \varepsilon], \partial_t)$ and $(\Sigma \times [1 - \varepsilon, 1 + \varepsilon], \partial_t)$ are collared neighborhoods of Σ_0 and Σ_1 and on which ∂_t is a contact vector field with dividing sets Γ and Γ' on Σ_0 and Σ_1.

Two contact structures ξ and ξ' are homotopic if there is a path $\{\xi_s\}_{s \in [0,1]}$ of contact structures on $\Sigma \times [0, 1]$ from ξ to ξ' satisfying (1) and (2) above.

The identity morphism $\Gamma \xrightarrow{id} \Gamma$ is (homotopy class of) the $[0, 1]$-invariant contact structure ξ on $\Sigma \times [0, 1]$ with dividing set Γ on $\Sigma \times \{t\}$, $t \in [0, 1]$.

To take the composition of

$$[\xi] \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma') \quad \text{and} \quad [\xi'] \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma', \Gamma''),$$

we choose representatives ξ and ξ' so they agree on collared neighborhoods of Σ and then glue. The composition $[\xi' \circ \xi]$ does not depend on the choices (see Remark 2.2.1.2). The associativity and unit axioms are easily verified.

Remark 2.2.1.2. The set of contact structures on $\Sigma \times [-\varepsilon, \varepsilon]$ which have dividing set Γ with respect to the contact vector field $X = \partial_t$ is contractible. For this reason, we will suppress the collar $\Sigma \times [-\varepsilon, \varepsilon]$ in the rest of the paper.

Notation 2.2.1.3. In what follows we abuse notation and write $\xi \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$ to mean the homotopy class of a contact structure ξ.

2.2.2. Isotopy of dividing curves and the weak identity morphism. Suppose $\Gamma_0, \Gamma_1 \in \mathfrak{ob}(\mathbf{Cont}(\Sigma))$ and Γ_t, $t \in [0, 1]$, is an isotopy of dividing curves from Γ_0 to Γ_1.

Definition 2.2.2.1. A contact structure ξ on $\Sigma \times [0,1]$ is a *weak identity morphism from* Γ_0 *to* Γ_1 if there exists a a contact vector field X for ξ such that X is transverse to all $\Sigma_t = \Sigma \times \{t\}$ and the dividing set of Σ_t with respect to (ξ, X) is Γ_t.

A weak identity morphism from Γ_0 to Γ_1 gives an *isomorphism* between Γ_0 and Γ_1, since one can similarly define its inverse morphism $\Gamma_1 \xrightarrow{\xi'} \Gamma_0$ which is also a weak identity morphism.

The space of dividing curves of a fixed isotopy type has trivial fundamental group, except when $\Sigma = S^2$ or T^2, or when Γ has a homotopically trivial component. Consider the situation where $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Suppose $\Gamma_0 = \Gamma_1$ consists of two parallel, homotopically nontrivial curves of slope ∞. Let ξ_0 be the $[0,1]$-invariant contact structure with dividing set Γ on $\Sigma \times \{t\}$ for all $t \in [0,1]$. If $\phi : T^2 \times [0,1] \xrightarrow{\sim} T^2 \times [0,1]$ is the diffeomorphism $(x, y, t) \mapsto (x + t, y, t)$, then let $\xi_1 = \phi^* \xi_0$. The contact structures ξ_0 and ξ_1 are not isotopic relative to the boundary. (However, they are isotopic when the dividing sets are allowed to move freely.) Similarly, when Γ has a homotopically trivial component, we can take the homotopically trivial component and isotop it around a nontrivial loop in Σ.

2.2.3. Bypass attachment. The most basic nontrivial morphism comes from a bypass attachment. When attaching a bypass along δ to (Σ, Γ) we need to Legendrian realize δ and ∂U, where $U \subset \Sigma$ is a disk neighborhood of δ which transversely intersects Γ along three arcs. This can be done using the Legendrian realization principle of [H1], which states that there exists a homotopy of contact structures $\{\xi_s\}_{s \in [0,1]}$ on $\Sigma \times [-\varepsilon, \varepsilon]$ such that:

1) ξ_0 is a given contact structure on $\Sigma \times [-\varepsilon, \varepsilon]$ which is t-invariant with dividing sets $\Gamma \times \{t\}$, $t \in [0,1]$,

2) ∂_t is a contact vector field on $\Sigma \times [-\varepsilon, \varepsilon]$ with dividing sets $\Gamma \times \{t\}$ for all ξ_s, $s \in [0,1]$, and $t \in [-\varepsilon, \varepsilon]$, and

3) δ and ∂U are Legendrian with respect to ξ_1.

Since we are taking homotopy classes of contact structures in the definition of $\mathbf{Cont}(\Sigma)$ in Section 2.2.1, we may assume that the Legendrian realization automatically takes place when attaching bypasses.

A bypass attachment of D from the front along δ depends on the choices of $U \supset \delta$ and $\phi : U \xrightarrow{\sim} D^2$ by (B1)–(B3) from Section 2.1.2. Two bypass attachments with the same δ and U are related by a weak identity morphism which is "supported on" U.

Every morphism $\zeta \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$ can be written as a composition of bypass attachment morphisms (or *bypass morphisms* for short), followed by a weak identity morphism; see [H2].

2.2.4. Connected components of Cont(Σ). Consider the following map ϕ which partitions the set of dividing curves according to their Euler class = first Chern class (we will often refer to ϕ as the "Spin^c-map"):

$$\phi : \mathfrak{ob}(\mathbf{Cont}(\Sigma)) \to \mathbb{Z},$$

$$\Gamma \mapsto \chi(R_+(\Gamma)) - \chi(R_-(\Gamma)).$$

Here $R_+(\Gamma)$ is the positively oriented subsurface of Σ whose boundary is Γ; $R_-(\Gamma)$ is the negatively oriented subsurface which is the complement of $R_+(\Gamma)$ in Σ; and χ is the Euler characteristic.

We leave it to the reader to verify that the set $\phi^{-1}(i)$ is connected, i.e., for any pair Γ, Γ' with the same ϕ value, there is a sequence of bypass morphisms from Γ to Γ'. We will write $\mathbf{Cont}(\Sigma, i)$ for the full subcategory of $\mathbf{Cont}(\Sigma)$ whose objects are $\phi^{-1}(i)$. Then we have

$$\mathbf{Cont}(\Sigma) = \coprod_{i \in \mathbb{Z}} \mathbf{Cont}(\Sigma, i).$$

We will often refer to $\mathbf{Cont}(\Sigma, i)$ as a *connected component* of $\mathbf{Cont}(\Sigma)$.

2.2.5. "Zero objects" and "zero morphisms". A "zero object" in $\mathbf{Cont}(\Sigma, i)$ is a dividing set Γ with an overtwisted neighborhood and a "zero morphism" is a homotopy class of overtwisted contact structures. Recall that, according to Giroux [Gi1], Γ is not a "zero object" if and only if either $\Sigma = S^2$ and Γ is connected, or $\Sigma \neq S^2$ and Γ has no homotopically trivial component.

Recall that, by Eliashberg's theorem [El1], there is a unique overtwisted contact structure in each homotopy class of 2-plane field. Hence there are as many "zero morphisms" as there are homotopy classes of 2-plane fields in each $\mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$; this problem is remedied when we pass to the universal cover of the contact category in Section 2.5.

2.2.6. The contact category $\mathcal{C}(\Sigma)$. We are now in a position to define the contact category

$$\mathcal{C}(\Sigma) = \coprod_{i \in \mathbb{Z}} \mathcal{C}(\Sigma, i),$$

which is an $\mathbb{F}_2$-linear (and in particular a pre-additive) category. The objects of $\mathcal{C}(\Sigma, i)$ are the same as those of $\mathbf{Cont}(\Sigma, i)$ and $\mathrm{Hom}_{\mathcal{C}(\Sigma,i)}(\Gamma, \Gamma')$ is the $\mathbb{F}_2$-vector space generated by the homotopy classes of tight contact structures of $\mathrm{Hom}_{\mathbf{Cont}(\Sigma,i)}(\Gamma, \Gamma')$.

We are identifying all the "zero morphisms" in $\mathrm{Hom}_{\mathbf{Cont}(\Sigma,i)}(\Gamma, \Gamma')$ into the unique zero morphism of $\mathrm{Hom}_{\mathcal{C}(\Sigma,i)}(\Gamma, \Gamma')$. Moreover, the "zero objects" in $\mathbf{Cont}(\Sigma, i)$ become genuine zero objects in $\mathcal{C}(\Sigma, i)$. They are isomorphic to each other; we choose one zero object and denote it by $\mathbf{0}$.

2.2.7. The categories $\mathbf{Cont}(\Sigma, F)$ and $\mathcal{C}(\Sigma, F)$. Let Σ be a compact oriented surface with boundary and let $F \subset \partial\Sigma$ be a finite set of points which divides $\partial\Sigma$ into alternating positive and negative regions $R_+(F)$ and $R_-(F)$, i.e., the signs on both sides of any point in F are opposite. (In other words, F is a set of *oriented* points which is the boundary of a 1-dimensional submanifold of $\partial\Sigma$.)

The objects of $\mathbf{Cont}(\Sigma, F)$ are dividing sets Γ with endpoints on F, subject to the condition that the signs on $\partial\Sigma - F$ and the signs on $\Sigma - \Gamma$ agree. The morphisms $\Gamma \xrightarrow{\xi} \Gamma'$ are homotopy classes of contact structures on $\Sigma \times [0, 1]$ so that the dividing set on $\Sigma \times \{0\}$ is Γ, the dividing set on $\Sigma \times \{1\}$ is Γ', and the dividing set on $\partial\Sigma \times [0, 1]$ is $\overline{F} \times [0, 1]$, where $\overline{F}$ is the set consisting of one point on each component of $\partial\Sigma - F$.

The contact category

$$\mathcal{C}(\Sigma, F) = \coprod_{i \in \mathbb{Z}} \mathcal{C}(\Sigma, F, i)$$

is defined in the same way as in Section 2.2.6: the objects of $\mathcal{C}(\Sigma, i)$ are the same as those of $\mathbf{Cont}(\Sigma, F, i)$ and $\mathrm{Hom}_{\mathcal{C}(\Sigma,F,i)}(\Gamma, \Gamma')$ is the $\mathbb{F}_2$-vector space generated by the tight contact structures of $\mathrm{Hom}_{\mathbf{Cont}(\Sigma,F,i)}(\Gamma, \Gamma')$.

Notation 2.2.7.1. From now on, Hom without subscripts will always mean $\mathrm{Hom}_{\mathcal{C}(\Sigma)}$ or $\mathrm{Hom}_{\mathcal{C}(\Sigma,F)}$.

2.2.8. Generators and relations. We now give a description of the generators and relations in $\mathcal{C}(\Sigma)$ or $\mathcal{C}(\Sigma, F)$. Recall that every $\xi \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$ can be written as a composition of bypass morphisms and weak identity morphisms. The description of the relations is due to Bin Tian.[2]

---

[2]Not to be confused with the second author.

Theorem 2.2.8.1 (Bin Tian). *Given $\xi \in \mathrm{Hom}_{\mathbf{Cont}(\Sigma)}(\Gamma, \Gamma')$, any two sequences of bypass attachments and weak identity morphisms from Γ to Γ' which compose to give ξ can be taken to one another via the following two types of operations:*

(R_1) *Far commutativity — given two disjoint bypass arcs of attachment, we can reverse the order in which the bypasses are attached.*

(R_2) *Adding a weak identity morphism.*

Theorem 2.2.8.1 is straightforward to prove for the contact category of a disk when ξ is tight and the proof will be sketched in Section 3.

It is easy to see that relations (R_1) and (R_2) imply (R_2'):

(R_2') Bypass rotation — referring to Figure 3, if we attach bypasses along δ_0 and then along δ_1, then the resulting contact structure is homotopic to the one obtained from attaching a bypass along δ_1 and followed by a weak identity morphism.

2.2.9. Opposite category. The opposite category $\mathcal{C}(\Sigma)^{op}$ of $\mathcal{C}(\Sigma)$ is obtained by reversing the arrows. It is not hard to see that $\mathcal{C}(\Sigma)^{op}$ is equivalent to $\mathcal{C}(-\Sigma)$ via the contravariant functor which sends Γ to $-\Gamma$ and the morphism $\Gamma \xrightarrow{\xi} \Gamma'$ to $-\Gamma \xleftarrow{\xi} -\Gamma'$. Observe that when we switch from Σ to $-\Sigma$, the positive and negative regions of $\Sigma - \Gamma$ get switched, i.e., Γ gets sent to $-\Gamma$.

2.3. Bypass exact triangles

A sequence of bypass attachments gives a triangle, called the *bypass exact triangle*, as follows: Suppose the initial configuration is Γ_1. Pick an arc of attachment $\delta \subset \Sigma$ and its neighborhood U. Apply a bypass attachment from the front along δ to obtain Γ_2. Now, inside U, there is a unique arc of attachment δ' which intersects all three arcs of $\Gamma_2 \cap U$. A bypass attachment from the front along δ' yields Γ_3. Similarly, a third bypass attachment from the front along δ'' yields Γ_1. This is summarized in Figure 4. For convenience, we say that the above bypass exact triangle *starts at* $(\Sigma, \Gamma_1, \delta)$.

We claim that attaching two bypasses in succession inside U creates an overtwisted contact structure. Indeed, δ' is the anti-bypass arc of the bypass arc δ. Hence the bypass along δ' from the back and the bypass along δ' from the front glue to give an overtwisted disk. Therefore, the bypass triangle will have the property that the composition of any two successive edges is the zero morphism.
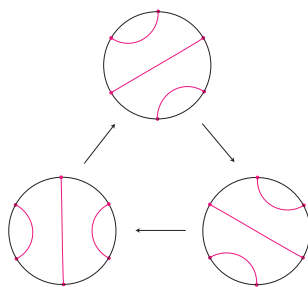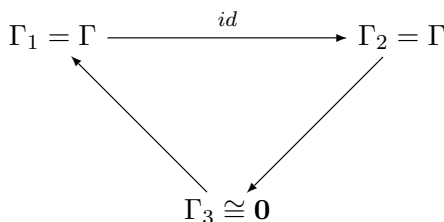
Figure 4: The top is Γ_1, the bottom right is Γ_2, and the bottom left is Γ_3.

Examples of bypass triangles.

(i) (Identity triangle) Consider the morphism $\Gamma_1 = \Gamma \overset{id}{\to} \Gamma = \Gamma_2$ which is equivalently obtained by attaching a *trivial bypass*. Now, attaching the next bypass yields Γ_3 which has a homotopically trivial component. Hence $\Gamma_3 \cong \mathbf{0}$.

(ii) (Fold-unfold triangle) Consider the morphism $\Gamma_1 = \Gamma \to \Gamma' = \Gamma_2$ corresponding to a bypass of fold type (i.e., a bypass such that Γ_2 is the disjoint union of Γ_1 and two parallel homotopically nontrivial curves). The next bypass attachment is an unfold type and the third bypass attachment is overtwisted. (The map $\Gamma \overset{0}{\to} \Gamma$ factors into $\Gamma \overset{fold}{\to} \Gamma'' \overset{unfold}{\longrightarrow} \Gamma$ which glues into an overtwisted contact structure.)

2.4. Octahedral axiom

One of the primary motivations for introducing the contact category was that the bypass triangles often satisfy the octahedral axiom. In other words, there was evidence that the contact category could be embedded inside some sort of "triangulated envelope" while still preserving the bypass triangles. Theorem 1.1 realizes this for the contact category of the disk.

We briefly review the octahedral axiom. Refer to Figure 6. If there are three exact triangles (A, B, C'), (B, C, A'), (A, C, B'), so that the face ABC
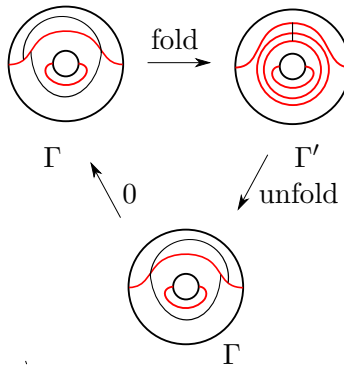
Figure 5: Fold-unfold triangle.

commutes, then there is a fourth exact triangle (C', B', A') (i) which makes the other three faces $A'BC'$, $A'CB'$, $AB'C'$ commute and (ii) such that the compositions $B' \xrightarrow{t} A \xrightarrow{j} B$, $B' \xrightarrow{q} A' \xrightarrow{m} B$ agree and the compositions $B \xrightarrow{k} C \xrightarrow{p} B'$, $B \xrightarrow{n} C' \xrightarrow{s} B'$ agree.
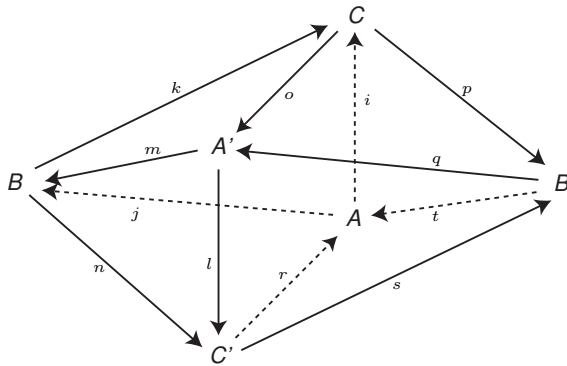

Figure 6.

We present some evidence for the octahedral axiom where $\Sigma = D^2$ and $\#F = 8$. The six dividing sets Γ with $\phi(\Gamma) = 1$, where ϕ is the Spin^c-map, form the octahedron given in Figure 7, where all the arrows are nontrivial bypass morphisms.

Using the labeling from Figure 6, let us consider the compositions $B' \xrightarrow{t} A \xrightarrow{j} B$ and $B' \xrightarrow{q} A' \xrightarrow{m} B$ in Figure 7. Both correspond to the same two bypass moves along disjoint arcs of attachment, and differ only in the order in which the attachment takes place. Hence both compositions give the same
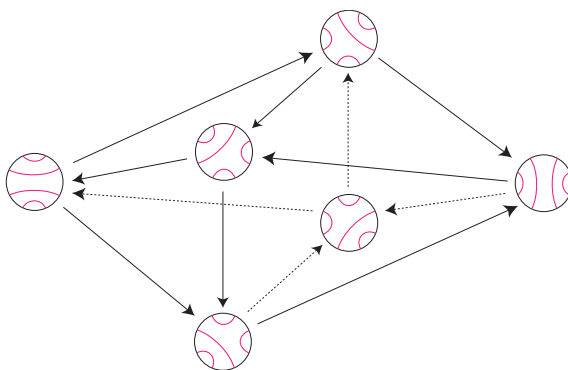
Figure 7.

contact structure up to isotopy, and agree as morphisms $B' \to B$. Similarly, the compositions sn and pk agree.

Let us also discuss the commutativity of the triangle $A'CB'$, for example. For this we use bypass rotation [HKM, Lemma 4.2]. The arc of attachment δ that gives rise to the morphism $C \overset{o}{\to} A'$ can be rotated to the left to give an arc of attachment δ' for the morphism $C \overset{p}{\to} B'$. More precisely, inside a small neighborhood of the union of $\Sigma = D^2$ and the bypass half-disk along δ, there exists a bypass half-disk along δ'. Moreover, the image of the arc δ on B' (after the bypass attachment along δ') is precisely the arc of attachment for $B' \overset{q}{\to} A'$. Therefore, $C \overset{o}{\to} A'$ can be factored into

$$C \overset{p}{\to} B' \overset{q}{\to} A' \overset{x}{\to} A'.$$

By Eliashberg's uniqueness theorem for tight contact structures on the 3-ball, $x = id$ and it follows that $o = qp$.

2.5. The universal cover of the contact category

In this subsection we describe the *universal covers* of the contact categories $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma, F)$.

2.5.1. The universal cover. Let $\mathcal{C}(\Sigma, i)$ be a connected component of $\mathcal{C}(\Sigma)$ and let $\Gamma_0 \in \mathfrak{ob}(\mathcal{C}(\Sigma, i))$. *The universal cover of $\mathcal{C}(\Sigma, i)$ with basepoint Γ_0 is the category* $\widetilde{\mathcal{C}}(\Sigma, i, \Gamma_0)$, together with the *covering functor*

$$\pi : \widetilde{\mathcal{C}}(\Sigma, i, \Gamma_0) \to \mathcal{C}(\Sigma, i),$$

defined as follows:

The objects are given by $(\Gamma_0 \overset{[\zeta]}{\rightsquigarrow} \Gamma)$, where $[\zeta]$ is a homotopy class of 2-plane fields which are contact near $\Sigma \times \{0, 1\}$ and have dividing sets Γ_0 and Γ. The objects are also denoted by $(\Gamma, [\zeta])$.

We define $\mathrm{Hom}_{\widetilde{\mathcal{C}}(\Sigma, i, \Gamma_0)}((\Gamma, [\zeta]), (\Gamma', [\zeta']))$ to be the $\mathbb{F}_2$-vector space generated by homotopy classes of tight contact structures $\xi \in \mathrm{Hom}(\Gamma, \Gamma')$ such that $[\xi \circ \zeta] = [\zeta']$, where $[\xi \circ \zeta]$ is the homotopy class of 2-plane fields on $\Sigma \times [0, 1]$ obtained by concatenating ζ and ξ.

The functor π takes $(\Gamma_0 \overset{[\zeta]}{\rightsquigarrow} \Gamma)$ to Γ and takes

$$(\Gamma_0 \overset{[\zeta]}{\rightsquigarrow} \Gamma) \overset{\xi}{\to} (\Gamma_0 \xrightarrow{[\zeta']=[\xi\circ\zeta]} \Gamma') \in \mathrm{Hom}_{\widetilde{\mathcal{C}}(\Sigma, i, \Gamma_0)}((\Gamma, [\zeta]), (\Gamma', [\zeta'])),$$

to $\Gamma \overset{\xi}{\to} \Gamma' \in \mathrm{Hom}_{\mathcal{C}(\Sigma, i)}(\Gamma, \Gamma')$.

The universal cover $\widetilde{\mathcal{C}}(\Sigma, F, i, \Gamma_0)$ of $\mathcal{C}(\Sigma, F, i)$ is defined similarly. Since Γ_0 determines the integer i, we will sometimes suppress the i and write $\widetilde{\mathcal{C}}(\Sigma, \Gamma_0)$ or $\widetilde{\mathcal{C}}(\Sigma, F, \Gamma_0)$.

2.5.2. 2-plane fields. Suppose that Σ is a closed surface. The preimage $\pi^{-1}(\Gamma)$ of $\Gamma \in \mathfrak{ob}(\mathcal{C}(\Sigma, i))$ is isomorphic to the $\mathbb{Z}$-module $\mathbb{Z} \oplus H_1(\Sigma; \mathbb{Z})$, albeit not naturally. Fix a trivialization of the tangent bundle of $\Sigma \times [0, 1]$ and a reference 2-plane field $(\Gamma_0 \overset{[\zeta_0]}{\rightsquigarrow} \Gamma)$.

We explain how to define the map

$$\Theta : \pi^{-1}(\Gamma) \to \mathbb{Z} \oplus H_1(\Sigma; \mathbb{Z}),$$

$$(\Gamma_0 \overset{[\zeta]}{\rightsquigarrow} \Gamma) \mapsto (\Theta_1(\zeta), \Theta_2(\zeta));$$

see [GH, Hu] for more details: Using a relative version of the Pontryagin-Thom construction, we can assign a framed tangle in $\Sigma \times [0, 1]$ to any ζ. Here the framed tangle is properly embedded and has endpoints on $\Sigma \times \{0, 1\}$. To the difference $\zeta - \zeta_0$ we can assign a framed link $\mathcal{L}$ in (the interior of) $\Sigma \times [0, 1]$.

Any framed link $\mathcal{L}$ in $\Sigma \times [0, 1]$ is the union of the following two types of links, up to framed cobordism: (i) a (not necessarily connected) 1-manifold C on $\Sigma \times \{1/2\}$, with framing coming from the surface, and (ii) a framed unknot. The proof is a slight generalization of the usual Pontryagin-Thom proof of $\pi_3(S^2) \simeq \mathbb{Z}$, whose elements are classified by framed unknots: Let $\pi_\Sigma : \Sigma \times [0, 1] \to \Sigma$ be the projection to Σ. Without loss of generality assume that $\pi_\Sigma(\mathcal{L})$ is an immersion with transverse crossings, and we resolve the crossings to obtain a 1-manifold C which we can view to be on $\Sigma \times \{1/2\}$.

Adjusting the framing on C and resolving the crossings are both equivalent in the framed cobordism category to adding a framed unknot. Finally a union of framed unknots is framed cobordant to a single framed unknot.

The framing of the unknot is $\Theta_1(\zeta)$ and is equal to the Hopf invariant and the homology class of $C \subset \Sigma \times \{1/2\}$ is $\Theta_2(\zeta)$. The class $\Theta_2(\zeta) \in H_1(\Sigma; \mathbb{Z})$ is dual to one-half of the first Chern class of the difference $\zeta - \zeta_0$.

Suppose that $\partial\Sigma \neq \emptyset$, $\partial\Sigma$ is connected, and $\#F > 0$. Let $\mathbf{b} = \{b_1, \ldots, b_{2g}\}$ be a *basis* for Σ, i.e., it is a collection of disjoint, properly embedded, oriented arcs which cut Σ up into a single polygon; in particular, $\mathbf{b}$ can be viewed as a basis for $H_1(\Sigma, \partial\Sigma)$. Let us also assume that $\mathbf{b}$ is transverse to Γ_0 and Γ. Given the basis $\mathbf{b}$, we can take disks $D_i = b_i \times [0, 1]$, whose orientation agrees with the boundary orientation given by that of $b_i \times \{1\}$. We can compute $\langle c_1(\zeta), D_i \rangle$ to be $\chi(R_+(\Gamma_{D_i})) - \chi(R_-(\Gamma_{D_i}))$ with respect to ζ. (Without loss of generality we may assume that ζ is an overtwisted contact structure by Eliashberg's classification of overtwisted contact structures [El1].) Then $\frac{1}{2}c_1(\zeta - \zeta_0)$ assigns an integer to the disks $D_1, \ldots, D_g$, and is dual to $\Theta_2(\zeta)$.

2.5.3. Change of basepoint.   Let $\Gamma_0, \Gamma_0' \in \mathfrak{ob}(\mathcal{C}(\Sigma, i))$ be two basepoints. If ζ is a homotopy class of 2-plane fields which is contact near $\Sigma \times \{0, 1\}$ and has dividing sets Γ_0 and Γ_0' that lie on $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, respectively, then ζ induces a *change-of-basepoint functor*

$$F_\zeta : \widetilde{\mathcal{C}}(\Sigma, \Gamma_0') \to \widetilde{\mathcal{C}}(\Sigma, \Gamma_0),$$

which is given by:

$$F((\Gamma_0' \overset{[\xi]}{\to} \Gamma_1) \overset{\xi'}{\to} (\Gamma_0' \xrightarrow{[\xi' \circ \xi]} \Gamma_2)) = (\Gamma_0 \overset{[\xi \circ \zeta]}{\to} \Gamma_1) \overset{\xi'}{\to} (\Gamma_0 \xrightarrow{[\xi' \circ \xi \circ \zeta]} \Gamma_2).$$

The functor F_ζ gives an equivalence of the two categories. Also, if ζ is a homotopy class from Γ_0 to $\Gamma_0' = \Gamma_0$, then the functor F_ζ is a *deck transformation* of $\widetilde{\mathcal{C}}(\Sigma, \Gamma_0)$.

2.5.4. Bypass exact triangles.   A bypass exact triangle

$$\ldots \to \Gamma_1 \overset{\xi_1}{\to} \Gamma_2 \overset{\xi_2}{\to} \Gamma_3 \overset{\xi_3}{\to} \Gamma_1 \to \ldots$$

lifts to a bypass exact triangle

$$\ldots \to (\Gamma_0 \overset{[\zeta]}{\to} \Gamma_1) \overset{\xi_1}{\to} (\Gamma_0 \xrightarrow{[\xi_1 \circ \zeta]} \Gamma_2) \overset{\xi_2}{\to} (\Gamma_0 \xrightarrow{[\xi_2 \circ \xi_1 \circ \zeta]} \Gamma_3)$$

$$\xrightarrow{\xi_3} (\Gamma_0 \xrightarrow{[\xi_3 \circ \xi_2 \circ \xi_1 \circ \zeta]} \Gamma_1) \to \dots.$$

For convenience, let us write $\zeta' = \xi_3 \circ \xi_2 \circ \xi_1 \circ \zeta$.

Let $T : \widetilde{\mathcal{C}}(\Sigma, \Gamma_0) \to \widetilde{\mathcal{C}}(\Sigma, \Gamma_0)$ be the deck transformation which, for each fiber $\pi^{-1}(\Gamma)$ and identification $\Theta : \pi^{-1}(\Gamma) \xrightarrow{\sim} \mathbb{Z} \oplus H_1(\Sigma; \mathbb{Z})$, sends $(m, x) \in \mathbb{Z} \oplus H_1(\Sigma; \mathbb{Z})$ to $(m - 1, x)$. In other words, the shift functor is a grading shift which drops the Hopf invariant (i.e., the linking number of the framed unknot) by one without changing the relative Spin^c-structure.

The following theorem, due to Huang [Hu], shows that the functor T is the shift functor for the bypass exact triangles in $\widetilde{\mathcal{C}}(\Sigma, \Gamma_0)$.

Theorem 2.5.4.1 (Huang [Hu]). *The homotopy classes of* $(\Gamma_0 \xrightarrow{\zeta} \Gamma_1)$ *and* $(\Gamma_0 \xrightarrow{\zeta'} \Gamma_1)$ *have the same relative Spin^c-structure, and the Hopf invariant of ζ is one higher than that of ζ'.*

3. Contact category of a disk

In the rest of this paper we restrict attention to contact categories of the disk. Letting $\Sigma = D^2$, $\#F = 2n + 2$, $0 \le e \le n$, we consider $\mathcal{C}(D^2, F, n - 2e)$ and $\widetilde{\mathcal{C}}(D^2, F, n - 2e)$. The basepoint Γ^0 is arbitrary at this point. If $\Gamma \in \mathfrak{ob}(\mathcal{C}(D^2, F, n - 2e))$, then

$$\chi_+(\Gamma) := \chi(R_+(\Gamma)) = n - e + 1, \quad \chi_-(\Gamma) := \chi(R_-(\Gamma)) = e + 1.$$

We also write $\chi_\pm$ if Γ is understood.

The $n + 1$ arcs of $R_+(F)$ (called "positive arcs") are labeled $\underline{0}, \underline{1}, \dots, \underline{n}$ in clockwise order around ∂D^2. The arc $\underline{0}$ is the "based arc", analogous to a basepoint. We will often write D_n^2 for (D^2, F) with $\#F = 2n + 2$ and a fixed labeling of $R_+(F)$. We also assume that the arcs of $R_+(F)$ are evenly spaced around ∂D^2.

Notation 3.1. The labels of the arcs of $R_+(F)$ will be underlined throughout the paper. We will write $\underline{s} < \underline{t}$ when we mean $s < t$.

3.1. Skeletal subcategory $\mathcal{C}_{n,e}$

The category $\mathcal{C}(D^2, F, n - 2e)$ has an uncountable number of objects since isotopic dividing sets are treated as different objects (cf. Remark 2.2.1.1). However,

- any two isotopic dividing sets are isomorphic in $\mathcal{C}(D^2, F, n - 2e)$ via a weak identity morphism and

- any dividing set with a contractible component is isomorphic to a zero object by Sections 2.2.5 and 2.2.6.

Hence $\mathcal{C}(D^2, F, n - 2e)$ has only finitely many isomorphism classes of objects. In particular, the set of isomorphism classes of nonzero objects in $\mathcal{C}(D^2, F, n - 2e)$ is in bijection with the set of isotopy classes of dividing sets without closed components, which in turn is in bijection with the set of crossingless matchings with $\chi_+ = n - e + 1$ and $\chi_- = e + 1$.

Let $\mathcal{C}_{n,e}$ be a skeletal subcategory of $\mathcal{C}(D^2, F, n - 2e)$, obtained by choosing one representative from each isomorphism class of objects of $\mathcal{C}(D^2, F, n - 2e)$ and taking the full subcategory of $\mathcal{C}(D^2, F, n - 2e)$ with these objects. Let $\widetilde{\mathcal{C}}_{n,e}$ be a skeletal subcategory of $\widetilde{\mathcal{C}}(D^2, F, n - 2e)$.

We now shift our perspective slightly and work with $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$ in the rest of the paper. At this point it would be convenient to slightly change the definition of a bypass from Γ to Γ' so that:

new bypass = old bypass, followed by a weak identity morphism.

Given nonzero objects Γ, Γ' of $\mathcal{C}_{n,e}$ with a (new) bypass from Γ to Γ', the (new) bypass does not depend on the choices of U and ϕ that appear in the definition of the old bypass as well as the weak identity morphism. In particular, the relation (R_2) in Theorem 2.2.8.1 can be rephrased as "adding a trivial bypass" in the case of a disk.

3.2. Compositions in $\mathcal{C}_{n,e}$

Given $\Gamma, \Gamma' \in \mathfrak{ob}(\mathcal{C}_{n,e})$, $\gamma_{\Gamma,\Gamma'}$ denotes the dividing set on $\partial(D^2 \times [0, 1])$ obtained by *edge rounding* Γ on $D^2 \times \{0\}$, Γ' on $D^2 \times \{1\}$, and the vertical dividing set on $\partial D^2 \times [0, 1]$. See [H1, Lemma 3.11] for the definition of *edge rounding* of two dividing sets along a common boundary Legendrian curve. We write $\#\gamma_{\Gamma,\Gamma'}$ for the number of components of $\gamma_{\Gamma,\Gamma'}$. If $\#\gamma_{\Gamma,\Gamma'} > 1$, then $\mathrm{Hom}(\Gamma, \Gamma') = 0$; if $\#\gamma_{\Gamma,\Gamma'} = 1$, then $\mathrm{Hom}(\Gamma, \Gamma') \simeq \mathbb{F}_2$ and we denote its generator by $\xi_{\Gamma,\Gamma'}$.

Convention 3.2.1. For the rest of the paper, if $\Gamma, \Gamma' \in \mathfrak{ob}(\mathcal{C}_{n,e})$, then $\mathrm{Hom}(\Gamma, \Gamma')$ is always understood to be $\mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma')$.

We study the composition $\mathrm{Hom}(\Gamma', \Gamma'') \times \mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}(\Gamma, \Gamma'')$ when all three spaces are nonzero. The following lemma is similar to [M1, Lemma 3.12]:

Lemma 3.2.2. *Suppose that* $\mathrm{Hom}(\Gamma, \Gamma'), \mathrm{Hom}(\Gamma', \Gamma''), \mathrm{Hom}(\Gamma, \Gamma'')$ *are nonzero and* $\Gamma' \neq \Gamma, \Gamma''$. *Then the following are equivalent:*

1) *The composition* $\mathrm{Hom}(\Gamma', \Gamma'') \times \mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}(\Gamma, \Gamma'')$ *is nontrivial.*

2) *There exists a sequence of dividing sets* Γ^i *for* $0 \leq i \leq k, k \geq 1$ *satisfying:*
   a) $\Gamma^0 = \Gamma, \Gamma^k = \Gamma'$;
   b) *each* $\mathrm{Hom}(\Gamma^i, \Gamma^{i+1})$, $0 \leq i \leq k-1$, *is nonzero and is generated by a bypass;*
   c) $\mathrm{Hom}(\Gamma^i, \Gamma'') \neq 0$ *for* $0 \leq i \leq k$.

3) *There exists a sequence of dividing sets* Γ^i *for* $0 \leq i \leq k, k \geq 1$ *satisfying:*
   a) $\Gamma^0 = \Gamma', \Gamma^k = \Gamma''$;
   b) *each* $\mathrm{Hom}(\Gamma^i, \Gamma^{i+1})$, $0 \leq i \leq k-1$, *is nonzero and is generated by a bypass;*
   c) $\mathrm{Hom}(\Gamma, \Gamma^i) \neq 0$ *for* $0 \leq i \leq k$.

Proof. We prove the equivalence of (1) and (2). The proof of the equivalence of (1) and (3) is similar.

$(1) \Rightarrow (2)$: The tight contact structure $\xi_{\Gamma, \Gamma'}$ can be written as a composition of bypasses $\xi_{\Gamma^{k-1}, \Gamma^k} \circ \cdots \circ \xi_{\Gamma^0, \Gamma^1}$, where $\Gamma^0 = \Gamma$ and $\Gamma^k = \Gamma'$. For $0 \leq i \leq k$, the contact structure $\xi_{\Gamma^k, \Gamma''} \circ \xi_{\Gamma^{k-1}, \Gamma^k} \circ \cdots \circ \xi_{\Gamma^i, \Gamma^{i+1}}$ is tight because it can be embedded in $\xi_{\Gamma', \Gamma''} \circ \xi_{\Gamma, \Gamma'} = \xi_{\Gamma, \Gamma''}$ which is tight. Hence $\mathrm{Hom}(\Gamma^i, \Gamma'') \neq 0$ for $0 \leq i \leq k$.

$(2) \Rightarrow (1)$: Suppose $k = 1$, i.e., $\mathrm{Hom}(\Gamma, \Gamma')$ is generated by a nontrivial bypass $\xi_{\Gamma, \Gamma'}$ and $\mathrm{Hom}(\Gamma, \Gamma'')$ and $\mathrm{Hom}(\Gamma', \Gamma'')$ are nonzero. The dividing sets Γ and Γ' only differ on a neighborhood of the bypass arc of attachment. Since $\mathrm{Hom}(\Gamma, \Gamma'')$ and $\mathrm{Hom}(\Gamma', \Gamma'')$ are nonzero, $\#\gamma_{\Gamma', \Gamma''} = \#\gamma_{\Gamma, \Gamma''} = 1$ and the portion of $\gamma_{\Gamma, \Gamma''}$ which is outside a neighborhood of the arc of attachment is given by the three black arcs in Figure 8. (There are a priori three possibilities for the black arcs by Euler class considerations and only one of them satisfies $\#\gamma_{\Gamma', \Gamma''} = \#\gamma_{\Gamma, \Gamma''} = 1$.) The tight contact structure $\xi_{\Gamma, \Gamma''}$ is obtained from $\xi_{\Gamma', \Gamma''}$ by attaching a bypass which is trivial when viewed as a bypass on $\partial(D^2 \times [0, 1])$. Hence (1) follows when $k = 1$.
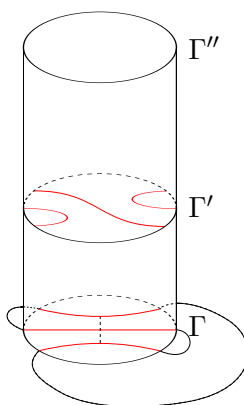
Figure 8: Peeling off the bypass $\xi_{\Gamma,\Gamma'}$ inside $\xi_{\Gamma,\Gamma''}$.

When $k > 1$, one can show by induction on $k - i$ that the compositions $\mathrm{Hom}(\Gamma', \Gamma'') \times \mathrm{Hom}(\Gamma^i, \Gamma') \to \mathrm{Hom}(\Gamma^i, \Gamma'')$ are nontrivial for all $0 \leq i \leq k - 1$. This implies (1) in general. $\qquad\square$

Lemma 3.2.3. *Let* $\Gamma, \Gamma', \Gamma''$ *be nonzero dividing sets. Suppose* $\xi \in \mathrm{Hom}(\Gamma, \Gamma')$ *is nonzero,* $\beta' \in \mathrm{Hom}(\Gamma', \Gamma'')$ *is a nontrivial bypass, and* $\beta' \circ \xi \in \mathrm{Hom}(\Gamma, \Gamma'')$ *is zero. Then* ξ *can be factored into* $\beta \circ \zeta$, *where* β, β' *are two consecutive bypasses of a bypass triangle.*

Proof. Since $\xi \in \mathrm{Hom}(\Gamma, \Gamma') \neq 0$, we have $\#\gamma_{\Gamma,\Gamma'} = 1$. Since attaching β' to $\partial(D^2 \times I)$ yields an overtwisted contact structure, there exists an anti-bypass along the same arc of attachment inside $(D^2 \times I, \xi)$. This implies that ξ can be factored into $\beta \circ \zeta$, where β, β' are two consecutive bypasses of a bypass triangle. $\qquad\square$

Using the same line of argument (details left to the reader), one can also prove the following:

Lemma 3.2.4. *Let* $\tilde{\Gamma}$ *be a nonzero dividing set. Then* $\mathrm{Hom}(\tilde{\Gamma}, -)$ *is an exact functor from* $\mathcal{C}_{n,e}$ *to the the category of* $\mathbb{F}_2$*-vector spaces, i.e., it takes bypass exact triangles to short exact sequences. Similarly,* $\mathrm{Hom}(-, \tilde{\Gamma})$ *is an exact functor from* $\mathcal{C}_{n,e}$ *to* $\mathbb{F}_2$*-vector spaces.*

We also sketch the proof of Theorem 2.2.8.1 for D_n^2 and a nonzero $\xi_{\Gamma,\Gamma'} \in \mathrm{Hom}(\Gamma, \Gamma')$. This will be used later in Section 6.4.

Sketch of proof of Theorem 2.2.8.1 for D_n^2 and nonzero $\xi_{\Gamma,\Gamma'} \in \mathrm{Hom}(\Gamma, \Gamma')$.
Let $\xi_{\Gamma,\Gamma'}$ be a tight contact structure on $D^2 \times [0, k]$. Suppose we are given
a sequence of bypasses $\xi_{\Gamma^0,\Gamma^1}, \dots, \xi_{\Gamma^{k-1},\Gamma^k}$ which compose to give $\xi_{\Gamma,\Gamma'}$. Here
$\Gamma^0 = \Gamma$, $\Gamma^k = \Gamma'$, and Γ^i is a dividing set on $D^2 \times \{i\}$. Let $\delta_i \subset D^2 \times \{i\}$ be
the arc of attachment for $\xi_{\Gamma^i,\Gamma^{i+1}}$.

Let κ_1 be a boundary parallel component of Γ' and let $\mathfrak{c}_1$ be a component
of $(D^2 \times \{k\}) - \Gamma'$ which is bounded by κ_1 and an arc d_1 of $\partial(D^2 \times \{k\})$.
(This is unique if $n \geq 1$, which we assume.) Extend d_1 in the clockwise
direction along $\partial(D^2 \times \{k\})$ until it reaches the next endpoint of Γ', and
call it d_1'. Let δ_1' be an arc of attachment obtained by slightly pushing d_1'
into $D^2 \times \{k\}$ and let β_1 be the corresponding trivial bypass. Now

$$\xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1} = \beta_1 \circ \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1},$$

where $=$ means equality as morphisms. Since δ_1' is close to ∂D^2, β_1 commutes
with all the $\xi_{\Gamma^i,\Gamma^{i+1}}$, and

$$\beta_1 \circ \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1} = \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1} \circ \beta_1.$$

Here we are abusing notation: there are analogous arcs d_1' and δ_1' on each
$D^2 \times \{i\}$ and we also refer to a bypass attached along $\delta_1' \subset D^2 \times \{i\}$ by β_1.

When we attach β_1 first along $D^2 \times \{0\}$, we obtain a boundary parallel
component of the dividing set which is unchanged through the attachments
of all other bypasses. Hence this boundary parallel component can be re-
moved from consideration, and the same construction can be applied to a
dividing set with fewer components. We can iteratively write down a se-
quence of bypasses $\beta_1, \dots, \beta_l$ so that

$$(3.2.5) \quad \xi_{\Gamma,\Gamma'} = \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1} = \beta_l \circ \cdots \circ \beta_1 \circ \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1}$$

$$(3.2.6) \qquad\qquad\qquad\qquad = \xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ \xi_{\Gamma^0,\Gamma^1} \circ \beta_l \circ \cdots \circ \beta_1,$$

and $\xi_{\Gamma,\Gamma'} = \beta_l \circ \cdots \circ \beta_1$. This means that the bypasses corresponding to
$\xi_{\Gamma^i,\Gamma^{i+1}}$ in Equation (3.2.6) are all trivial. The theorem then follows. $\square$

3.3. Serre functors

In this subsection we define endofunctors $\mathcal{S}$ of $\mathcal{C}_{n,e}$ and $\mathcal{S}_{\widetilde{\mathcal{C}}}$ of $\widetilde{\mathcal{C}}_{n,e}$ which we
call the *Serre functors* of $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$ by analogy with the Serre functors
of triangulated categories introduced by Bondal and Kapranov [BK]. The
reader is referred to [Ke] for more details on Serre functors.

3.3.1. Serre functor of $\mathcal{C}_{n,e}$.

Definition 3.3.1.1 (Serre functor $\mathcal{S}$). The *Serre functor* $\mathcal{S}$ is an end-ofunctor of $\mathcal{C}_{n,e}$ which rotates dividing sets and contact structures by a counterclockwise angle of $\frac{2\pi}{n+1}$. (Recall that we are assuming that $R_+(F)$ is evenly spaced around ∂D^2.)

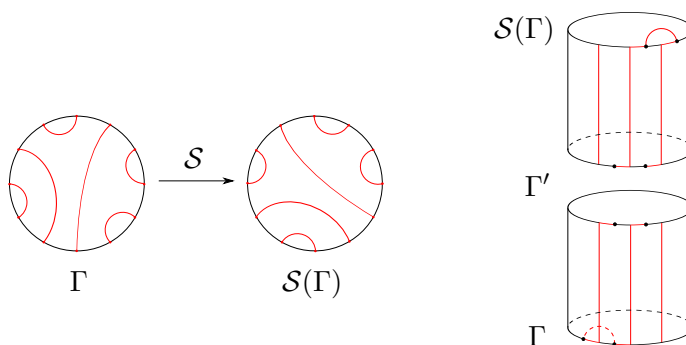See Figure 9 for an example of $\mathcal{S}$ acting on a dividing set Γ.

Figure 9: An example of $\mathcal{S}(\Gamma)$ on the left; dividing sets $\gamma_{\Gamma,\Gamma'}$ and $\gamma_{\Gamma',\mathcal{S}(\Gamma)}$ on the right, used in the proof of Lemma 3.3.1.3.

Remark 3.3.1.2. This rotation operation was first studied in [M1, M2].

Lemma 3.3.1.3. $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ *if and only if* $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)) \neq 0$. *Hence,* $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma)) \neq 0$.

We denote the generator of $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma))$ by $\zeta(\Gamma)$.

Proof. Consider the dividing sets $\gamma_{\Gamma,\Gamma'}$ and $\gamma_{\Gamma',\mathcal{S}(\Gamma)}$. For any boundary parallel component Γ_0 of Γ, there is a corresponding boundary parallel component $\mathcal{S}(\Gamma_0)$ of $\mathcal{S}(\Gamma)$. The results of edge rounding Γ_0 in $\gamma_{\Gamma,\Gamma'}$ and $\mathcal{S}(\Gamma_0)$ in $\gamma_{\Gamma',\mathcal{S}(\Gamma)}$ are the same; see Figure 9. By iterating the above procedure, we obtain $\#\gamma_{\Gamma,\Gamma'} = \#\gamma_{\Gamma',\mathcal{S}(\Gamma)}$, which implies the lemma. $\square$

Lemma 3.3.1.4. *If* $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$, *then the composition*

$$\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)) \times \mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma))$$

is nontrivial.

Proof. We decompose $\xi_{\Gamma,\Gamma'}$ into a composition of bypasses $\xi_{\Gamma^{k-1},\Gamma^k} \circ \cdots \circ$ ξ_{Γ^1,Γ^2}, where $\Gamma^1 = \Gamma$ and $\Gamma^k = \Gamma'$. Then $\mathrm{Hom}(\Gamma, \Gamma^i) \neq 0$ for $1 \leq i \leq k$. By Lemma 3.3.1.3, $\mathrm{Hom}(\Gamma^i, \mathcal{S}(\Gamma)) \neq 0$. The lemma then follows from Lemma 3.2.2. $\qquad\square$

3.3.2. Serre functor of $\widetilde{\mathcal{C}}_{n,e}$.

Definition 3.3.2.1 (Serre functor $\mathcal{S}_{\widetilde{\mathcal{C}}}$). The *Serre functor* $\mathcal{S}_{\widetilde{\mathcal{C}}}$ is an endofunctor of $\widetilde{\mathcal{C}}_{n,e}$ which is defined on objects by $\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma, [\xi]) = (\mathcal{S}(\Gamma), [\zeta(\Gamma) \circ \xi])$ and on morphisms by rotating contact structures by a counterclockwise angle of $\frac{2\pi}{n+1}$.

Claim 3.3.2.2. *The Serre functor $\mathcal{S}_{\widetilde{\mathcal{C}}}$ is well-defined.*

Proof. It suffices to show the following diagram commutes:

$$
\begin{array}{ccc}
(\Gamma, [\xi_0]) & \xrightarrow{\zeta(\Gamma)} & (\mathcal{S}(\Gamma), [\zeta(\Gamma) \circ \xi_0]) \\
\xi \downarrow & & \downarrow \mathcal{S}_{\widetilde{\mathcal{C}}}(\xi) \\
(\Gamma', [\xi \circ \xi_0]) & \xrightarrow{\zeta(\Gamma')} & (\mathcal{S}(\Gamma'), [\zeta(\Gamma') \circ \xi \circ \xi_0])
\end{array}
$$

i.e., $[\zeta(\Gamma') \circ \xi \circ \xi_0] = [\mathcal{S}_{\widetilde{\mathcal{C}}}(\xi) \circ \zeta(\Gamma) \circ \xi_0]$. Here we are assuming that $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$. By Lemma 3.3.1.3, $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)) \neq 0$ and is generated by $\xi_{\Gamma',\mathcal{S}(\Gamma)}$. By applying Lemma 3.3.1.4 to the lower and upper triangles in the diagram, we obtain

$$[\zeta(\Gamma') \circ \xi \circ \xi_0] = [\mathcal{S}_{\widetilde{\mathcal{C}}}(\xi) \circ \xi_{\Gamma',\mathcal{S}(\Gamma)} \circ \xi \circ \xi_0] = [\mathcal{S}_{\widetilde{\mathcal{C}}}(\xi) \circ \zeta(\Gamma) \circ \xi_0].$$

This proves the claim. $\qquad\square$

3.3.3. Calabi-Yau property.

According to [Ke], a triangulated category $\mathcal{T}$ is *weakly d-Calabi-Yau* if it admits a Serre functor $\mathcal{S}'$ and there is an isomorphism of functors $T^d \xrightarrow{\sim} \mathcal{S}'$, where d is an integer and T is the shift functor on $\mathcal{T}$. The analogous result for $\mathcal{S}_{\widetilde{\mathcal{C}}}$ on $\widetilde{\mathcal{C}}_{n,e}$ is the following:

Lemma 3.3.3.1. *The endofunctor $\mathcal{S}_{\widetilde{\mathcal{C}}}^{n+1}$ is isomorphic to $T^{e(n-e)}$ on $\widetilde{\mathcal{C}}_{n,e}$, i.e., $\widetilde{\mathcal{C}}_{n,e}$ is "d-Calabi-Yau" for a fraction $d = \frac{e(n-e)}{n+1}$.*

Proof. For any dividing set Γ, $\mathcal{S}^{n+1}(\Gamma) = \Gamma$ since $\mathcal{S}(\Gamma)$ rotates Γ by $\frac{2\pi}{n+1}$. Also

$$\mathcal{S}_{\widetilde{\mathcal{C}}}^{n+1}(\Gamma, [\xi]) = (\Gamma, [\xi_n(\Gamma) \circ \cdots \circ \xi_0(\Gamma) \circ \xi]),$$

where $\xi_i(\Gamma) = \xi_{\mathcal{S}^i(\Gamma), \mathcal{S}^{i+1}(\Gamma)}$ for $0 \leq i \leq n$. Let $k(\Gamma)$ be minus the Hopf invariant of $\xi_n(\Gamma) \circ \cdots \circ \xi_0(\Gamma)$. The proof of Claim 3.3.2.2 implies that $k(\Gamma)$ is independent of Γ. Hence $\mathcal{S}_{\widetilde{\mathcal{C}}}^{n+1}$ is isomorphic to some T^k, where $k = k(\Gamma)$ for any $\Gamma \in \mathfrak{ob}(\mathcal{C}_{n,e})$.

We compute k by choosing a special $\Gamma \in \mathfrak{ob}(\mathcal{C}_{n,e})$ which has n boundary parallel components; such a Γ is unique in $\mathfrak{ob}(\mathcal{C}_{n,e})$ up to rotation. The case of $n = 5, e = 3$ is depicted in Figure 10. We can write $\xi_0(\Gamma)$ as a composition of $n - e$ bypasses, illustrated as in the upper left diagram of Figure 10. The bypasses of $\xi_i(\Gamma)$ are obtained from those of $\xi_{i-1}(\Gamma)$ by a $\frac{2\pi}{n+1}$ rotation, followed by an isotopy in the radial direction so that they are closer to ∂D^2_{n+1}; the $(n - e)(n + 1)$ bypasses are then mutually disjoint. The $(n - e)(n + 1)$ bypasses can be grouped into $n - e$ copies of $n + 1$ bypasses which are arranged in a circle; see the upper right diagram of Figure 10.
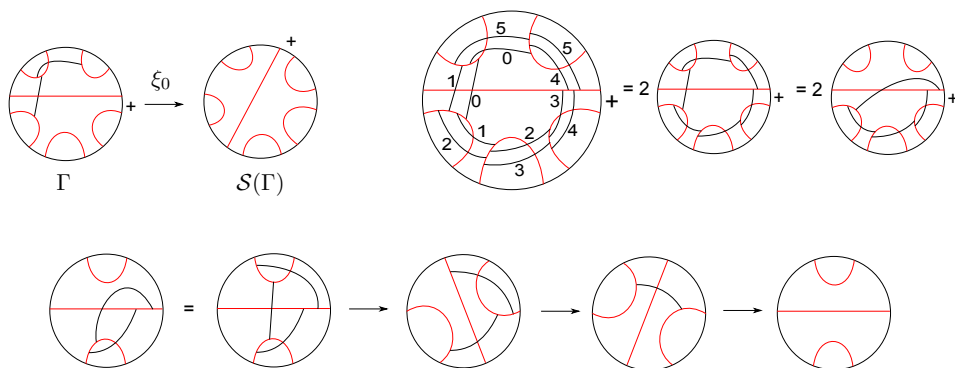

Figure 10: ξ_0 is written as a composition of $n - e = 2$ bypasses on the upper left. The $(n - e)(n + 1) = 2(5 + 1)$ bypasses for $\xi_n(\Gamma) \circ \cdots \circ \xi_0(\Gamma)$ are drawn on the upper right, where the black arcs with label i denote the bypasses of $\xi_i(\Gamma)$ for $0 \leq i \leq n = 5$. The computation of the Hopf invariant is given on the bottom row for $e = 1$.

It suffices to show that the Hopf invariant of the composition of the $n + 1$ bypasses is $-e$. Among the $n + 1$ bypasses, $n - e$ of them are trivial. After attaching the trivial bypasses we are left with $e + 1$ overtwisted bypasses arranged in a pinwheel [HKM, Section 1]; see the upper right diagram of Figure 10. When $e = 1$ the bottom row of Figure 10 shows that attaching the $e + 1 = 2$ overtwisted bypasses in pinwheel position is equivalent to the

composition of the three bypasses in a bypass triangle. Hence the Hopf invariant of the composition is -1 by Theorem 2.5.4.1. We can inductively write the pinwheel with $e+1$ bypasses into two pinwheels, one with two bypasses and another with e bypasses (left to the reader). Hence the Hopf invariant of $\xi_n(\Gamma) \circ \cdots \circ \xi_0(\Gamma)$ is $-e$. $\qquad\square$

Remark 3.3.3.2. Fractional Calabi-Yau categories have recently been studied by Kuznetsov [Ku].

In Section 8.2 we will show that $\widetilde{\mathcal{C}}_{n,e}$ can be embedded into a triangulated category $\widetilde{\mathcal{D}}_{n,e}$ which admits a Serre functor. Moreover, the Serre functors commute with the embedding as proved in Proposition 8.2.3.

4. Algebraic description of $\mathcal{C}_{n,e}$

The contact category $\mathcal{C}_{n,e}$ is defined over a disk D_n^2 with $2(n+1)$ marked points on the boundary, and the Euler number equal to $n-2e$. In particular, $\chi_+ = n - e + 1$, and $\chi_- = e + 1$.

4.1. Notation for dividing sets

In this subsection let $\Gamma \in \mathfrak{ob}(\mathcal{C}_{n,e})$ be a *nonzero* dividing set. Such Γ is a crossingless matching of $2(n+1)$ marked points on ∂D_n^2. We introduce notation to algebraically encode Γ.

A dividing set Γ is determined by its positive region $R_+(\Gamma)$. Let $\pi_0(R_+(\Gamma))$ be the set of components of $R_+(\Gamma)$. Each component $\mathfrak{c}$ of $R_+(\Gamma)$ is a (partially open) disk which intersects ∂D_n^2 at one or more positive arcs and the labels of $\mathfrak{c} \cap \partial D_n^2$ form the set of labels of $\mathfrak{c}$. Observe that a component $\mathfrak{c}$ is determined by its set of labels. A component which has only one label is said to be *boundary parallel*.

The relative positions of the components of $R_+(\Gamma)$ are described by the following *nesting* and *adjacency* relations:

Definition 4.1.1 (Nesting and adjacency). Let $\mathfrak{c}$ and $\mathfrak{c}'$ be components of $R_+(\Gamma)$. Then:

1) $\mathfrak{c}$ *nests inside* $\mathfrak{c}'$ if any path $[0,1] \to D_n^2$ from $\mathfrak{c}$ to the component of $R_+(F)$ corresponding to the label $\underline{0}$ nontrivially intersects $\mathfrak{c}'$.

2) $\mathfrak{c}$ and $\mathfrak{c}'$ are *adjacent* if there is a path in D_n^2 between $\mathfrak{c}$ and $\mathfrak{c}'$ which does not intersect any other component of $R_+(\Gamma)$.

3) $\mathfrak{c}$ *directly nests inside* $\mathfrak{c}'$ if $\mathfrak{c}$ nests inside $\mathfrak{c}'$ and $\mathfrak{c}$ and $\mathfrak{c}'$ are adjacent.

In particular, the component containing $\underline{0}$ does not nest inside any other component and there is no component which nests inside a boundary parallel component.

Let $\mathbb{Z}_+$ be the set of positive integers. Define

$$\mathbb{V} = \bigsqcup_{k \geq 0} \mathbb{Z}_+^k,$$

where $\mathbb{Z}_+^0 = \{*\}$ is a set of one special element. The *dimension* $\dim(\mathbf{v})$ of $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{Z}_+^k$ is k and the dimension of $*$ is 0.

Definition 4.1.2 (Direct nesting of vectors). For $\mathbf{v} \in \mathbb{V}, t \in \mathbb{Z}_+$, define $\mathbf{v} \sqcup t \in \mathbb{V}$ by $\dim(\mathbf{v} \sqcup t) = \dim(\mathbf{v}) + 1$ and

$$(\mathbf{v} \sqcup t)_j = \begin{cases} v_j & \text{if } 1 \leq j \leq \dim(\mathbf{v}), \\ t & \text{if } j = \dim(\mathbf{v}) + 1. \end{cases}$$

The vector $\mathbf{v} \sqcup t$ is said to *directly nest inside* $\mathbf{v}$.

The assignment Φ_Γ. We label regions of $R_+(\Gamma)$ by some vectors in $\mathbb{V}$. More precisely, we inductively define an injective map

$$\Phi_\Gamma : \pi_0(R_+(\Gamma)) \to \mathbb{V}.$$

We use the notation $\Gamma_\mathbf{v} = \Phi_\Gamma^{-1}(\mathbf{v})$ for $\mathbf{v} \in \text{Im}(\Phi_\Gamma)$. The component which contains $\underline{0}$ is defined to be Γ_* and is called the *based component*. Next, given a component $\Gamma_\mathbf{v}$, suppose there are k components which directly nest inside $\Gamma_\mathbf{v}$, arranged in clockwise order with respect to the label $\underline{0}$. The t-th component is then defined to be $\Gamma_{\mathbf{v} \sqcup t}$ for $1 \leq t \leq k$. Note that the two notions of direct nesting — for vectors in $\mathbb{V}$ and for regions of $R_+(\Gamma)$ — agree under the map Φ_Γ. We also sometimes mix up the notation and say that a region directly nests inside a vector.

We now define

$$V(\Gamma) = \text{Im}(\Phi_\Gamma), \quad V^+(\Gamma) = V(\Gamma) \backslash \{*\}, \quad V_{nb}^+(\Gamma) = \{\mathbf{v} \in V^+(\Gamma) \mid |\Gamma_\mathbf{v}| > 1\}.$$

By abuse of notation, we are using $\Gamma_\mathbf{v}$ to denote its set of labels (a subset of $\{\underline{0}, \ldots, \underline{n}\}$) and $|\Gamma_\mathbf{v}|$ to denote its cardinality. Observe that the component $\Gamma_\mathbf{v}$ is not boundary parallel for any $\mathbf{v} \in V_{nb}^+(\Gamma)$. The cardinality $|V(\Gamma)|$ is

the number of components of $R_+(\Gamma)$, which is equal to $\chi_+(\Gamma) = n - e + 1$ for Γ in $\mathcal{C}_{n,e}$.

Definition 4.1.3. A (nonzero) dividing set Γ is *basic* if $V_{nb}^+(\Gamma) = \emptyset$, i.e., every component $\Gamma_{\mathbf{v}} \neq \Gamma_*$ is boundary parallel.

The set of all basic dividing sets in $\mathcal{C}_{n,e}$ is denoted by $B_{n,e}$. For $\underline{0} < \underline{s_1} < \cdots < \underline{s_e} \leq \underline{n}$, let $\Gamma(\underline{s_1}, \ldots, \underline{s_e})$ denote the basic dividing set $\Gamma \in B_{n,e}$ such that $\Gamma_* = \{\underline{0}, \underline{s_1}, \ldots, \underline{s_e}\}$. Any $\Gamma \in B_{n,e}$ is determined by its based component Γ_* which is a subset of $\{\underline{0}, \ldots, \underline{n}\}$ containing $\underline{0}$. Moreover, $(n + 1 - |\Gamma_*|) + 1 = \chi_+(\Gamma) = n - e + 1$. Hence $|\Gamma_*| = e + 1$ and $|B_{n,e}| = \binom{n}{e}$.

Let $l_{\Gamma_{\mathbf{v}}} = |\Gamma_{\mathbf{v}}| - 1$. We order the elements of $\Gamma_{\mathbf{v}}$ so that

$$\Gamma_{\mathbf{v}} = \{\Gamma_{\mathbf{v}}(0), \ldots, \Gamma_{\mathbf{v}}(l_{\Gamma_{\mathbf{v}}})\} \quad \text{and} \quad \Gamma_{\mathbf{v}}(0) < \cdots < \Gamma_{\mathbf{v}}(l_{\Gamma_{\mathbf{v}}}).$$

Example 4.1.4. Let Γ be the dividing set in $\mathcal{C}_{n,e}$ for $n = 7, e = 4$ as shown in Figure 11. There are 4 components of $R_+(\Gamma)$:

$$\Gamma_* = \{\underline{0}, \underline{4}\}; \quad \Gamma_{(1)} = \{\underline{1}, \underline{3}\}, \quad \Gamma_{(2)} = \{\underline{5}, \underline{6}, \underline{7}\}; \quad \Gamma_{(1,1)} = \{\underline{2}\}.$$

The elements of $\Gamma_{(2)}$ satisfy $\Gamma_{(2)}(0) = \underline{5}$, $\Gamma_{(2)}(1) = \underline{6}$, and $\Gamma_{(2)}(2) = \underline{7}$.
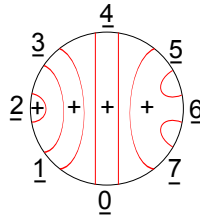

Figure 11: The four components of $R_+(\Gamma)$.

To summarize, we describe a dividing set Γ by a partition $\{\Gamma_{\mathbf{v}} \mid \mathbf{v} \in V(\Gamma)\}$ of $\{\underline{0}, \ldots, \underline{n}\}$, where each $\Gamma_{\mathbf{v}}$ is a component of $R_+(\Gamma)$. We will write $\Gamma = \{\Gamma_{\mathbf{v}}\}$ for simplicity. The collection $\{\Gamma_{\mathbf{v}}\}$ satisfies the following:

1) $V(\Gamma)$ is a finite subset of $\mathbb{V}$ such that $|V(\Gamma)| = n - e + 1$.

2) $* \in V(\Gamma)$ for any Γ and $\underline{0} \in \Gamma_*$.

3) If $\mathbf{v}, \mathbf{v} \sqcup t \in V(\Gamma)$, then there exists unique $i \in \{0, 1, \ldots, l_{\Gamma_{\mathbf{v}}}\}$ such that $\Gamma_{\mathbf{v} \sqcup t}$ is a subset of an open interval $(\Gamma_{\mathbf{v}}(i), \Gamma_{\mathbf{v}}(i + 1))$. Here $\Gamma_{\mathbf{v}}(l_{\Gamma_{\mathbf{v}}} + 1)$ is understood to be $\underline{n + 1}$.

4) If $\mathbf{v}, \mathbf{v} \sqcup t_0 \in V(\Gamma)$, then $\mathbf{v} \sqcup t \in V(\Gamma)$ for $1 \leq t \leq t_0$, and $\Gamma_{\mathbf{v} \sqcup t}(l_{\Gamma_{\mathbf{v} \sqcup t}}) < \Gamma_{\mathbf{v} \sqcup t'}(0)$ for $1 \leq t < t' \leq t_0$.

5) $\displaystyle\bigsqcup_{\mathbf{v} \in V(\Gamma)} \Gamma_{\mathbf{v}} = \{\underline{0}, \ldots, \underline{n}\}$.

Remark 4.1.5. Properties (3) and (4) follow from the fact that a dividing set Γ is properly embedded in D^2.

4.2. Notation for bypasses

In this subsection we introduce notation to describe bypasses.

Let Γ, Γ' be nonzero objects of $\mathcal{C}_{n,e}$, let $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ be a nontrivial bypass, and let $\delta = \delta_+ \cup \delta_-$ be the arc of attachment for β. Since β is nontrivial, δ intersects three distinct components of Γ. We position δ and the three components of Γ as in Figure 12 so that δ is vertical, $int(\delta_+) \subset R_+(\Gamma)$ is the lower subarc, and $int(\delta_-) \subset R_-(\Gamma)$ is the upper subarc.

Notation 4.2.1.

1) $\underline{\mathbf{b}}(\beta)$ is the vector in $V(\Gamma)$ such that $int(\delta_+)$ is contained in the component $\Gamma_{\underline{\mathbf{b}}(\beta)}$.

2) $\overline{\mathbf{b}}(\beta)$ is the vector in $V(\Gamma)$ such that $int(\delta_-)$ connects the components $\Gamma_{\underline{\mathbf{b}}(\beta)}$ and $\Gamma_{\overline{\mathbf{b}}(\beta)}$.

3) $x(\beta), y(\beta)$ are elements of $\{0, \ldots, l_{\Gamma_{\underline{\mathbf{b}}(\beta)}}\}$ such that labels $\Gamma_{\underline{\mathbf{b}}(\beta)}(x(\beta))$ and $\Gamma_{\underline{\mathbf{b}}(\beta)}(y(\beta))$ appear at the bottom left and top left corners of $\Gamma_{\underline{\mathbf{b}}(\beta)}$, respectively.

4) $z(\beta)$ is the element of $\{0, \ldots, l_{\Gamma_{\overline{\mathbf{b}}(\beta)}}\}$ such that the label $\Gamma_{\overline{\mathbf{b}}(\beta)}(z(\beta))$ appears at the bottom left corner of $\Gamma_{\overline{\mathbf{b}}(\beta)}$.

Refer to the left-hand side of Figure 12 for an illustration.

Observe that $\underline{\mathbf{b}}(\beta) \neq \overline{\mathbf{b}}(\beta)$ since β is nontrivial. We will omit the variable β and write $\underline{\mathbf{b}}, \overline{\mathbf{b}}, x, y, z$ for simplicity when β is understood.

Depending on the position of the label $\underline{0}$, both $x \leq y$ and $x > y$ are possible. We use the notation $[[x, y]]$ for the generalized interval between $x, y \in \mathbb{Z}$ given by:

$$[[x, y]] := \begin{cases} [x, y] & \text{if } x \leq y; \\ (-\infty, y] \cup [x, +\infty) & \text{otherwise.} \end{cases}$$

Notation 4.2.2. The component $\Gamma_{\underline{\mathbf{b}}}$ is cut into two parts $\Gamma_{\underline{\mathbf{b}}}^l$ and $\Gamma_{\underline{\mathbf{b}}}^r$ by δ_+, where:

1) $\Gamma_{\underline{\mathbf{b}}}^l := \{\Gamma_{\underline{\mathbf{b}}}(i) \mid i \in [[x, y]]\}$ is the subset of $\Gamma_{\underline{\mathbf{b}}}$ which consists of labels to the left of δ_+.

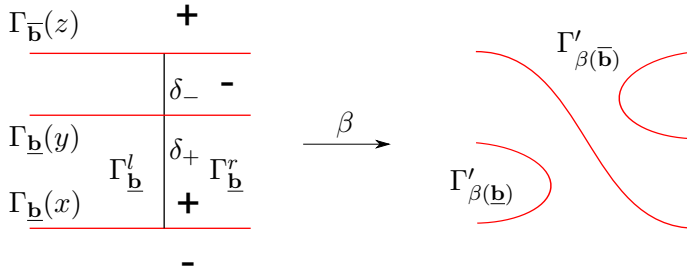2) $\Gamma_{\underline{\mathbf{b}}}^r := \Gamma_{\underline{\mathbf{b}}} \backslash \Gamma_{\underline{\mathbf{b}}}^l$.

Figure 12: Definitions of $\underline{\mathbf{b}}, \overline{\mathbf{b}}, x, y, z$, and the change of components of R_+ under a bypass β.

The map β. Given a nontrivial $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$, by abuse of notation we write

$$(4.2.3) \qquad\qquad \beta : V(\Gamma) \to V(\Gamma')$$

for the map which satisfies

$$\Gamma'_{\beta(\mathbf{v})} = \begin{cases} \Gamma_{\underline{\mathbf{b}}}^l & \text{if } \mathbf{v} = \underline{\mathbf{b}}, \\ \Gamma_{\overline{\mathbf{b}}} \sqcup \Gamma_{\underline{\mathbf{b}}}^r & \text{if } \mathbf{v} = \overline{\mathbf{b}}, \\ \Gamma_{\mathbf{v}} & \text{otherwise}, \end{cases}$$

as subsets of $\{\underline{0}, \dots, \underline{n}\}$ for $\mathbf{v} \in V(\Gamma)$. The map β is a bijection.

Remark 4.2.4. We have $\beta(*) = * \in V(\Gamma')$ unless $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^r$; in that case $\beta(\overline{\mathbf{b}}) = * \in V(\Gamma')$.

5. Definition of $\widetilde{\mathcal{D}}_{n,e}$

Recall from Definition 4.1.3 that $B_{n,e}$, $0 \le e \le n$, is the set of basic dividing sets Γ in $\mathcal{C}_{n,e}$ and that each basic dividing set is determined by Γ_*. Let (Γ) and $(\Gamma|\Gamma')$ denote the generators of $\mathrm{End}(\Gamma)$ and $\mathrm{Hom}(\Gamma, \Gamma')$ which are 1-dimensional when $\mathrm{Hom}(\Gamma, \Gamma') \ne 0$.

For $0 \le e \le n$, define the $\mathbb{F}_2$-algebra

$$R_{n,e} = \bigoplus_{\Gamma, \Gamma' \in B_{n,e}} \mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma'),$$

where the multiplication $a \cdot b$ is given by the composition $b \circ a$ in $\mathcal{C}_{n,e}$ for $a, b \in \{(\Gamma), (\Gamma'|\Gamma'')\}$ if they are composable and zero otherwise.

Proposition 5.1 (Tightness criterion for basic dividing sets). *For* $\Gamma, \Gamma' \in B_{n,e}, \Gamma \ne \Gamma'$, *the following are equivalent:*

1) $\mathrm{Hom}(\Gamma, \Gamma') \ne 0$.

2) *There exists a sequence of labels* $\underline{0} < \underline{s_1} < \underline{s_1'} < \cdots < \underline{s_k} < \underline{s_k'}$ *such that*

$$\Gamma_* \cap [\underline{s_i}, \underline{s_i'}] = \{\underline{s_i}\} \quad for \ 1 \le i \le k$$

and

$$\Gamma_*' = (\Gamma_* \backslash \{\underline{s_1}, \ldots, \underline{s_k}\}) \cup \{\underline{s_1'}, \ldots, \underline{s_k'}\}.$$

We will refer to (2) as the *tightness condition*.

Proof. $(2) \Rightarrow (1)$: The dividing set Γ' can be obtained from Γ by attaching k disjoint bypasses corresponding to k disjoint closed intervals $[\underline{s_i}, \underline{s_i'}]$ for $1 \le i \le k$; see the left-hand side of Figure 13. The resulting contact structure is tight by [HKM, Theorem 1.2].
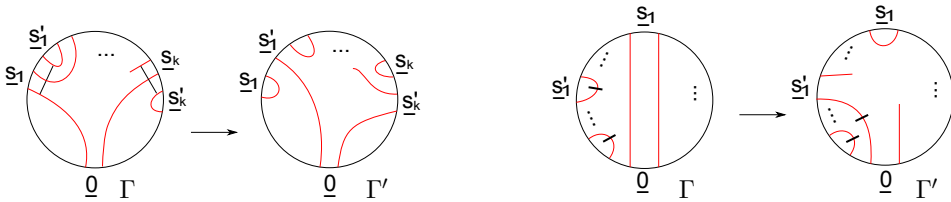

Figure 13: The k disjoint bypasses in the proof of $(2) \Rightarrow (1)$ on the left. The case of $\underline{s_1} > \underline{s_1'}$ in the proof of $(1) \Rightarrow (2)$ on the right. The components of Γ and Γ' with markings form a closed loop after edge rounding.

$(1) \Rightarrow (2)$: Let $\underline{s_1} = \min(\Gamma_* \backslash \Gamma_*'), s_1' = \min(\Gamma_*' \backslash \Gamma_*)$.

We first prove that $\underline{s_1} < \underline{s_1'}$. Arguing by contradiction, suppose that $\underline{s_1} > \underline{s_1'}$. (Note that $\underline{s_1} = s_1'$ is not possible by definition.) Let $\underline{s_1} = \Gamma_*(r_0), \overline{s_1'} = \overline{\Gamma_*'}(r_0)$, and $\Gamma_*(r) = \overline{\Gamma_*'}(r)$ for $r \le r_0$. The case of $r_0 = 1$ is depicted on the right-hand side of Figure 13. The dividing curve on $\partial(D^2 \times [0, 1])$ obtained

by edge rounding (i) the boundary parallel components $\Gamma_{(t)} = \{\underline{t}\}$ for $1 \leq t \leq s'_1$, (ii) the boundary parallel components $\Gamma'_{(t')} = \{\underline{t'}\}$ for $1 \leq t' \leq s'_1 - 1$, and (iii) the arc of Γ' joining the labels $\underline{0}$ and $\underline{s'_1}$, forms a closed loop. Hence $\gamma_{\Gamma,\Gamma'}$ has more than one component and $\mathrm{Hom}(\overline{\Gamma},\Gamma') = 0$, which is a contradiction. The case of r_0 in general is no more difficult and is left to the reader.

Next we prove that $\Gamma_* \cap [\underline{s_1}, \underline{s'_1}] = \{\underline{s_1}\}$ and $\Gamma'_* \cap [\underline{s_1}, \underline{s'_1}] = \{\underline{s'_1}\}$. Suppose that $\Gamma_* \cap [\underline{s_1}, \underline{s'_1}] \neq \{\underline{s_1}\}$. Let $\underline{t_1} = \min(\Gamma_* \cap (\underline{s_1}, \underline{s'_1}))$. The dividing curve on $\partial(D^2 \times [0,1])$ obtained by edge rounding (i) the components of Γ with both endpoints on arcs of $R_+(F)$ labeled $\underline{s_1}$ to $\underline{t_1}$ and (ii) the boundary parallel components of Γ' with both endpoints on arcs labeled $\underline{s_1}$ to $\underline{t_1} - 1$ forms a closed loop. Hence $\mathrm{Hom}(\Gamma, \Gamma') = 0$ and we have a contradiction. This implies that $\Gamma_* \cap [\underline{s_1}, \underline{s'_1}] = \{\underline{s_1}\}$. The proof of $\Gamma'_* \cap [\underline{s_1}, \underline{s'_1}] = \{\underline{s'_1}\}$ is similar.

Let $\Gamma^1 \in B_{n,e}$ such that $\Gamma^1_* = \Gamma_* \backslash \{\underline{s_1}\} \cup \{\underline{s'_1}\}$. We claim that $\mathrm{Hom}(\Gamma^1, \Gamma') \neq 0$. From the proof of $(2) \Rightarrow (1)$ above, $\mathrm{Hom}(\Gamma, \Gamma^1) \neq 0$ and is generated by a bypass that we denote by $(\Gamma|\Gamma^1)$. By an argument similar to the proof of $(2) \Rightarrow (1)$ in Lemma 3.2.2, the bypass $(\Gamma|\Gamma^1)$ is a trivial bypass when viewed as a bypass on $\partial(D^2 \times [0,1])$ with dividing set $\gamma_{\Gamma,\Gamma'}$. By peeling off the bypass $(\Gamma|\Gamma^1)$ from the contact structure which generates $\mathrm{Hom}(\Gamma, \Gamma')$, we obtain a tight contact structure which generates $\mathrm{Hom}(\Gamma^1, \Gamma')$. In particular, this implies the claim.

If $\Gamma^1 = \Gamma'$, then we are done. Otherwise, we inductively define $\underline{s_i} = \min(\Gamma^{i-1}_* \backslash \Gamma'_*)$, $\underline{s'_i} = \min(\Gamma'_* \backslash \Gamma^{i-1}_*)$, and $\Gamma^i \in B_{n,e}$ such that $\Gamma^i_* = \Gamma^{i-1}_* \backslash \{\underline{s_i}\} \cup \{\underline{s'_i}\}$. After finitely many steps, we have $\Gamma^k = \Gamma'$ for some k, which implies (2). $\qquad \square$

Corollary 5.2. *For $\Gamma, \Gamma', \Gamma'' \in B_{n,e}$, if $\mathrm{Hom}(\Gamma, \Gamma'), \mathrm{Hom}(\Gamma', \Gamma'')$, and $\mathrm{Hom}(\Gamma, \Gamma'')$ are all nonzero, then $(\Gamma|\Gamma')(\Gamma'|\Gamma'') = (\Gamma|\Gamma'')$.*

Corollary 5.3. *Suppose $\mathrm{Hom}(\Gamma, \Gamma')$ and $\mathrm{Hom}(\Gamma', \Gamma'')$ are both nonzero for $\Gamma, \Gamma', \Gamma'' \in B_{n,e}$. If there exists $\underline{s}$ such that $\underline{s} \in \Gamma_* \cap \Gamma''_*$ but $\underline{s} \notin \Gamma'_*$, then $\mathrm{Hom}(\Gamma, \Gamma'') = 0$.*

Given $\widetilde{\Gamma} \in B_{n,e}$, let us define $\#(\widetilde{\Gamma}, \underline{s}) = |\{\underline{t} \in \widetilde{\Gamma}_* \mid \underline{t} > \underline{s}\}|$.

Proof. Suppose that $\underline{s} \in \Gamma_* \cap \Gamma''_*$, $\underline{s} \notin \Gamma'_*$, and $\mathrm{Hom}(\Gamma, \Gamma'') \neq 0$. Let $\underline{0} < \underline{s_1} < \underline{s''_1} < \cdots < \underline{s_k} < \underline{s''_k}$ be the sequence of labels in Proposition 5.1 which correspond to $\mathrm{Hom}(\Gamma, \Gamma'')$. Since $\underline{s} \in \Gamma_* \cap \Gamma''_*$, $\underline{s} \notin [\underline{s_i}, \underline{s''_i}]$ for all i. Then we immediately have $\#(\Gamma, \underline{s}) = \#(\Gamma'', \underline{s})$.

On the other hand, $\#(\Gamma', \underline{s}) = \#(\Gamma, \underline{s}) + 1$ since $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ and $\underline{s} \in \Gamma_*, \underline{s} \notin \Gamma'_*$; and $\#(\Gamma'', \underline{s}) \geq \#(\Gamma', \underline{s})$ since $\mathrm{Hom}(\Gamma', \Gamma'') \neq 0$. Hence $\#(\Gamma'', \underline{s}) > \#(\Gamma, \underline{s})$, a contradiction. $\qquad \square$

Notation 5.4. We write $\Gamma \xrightarrow{s} \Gamma'$ for $\Gamma, \Gamma' \in B_{n,e}, s \in \{1, \dots, n-1\}$ if $\underline{s} \in \Gamma_*, \underline{s+1} \notin \Gamma_*$ and $\Gamma'_* = \Gamma\backslash\{\underline{s}\} \cup \{\underline{s+1}\}$. In this case $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$.

The bypasses $\Gamma \xrightarrow{s} \Gamma'$ are the elementary blocks of tight contact structures between basic dividing sets.

Lemma 5.5. *The algebra $R_{n,e}$ has idempotents (Γ), generators $(\Gamma|\Gamma')$ where $\Gamma, \Gamma' \in B_{n,e}, \Gamma \xrightarrow{s} \Gamma'$ for some s, and relations:*

$$(5.6) \qquad\qquad (\Gamma)(\Gamma') = \delta_{\Gamma,\Gamma'};$$

$$(5.7) \qquad\qquad (\Gamma)(\Gamma|\Gamma') = (\Gamma|\Gamma')(\Gamma') = (\Gamma|\Gamma');$$

$$(5.8) \qquad\qquad (\Gamma|\Gamma')(\Gamma'|\Gamma'') = 0 \quad \textit{if } \Gamma \xrightarrow{s} \Gamma', \Gamma' \xrightarrow{s-1} \Gamma'';$$

$$(5.9) \qquad\qquad (\Gamma|\Gamma')(\Gamma'|\Gamma'') = (\Gamma|\Gamma''')(\Gamma'''|\Gamma'')$$

$$\textit{if } \Gamma \xrightarrow{s} \Gamma', \Gamma' \xrightarrow{t} \Gamma'', \Gamma \xrightarrow{t} \Gamma''', \Gamma''' \xrightarrow{s} \Gamma'' \textit{ for } |s-t| > 1.$$

Proof. Suppose that $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ and $\Gamma \neq \Gamma'$. Let $\underline{s_1} = \min(\Gamma_*\backslash\Gamma'_*)$. Define $\widetilde{\Gamma} \in B_{n,e}$ such that $\Gamma \xrightarrow{s_1} \widetilde{\Gamma}$. By Proposition 5.1, $\mathrm{Hom}(\Gamma, \widetilde{\Gamma})$ and $\mathrm{Hom}(\widetilde{\Gamma}, \Gamma')$ are nonzero. By Lemma 3.2.2, the composition $\mathrm{Hom}(\widetilde{\Gamma}, \Gamma') \times \mathrm{Hom}(\Gamma, \widetilde{\Gamma}) \to \mathrm{Hom}(\Gamma, \Gamma')$ is nontrivial since the generator $(\Gamma|\widetilde{\Gamma})$ is a bypass. By an iterated peeling off of bypasses, one can prove that $\{(\Gamma), (\Gamma|\Gamma') \mid \Gamma \xrightarrow{s} \Gamma'$ for some $s\}$ generate $R_{n,e}$ as an algebra.

The first two relations of $R_{n,e}$ are immediate from the definition of $R_{n,e}$.

For a composition of two bypasses $\Gamma \xrightarrow{s} \Gamma', \Gamma' \xrightarrow{t} \Gamma''$, there are 3 possibilities:

1) If $t = s - 1$, then $\Gamma_* \cap [s-1, s+1] = \{\underline{s-1}, \underline{s}\}$ and $\Gamma''_* \cap [s-1, s+1] = \{\underline{s}, \underline{s+1}\}$. Hence $\mathrm{Hom}(\Gamma, \Gamma'') = 0$ by Corollary 5.3, implying the third relation of $R_{n,e}$.

2) If $t = s + 1$, then $\mathrm{Hom}(\Gamma, \Gamma'') \neq 0$ and the product $(\Gamma|\Gamma')(\Gamma'|\Gamma'')$ is the generator of $\mathrm{Hom}(\Gamma, \Gamma'')$. $\Gamma_* \cap [\underline{s}, \underline{s+2}] = \{\underline{s}\}$ and $\Gamma''_* \cap [\underline{s}, \underline{s+2}] = \{\underline{s+2}\}$ and there is no relation in this case.

3) If $|s - t| > 1$, then $\Gamma' \xrightarrow{t} \Gamma''$ induces a bypass $\Gamma \xrightarrow{t} \Gamma'''$ on Γ which is disjoint from the bypass $\Gamma \xrightarrow{s} \Gamma'$. We have $\Gamma_* \cap [\underline{s}, \underline{s+1}] = \{\underline{s}\}$, $\Gamma_* \cap [\underline{t}, \underline{t+1}] = \{\underline{t}\}$, $\Gamma''_* \cap [\underline{s}, \underline{s+1}] = \{\underline{s+1}\}$, and $\Gamma''_* \cap [\underline{t}, \underline{t+1}] = \{\underline{t+1}\}$. The last relation of $R_{n,e}$ follows from the commutativity of a pair of disjoint bypasses.

Now let $R_{n,e}^{\mathrm{alg}}$ denote the algebra with the generators and defining relations as in the lemma. The discussion above gives a homomorphism of

algebras $\phi : R_{n,e}^{\mathrm{alg}} \to R_{n,e}$, which is obviously surjective. To prove the injectivity, it suffices to show that

$$(\Gamma) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma') \cong (\Gamma) \cdot R_{n,e} \cdot (\Gamma') \quad \forall \Gamma, \Gamma'.$$

By Proposition 5.1, $(\Gamma) \cdot R_{n,e} \cdot (\Gamma')$ is one-dimensional if the tightness condition (Proposition 5.1(2)) holds; otherwise, it is zero. If the tightness condition does not hold, then either

(i) there is no path from Γ to Γ', i.e., a sequence $\Gamma = \Gamma_1, \dots, \Gamma_n = \Gamma'$ such that $(\Gamma_i | \Gamma_{i+1})$ is a generator (this is the case if and only if there exists r_0 such that $\Gamma_*(r) \leq \Gamma'_*(r)$ for $r = 0, \dots, r_0 - 1$ and $\Gamma_*(r_0) > \Gamma'_*(r_0)$) or

(ii) there exist $\Gamma_i \xrightarrow{s} \Gamma_{i+1}, \Gamma_{i+1} \xrightarrow{s-1} \Gamma_{i+2}$ such that

$$(5.10) \qquad (\Gamma) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma') \cong \left((\Gamma) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma_i)\right) \cdot \left((\Gamma_i) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma_{i+2})\right)$$
$$\cdot \left((\Gamma_{i+2}) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma')\right).$$

But then (5.10) is zero since $(\Gamma_i) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma_{i+2}) = 0$ by (5.8). If the tightness condition holds, then no factorization of $(\Gamma) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma')$ contains $(\Gamma_i) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma_{i+2})$ satisfying $\Gamma_i \xrightarrow{s} \Gamma_{i+1}, \Gamma_{i+1} \xrightarrow{s-1} \Gamma_{i+2}$. Hence $(\Gamma) \cdot R_{n,e}^{\mathrm{alg}} \cdot (\Gamma')$ is nonzero and one-dimensional. $\qquad \square$

Remark 5.11. The algebra $R_{n,e}$ is isomorphic to the homology of a *strands algebra* of a disk which is a differential graded algebra. Relationships between the contact category and bordered/sutured Heegaard Floer homology have been studied in [Za, M1, M2, M3, M4, Co].

The quiver $Q_{n,e}$. Let $Q_{n,e}$ be the oriented quiver whose set of vertices is $V(Q_{n,e}) = B_{n,e}$ and whose set of arrows is $I(Q_{n,e}) = \{\Gamma \xrightarrow{s} \Gamma' \text{ for some } s\}$. A path in $Q_{n,e}$ from Γ to Γ' is said to be *nonzero* if $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ and a nonzero path is denoted by $\Gamma \to \Gamma'$. We define a partial order "$\leq$" on the set of all nonzero paths: $(\Gamma_1 \to \Gamma'_1) \leq (\Gamma_2 \to \Gamma'_2)$ if $\Gamma_1 \to \Gamma'_1$ can be extended to $\Gamma_2 \to \Gamma'_2$ in $Q_{n,e}$. The partial order motivates the constructions in Section 6; see Remarks 6.2.15 and 6.3.2.7.

The finite dimensional algebra $R_{n,e}$ is isomorphic to a quotient of the path algebra $\mathbb{F}_2 Q_{n,e}$ of $Q_{n,e}$. We refer to [ASS] for an introduction to the representation theory of finite dimensional algebras and quivers. In particular,

by [ASS, Section I.4], $\{(\Gamma) \mid \Gamma \in B_{n,e}\}$ is a complete set of primitive orthogonal idempotents in $R_{n,e}$ and $\{P(\Gamma) = R_{n,e}(\Gamma) \mid \Gamma \in B_{n,e}\}$ forms a complete set of non-isomorphic indecomposable projective left $R_{n,e}$-modules. A nice property of the finite quiver $Q_{n,e}$ is that it has no oriented cycles. It implies that any simple module has a finite projective resolution. Hence the algebra $R_{n,e}$ has finite global dimension.

Define $\widetilde{\mathcal{D}}_{n,e}$ as the homotopy category of bounded cochain complexes of finitely generated projective left $R_{n,e}$-modules. By a standard result in homological algebra $\widetilde{\mathcal{D}}_{n,e}$ is equivalent to the bounded derived category $\mathbf{D}^b(R_{n,e})$ of finitely generated left $R_{n,e}$-modules as triangulated categories. The Grothendieck group $K_0(\widetilde{\mathcal{D}}_{n,e})$ is isomorphic to $\mathbb{Z}^{\oplus \binom{n}{e}}$.

Let $\mathcal{D}_{n,e}$ be the ungraded version of $\widetilde{\mathcal{D}}_{n,e}$, whose objects are the same as $\widetilde{\mathcal{D}}_{n,e}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{D}_{n,e}}(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(M, N[n]).$$

6. The functors $\mathcal{F}_{n,e}$

In this section, we define a family of functors $\mathcal{F}_{n,e} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ for $0 \le e \le n$. We write $\mathcal{F}$ for $\mathcal{F}_{n,e}$ when n, e are understood. Since the definition of $\mathcal{F}$ is highly technical, we first give some motivating examples in Section 6.1. In Section 6.2, we define a complex $\mathcal{F}(\Gamma)$ in $\mathcal{D}_{n,e}$ for each dividing set Γ in $\mathcal{C}_{n,e}$. In Section 6.3, we define a chain map $\mathcal{F}(\beta) \in \mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$ for any nontrivial bypass morphism $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ and then define $\mathcal{F}(\xi)$ in general as a composition of chain maps corresponding to bypasses. In Section 6.4, we show that the functor $\mathcal{F}$ is well-defined.

6.1. Motivation from $\mathcal{C}_{n,e}$

The goal of this subsection is to give some motivating examples.

We say that Γ is *represented* by Γ', Γ'' if there exists a bypass triangle $\Gamma \to \Gamma' \to \Gamma''$ in $\mathcal{C}_{n,e}$. The idea for constructing $\mathcal{F}(\Gamma)$ is to iteratively represent Γ by basic dividing sets using iterated bypass triangles, and then form a complex of (left) projective $R_{n,e}$-modules corresponding to the basic dividing sets.

Recall that the indecomposable projective $R_{n,e}$-modules are of the form $R_{n,e}(\Gamma)$ for $\Gamma \in B_{n,e}$. Using the notation $\Gamma(\underline{s_1}, \dots, \underline{s_e})$ for the basic dividing set in $B_{n,e}$ satisfying $\Gamma(\underline{s_1}, \dots, \underline{s_e})_* = \{\underline{0}, \underline{s_1}, \dots, \underline{s_e}\}$, we write $P(\underline{s_1}, \dots, \underline{s_e})$ for the projective module corresponding to $\Gamma(\underline{s_1}, \dots, \underline{s_e})$.

Example 6.1.1. There are three dividing sets $\Gamma, \Gamma(\underline{1}), \Gamma(\underline{2})$ in $\mathfrak{ob}(\mathcal{C}_{2,1})$ as shown in Figure 14. Among them Γ is not basic since $\Gamma_* = \{\underline{0}\}$ and $\Gamma_{(1)} = \{\underline{1}, \underline{2}\}$. There is a bypass triangle $\Gamma \to \Gamma(\underline{1}) \to \Gamma(\underline{2})$ in $\mathcal{C}_{2,1}$ which is induced by a bypass $\beta(\Gamma) \in \mathrm{Hom}(\Gamma, \Gamma(\underline{1}))$. Here $\beta(\Gamma)$ is the unique nontrivial bypass on Γ whose arc of attachment δ intersects Γ_* at one point and $\delta_+ \subset \Gamma_{(1)}$. Hence Γ is represented by the basic dividing sets $\Gamma(\underline{1}), \Gamma(\underline{2})$. The bypass in $\mathrm{Hom}(\Gamma(\underline{1}), \Gamma(\underline{2}))$ gives a generator of $R_{2,1}$. We define $\mathcal{F}(\Gamma) \in \mathcal{D}_{2,1}$ as the cochain complex $P(\underline{1}) \to P(\underline{2})$, where the differential is the multiplication by the generator of $R_{2,1}$ and $P(\underline{2})$ is at degree 0.
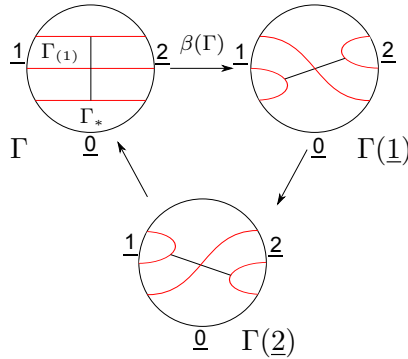

Figure 14: $\mathcal{F}(\Gamma)$ for a non-basic Γ in $\mathcal{C}_{2,1}$.

Representing non-basic dividing sets. Before proceeding to the next examples, we describe the bypass triangles we choose to iteratively represent a non-basic dividing set by basic ones. Given a non-basic Γ, the set $\{i \in \mathbb{Z}_+ \mid (i) \in V_{nb}^+(\Gamma)\}$ is nonempty. Let i_0 be the smallest element of this set. Let $By(\Gamma)$ denote the set of nontrivial bypasses on Γ whose arcs of attachment δ intersect the closure of Γ_* at one point and satisfy $\delta_+ \subset int(\Gamma_{(i_0)})$. The set $By(\Gamma)$ is nonempty since the component $\Gamma_{(i_0)}$ is not boundary parallel. We make the following choice:

Definition 6.1.2 (Choice of $\beta(\Gamma)$). Given a non-basic Γ, let $\beta(\Gamma)$ be the first bypass in the clockwise direction starting from $\underline{0}$ in $By(\Gamma)$. As the based arc $\underline{0}$ is usually put at the bottom, $\beta(\Gamma)$ is called the *leftmost* bypass in $By(\Gamma)$.

Let $\Gamma \xrightarrow{\beta(\Gamma)} \Gamma' \to \Gamma''$ be the triangle induced by $\beta(\Gamma)$. One immediately sees that Γ_* can be viewed as a proper subset of Γ'_* and Γ''_* (in fact this

holds for any bypass in $By(\Gamma)$). In other words, Γ' and Γ'' are "closer" to being basic. If Γ' or Γ'' is not basic, we can further represent it using a triangle induced by $\beta(\Gamma')$ or $\beta(\Gamma'')$. After finitely many steps we can iteratively represent Γ by basic dividing sets.

In Example 6.1.1, Γ is not basic, $V^+(\Gamma) = \{(1)\}$, and $l_{\Gamma_{(1)}} = 1$. In the next example we have $V^+(\Gamma) = \{(1)\}$ and $l_{\Gamma_{(1)}} = 2$.

Example 6.1.3. Let $\Gamma \in \mathfrak{ob}(\mathcal{C}_{4,3})$ such that $\Gamma_* = \{\underline{0}, \underline{4}\}, \Gamma_{(1)} = \{\underline{1}, \underline{2}, \underline{3}\}$; see Figure 15. The bypasses $\beta(\Gamma)$ and $\beta(\Gamma')$ induce two triangles:

$$\Gamma \xrightarrow{\beta(\Gamma)} \Gamma' \to \Gamma(\underline{2}, \underline{3}, \underline{4}), \quad \Gamma' \xrightarrow{\beta(\Gamma')} \Gamma(\underline{1}, \underline{2}, \underline{4}) \to \Gamma(\underline{1}, \underline{3}, \underline{4}),$$

where Γ' is not basic: $\Gamma'_* = \{\underline{0}, \underline{1}, \underline{4}\}, \Gamma'_{(1)} = \{\underline{2}, \underline{3}\}$. The non-basic dividing set Γ is iteratively represented by basic dividing sets $\Gamma(\underline{1}, \underline{2}, \underline{4}), \Gamma(\underline{1}, \underline{3}, \underline{4})$, and $\Gamma(\underline{2}, \underline{3}, \underline{4})$.

Each of $\mathrm{Hom}(\Gamma(\underline{1}, \underline{2}, \underline{4}), \Gamma(\underline{1}, \underline{3}, \underline{4}))$ and $\mathrm{Hom}(\Gamma(\underline{1}, \underline{3}, \underline{4}), \Gamma(\underline{2}, \underline{3}, \underline{4}))$ is generated by a nontrivial bypass and their composition is zero by the tightness criterion (Proposition 5.1). We define $\mathcal{F}(\Gamma) \in \mathcal{D}_{4,3}$ as the cochain complex

$$P(\underline{1}, \underline{2}, \underline{4}) \to P(\underline{1}, \underline{3}, \underline{4}) \to P(\underline{2}, \underline{3}, \underline{4}),$$

where the differentials are given by the bypasses and $P(\underline{2}, \underline{3}, \underline{4})$ is at degree 0.
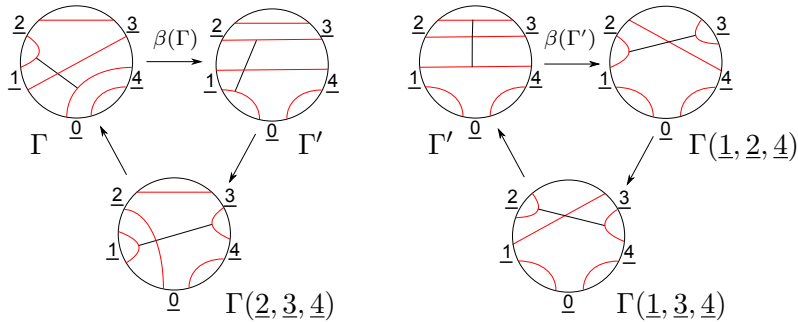

Figure 15: Two triangles to represent a non-basic Γ in $\mathcal{C}_{4,3}$.

In Examples 6.1.1 and 6.1.3, $|V^+(\Gamma)| = 1$. In the following two examples, we consider the case $|V^+(\Gamma)| = 2$.

Example 6.1.4. Consider $\Gamma \in \mathfrak{ob}(\mathcal{C}_{4,2})$ such that $\Gamma_* = \{\underline{0}\}$, $\Gamma_{(1)} = \{\underline{1}, \underline{2}\}$, $\Gamma_{(2)} = \{\underline{3}, \underline{4}\}$; see Figure 16. The bypass $\beta(\Gamma)$ induces a triangle $\Gamma \xrightarrow{\beta(\Gamma)} \Gamma' \to \Gamma''$, where

$$\Gamma'_* = \{\underline{0}, \underline{1}\}, \Gamma'_{(1)} = \{\underline{2}\}, \Gamma'_{(2)} = \{\underline{3}, \underline{4}\}; \quad \Gamma''_* = \{\underline{0}, \underline{2}\}, \Gamma''_{(1)} = \{\underline{1}\}, \Gamma''_{(2)} = \{\underline{3}, \underline{4}\}.$$

Since Γ' and Γ'' are not basic, there are two more triangles induced by $\beta(\Gamma')$ and $\beta(\Gamma'')$:

$$\Gamma' \xrightarrow{\beta(\Gamma')} \Gamma(\underline{1}, \underline{3}) \to \Gamma(\underline{1}, \underline{4}), \quad \Gamma'' \xrightarrow{\beta(\Gamma'')} \Gamma(\underline{2}, \underline{3}) \to \Gamma(\underline{2}, \underline{4}).$$

Each of the nontrivial morphisms $\Gamma(\underline{1}, \underline{3}) \to \Gamma(\underline{1}, \underline{4})$, $\Gamma(\underline{1}, \underline{3}) \to \Gamma(\underline{2}, \underline{3})$, $\Gamma(\underline{1}, \underline{4}) \to \Gamma(\underline{2}, \underline{4})$, $\Gamma(\underline{2}, \underline{3}) \to \Gamma(\underline{2}, \underline{4})$ is given by a nontrivial bypass and the two ways of composing the bypasses in $\mathrm{Hom}(\Gamma(\underline{1}, \underline{3}), \Gamma(\underline{2}, \underline{4}))$ commute. We define $\mathcal{F}(\Gamma) \in \mathcal{D}_{4,2}$ as the cochain complex

$$P(\underline{1}, \underline{3}) \to (P(\underline{1}, \underline{4}) \oplus P(\underline{2}, \underline{3})) \to P(\underline{2}, \underline{4}),$$

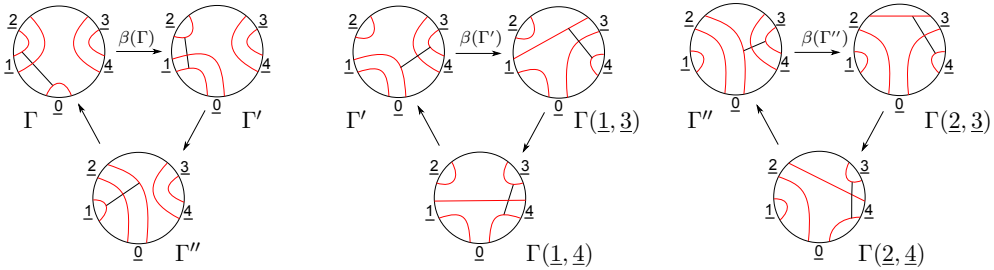where the differentials are induced by the bypasses and $P(\underline{2}, \underline{4})$ is at degree 0.

Figure 16: Three triangles to represent a non-basic Γ in $\mathcal{C}_{4,2}$.

Example 6.1.5. Consider $\Gamma \in \mathfrak{ob}(\mathcal{C}_{4,2})$ such that $\Gamma_* = \{\underline{0}\}$, $\Gamma_{(1)} = \{\underline{1}, \underline{4}\}$, $\Gamma_{(1,1)} = \{\underline{2}, \underline{3}\}$; see Figure 17. The bypass $\beta(\Gamma)$ induces a triangle $\Gamma \xrightarrow{\beta(\Gamma)} \Gamma' \to \Gamma''$, where

$$\Gamma'_* = \{\underline{0}, \underline{1}\}, \Gamma'_{(1)} = \{\underline{2}, \underline{3}\}, \Gamma'_{(2)} = \{\underline{4}\}; \quad \Gamma''_* = \{\underline{0}, \underline{4}\}, \Gamma''_{(1)} = \{\underline{1}\}, \Gamma''_{(2)} = \{\underline{2}, \underline{3}\}.$$

Since Γ' and Γ'' are not basic, there are two more triangles induced by $\beta(\Gamma')$ and $\beta(\Gamma'')$:

$$\Gamma' \xrightarrow{\beta(\Gamma')} \Gamma(\underline{1}, \underline{2}) \to \Gamma(\underline{1}, \underline{3}), \quad \Gamma'' \xrightarrow{\beta(\Gamma'')} \Gamma(\underline{2}, \underline{4}) \to \Gamma(\underline{3}, \underline{4}).$$

There is a tight contact structure in $\mathrm{Hom}(\Gamma(\underline{1},\underline{3}),\Gamma(\underline{2},\underline{4}))$ which is a composition of two bypasses. The spaces $\mathrm{Hom}(\Gamma(\underline{1},\underline{2}),\Gamma(\underline{2},\underline{4}))$ and $\mathrm{Hom}(\Gamma(\underline{1},\underline{3}),\Gamma(\underline{3},\underline{4}))$ are zero. We then define $\mathcal{F}(\Gamma) \in \mathcal{D}_{4,2}$ as the cochain complex

$$(6.1.6) \qquad \mathcal{N} := (P(\underline{1},\underline{2}) \to P(\underline{1},\underline{3}) \to P(\underline{2},\underline{4}) \to P(\underline{3},\underline{4})),$$

where the differentials are induced by the tight contact structures and $P(\underline{3},\underline{4})$ is at degree 0.
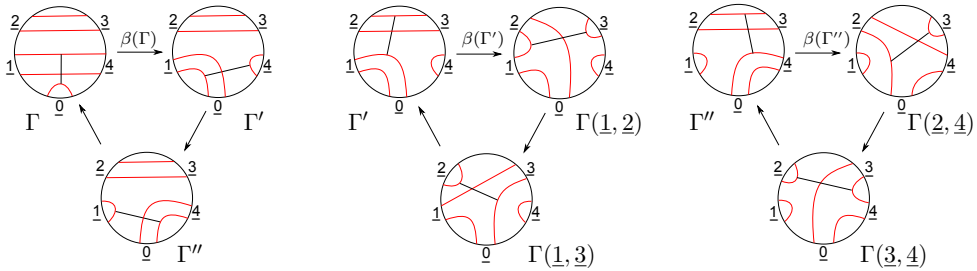

Figure 17: Three triangles to represent a non-basic Γ in $\mathcal{C}_{4,2}$.

In Example 6.1.4, $V(\Gamma) = \{*, (1), (2)\}$, where $\Gamma_{(1)}$ and $\Gamma_{(2)}$ directly nest inside Γ_*. In Example 6.1.5, $V(\Gamma) = \{*, (1), (1,1)\}$ and $\Gamma_{(1,1)}$ directly nests inside $\Gamma_{(1)}$ which in turn directly nests inside Γ_*. As we will see in Equation (6.2.16), the differentials in $\mathcal{F}(\Gamma)$ are defined differently for the two examples.

6.2. Definition of $\mathcal{F}(\Gamma)$

In this subsection we define $\mathcal{F}(\Gamma)$ for $\Gamma \in \mathfrak{ob}(\mathcal{C}_{n,e})$. If $\Gamma \in B_{n,e}$, then we set

$$(6.2.1) \qquad\qquad \mathcal{F}(\Gamma) := P(\Gamma) \in \mathfrak{ob}(\mathcal{D}_{n,e}),$$

viewed as a complex centered at degree 0, and if Γ is a zero object, then we set $\mathcal{F}(\Gamma) = 0$.

In the rest of this subsection suppose Γ is nonzero. We use bypass triangles in $\mathcal{C}_{n,e}$ and construct $\mathcal{F}(\Gamma) \in \mathfrak{ob}(\mathcal{D}_{n,e})$ in 3 steps:

Step 1. Make a list of projective $R_{n,e}$-modules that appear in $\mathcal{F}(\Gamma)$.

Step 2. Define the cohomological degree for each term in Step 1.

Step 3. Define the differential using Steps 1 and 2.

The following definition generalizes the ad hoc definitions in Examples 6.1.1, 6.1.3, 6.1.4, 6.1.5.

Step 1. The list of projective $R_{n,e}$-modules that appear in $\mathcal{F}(\Gamma)$ is given by $\{\Gamma(\mathbf{i}) \in B_{n,e} \mid \mathbf{i} \in OI(\Gamma)\}$, which we describe now.

In the examples from Section 6.1, every Γ was represented by an iterated cone of certain basic dividing sets. The key observation is that the based component of each of these basic dividing sets was obtained from the total set $\{\underline{0}, \ldots, \underline{n}\}$ by omitting one label from each component in $V^+(\Gamma)$.

Definition 6.2.2 (Omitting index). Let

$$OI(\Gamma) = \prod_{\mathbf{v} \in V^+(\Gamma)} \{0, \ldots, l_{\Gamma_{\mathbf{v}}}\}.$$

An element $\mathbf{i} = \langle \mathbf{i_v} \rangle \in OI(\Gamma)$ is called an *omitting index* of Γ and $\mathbf{i_v}$ is called the *entry of* $\mathbf{i}$ *corresponding to* $\mathbf{v}$ (or the "$\mathbf{v}$-entry of $\mathbf{i}$", for short). Also, the set $\{\Gamma_{\mathbf{v}}(\mathbf{i_v})\}_{\mathbf{v} \in V^+(\Gamma)}$ is called the *set of omitting labels for* $\mathbf{i}$.

We define $\mathbf{0} \in OI(\Gamma)$ to be the omitting index such that $\mathbf{0_v} = 0$ for any $\mathbf{v} \in V^+(\Gamma)$.

Remark 6.2.3. Any $\mathbf{i} \in OI(\Gamma)$ is determined by $\{\mathbf{i_v} \mid \mathbf{v} \in V_{nb}^+(\Gamma)\}$ since $\mathbf{i_v} = 0$ for all $\mathbf{v} \in V^+(\Gamma) \backslash V_{nb}^+(\Gamma)$.

Given $\mathbf{i} \in OI(\Gamma)$, define $\Gamma(\mathbf{i}) \in B_{n,e}$ such that

(6.2.4) $\Gamma(\mathbf{i})_* = \Gamma_* \cup \bigsqcup_{\mathbf{v} \in V^+(\Gamma)} (\Gamma_{\mathbf{v}} \backslash \{\Gamma_{\mathbf{v}}(\mathbf{i_v})\}) = \{\underline{0}, \ldots, \underline{n}\} \backslash \{\Gamma_{\mathbf{v}}(\mathbf{i_v})\}_{\mathbf{v} \in V^+(\Gamma)}.$

Since $|V(\Gamma)| = n - e + 1, |V^+(\Gamma)| = n - e$, and $\bigsqcup_{\mathbf{v} \in V(\Gamma)} \Gamma_{\mathbf{v}} = \{\underline{0}, \ldots, \underline{n}\}$, it follows that $|\Gamma(\mathbf{i})_*| = e + 1$. Observe that (i) if $\Gamma \in B_{n,e}$, then we have $OI(\Gamma) = \{\mathbf{0}\}$ and $\Gamma(\mathbf{0}) = \Gamma$, since $l_{\Gamma_{\mathbf{v}}} = 0$ for all $\mathbf{v} \in V^+(\Gamma)$, and (ii) if $\Gamma \notin B_{n,e}$ and $\mathbf{i} \in OI(\Gamma)$, then $\Gamma(\mathbf{i})_*$ always contains Γ_* as a proper subset.

Remark 6.2.5. We have two ways of describing basic dividing sets; they are complementary in some sense. The first one $\Gamma(\underline{s_1}, \ldots, \underline{s_e})$ describes the labels that are contained in the based component and is mainly used in

examples. The second one $\Gamma(\mathbf{i})$ emphasizes the labels that are omitted (i.e., the set $\mathbf{i} \in OI(\Gamma)$) and is our choice for most of the paper.[3]

Step 2. We define the cohomological degree $h(\mathbf{i})$ for each $\mathbf{i} \in OI(\Gamma)$.

Definition 6.2.6.

(i) Given $\mathbf{v} \in V(\Gamma)$ and $0 \le i \le l_{\Gamma_{\mathbf{v}}}$, a vector $\mathbf{w} \in V_{nb}^+(\Gamma)$ *nests inside* $\mathbf{v}$ *up to* i if $\Gamma_{\mathbf{w}} \subset (\Gamma_{\mathbf{v}}(0), \Gamma_{\mathbf{v}}(i))$.

(ii) Given $\mathbf{v} \in V(\Gamma)$ and $0 < i \le l_{\Gamma_{\mathbf{v}}}$, a vector $\mathbf{w} \in V_{nb}^+(\Gamma)$ is a *direct nesting vector* of $(\mathbf{v}, i)$ if $\Gamma_{\mathbf{w}}$ directly nests inside $\Gamma_{\mathbf{v}}$ and $\Gamma_{\mathbf{w}} \subset (\Gamma_{\mathbf{v}}(i-1), \Gamma_{\mathbf{v}}(i))$.

Let $NV(\mathbf{v}, i)$ denote the set of vectors in $V_{nb}^+(\Gamma)$ that nest inside $\mathbf{v}$ up to i and let $DNV(\mathbf{v}, i)$ denote the set of direct nesting vectors of $(\mathbf{v}, i)$.

Definition 6.2.7 (Nesting degree). Given $\mathbf{v} \in V(\Gamma)$ and $0 \le i \le l_{\Gamma_{\mathbf{v}}}$, the *nesting degree* $c_{\mathbf{v}}(i)$ is given by

$$c_{\mathbf{v}}(i) = \sum_{\mathbf{w} \in NV(\mathbf{v},i)} l_{\Gamma_{\mathbf{w}}} = \sum_{\mathbf{w} \in NV(\mathbf{v},i)} (|\Gamma_{\mathbf{w}}| - 1),$$

if $NV(\mathbf{v}, i) \ne \emptyset$ and is zero otherwise.

Definition 6.2.8 (Cohomological degree). The *cohomological degree* $h(\mathbf{i})$ of $\mathbf{i} \in OI(\Gamma)$ is given by

$$h(\mathbf{i}) = \sum_{\mathbf{v} \in V^+(\Gamma)} h(\mathbf{i}, \mathbf{v}), \quad h(\mathbf{i}, \mathbf{v}) = \mathbf{i}_{\mathbf{v}} + c_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}}).$$

Remark 6.2.9. Since $NV(\mathbf{v}, 0) = \emptyset$, we have $c_{\mathbf{v}}(0) = 0$ and $h(\mathbf{0}) = 0$.

The nesting degree is trivial in Examples 6.1.1, 6.1.3, and 6.1.4. Hence $h(\mathbf{i})$ is simply the sum of all the entries of $\mathbf{i}$.

Example 6.1.5 revisited. We have $V^+(\Gamma) = \{(1), (1, 1)\}$ and $l_{\Gamma_{\mathbf{v}}} = 1$ for $\mathbf{v} \in V^+(\Gamma)$. The only nonzero nesting degree is $c_{(1)}(1) = 1$ since $\Gamma_{(1,1)}$ nests inside

---

[3]The notations are similar, but we note that the former has entries that are underlined.

$\Gamma_{(1)}$ up to 1:

$$\Gamma_{(1,1)} = \{\underline{2}, \underline{3}\} \subset (\underline{1}, \underline{4}) = (\Gamma_{(1)}(0), \Gamma_{(1)}(1)).$$

Writing $\mathbf{i} = \langle \mathbf{i}_{(1)}, \mathbf{i}_{(1,1)} \rangle$ for $\mathbf{i} \in OI(\Gamma)$, we have:

$$h\langle 1, 1 \rangle = 1 + 1 + c_{(1)}(1) = 3, \quad h\langle 1, 0 \rangle = 1 + 0 + c_{(1)}(1) = 2,$$
$$h\langle 0, 1 \rangle = 0 + 1 = 1, \quad h\langle 0, 0 \rangle = 0,$$

which agree with the negatives of the degrees in the complex $\mathcal{N}$ from (6.1.6).

Step 3. We define the differential, which is induced by the morphisms between basic dividing sets in $\mathcal{C}_{n,e}$.

Example 6.1.5 revisited. We have

$$\Gamma\langle 1, 1 \rangle = \Gamma(\underline{1}, \underline{2}), \quad \Gamma\langle 1, 0 \rangle = \Gamma(\underline{1}, \underline{3}), \quad \Gamma\langle 0, 1 \rangle = \Gamma(\underline{2}, \underline{4}), \quad \Gamma\langle 0, 0 \rangle = \Gamma(\underline{3}, \underline{4}).$$

There are two types of morphisms between the $\Gamma(\mathbf{i})$'s for $\mathbf{i} = \langle \mathbf{i}_{(1)}, \mathbf{i}_{(1,1)} \rangle \in OI(\Gamma)$:

(SL) $\mathrm{Hom}(\Gamma\langle i, 1 \rangle, \Gamma\langle i, 0 \rangle)$ for $i = 0, 1$;

(SH) $\mathrm{Hom}(\Gamma\langle 1, 0 \rangle, \Gamma\langle 0, 1 \rangle)$.

For Type (SL), only the $(1,1)$-entry of $\mathbf{i} = \langle i, 1 \rangle$ decreases by 1 and the other entry is left unchanged. In Definition 6.2.10 the vector $(1,1)$ is called an *sliding vector* of $\mathbf{i}$.

For Type (SH), the (1)-entry of $\mathbf{i} = \langle 1, 0 \rangle$ decreases from 1 to 0 and the $(1,1)$-entry increases from 0 to $1 = l_{\Gamma_{(1,1)}}$. In this case $(1,1)$ directly nests inside (1). In Definition 6.2.10 the vector (1) is called a *shuffling vector* of $\mathbf{i} = \langle 1, 0 \rangle$.

Definition 6.2.10 (Sliding and shuffling vectors).

  1) A vector $\mathbf{v} \in V^+(\Gamma)$ is a *sliding vector* of $\mathbf{i} \in OI(\Gamma)$ if $\mathbf{i_v} > 0$ and $DNV(\mathbf{v}, \mathbf{i_v}) = \emptyset$.

  2) A vector $\mathbf{v} \in V^+(\Gamma)$ is a *shuffling vector* of $\mathbf{i} \in OI(\Gamma)$ if $\mathbf{i_v} > 0$, $DNV(\mathbf{v}, \mathbf{i_v}) \neq \emptyset$ and $\mathbf{i_w} = 0$ for all $\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})$.

Let $SLV(\mathbf{i})$ denote the set of sliding vectors of $\mathbf{i}$, let $SHV(\mathbf{i})$ denote the set of shuffling vectors of $\mathbf{i}$, and let $SV(\mathbf{i}) = SLV(\mathbf{i}) \cup SHV(\mathbf{i})$. Note that not every vector in $V^+(\Gamma)$ is a sliding vector or a shuffling vector.

Definition 6.2.11 (Modified omitting index). Given $\mathbf{i} \in OI(\Gamma)$ and $\mathbf{v} \in SV(\mathbf{i})$, the $\mathbf{v}$-*modified omitting index* $\mathbf{v}|\mathbf{i} \in OI(\Gamma)$ satisfies

$$(\mathbf{v}|\mathbf{i})_{\mathbf{w}} = \begin{cases} \mathbf{i_v} - 1 & \text{if } \mathbf{w} = \mathbf{v}, \\ \mathbf{i_w} & \text{otherwise}, \end{cases}$$

if $\mathbf{v} \in SLV(\mathbf{i})$;

$$(\mathbf{v}|\mathbf{i})_{\mathbf{w}} = \begin{cases} \mathbf{i_v} - 1 & \text{if } \mathbf{w} = \mathbf{v}, \\ l_{\Gamma_{\mathbf{w}}} & \text{if } \mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v}), \\ \mathbf{i_w} & \text{otherwise}, \end{cases}$$

if $\mathbf{v} \in SHV(\mathbf{i})$.

Vectors in $DNV(\mathbf{v}, \mathbf{i_v}) \subset V_{nb}^+(\Gamma)$ must be indices of non-boundary-parallel regions, so that the $l_{\Gamma_{\mathbf{w}}}$ are positive for all $\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})$.

Remark 6.2.12. As we change from $\mathbf{i}$ to $\mathbf{v}|\mathbf{i}$,

(i) the $\mathbf{v}$-entry of $\mathbf{i}$ is the only entry which decreases;

(ii) a $\mathbf{w}$-entry of $\mathbf{i}$ increases if and only if $\mathbf{v} \in SHV(\mathbf{i})$ and $\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})$; and

(iii) all other entries of $\mathbf{i}$ are left unchanged.

Lemma 6.2.13. *If $\mathbf{v} \in SV(\mathbf{i})$, then $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma(\mathbf{v}|\mathbf{i})) \neq 0$.*

Given $\mathbf{v} \in SV(\mathbf{i})$, let $r(\mathbf{i}, \mathbf{v}) \in R_{n,e}$ be the generator of $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma(\mathbf{v}|\mathbf{i}))$.

Proof. If $\mathbf{v} \in SLV(\mathbf{i})$, then $\Gamma(\mathbf{i})_* \cap [\Gamma_{\mathbf{v}}(\mathbf{i_v} - 1), \Gamma_{\mathbf{v}}(\mathbf{i_v})] = \{\Gamma_{\mathbf{v}}(\mathbf{i_v} - 1)\}$ and

$$\Gamma(\mathbf{v}|\mathbf{i})_* = \Gamma(\mathbf{i})_* \backslash \{\Gamma_{\mathbf{v}}(\mathbf{i_v} - 1)\} \cup \{\Gamma_{\mathbf{v}}(\mathbf{i_v})\}.$$

Then $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma(\mathbf{v}|\mathbf{i})) \neq 0$ by Proposition 5.1.

If $\mathbf{v} \in SHV(\mathbf{i})$, then we can write $DNV(\mathbf{v}, \mathbf{i_v}) = \{\mathbf{u}^1, \dots, \mathbf{u}^k\}$ so that

$$\Gamma_{\mathbf{v}}(\mathbf{i_v} - 1) < \Gamma_{\mathbf{u}^1}(0); \quad \Gamma_{\mathbf{u}^j}(l_{\Gamma_{\mathbf{u}^j}}) < \Gamma_{\mathbf{u}^{j+1}}(0), \ 1 \leq j \leq k - 1;$$
$$\Gamma_{\mathbf{u}^k}(l_{\Gamma_{\mathbf{u}^k}}) < \Gamma_{\mathbf{v}}(\mathbf{i_v}).$$

If $[a, b]$ is any of the corresponding $k + 1$ closed disjoint intervals:

$$(6.2.14) \qquad [\Gamma_{\mathbf{v}}(\mathbf{i_v} - 1), \Gamma_{\mathbf{u}^1}(0)]; \quad [\Gamma_{\mathbf{u}^j}(l_{\Gamma_{\mathbf{u}^j}}), \Gamma_{\mathbf{u}^{j+1}}(0)], \ 1 \leq j \leq k - 1;$$
$$[\Gamma_{\mathbf{u}^k}(l_{\Gamma_{\mathbf{u}^k}}), \Gamma_{\mathbf{v}}(\mathbf{i_v})],$$

the intersections $\Gamma(\mathbf{i})_* \cap [a, b] = \{a\}$. We have

$$\Gamma(\mathbf{v}|\mathbf{i})_* = \Gamma(\mathbf{i})_* \setminus \left(\{\Gamma_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}} - 1)\} \cup \{\Gamma_{\mathbf{u}^j}(l_{\Gamma_{\mathbf{u}^j}})\}_{j=1}^k \right)$$
$$\cup \left(\{\Gamma_{\mathbf{u}^j}(0)\}_{j=1}^k \cup \{\Gamma_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}})\} \right).$$

Hence $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma(\mathbf{v}|\mathbf{i})) \neq 0$ by Proposition 5.1. □

Remark 6.2.15. With a little more work one can show that $\{\Gamma(\mathbf{i}) \to \Gamma(\mathbf{v}|\mathbf{i}) \mid \mathbf{v} \in SV(\mathbf{i})\}$ coincides with the set of minimal elements of $\{$nonzero path $\Gamma(\mathbf{i}) \to \Gamma(\mathbf{j}) \mid \mathbf{i}, \mathbf{j} \in OI(\Gamma)\}$ with respect to the partial order $\leq$ from Section 5. This is actually the motivation behind Definitions 6.2.10 and 6.2.11.

We now define $\mathcal{F}(\Gamma)$ for Γ in general:

$$(6.2.16) \qquad \mathcal{F}(\Gamma) = \left(\bigoplus_{\mathbf{i} \in OI(\Gamma)} P(\Gamma(\mathbf{i}))[h(\mathbf{i})], \quad d_\Gamma = \sum_{\mathbf{i} \in OI(\Gamma)} \sum_{\mathbf{v}} d(\mathbf{i}, \mathbf{v}) \right),$$

where the second summation is taken over $\mathbf{v} \in SV(\mathbf{i})$ and $d(\mathbf{i}, \mathbf{v}) : P(\Gamma(\mathbf{i})) \to P(\Gamma(\mathbf{v}|\mathbf{i}))$ is given by right multiplication by $r(\mathbf{i}, \mathbf{v})$.

Remark 6.2.17.

1) If $\Gamma \in B_{n,e}$, then $OI(\Gamma) = \{\mathbf{0}\}, \Gamma(\mathbf{0}) = \Gamma$, and $h(\mathbf{0}) = 0$. Hence $\mathcal{F}(\Gamma) = (P(\Gamma), d_\Gamma = 0)$, which agrees with $\mathcal{F}(\Gamma)$ from Equation (6.2.1).

2) By the usual grading shift convention, $P(\Gamma(\mathbf{i}))[h(\mathbf{i})]$ is at degree $-h(\mathbf{i})$. Since $h(\mathbf{i})$ is nonnegative, the highest degree term of $\mathcal{F}(\Gamma)$ has degree 0.

We will write d for d_Γ when Γ is understood. By definition $d : \mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma)$ is a map of $R_{n,e}$-modules. It remains to verify that $d^2 = 0$ and d is homogeneous of degree 1; they are proved in Lemmas 6.2.20 and 6.4.1.3.

Interpretation in terms of negative regions. We now give a slightly more unified way of describing $d = d_\Gamma$ in terms of the negative region $R_-(\Gamma)$. Let $\mathbf{i} \in OI(\Gamma)$ and let $\mathfrak{c}$ be a component of $R_-(\Gamma)$ such that:

(*) if the component $\Gamma_{\mathbf{v}}$ of $R_+(\Gamma)$ has (nonempty) boundary $\gamma_{\mathbf{v}}$ in common with $\mathfrak{c}$, then $\mathbf{v} \neq *$.

Then we say $\mathbf{i}$ is $\mathfrak{c}$-*admissible* if

(**) the omitting label of $\mathbf{v}$ satisfying (*) is the label of the interval of ∂D^2 which is adjacent to the initial point of $\gamma_{\mathbf{v}}$. (Recall that Γ is oriented as the boundary of $R_+(\Gamma)$.)

If $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathfrak{c}|\mathbf{i}$ is obtained from $\mathbf{i}$ by replacing the label of each $\mathbf{v}$ satisfying (*) by the label which is adjacent to the terminal point of $\gamma_{\mathbf{v}}$; see Figure 18. Observe that if $\mathbf{i} \in OI(\Gamma)$ and $\mathbf{v} \in SV(\mathbf{i})$, then there is a unique $\mathfrak{c}$ such that $\mathbf{i}$ is $\mathfrak{c}$-admissible and $\mathbf{v}|\mathbf{i} = \mathfrak{c}|\mathbf{i}$. In such a case we write $r(\mathbf{i}, \mathfrak{c}) = r(\mathbf{i}, \mathbf{v})$ and $d(\mathbf{i}, \mathfrak{c}) = d(\mathbf{i}, \mathbf{v})$. Hence $d(\mathbf{i}, \mathfrak{c})$ can be viewed as a refinement of $d(\mathbf{i}, \mathbf{v})$.
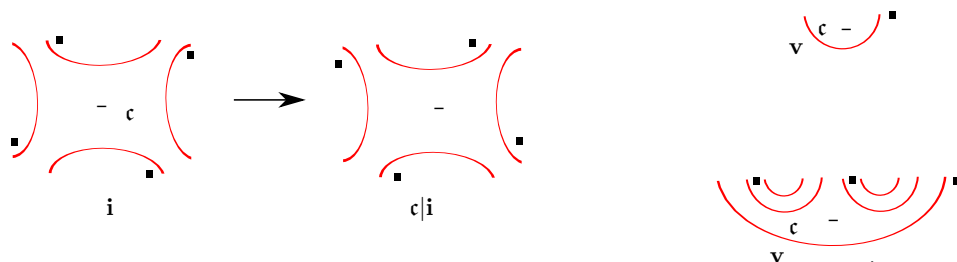

Figure 18: The picture on the left describes $d(\mathbf{i}, \mathfrak{c})$, where dividing sets are $\gamma_{\mathbf{v}}$, and black boxes are locations of omitting labels for $\mathbf{i}$ and $\mathfrak{c}|\mathbf{i}$. The pictures on the right are $\mathfrak{c}$-admissible omitting indices $\mathbf{i}$ such that $\mathbf{v}|\mathbf{i} = \mathfrak{c}|\mathbf{i}$, where $\mathbf{v} \in SLV(\mathbf{i})$ and $SHV(\mathbf{i})$, respectively.

Given $\mathfrak{c} \in \pi_0(R_-(\Gamma))$, let us define

$$(6.2.18) \qquad\qquad d_{\mathfrak{c}} = \sum_{\mathbf{i}} d(\mathbf{i}, \mathfrak{c}),$$

where the summation in the first equation is taken over $\mathbf{i} \in OI(\Gamma)$ such that $\mathbf{i}$ is $\mathfrak{c}$-admissible. We immediately see that $d = \sum_{\mathfrak{c} \in \pi_0(R_-(\Gamma))} d_{\mathfrak{c}}$. Let us also write

$$(6.2.19) \qquad\qquad d_{\mathbf{v}} = \sum_{\mathfrak{c}} d_{\mathfrak{c}},$$

where $\mathfrak{c}$ has a boundary component in common with $\mathbf{v}$ and $\mathbf{v}$ is closer to the based component.

Lemma 6.2.20.

1) *If $\mathfrak{c} \in \pi_0(R_-(\Gamma))$, then $d_{\mathfrak{c}}^2 = 0$.*

2) *If* $\mathfrak{c}, \mathfrak{c}' \in \pi_0(R_-(\Gamma))$ *and* $\mathfrak{c} \neq \mathfrak{c}'$, *then* $d_{\mathfrak{c}} d_{\mathfrak{c}'} = d_{\mathfrak{c}} d_{\mathfrak{c}'}$.

3) $d^2 = 0$.

Proof. (1) This is immediate from observing that $\mathfrak{c}|\mathbf{i}$ is not $\mathfrak{c}$-admissible.

(2) Suppose $\mathfrak{c}$ and $\mathfrak{c}'$ are adjacent, i.e., there is a component of $R_+(\Gamma)$ labeled by $\mathbf{v}$ which has boundary in common with both $\mathfrak{c}$ and $\mathfrak{c}'$. If $\mathbf{i}$ is not $\mathfrak{c}$- or $\mathfrak{c}'$-admissible, then $d(\mathbf{i}, \mathfrak{c}) = d(\mathbf{i}, \mathfrak{c}') = 0$. If $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathbf{i}$ cannot be $\mathfrak{c}'$-admissible and $d(\mathbf{i}, \mathfrak{c}') = 0$. If $\mathfrak{c}|\mathbf{i}$ is $\mathfrak{c}'$-admissible, then there are components of $\partial D^2 \cap \mathfrak{c}$ and $\partial D^2 \cap \mathfrak{c}'$ that are adjacent to a label of $\mathbf{v}$; however, $d(\mathfrak{c}|\mathbf{i}, \mathfrak{c}') \circ d(\mathbf{i}, \mathfrak{c}) = 0$ by Corollary 5.3. In any case $d_{\mathfrak{c}} d_{\mathfrak{c}'} = d_{\mathfrak{c}'} d_{\mathfrak{c}} = 0$.

Suppose $\mathfrak{c}$ and $\mathfrak{c}'$ are not adjacent. Then clearly $\mathfrak{c}|(\mathfrak{c}'|\mathbf{i}) = \mathfrak{c}'|(\mathfrak{c}|\mathbf{i})$ (if either side exists) and $d_{\mathfrak{c}} d_{\mathfrak{c}'} = d_{\mathfrak{c}'} d_{\mathfrak{c}}$.

(3) It follows from (1), (2), and $d = \sum_{\mathfrak{c}} d_{\mathfrak{c}}$. $\square$

Lemma 6.2.21. *If* $\mathbf{v} \in V(\Gamma)$ *and* $0 < i \leq l_{\Gamma_{\mathbf{v}}}$, *then the following holds as subsets of* $V_{nb}^+(\Gamma)$:

$$NV(\mathbf{v}, i) = NV(\mathbf{v}, i-1) \sqcup DNV(\mathbf{v}, i) \sqcup \bigsqcup_{\mathbf{w} \in DNV(\mathbf{v}, i)} NV(\mathbf{w}, l_{\Gamma_{\mathbf{w}}}).$$

Proof. By Definition 6.2.6,

$$NV(\mathbf{v}, i) = NV(\mathbf{v}, i-1) \sqcup \{\mathbf{u} \in V_{nb}^+(\Gamma) \mid \Gamma_{\mathbf{u}} \subset (\Gamma_{\mathbf{v}}(i-1), \Gamma_{\mathbf{v}}(i))\}.$$

For any $\mathbf{u}$ satisfying $\Gamma_{\mathbf{u}} \subset (\Gamma_{\mathbf{v}}(i-1), \Gamma_{\mathbf{v}}(i))$, either $\Gamma_{\mathbf{u}}$ directly nests inside $\Gamma_{\mathbf{v}}$, i.e., $\mathbf{u} \in DNV(\mathbf{v}, i)$; or $\Gamma_{\mathbf{u}}$ nests inside $\Gamma_{\mathbf{w}}$ for a unique $\mathbf{w} \in DNV(\mathbf{v}, i)$, i.e., $\mathbf{u} \in NV(\mathbf{w}, l_{\Gamma_{\mathbf{w}}})$. $\square$

Lemma 6.2.22. *The degree of d is* 1.

Proof. Since the term $P(\Gamma(\mathbf{i}))[h(\mathbf{i})]$ is at cohomological degree $-h(\mathbf{i})$, it suffices to show that $h(\mathbf{v}|\mathbf{i}) = h(\mathbf{i}) - 1$ for $\mathbf{v} \in SV(\mathbf{i})$.

If $\mathbf{v} \in SLV(\mathbf{i})$, then $\mathbf{i} \mapsto \mathbf{v}|\mathbf{i}$ leaves all the entries of $\mathbf{i}$ unchanged except for the $\mathbf{v}$-entry. In particular, $h(\mathbf{v}|\mathbf{i}, \mathbf{w}) = h(\mathbf{i}, \mathbf{w})$ for $\mathbf{w} \neq \mathbf{v}$. Since $DNV(\mathbf{v}, \mathbf{i}_{\mathbf{v}}) = \emptyset$, $NV(\mathbf{v}, \mathbf{i}_{\mathbf{v}} - 1) = NV(\mathbf{v}, \mathbf{i}_{\mathbf{v}})$ by Lemma 6.2.21. Hence $c_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}} - 1) = c_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}})$ and $h(\mathbf{v}|\mathbf{i}, \mathbf{v}) = h(\mathbf{i}, \mathbf{v}) - 1$. This implies that $h(\mathbf{v}|\mathbf{i}) = h(\mathbf{i}) - 1$.

If $\mathbf{v} \in SHV(\mathbf{i})$, then the entries of $\mathbf{i}$ that are unchanged by $\mathbf{i} \mapsto \mathbf{v}|\mathbf{i}$ are those of $\mathbf{w} \notin \{\mathbf{v}\} \cup DNV(\mathbf{v}, \mathbf{i}_{\mathbf{v}})$. Hence $h(\mathbf{v}|\mathbf{i}, \mathbf{w}) = h(\mathbf{i}, \mathbf{w})$ for $\mathbf{w} \notin \{\mathbf{v}\} \cup$

$DNV(\mathbf{v}, \mathbf{i_v})$. It remains to show that

$$\left(\sum_{\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})} h(\mathbf{v}|\mathbf{i}, \mathbf{w}) \right) + h(\mathbf{v}|\mathbf{i}, \mathbf{v}) = \left(\sum_{\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})} h(\mathbf{i}, \mathbf{w}) \right) + h(\mathbf{i}, \mathbf{v}) - 1.$$

By Definitions 6.2.7 and 6.2.8 this can be rewritten as

$$\sum_{\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})} (c_{\mathbf{w}}(l_{\Gamma_{\mathbf{w}}}) + l_{\Gamma_{\mathbf{w}}}) + c_{\mathbf{v}}(\mathbf{i_v} - 1) + \mathbf{i_v} - 1$$

$$= \sum_{\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})} (c_{\mathbf{w}}(0) + 0) + c_{\mathbf{v}}(\mathbf{i_v}) + \mathbf{i_v} - 1,$$

$$\sum_{\mathbf{w} \in DNV(\mathbf{v}, \mathbf{i_v})} (c_{\mathbf{w}}(l_{\Gamma_{\mathbf{w}}}) + l_{\Gamma_{\mathbf{w}}}) + c_{\mathbf{v}}(\mathbf{i_v} - 1) = c_{\mathbf{v}}(\mathbf{i_v}).$$

The last equation follows from Lemma 6.2.21. □

6.3. Definition of $\mathcal{F}(\beta)$

In this subsection we define $\mathcal{F}(\beta) \in \mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$ for any nontrivial bypass $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$. Recall from Notation 4.2.1 and Notation 4.2.2 (see also Figure 12) that a bypass β is described by two vectors $\underline{\mathbf{b}}, \overline{\mathbf{b}} \in V(\Gamma)$, three integers x, y, z, and a partition $\Gamma_{\underline{\mathbf{b}}} = \Gamma_{\underline{\mathbf{b}}}^l \sqcup \Gamma_{\underline{\mathbf{b}}}^r$.

6.3.1. Identity and shuffling indices. Given $\mathbf{i} \in OI(\Gamma)$ and $\mathbf{j} \in OI(\Gamma')$, the restriction of $\mathcal{F}(\beta)$ to $P(\Gamma(\mathbf{i})) \to P(\Gamma'(\mathbf{j}))$ will be the zero map or one of two types:

(Id) the identity map $P(\Gamma(\mathbf{i})) \to P(\Gamma'(\mathbf{j}))$ for a unique $\mathbf{j} \in OI(\Gamma')$;

(Sh) a nonzero map $P(\Gamma(\mathbf{i})) \to P(\Gamma'(\mathbf{j}))$ for a unique $\mathbf{j} \in OI(\Gamma')$.

For each $\mathbf{i} \in OI(\Gamma)$, the unique $\mathbf{j} \in OI(\Gamma')$, if it exists, satisfies the condition that the path from $\Gamma(\mathbf{i})$ to $\Gamma'(\mathbf{j})$ (possibly the identity path) is the shortest nonzero path in $Q_{n,e}$ starting from $\Gamma(\mathbf{i})$. The main distinction between Types (Id) and (Sh) is whether there exists $\mathbf{j} \in OI(\Gamma')$ such that $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$. In each type there are two subcases $\underline{\mathbf{b}} = *$ or $\underline{\mathbf{b}} \neq *$; see Figures 19 and 20.

Type (Id).

Definition 6.3.1.1 (Identity index). An omitting index $\mathbf{i} \in OI(\Gamma)$ is called an *identity index* of β if either

(a) $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^{l}$; or

(b) $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}$ and $\mathbf{i}_{\underline{\mathbf{b}}} \in [[x,y]]$ (i.e., $\Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) \in \Gamma_{\underline{\mathbf{b}}}^{l}$).

Let $II(\beta) \subset OI(\Gamma)$ denote the set of identity indices of β.

If $\mathbf{i} \in II(\beta)$, then there exists a unique $\mathbf{j} \in OI(\Gamma')$ such that $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$.
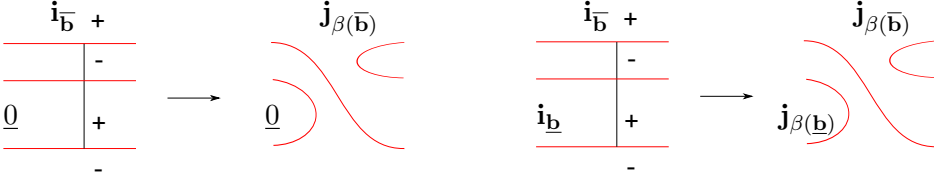

Figure 19: Type (Id). The case $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^{l}$ is on the left and the case $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}$ and $\mathbf{i}_{\underline{\mathbf{b}}} \in [[x,y]]$ is on the right. Here $\mathbf{i}_{\mathbf{v}}, \mathbf{j}_{\mathbf{w}}$ denote entries of $\mathbf{i} \in OI(\Gamma), \mathbf{j} \in OI(\Gamma')$ and $\underline{0}$ is the label on ∂D^2.

Type (Sh). We first require that $\mathbf{i} \in OI(\Gamma)$ satisfies: (1) either $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^{r}$; or $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}$ and $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x,y]]$; and (2A) $\overline{\mathbf{b}} \in V^+(\Gamma)$. Additional conditions are more involved and will be motivated in the following several paragraphs; the full description will then be given in Definition 6.3.1.5.

We first observe that if $\mathbf{i} \in OI(\Gamma)$ satisfies (1), then there is no $\mathbf{j} \in OI(\Gamma')$ such that $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$: this is because there is no $\Gamma'_{\beta(\underline{\mathbf{b}})}(\mathbf{j}_{\beta(\underline{\mathbf{b}})})$ if $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$.
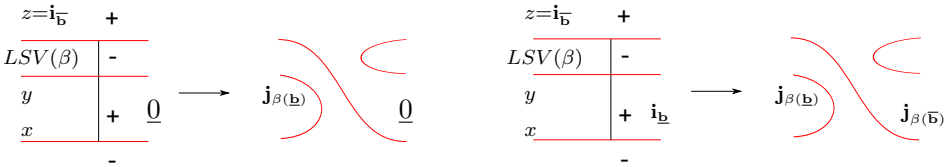

Figure 20: Type (Sh). The case $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^{r}$ is on the left and the case $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}$ and $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x,y]]$ is on the right. In both cases $\overline{\mathbf{b}} \neq *$. See Definition 6.3.1.2 for $LSV(\beta)$.

If $\mathbf{i} \in OI(\Gamma)$ satisfies (1) and (2A), the most efficient way to move omitting labels of $\mathbf{i}$ to omitting labels of $\mathbf{j}$ is to send $\Gamma_{\overline{\mathbf{b}}}(z)$ to $\Gamma_{\underline{\mathbf{b}}}(y)$ and leave the other labels intact. In particular, this means that $\mathbf{i}_{\overline{\mathbf{b}}} = z$ and $\Gamma'_{\beta(\mathbf{b})}(\mathbf{j}_{\beta(\mathbf{b})}) = \Gamma_{\underline{\mathbf{b}}}(y)$; at the same time the omitting labels in $\Gamma_{\mathbf{w}}$ may be moved for $\Gamma_{\mathbf{w}}$ lying between $\Gamma_{\underline{\mathbf{b}}}(y)$ and $\Gamma_{\overline{\mathbf{b}}}(z)$.

Definition 6.3.1.2. A vector $\mathbf{w} \in V^+(\Gamma)$ is called a *left shuffling vector* of β if $l_{\Gamma_{\mathbf{w}}} > 0$, the component $\Gamma_{\mathbf{w}}$ is adjacent to $\Gamma_{\underline{\mathbf{b}}}$ and $\Gamma_{\overline{\mathbf{b}}}$, and $\Gamma_{\mathbf{w}} \subset [[\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\overline{\mathbf{b}}}(z)]]$. Let $LSV(\beta)$ denote the set of left shuffling vectors of β.

The condition $\Gamma_{\mathbf{w}} \subset [[\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\overline{\mathbf{b}}}(z)]]$ implies that the component $\Gamma_{\mathbf{w}}$ is to the left to the arc of attachment δ, assuming that δ is positioned as in Figure 12.

Definition 6.3.1.3 (Shuffling types).

(Y) A bypass $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ is of *shuffling type* (Y) if $\Gamma_{\underline{\mathbf{b}}}(y) < \Gamma_{\overline{\mathbf{b}}}(z)$.

(Z) A bypass $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ is of *shuffling type* (Z) if $\Gamma_{\underline{\mathbf{b}}}(y) > \Gamma_{\overline{\mathbf{b}}}(z)$ and there exist $\mathbf{w}(\beta) \in LSV(\beta)$ and $0 < k(\beta) \leq l_{\Gamma_{\mathbf{w}(\beta)}}$ such that $\Gamma_{\underline{\mathbf{b}}} \cup \Gamma_{\overline{\mathbf{b}}} \subset (\Gamma_{\mathbf{w}(\beta)}(k(\beta) - 1), \Gamma_{\mathbf{w}(\beta)}(k(\beta)))$.

For type (Y), neither $\Gamma_{\underline{\mathbf{b}}}$ nor $\Gamma_{\overline{\mathbf{b}}}$ directly nests inside any $\mathbf{w} \in LSV(\beta)$. For type (Z), $\Gamma_{\underline{\mathbf{b}}}$ and $\Gamma_{\overline{\mathbf{b}}}$ directly nest inside a unique $\mathbf{w}(\beta) \in LSV(\beta)$.

Remark 6.3.1.4.

1) The two shuffling types (Y) and (Z) are mutually exclusive but *some bypasses do not belong to either type* when the conditions $\mathbf{w} \in V^+(\Gamma)$ and $l_{\Gamma_{\mathbf{w}}} > 0$ in the definition of a left shuffling vector are not met. This happens when Γ_* is adjacent to $\Gamma_{\underline{\mathbf{b}}}$ and $\Gamma_{\overline{\mathbf{b}}}$ and $\Gamma_* \subset [[\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\overline{\mathbf{b}}}(z)]]$.

2) If β is of shuffling type (Z), then the pair $(\mathbf{w}(\beta), k(\beta))$ is unique. We use $(\mathbf{w}(\beta), k(\beta))$ to denote this pair.

Definition 6.3.1.5 (Shuffling index). An omitting index $\mathbf{i} \in OI(\Gamma)$ is a *shuffling index* of β if the following conditions hold:

1) either
   a) $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^r$; or
   b) $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}$ and $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x, y]]$ (i.e., $\Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) \in \Gamma_{\underline{\mathbf{b}}}^r$);

2) $\overline{\mathbf{b}} \in V^+(\Gamma)$ and $\mathbf{i}_{\overline{\mathbf{b}}} = z$;

3) $\mathbf{i}_{\mathbf{w}} = 0$ for all $\mathbf{w} \in LSV(\beta)$ if β is of shuffling type (Y);

4) $\mathbf{i}_{\mathbf{w}(\beta)} = k(\beta)$ and $\mathbf{i}_{\mathbf{w}} = 0$ for all $\mathbf{w} \in LSV(\beta)\backslash\{\mathbf{w}(\beta)\}$ if β is of shuffling type (Z).

Let $SI(\beta) \subset OI(\Gamma)$ denote the set of shuffling indices of β.
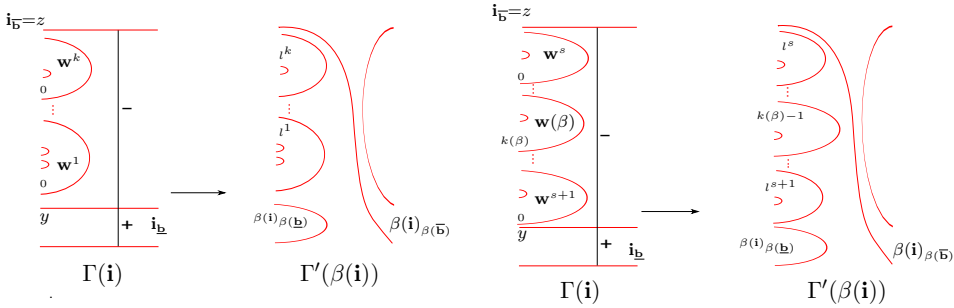
Figure 21: Examples of $SI(\beta)$, where $\beta(\mathbf{i})$ is defined in Definition 6.3.2.2. A bypass of shuffling type (Y), $\Gamma_{\underline{\mathbf{b}}}(y) < \Gamma_{\overline{\mathbf{b}}}(z)$, is on the left, where $LSV(\beta) = \{\mathbf{w}^1, \ldots, \mathbf{w}^k\}$ and $l^t = l_{\Gamma_{\mathbf{w}^t}}$. A bypass of shuffling type (Z), $\Gamma_{\underline{\mathbf{b}}}(y) > \Gamma_{\overline{\mathbf{b}}}(z)$, with the pair $(\mathbf{w}(\beta), k(\beta))$ is on the right, where $LSV(\beta) = \{\mathbf{w}(\beta), \mathbf{w}^1, \ldots, \mathbf{w}^s, \ldots, \mathbf{w}^{k-1}\}$ and $l^t = l_{\Gamma_{\mathbf{w}^t}}$. The entries $\mathbf{i}_\mathbf{v}$ and $\beta(\mathbf{i})_{\beta(\mathbf{v})}$ are drawn for $\mathbf{v} \in LSV(\beta) \sqcup \{\underline{\mathbf{b}}, \overline{\mathbf{b}}\}$ for both types of shuffling.

In the special case where $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}$ or $\Gamma_{\overline{\mathbf{b}}}$, the following descriptions of $II(\beta)$ and $SI(\beta)$ are straightforward.

Lemma 6.3.1.6.

1) If $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^l$ then $II(\beta) = OI(\Gamma)$ and $SI(\beta) = \emptyset$.

2) If $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^r$, then $II(\beta) = \emptyset$.

3) If $\underline{0} \in \Gamma_{\overline{\mathbf{b}}}$, then $SI(\beta) = \emptyset$.

By Definitions 6.3.1.1 and 6.3.1.5(1), $II(\beta) \cap SI(\beta) = \emptyset$. The disjoint union $II(\beta) \sqcup SI(\beta)$ is always nonempty, but is not equal to $OI(\Gamma)$ in general.

6.3.2. Definition of the chain map. The bypass β changes omitting indices in $II(\beta) \sqcup SI(\beta) \subset OI(\Gamma)$ to those in $OI(\Gamma')$.

Notation 6.3.2.1. We abuse notation and use β to denote three related things:

1) a bypass;

2) the map $V(\Gamma) \to V(\Gamma')$ from Equation (4.2.3); and

3) the map $II(\beta) \sqcup SI(\beta) \to OI(\Gamma')$, defined below.

Definition 6.3.2.2. For $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$, define $\beta : II(\beta) \sqcup SI(\beta) \to OI(\Gamma')$ by:

(C1). If $\mathbf{i} \in II(\beta)$, then define $\beta(\mathbf{i}) \in OI(\Gamma')$ by $\beta(\mathbf{i})_{\beta(\mathbf{v})} \in \mathbb{Z}_+$ for $\beta(\mathbf{v}) \neq *$ such that

$$\Gamma'_{\beta(\mathbf{v})}(\beta(\mathbf{i})_{\beta(\mathbf{v})}) = \Gamma_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}}).$$

(C2). If $\mathbf{i} \in SI(\beta)$ and β is of shuffling type (Y), then define $\beta(\mathbf{i}) \in OI(\Gamma')$ by $\beta(\mathbf{i})_{\beta(\mathbf{v})} \in \mathbb{Z}_+$ for $\beta(\mathbf{v}) \neq *$ such that

$$\Gamma'_{\beta(\mathbf{v})}(\beta(\mathbf{i})_{\beta(\mathbf{v})}) = \begin{cases} \Gamma_{\underline{\mathbf{b}}}(y) & \text{if } \mathbf{v} = \underline{\mathbf{b}}, \\ \Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) & \text{if } \mathbf{v} = \overline{\mathbf{b}} \text{ and } \beta(\overline{\mathbf{b}}) \neq *, \\ \Gamma_{\mathbf{v}}(l_{\Gamma_{\mathbf{v}}}) & \text{if } \mathbf{v} \in LSV(\beta), \\ \Gamma_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}}) & \text{otherwise.} \end{cases}$$

(C3). If $\mathbf{i} \in SI(\beta)$ and β is of shuffling type (Z), then define $\beta(\mathbf{i}) \in OI(\Gamma')$ by $\beta(\mathbf{i})_{\beta(\mathbf{v})} \in \mathbb{Z}_+$ for $\beta(\mathbf{v}) \neq *$ such that

$$\Gamma'_{\beta(\mathbf{v})}(\beta(\mathbf{i})_{\beta(\mathbf{v})}) = \begin{cases} \Gamma_{\underline{\mathbf{b}}}(y) & \text{if } \mathbf{v} = \underline{\mathbf{b}}, \\ \Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) & \text{if } \mathbf{v} = \overline{\mathbf{b}} \text{ and } \beta(\overline{\mathbf{b}}) \neq *, \\ \Gamma_{\mathbf{w}(\beta)}(k(\beta) - 1) & \text{if } \mathbf{v} = \mathbf{w}(\beta), \\ \Gamma_{\mathbf{v}}(l_{\Gamma_{\mathbf{v}}}) & \text{if } \mathbf{v} \in LSV(\beta), \mathbf{v} \neq \mathbf{w}(\beta), \\ \Gamma_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}}) & \text{otherwise.} \end{cases}$$

See Figure 21 for examples. We verify that Definition 6.3.2.2 is well-defined, i.e., $\mathbf{i}_{\mathbf{v}}$ exists (equivalently $\mathbf{v} \neq *$) on the right-hand side of the equations. For (C1), $\underline{0} \notin \Gamma_{\underline{\mathbf{b}}}^r$ since $\mathbf{i} \in II(\beta)$. Hence $\beta(*) = *$ by Remark 4.2.4. Then $\beta(\mathbf{v}) \neq *$ implies that $\mathbf{v} \neq *$. For the second rows of (C2) and (C3), $0 \notin \Gamma_{\underline{\mathbf{b}}}^r$ since $\beta(\overline{\mathbf{b}}) \neq *$; and $0 \notin \Gamma_{\underline{\mathbf{b}}}^l$ since $\mathbf{i} \in SI(\beta)$. Hence $\underline{\mathbf{b}} \neq *$.

Remark 6.3.2.3.

(i) An effective way to understand β is to track the movement of the omitting labels from $\Gamma(\mathbf{i})$ to $\Gamma'(\beta(\mathbf{i}))$.

(ii) Since $\beta : V(\Gamma) \to V(\Gamma')$ only changes the two components $\Gamma_{\underline{\mathbf{b}}}, \Gamma_{\overline{\mathbf{b}}}$ by definition, we have

$$\beta(\mathbf{i})_{\beta(\mathbf{v})} = \mathbf{i}_{\mathbf{v}}$$

for $\mathbf{v} \notin \{\underline{\mathbf{b}}, \overline{\mathbf{b}}\}$ if $\mathbf{i} \in II(\beta)$, and for $\mathbf{v} \notin \{\underline{\mathbf{b}}, \overline{\mathbf{b}}\} \cup LSV(\beta)$ if $\mathbf{i} \in SI(\beta)$.

(iii) The map $\beta : II(\beta) \sqcup SI(\beta) \to OI(\Gamma')$ is injective.

Interpretation in terms of negative regions. For $\mathbf{i} \in SI(\beta)$ we give a description of $\beta(\mathbf{i})$ in terms of the component $\bar{\mathfrak{c}}$ of $R_-(\Gamma)$ that lies between $\underline{\mathbf{b}}$ and $\overline{\mathbf{b}}$. Let $\bar{\mathfrak{c}}^l$ be the left-hand side of $\bar{\mathfrak{c}}$ cut along the arc of attachment for β, i.e., the region that lies between $\Gamma^l_{\underline{\mathbf{b}}}$ and $\Gamma_{\overline{\mathbf{b}}}$ as in Figure 12. Let $\mathbf{i} \in OI(\Gamma)$. Then $\mathbf{i} \in SI(\beta)$ if and only if the following hold:

(N1) if $\Gamma_{\mathbf{v}} \neq \Gamma_{\underline{\mathbf{b}}}$ is a component of $R_+(\Gamma)$ which has boundary $\gamma_{\mathbf{v}}$ in common with $\bar{\mathfrak{c}}^l$, then $\mathbf{v} \neq *$;

(N2) the omitting label of any $\mathbf{v}$ satisfying (N1) is the label of the interval of ∂D^2 which is adjacent to the initial point of $\gamma_{\mathbf{v}}$;

(N3) neither $\underline{0}$ nor the omitting label of $\underline{\mathbf{b}}$ is on the left-hand side of $\underline{\mathbf{b}}$.

If $\mathbf{i} \in OI(\Gamma)$, then $\beta(\mathbf{i})$ is obtained from $\mathbf{i}$ by removing the label of each $\mathbf{v}$ satisfying (N1) and, for each $\mathbf{v} \neq \overline{\mathbf{b}}$ which has boundary $\gamma_{\mathbf{v}}$ in common with $\bar{\mathfrak{c}}^l$, adding the label which is adjacent to the terminal point of $\gamma_{\mathbf{v}}$.

A closer look at $SI(\beta)$. The conditions $\mathbf{i}_{\mathbf{w}} = 0$ in Definitions 6.2.10(2) and 6.3.1.5(3),(4) are similar, and there is a good reason for this. Let Γ'' be the third dividing set in the bypass triangle

$$\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma''$$

induced by β. For each $\mathbf{i} \in SI(\beta)$ there exists a unique $\mathbf{k} \in OI(\Gamma'')$ such that $\Gamma''(\mathbf{k}) = \Gamma(\mathbf{i})$; see Lemma 6.4.1.1. We will see in the proof of Lemma 6.4.1.3 that there exists $\mathbf{u} \in SV_{\Gamma''}(\mathbf{k})$ such that $\Gamma''(\mathbf{u}|\mathbf{k}) = \Gamma'(\beta(\mathbf{i}))$. As a result, the restriction $\mathcal{F}(\beta)|_{\Gamma(\mathbf{i})} : P(\Gamma(\mathbf{i})) \to P(\Gamma'(\beta(\mathbf{i})))$ coincides with part of the differential $d_{\Gamma''}(\mathbf{k}, \mathbf{u}) : P(\Gamma''(\mathbf{k})) \to P(\Gamma''(\mathbf{u}|\mathbf{k}))$. This is the key to proving that

$$\mathcal{F}(\Gamma) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(\Gamma') \xrightarrow{\mathcal{F}(\beta')} \mathcal{F}(\Gamma'')$$

is a distinguished triangle in Proposition 7.3.1; see Examples 6.3.3.1 and 6.3.3.2.

Lemma 6.3.2.4.

1) *If $\mathbf{i} \in II(\beta)$, then $\Gamma'(\beta(\mathbf{i})) = \Gamma(\mathbf{i})$.*

2) *If $\mathbf{i} \in SI(\beta)$, then $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma'(\beta(\mathbf{i}))) \neq 0$.*

Proof. (1) is immediate from Definition 6.3.2.2(C1).

(2) Suppose that β is of shuffling type (Y). Let $LSV(\beta) = \{\mathbf{w}^1, \ldots, \mathbf{w}^k\}$ such that

$$\Gamma_{\underline{\mathbf{b}}}(y) < \Gamma_{\mathbf{w}^1}(0); \quad \Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}}) < \Gamma_{\mathbf{w}^{j+1}}(0), \ 1 \leq j \leq k-1;$$
$$\Gamma_{\mathbf{w}^k}(l_{\Gamma_{\mathbf{w}^k}}) < \Gamma_{\overline{\mathbf{b}}}(z).$$

There are $k+1$ disjoint closed intervals:

(6.3.2.5) $[\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\mathbf{w}^1}(0)]; \quad [\Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}}), \Gamma_{\mathbf{w}^{j+1}}(0)], \ 1 \leq j \leq k-1;$
$\qquad\qquad [\Gamma_{\mathbf{w}^k}(l_{\Gamma_{\mathbf{w}^k}}), \Gamma_{\overline{\mathbf{b}}}(z)].$

The intersections $\Gamma(\mathbf{i})_* \cap [a, b] = \{a\}$, where $[a, b]$ is any of the $k+1$ intervals. We have

$$\Gamma'(\beta(\mathbf{i}))_* = \Gamma(\mathbf{i})_* \setminus \left(\{\Gamma_{\underline{\mathbf{b}}}(y)\} \cup \{\Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}})\}_{j=1}^{k} \right) \cup \left(\{\Gamma_{\mathbf{w}^j}(0)\}_{j=1}^{k} \cup \{\Gamma_{\overline{\mathbf{b}}}(z)\} \right)$$

and $\operatorname{Hom}(\Gamma(\mathbf{i}), \Gamma'(\beta(\mathbf{i}))) \neq 0$ by Proposition 5.1.

Suppose that β is of shuffling type (Z). Let $LSV(\beta) = \{\mathbf{w}(\beta), \mathbf{w}^1, \ldots, \mathbf{w}^s, \ldots, \mathbf{w}^{k-1}\}$ such that $\Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}}) < \Gamma_{\mathbf{w}^{j+1}}(0)$ for $1 \leq j \leq k-2$, $\Gamma_{\mathbf{w}^s}(l_{\Gamma_{\mathbf{w}^s}}) < \Gamma_{\overline{\mathbf{b}}}(z)$, and $\Gamma_{\underline{\mathbf{b}}}(y) < \Gamma_{\mathbf{w}^{s+1}}(0)$. Moreover, $\Gamma_{\mathbf{w}(\beta)}(k(\beta)-1) < \Gamma_{\mathbf{w}^1}(0)$, and $\Gamma_{\mathbf{w}^{k-1}}(l_{\Gamma_{\mathbf{w}^{k-1}}}) < \Gamma_{\mathbf{w}(\beta)}(k(\beta))$. Then there are $k+1$ disjoint closed intervals:

(6.3.2.6) $[\Gamma_{\mathbf{w}(\beta)}(k(\beta)-1), \Gamma_{\mathbf{w}^1}(0)]; \quad [\Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}}), \Gamma_{\mathbf{w}^{j+1}}(0)], \ 1 \leq j \leq s-1;$
$\qquad\quad [\Gamma_{\mathbf{w}^s}(l_{\Gamma_{\mathbf{w}^s}}), \Gamma_{\overline{\mathbf{b}}}(z)]; \quad [\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\mathbf{w}^{s+1}}(0)];$
$\qquad\quad [\Gamma_{\mathbf{w}^j}(l_{\Gamma_{\mathbf{w}^j}}), \Gamma_{\mathbf{w}^{j+1}}(0)], \ s+1 \leq j \leq k-2;$
$\qquad\quad [\Gamma_{\mathbf{w}^{k-1}}(l_{\Gamma_{\mathbf{w}^{k-1}}}), \Gamma_{\mathbf{w}(\beta)}(k(\beta))].$

Again $\operatorname{Hom}(\Gamma(\mathbf{i}), \Gamma'(\beta(\mathbf{i}))) \neq 0$ follows from Proposition 5.1. $\square$

Remark 6.3.2.7. With a little more work one can show that the nonzero path $\Gamma(\mathbf{i}) \to \Gamma'(\beta(\mathbf{i}))$ is the unique minimal element of $\{$nonzero path $\Gamma(\mathbf{i}) \to \Gamma'(\mathbf{j}) \mid \mathbf{i} \in OI(\Gamma), \mathbf{j} \in OI(\Gamma')\}$. This is actually the motivation behind Definitions 6.3.1.5 and 6.3.2.2.

The $k+1$ closed intervals in Equation (6.3.2.5) or (6.3.2.6) are called the *chain intervals* of β.

Let $t(\beta, \mathbf{i}) \in R_{n,e}$ denote the idempotent of $\Gamma(\mathbf{i})$ if $\mathbf{i} \in II(\beta)$, or the generator of $\operatorname{Hom}(\Gamma(\mathbf{i}), \Gamma'(\beta(\mathbf{i})))$ if $\mathbf{i} \in SI(\beta)$.

Definition 6.3.2.8. For a nontrivial bypass morphism $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$, define a map of $R_{n,e}$-modules $\mathcal{F}(\beta) : \mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma')$ by

$$\mathcal{F}(\beta) = \sum_{\mathbf{i} \in II(\beta) \sqcup SI(\beta)} \mathcal{F}(\beta, \mathbf{i}),$$

where $\mathcal{F}(\beta, \mathbf{i}) : P(\Gamma(\mathbf{i})) \to P(\Gamma'(\beta(\mathbf{i})))$ is the right multiplication by $t(\beta, \mathbf{i})$ for $\mathbf{i} \in II(\beta) \sqcup SI(\beta)$.

6.3.3. Some examples. Before proving that $\mathcal{F}(\beta)$ is a chain map, we look at some examples.

Example 6.3.3.1. Consider the bypass triangle $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma'' \xrightarrow{\beta''} \Gamma$ in Figure 22. By definition,

$$
\begin{array}{lcccc}
\mathcal{F}(\Gamma): & P(\underline{1},\underline{2}) \longrightarrow P(\underline{1},\underline{3}) \longrightarrow P(\underline{2},\underline{4}) \longrightarrow P(\underline{3},\underline{4}) \\
\mathcal{F}(\beta) \downarrow & & (A),\underline{\mathfrak{c}} \quad\quad (B),\overline{\mathfrak{c}} \\
\mathcal{F}(\Gamma'): & & P(\underline{1},\underline{2}) \longrightarrow P(\underline{1},\underline{4}) \longrightarrow P(\underline{2},\underline{4}) \\
\mathcal{F}(\beta') \downarrow & & (B),\overline{\mathfrak{c}} \quad (A),\underline{\mathfrak{c}} \\
\mathcal{F}(\Gamma''): & & P(\underline{1},\underline{3}) \longrightarrow P(\underline{1},\underline{4}) \longrightarrow P(\underline{3},\underline{4}) \\
\mathcal{F}(\beta'') \downarrow & & (A),\underline{\mathfrak{c}} \quad (B),\overline{\mathfrak{c}} \\
\mathcal{F}(\Gamma): & P(\underline{1},\underline{2}) \longrightarrow P(\underline{1},\underline{3}) \longrightarrow P(\underline{2},\underline{4}) \longrightarrow P(\underline{3},\underline{4})
\end{array}
$$

The labels (e.g., (A), $\underline{\mathfrak{c}}$) above correspond to the cases (e.g., Case (A), $\mathfrak{c} = \underline{\mathfrak{c}}$) in the proof of Lemma 6.3.4.2. All the squares are commutative by the commutativity relation of $R_{n,e}$. Therefore, $\mathcal{F}(\beta)$, $\mathcal{F}(\beta')$, and $\mathcal{F}(\beta'')$ are all chain maps and the sum of their degrees is 1. Moreover, $\mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma') \to \mathcal{F}(\Gamma'')$ is a distinguished triangle in $\mathcal{D}_{n,e}$ up to grading shift.

We now check the definition of $\mathcal{F}(\beta)$ in more detail. We have $\Gamma^l_{\underline{\mathbf{b}}} = \{\underline{3}\}, \Gamma^r_{\underline{\mathbf{b}}} = \{\underline{2}\}, \Gamma_{\overline{\mathbf{b}}} = \{\underline{1},\underline{4}\}$, and $x = y = z = 1$. By definition β is of shuffling type $(\overline{Y})$ since $\Gamma_{\underline{\mathbf{b}}}(1) < \Gamma_{\overline{\mathbf{b}}}(1)$.

1) For $P(\underline{1},\underline{2}), P(\underline{2},\underline{4}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \in II(\beta)$ since $\Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) = \underline{3} \in \Gamma^l_{\underline{\mathbf{b}}}$.

2) For $P(\underline{1},\underline{3}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \in SI(\beta)$ since $\Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}}) = \underline{2} \in \Gamma^r_{\underline{\mathbf{b}}}$, $\Gamma_{\overline{\mathbf{b}}}(\mathbf{i}_{\overline{\mathbf{b}}}) = \underline{4} = \Gamma_{\overline{\mathbf{b}}}(z)$ and $LSV(\beta) = \emptyset$. The restriction $\mathcal{F}(\beta)|_{P(\underline{1},\underline{3})} : P(\underline{1},\underline{3}) \to P(\underline{1},\underline{4})$ corresponds to $d_{\Gamma''}(\mathbf{k}, \mathbf{u})$, where $\Gamma''(\mathbf{k}) = \Gamma(\mathbf{i})$ and $\mathbf{u} = \beta''^{-1}(\overline{\mathbf{b}})$.
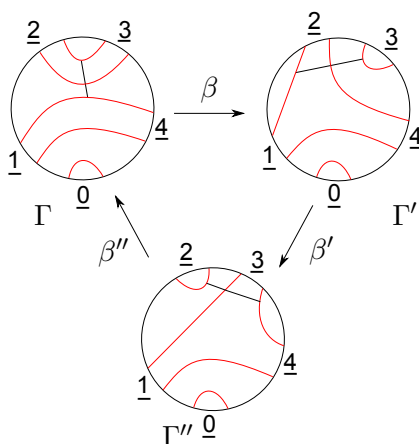
Figure 22.

3) For $P(\underline{3}, \underline{4}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \notin II(\beta)$ since $\Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\mathbf{b}}) = \underline{2} \in \Gamma^r_{\underline{\mathbf{b}}}$, and $\mathbf{i} \notin SI(\beta)$ since $\Gamma_{\overline{\mathbf{b}}}(\mathbf{i}_{\overline{\mathbf{b}}}) = \underline{1} \neq \Gamma_{\overline{\mathbf{b}}}(z)$. Hence $\mathbf{i} \notin II(\beta) \sqcup SI(\beta)$.

All the three bypasses in Example 6.3.3.1 are of shuffling type (Y). We consider a bypass of shuffling type (Z) in the next example.

Example 6.3.3.2. Consider the bypass triangle $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma'' \xrightarrow{\beta''} \Gamma$ in Figure 23. By definition,

$$
\begin{array}{llll}
\mathcal{F}(\Gamma): & P(\underline{1}, \underline{3}) \longrightarrow P(\underline{1}, \underline{4}) \longrightarrow P(\underline{3}, \underline{5}) \longrightarrow P(\underline{4}, \underline{5}) \\
\mathcal{F}(\beta)\downarrow & & & \\
\mathcal{F}(\Gamma'): & P(\underline{1}, \underline{2}) \longrightarrow P(\underline{1}, \underline{3}) \longrightarrow P(\underline{2}, \underline{5}) \longrightarrow P(\underline{3}, \underline{5}) \\
\mathcal{F}(\beta')\downarrow & & & \\
\mathcal{F}(\Gamma''): & P(\underline{1}, \underline{2}) \longrightarrow P(\underline{1}, \underline{4}) \longrightarrow P(\underline{2}, \underline{5}) \longrightarrow P(\underline{4}, \underline{5}) \\
\mathcal{F}(\beta'')\downarrow & & & \\
\mathcal{F}(\Gamma): & P(\underline{1}, \underline{3}) \longrightarrow P(\underline{1}, \underline{4}) \longrightarrow P(\underline{3}, \underline{5}) \longrightarrow P(\underline{4}, \underline{5})
\end{array}
$$

The maps $\mathcal{F}(\beta), \mathcal{F}(\beta'), \mathcal{F}(\beta'')$ are chain maps, the sum of their degrees is 1, and $\mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma') \to \mathcal{F}(\Gamma'')$ is a distinguished triangle in $\mathcal{D}_{n,e}$ up to grading shift.
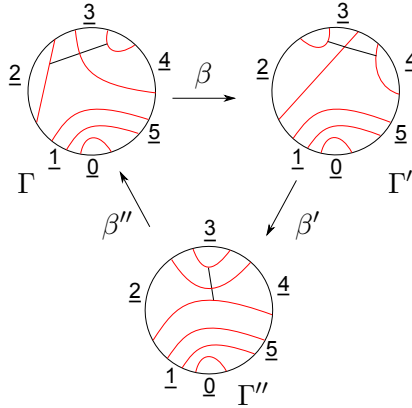
Figure 23.

For the bypass $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$, we have $\Gamma_{\mathbf{b}}^l = \{\underline{4}\}, \Gamma_{\mathbf{b}}^r = \{\underline{3}\}, \Gamma_{\overline{\mathbf{b}}} = \{\underline{2}\}$, and $x = y = 1, z = 0$. Also $\Gamma_{\mathbf{b}}, \Gamma_{\overline{\mathbf{b}}} \subset (\underline{1}, \underline{5}) = (\overline{\Gamma}_{\mathbf{w}(\beta)}(0), \overline{\Gamma}_{\mathbf{w}(\beta)}(1))$. Hence β is of shuffling type (Z), where $\Gamma_{\mathbf{w}(\beta)} = \{\underline{1}, \underline{5}\}$ and $k(\beta) = 1$.

1) For $P(\underline{1}, \underline{3}), P(\underline{3}, \underline{5}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \in II(\beta)$ since $\Gamma_{\mathbf{b}}(\mathbf{i_b}) = \underline{4} \in \Gamma_{\mathbf{b}}^l$.

2) For $P(\underline{1}, \underline{4}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \in SI(\beta)$ since $\Gamma_{\mathbf{b}}(\mathbf{i_b}) = \underline{3} \in \Gamma_{\mathbf{b}}^r$, $\Gamma_{\overline{\mathbf{b}}}(\mathbf{i_{\overline{b}}}) = \underline{2} = \Gamma_{\overline{\mathbf{b}}}(z)$, $\Gamma_{\mathbf{w}(\beta)}(\mathbf{i_{w(\beta)}}) = \underline{5} = \Gamma_{\mathbf{w}(\beta)}(k(\beta))$, and $LSV(\beta) \backslash \mathbf{w}(\beta) = \emptyset$. The restriction $\mathcal{F}(\beta)|_{P(\underline{1},\underline{4})} : P(\underline{1}, \underline{4}) \to P(\underline{2}, \underline{5})$ corresponds to $d_{\Gamma''}(\mathbf{k}, \mathbf{u})$, where $\Gamma''(\mathbf{k}) = \Gamma(\mathbf{i})$ and $\mathbf{u} = \beta''^{-1}(\mathbf{w}(\beta))$.

3) For $P(\underline{4}, \underline{5}) \in \mathcal{F}(\Gamma)$, the corresponding $\mathbf{i} \notin II(\beta)$ since $\Gamma_{\mathbf{b}}(\mathbf{i_b}) = \underline{3} \in \Gamma_{\mathbf{b}}^r$, and $\mathbf{i} \notin SI(\beta)$ since $\Gamma_{\mathbf{w}(\beta)}(\mathbf{i_{w(\beta)}}) = 1 \neq \Gamma_{\mathbf{w}(\beta)}(k(\beta))$. Hence $\mathbf{i} \notin II(\beta) \sqcup SI(\beta)$.

6.3.4. Proof that $\mathcal{F}(\beta)$ is a chain map.

Proposition 6.3.4.1. *If $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ is a nontrivial bypass morphism, then $d_{\Gamma'} \circ \mathcal{F}(\beta) = \mathcal{F}(\beta) \circ d_\Gamma$.*

Proof. Observe that $\beta : II(\beta) \sqcup SI(\beta) \to OI(\Gamma')$ is injective by Remark 6.3.2.3(iii) and any two paths with the same endpoints in $Q_{n,e}$ give the same element of $R_{n,e}$. Hence the proposition is a consequence of the following lemma. $\square$

Lemma 6.3.4.2. *Let $\mathbf{i} \in OI(\Gamma)$.*

1) *If* $\mathbf{i}$ *is* $\mathfrak{c}$-*admissible for* $\mathfrak{c} \in \pi_0(R_-(\Gamma))$, $\mathfrak{c}|\mathbf{i} \in II(\beta) \sqcup SI(\beta)$, *and the path* $\Gamma(\mathbf{i}) \to \Gamma(\mathfrak{c}|\mathbf{i}) \to \Gamma'(\beta(\mathfrak{c}|\mathbf{i}))$ *is nonzero, then* $\mathbf{i} \in II(\beta) \sqcup SI(\beta)$ *and there exists a unique* $\mathfrak{c}' \in \pi_0(R_-(\Gamma'))$ *such that* $\beta(\mathbf{i})$ *is* $\mathfrak{c}'$-*admissible and* $\mathfrak{c}'|\beta(\mathbf{i}) = \beta(\mathfrak{c}|\mathbf{i})$.

2) *If* $\mathbf{i} \in II(\beta) \sqcup SI(\beta)$, *there exists* $\mathfrak{c}' \in \pi_0(R_-(\Gamma'))$ *such that* $\beta(\mathbf{i})$ *is* $\mathfrak{c}'$-*admissible, and the path* $\Gamma(\mathbf{i}) \to \Gamma'(\beta(\mathbf{i})) \to \Gamma'(\mathfrak{c}'|\beta(\mathbf{i}))$ *is nonzero, then there exists* $\mathfrak{c} \in \pi_0(R_-(\Gamma))$ *such that* $\mathbf{i}$ *is* $\mathfrak{c}$-*admissible,* $\mathfrak{c}|\mathbf{i} \in II(\beta) \sqcup SI(\beta)$, *and* $\beta(\mathfrak{c}|\mathbf{i}) = \mathfrak{c}'|\beta(\mathbf{i})$.

Proof. (1) We assume that $\underline{\mathbf{b}}, \overline{\mathbf{b}} \neq *$. The proofs of other cases are similar. The main idea is to track the movements of the omitting labels under $\mathcal{F}(\beta)$ and the differentials. Note that the uniqueness of $\mathfrak{c}' \in \pi_0(R_-(\Gamma'))$ follows immediately from the existence by Remark 6.2.12(i).

Let $\overline{\mathfrak{c}}(\beta)$ be the component of $R_-(\beta)$ that lies between $\underline{\mathbf{b}}(\beta)$ and $\overline{\mathbf{b}}(\beta)$ and let $\underline{\mathfrak{c}}(\beta)$ be the component of $R_-(\Gamma)$ that lies directly below $\underline{\mathbf{b}}(\beta)$. If we omit β, it is understood that $\overline{\mathfrak{c}} = \overline{\mathfrak{c}}(\beta)$, etc. Let $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma''$ be the bypass triangle starting with β.

Suppose $\mathfrak{c}$ satisfies the assumptions of (1).

Case (A). Suppose that $\mathbf{i}_{\underline{\mathbf{b}}} \in [[x, y]]$, i.e., the label is on the left-hand side of $\underline{\mathbf{b}}$. Then $\mathbf{i} \in II(\beta)$ and $\Gamma'(\beta(\mathbf{i})) = \Gamma(\mathbf{i})$.

If $\mathfrak{c} \neq \underline{\mathfrak{c}}, \overline{\mathfrak{c}}$, then $(\mathfrak{c}|\mathbf{i})_{\underline{\mathbf{b}}} \in [[x, y]]$ and $\mathfrak{c}|\mathbf{i} \in II(\beta)$. We can take $\mathfrak{c}' = \mathfrak{c}$, viewed as an element of $\pi_0(R_-(\Gamma'))$, and it is immediate that $\mathfrak{c}'|\beta(\mathbf{i}) = \beta(\mathfrak{c}|\mathbf{i})$. See Figure 24 for two possible locations for $\mathfrak{c}$.

If $\mathfrak{c} = \underline{\mathfrak{c}}$ and $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathbf{i}_{\underline{\mathbf{b}}} = x$ and $(\mathfrak{c}|\mathbf{i})_{\underline{\mathbf{b}}} \notin [[x, y]]$. Since we are assuming that $\mathfrak{c}|\mathbf{i} \in SI(\beta)$, (N1)–(N3) from Section 4.2 must hold. If we take $\mathfrak{c}' = \overline{\mathfrak{c}}(\beta')$ (this is $\overline{\mathfrak{c}}$ with respect to β'), then $\beta(\mathbf{i})$ is $\mathfrak{c}'$-admissible and $\mathfrak{c}'|\beta(\mathbf{i}) = \beta(\mathfrak{c}|\mathbf{i})$.

If $\mathfrak{c} = \overline{\mathfrak{c}}$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible.

Case (B). Suppose that $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x, y]]$. Then $\mathbf{i} \in SI(\beta)$.

If $\mathfrak{c} = \underline{\mathfrak{c}}$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible.

If $\mathfrak{c} = \overline{\mathfrak{c}}$ and $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathbf{i}_{\overline{\mathbf{b}}} = z$ and $(\mathfrak{c}|\mathbf{i})_{\underline{\mathbf{b}}} = y \in [[x, y]]$. Hence $\mathfrak{c}|\mathbf{i} \in II(\beta)$. If we take $\mathfrak{c}' = \underline{\mathfrak{c}}(\beta')$, then $\beta(\mathbf{i})$ is $\mathfrak{c}'$-admissible and $\mathfrak{c}'|\beta(\mathbf{i}) = \beta(\mathfrak{c}|\mathbf{i})$; see Figure 25. In this case, we say that $\mathfrak{c} = \overline{\mathfrak{c}}$ lies below and shares a common boundary with $\overline{\mathbf{b}}$. This convention on the relative positions of the different regions (i.e., the positioning as in Figure 25) will be used for the rest of the proof.

If $\mathfrak{c} \neq \underline{\mathfrak{c}}, \overline{\mathfrak{c}}$, then $(\mathfrak{c}|\mathbf{i})_{\underline{\mathbf{b}}} \notin [[x, y]]$ and $\mathfrak{c}|\mathbf{i} \in SI(\beta)$. Suppose $\mathfrak{c}$ lies above and shares a common boundary with $\overline{\mathbf{b}}$, or $\mathfrak{c}$ is to the left of $\overline{\mathfrak{c}}$ and shares a
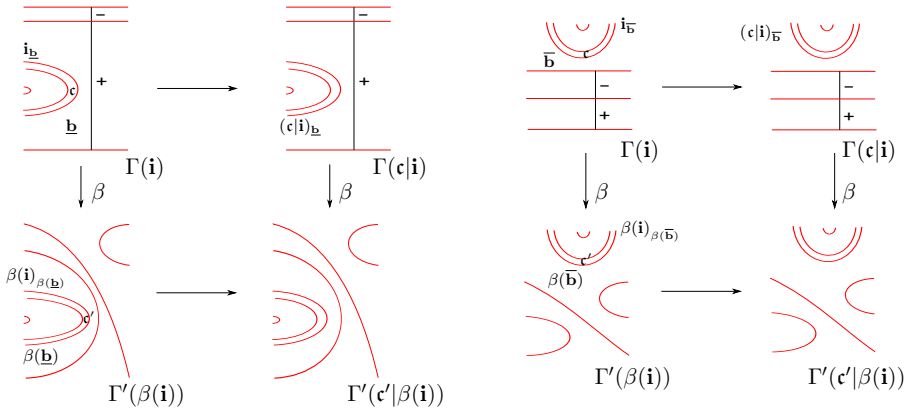
Figure 24: $\mathfrak{c}$ is to the left of $\underline{\mathbf{b}}$ in the left-hand figure and is above $\overline{\mathbf{b}}$ in the right-hand figure.

common boundary with some $\mathbf{w} \in LSV(\beta)$. Since $\mathbf{i}$ is $\mathfrak{c}$-admissible, we have $\mathbf{i} \notin SI(\beta)$ by (N1)–(N3), a contradiction; on the other hand, by Corollary 5.3, $\Gamma(\mathbf{i}) \to \Gamma(\mathfrak{c}|\mathbf{i}) \to \Gamma'(\beta(\mathfrak{c}|\mathbf{i}))$ is zero, which is consistent. If $\mathfrak{c}$ is to the left of and shares a common boundary with $\underline{\mathbf{b}}$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible since $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x, y]]$. In the remaining cases of $\mathbf{i}$-admissible $\mathfrak{c} \neq \underline{\mathfrak{c}}, \overline{\mathfrak{c}}$, we can take $\mathfrak{c}' = \mathfrak{c}$, viewed as an element of $\pi_0(R_-(\Gamma'))$, and it is immediate that $\mathfrak{c}'|\beta(\mathbf{i}) = \beta(\mathfrak{c}|\mathbf{i})$.
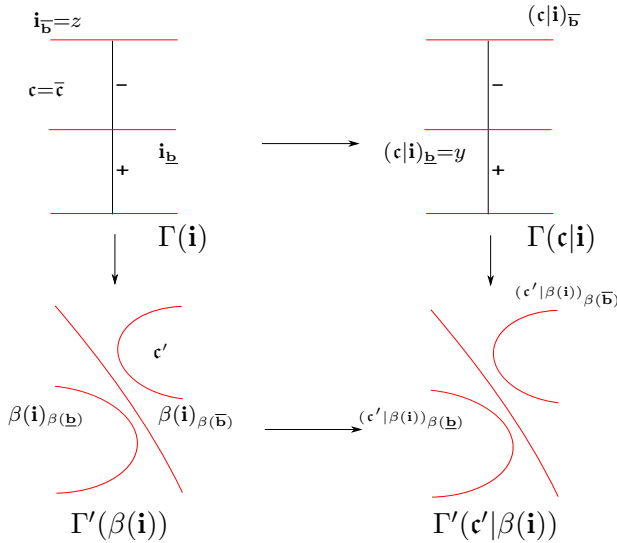

Figure 25.

(2) The proof is similar to that of (1) and is left to the reader. □

Remark 6.3.4.3. Observe that we set $\mathfrak{c}' = \mathfrak{c}$ except in Case (A) when $\mathfrak{c} = \underline{\mathfrak{c}}$ and Case (B) when $\mathfrak{c} = \bar{\mathfrak{c}}$; see the labels of the commuting squares in Example 6.3.3.1 for example.

We now complete the definition of $\mathcal{F} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$. If $\Gamma \xrightarrow{\xi} \Gamma'$ is a zero morphism, then $\mathcal{F}(\xi)$ is defined to be the zero morphism; in particular this is the case when Γ or Γ' is the zero object and $\mathcal{F}(\Gamma)$ or $\mathcal{F}(\Gamma') = 0$. Any nonzero morphism ξ in $\mathcal{C}_{n,e}$ can be written as a composition $\beta_k \circ \cdots \circ \beta_1$ of nontrivial bypass morphisms and we define $\mathcal{F}(\xi)$ as the composition $\mathcal{F}(\beta_k) \circ \cdots \circ \mathcal{F}(\beta_1)$. If ξ is an identity morphism (i.e., induced by a trivial bypass), then we set $\mathcal{F}(\xi) = \mathrm{id}$. In Section 6.4 we will show that $\mathcal{F}$ is well-defined.

6.4. Well-definition of the composition

In order to prove that $\mathcal{F} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ is well-defined it suffices to show the following:

1) $\mathcal{F}(\xi)$ is independent of the choice of decomposition of any nonzero morphism ξ into a sequence of nontrivial bypass morphisms.

2) If the composition of a sequence $\beta_1, \ldots, \beta_k$ of nontrivial bypasses is a zero morphism, then the composition $\mathcal{F}(\beta_k) \circ \cdots \circ \mathcal{F}(\beta_1) = 0$ in $\mathcal{D}_{n,e}$.

By Theorem 2.2.8.1, (1) can be reduced to the case where ξ is a composition of two disjoint bypasses. Here the bypasses may be trivial.

By Lemma 3.2.3, (2) can be reduced to the case where the composition is $\beta' \circ \beta$ for two consecutive (nonzero, non-identity) bypasses β, β' in any bypass triangle: Let $\beta' = \beta_k$ and $\xi = \beta_{k-1} \circ \cdots \circ \beta_1$. If ξ is a zero morphism, then we can replace $\beta_1, \ldots, \beta_k$ by $\beta_1, \ldots, \beta_{k-1}$. If ξ is nonzero, then Lemma 3.2.3 implies that ξ can be factored into $\beta \circ \zeta$, where β, β' are two consecutive bypasses of a bypass triangle.

6.4.1. Composition in bypass triangles. We fix the notation $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma'' \xrightarrow{\beta''} \Gamma$ for a bypass triangle throughout this section. We also use the notation $\underline{\mathbf{b}}(\beta)$ and $\underline{\mathfrak{c}}(\beta)$ to mean $\underline{\mathbf{b}}$ and $\underline{\mathfrak{c}}$ for β.

In Examples 6.3.3.1 and 6.3.3.2, each projective $R_{n,e}$-module in $\mathcal{F}(\Gamma)$ appears either in $\mathcal{F}(\Gamma')$ or in $\mathcal{F}(\Gamma'')$. This observation can be generalized as follows.

Lemma 6.4.1.1. *For any* $\mathbf{i} \in OI(\Gamma)$, *one of the following holds:*

1) *if* $\mathbf{i} \in II(\beta)$, *then* $\mathbf{j} := \beta(\mathbf{i})$ *is not in* $II(\beta')$ *and satisfies* $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$;

2) *if* $\mathbf{i} \notin II(\beta)$, *then there exists a unique* $\mathbf{k} \in II(\beta'')$ *such that* $\Gamma''(\mathbf{k}) = \Gamma(\mathbf{i})$.

Proof. First observe that $\Gamma_{\overline{\mathbf{b}}(\beta)} = \Gamma'^{r}_{\underline{\mathbf{b}}(\beta')} = \Gamma''^{l}_{\underline{\mathbf{b}}(\beta'')}$.

If $\mathbf{i} \in II(\beta)$, then $\mathbf{j} := \beta(\mathbf{i})$ satisfies $\Gamma'(\mathbf{j}) = \Gamma(\mathbf{i})$ by definition of $II(\beta)$. If $\overline{\mathbf{b}}(\beta) \neq *$, then

$$\Gamma'_{\underline{\mathbf{b}}(\beta')}(\mathbf{j}_{\underline{\mathbf{b}}(\beta')}) = \Gamma_{\overline{\mathbf{b}}(\beta)}(\mathbf{i}_{\overline{\mathbf{b}}(\beta)}) \in \Gamma_{\overline{\mathbf{b}}(\beta)} = \Gamma'^{r}_{\underline{\mathbf{b}}(\beta')},$$

and, if $\overline{\mathbf{b}}(\beta) = *$, then $\underline{0} \in \Gamma'^{r}_{\underline{\mathbf{b}}(\beta')}$. In both cases $\mathbf{j} \notin II(\beta')$.

If $\mathbf{i} \notin II(\beta)$, then each label which is omitted in $\Gamma(\mathbf{i})_*$ appears exactly once in $\Gamma''_{\mathbf{w}}$ for some $\mathbf{w} \neq *$. Hence there exists $\mathbf{k} \in OI(\Gamma'')$ such that $\Gamma''(\mathbf{k}) = \Gamma(\mathbf{i})$. If $\overline{\mathbf{b}}(\beta) \neq *$, then

$$\Gamma''_{\underline{\mathbf{b}}(\beta'')}(\mathbf{k}_{\underline{\mathbf{b}}(\beta'')}) = \Gamma_{\overline{\mathbf{b}}(\beta)}(\mathbf{i}_{\overline{\mathbf{b}}(\beta)}) \in \Gamma_{\overline{\mathbf{b}}(\beta)} = \Gamma''^{l}_{\underline{\mathbf{b}}(\beta'')},$$

and, if $\overline{\mathbf{b}}(\beta) = *$, then $\underline{0} \in \Gamma''^{l}_{\underline{\mathbf{b}}(\beta'')}$. In both cases $\mathbf{k} \in II(\beta'')$. $\square$

In view of Lemma 6.4.1.1(2), there exists a map

$$\gamma(\beta'') : OI(\Gamma) \backslash II(\beta) \to II(\beta'') \subset OI(\Gamma'')$$

such that $\Gamma''(\gamma(\beta'')(\mathbf{i})) = \Gamma(\mathbf{i})$ for $\mathbf{i} \notin II(\beta)$. Then define the map

(6.4.1.2) $$\mathcal{F}(\gamma(\beta'')) : \mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma'')$$

of $R_{n,e}$-modules as the direct sum of identity morphisms

$$P(\Gamma(\mathbf{i})) \to P(\Gamma''(\gamma(\beta'')(\mathbf{i}))) \quad \text{for } \mathbf{i} \notin II(\beta).$$

The following lemma implies that

$$\mathcal{F}(\beta') \circ \mathcal{F}(\beta) = 0 \in \mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'')),$$

i.e., in the (ungraded) homotopy category.

Lemma 6.4.1.3. $d\mathcal{F}(\gamma(\beta'')) = \mathcal{F}(\beta') \circ \mathcal{F}(\beta)$ *as* $R_{n,e}$-*module maps from* $\mathcal{F}(\Gamma)$ *to* $\mathcal{F}(\Gamma'')$.

Proof. We write γ for $\gamma(\beta'')$ during the proof. By definition $d\mathcal{F}(\gamma) = d_{\Gamma''} \circ \mathcal{F}(\gamma) + \mathcal{F}(\gamma) \circ d_{\Gamma}$. We assume that $\underline{\mathbf{b}}(\beta), \overline{\mathbf{b}}(\beta) \neq *$; the proofs of the other cases are easier.

We will show that

$$(6.4.1.4) \qquad (\mathcal{F}(\beta') \circ \mathcal{F}(\beta) + d_{\Gamma''} \circ \mathcal{F}(\gamma) + \mathcal{F}(\gamma) \circ d_{\Gamma})|_{P(\Gamma(\mathbf{i}))} = 0$$

for any $\mathbf{i} \in OI(\Gamma)$.

Case A. Suppose $\mathbf{i} \in II(\beta)$, i.e., $\mathbf{i}_{\underline{\mathbf{b}}} \in [[x(\beta), y(\beta)]]$. Then $\mathcal{F}(\gamma)|_{P(\Gamma(\mathbf{i}))} = 0$ and Equation (6.4.1.4) becomes $(\mathcal{F}(\beta') \circ \mathcal{F}(\beta) + \mathcal{F}(\gamma) \circ d_{\Gamma})|_{P(\Gamma(\mathbf{i}))} = 0$. Also $\beta(\mathbf{i}) \notin II(\beta')$ by Lemma 6.4.1.1(1).

We first consider $\mathcal{F}(\beta') \circ \mathcal{F}(\beta)|_{P(\Gamma(\mathbf{i}))}$. If $\mathcal{F}(\beta') \circ \mathcal{F}(\beta)|_{P(\Gamma(\mathbf{i}))} \neq 0$, then $\beta(\mathbf{i}) \in SI(\beta')$ since $\beta(\mathbf{i}) \notin II(\beta')$. This forces $\mathbf{i}_{\underline{\mathbf{b}}(\beta)} = x(\beta)$ and $\mathbf{i}$ to be $\underline{\mathfrak{c}}(\beta)$-admissible, and we have $\Gamma''(\beta'(\beta(\mathbf{i}))) = \Gamma(\underline{\mathfrak{c}}(\beta)|\mathbf{i})$. Note that, by Remark 6.3.1.4(1), $\underline{\mathfrak{c}}(\beta)$ cannot share a common boundary with $*$.

Recall that $d_{\Gamma} = \sum_{\mathfrak{c}} d_{\mathfrak{c}}$, where the summation is over $\pi_0(R_-(\Gamma))$. Hence $\mathcal{F}(\gamma) \circ d_{\Gamma}|_{P(\Gamma(\mathbf{i}))}$ is the sum of $\mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))}$, where $\mathbf{i}$ is $\mathfrak{c}$-admissible. If $\mathfrak{c} \neq \underline{\mathfrak{c}}(\beta), \overline{\mathfrak{c}}(\beta)$, then $\mathfrak{c}|\mathbf{i} \in II(\beta)$ and $\mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))} = 0$. If $\mathfrak{c} = \overline{\mathfrak{c}}(\beta)$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible. If $\mathfrak{c} = \underline{\mathfrak{c}}(\beta)$, then the $\mathfrak{c}$-admissibility of $\mathbf{i}$ implies that $\mathbf{i}_{\underline{\mathbf{b}}(\beta)} = x(\beta)$ and $\Gamma''(\beta'(\beta(\mathbf{i}))) = \Gamma(\mathfrak{c}|\mathbf{i})$. See Figure 26.
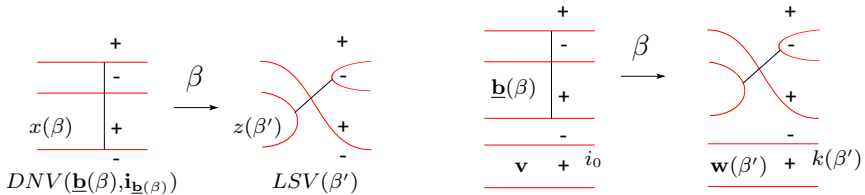

Figure 26: Two subcases of $\mathfrak{c} = \underline{\mathfrak{c}}(\beta)$: $\underline{\mathbf{b}}(\beta) = \mathbf{v}$ on the left and $\underline{\mathbf{b}}(\beta) \in DNV(\mathbf{v}, i_0)$ on the right.

Case B. Suppose $\mathbf{i} \notin II(\beta)$, i.e., $\mathbf{i}_{\underline{\mathbf{b}}} \notin [[x(\beta), y(\beta)]]$.

We first consider $\mathcal{F}(\beta') \circ \mathcal{F}(\beta)|_{P(\Gamma(\mathbf{i}))}$. Note that $\mathcal{F}(\beta') \circ \mathcal{F}(\beta)|_{P(\Gamma(\mathbf{i}))} \neq 0$ if and only if $\mathbf{i} \in SI(\beta)$, since $\beta(\mathbf{i}) \in II(\beta')$ is automatic.

Next let $\mathfrak{c} \in \pi_0(R_-(\Gamma))$. If $\mathfrak{c} \neq \underline{\mathfrak{c}}(\beta), \overline{\mathfrak{c}}(\beta)$, then there exists $\mathfrak{c}'' = \mathfrak{c}$, viewed as an element of $\pi_0(R_-(\Gamma''))$, such that $d_{\mathfrak{c}''} \circ \mathcal{F}(\gamma)|_{P(\Gamma(\mathbf{i}))} = \mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))}$. (This takes care of all $\mathfrak{c}'' \neq \underline{\mathfrak{c}}(\beta''), \overline{\mathfrak{c}}(\beta'')$.) If $\mathfrak{c} = \overline{\mathfrak{c}}(\beta)$ and $\mathbf{i}$ is not $\mathfrak{c}$-admissible, then $\mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))} = 0$. If $\mathfrak{c} = \overline{\mathfrak{c}}(\beta)$ and $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathfrak{c}|\mathbf{i} \in II(\beta)$ and $\mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))} = 0$. If $\mathfrak{c} = \underline{\mathfrak{c}}(\beta)$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible and $\mathcal{F}(\gamma) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))} = 0$. If $\mathfrak{c}'' = \overline{\mathfrak{c}}(\beta'')$, then $\gamma(\beta'')(\mathbf{i})$ is not $\mathfrak{c}''$-admissible and $d_{\mathfrak{c}''} \circ$

$\mathcal{F}(\gamma)|_{P(\Gamma(\mathbf{i}))} = 0$. Finally, if $\mathfrak{c}'' = \underline{\mathfrak{c}}(\beta'')$, then $d_{\mathfrak{c}''} \circ \mathcal{F}(\gamma)|_{P(\Gamma(\mathbf{i}))} \neq 0$ if and only if $\mathbf{i} \in SI(\beta)$. Moreover, when this happens,

$$\beta(\mathbf{i}) \in II(\beta') \quad \text{and} \quad d_{\mathfrak{c}''} \circ \mathcal{F}(\gamma)|_{P(\Gamma(\mathbf{i}))} = \mathcal{F}(\beta') \circ \mathcal{F}(\beta)|_{P(\Gamma(\mathbf{i}))}. \qquad \square$$

Lemma 6.4.1.3 will be the key to proving that $\mathcal{F}$ is exact, i.e., maps bypass triangles to distinguished triangles; see Proposition 7.3.1.

6.4.2. Disjoint pairs. For $s = 0, 1$, let $\beta^s \in \mathrm{Hom}(\Gamma, \Gamma^s)$ be a pair of bypass morphisms whose arcs of attachment are disjoint. Let $\widetilde{\Gamma}$ be the resulting dividing set after attaching both bypasses to Γ. We assume that the composition of the two bypass morphisms is nonzero; in particular we are assuming that $\widetilde{\Gamma}$ does not contain contractible components. Let $\widetilde{\beta}^s \in \mathrm{Hom}(\Gamma^{1-s}, \widetilde{\Gamma})$, $s = 0, 1$, be bypass morphisms such that $\widetilde{\beta}^1 \circ \beta^0 = \widetilde{\beta}^0 \circ \beta^1 \in \mathrm{Hom}(\Gamma, \widetilde{\Gamma})$.

The goal of this subsection is to prove:

Lemma 6.4.2.1. *Given a pair of disjoint bypasses β^s, $s = 0, 1$, on Γ, and $\widetilde{\Gamma}, \widetilde{\beta}^s$, $s = 0, 1$, as above, we have*

$$\mathcal{F}(\widetilde{\beta}^1) \circ \mathcal{F}(\beta^0) = \mathcal{F}(\widetilde{\beta}^0) \circ \mathcal{F}(\beta^1) \in \mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\widetilde{\Gamma})).$$

Since the reduction to disjoint pairs of bypasses at the beginning of Section 6.4 allows for any of $\beta^s, \widetilde{\beta}^s$ to be trivial, we first consider the situation where at least one of $\beta^s, \widetilde{\beta}^s$ is trivial. We can enumerate all the possible relative positions of β^0 and β^1, assuming β^0 is a fixed trivial bypass. The enumeration is left to the reader, but we almost always have β^0 and $\widetilde{\beta}^0$ trivial and $\beta^1 = \widetilde{\beta}^1$. The only (nontrivial, nonzero) exception is given in Figure 27, which is equivalent to a bypass rotation relation (R_2') right below Theorem 2.2.8.1.
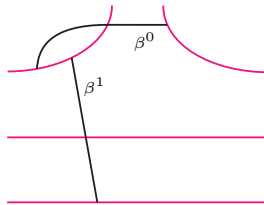

Figure 27.

Next we enumerate all the cases where all of β^s, $\widetilde{\beta}^s$ are nontrivial and nonzero. For any nontrivial morphism β, $\mathcal{F}(\beta)$ is determined by the maps

$\beta : II(\beta) \sqcup SI(\beta) \to OI(\Gamma)$. Let

$$BV(\beta) = \{\underline{\mathbf{b}}(\beta), \overline{\mathbf{b}}(\beta)\} \cup LSV(\beta).$$

Then $\beta(\mathbf{i})_{\beta(\mathbf{v})} = \mathbf{i_v}$ for $\mathbf{v} \notin BV(\beta)$ by Remark 6.3.2.3(ii). Our proof is based on a case-by-case analysis of the relative positions of $BV(\beta^0)$ and $BV(\beta^1)$. The following cases cover all the possibilities, after possibly switching β^0 and β^1:

1) $BV(\beta^0) \cap BV(\beta^1) = \emptyset$;

2) $\overline{\mathbf{b}}(\beta^0) = \underline{\mathbf{b}}(\beta^1)$;

3) $\overline{\mathbf{b}}(\beta^0) = \overline{\mathbf{b}}(\beta^1)$;

4) $\underline{\mathbf{b}}(\beta^0) = \underline{\mathbf{b}}(\beta^1)$;

5) $LSV(\beta^0) \cap BV(\beta^1) \neq \emptyset$.

See Figure 28 for a full list of Cases (2)-(5). Each red arc is assumed to be a distinct arc of the dividing set, except for Case (4) where it is only required that the two positive regions belong to the same component. The list for Case (5) does not include cases that were already listed (e.g., Case (2-1)). Note that the various cases may have overlaps.
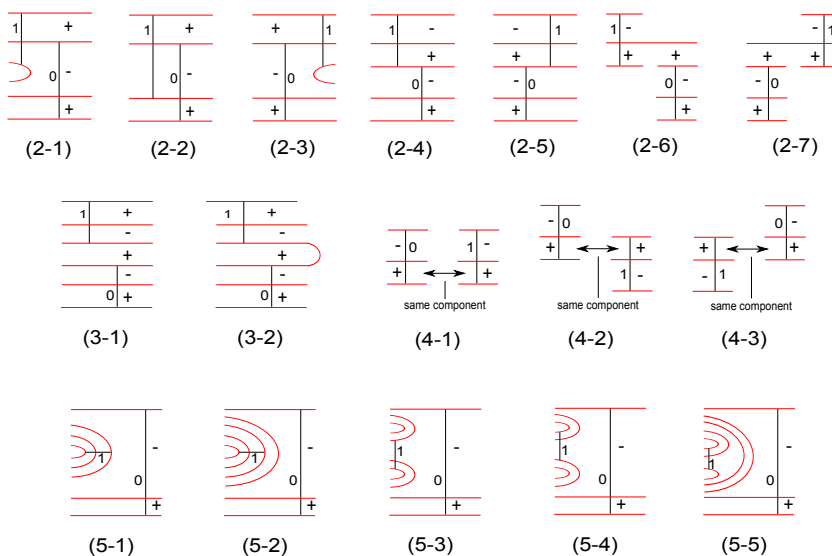

Figure 28: Black arcs with labels $s = 0, 1$ are the arcs where bypasses β^s are attached.

Before we discuss the general case, we look at an example in Case (2-1) where $\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) \neq \mathcal{F}(\widetilde{\beta^0}) \circ \mathcal{F}(\beta^1)$ as $R_{n,e}$-linear maps from $\mathcal{F}(\Gamma)$ to $\mathcal{F}(\widetilde{\Gamma})$. This illustrates the necessity of working in the homotopy category $\mathcal{D}_{n,e}$.

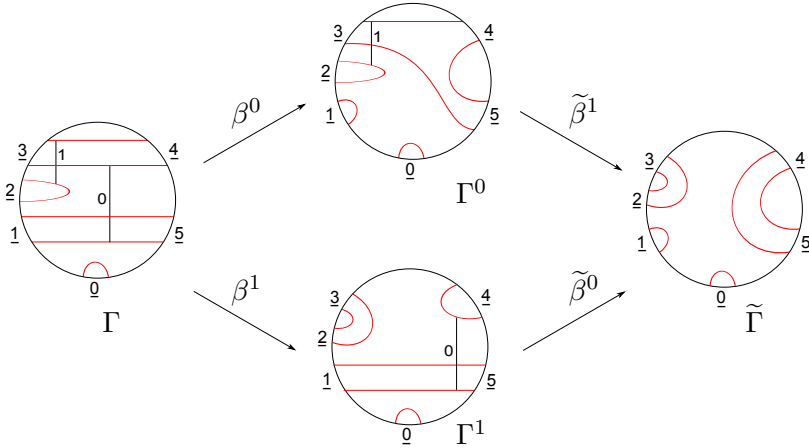**Example 6.4.2.2.** Consider the bypasses in Figure 29, where $\overline{\mathbf{b}}(\beta^0) = \underline{\mathbf{b}}(\beta^1)$.

Figure 29: A pair of disjoint bypasses in Case (2-1).

By definition, the two compositions are:

For $P(\underline{3}, \underline{5})$, the corresponding index $\mathbf{i}$ is in $II(\beta^0) \cap II(\beta^1)$. Hence both compositions are identity morphisms when restricted to $P(\underline{3}, \underline{5})$.

Let $\mathcal{F}(h)$ be the following map:

$$
\begin{array}{ccccccc}
\mathcal{F}(\Gamma): & P(\underline{1}, \underline{3}) \longrightarrow & P(\underline{1}, \underline{4}) & \longrightarrow & P(\underline{3}, \underline{5}) & \longrightarrow & P(\underline{4}, \underline{5}) \\
\mathcal{F}(h) \downarrow & & \downarrow & & & & \\
\mathcal{F}(\widetilde{\Gamma}): & & P(\underline{2}, \underline{4}) \longrightarrow & (P(\underline{3}, \underline{4}) \oplus P(\underline{2}, \underline{5})) & \longrightarrow & P(\underline{3}, \underline{5})
\end{array}
$$

We can easily verify that $\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) + \mathcal{F}(\widetilde{\beta^0}) \circ \mathcal{F}(\beta^1) = d\mathcal{F}(h)$ as maps. Hence $\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) = \mathcal{F}(\widetilde{\beta^0}) \circ \mathcal{F}(\beta^1) \in \mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\widetilde{\Gamma}))$.

Proof of Lemma 6.4.2.1. The following claims imply Lemma 6.4.2.1:

Claim A. *With the exception of Cases (2-1), (2-2), and (5-4), in Cases (1)–(5),*

$$\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) = \mathcal{F}(\widetilde{\beta^0}) \circ \mathcal{F}(\beta^1)$$

as maps from $\mathcal{F}(\Gamma)$ to $\mathcal{F}(\widetilde{\Gamma})$.

Claim B. *In Cases (2-1), (2-2), and (5-4), there exists $\mathcal{F}(h): \mathcal{F}(\Gamma) \to \mathcal{F}(\widetilde{\Gamma})$ such that*

(H) $\quad (\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) + \mathcal{F}(\widetilde{\beta^0}) \circ \mathcal{F}(\beta^1) + d\mathcal{F}(h))|_{P(\Gamma(\mathbf{i}))} = 0,$

for any $\mathbf{i} \in OI(\Gamma)$.

Claim C. *Referring to Figure 3, if β^0 and β^1 are bypass morphisms corresponding to δ_0 and δ_1, then*

$$\mathcal{F}(\widetilde{\beta^1}) \circ \mathcal{F}(\beta^0) = \mathcal{F}(\beta^1).$$

We only prove Claim A for Case (1) and Claim B for Case (2-2). The proofs of the other cases are similar and are left to the reader.

Case (1). Since $BV(\beta^0) \cap BV(\beta^1) = \emptyset$, the maps $\mathcal{F}(\beta^0)$ and $\mathcal{F}(\beta^1)$ do not affect each other. More precisely, for $\mathbf{i} \in XI(\beta^0) \cap YI(\beta^1)$ where $XI, YI \in \{II, SI\}$, we have $\beta^0(\mathbf{i}) \in YI(\widetilde{\beta^1}), \beta^1(\mathbf{i}) \in XI(\widetilde{\beta^0})$, and $\widetilde{\beta^1}(\beta^0(\mathbf{i})) = \widetilde{\beta^0}(\beta^1(\mathbf{i}))$.

For $\mathbf{i} \notin (II(\beta^0) \sqcup SI(\beta^0)) \cap (II(\beta^1) \sqcup SI(\beta^1))$,

$$\mathcal{F}(\widetilde{\beta}^1) \circ \mathcal{F}(\beta^0)|_{P(\Gamma(\mathbf{i}))} = \mathcal{F}(\widetilde{\beta}^0) \circ \mathcal{F}(\beta^1)|_{P(\Gamma(\mathbf{i}))} = 0.$$

Case (2-2). Suppose that $\underline{\mathbf{b}}(\beta^s) = \overline{\mathbf{b}}(\beta^{1-s})$. Assume that $\underline{\mathbf{b}}, \overline{\mathbf{b}} \neq *$ for simplicity. Let $\underline{\mathbf{b}}^s$, $\overline{\mathbf{b}}^s$, $\underline{\mathbf{c}}^s$, $\overline{\mathbf{c}}^s$ denote $\underline{\mathbf{b}}$, $\overline{\mathbf{b}}$, $\underline{\mathbf{c}}$, $\overline{\mathbf{c}}$ for β^s, where $s = 0, 1$. In this case $\overline{\mathbf{c}}^0 = \overline{\mathbf{c}}^1$. We say $\mathbf{i} \in OI(\Gamma)$ is of *type* (a^0, a^1) for $a^s \in \{l, r\}$ if $\Gamma_{\underline{\mathbf{b}}^s}(i_{\underline{\mathbf{b}}^s}) \in \Gamma_{\underline{\mathbf{b}}^s}^{a^s}$; see Figure 30.

For $\mathbf{i}$ of type (r, r), there exists $\mathbf{j} \in OI(\widetilde{\Gamma})$ such that $\widetilde{\Gamma}(\mathbf{j}) = \Gamma(\mathbf{i})$. We denote $\mathbf{j}$ by $h(\mathbf{i})$ for $\mathbf{i}$ of type (r, r). Define $\mathcal{F}(h) : \mathcal{F}(\Gamma) \to \mathcal{F}(\widetilde{\Gamma})$ as the sum of identity morphisms $P(\Gamma(\mathbf{i})) \to P(\widetilde{\Gamma}(h(\mathbf{i})))$ for $\mathbf{i}$ of type (r, r).
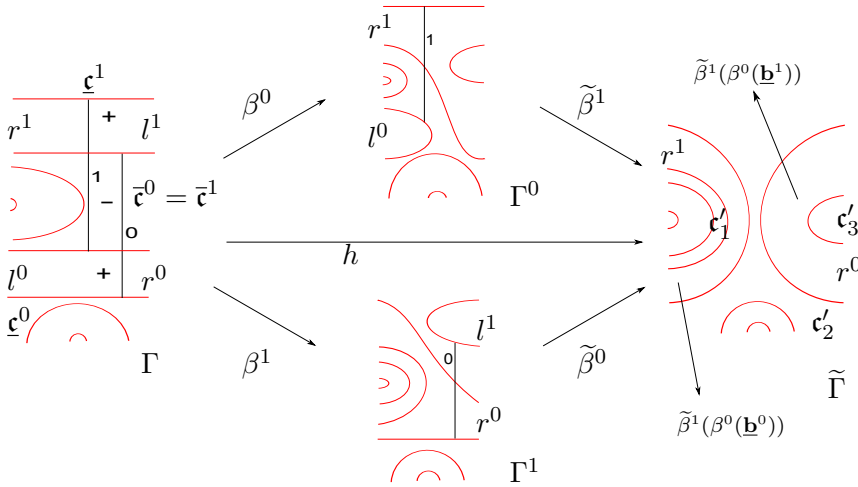
Figure 30: The letters l^s, r^s on Γ indicate types of $\mathbf{i} \in OI(\Gamma)$; l^s, r^{1-s} on Γ^s indicate $\beta^s(\mathbf{i})$ for $\mathbf{i} \in II(\beta^s)$; r^0, r^1 on $\widetilde{\Gamma}$ indicate $h(\mathbf{i})$ for $\mathbf{i}$ of type (r, r).

Case (2-2-A). Suppose $\mathbf{i}$ is of type (l, l). Then $\mathbf{i} \in II(\beta^0) \cap II(\beta^1)$ and $\beta^s(\mathbf{i}) \in II(\widetilde{\beta}^{1-s})$ and $\widetilde{\beta}^1(\beta^0(\mathbf{i})) = \widetilde{\beta}^0(\beta^1(\mathbf{i}))$. Hence $(\mathcal{F}(\widetilde{\beta}^1) \circ \mathcal{F}(\beta^0) + \mathcal{F}(\widetilde{\beta}^0) \circ \mathcal{F}(\beta^1))|_{P(\Gamma(\mathbf{i}))} = 0$ by the commutativity.

We have $\mathcal{F}(h)|_{P(\Gamma(\mathbf{i}))} = 0$ by the definition of $\mathcal{F}(h)$; hence $d_{\widetilde{\Gamma}} \circ \mathcal{F}(h)|_{P(\Gamma(\mathbf{i}))} = 0$.

It remains to show that $\mathcal{F}(h) \circ d_\Gamma|_{P(\Gamma(\mathbf{i}))} = 0$. This follows from observing that there is no $\mathfrak{c} \in \pi_0(R_-(\Gamma))$ such that $\mathbf{i}$ is $\mathfrak{c}$-admissible and $\mathfrak{c}|\mathbf{i}$ is of type (r, r): This is clear if $\mathfrak{c} \neq \overline{\mathfrak{c}}^0, \underline{\mathfrak{c}}^0, \underline{\mathfrak{c}}^1$ since the labels in $\underline{\mathbf{b}}^0$ and $\overline{\mathbf{b}}^0$ are not moved. If $\mathfrak{c} = \overline{\mathfrak{c}}^0 = \overline{\mathfrak{c}}^1$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible. If $\mathfrak{c} = \underline{\mathfrak{c}}^0$, then the label of $\overline{\mathbf{b}}^0$ is not moved, and if $\mathfrak{c} = \underline{\mathfrak{c}}^1$, then the label of $\underline{\mathbf{b}}^0$ is not moved.

Summing all the above terms gives (H) for $\mathbf{i}$ of type (l, l).

Case (2-2-B). Suppose $\mathbf{i}$ is of type (l, r). Then $\mathbf{i} \in II(\beta^0)$, $\mathbf{i} \notin II(\beta^1)$ and $\beta^0(\mathbf{i}) \notin II(\widetilde{\beta}^1)$. Since $\Gamma_{\underline{\mathbf{b}}^0}(\mathbf{i}_{\underline{\mathbf{b}}^0}) \in \Gamma^l_{\underline{\mathbf{b}}^0}$, we have $\mathbf{i} \notin SI(\beta^1)$ and $\mathcal{F}(\beta^1)|_{P(\Gamma(\mathbf{i}))} = 0$.

We have $\mathcal{F}(h)|_{P(\Gamma(\mathbf{i}))} = 0$ by the definition of $\mathcal{F}(h)$; hence $d_{\widetilde{\Gamma}} \circ \mathcal{F}(h)|_{P(\Gamma(\mathbf{i}))} = 0$.

Let $\mathfrak{c} \in \pi_0(R_-(\Gamma))$. If $\mathfrak{c} \neq \bar{\mathfrak{c}}^0, \underline{\mathfrak{c}}^0, \underline{\mathfrak{c}}^1$, then $\mathfrak{c}|\mathbf{i}$ cannot be of type (r, r) even if it exists. If $\mathfrak{c} = \bar{\mathfrak{c}}^0 = \bar{\mathfrak{c}}^1$ or $\underline{\mathfrak{c}}^1$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible. Hence $\mathcal{F}(h) \circ d_{\mathfrak{c}}|_{P(\Gamma(\mathbf{i}))} = 0$ for $\mathfrak{c} \neq \underline{\mathfrak{c}}^0$.

Finally, if $\mathfrak{c} = \underline{\mathfrak{c}}^0$, then $\mathbf{i}$ is $\mathfrak{c}$-admissible if and only if $\mathfrak{c}|\mathbf{i}$ is of type (r, r). This holds precisely when $\beta^0(\mathbf{i}) \in SI(\widetilde{\beta}^1)$. Hence $(\mathcal{F}(\widetilde{\beta}^1) \circ \mathcal{F}(\beta^0) + \mathcal{F}(h) \circ d_{\underline{\mathfrak{c}}^0})|_{P(\Gamma(\mathbf{i}))} = 0$.

Summing all the above terms gives (H) for $\mathbf{i}$ of type (l, r).

Case (2-2-C). Suppose $\mathbf{i}$ is of type (r, l). The proof is the same as that of type (l, r), with l and r reversed.

Case (2-2-D). Suppose $\mathbf{i}$ is of type (r, r). Then $\mathbf{i} \notin II(\beta^0) \cup II(\beta^1)$.

Let $\mathfrak{c} \in \pi_0(R_-(\Gamma))$. If $\mathfrak{c} \neq \bar{\mathfrak{c}}^0, \underline{\mathfrak{c}}^0, \underline{\mathfrak{c}}^1$, then the labels in $\underline{\mathbf{b}}^0$ and $\overline{\mathbf{b}}^0$ are not moved and there exists $\mathfrak{c}' = \mathfrak{c}$, viewed as an element of $\pi_0(R_-(\widetilde{\Gamma}))$, such that $\mathcal{F}(h) \circ d_{\mathfrak{c}} + d_{\mathfrak{c}'} \circ \mathcal{F}(h) = 0$ on $P(\Gamma(\mathbf{i}))$. If $\mathfrak{c} = \underline{\mathfrak{c}}^0$ or $\underline{\mathfrak{c}}^1$, then $\mathbf{i}$ is not $\mathfrak{c}$-admissible. If $\mathfrak{c} = \bar{\mathfrak{c}}^0$ and $\mathbf{i}$ is $\mathfrak{c}$-admissible, then $\mathfrak{c}|\mathbf{i}$ is of type (l, l) and $\mathcal{F}(h) \circ d_{\mathfrak{c}} = 0$ on $P(\Gamma(\mathbf{i}))$.

There are three components $\mathfrak{c}'$ of $\pi_0(R_-(\widetilde{\Gamma}))$ that are not of the form $\mathfrak{c}' = \mathfrak{c}$; they will be denoted by $\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{c}'_3$, in order from left to right in the right-hand diagram of Figure 30. $h(\mathbf{i})$ is not $\mathfrak{c}'_2$-admissible. One easily checks that

$$(d_{\mathfrak{c}'_1} \circ \mathcal{F}(h) + \mathcal{F}(\widetilde{\beta}^1) \circ \mathcal{F}(\beta^0))|_{P(\Gamma(\mathbf{i}))} = 0,$$
$$(d_{\mathfrak{c}'_3} \circ \mathcal{F}(h) + \mathcal{F}(\widetilde{\beta}^0) \circ \mathcal{F}(\beta^1))|_{P(\Gamma(\mathbf{i}))} = 0.$$

Summing all the above terms gives (H) for $\mathbf{i}$ of type (r, r). $\qquad\square$

The well-definition of the functor $\mathcal{F} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ follows from Lemmas 6.4.1.3 and 6.4.2.1.

7. The functors $\widetilde{\mathcal{F}}_{n,e}$

In this section we extend $\mathcal{F} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ to $\widetilde{\mathcal{F}} : \widetilde{\mathcal{C}}_{n,e} \to \widetilde{\mathcal{D}}_{n,e}$ by relating the homotopy gradings on both sides.

7.1. Degree of $\mathcal{F}(\beta)$

Let $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ be a nontrivial, nonzero bypass. If $h(\mathbf{i}) - h(\beta(\mathbf{i}))$ is the same for all $\mathbf{i} \in II(\beta) \sqcup SI(\beta)$, then we say that $\mathcal{F}(\beta)$ is *homogeneous* and define the *degree of* $\mathcal{F}(\beta)$ to be $\deg(\mathcal{F}(\beta)) = h(\mathbf{i}) - h(\beta(\mathbf{i}))$ for any $\mathbf{i}$. The goal of this subsection is to show that $\mathcal{F}(\beta)$ is homogeneous and to compute $\deg(\mathcal{F}(\beta))$.

The arc of attachment of a nontrivial bypass $0 \neq \beta \in \mathrm{Hom}(\Gamma, \Gamma')$, together with the three components of Γ that it intersects, cuts the disk D^2 into 6 components. The components are labeled $P_i(\beta)$, $i = 1, \dots, 6$, where $P_1(\beta)$ is the bottom component and i increases as we go clockwise around ∂D^2; see the top left diagram of Figure 31. $P_i(\beta')$ and $P_i(\beta'')$ are defined analogously; see the top right and bottom diagrams of Figure 31. $P_i(\beta)$ will be abbreviated as P_i if β is understood. We write $\underline{0} \in P_i$ if the boundary arc with label $\underline{0}$ is contained in P_i.
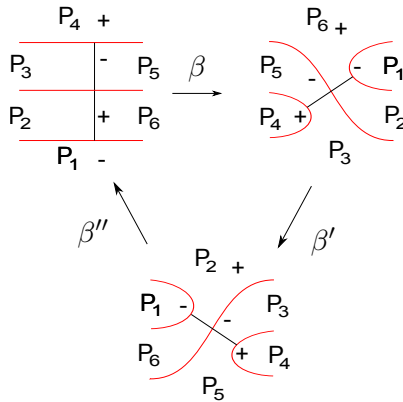

Figure 31.

For convenience we will write

$$(7.1.1) \qquad l_\Gamma(A) = \sum_{\Gamma_\mathbf{v} \subset A} l_{\Gamma_\mathbf{v}},$$

where A is a subset of $\mathbb{R}$.

Lemma 7.1.2.

1) *If $\beta \in \operatorname{Hom}(\Gamma, \Gamma')$ is a nontrivial bypass, then $\mathcal{F}(\beta)$ is homogeneous and*

$$(7.1.3) \qquad \deg(\mathcal{F}(\beta)) = \begin{cases} 0 & \text{if } \underline{0} \in P_1, P_2; \\ |\Gamma_{\underline{\mathbf{b}}}^r| + l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\underline{\mathbf{b}}}(x))) & \text{if } \underline{0} \in P_3, P_4; \\ 1 - |\Gamma_{\underline{\mathbf{b}}}^l| - l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(x), \Gamma_{\overline{\mathbf{b}}}(z))) & \text{if } \underline{0} \in P_5, P_6. \end{cases}$$

2) *The sum of degrees of the three bypasses in a bypass triangle is 1.*

Proof. (1) Suppose $\underline{0} \in P_1$. The other cases are similar and are left to the reader.

If $\mathbf{i} \in II(\beta)$, then $\beta(\mathbf{i})_{\beta(\mathbf{v})} = \mathbf{i}_{\mathbf{v}}$ for any $\mathbf{v} \in V^+(\Gamma)$. Moreover, the nesting degree is unchanged: $c_{\beta(\mathbf{v})}(\beta(\mathbf{i})_{\beta(\mathbf{v})}) = c_{\mathbf{v}}(\mathbf{i}_{\mathbf{v}})$ for any $\mathbf{v}$ since $\underline{0} \in P_1$. Hence $h(\beta(\mathbf{i})) = h(\mathbf{i})$ and $\deg(\mathcal{F}(\beta)) = 0$.

If $\mathbf{i} \in SI(\beta)$, then the bypass is of shuffling type (Y); see Figure 32. Note that

$$(7.1.4)$$
$$h(\beta(\mathbf{i})) - h(\mathbf{i}) = (h(\beta(\mathbf{i}), \beta(\overline{\mathbf{b}})) - h(\mathbf{i}, \underline{\mathbf{b}})) + (h(\beta(\mathbf{i}), \beta(\underline{\mathbf{b}})) - h(\mathbf{i}, \overline{\mathbf{b}}))$$
$$+ \sum_{\mathbf{w} \in LSV(\beta)} (h(\beta(\mathbf{i}), \beta(\mathbf{w})) - h(\mathbf{i}, \mathbf{w})),$$

since the only regions that are modified are $\overline{\mathbf{b}}$, $\underline{\mathbf{b}}$, and $\mathbf{w} \in LSV(\beta)$. We have $x = z = 0, y = |\Gamma_{\underline{\mathbf{b}}}^l| - 1$, and

(a) $\beta(\mathbf{i})_{\beta(\overline{\mathbf{b}})} - \mathbf{i}_{\underline{\mathbf{b}}} = l_{\Gamma_{\overline{\mathbf{b}}}} - |\Gamma_{\underline{\mathbf{b}}}^l| + 1$;

(b) $\mathbf{i}_{\overline{\mathbf{b}}} = 0$ and $\beta(\mathbf{i})_{\beta(\underline{\mathbf{b}})} = |\Gamma_{\underline{\mathbf{b}}}^l| - 1$;

(c) $\mathbf{i}_{\mathbf{w}} = 0$ and $\beta(\mathbf{i})_{\beta(\mathbf{w})} = l_{\Gamma_{\mathbf{w}}}$ for $\mathbf{w} \in LSV(\beta)$.

Using (a)–(c) we compute each of the three terms on the right-hand side of Equation (7.1.4):

$$h(\beta(\mathbf{i}), \beta(\overline{\mathbf{b}})) - h(\mathbf{i}, \underline{\mathbf{b}}) = (\beta(\mathbf{i})_{\beta(\overline{\mathbf{b}})} - \mathbf{i}_{\underline{\mathbf{b}}}) + l_\Gamma((\Gamma_{\overline{\mathbf{b}}}(0), \Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}})))$$
$$- l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\underline{\mathbf{b}}}(\mathbf{i}_{\underline{\mathbf{b}}})))$$
$$= (l_{\Gamma_{\overline{\mathbf{b}}}} - |\Gamma_{\underline{\mathbf{b}}}^l| + 1) - (l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\overline{\mathbf{b}}}(0))) + l_{\Gamma_{\overline{\mathbf{b}}}})$$
$$= 1 - |\Gamma_{\underline{\mathbf{b}}}^l| - l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\overline{\mathbf{b}}}(0))),$$

$$h(\beta(\mathbf{i}), \beta(\underline{\mathbf{b}})) - h(\mathbf{i}, \overline{\mathbf{b}}) = |\Gamma_{\underline{\mathbf{b}}}^l| - 1 + l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\underline{\mathbf{b}}}(y))),$$

$$\sum_{\mathbf{w} \in LSV(\beta)} (h(\beta(\mathbf{i}), \beta(\mathbf{w})) - h(\mathbf{i}, \mathbf{w})) = l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\overline{\mathbf{b}}}(0))).$$

Summing the three terms gives:

$$h(\beta(\mathbf{i})) - h(\mathbf{i}) = l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\underline{\mathbf{b}}}(y))) + l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(y), \Gamma_{\overline{\mathbf{b}}}(0)))$$
$$- l_\Gamma((\Gamma_{\underline{\mathbf{b}}}(0), \Gamma_{\overline{\mathbf{b}}}(0)))$$
$$= 0.$$
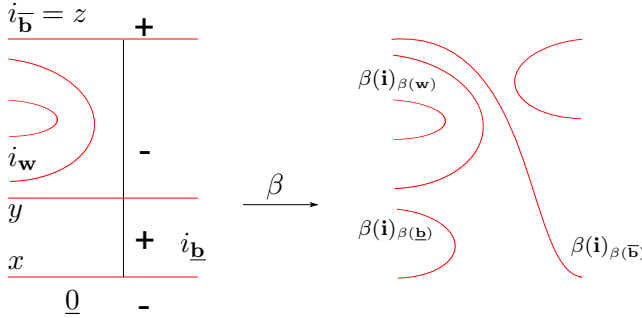
Figure 32: A shuffling index $\mathbf{i} \in SI(\beta)$, where $\underline{0} \in P_1$.

(2) For a bypass triangle $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma'' \xrightarrow{\beta''} \Gamma$, $P_i(\beta) = P_{i+2}(\beta') = P_{i+4}(\beta'')$ where the subscripts are viewed mod 6. Since any triangle is invariant under rotation, we may assume that $\underline{0} \in P_1(\beta)$ or $P_2(\beta)$. By Equation (7.1.3), we may further assume that $\underline{0} \in P_1(\beta)$. Then $\underline{0} \in P_3(\beta') = P_5(\beta'')$. Note that the terms $\underline{\mathbf{b}}, \overline{\mathbf{b}}, x, y, z$ that appear in Equation (7.1.3) are those for β. We can verify that:

$$\Gamma^r_{\underline{\mathbf{b}}(\beta')} = \Gamma^l_{\underline{\mathbf{b}}(\beta'')}, \quad \Gamma_{\underline{\mathbf{b}}(\beta')}(0) = \Gamma_{\underline{\mathbf{b}}(\beta'')}(x(\beta'')), \quad \Gamma_{\underline{\mathbf{b}}(\beta')}(x(\beta')) = \Gamma_{\overline{\mathbf{b}}(\beta'')}(z(\beta''));$$
$$l_{\Gamma'}((\Gamma_{\underline{\mathbf{b}}(\beta')}(0), \Gamma_{\underline{\mathbf{b}}(\beta')}(x(\beta')))) = l_{\Gamma''}((\Gamma_{\underline{\mathbf{b}}(\beta'')}(x(\beta'')), \Gamma_{\overline{\mathbf{b}}(\beta'')}(z(\beta'')))).$$

Hence $\deg(\mathcal{F}(\beta)) + \deg(\mathcal{F}(\beta')) + \deg(\mathcal{F}(\beta'')) = 1$ by Equation (7.1.3). $\square$

7.2. Definition of $\widetilde{\mathcal{F}}$

Each indecomposable object of $\widetilde{\mathcal{C}}_{n,e}$ is a pair $(\Gamma, [\xi])$ consisting of a dividing set Γ in $\mathcal{C}_{n,e}$ and a homotopy grading. *From now on the source $\Gamma^0 \in B_{n,e}$ of the quiver $Q_{n,e}$, i.e., $\Gamma^0_* = \{0, 1, \ldots, e\}$, will be the basepoint of $\widetilde{\mathcal{C}}_{n,e}$.*

7.2.1. Definition of $[\xi(\Gamma)]$. We first choose a "canonical" homotopy grading $[\xi(\Gamma)]$ for each $\Gamma \in \mathcal{C}_{n,e}$. It is defined by induction on $m(\Gamma) = e + 1 - |\Gamma_*|$. Note that $m(\Gamma) = 0$ if and only if $\Gamma \in B_{n,e}$ is basic.

For any $\Gamma^b \in B_{n,e}$, choose a path from Γ^0 to Γ^b in $Q_{n,e}$ and let ξ^b denote the composition of bypasses corresponding to the path. The homotopy grading $[\xi^b]$ is independent of the choice of path and we define $[\xi(\Gamma^b)] = [\xi^b]$. (Note that ξ^b may not be a tight contact structure.)

Next suppose that $m(\Gamma) > 0$. Recall $\beta(\Gamma)$ from Definition 6.1.2 for any non-basic Γ. There is a bypass triangle

$$(7.2.1.1) \qquad \Gamma \xrightarrow{\beta(\Gamma)} \Gamma' \to \Gamma'' \xrightarrow{\overline{\beta}(\Gamma)} \Gamma.$$

By the construction of $\beta(\Gamma)$, we have $m(\Gamma'), m(\Gamma'') < m(\Gamma)$. Let $\overline{\beta}(\Gamma)$ denote the bypass from Γ'' to Γ. We then define $[\xi(\Gamma)] = [\overline{\beta}(\Gamma) \circ \xi(\Gamma'')]$.

7.2.2. Definition of $\widetilde{\mathcal{F}}$. We define $\widetilde{\mathcal{F}} : \widetilde{\mathcal{C}}_{n,e} \to \widetilde{\mathcal{D}}_{n,e}$ as follows: We set $\widetilde{\mathcal{F}}(\Gamma, [\xi(\Gamma)]) = \mathcal{F}(\Gamma)$ and extend $\widetilde{\mathcal{F}}$ to any object so that it commutes with the shift functors on both sides, i.e., $\widetilde{\mathcal{F}}((\Gamma, [\xi])[i]) = \widetilde{\mathcal{F}}(\Gamma, [\xi])[i]$. Here $(\Gamma, [\xi])[i] = T^i(\Gamma, [\xi])$, where T is the shift functor on $\widetilde{\mathcal{C}}_{n,e}$. Next, suppose $\beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$ is nonzero, where β *is not necessarily a bypass.* Let $c(\beta)$ be the integer encoding minus the Hopf invariant and satisfying

$$(7.2.2.1) \qquad [\beta \circ \xi(\widetilde{\Gamma})] = [\xi(\Gamma)][c(\beta)],$$

where $[c(\beta)]$ refers to shifting by $c(\beta)$. Then we define

$$\widetilde{\mathcal{F}}((\widetilde{\Gamma}, [\xi(\widetilde{\Gamma})]) \xrightarrow{\beta} (\Gamma, [\beta \circ \xi(\widetilde{\Gamma})])) = (\mathcal{F}(\widetilde{\Gamma}) \xrightarrow{\mathcal{F}(\beta)[c(\beta)]} \mathcal{F}(\Gamma)[c(\beta)]),$$

where $\mathcal{F}(\beta)[c(\beta)]$ is $\mathcal{F}(\beta)$ postcomposed with the shift $[c(\beta)]$.

7.2.3. Well-definition. In this subsection we will abuse notation and not distinguish between $\beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$ and $\beta \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_{n,e}}((\widetilde{\Gamma}, [\xi]), (\Gamma, [\beta \circ \xi]))$.

Proposition 7.2.3.1. *The functor $\widetilde{\mathcal{F}}$ is well-defined, i.e., $\deg(\widetilde{\mathcal{F}}(\beta)) = 0$ for any nonzero $\beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$.*

Proof. It suffices to prove that

$$(7.2.3.2) \qquad [\beta \circ \xi(\widetilde{\Gamma})] = [\xi(\Gamma)][\deg(\mathcal{F}(\beta))]$$

for a nontrivial bypass β. Comparing with Equation (7.2.2.1), $c(\beta) = \deg(\mathcal{F}(\beta))$ and $\deg(\widetilde{\mathcal{F}}(\beta)) = \deg(\mathcal{F}(\beta)) - c(\beta) = 0$.

We first prove Equation (7.2.3.2) for $\beta = \overline{\beta}(\Gamma) \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$. By Definition 6.1.2, $\underline{0} \in \Gamma_{\underline{\mathbf{b}}}^l$, where $\underline{\mathbf{b}} = \underline{\mathbf{b}}(\beta)$. Hence $\underline{0} \in P_2(\beta)$, which implies that

$\deg(\mathcal{F}(\beta)) = 0$ by Lemma 7.1.2(1). Equation (7.2.3.2) is immediate from the definition of $[\xi(\Gamma)]$ as $[\xi(\Gamma)] = [\overline{\beta}(\Gamma) \circ \xi(\widetilde{\Gamma})] = [\beta \circ \xi(\widetilde{\Gamma})]$.

For a general nontrivial bypass β, we use $\overline{\beta}(\Gamma)$ and $\overline{\beta}(\widetilde{\Gamma})$, defined as in Equation (7.2.1.1), to simplify β.

Case 1. $\underline{0} \in P_1(\beta) \cup P_2(\beta)$. Equation (7.2.3.2) will be proved by induction on

$$\kappa(\beta) := \dim \underline{\mathbf{b}}(\beta) + l_{\widetilde{\Gamma}}((0, \widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}(0))).$$

If $\kappa(\beta) \geq 1$, then there exists a commutative diagram:

(7.2.3.3)
$$
\begin{array}{ccc}
\widetilde{\Gamma} & \xrightarrow{\beta} & \Gamma \\
{\scriptstyle\overline{\beta}(\widetilde{\Gamma})}\Big\uparrow & & \Big\uparrow{\scriptstyle\overline{\beta}(\Gamma)} \\
\widetilde{\Gamma}'' & \xrightarrow{\beta''} & \Gamma''
\end{array}
$$

where $\kappa(\beta'') < \kappa(\beta)$ and $\underline{0} \in P_1(\beta'') \cup P_2(\beta'')$. (In some cases, e.g., Figure 33, there may be degeneracies.) Since we have already shown that Equation (7.2.3.2) holds for $\overline{\beta}(\Gamma)$ and $\overline{\beta}(\widetilde{\Gamma})$, it suffices to prove Equation (7.2.3.2) for β''. We then reduce to the case where $\kappa(\beta) = 0$, i.e., $l_{\widetilde{\Gamma}}((0, \widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}(0))) = 0$, $\dim \underline{\mathbf{b}}(\beta) = 0$, and $\underline{0} \in \widetilde{\Gamma}^l_{\underline{\mathbf{b}}(\beta)}$.
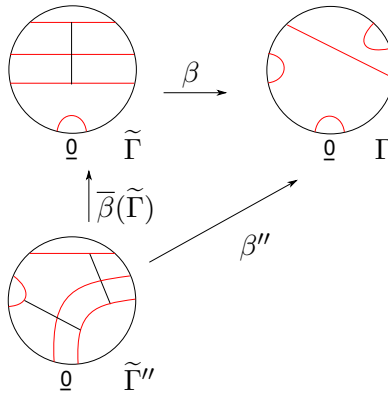
Figure 33: The case $\underline{0} \in P_1(\beta)$ and $\dim \underline{\mathbf{b}}(\beta) = 1$.

Let $\widetilde{\Gamma} \xrightarrow{\beta} \Gamma$ be a bypass satisfying $\kappa(\beta) = 0$. If $l_{\widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}} = 0$, then $\beta = \overline{\beta}(\Gamma)$ and Equation (7.2.3.2) holds. If $l_{\widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}} > 0$, then we apply the same commutative diagram (7.2.3.3) with β'' trivial to iteratively reduce to the case where $\beta = \overline{\beta}(\Gamma)$.

Case 2. $\underline{0} \in P_5(\beta) \cup P_6(\beta)$. Then $\deg(\mathcal{F}(\beta)) \leq 0$ by Lemma 7.1.2(1). We use bypass triangles to reduce to the case where $\deg(\mathcal{F}(\beta)) = 0$.

Suppose that $\deg(\mathcal{F}(\beta)) < 0$ for $\beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$. Then by Lemma 7.1.2 there exists a nontrivial bypass $\alpha \in \mathrm{Hom}(\Gamma, \Gamma')$ such that $\overline{\mathbf{b}}(\alpha) = \beta(\overline{\mathbf{b}}(\beta))$ and $\Gamma_{\underline{\mathbf{b}}(\alpha)} \subset [\widetilde{\Gamma}_{\underline{\mathbf{b}}(\beta)}(x), \widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}(z)]$; see Figure 34 for an example. There exists a nontrivial bypass $\widetilde{\alpha} \in \mathrm{Hom}(\widetilde{\Gamma}, \widetilde{\Gamma}')$ whose arc of attachment is disjoint from that of β; attaching the bypasses in different orders, we obtain a bypass $\beta' \in \mathrm{Hom}(\widetilde{\Gamma}', \Gamma')$ such that $\beta' \circ \widetilde{\alpha} = \alpha \circ \beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma')$. Note that β' is a trivial bypass if $\underline{\mathbf{b}}(\alpha) = \beta(\underline{\mathbf{b}}(\beta))$. We also use the bypass triangle $\Gamma \xrightarrow{\alpha} \Gamma' \xrightarrow{\alpha'} \Gamma'' \xrightarrow{\alpha''} \Gamma$.
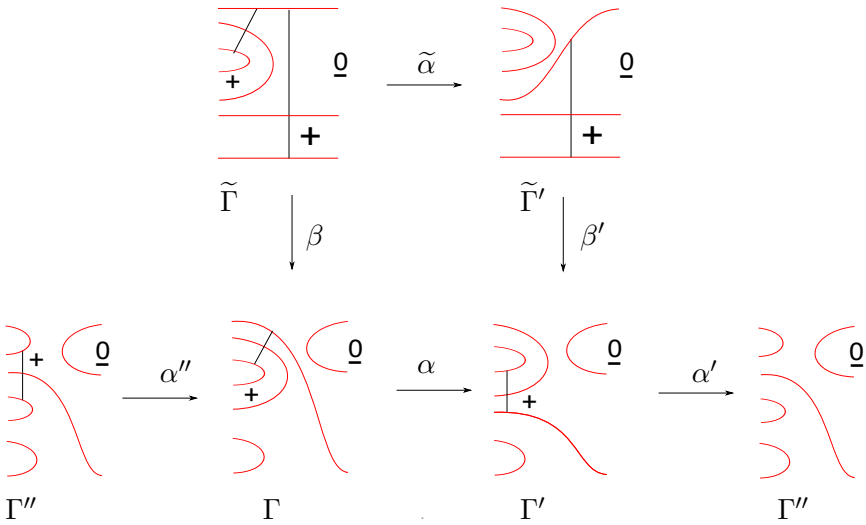
Figure 34: The case $\underline{0} \in P_5(\beta)$ and $\deg(\mathcal{F}(\beta)) < 0$.

By the definition of α, we have $\underline{0} \in P_4(\alpha)$, which implies that $\underline{0} \in P_2(\alpha'')$ and $\underline{0} \in P_6(\alpha')$. Note that $\underline{0} \in P_5(\widetilde{\alpha}) \cup P_6(\widetilde{\alpha})$ and $\underline{0} \in P_5(\beta') \cup P_6(\beta')$. By Lemma 7.1.2(1),(2),

$$\begin{aligned}
\deg(\mathcal{F}(\beta)) &= \deg(\mathcal{F}(\widetilde{\alpha})) + \deg(\mathcal{F}(\beta')) - \deg(\mathcal{F}(\alpha)) \\
&= \deg(\mathcal{F}(\widetilde{\alpha})) + \deg(\mathcal{F}(\beta')) + \deg(\mathcal{F}(\alpha')) + \deg(\mathcal{F}(\alpha'')) - 1 \\
&= \deg(\mathcal{F}(\widetilde{\alpha})) + \deg(\mathcal{F}(\beta')) + \deg(\mathcal{F}(\alpha')) - 1.
\end{aligned}$$

Suppose by induction Equation (7.2.3.2) holds for any $\widetilde{\beta}$ such that $\deg(\mathcal{F}(\widetilde{\beta})) > \deg(\mathcal{F}(\beta))$ and $\underline{0} \in P_5(\widetilde{\beta}) \cup P_6(\widetilde{\beta})$. In particular, it holds for

$\widetilde{\beta} \in \{\widetilde{\alpha}, \beta', \alpha'\}$. Then

$$[\alpha \circ \beta \circ \xi(\widetilde{\Gamma})] = [\beta' \circ \widetilde{\alpha} \circ \xi(\widetilde{\Gamma})] = [\xi(\Gamma')][\deg(\mathcal{F}(\beta')) + \deg(\mathcal{F}(\widetilde{\alpha}))]$$
$$= [\xi(\Gamma')][\deg(\mathcal{F}(\alpha)) + \deg(\mathcal{F}(\beta))].$$

By composing with $\alpha'' \circ \alpha'$, we obtain

$$[\beta \circ \xi(\widetilde{\Gamma})][1] = [\alpha'' \circ \alpha' \circ \alpha \circ \beta \circ \xi(\widetilde{\Gamma})]$$
$$= [\alpha'' \circ \alpha' \circ \xi(\Gamma')][\deg(\mathcal{F}(\alpha)) + \deg(\mathcal{F}(\beta))]$$
$$= [\xi(\Gamma')][\deg(\mathcal{F}(\alpha'')) + \deg(\mathcal{F}(\alpha')) + \deg(\mathcal{F}(\alpha)) + \deg(\mathcal{F}(\beta))]$$
$$= [\xi(\Gamma')][1 + \deg(\mathcal{F}(\beta))],$$

where the third line uses Equation (7.2.3.2) for α'' and α'. Equation (7.2.3.2) holds for α'', since $\underline{0} \in P_2(\alpha'')$ was treated in Case 1, and for α' by the inductive hypothesis. This proves Equation (7.2.3.2) for β.

By induction on $\deg(\mathcal{F}(\beta))$, we reduce to the case where $\deg(\mathcal{F}(\beta)) = 0$. Then, by the procedure used in Case 1, we reduce to the case where $l_{\widetilde{\Gamma}}((0, \widetilde{\Gamma}_{\overline{\mathbf{b}}(\beta)}(0))) = 0$ and $\dim \underline{\mathbf{b}}(\beta) = 0$. A further reduction (details left to the reader) gets us to the case where $\beta \in \mathrm{Hom}(\widetilde{\Gamma}, \Gamma)$ and $\widetilde{\Gamma}, \Gamma$ are basic dividing sets. Hence Equation (7.2.3.2) holds since $[\xi(\Gamma)] = [\beta \circ \xi(\widetilde{\Gamma})]$ by definition.

Case 3. $\underline{0} \in P_3(\beta) \cup P_4(\beta)$. We have a bypass triangle $\widetilde{\Gamma} \xrightarrow{\beta} \Gamma \xrightarrow{\beta'} \Gamma' \xrightarrow{\beta''} \widetilde{\Gamma}$ such that $\underline{0} \in P_5(\beta') \cup P_6(\beta')$ and $\underline{0} \in P_1(\beta'') \cup P_2(\beta'')$. Equation (7.2.3.2) follows from Lemma 7.1.2(2) and Cases 1 and 2. $\qquad\square$

7.3. Bypass triangles

In this subsection we show that $\widetilde{\mathcal{F}}$ takes bypass triangles in $\widetilde{\mathcal{C}}_{n,e}$ to distinguished triangles in $\widetilde{\mathcal{D}}_{n,e}$.

Proposition 7.3.1. *Let* $(\Gamma, [\xi]) \xrightarrow{\beta} (\Gamma', [\beta \circ \xi]) \xrightarrow{\beta'} (\Gamma'', [\beta' \circ \beta \circ \xi]) \xrightarrow{\beta''} (\Gamma, [\xi][1])$ *be a bypass triangle in* $\widetilde{\mathcal{C}}_{n,e}$. *Then its image under* $\widetilde{\mathcal{F}}$ *is a distinguished triangle in* $\widetilde{\mathcal{D}}_{n,e}$.

Proof. We will omit the gradings in the objects and write $\widetilde{\mathcal{F}}(\Gamma)$, etc. for simplicity. Let $\mathcal{F}(\gamma'')$ denote $\mathcal{F}(\gamma(\beta''))$ from Equation (6.4.1.2). Similarly, let $\mathcal{F}(\gamma)$ and $\mathcal{F}(\gamma')$ denote $\mathcal{F}(\gamma(\beta))$ and $\mathcal{F}(\gamma(\beta'))$, respectively.

In order to show that $\mathrm{cone}(\widetilde{\mathcal{F}}(\beta))$ and $\widetilde{\mathcal{F}}(\Gamma'')$ are homotopy equivalent, we define the following two maps

$$\widetilde{\mathcal{F}}(\eta) = \widetilde{\mathcal{F}}(\gamma'') \oplus \widetilde{\mathcal{F}}(\beta') : \mathrm{cone}(\widetilde{\mathcal{F}}(\beta)) = \widetilde{\mathcal{F}}(\Gamma) \oplus \widetilde{\mathcal{F}}(\Gamma') \to \widetilde{\mathcal{F}}(\Gamma''),$$
$$\widetilde{\mathcal{F}}(\epsilon) = \widetilde{\mathcal{F}}(\beta'') + \widetilde{\mathcal{F}}(\gamma') : \widetilde{\mathcal{F}}(\Gamma'') \to \mathrm{cone}(\widetilde{\mathcal{F}}(\beta)) = \widetilde{\mathcal{F}}(\Gamma) \oplus \widetilde{\mathcal{F}}(\Gamma').$$

They fit into the following diagram:

$$\widetilde{\mathcal{F}}(\Gamma') \xrightarrow{\widetilde{\mathcal{F}}(\beta')} \widetilde{\mathcal{F}}(\Gamma'') \xrightarrow{\widetilde{\mathcal{F}}(\gamma')} \widetilde{\mathcal{F}}(\Gamma')$$

where $\widetilde{\mathcal{F}}(\alpha) = \mathcal{F}(\alpha)$ up to grading shifts for $\alpha \in \{\beta, \beta', \beta'', \gamma', \gamma''\}$. By the method of Lemma 7.1.2, we can show that $\mathcal{F}(\gamma)$, $\mathcal{F}(\gamma')$, and $\mathcal{F}(\gamma'')$ are homogeneous. By Lemma 6.4.1.3, $\widetilde{\mathcal{F}}(\eta)$ and $\widetilde{\mathcal{F}}(\epsilon)$ are chain maps and are therefore morphisms in $\widetilde{\mathcal{D}}_{n,e}$.

We now show that they are homotopy inverses. By Lemma 6.4.1.1 and Equation (6.4.1.2),

$$\widetilde{\mathcal{F}}(\gamma) \circ \widetilde{\mathcal{F}}(\beta) + \widetilde{\mathcal{F}}(\beta'') \circ \widetilde{\mathcal{F}}(\gamma'') = \mathrm{id}_{\widetilde{\mathcal{F}}(\Gamma)},$$
$$\widetilde{\mathcal{F}}(\gamma') \circ \widetilde{\mathcal{F}}(\beta') + \widetilde{\mathcal{F}}(\beta) \circ \widetilde{\mathcal{F}}(\gamma) = \mathrm{id}_{\widetilde{\mathcal{F}}(\Gamma')},$$
$$\widetilde{\mathcal{F}}(\gamma'') \circ \widetilde{\mathcal{F}}(\beta'') + \widetilde{\mathcal{F}}(\beta') \circ \widetilde{\mathcal{F}}(\gamma') = \mathrm{id}_{\widetilde{\mathcal{F}}(\Gamma'')}.$$

We have $\widetilde{\mathcal{F}}(\eta) \circ \widetilde{\mathcal{F}}(\epsilon) = \mathrm{id}_{\widetilde{\mathcal{F}}(\Gamma'')}$ from the third equation. It follows from the first two equations and Lemma 6.4.1.1 that

$$\widetilde{\mathcal{F}}(\epsilon) \circ \widetilde{\mathcal{F}}(\eta) = \mathrm{id}_{\mathrm{cone}(\widetilde{\mathcal{F}}(\beta))} + d\widetilde{\mathcal{F}}(\gamma),$$

where $\widetilde{\mathcal{F}}(\gamma) : \widetilde{\mathcal{F}}(\Gamma') \to \widetilde{\mathcal{F}}(\Gamma)$. Here we are also using $\widetilde{\mathcal{F}}(\gamma') \circ \widetilde{\mathcal{F}}(\gamma'') = 0$ which is immediate from the definition. $\qquad\square$

8. $\widetilde{\mathcal{D}}_{n,e}$ as a triangulated envelope of $\widetilde{\mathcal{C}}_{n,e}$

The image of $\widetilde{\mathcal{F}}_{n,e}$ generates $\widetilde{\mathcal{D}}_{n,e}$ under taking iterated cones since all the $P(\Gamma)$, $\Gamma \in B_{n,e}$, are in the image. The goal of this section is to show that

$\widetilde{\mathcal{F}}_{n,e}$ is faithful, i.e.,

(F) $\widetilde{\mathcal{F}}_{n,e} : \mathrm{Hom}_{\widetilde{\mathcal{C}}_{n,e}}((\Gamma, [\xi]), (\Gamma', [\xi'])) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\widetilde{\mathcal{F}}(\Gamma, [\xi]), \widetilde{\mathcal{F}}(\Gamma', [\xi'])).$

We first prove (F) in the following three basic cases: (i) $\Gamma = \Gamma'$; (ii) there exists a bypass $\Gamma \xrightarrow{\beta} \Gamma'$; and (iii) there exists a bypass $\Gamma' \xrightarrow{\beta} \Gamma$. Using the calculations for the basic cases we show in Proposition 8.2.3 that the Serre functors $\mathcal{S}_{\widetilde{\mathcal{C}}}$ of $\widetilde{\mathcal{C}}_{n,e}$ and $\mathcal{S}_{\widetilde{\mathcal{D}}}$ of $\widetilde{\mathcal{D}}_{n,e}$ commute with $\widetilde{\mathcal{F}}_{n,e}$. We finally prove the faithfulness in general by combining the results for the basic cases and the Serre functors.

8.1. Basic cases

8.1.1. The case $\Gamma = \Gamma'$.   The goal is to prove (F) for $\Gamma = \Gamma'$. Since $\mathrm{End}_{\mathcal{C}_{n,e}}(\Gamma) = \mathbb{F}_2\langle \mathrm{id}_\Gamma \rangle$, it suffices to prove the following.

Proposition 8.1.1.1.   *For any Γ in $\mathcal{C}_{n,e}$, $\mathrm{End}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma)) = \mathbb{F}_2\langle \mathrm{id}_{\mathcal{F}(\Gamma)} \rangle.$*

Before proving Proposition 8.1.1.1 we introduce some definitions. Recall that $d_\Gamma = \sum_{\mathbf{v} \in V^+(\Gamma)} d_\mathbf{v}$, where $d_\mathbf{v}$ is given in Equation (6.2.19). Lemma 6.2.20 implies that:

(D) $d_\mathbf{v}^2 = 0, \quad d_\Gamma d_\mathbf{v} + d_\mathbf{v} d_\Gamma = 0,$

for any $\mathbf{v} \in V^+(\Gamma)$.

For Γ, Γ' in $\mathcal{C}_{n,e}$, let $\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$ denote the space of $R_{n,e}$-module maps, where $\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')$ are viewed as $R_{n,e}$-modules by ignoring the differentials. We then define maps

(8.1.1.2) $d_{\mathbf{w},\mathbf{v}}, d_{\emptyset,\mathbf{v}}, d_{\mathbf{w},\emptyset}, d_{\Gamma',\Gamma} : \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')) \to \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')),$

where $f \in \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$, $\mathbf{v} \in OI(\Gamma)$, $\mathbf{w} \in OI(\Gamma')$, and

$$d_{\mathbf{w},\mathbf{v}}f = d_\mathbf{w} \circ f + f \circ d_\mathbf{v},$$
$$d_{\emptyset,\mathbf{v}}f = f \circ d_\mathbf{v},$$
$$d_{\mathbf{w},\emptyset}f = d_\mathbf{w} \circ f,$$
$$d_{\Gamma',\Gamma}f = d_{\Gamma'} \circ f + f \circ d_\Gamma.$$

The lemma below follows from (D).

Lemma 8.1.1.3. *If $d_0, d_1 \in \{d_{\mathbf{w},\mathbf{v}}, d_{\emptyset,\mathbf{v}}, d_{\mathbf{w},\emptyset}, d_{\Gamma',\Gamma}\}$, then*

$$d_0^2 = 0, \qquad d_0 d_1 = d_1 d_0,$$

in $\mathrm{End}(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')))$.

Note that $\mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$ is the cohomology $H_{d_{\Gamma',\Gamma}}(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')))$ by definition.

Recall that to compute $\mathrm{End}_{\mathcal{C}_{n,e}}(\Gamma)$ on the topological side, we observe that $\#\gamma_{\Gamma,\Gamma} = \#\gamma_{\Gamma',\Gamma'}$, where Γ' is obtained from Γ by removing a boundary parallel component. By repeating this reduction, we eventually obtain $\#\gamma_{\Gamma,\Gamma} = \#\gamma_{\Gamma^0,\Gamma^0} = 1$, where Γ^0 is the unique dividing set in $\mathcal{C}_{0,0}$. To compute $\mathrm{End}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma))$ on the algebraic side we consider a similar reduction on

$$|V_{nb}^+(\Gamma)| = |\{\mathbf{v} \in V^+(\Gamma) \mid l_{\Gamma_{\mathbf{v}}} > 0\}|.$$

The following example illustrates the idea of the reduction.

Example 8.1.1.4. We compute $\mathrm{End}(\mathcal{F}(\Gamma))$ for Γ from Example 6.1.5. Recall that $V_{nb}^+(\Gamma) = \{(1), (1,1)\}$, where $\Gamma_{(1,1)} = \{\underline{2}, \underline{3}\}$ directly nests inside $\Gamma_{(1)} = \{\underline{1}, \underline{4}\}$ and no vector in $V_{nb}^+(\Gamma)$ nests inside $\Gamma_{(1,1)}$. Let $\mathbf{w} = (1)$ and $\mathbf{v} = (1,1)$. Let $\widehat{\Gamma}^{\mathbf{v}}$ denote a dividing set which is obtained from Γ by replacing the component $\Gamma_{\mathbf{v}}$ with $l_{\Gamma_{\mathbf{v}}} + 1$ boundary parallel components corresponding to the $l_{\Gamma_{\mathbf{v}}} + 1$ elements of $\Gamma_{\mathbf{v}}$. In particular, $V_{nb}^+(\widehat{\Gamma}^{\mathbf{v}}) = \{(1)\}$ and $\widehat{\Gamma}^{\mathbf{v}}_{(1)} = \{\underline{1}, \underline{4}\}$.

The complex $\mathcal{N}$ (given by (6.1.6)) for $\mathcal{F}(\Gamma)$ is

$$P(\underline{1}, \underline{2}) \xrightarrow{d_{\mathbf{v}}} P(\underline{1}, \underline{3}) \xrightarrow{d_{\mathbf{w}}} P(\underline{2}, \underline{4}) \xrightarrow{d_{\mathbf{v}}} P(\underline{3}, \underline{4}),$$

where $d_{\mathbf{v}}$ is given in (6.2.19). As an $\mathbb{F}_2$-vector space $\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma))$ is 7-dimensional. The differential $d_{\mathbf{v},\mathbf{v}}$ acts on $\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma))$ by:

$$
\begin{array}{rccl}
d_{\mathbf{v},\mathbf{v}} : & \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)) & \to & \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)) \\
& \mathrm{id}_{P(\underline{1},\underline{2})} & \mapsto & d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{1},\underline{2})}, \\
& \mathrm{id}_{P(\underline{1},\underline{3})} & \mapsto & d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{1},\underline{2})}, \\
& \mathrm{id}_{P(\underline{2},\underline{4})} & \mapsto & d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{2},\underline{4})}, \\
& \mathrm{id}_{P(\underline{3},\underline{4})} & \mapsto & d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{2},\underline{4})}, \\
d_{\mathbf{w}} \circ \mathrm{id}_{P(\underline{1},\underline{3})}, \quad d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{1},\underline{2})}, \quad d_{\mathbf{v}} \circ \mathrm{id}_{P(\underline{2},\underline{4})} & & \mapsto & 0.
\end{array}
$$

There is a double complex $(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\mathbf{v},\mathbf{v}}, d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$. Its first page $(H_{d_{\mathbf{v},\mathbf{v}}}; d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$ is given by:

$$d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}} : \qquad H_{d_{\mathbf{v},\mathbf{v}}} \qquad \rightarrow \qquad H_{d_{\mathbf{v},\mathbf{v}}}$$
$$[\mathrm{id}_{P(\underline{1},\underline{2})} + \mathrm{id}_{P(\underline{1},\underline{3})}] \quad \mapsto \quad [d_{\mathbf{w}} \circ \mathrm{id}_{P(\underline{1},\underline{3})}],$$
$$[\mathrm{id}_{P(\underline{2},\underline{4})} + \mathrm{id}_{P(\underline{3},\underline{4})}] \quad \mapsto \quad [d_{\mathbf{w}} \circ \mathrm{id}_{P(\underline{1},\underline{3})}].$$

It is easy to see that $(H_{d_{\mathbf{v},\mathbf{v}}}; d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$ is isomorphic to $(\mathrm{Map}(\widehat{\Gamma}^{\mathbf{v}}, \widehat{\Gamma}^{\mathbf{v}}); d_{\widehat{\Gamma}^{\mathbf{v}}, \widehat{\Gamma}^{\mathbf{v}}})$ whose cohomology is $\mathrm{End}(\widehat{\Gamma}^{\mathbf{v}})$. By the spectral sequence associated to the double complex, $\mathrm{End}(\widehat{\Gamma}^{\mathbf{v}})$ converges to $\mathrm{End}(\Gamma)$.

Returning to the discussion of $\mathrm{End}(\mathcal{F}(\Gamma))$ in general, suppose $\Gamma \notin B_{n,e}$ and $\mathbf{v} \in V_{nb}^{+}(\Gamma)$ such that $NV(\mathbf{v}, l_{\Gamma_{\mathbf{v}}}) = \emptyset$, i.e., no vector in $V_{nb}^{+}(\Gamma)$ directly nests inside $\mathbf{v}$. (If $\Gamma \notin B_{n,e}$, then such a vector always exists.) We define $\widehat{\Gamma}^{\mathbf{v}}$ as the dividing set in $\mathfrak{ob}(\mathcal{C}_{n,e-l_{\Gamma_{\mathbf{v}}}})$ obtained from Γ by replacing the component $\Gamma_{\mathbf{v}}$ with $l_{\Gamma_{\mathbf{v}}} + 1$ boundary parallel components corresponding to the $l_{\Gamma_{\mathbf{v}}} + 1$ elements of $\Gamma_{\mathbf{v}}$. We will write $\widehat{\Gamma}$ for $\widehat{\Gamma}^{\mathbf{v}}$ and l for $l_{\Gamma_{\mathbf{v}}}$ when $\mathbf{v}$ is understood.

There is a bijection

$$\widehat{\mathbf{v}} : V_{nb}^{+}(\Gamma) \backslash \{\mathbf{v}\} \xrightarrow{\sim} V_{nb}^{+}(\widehat{\Gamma})$$

satisfying $\widehat{\Gamma}_{\widehat{\mathbf{v}}(\mathbf{w})} = \Gamma_{\mathbf{w}}$ for $\mathbf{w} \in V_{nb}^{+}(\Gamma) \backslash \{\mathbf{v}\}$. There is also an induced map

$$\widehat{\mathbf{v}} : OI(\Gamma) \to OI(\widehat{\Gamma})$$

given by $\widehat{\mathbf{v}}(\mathbf{i})_{\widehat{\mathbf{v}}(\mathbf{w})} = \mathbf{i}_{\mathbf{w}}$ for $\mathbf{w} \in V_{nb}^{+}(\Gamma) \backslash \{\mathbf{v}\}$. (It is also called $\widehat{\mathbf{v}}$ by abuse of notation.) The map $\widehat{\mathbf{v}} : OI(\Gamma) \to OI(\widehat{\Gamma})$ is surjective and is an $(l+1)$-to-1 map. Given $\widehat{\mathbf{i}} \in OI(\widehat{\Gamma})$ and $0 \le s \le l$, define $\mathbf{i}^{s} \in \widehat{\mathbf{v}}^{-1}(\widehat{\mathbf{i}})$ such that $(\mathbf{i}^{s})_{\mathbf{v}} = s$.

The following lemma relates $\mathrm{End}(\mathcal{F}(\Gamma))$ and $\mathrm{End}(\mathcal{F}(\widehat{\Gamma}))$.

Lemma 8.1.1.5. *Suppose that* $\mathbf{v} \in V_{nb}^{+}(\Gamma)$ *such that* $NV(\mathbf{v}, l_{\Gamma_{\mathbf{v}}}) = \emptyset$. *Then there is a finite double complex* $(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\mathbf{v},\mathbf{v}}, d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$ *whose first page* $(H_{d_{\mathbf{v},\mathbf{v}}}; d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$ *is isomorphic to the complex* $(\mathrm{Map}(\mathcal{F}(\widehat{\Gamma}), \mathcal{F}(\widehat{\Gamma})); d_{\widehat{\Gamma}, \widehat{\Gamma}})$.

Proof. Since $d_{\mathbf{v},\mathbf{v}}$ and $d_{\Gamma,\Gamma}$ are two commuting differentials by Lemma 8.1.1.3,

$$(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\mathbf{v},\mathbf{v}}, d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}})$$

is a finite double complex.

Since $d_{\mathbf{u}} = 0$ for $\mathbf{u} \notin V_{nb}^+(\Gamma)$, the space $\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma))$ has an $\mathbb{F}_2$-basis

$$\left\{ \prod_{t=1}^{k} d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i})} \;\middle|\; \mathbf{i} \in OI(\Gamma), \mathbf{u}^t \in V_{nb}^+(\Gamma), k \geq 0 \right\},$$

where $\prod_{t=1}^{0} d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i})}$ is understood to be $\mathrm{id}_{\Gamma(\mathbf{i})}$. In this basis

- the composition is independent of the order of the $d_{\mathbf{u}^t}$ since they pairwise commute; and

- each $d_{\mathbf{u}^t}$ appears at most once since $d_{\mathbf{u}^t}^2 = 0$ by Lemma 6.2.20.

Let $\mathbf{w} \in V(\Gamma)$ such that $\mathbf{v}$ directly nests inside $\mathbf{w}$. We consider the case $\mathbf{w} \neq *$. (When $\mathbf{w} = *$, there is no $d_{\mathbf{w}}$ and the same proof holds by setting $d_{\mathbf{w}} = 0$ and $d_{\widehat{\mathbf{v}}(\mathbf{w})} = 0$ below.)

We say that a generator f is of Type (1) if it has the form:

$$f = \prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i})} \quad \text{or} \quad \prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i})}, \text{ where } \mathbf{u}^t \neq \mathbf{w}, \mathbf{v}.$$

For each $\prod_t d_{\mathbf{u}^t}$, $\mathbf{u}^t \neq \mathbf{w}, \mathbf{v}$, there is a subcomplex of $(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\mathbf{v}, \mathbf{v}})$:

Type (1) : $\displaystyle\prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i}^s)}$

$$\xrightarrow{d_{\mathbf{v}, \mathbf{v}}} \begin{cases} \displaystyle\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^1)} & s = 0, \\[2mm] \displaystyle\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^s)} + \prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^{s+1})} & 0 < s < l, \\[2mm] \displaystyle\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^l)} & s = l. \end{cases}$$

Such a subcomplex is said to be of Type (1). As an $\mathbb{F}_2$-vector space, this subcomplex has dimension $2l + 1$ and any two subcomplexes of Type (1) are either equal or intersect trivially.

We say that a generator f is of Type (2) if it has the form:

$$f = \prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{w}} \circ \mathrm{id}_{\Gamma(\mathbf{i})}, \text{ where } \mathbf{u}^t \neq \mathbf{w}, \mathbf{v}.$$

If $f \neq 0$, then $\mathbf{i}_{\mathbf{v}} = 0$, i.e., $\mathbf{i} = \mathbf{i}^0$ for some $\widehat{\mathbf{i}} \in OI(\widehat{\Gamma})$. Each nonzero f of Type (2) generates a 1-dimensional subcomplex:

$$\text{Type (2) : } \quad f \xrightarrow{d_{\mathbf{v}, \mathbf{v}}} 0.$$

The complex $(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\mathbf{v},\mathbf{v}})$ is a direct sum of its subcomplexes of Types (1) and (2); note that there are no generators of type $\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{w}} \circ d_{\mathbf{v}} \circ \mathrm{id}_{\Gamma(\mathbf{i})}$ since $d_{\mathbf{w}} \circ d_{\mathbf{v}} = 0$.

We compute the cohomology $H_{d_{\mathbf{v},\mathbf{v}}}$. For a subcomplex of Type (1), $d_{\mathbf{v},\mathbf{v}}$ is a surjective map from $\mathbb{F}_2^{l+1}$ to $\mathbb{F}_2^l$ and a generator of $H_{d_{\mathbf{v},\mathbf{v}}}$ is given by $\sum_{s=0}^l \prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i}^s)}$. For a subcomplex of Type (2), a generator of $H_{d_{\mathbf{v},\mathbf{v}}}$ is given by $\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{w}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^0)}$.

Define an $\mathbb{F}_2$-linear map:

$$
\begin{array}{rccc}
G: & H_{d_{\mathbf{v},\mathbf{v}}} & \to & \mathrm{Map}(\mathcal{F}(\widehat{\Gamma}), \mathcal{F}(\widehat{\Gamma})) \\[2mm]
& \displaystyle\sum_{s=0}^l \prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma(\mathbf{i}^s)} & \mapsto & \displaystyle\prod_t d_{\widehat{\mathbf{v}}(\mathbf{u}^t)} \circ \mathrm{id}_{\widehat{\Gamma}(\widehat{\mathbf{i}})}, \\[2mm]
& \displaystyle\prod_t d_{\mathbf{u}^t} \circ d_{\mathbf{w}} \circ \mathrm{id}_{\Gamma(\mathbf{i}^0)} & \mapsto & \displaystyle\prod_t d_{\widehat{\mathbf{v}}(\mathbf{u}^t)} \circ d_{\widehat{\mathbf{v}}(\mathbf{w})} \circ \mathrm{id}_{\widehat{\Gamma}(\widehat{\mathbf{i}})}.
\end{array}
$$

It is an isomorphism of $\mathbb{F}_2$-vector spaces since $\widehat{\mathbf{v}}: V_{nb}^+(\Gamma)\backslash\{\mathbf{v}\} \xrightarrow{\sim} V_{nb}^+(\widehat{\Gamma})$ is a bijection and $\widehat{\mathbf{v}}: OI(\Gamma) \to OI(\widehat{\Gamma})$ is surjective.

For any $\mathbf{u} \in V_{nb}^+(\Gamma)\backslash\{\mathbf{v}\}$, we have $G(d_{\mathbf{u}} \circ f) = d_{\widehat{\mathbf{v}}(\mathbf{u})} \circ G(f)$ and $G(f \circ d_{\mathbf{u}}) = G(f) \circ d_{\widehat{\mathbf{v}}(\mathbf{u})}$ for $f \in H_{d_{\mathbf{v},\mathbf{v}}}$. Since

$$
d_\Gamma - d_{\mathbf{v}} = \sum_{\mathbf{u} \in V_{nb}^+(\Gamma)\backslash\{\mathbf{v}\}} d_{\mathbf{u}}, \qquad d_{\widehat{\Gamma}} = \sum_{\widehat{\mathbf{u}} \in V_{nb}^+(\widehat{\Gamma})} d_{\widehat{\mathbf{u}}},
$$

it follows that G commutes with $d_{\Gamma,\Gamma} - d_{\mathbf{v},\mathbf{v}}$ and $d_{\widehat{\Gamma},\widehat{\Gamma}}$. Hence the two complexes are isomorphic. $\square$

Proof of Proposition 8.1.1.1. Since $\mathcal{F}(\Gamma)$ is not isomorphic to the zero object of $\mathcal{D}_{n,e}$, it follows that $\mathrm{id}_{\mathcal{F}(\Gamma)} \in \mathrm{End}(\mathcal{F}(\Gamma))$ is nonzero. It then remains to prove that $\dim(\mathrm{End}(\mathcal{F}(\Gamma))) \le 1$, which is proved by induction on $|V_{nb}^+(\Gamma)|$.

If $|V_{nb}^+(\Gamma)| = 0$, then $\Gamma \in B_{n,e}$. Hence $\mathcal{F}(\Gamma) = P(\Gamma)$ and $\dim(\mathrm{End}(P(\Gamma))) \le 1$.

If $|V_{nb}^+(\Gamma)| > 0$, there exists $\mathbf{v} \in V_{nb}^+(\Gamma)$ such that $NV(\mathbf{v}, l_{\Gamma_{\mathbf{v}}}) = \emptyset$. By definition $|V_{nb}^+(\widehat{\Gamma}^{\mathbf{v}})| = |V_{nb}^+(\Gamma)| - 1$. By Lemma 8.1.1.5, the cohomology of $(\mathrm{Map}(\mathcal{F}(\widehat{\Gamma}), \mathcal{F}(\widehat{\Gamma})); d_{\widehat{\Gamma},\widehat{\Gamma}})$ converges to the cohomology of $(\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma)); d_{\Gamma,\Gamma})$. Hence

$$
\dim(\mathrm{End}(\mathcal{F}(\Gamma))) \le \dim(\mathrm{End}(\mathcal{F}(\widehat{\Gamma})))
$$

and $\dim(\mathrm{End}(\mathcal{F}(\Gamma))) \le 1$ by induction. $\square$

8.1.2. The case of a bypass. Let $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$ be a nontrivial bypass. Then $\mathrm{Hom}(\Gamma, \Gamma') = \mathbb{F}_2\langle\beta\rangle$ and $\mathrm{Hom}(\Gamma', \Gamma) = 0$.

Proposition 8.1.2.1. *For any nontrivial bypass* $\beta \in \mathrm{Hom}(\Gamma, \Gamma')$, *we have*

1) $\mathrm{Hom}(\widetilde{\mathcal{F}}(\Gamma, [\xi]), \widetilde{\mathcal{F}}(\Gamma', [\widetilde{\xi'}])) = \mathbb{F}_2 \langle \widetilde{\mathcal{F}}(\beta) \rangle$, *if* $[\widetilde{\xi'}] = [\beta \circ \xi]$; *otherwise, it is zero.*

2) $\mathrm{Hom}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)) = 0$.

Proof. Consider a bypass triangle $(\Gamma, [\xi]) \xrightarrow{\beta} (\Gamma', [\xi']) \xrightarrow{\beta'} (\Gamma'', [\xi'']) \xrightarrow{\beta''}$ $(\Gamma', [\xi][1])$ in $\widetilde{\mathcal{C}}_{n,e}$. By Proposition 7.3.1, it is mapped to a distinguished triangle in $\widetilde{\mathcal{D}}_{n,e}$. By applying the exact functor $\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\widetilde{\mathcal{F}}(\Gamma, [\widetilde{\xi}]), -)$ to the distinguished triangle, we obtain a long exact sequence:

$$\mathrm{Hom}(\widetilde{\mathcal{F}}(\Gamma, [\widetilde{\xi}]), \widetilde{\mathcal{F}}(\Gamma, [\xi])) \xrightarrow{\widetilde{\mathcal{F}}(\beta)} \mathrm{Hom}(\widetilde{\mathcal{F}}(\Gamma, [\widetilde{\xi}]), \widetilde{\mathcal{F}}(\Gamma', [\xi']))$$

$$\xrightarrow{\widetilde{\mathcal{F}}(\beta')} \mathrm{Hom}(\widetilde{\mathcal{F}}(\Gamma, [\widetilde{\xi}]), \widetilde{\mathcal{F}}(\Gamma'', [\xi''])),$$

where the subscripts $\widetilde{\mathcal{D}}_{n,e}$ are omitted. Hence (1) is equivalent to $\mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'')) = 0$.

Similarly, by applying the exact functor $\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(-, \widetilde{\mathcal{F}}(\Gamma', [\widetilde{\xi'}]))$, one can see that (1) is equivalent to $\mathrm{Hom}(\mathcal{F}(\Gamma''), \mathcal{F}(\Gamma')) = 0$. By rotating the bypass triangle, the proposition is equivalent to any one of the following three statements:

$$(i)\, \mathrm{Hom}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)) = 0; \quad (ii)\, \mathrm{Hom}(\mathcal{F}(\Gamma''), \mathcal{F}(\Gamma')) = 0;$$
$$(iii)\, \mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'')) = 0.$$

Without loss of generality we can assume that $\underline{0} \in P_1(\beta) \cup P_2(\beta)$. Proposition 8.1.2.1 is a consequence of the following lemma. $\square$

Lemma 8.1.2.2. *We have* $\mathrm{Hom}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)) = 0$ *if* $\mathrm{Hom}(\Gamma, \Gamma')$ *is generated by a bypass* β *and* $\underline{0} \in P_1(\beta) \cup P_2(\beta)$.

Proof. There is a bypass triangle $\Gamma \xrightarrow{\beta} \Gamma' \xrightarrow{\beta'} \Gamma''$ starting with β. Let us write $\mathbf{b}, x, y, \mathbf{w}$ for $\underline{\mathbf{b}}(\beta'), x(\beta'), y(\beta'), \overline{\mathbf{b}}(\beta)$, as in Notation 4.2.1; see Figure 35. Since $\underline{0} \in P_1(\beta) \cup P_2(\beta) = P_3(\beta') \cup P_4(\beta')$, we have $0 < x \le y = l_{\Gamma'_{\underline{\mathbf{b}}}}$ and $\underline{\mathbf{b}} \ne *, \mathbf{w} \ne *$. We write

$$\mathrm{Map}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)) = \bigoplus_{\mathbf{j} \in OI(\Gamma'), \mathbf{i} \in OI(\Gamma)} \mathrm{Hom}(P(\Gamma'(\mathbf{j}))[h(\mathbf{j})], P(\Gamma(\mathbf{i}))[h(\mathbf{i})]).$$

We prove the lemma by induction on $l_{\Gamma'}((\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y)))$, defined in Equation (7.1.1).
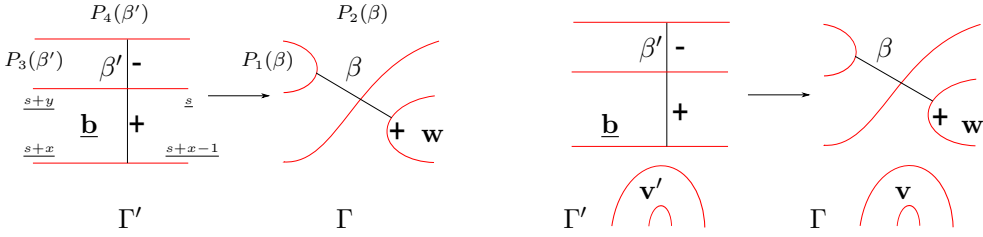
Figure 35: The left-hand side describes Case 1 after removing boundary parallel components nesting inside $(\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))$; the right-hand side depicts $\mathbf{v}, \mathbf{v}'$ in Case 2.

Case 1. If $l_{\Gamma'}((\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))) = 0$, then no vector in $V^+_{nb}(\Gamma')$ nests inside $\underline{\mathbf{b}}$. We view $\mathrm{Map}(\mathcal{F}(\overline{\Gamma}'), \mathcal{F}(\overline{\Gamma}))$ as a finite double complex

$$(\mathrm{Map}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)); d_{\emptyset, \underline{\mathbf{b}}}, d_{\Gamma, \Gamma'} - d_{\emptyset, \underline{\mathbf{b}}}),$$

whose first page $(H_{d_{\emptyset, \underline{\mathbf{b}}}}; d_{\Gamma, \Gamma'} - d_{\emptyset, \underline{\mathbf{b}}})$ converges to $H_{d_{\Gamma, \Gamma'}} = \mathrm{Hom}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma))$. Here $d_{\emptyset, \underline{\mathbf{b}}} f := f \circ d_{\underline{\mathbf{b}}}$ for $f \in \mathrm{Map}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma))$. The following claim implies the lemma for Case 1.

Claim. *The cohomology of the complex* $(\mathrm{Map}(\mathcal{F}(\Gamma'), P(\Gamma(\mathbf{i}))); d_{\emptyset, \underline{\mathbf{b}}})$ *is zero for any* $\mathbf{i} \in OI(\Gamma)$.

Proof of Claim. We can ignore boundary parallel components $\Gamma'_{\mathbf{v}} \subset (\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))$ in the computation since those labels do not appear in either $P(\overline{\Gamma}'(\mathbf{j}))$ or $P(\Gamma(\mathbf{i}))$ for any $\mathbf{j} \in OI(\Gamma'), \mathbf{i} \in OI(\Gamma)$. Since we are assuming that $l_{\Gamma'}((\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))) = 0$,

$$\Gamma'_{\underline{\mathbf{b}}}(0) = \underline{s}, \quad \Gamma'_{\underline{\mathbf{b}}}(x) = \underline{s+x}, \quad \Gamma'_{\underline{\mathbf{b}}}(y) = \underline{s+y};$$

$$\Gamma_{\mathbf{w}}(0) = \underline{s}, \quad \Gamma_{\mathbf{w}}(l_{\Gamma_{\mathbf{w}}}) = \underline{s+x-1}.$$

Let $f \in \mathrm{Map}(P(\Gamma'(\mathbf{j})), P(\Gamma(\mathbf{i})))$ be a generator such that $f \circ d_{\underline{\mathbf{b}}} = 0$. If $\mathbf{j}_{\underline{\mathbf{b}}} < y$, then $f \circ d_{\underline{\mathbf{b}}}$ is a map $P(\Gamma'(\mathbf{j}')) \to P(\Gamma(\mathbf{i}))$, where $\mathbf{j} = \underline{\mathbf{b}}|\mathbf{j}'$. We have

$$\mathrm{Hom}(\Gamma'(\mathbf{j}), \Gamma(\mathbf{i})) \neq 0, \quad \mathrm{Hom}(\Gamma'(\mathbf{j}'), \Gamma(\mathbf{i})) = 0; \qquad \Gamma'(\mathbf{j}') \xrightarrow{s+\mathbf{j}_{\underline{\mathbf{b}}}} \Gamma'(\mathbf{j}).$$

By the tightness criterion (Proposition 5.1), in order for $\mathrm{Hom}(\Gamma'(\mathbf{j}'), \Gamma(\mathbf{i})) = 0$, $\mathrm{Hom}(\Gamma'(\mathbf{j}), \Gamma(\mathbf{i}))$ must factor through $\mathrm{Hom}(\Gamma'(\mathbf{j}), \Gamma'(\mathbf{j}''))$, where $\mathbf{j}'' = \underline{\mathbf{b}}|\mathbf{j} \in OI(\Gamma')$ such that $\Gamma'(\mathbf{j}) \xrightarrow{s+\mathbf{j}_{\underline{\mathbf{b}}}-1} \Gamma'(\mathbf{j}'')$. Hence $\mathrm{Hom}(\Gamma'(\mathbf{j}''), \Gamma(\mathbf{i})) \neq 0$ is generated by g such that $f = g \circ d_{\underline{\mathbf{b}}}$.

If $\mathbf{j_b} = y$ and $\mathrm{Hom}(\Gamma'(\underline{\mathbf{b}}|\mathbf{j}), \Gamma(\mathbf{i})) \neq 0$ is generated by g, then $f = g \circ d_{\underline{\mathbf{b}}}$. If $\mathbf{j_b} = y$ and $\mathrm{Hom}(\Gamma'(\underline{\mathbf{b}}|\mathbf{j}), \Gamma(\mathbf{i})) = 0$, then $\{s, s+1, \ldots, s+y-1\} \subset \Gamma(\mathbf{i})_*$. In particular $\Gamma_{\mathbf{w}} \subset \Gamma(\mathbf{i})_*$ which is not possible since $\mathbf{w} \neq *$. This proves the claim. $\qquad\square$

Case 2. If $l_{\Gamma'}((\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))) > 0$, then there exists $\mathbf{v}' \in V^+_{nb}(\Gamma')$ such that $\Gamma'_{\mathbf{v}'} \subset (\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))$ and no vector in $V^+_{nb}(\Gamma')$ nests inside $\mathbf{v}'$ (i.e., $\mathbf{v}'$ is "outermost"). There is a corresponding vector $\mathbf{v} \in V^+_{nb}(\Gamma)$ such that $\Gamma_{\mathbf{v}} = \Gamma'_{\mathbf{v}'}$ and no vector in $V^+_{nb}(\Gamma)$ nests inside $\mathbf{v}$; see the right-hand side of Figure 35. Let $\widehat{\Gamma}, \widehat{\Gamma}'$ denote $\widehat{\Gamma}^{\mathbf{v}}, \widehat{\Gamma}'^{\mathbf{v}'}$. There is a bypass triangle $\widehat{\Gamma} \xrightarrow{\widehat{\beta}} \widehat{\Gamma}' \xrightarrow{\widehat{\beta}'} \widehat{\Gamma}''$. Let $\underline{\widehat{\mathbf{b}}}, \widehat{x}, \widehat{y}$ denote $\underline{\mathbf{b}}(\widehat{\beta}'), x(\widehat{\beta}'), y(\widehat{\beta}')$. Then

$$l_{\widehat{\Gamma}'}((\widehat{\Gamma}'_{\underline{\widehat{\mathbf{b}}}}(0), \widehat{\Gamma}'_{\underline{\widehat{\mathbf{b}}}}(\widehat{y}))) < l_{\Gamma'}((\Gamma'_{\underline{\mathbf{b}}}(0), \Gamma'_{\underline{\mathbf{b}}}(y))).$$

The following claim implies the lemma for Case 2.

Claim. *There is a finite double complex* $(\mathrm{Map}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma)); d_{\mathbf{v}, \mathbf{v}'}, d_{\Gamma, \Gamma'} - d_{\mathbf{v}, \mathbf{v}'})$, *where* $d_{\mathbf{v}, \mathbf{v}'} f = d_{\mathbf{v}} \circ f + f \circ d_{\mathbf{v}'}$, *whose first page with respect to* $d_{\mathbf{v}, \mathbf{v}'}$ *is isomorphic to* $(\mathrm{Map}(\mathcal{F}(\widehat{\Gamma}'), \mathcal{F}(\widehat{\Gamma})); d_{\widehat{\Gamma}, \widehat{\Gamma}'})$.

Proof of Claim. If $\mathbf{j_b} < x$, then there exists $\mathbf{i}' \in OI(\Gamma)$ such that $\Gamma(\mathbf{i}') = \Gamma'(\mathbf{j})$. Such $\mathbf{j} \in OI(\Gamma')$ is said to be of Type (1). Any $f \in \mathrm{Hom}(P(\Gamma'(\mathbf{j})), P(\Gamma(\mathbf{i})))$ can be written as $\prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma'(\mathbf{j})}$ for $\mathbf{u}^t \in V^+_{nb}(\Gamma)$.

If $\mathbf{j_b} \geq x$ and $\mathrm{Hom}(P(\Gamma'(\mathbf{j})), P(\Gamma(\mathbf{i}))) \neq 0$ for some $\mathbf{i}$, then $\mathbf{j_b} = x$ and $\mathbf{b} \in SV(\mathbf{j})$ such that $(\underline{\mathbf{b}}|\mathbf{j})_{\mathbf{b}} = x - 1$. Such $\mathbf{j} \in OI(\Gamma')$ is said to be of Type (2). Then there exists $\mathbf{i}' \in OI(\Gamma)$ such that $\Gamma(\mathbf{i}') = \Gamma'(\underline{\mathbf{b}}|\mathbf{j})$ and any $f \in \mathrm{Hom}(P(\Gamma'(\mathbf{j})), P(\Gamma(\mathbf{i})))$ can be written as $\prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma'(\mathbf{j})} \circ d_{\underline{\mathbf{b}}}$ for $\mathbf{u}^t \in V^+_{nb}(\Gamma)$.

Summarizing, $\mathrm{Map}(\mathcal{F}(\Gamma'), \mathcal{F}(\Gamma))$ has an $\mathbb{F}_2$-basis:

$$\left\{ \prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma'(\mathbf{j})} \,\middle|\, \mathbf{j} \in OI(\Gamma') \text{ of Type (1)}, \mathbf{u}^t \in V^+_{nb}(\Gamma) \right\}$$

$$\bigcup \left\{ \prod_t d_{\mathbf{u}^t} \circ \mathrm{id}_{\Gamma'(\mathbf{j})} \circ d_{\underline{\mathbf{b}}} \,\middle|\, \mathbf{j} \in OI(\Gamma') \text{ of Type (2)}, \mathbf{u}^t \in V^+_{nb}(\Gamma) \right\}.$$

The rest of the proof is similar to that of Lemma 8.1.1.5 and is left to the reader. $\qquad\square$

This completes the proof of Lemma 8.1.2.2. $\qquad\square$

8.2. Serre functors of $\widetilde{\mathcal{D}}_{n,e}$

In this subsection we write $R = R_{n,e}$ for simplicity. According to [Ke, Theorem 3.1], $\mathbf{D}^b(R)$ admits a Serre functor since R has finite global dimension and the Serre functor is the left derived functor

$$M \to DR \otimes_R^{\mathbb{L}} M,$$

where M is a left R-module and DR denotes the R-bimodule $\mathrm{Hom}_{\mathbb{F}_2}(R, \mathbb{F}_2)$. Note that this means that if $r_1, r_2, r \in R$ and $\phi \in \mathrm{Hom}_{\mathbb{F}_2}(R, \mathbb{F}_2)$, then $r_1 \phi r_2(r) = \phi(r_2 r r_1)$. Since $\widetilde{\mathcal{D}}_{n,e}$ is equivalent to $\mathbf{D}^b(R)$, it admits an induced Serre functor which we denote by $\mathcal{S}_{\widetilde{\mathcal{D}}}$. For any projective R-module $P(\Gamma)$, $\Gamma \in B_{n,e}$, $\mathcal{S}_{\widetilde{\mathcal{D}}}(P(\Gamma))$ is isomorphic to a projective resolution of the tensor product $DR \otimes_R P(\Gamma)$.

By definition, DR has a dual $\mathbb{F}_2$-basis $\{[\Gamma'|\Gamma] \mid \mathrm{Hom}(\Gamma, \Gamma') \neq 0, \Gamma, \Gamma' \in B_{n,e}\}$, where the linear map $[\Gamma'|\Gamma] : R \to \mathbb{F}_2$ sends the generator of $\mathrm{Hom}(\Gamma, \Gamma')$ to 1 and other generators to zero. As an R-bimodule, DR has the defining relations:

$$(\Gamma')[\Gamma'|\Gamma] = [\Gamma'|\Gamma](\Gamma) = [\Gamma'|\Gamma].$$

$$(\widetilde{\Gamma}'|\Gamma')[\Gamma'|\Gamma] = \begin{cases} [\widetilde{\Gamma}'|\Gamma] & \text{if } \mathrm{Hom}(\Gamma, \widetilde{\Gamma}') \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

$$[\Gamma'|\Gamma](\Gamma|\widetilde{\Gamma}) = \begin{cases} [\Gamma'|\widetilde{\Gamma}] & \text{if } \mathrm{Hom}(\widetilde{\Gamma}, \Gamma') \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $DR \otimes_R P(\Gamma)$ has an $\mathbb{F}_2$-basis $\{[\Gamma'|\Gamma] \mid \mathrm{Hom}(\Gamma, \Gamma') \neq 0, \Gamma' \in B_{n,e}\}$.

For $\Gamma \in B_{n,e}$, we compute $\mathcal{S}_{\widetilde{\mathcal{D}}}(P(\Gamma)) \cong DR \otimes_R P(\Gamma)$ in terms of $\mathcal{S}(\Gamma)$. If $\underline{1} \in \Gamma_*$ then $V_{nb}^+(\mathcal{S}(\Gamma)) = \emptyset$. If $\underline{1} \notin \Gamma_*$, then $V_{nb}^+(\mathcal{S}(\Gamma)) = \{\mathbf{w}\}, l_{\mathcal{S}(\Gamma)_{\mathbf{w}}} = e$ and $\mathcal{S}(\Gamma)_{\mathbf{w}} = \{\Gamma_*(1) - \underline{1}, \ldots, \Gamma_*(e) - \underline{1}, \underline{n}\}$, and we write

$$(8.2.1) \qquad \mathcal{S}(\Gamma)^i = \mathcal{S}(\Gamma)(\mathbf{i}) \in B_{n,e} \quad \text{for} \quad 0 \leq i \leq e,$$

where $\mathbf{i} \in OI(\mathcal{S}(\Gamma))$ such that $\mathbf{i_w} = i$.

Lemma 8.2.2. *For any $\Gamma \in B_{n,e}$, $\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma)) = \mathcal{S}_{\widetilde{\mathcal{D}}}(P(\Gamma))$ is isomorphic to $\mathcal{F}(\mathcal{S}(\Gamma))$ in $\widetilde{\mathcal{D}}_{n,e}$.*

Proof. The tensor product $DR \otimes_R P(\Gamma)$ has an $\mathbb{F}_2$-basis $\{[\Gamma'|\Gamma] \mid \mathrm{Hom}(\Gamma, \Gamma') \neq 0, \Gamma' \in B_{n,e}\}$.

If $\underline{1} \in \Gamma_*$, then $\mathcal{S}(\Gamma) \in B_{n,e}$. By Lemma 3.3.1.3, $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ if and only if $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)) \neq 0$. Hence $DR \otimes_R P(\Gamma)$ is isomorphic to $P(\mathcal{S}(\Gamma))$, i.e., $\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma))$ is isomorphic to $\mathcal{F}(\mathcal{S}(\Gamma))$.

Assume $\underline{1} \notin \Gamma_*$ from now on. Our proof makes repeated use of Proposition 5.1 and Corollary 5.2. Consider the complex $\mathcal{F}(\mathcal{S}(\Gamma))$:

$$P(\mathcal{S}(\Gamma)^e) \xrightarrow{\mathrm{pr}_e} \cdots \xrightarrow{\mathrm{pr}_2} P(\mathcal{S}(\Gamma)^1) \xrightarrow{\mathrm{pr}_1} P(\mathcal{S}(\Gamma)^0).$$

We have $\mathcal{S}(\Gamma)_*^0 = \{\underline{0}, \Gamma_*(2) - \underline{1}, \ldots, \Gamma_*(e) - \underline{1}, \underline{n}\}$. By Proposition 5.1, $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma)^0) \neq 0$ so that $[\mathcal{S}(\Gamma)^0 | \Gamma] \in DR \otimes_R P(\Gamma)$ exists. Define a map of left R-modules

$$\mathrm{pr}_0 : P(\mathcal{S}(\Gamma)^0) \to DR \otimes_R P(\Gamma)$$

by $\mathrm{pr}_0(\mathcal{S}(\Gamma)^0) = [\mathcal{S}(\Gamma)^0 | \Gamma]$. Moreover, the path from Γ to $\mathcal{S}(\Gamma)^0$ is the longest nonzero path starting from Γ. In other words, $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)^0) \neq 0$ if $\mathrm{Hom}(\Gamma, \Gamma') \neq 0$ for $\Gamma' \in B_{n,e}$. Then $\mathrm{pr}_0(\Gamma' | \mathcal{S}(\Gamma)^0) = [\Gamma' | \Gamma]$ for any generator $[\Gamma' | \Gamma] \in DR \otimes_R P(\Gamma)$. Hence pr_0 is a surjection.

Claim. $\mathrm{Ker}(\mathrm{pr}_0) = \mathrm{Im}(\mathrm{pr}_1)$.

Proof of Claim. Since

$$\mathcal{S}(\Gamma)_*^1 = \{\underline{0}, \Gamma_*(1) - \underline{1}, \Gamma_*(3) - \underline{1}, \ldots, \Gamma_*(e) - \underline{1}, \underline{n}\},$$

it follows that $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma)^1) = 0$ and $\mathrm{Im}(\mathrm{pr}_1) \subset \mathrm{Ker}(\mathrm{pr}_0)$.

To prove $\mathrm{Ker}(\mathrm{pr}_0) \subset \mathrm{Im}(\mathrm{pr}_1)$, it suffices to show that if $\mathrm{Hom}(\Gamma, \Gamma') = 0$ and $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)^0) \neq 0$ for $\Gamma' \in B_{n,e}$, then $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)^1) \neq 0$. Since $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)^0) \neq 0$ we have

$$\Gamma_*(i) - \underline{1} = \mathcal{S}(\Gamma)_*^0(i-1) < \Gamma_*'(i) \leq \mathcal{S}(\Gamma)_*^0(i) = \Gamma_*(i+1) - \underline{1},$$

for $1 < i \leq e$. This implies that $\Gamma_*'(1) < \Gamma_*(1)$ since $\mathrm{Hom}(\Gamma, \Gamma') = 0$. We have

$$\Gamma_*'(1) \leq \Gamma_*(1) - \underline{1},$$
$$\mathcal{S}(\Gamma)_*^1(i-1) \leq \mathcal{S}(\Gamma)_*^0(i-1) < \Gamma_*'(i) \leq \mathcal{S}(\Gamma)_*^0(i) = \mathcal{S}(\Gamma)_*^1(i),$$

for $1 < i \leq e$. Hence $\mathrm{Hom}(\Gamma', \mathcal{S}(\Gamma)^1) \neq 0$. $\square$

The proofs of $\mathrm{Ker}(\mathrm{pr}_i) = \mathrm{Im}(\mathrm{pr}_{i+1})$ for $0 < i \leq e$ are similar, where $\mathrm{Im}(\mathrm{pr}_{e+1})$ is understood to be 0. Hence $\mathcal{F}(\mathcal{S}(\Gamma))$ is a projective resolution of $DR \otimes_R P(\Gamma)$. $\square$

Proposition 8.2.3. *The Serre functors $\mathcal{S}_{\widetilde{\mathcal{C}}}$ and $\mathcal{S}_{\widetilde{\mathcal{D}}}$ commute with $\widetilde{\mathcal{F}}$.*

Proof.

Step 1. We first show that $\mathcal{S}_{\widetilde{\mathcal{C}}}$ and $\mathcal{S}_{\widetilde{\mathcal{D}}}$ commute with $\widetilde{\mathcal{F}}$ on the level of objects, i.e.,

$$(8.2.4) \qquad \mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma, [\xi])) \cong \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma, [\xi])).$$

We prove this by induction on $m(\Gamma) = e + 1 - |\Gamma_*|$.

Suppose $m(\Gamma) = 0$, i.e., $\Gamma \in B_{n,e}$. It suffices to prove Equation (8.2.4) for $\xi = \xi(\Gamma)$ since both sides commute with shift functors. By Lemma 8.2.2,

$$\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma, [\xi(\Gamma)])) = \mathcal{S}_{\widetilde{\mathcal{D}}}(\mathcal{F}(\Gamma)) \cong \mathcal{F}(\mathcal{S}(\Gamma)) = \widetilde{\mathcal{F}}(\mathcal{S}(\Gamma), [\xi(\mathcal{S}(\Gamma))]).$$

By Definition 3.3.2.1, $\widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma, [\xi(\Gamma)])) = \widetilde{\mathcal{F}}(\mathcal{S}(\Gamma), [\zeta(\Gamma) \circ \xi(\Gamma)])$, where $\zeta(\Gamma)$ is the generator of $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma))$. It remains to show that

$$(8.2.5) \qquad [\zeta(\Gamma) \circ \xi(\Gamma)] = [\xi(\mathcal{S}(\Gamma))],$$

for $\Gamma \in B_{n,e}$. If $\mathcal{S}(\Gamma) \in B_{n,e}$, then $\zeta(\Gamma)$ corresponds to a nonzero path from Γ to $\mathcal{S}(\Gamma)$ in $Q_{n,e}$ and Equation (8.2.5) follows from the definition of $[\xi(\mathcal{S}(\Gamma))]$. If $\mathcal{S}(\Gamma) \notin B_{n,e}$, then consider $\overline{\beta}(\mathcal{S}(\Gamma)) \in \mathrm{Hom}(\mathcal{S}(\Gamma)^0, \mathcal{S}(\Gamma))$ as defined in Equation (7.2.1.1). By Lemma 3.3.1.4, the composition $\mathrm{Hom}(\mathcal{S}(\Gamma)^0, \mathcal{S}(\Gamma)) \times \mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma)^0) \to \mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma))$ is nontrivial. Hence

$$\begin{aligned} [\xi(\mathcal{S}(\Gamma))] &= [\overline{\beta}(\mathcal{S}(\Gamma)) \circ \xi(\mathcal{S}(\Gamma)^0)] \\ &= [\overline{\beta}(\mathcal{S}(\Gamma)) \circ \xi_{\Gamma, \mathcal{S}(\Gamma)^0} \circ \xi(\Gamma)] = [\zeta(\Gamma) \circ \xi(\Gamma)], \end{aligned}$$

where $\xi_{\Gamma, \mathcal{S}(\Gamma)^0}$ is the generator of $\mathrm{Hom}(\Gamma, \mathcal{S}(\Gamma)^0)$.

Suppose $m(\Gamma) > 0$. Consider the bypass triangle

$$(\Gamma, [\xi][-1]) \xrightarrow{\beta(\Gamma)} (\Gamma', [\xi']) \xrightarrow{\beta} (\Gamma'', [\xi'']) \to (\Gamma, [\xi])$$

in $\widetilde{\mathcal{C}}_{n,e}$ which contains $\beta(\Gamma)$ as defined in Definition 6.1.2. We have $m(\Gamma'), m(\Gamma'') < m(\Gamma)$. Let $\beta \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_{n,e}}((\Gamma', [\xi']), (\Gamma'', [\xi'']))$ be the second morphism in the triangle. By Proposition 8.1.2.1,

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\widetilde{\mathcal{F}}(\Gamma', [\xi']), \widetilde{\mathcal{F}}(\Gamma'', [\xi''])) = \mathbb{F}_2 \langle \widetilde{\mathcal{F}}(\beta) \rangle.$$

Since $\mathcal{S}_{\widetilde{\mathcal{C}}}$ is the rotation endofunctor of $\widetilde{\mathcal{C}}_{n,e}$, it maps bypass triangles to bypass triangles and

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma', [\xi'])), \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma'', [\xi'']))) = \mathbb{F}_2\langle\widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\beta))\rangle.$$

Also, since $\mathcal{S}_{\widetilde{\mathcal{D}}}$ is an auto-equivalence of $\widetilde{\mathcal{D}}_{n,e}$,

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma', [\xi'])), \mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma'', [\xi'']))) = \mathbb{F}_2\langle\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\beta))\rangle.$$

Finally, since

$$\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma', [\xi'])) \cong \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma', [\xi']))$$

and

$$\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma'', [\xi''])) \cong \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma'', [\xi'']))$$

by induction, we have $\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\beta)) = \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\beta))$. Hence,

$$\begin{aligned}
\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma, [\xi])) &\cong \mathcal{S}_{\widetilde{\mathcal{D}}}(\mathrm{Cone}(\widetilde{\mathcal{F}}(\beta))) \\
&= \mathrm{Cone}(\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\beta))) = \mathrm{Cone}(\widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\beta))) \cong \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma, [\xi])),
\end{aligned}$$

where the first and last isomorphisms follow from Proposition 7.3.1.

Step 2. Since the morphisms of $\widetilde{\mathcal{C}}_{n,e}$ are generated by bypasses, it suffices to prove that $\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\beta)) \cong \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\beta))$ for any bypass $\beta \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_{n,e}}((\Gamma, [\xi]), (\Gamma', [\xi']))$. This in turn follows from observing that both are generators of

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}&(\widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma, [\xi])), \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}(\Gamma', [\xi']))) \\
&= \mathrm{Hom}_{\widetilde{\mathcal{D}}_{n,e}}(\mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma, [\xi])), \mathcal{S}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{F}}(\Gamma', [\xi']))).
\end{aligned}$$

$\square$

We prove the analogue of Lemma 3.3.3.1 for the Serre functor $\mathcal{S}_{\widetilde{\mathcal{D}}}$ of $\widetilde{\mathcal{D}}_{n,e}$.

Proposition 8.2.6. *There is an isomorphism of endofunctors of $\widetilde{\mathcal{D}}_{n,e}$:* $\mathcal{S}_{\widetilde{\mathcal{D}}}^{n+1} \cong T^{e(n-e)}$.

Proof. Since $\widetilde{\mathcal{D}}_{n,e}$ is generated by the image of $\widetilde{\mathcal{F}}$ (and in particular the projectives $P(\Gamma)$, $\Gamma \in B_{n,e}$, and morphisms between projectives), it suffices to show that $\mathcal{S}_{\widetilde{\mathcal{D}}}^{n+1}(\widetilde{\mathcal{F}}(\Gamma, [\xi])) = \widetilde{\mathcal{F}}(\Gamma, [\xi])[e(n-e)]$ for any $(\Gamma, [\xi])$. By Lemma

3.3.3.1 and Proposition 8.2.3,

$$\mathcal{S}_{\widetilde{\mathcal{D}}}^{n+1}(\widetilde{\mathcal{F}}(\Gamma, [\xi])) = \widetilde{\mathcal{F}}(\mathcal{S}_{\widetilde{\mathcal{C}}}^{n+1}(\Gamma, [\xi]))$$
$$= \widetilde{\mathcal{F}}(\Gamma, [\xi][e(n-e)]) = \widetilde{\mathcal{F}}(\Gamma, [\xi])[e(n-e)].$$

$\square$

8.3. General cases

Since $\mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma')$ is at most one-dimensional, $\widetilde{\mathcal{F}}_{n,e}$ is faithful if and only if $\mathcal{F}_{n,e}$ is faithful, i.e.,

(F′) $\mathcal{F}_{n,e} : \mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')).$

By Proposition 8.2.3, (F′) holds for Γ, Γ' if and only if it holds for $\mathcal{S}^k(\Gamma)$, $\mathcal{S}^k(\Gamma')$ for some k.

   We prove Equation (F′) for Γ, Γ' in $\mathcal{C}_{n,e}$ by induction on n. If Γ and Γ' have a common boundary parallel component, then it is either a positive region or a negative region.

Case 1. Suppose that $R_+(\Gamma)$ and $R_+(\Gamma')$ have a common boundary parallel component, i.e., there exist $\mathbf{v} \in V(\Gamma)$ and $\mathbf{v}' \in V(\Gamma')$ such that $\Gamma_{\mathbf{v}} = \Gamma'_{\mathbf{v}'} = \{\underline{t}\}$. By applying the Serre functor $t+1$ times, we can assume that $\Gamma_{\mathbf{v}} = \Gamma'_{\mathbf{v}'} = \{\underline{n}\}$. Let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ denote dividing sets in $\mathcal{C}_{n-1,e}$ obtained from Γ and Γ' by removing $\Gamma_{\mathbf{v}}$ and $\Gamma'_{\mathbf{v}'}$, respectively.

Case 2. Suppose that $R_-(\Gamma)$ and $R_-(\Gamma')$ have a common boundary parallel component, i.e., there exist $\mathbf{v} \in V(\Gamma)$ and $\mathbf{v}' \in V(\Gamma')$ such that $\{\underline{t}, \underline{t+1}\} \subset \Gamma_{\mathbf{v}}, \Gamma'_{\mathbf{v}'}$. By applying the Serre functor $t+1$ times, we can assume that $\{\underline{0}, \underline{n}\} \subset \Gamma_*, \Gamma'_*$. Let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ denote dividing sets in $\mathcal{C}_{n-1,e-1}$ obtained from Γ and Γ' by removing $\underline{n}$ from Γ_* and Γ'_*, respectively.

Lemma 8.3.1. (F′) *holds for* Γ, Γ' *if and only if it holds for* $\widetilde{\Gamma}, \widetilde{\Gamma}'$ *in both Cases 1 and 2.*

Proof. It suffices to prove that there exist canonical isomorphisms:

(8.3.2) $\mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma') \cong \mathrm{Hom}_{\mathcal{C}_{n-1,e}}(\widetilde{\Gamma}, \widetilde{\Gamma}');$

(8.3.3) $\mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')) \cong \mathrm{Hom}_{\mathcal{D}_{n-1,e}}(\mathcal{F}(\widetilde{\Gamma}), \mathcal{F}(\widetilde{\Gamma}')).$

   The first isomorphism (8.3.2) follows from observing that $\gamma_{\Gamma, \Gamma'}$ is isomorphic to $\gamma_{\widetilde{\Gamma}, \widetilde{\Gamma}'}$.

Consider two full sub-quivers $Q'_{n,e}$ and $Q''_{n,e}$ of $Q_{n,e}$, where

$$V(Q'_{n,e}) = \{\Gamma \in B_{n,e} \mid \underline{n} \notin \Gamma_*\}, \quad V(Q''_{n,e}) = \{\Gamma \in B_{n,e} \mid \underline{n} \in \Gamma_*\}.$$

There are two subalgebras $R'_{n,e}$ and $R''_{n,e}$ of $R_{n,e}$ which are generated by $Q'_{n,e}$ and $Q''_{n,e}$, respectively. Let $\mathcal{D}'_{n,e}$ and $\mathcal{D}''_{n,e}$ be the corresponding full subcategories of $\mathcal{D}_{n,e}$. By Lemma 5.5, $R'_{n,e}$ is canonically isomorphic to $R_{n-1,e}$, and $R''_{n,e}$ is canonically isomorphic to $R_{n-1,e-1}$. The second isomorphism (8.3.3) follows from compositions of functors: $\mathcal{D}_{n-1,e} \xrightarrow{\sim} \mathcal{D}'_{n,e} \hookrightarrow \mathcal{D}_{n,e}$ and $\mathcal{D}_{n-1,e-1} \xrightarrow{\sim} \mathcal{D}''_{n,e} \hookrightarrow \mathcal{D}_{n,e}$.   □

Before proving (F′) in general, consider the special cases described in Figure 36. There are two boundary parallel components, one in $R_{\pm}(\Gamma)$ and the other in $R_{\mp}(\Gamma')$. The boundary parallel component of Γ' is obtained by rotating that of Γ through a counterclockwise angle of $\frac{\pi}{n+1}$. We say that the pair (Γ, Γ') is in *local annihilation position*.

Figure 36: A pair (Γ, Γ') in local annihilation position, normalized using the Serre functor.

Lemma 8.3.4. *If the pair (Γ, Γ') is in local annihilation position, then*

$$\mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma') = 0, \qquad \mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')) = 0.$$

Proof. Since the two boundary parallel components form a loop after edge rounding, $\#\gamma_{\Gamma,\Gamma'} > 1$ and $\mathrm{Hom}_{\mathcal{C}_{n,e}}(\Gamma, \Gamma') = 0$.

By applying the Serre functor we are in one of the following two cases as in Figure 36:

1) The boundary parallel components are in $R_+(\Gamma)$ and $R_-(\Gamma')$: there exists $\mathbf{v} \in V(\Gamma)$ such that $\Gamma_\mathbf{v} = \{\underline{1}\}$; and $\underline{1} \in \Gamma'_*$.

2) The boundary parallel components are in $R_-(\Gamma)$ and $R_+(\Gamma')$: there exists $\mathbf{v}' \in V(\Gamma')$ such that $\Gamma'_{\mathbf{v}'} = \{\underline{n}\}$; and $\underline{n} \in \Gamma_*$.

For any $\mathbf{i} \in OI(\Gamma), \mathbf{j} \in OI(\Gamma')$, $\underline{1} \notin \Gamma(\mathbf{i})$ and $\underline{1} \in \Gamma'(\mathbf{j})$ in the first case and $\underline{n} \in \Gamma(\mathbf{i})$ and $\underline{n} \notin \Gamma'(\mathbf{j})$ in the second case. In either case $\mathrm{Hom}(\Gamma(\mathbf{i}), \Gamma'(\mathbf{j})) = 0$ by Proposition 5.1. Hence $\mathrm{Hom}_{\mathcal{D}_{n,e}}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')) = 0$.   □

We are finally in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We show that (F′) holds for any Γ, Γ' by induction on n. For any boundary parallel component of Γ', consider the neighborhood of the component in Γ' as on the right-hand side of Figure 37: there are three endpoints r, s, t of Γ' in clockwise order around ∂D^2 and Γ' connects r and s. We may assume that Γ does not connect r and s since if Γ and Γ' have a common boundary parallel component then we can reduce n by Lemma 8.3.1; and that Γ does not connect s and t since if (Γ, Γ') is in local annihilation position then we are done by Lemma 8.3.4. Hence there exists a nontrivial bypass triangle $\Gamma \xrightarrow{\beta} \widetilde{\Gamma} \to \Gamma^0 \to \Gamma$ such that (Γ^0, Γ') is in local annihilation position; see Figure 37.

By applying exact functors $\mathrm{Hom}(-, \Gamma')$ (this is exact by Lemma 3.2.4) and $\mathrm{Hom}(-, \mathcal{F}(\Gamma'))$, we have two isomorphisms:

$$\mathrm{Hom}(\widetilde{\Gamma}, \Gamma') \xrightarrow{\circ\beta} \mathrm{Hom}(\Gamma, \Gamma'), \quad \mathrm{Hom}(\mathcal{F}(\widetilde{\Gamma}), \mathcal{F}(\Gamma')) \xrightarrow{\circ\mathcal{F}(\beta)} \mathrm{Hom}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma')),$$

since $\mathrm{Hom}(\Gamma^0, \Gamma') = 0$ and $\mathrm{Hom}(\mathcal{F}(\Gamma^0), \mathcal{F}(\Gamma')) = 0$ by Lemma 8.3.4.
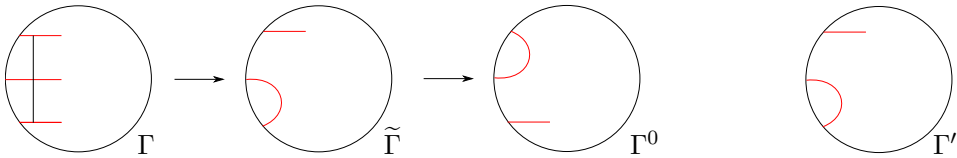

Figure 37.

Since $\widetilde{\Gamma}$ and Γ' have a common boundary parallel component we can reduce n by Lemma 8.3.1. In the case where $n = 2$, there are 5 dividing sets in $\mathcal{C}_2$, and Γ and Γ' are either the same or in the unique bypass triangle in $\mathcal{C}_{2,1}$. The first part of Theorem 1.1 follows from Propositions 8.1.1.1 and 8.1.2.1. The assertion about exact triangles was the content of Proposition 7.3.1. □

Index of notation

Section 2

Γ : a dividing set

$R_+(\Gamma)$: the positive region of a convex surface Σ with dividing set Γ

$R_+(F)$: the positive region of $\partial\Sigma$ with respect to the marked points $F \subset \partial\Sigma$

$\delta = \delta_+ \cup \delta_-$: the arc of a bypass attachment as a union of its positive and negative parts

Section 3

$\mathcal{C}_{n,e}$: the (skeletal version of the) contact category of a disk

$\widetilde{\mathcal{C}}_{n,e}$: the universal cover of the (skeletal version of the) contact category of a disk

$\chi_+ = \chi_+(\Gamma), \chi_- = \chi_-(\Gamma)$: the Euler characteristics of $R_+(\Gamma)$ and $R_-(\Gamma)$.

$\underline{s}$: the label s of a positive arc in ∂D^2

$\mathcal{S}, \mathcal{S}_{\widetilde{\mathcal{C}}}$: Serre functors of $\mathcal{C}_{n,e}$ and $\widetilde{\mathcal{C}}_{n,e}$

Section 4

$\pi_0(R_+(\Gamma)), \pi_0(R_-(\Gamma))$: the set of components of $R_+(\Gamma)$ and $R_-(\Gamma)$

$\mathbb{V}$: the set of vectors of positive integers

$*$: the special element of $\mathbb{V}$

$\Phi_\Gamma : \pi_0(R_+(\Gamma)) \to \mathbb{V}$ the assignment of components of $R_+(\Gamma)$ by vectors in $\mathbb{V}$

$V(\Gamma) = \mathrm{Im}(\Phi_\Gamma)$: the set of vectors of Γ

$V^+(\Gamma) = V(\Gamma)\backslash\{*\}$

$V_{nb}^+(\Gamma)$: the subset of non-boundary-parallel components of $V^+(\Gamma)$

$B_{n,e}$: the set of basic dividing sets

$\Gamma(\underline{s_1}, \ldots, \underline{s_e})$: the basic dividing set $\Gamma \in B_{n,e}$ such that $\Gamma_* = \{\underline{0}, \underline{s_1}, \ldots, \underline{s_e}\}$.

$\Gamma_{\mathbf{v}}$: the $\mathbf{v}$-component of $R_+(\Gamma)$, where $\mathbf{v} \in V(\Gamma) = \mathrm{Im}(\Phi_\Gamma)$; the set of labels contained in the
    $\mathbf{v}$-component of $R_+(\Gamma)$

Γ_* : the based component of $R_+(\Gamma)$ containing the label $\underline{0}$

$\Gamma_{\mathbf{v}}(i)$: the ith element of $\Gamma_{\mathbf{v}}$

$l_{\Gamma_{\mathbf{v}}} = |\Gamma_{\mathbf{v}}| - 1$

β : a bypass attachment; later β will also be a map $V(\Gamma) \to V(\Gamma')$ (cf. Equation 4.2.3) and a
    map $II(\beta) \sqcup SI(\beta) \to OI(\Gamma')$ (cf. Definition 6.3.2.2)

$\underline{\mathbf{b}}(\beta), \overline{\mathbf{b}}(\beta)$: elements of $V(\Gamma)$, cf. Notation 4.2.1

$x(\beta), y(\beta)$: elements of $\{0, \ldots, l_{\Gamma_{\underline{\mathbf{b}}(\beta)}}\}$, cf. Notation 4.2.1

$z(\beta)$: element of $\{0, \ldots, l_{\Gamma_{\overline{\mathbf{b}}(\beta)}}\}$, cf. Notation 4.2.1

$\Gamma^l_{\underline{\mathbf{b}}(\beta)}, \Gamma^r_{\underline{\mathbf{b}}(\beta)}$: left and right subsets of $\Gamma_{\underline{\mathbf{b}}(\beta)}$, cf. Notation 4.2.2

Section 5

$Q_{n,e}$: the quiver

$R_{n,e}$: the $\mathbb{F}_2$-algebra

$\Gamma \xrightarrow{i} \Gamma'$: an arrow in the quiver $Q_{n,e}$

$(\Gamma), (\Gamma|\Gamma')$: generators of the algebra $R_{n,e}$

$P(\Gamma)$: left projective $R_{n,e}$-module corresponding to $\Gamma \in B_{n,e}$

$\widehat{\mathcal{D}}_{n,e}$: the homotopy category of bounded complexes of finitely projective $R_{n,e}$-modules

$\mathcal{D}_{n,e}$: ungraded version of $\widetilde{\mathcal{D}}_{n,e}$

Section 6

$\mathcal{F}_{n,e} : \mathcal{C}_{n,e} \to \mathcal{D}_{n,e}$ and $\widetilde{\mathcal{F}}_{n,e} : \widetilde{\mathcal{C}}_{n,e} \to \widetilde{\mathcal{D}}_{n,e}$ functors

$\beta(\Gamma)$: the leftmost bypass on Γ

$\mathbf{i} = \langle \mathbf{i_v} \rangle$: an omitting index with its $\mathbf{v}$-entries

$OI(\Gamma)$: the set of omitting indices of Γ

$\Gamma(\mathbf{i})$: the basic dividing set corresponding to $\mathbf{i} \in OI(\Gamma)$

$c_{\mathbf{v}}(i)$: the nesting degree of i for $\mathbf{v} \in V(\Gamma)$

$h(\mathbf{i})$: the cohomological degree of $\mathbf{i} \in OI(\Gamma)$

$NV(\mathbf{v}, i)$: the set of nesting vectors inside $\mathbf{v}$ up to i

$DNV(\mathbf{v}, i)$: the set of direct nesting vectors inside $\mathbf{v}$ between $i - 1$ and i

$SLV(\mathbf{i})$: the set of sliding vectors of $\mathbf{i}$

$SHV(\mathbf{i})$: the set of shuffling vectors of $\mathbf{i}$

$SV(\mathbf{i})$: $SLV(\mathbf{i}) \cup SHV(\mathbf{i})$

$\mathbf{v}|\mathbf{i}$: $\mathbf{v}$-modified omitting index, cf. Definition 6.2.11

$\mathfrak{c}|\mathbf{i}$: $\mathfrak{c}$-modified omitting index, where $\mathfrak{c}$ is a component of $\pi_0(R_-(\Gamma))$

$r(\mathbf{i}, \mathbf{v})$: a nonzero element of $R_{n,e}$ corresponding to a path from $\Gamma(\mathbf{i})$ to $\Gamma(\mathbf{v}|\mathbf{i})$ in $Q_{n,e}$

$d(\mathbf{i}, \mathbf{v}) : P(\Gamma(\mathbf{i})) \to P(\Gamma(\mathbf{v}|\mathbf{i}))$ given by right multiplication by $r(\mathbf{i}, \mathbf{v})$

$r(\mathbf{i}, \mathfrak{c}), d(\mathbf{i}, \mathfrak{c})$ defined analogously

$d_{\mathbf{v}}, d_{\mathfrak{c}}$: components of differential $d = d_{\Gamma}$ for $\mathcal{F}(\Gamma)$

$LSV(\beta)$: the set of left shuffling vectors of β

$II(\beta)$: the set of omitting indices of type (Id) for β

$SI(\beta)$: the set of omitting indices of type (Sh) for β

$t(\beta, \mathbf{i})$: a nonzero element of $R_{n,e}$ corresponding to a path from $\Gamma(\mathbf{i})$ to $\Gamma'(\beta(\mathbf{i}))$ in $Q_{n,e}$

Section 7

$[\xi(\Gamma)]$: a homotopy class of Γ

$\overline{\beta}(\Gamma)$: a nontrivial bypass to Γ

$l_\Gamma(A)$: the sum of $l_{\Gamma_\mathbf{v}}$ for $\Gamma_\mathbf{v} \subset A$

$P_i(\beta)$: six parts of ∂D^2 for β

Section 8

$\widehat{\Gamma}^\mathbf{v}$: a dividing set associated to Γ and $\mathbf{v} \in V_{nb}^+(\Gamma)$; also written as $\widehat{\Gamma}$ if $\mathbf{v}$ is understood

$\widehat{\mathbf{v}}$: bijection $V_{nb}^+(\Gamma)\backslash\{\mathbf{v}\} \xrightarrow{\sim} V_{nb}^+(\widehat{\Gamma})$; also denotes the induced map $\widehat{\mathbf{v}}$: $OI(\Gamma) \to OI(\widehat{\Gamma})$

$\mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$: $R_{n,e}$-module maps where $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Gamma')$ are viewed as $R_{n,e}$-modules

$d_{\mathbf{w},\mathbf{v}}f = d_\mathbf{w} \circ f + f \circ d_\mathbf{v}$, where $f \in \mathrm{Map}(\mathcal{F}(\Gamma), \mathcal{F}(\Gamma'))$ and $\mathbf{v} \in OI(\Gamma), \mathbf{w} \in OI(\Gamma')$

$d_{\emptyset,\mathbf{v}}f = f \circ d_\mathbf{v}$, $d_{\mathbf{w},\emptyset}f = d_\mathbf{w} \circ f$, $d_{\Gamma',\Gamma}f = d_{\Gamma'} \circ f + f \circ d_\Gamma$

$\mathcal{S}_{\widetilde{\mathcal{D}}}$: Serre functor of $\widetilde{\mathcal{D}}_{n,e}$

DR : R-bimodule $\mathrm{Hom}_{\mathbb{F}_2}(R, \mathbb{F}_2)$

$[\Gamma'|\Gamma]$: generators of the R-bimodule DR

$\mathcal{S}(\Gamma)^i$: a basic dividing set representing $\mathcal{S}(\Gamma)$

References

[ASS] I. Assem, D. Simson and A. Skowronski, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[BK] A. Bondal and M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk. SSSR Ser. Mat. **53** (1989), 1183–1205.

[Co] B. Cooper, *Formal contact categories*, preprint 2015. `arXiv:1511.04765`.

[El1] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623–637.

[Gi1] E. Giroux, *Structures de contact sur les variétés fibrées en cercles au-dessus d'une surface*, Comment. Math. Helv. **76** (2001), 218–262.

[Gi2] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), 615–689.

[H1] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368.

[H2] K. Honda, *Gluing tight contact structures*, Duke Math. J. **115** (2002), 435–478.

[HKM] K. Honda, W. Kazez and G. Matić, *Pinwheels and bypasses*, Algebr. Geom. Topol. **5** (2005), 769–784.

[Hu] Y. Huang, *Bypass attachments and homotopy classes of 2-plane fields in contact topology*, J. Symplectic Geom. **12** (2014), 599–617.

[GH] V. Gripp and Y. Huang, *A topological grading on bordered Heegaard Floer homology*, Quantum Topol. **6** (2015), 403–449.

[Ke] B. Keller, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, 467–489, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008.

[Ku] A. Kuznetsov, *Calabi-Yau and fractional Calabi-Yau categories*, J. Reine Angew. Math. **753** (2019), 239–267.

[LOT] R. Lipshitz, P. Ozsváth and D. Thurston, *Bordered Heegaard Floer homology*, Mem. Amer. Math. Soc. **254** (2018), no. 1216.

[M1] D. Mathews, *Chord diagrams, contact-topological quantum field theory and contact categories*, Algebr. Geom. Topol. **10** (2010), 2091–2189.

[M2] D. Mathews, *Sutured Floer homology, sutured TQFT and noncommutative QFT*, Algebr. Geom. Topol. **11** (2011), 2681–2739.

[M3] D. Mathews, *Itsy bitsy topological field theory*, Ann. Henri Poincaré **15** (2014), 1801–1865.

[M4] D. Mathews, *Strand algebras and contact categories*, Geom. Topol. **23** (2019), 637–683.

[Ti] B. Tian, in preparation.

[T1] Y. Tian, *A categorification of $\mathbf{U}_T(\mathfrak{sl}(1|1))$ and its tensor product representations*, Geom. Topol. **18** (2014), 1635–1718.

[T2] Y. Tian, *A diagrammatic categorification of a Clifford algebra*, Int. Math. Res. Notices (2015), **21**, 10872–10928.

[T3] Y. Tian, *Categorification of Clifford algebras and $\mathbf{U}_q(\mathfrak{sl}(1|1))$*, J. Symplectic Geom. (2016) **14**, 541–585.

[T4] Y. Tian, *Towards a categorical boson-fermion correspondence*, Adv. Math. **365** (2020), 107034.

[Za] R. Zarev, *Bordered Floer homology for sutured manifolds*, preprint 2009. `arXiv:0908.1106`.

Department of Mathematics, University of California, Los Angeles
Los Angeles, CA 90095, USA
E-mail address: `honda@math.ucla.edu`

Yau Mathematical Sciences Center, Tsinghua University
Beijing 100084, China
E-mail address: `yintian@tsinghua.edu.cn`