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Twisted cyclic group actions on Fukaya categories and mirror symmetry

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Let (X,ω) be a compact symplectic manifold whose first Chern class $c_1(X)$ is divisible by a positive integer n. We construct a twisted $\mathbb{Z}_{2n}$-action on its Fukaya category $Fuk(X)$ and a $\mathbb{Z}_n$-action on the local models of its moduli of Lagrangian branes. We show that this action is compatible with the gluing functions for different local models.

1. Introduction

Let (X,ω) be a compact symplectic manifold. We study the following objects:

1) the Fukaya category $Fuk(X)$ of X; and

2) the moduli $\mathcal{M}$ of Lagrangian branes on X with the superpotential W.

This paper establishes results about cyclic group actions on these objects which arise from the divisibility of the first Chern class $c_1(X)$.

Assumption (A). The first Chern class $c_1(X) \in H^2(X;\mathbb{Z})$ is divisible by a positive integer n, i.e. there exists $\alpha \in H^2(X;\mathbb{Z})$ such that $c_1(X) = n\cdot\alpha$.

Let us begin with (1). Let ζ be a complex number. Define a ζ-*twisted* A_∞ *functor* on $Fuk(X)$, or simply a *twisted* A_∞ *functor*, to be an A_∞ functor of the form

$$\Phi : Fuk(X) \to Fuk(X)_{(\zeta)}$$

where $Fuk(X)_{(\zeta)}$ is the A_∞ category whose objects and morphism spaces are the same as those of $Fuk(X)$, and whose A_∞ product $(m_{(\zeta)})_k$ is defined by

$$(m_{(\zeta)})_k := \zeta^{k-2} m_k,\ k \geqslant 0$$

where m_k is the A_∞ product of $Fuk(X)$. Clearly, a twisted A_∞ functor can also be regarded as an A_∞ functor $Fuk(X)_{(\zeta^i)} \to Fuk(X)_{(\zeta^{i+1})}$ for any $i \in \mathbb{Z}$.

Theorem 1.1. *Put $\zeta = e^{\frac{2\pi i}{2n}}$. A choice of α in Assumption (A) induces a ζ-twisted A_∞ functor Φ on $Fuk(X)$ whose $(2n)$-th power is A_∞ homotopic to the identity functor $id_{Fuk(X)}$.*

Next we consider (2). Recall that $\mathcal{M}$ comes with the *superpotential* W which arises from the effect by quantum corrections. It is known that local models for $(\mathcal{M}, W)$ are the *weak Maurer-Cartan schemes* $\mathcal{M}_{weak}(L)$ associated to Lagrangians L of X [12, 17, 19]. The deformed m_0 of each $\mathbf{b} \in \mathcal{M}_{weak}(L)$ is by definition equal to the unit class of $CF(L, L)$ multiplied by a constant which is the value of W evaluated at $\mathbf{b}$.

Proposition 1.2. *A choice of α in Assumption (A) induces a $\mathbb{Z}_n$-action on $(\mathcal{M}_{weak}(L), W)$, i.e.*

$$W(\tau \cdot \mathbf{b}) = e^{\frac{2\pi i}{n}} W(\mathbf{b}) \text{ for any } \mathbf{b} \in \mathcal{M}_{weak}(L)$$

where τ is a generator of the $\mathbb{Z}_n$-action on $\mathcal{M}_{weak}(L)$.

In order to give a reasonable structure on $\mathcal{M}$, one has to define gluing functions between the weak Maurer-Cartan schemes associated to two different Lagrangians. This problem has been studied by a lot of people [5, 8, 10, 11, 13, 21, 23, 27, 29]. In this paper, we consider the following effective approach by Fukaya [13] whose idea is now known as the *Fukaya's trick*. For any two Lagrangians L and L' which can be brought from one to the other by an isotopy φ_t. Fix an ω-tame almost complex structure J. Then the count of $(\varphi_t^{-1})_* J$-holomorphic disks bounding L yields the desired gluing function

$$\Psi_{L,L'} : \mathcal{M}_{weak}(L) \dashrightarrow \mathcal{M}_{weak}(L') \tag{1.1}$$

where the dash arrow indicates that this function is defined only on an open subset of the domain. See Section 3.1 for more detail.

Proposition 1.3. *The $\mathbb{Z}_n$-actions on $\mathcal{M}_{weak}(L)$ and $\mathcal{M}_{weak}(L')$ commute with $\Psi_{L,L'}$ in (1.1).*

In other words, our $\mathbb{Z}_n$-action on each $(\mathcal{M}_{weak}(L), W)$ is compatible with the gluing of these local models. Hence, we obtain

Theorem 1.4. *There exists a $\mathbb{Z}_n$-action on $(\mathcal{M}, W)$.*

As an example, assume X is Kähler and has an anticanonical divisor D. Consider an SYZ fibration defined on the complement $X - D$, i.e. a special Lagrangian torus fibration with singularities [30]. By gluing the local models associated to the smooth Lagrangian torus fibers based on the Fukaya's trick, the moduli $\mathcal{M}$ of these Lagrangian tori can thus be given the structure of an analytic variety[1]. The details can be found in the work of Tu [31] under the assumption $W = 0$ and the recent work by Yuan [32] for the general case. See also the work of Abouzaid [1–3] using another approach. In this case $(\mathcal{M}, W)$ is the (uncompactified) SYZ mirror of the pair (X, D) which we denote by $(\check{X}^\circ, W)$.

Corollary 1.5. *There exists a* $\mathbb{Z}_n$*-action on* $(\check{X}^\circ, W)$*.*

Remark 1.6. If X is Fano, then $(\check{X}^\circ, W)$ is usually defined over $\mathbb{C}$ and can be compactified to the complete mirror $(\check{X}, W)$ which is an affine variety by adjoining a codimension-two subvariety (those points arising from the singular fibers). We point out that in this case our $\mathbb{Z}_n$-action on $(\check{X}^\circ, W)$ can be extended to a $\mathbb{Z}_n$-action on the complete mirror $(\check{X}, W)$ by the *second Riemann extension theorem*[2]. See Section 4 for more detail.

Remark 1.7. As pointed out by Kuznetsov and Smirnov [24, 25], the existence of a $\mathbb{Z}_n$-action on $(\check{X}, W)$ may be interpreted, via the homological mirror symmetry [22], as the mirror of the existence of a $\mathbb{Z}_n$-action on the *Lefschetz decomposition* of the derived category $D^b(X)$ of coherent sheaves on X. Our results give an A-side interpretation of this phenomenon.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we recall the Fukaya's trick and the definition of $\mathcal{M}_{weak}(L)$, and prove Proposition 1.2 and 1.3. In Section 4, we fill in the details for the claim made in Remark 1.6.

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[1]In general, $\mathcal{M}$ is a *rigid analytic space*, a notion first brought by Kontsevich and Soibelman [23] into the picture of mirror symmetry.

[2]It states that if U is an open subset of a normal complex analytic variety Y whose complement $Y - U$ can be locally covered by closed analytic subvarieties of codimension at least two, then every holomorphic function on U extends to a unique holomorphic function on Y.

26]. C. H. Chow also thanks Kaoru Ono for useful comments and for teaching him a lot from the monumental book [17], Cheuk Yu Mak and Weiwei Wu for helpful discussions as well as Junwu Tu and Siu-Cheong Lau for useful email conversations.

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2. Action on $Fuk(X)$

In this paper, we will not work with a particular version of Fukaya category because, as we will see, our twisted A_∞ functor Φ modifies only the local system carried by each object, and hence it does not depend on which approach adopted to handle the issues arising from the moduli spaces of holomorphic disks. We believe that readers can easily apply our ideas to the version they are using.

Nevertheless, we will recall in Appendix A the least amount of features of $Fuk(X)$ which are necessary in order to explain the construction of Φ. For example, the objects of $Fuk(X)$ consist of $\mathbb{L} = (L, \mathcal{E})$ where L is an immersed Lagrangian of X with clean self-intersection and $\mathcal{E}$ is a $\mathbb{C}^\times$-local system on L, and the morphism space between two cleanly intersecting objects $\mathbb{L}_i = (L_i, \mathcal{E}_i),\ i = 0, 1$ is defined by

$$Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_1) := \bigoplus_{C \in \pi_0(L_0 \times_\iota L_1)} \Omega^\bullet(C; Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C)) \tag{2.1}$$

where $\Omega^\bullet(C; \mathcal{E})$ is any of the standard models (de Rham, singular cochain, etc) of the cohomology group $H^\bullet(C; \mathcal{E})$ with local coefficient $\mathcal{E}$.

Here is the idea of the construction of Φ. We assign to each Lagrangian L of X a particular $\mathbb{Z}_{2n}$-local system $\mathcal{E}_L$ and to each connected component C of the fiber product $L_0 \times_\iota L_1$ a flat section f_C of $Hom(\mathcal{E}_{L_0}|_C, \mathcal{E}_{L_1}|_C)$. Then $\mathcal{E}_L$ and f_C will contribute to the object and morphism parts of Φ respectively. The key is to show that Φ satisfies the twisted version of A_∞ equations.

Remark 2.1. The idea of twisting objects by particular $\mathbb{C}^\times$-local systems has been used by Fukaya in his early work [14]. They are the restriction of a prequantum line bundle on the ambient manifold. See also [18] and [26] for other applications of these local systems. Recently, Auroux and Smith [7] constructed group actions on the Fukaya categories of Riemann surfaces using ambient local systems.

We point out that our local system $\mathcal{E}_L$ is different from theirs, as it cannot be extended to an ambient one and it is equal to Fukaya's one only when X and L are *monotone*.

Remark 2.2. Regarding group actions on Fukaya categories, we also mention the work [9] of Cho and Hong. But their action is not constructed in the above fashion, i.e. twisting objects by local systems.

2.1. Object level

Let $m = \frac{1}{2}\dim(X)$. Notice that the unitary group $U(m)$ admits a unique $\mathbb{Z}_n$-covering group $\widetilde{U(m)}$, i.e. there is a short exact sequence

$$0 \to \mathbb{Z}_n \to \widetilde{U(m)} \to U(m) \to 1.$$

By Assumption (A), $c_1(X) = n \cdot \alpha$ for some $\alpha \in H^2(X;\mathbb{Z})$. It follows that α induces a reduction of the structure group of the frame bundle of X from $U(m)$ to $\widetilde{U(m)}$. Let $LG(m)$ be the Lagrangian Grassmannian of $(\mathbb{C}^m, \omega_{std})$ which has fundamental group $\mathbb{Z}$. We know that

$$LG(m) \simeq U(m)/O(m)$$

is a symmetric space of $U(m)$, or $U(m)/\{\pm 1\}$. As $\widetilde{U(m)}$ is a $\mathbb{Z}_{2n}$-cover of $U(m)/\{\pm 1\}$, it acts on the $\mathbb{Z}_{2n}$-cover $LG'(m)$ of $LG(m)$. Since the reduced frame bundle has structure group $\widetilde{U(m)}$, we have

Lemma 2.3. *[28, Lemma 2.2] The Lagrangian Grassmannian bundle $\mathcal{L}_X := LG(TX, \omega)$ on X admits a fiberwise cover $\mathcal{L}'_X \to \mathcal{L}_X$ with deck transformation group isomorphic to $\mathbb{Z}_{2n}$. It depends on the choice of α in Assumption (A).*

Let $\iota : L \to X$ be a Lagrangian immersion with clean self-intersection. The *Gauss map* of L is a section θ_L of $\iota^*\mathcal{L}_X$ defined by

$$\theta_L(x) := d\iota(T_xL) \in (\mathcal{L}_X)_{\iota(x)}, \ x \in L. \tag{2.2}$$

The inverse image of the subspace $\theta_L(L) \subseteq \iota^*\mathcal{L}_X$ under the fiberwise covering map $\iota^*\mathcal{L}'_X \to \iota^*\mathcal{L}_X$ is then a $\mathbb{Z}_{2n}$-local system on L, which we denote by $\mathcal{E}_L$. We may regard $\mathcal{E}_L$ as a $\mathbb{C}^\times$-local system via the inclusion $\mathbb{Z}_{2n} \hookrightarrow \mathbb{C}^\times$: $1 \pmod{2n} \mapsto \zeta := e^{\frac{2\pi i}{2n}}$.

Definition 2.4. Let $\mathbb{L} = (L, \mathcal{E})$ be an object of $Fuk(X)$. Define

$$\Phi(\mathbb{L}) := (L, \mathcal{E} \otimes \mathcal{E}_L).$$

Remark 2.5. If L is oriented, then θ_L has a lift in the fiberwise double cover of $\iota^*\mathcal{L}_X$ which lies between $\iota^*\mathcal{L}_X$ and $\iota^*\mathcal{L}'_X$. It follows that the $\mathbb{Z}_{2n}$-local system $\mathcal{E}_L$ is reduced to a $\mathbb{Z}_n$-local system, and hence $\Phi^n(\mathbb{L}) = \mathbb{L}$. Similarly, if L is $\mathbb{Z}_{2n}$-graded with respect to the fiberwise covering map $\mathcal{L}'_X \to \mathcal{L}_X$, i.e. θ_L has a lift in $\iota^*\mathcal{L}'_X$, then $\Phi(\mathbb{L}) = \mathbb{L}$.

Remark 2.6. Suppose X is Kähler and L is a special Lagrangian in the complement of an anti-canonical divisor D (say defined by a section s of K_X^{-1}). Then $\mathcal{E}_L$ is the restriction of a local system on $X \setminus D$, namely, the inverse image of $s|_{X\setminus D}$ with respect to the fiberwise covering map $K' \to K_X^{-1}$ (branched along the zero section) where K' is the complex line bundle with $c_1(K') = \alpha$ where α is given in Assumption (A). The reason is that since L is a special Lagrangian, the two non-vanishing sections of K_X^{-1}, $s|_L$ and the Gauss map of L, are homotopic through non-vanishing sections.

2.2. Morphism level

The linear part Φ_1 of our twisted A_∞ functor Φ is of the form

$$\Phi_1 : Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_1) \to Hom_{Fuk(X)}(\Phi(\mathbb{L}_0), \Phi(\mathbb{L}_1)).$$

By (A.1), the morphism space between two objects $\mathbb{L}_i = (L_i, \mathcal{E}_i),\ i = 0, 1$ of $Fuk(X)$ is given by

$$Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_1) = \bigoplus_{C \in \pi_0(L_0 \times_\iota L_1)} \Omega^\bullet(C; Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C)). \tag{2.3}$$

Then

$$\begin{aligned} &Hom_{Fuk(X)}(\Phi(\mathbb{L}_0), \Phi(\mathbb{L}_1)) \\ &= \bigoplus_{C \in \pi_0(L_0 \times_\iota L_1)} \Omega^\bullet(C; Hom((\mathcal{E}_0 \otimes \mathcal{E}_{L_0})|_C, (\mathcal{E}_1 \otimes \mathcal{E}_{L_1})|_C)) \\ &\simeq \bigoplus_{C \in \pi_0(L_0 \times_\iota L_1)} \Omega^\bullet(C; Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C) \otimes Hom(\mathcal{E}_{L_0}|_C, \mathcal{E}_{L_1}|_C)). \end{aligned} \tag{2.4}$$

Define the family version of the *canonical short path* [6], θ_t^C, taking $\theta_{L_0}|_C$ to $\theta_{L_1}|_C$ through sections of $\iota^*\mathcal{L}_X$ over C as follows. Consider the symplectic vector bundle $V_C := TC^{\perp\omega}/TC$ defined on C with the induced symplectic form $[\omega]$. Its associated Lagrangian Grassmannian bundle $\mathcal{L}_C := LG(V_C, [\omega])$ embeds canonically into $\mathcal{L}_X|_C$ through the quotient map $TC^{\perp\omega} \twoheadrightarrow V_C$. Notice that the images of $\theta_{L_0}|_C$ and $\theta_{L_1}|_C$ lie in $\mathcal{L}_C$. Choose a compatible almost complex structure J_C on $(V_C, [\omega])$ such that $J_C \cdot TL_0/TC = TL_1/TC$. Then the desired path θ_t^C is defined by

$$\theta_t^C := e^{-\frac{\pi t}{2}J_C} \cdot TL_0/TC,\ t \in [0,1]. \tag{2.5}$$

Remark 2.7. It can be shown that θ_t^C is independent of J_C up to homotopy.

The lift of θ_t^C with respect to the fiberwise covering map $\mathcal{L}'_X|_C \to \mathcal{L}_X|_C$ gives an isomorphism $s_C : \mathcal{E}_{L_0}|_C \to \mathcal{E}_{L_1}|_C$ of local systems and hence a flat section of $Hom(\mathcal{E}_{L_0}|_C, \mathcal{E}_{L_1}|_C)$ which is denoted by the same notation s_C.

By (2.3) and (2.4), s_C induces a chain isomorphism

$$\begin{aligned} f_C : (\Omega^\bullet(C; Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C)), m_{1,0}) \\ \to (\Omega^\bullet(C; Hom((\mathcal{E}_0 \otimes \mathcal{E}_{L_0})|_C, (\mathcal{E}_1 \otimes \mathcal{E}_{L_1})|_C)), m_{1,0}) . \end{aligned}$$

For later use, we denote by C' the connected component C regarded as an element of $\pi_0(L_1 \times_\iota L_0)$, i.e. considered by interchanging L_0 and L_1. Define $s_{C'}$ and $f_{C'}$ similarly. Notice that in this case, we should start with the canonical short path taking $\theta_{L_1}|_C$ to $\theta_{L_0}|_C$.

Before defining Φ_1, recall that our goal is not to define an A_∞ functor but a twisted one. That means Φ_1 should not commute with $m_{1,0}$ exactly, but commute with it up to a twist. Hence it is natural to introduce the following operator

Definition 2.8. Let $A^\bullet = \bigoplus_{r\in\mathbb{Z}} A^r$ and $B^\bullet = \bigoplus_{r\in\mathbb{Z}} B^r$ be two cochain complexes and $f : A^\bullet \to B^\bullet$ be a chain map. Define a map $f^\zeta : A^\bullet \to B^\bullet$ by

$$f^\zeta|_{A^r} := \zeta^{-r}\mathrm{id}_{A^r}$$

for any $r \in \mathbb{Z}$. Notice that f^ζ is no longer a chain map.

Definition 2.9. Define

$$\Phi_1 := \bigoplus_{C\in\pi_0(L_0\times_\iota L_1)} f_C^\zeta$$

and $\Phi_k := 0$ for $k \geqslant 2$.

2.3. Proof of Theorem 1.1

It is clear from the construction that Φ has order $2n$. Thus it remains to verify that Φ satisfies the twisted A_∞ equations

$$m_k \circ \Phi_1^{\otimes k} = \zeta^{2-k}\Phi_1 \circ m_k,\ k \geqslant 0. \tag{2.6}$$

Let $\overrightarrow{\mathbb{L}} = (\mathbb{L}_0, \ldots, \mathbb{L}_k)$ be a sequence of $k+1$ objects of $Fuk(X)$. Put $\mathbb{L}_{-1} := \mathbb{L}_k$. Suppose for each $i = 0, \ldots, k$, L_{i-1} and L_i intersect cleanly. By (A.3), the A_∞ product map

$$m_k : \bigotimes_{i=1}^{k} Hom_{Fuk(X)}(\mathbb{L}_{i-1}, \mathbb{L}_i) \to Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_k)$$

is equal to the sum $\displaystyle\sum_{\overrightarrow{C},\beta} T^{E(\beta)} m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}$ where

$$m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}} : \bigotimes_{i=1}^{k} \Omega^\bullet(C_i; Hom(\mathcal{E}_{i-1}|_{C_i}, \mathcal{E}_i|_{C_i})) \to \Omega^\bullet(C_0'; Hom(\mathcal{E}_0|_{C_0'}, \mathcal{E}_k|_{C_0'}))$$

is a multilinear map, $\overrightarrow{C} = (C_0, \ldots, C_k)$ with $C_i \in \pi_0(L_{i-1} \times_\iota L_i)$, $\beta \in \pi_2\left(X, \bigcup_{i=0}^k \iota(L_i)\right)$ such that $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J) \neq \emptyset$ and $E(\beta) = \int_\beta \omega$ is the symplectic area.

Put $\Phi(\overrightarrow{\mathbb{L}}) := (\Phi(\mathbb{L}_0), \ldots, \Phi(\mathbb{L}_k))$ and fix $\overrightarrow{C}$. Equation (2.6) will be verified if we show that

$$m_{k,\beta,\Phi(\overrightarrow{\mathbb{L}}),\overrightarrow{C}} \circ \left(\bigotimes_{i=1}^{k} f_{C_i}^\zeta\right) = \zeta^{2-k} f_{C_0'}^\zeta \circ m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}. \tag{2.7}$$

Consider inputs $a_i \in \Omega^{r_i}(C_i, Hom(\mathcal{E}_{i-1}|_{C_i}, \mathcal{E}_i|_{C_i}))$. Then $f_{C_i}^\zeta(a_i) = \zeta^{-r_i} f_{C_i}(a_i)$.

<u>*Case* $(k,\beta) = (1,0)$</u>

In this case, $\overrightarrow{\mathbb{L}} = (\mathbb{L}_0, \mathbb{L}_1)$, $\overrightarrow{C} = (C_0, C_1)$, and it is necessary that $C_0' = C_1$. The LHS of (2.7) is equal to $\zeta^{-r_1} m_{1,0,\Phi(\overrightarrow{\mathbb{L}}),\overrightarrow{C}}(f_{C_1}(a_1))$ and the RHS of (2.7) is equal to $\zeta^{2-1} \cdot \zeta^{-(r_1+1)} f_{C_0'}(m_{1,0,\overrightarrow{\mathbb{L}},\overrightarrow{C}}(a_1))$.

Since $-r_1 = 2 - 1 - (r_1 + 1)$, $C_0' = C_1$ and f_{C_1} is a chain map, (2.7) holds.

<u>*Case* $(k,\beta) \neq (1,0)$</u>

The LHS of (2.7) is equal to $\zeta^{-\sum_{i=1}^{k} r_i} m_{k,\beta,\Phi(\overrightarrow{\mathbb{L}}),\overrightarrow{C}}(f_{C_1}(a_1),\ldots,f_{C_k}(a_k))$ and the RHS of (2.7) is equal to $\zeta^{2-k-r_0} f_{C_0'}(m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}(a_1,\ldots,a_k))$ where r_0 is the degree of $m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}(a_1,\ldots,a_k)$. By A.2,

$$r_0 = \left(\dim(C_0) - \frac{1}{2}\dim(X)\right) + 2 - k - \mu(\beta) + \sum_{i=1}^{k} r_i.$$

In other words, we have to show

$$\begin{aligned} &m_{k,\beta,\Phi(\overrightarrow{\mathbb{L}}),\overrightarrow{C}}(f_{C_1}(a_1),\ldots,f_{C_k}(a_k)) \\ &\quad = \zeta^{\mu(\beta)-\left(\dim(C_0)-\frac{1}{2}\dim(X)\right)} f_{C_0'}(m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}(a_1,\ldots,a_k)). \end{aligned}$$

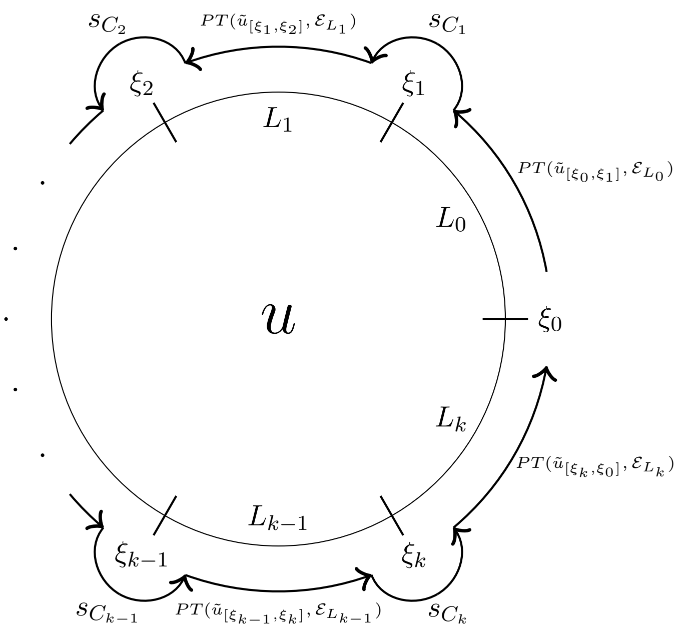
Observe that the holonomy contribution of a J-holomorphic polygon $u \in \overline{\mathcal{M}}_{k+1}\left(\overrightarrow{L},\overrightarrow{C},\beta,J\right)$ to $m_{k,\beta,\Phi(\overrightarrow{\mathbb{L}}),\overrightarrow{C}}$ is equal to the tensor product of the holonomy contribution of the same polygon to $m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}}$ and

$$\begin{aligned} &PT(\tilde{u}_{[\xi_k,\xi_0]},\mathcal{E}_{L_k}) \circ (s_{C_k})_{\tilde{u}(\xi_k)} \circ \cdots \circ (s_{C_1})_{\tilde{u}(\xi_1)} \circ PT(\tilde{u}_{[\xi_0,\xi_1]},\mathcal{E}_{L_0}) \\ &\quad \in Hom\left((\mathcal{E}_{L_0})_{\tilde{u}(\xi_0^+)},(\mathcal{E}_{L_k})_{\tilde{u}(\xi_0^-)}\right) \end{aligned}$$

where $PT(\gamma,\mathcal{E})$ is the parallel transport of the local system $\mathcal{E}$ along the path γ and

$$\begin{aligned} \tilde{u}(\xi_i) &:= (\tilde{u}(\xi_i^-),\tilde{u}(\xi_i^+)) \\ &:= \left(\lim_{\substack{\xi\to\xi_i \\ \xi\in(\xi_{i-1},\xi_i)}} \tilde{u}_{[\xi_{i-1},\xi_i]}(\xi),\ \lim_{\substack{\xi\to\xi_i \\ \xi\in(\xi_i,\xi_{i+1})}} \tilde{u}_{[\xi_i,\xi_{i+1}]}(\xi)\right) \in C_i. \end{aligned}$$

The last expression is illustrated schematically as follows.



Hence it suffices to show that this expression is equal to

$$\zeta^{\mu(\beta)-\left(\dim(C_0)-\frac{1}{2}\dim(X)\right)} s_{C_0'}.$$

Since the concatenation of two canonical short paths in $\mathcal{L}_{C_0}$, from TL_k/TC_0 to TL_0/TC_0 and from TL_0/TC_0 back to TL_k/TC_0 has Maslov index $-\dim\left(TC_0^{\perp\omega}/TC_0\right) = \dim(C_0) - \frac{1}{2}\dim(X)$, we have

$$s_{C_0'} \circ s_{C_0} = \zeta^{\dim(C_0)-\frac{1}{2}\dim(X)} \mathrm{id}_{\mathcal{E}_{L_0}}.$$

Hence the last claim is equivalent to

$$\begin{aligned} s_{C_0} \circ PT(\tilde{u}_{[\xi_k,\xi_0]}, \mathcal{E}_{L_k}) \circ (s_{C_k})_{\tilde{u}(\xi_k)} \circ \cdots \\ \circ (s_{C_1})_{\tilde{u}(\xi_1)} \circ PT(\tilde{u}_{[\xi_0,\xi_1]}, \mathcal{E}_{L_0}) = \zeta^{\mu(\beta)} \mathrm{id}_{\mathcal{E}_{L_0}}. \end{aligned} \tag{2.8}$$

To show (2.8), notice that the domain of u is contractible[3], and hence the bundle $u^*\mathcal{L}'_X$ has a fiberwise infinite cover $\mathcal{L}'' \to u^*\mathcal{L}'_X$, i.e. its deck transformation group is isomorphic to $\mathbb{Z}$.

[3]In case u contains sphere bubbles, we still regard the domain of u as the punctured disk by contracting a finite collection of loops. The argument which follows will also work because Maslov index is additive with respect to cut-and-paste operation.

Consider the loop η in $u^*\mathcal{L}_X$ which is the concatenation of following paths (in the given order)

$$\theta_{L_0}(\tilde{u}_{[\xi_0,\xi_1]}), \theta_t^{C_1}(\tilde{u}(\xi_1)), \ldots, \theta_t^{C_k}(\tilde{u}(\xi_k)), \theta_{L_k}(\tilde{u}_{[\xi_k,\xi_0]}), \theta_t^{C_0}(\tilde{u}(\xi_0)).$$

Then the lifts of η in $u^*\mathcal{L}'_M$ and in $\mathcal{L}''$ are paths whose end points are related by some group elements $a \in \mathbb{Z}_{2n}$ and $b \in \mathbb{Z}$ respectively. It is easy to see that $\zeta^a = \zeta^b$. By definition, $\zeta^a \mathrm{id}_{\mathcal{E}_{L_0}}$ is equal to the LHS of (2.8) and $b = \mu(\beta)$. This shows (2.8) and hence completes the proof of Theorem 1.1.

Remark 2.10. The construction of the A_∞ structure m_k is usually supplied with some algebraic tools. For example, the perturbation theory of Kuranishi spaces only gives "$m_{k,\beta}$ modulo T^E" [15] or an "$A_{N,K}$ structure" [4, 17]. In order to enhance them to an A_∞ structure, algebraic arguments such as the homological perturbation and the approximate A_∞ Whitehead's theorem are used. It is straightforward to keep track of these arguments and show that $\Phi : Fuk(X) \to Fuk(X)_{(\zeta)}$ is an A_∞ functor. The key point is that Φ_1 is a (twisted) chain isomorphism which allows us to transport the data used in the construction of the A_∞ structure, such as the inputs for the homological perturbation and the homotopy inverses given by the approximate A_∞ Whitehead's theorem, from the source $Fuk(X)$ to the target $Fuk(X)_{(\zeta)}$ of Φ. Since we are using different data for the source and the target, we should consider Φ only well-defined up to A_∞ homotopy. In particular, Φ^{2n} is only A_∞ homotopic to the identity.

3. Action on $(\mathcal{M}, W)$

Define

$$\begin{aligned}
\Lambda &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_0 \leqslant \lambda_1 \leqslant \cdots,\ \lim_{i \to +\infty} \lambda_i = +\infty \right\} \\
\Lambda_0 &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ 0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots,\ \lim_{i \to +\infty} \lambda_i = +\infty \right\} \\
\Lambda_+ &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ 0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots,\ \lim_{i \to +\infty} \lambda_i = +\infty \right\} \\
U_\Lambda &:= \mathbb{C} \oplus \Lambda_+.
\end{aligned}$$

Define the valuation $val : \Lambda \to \mathbb{R} \cup \{+\infty\}$ by $val(0) := +\infty$ and

$$val\left(\sum_{i=0}^{\infty} a_i T^{\lambda_i}\right) := \lambda_{\min\{i|a_i \neq 0\}}.$$

Similarly, define $val : V \otimes_{\mathbb{C}} \Lambda \to \mathbb{R}$ for any $\mathbb{C}$-vector space V.

We start by recalling the Fukaya's trick and the definition of $\mathcal{M}_{weak}(L)$. Details can be found in [13, 31, 32].

3.1. The Fukaya's trick and $\mathcal{M}_{weak}(L)$

Let L be a compact, oriented, relatively spin Lagrangian of X. In what follows, unless otherwise specified, the coefficient ring of the de Rham complex $\Omega^\bullet(L)$ and the cohomology $H^\bullet(L)$ is taken to be Λ_0. In [13], Fukaya constructed, for any ω-tame almost complex structure J on X, a *cyclic unital filtered* A_∞ structure $m^J = (m^J_{k,\beta})$ on $\Omega^\bullet(L)$ which satisfies the open string analogue of the *divisor axiom* originating from the closed string Gromov-Witten theory:

$$\begin{aligned} \sum_{m_0+\cdots+m_k=m} & m^J_{k+m,\beta}(b^{\otimes m_0}, x_1, b^{\otimes m_1}, \ldots, b^{\otimes m_{k-1}}, x_k, b^{\otimes m_k}) \\ &= \frac{\langle \partial\beta, b\rangle^m}{m!} m^J_{k,\beta}(x_1, \ldots, x_k) \end{aligned} \tag{3.1}$$

for any β with $\overline{\mathcal{M}}_1(L,\beta,J) \neq \emptyset$, $k, m \geqslant 0$ with $(k,m,\beta) \neq (0,1,0)$; $b \in \Omega^1(L)$ and $x_1, \ldots, x_k \in \Omega^\bullet(L)$.

Moreover, for any smooth family $\mathcal{J} = \{J_t\}_{t\in[0,1]}$ of ω-tame almost complex structures on X, the A_∞ quasi-isomorphism $F^{\mathcal{J}} : (\Omega^\bullet(L), m^{J_0}) \to (\Omega^\bullet(L), m^{J_1})$ induced by the associated pseudo-isotopy is cyclic, unital and satisfies the similar divisor axiom:

$$\begin{aligned} \sum_{m_0+\cdots+m_k=m} & F^{\mathcal{J}}_{k+m,\beta}(b^{\otimes m_0}, x_1, b^{\otimes m_1}, \ldots, b^{\otimes m_{k-1}}, x_k, b^{\otimes m_k}) \\ &= \frac{\langle \partial\beta, b\rangle^m}{m!} F^{\mathcal{J}}_{k,\beta}(x_1, \ldots, x_k) \end{aligned} \tag{3.2}$$

for any β with $\overline{\mathcal{M}}_1(L,\beta,J) \neq \emptyset$, $k, m \geqslant 0$ with $(k,m,\beta) \neq (0,1,0)$; $b \in \Omega^1(L)$ and $x_1, \ldots, x_k \in \Omega^\bullet(L)$.

Notice that $m^J_{1,0}$ is equal to the de Rham differential and $F^{\mathcal{J}}_{1,0}$ is equal to the identity.

By passing to the *canonical model* via homological perturbation, the above story holds with $\Omega^\bullet(L)$ replaced by $H^\bullet(L)$, except that $m^J_{1,0}$ is now equal to zero.

An immediate consequence of (3.1) is the following. Let $b \in H^1(L)$. Define $m^{J,b} = (m^{J,b}_{k,\beta})$ where $m^{J,b}_{k,\beta} : H^\bullet(L)^{\otimes k} \to H^\bullet(L)$ is given by

$$\begin{aligned} &m^{J,b}_{k,\beta}(x_1, \ldots, x_k) \\ &\quad := \sum_{m=0}^{\infty} \sum_{m_0+\cdots+m_k=m} m^J_{k+m,\beta}(b^{\otimes m_0}, x_1, b^{\otimes m_1}, \ldots, b^{\otimes m_{k-1}}, x_k, b^{\otimes m_k}). \end{aligned}$$

By (3.1), $m^{J,b}_{k,\beta} = e^{\langle \partial\beta, b\rangle} m^J_{k,\beta}$. In other words, $m^{J,b}$ is equal to m^J twisted by the local system $\mathcal{E}_b$ with holonomy $e^{\langle -, b\rangle}$. This allows us to identify the Lagrangian brane $(L, \mathcal{E}_b)$ with the Lagrangian brane (L, b) where the latter does not carry any local system but an element $b \in H^1(L)$ which will play the role of weak bounding cochain.

Take a basis $\{e_1, \ldots, e_\ell\}$ of $H^1(L;\mathbb{Z})/\text{torsion}$. Then every element $b \in H^1(L)$ can be written uniquely as $b = x_1 e_1 + \cdots + x_\ell e_\ell$ with $x_1, \ldots, x_\ell \in \Lambda_0$. Put $y_i = e^{x_i}$. Then $y_1, \ldots, y_\ell$ are coordinates of the ℓ-torus $H^1(L;\Lambda^\times)$ over the Novikov field[4] ($\Lambda^\times := \Lambda - \{0\}$).

Consider a formal power series P on $H^1(L;\Lambda^\times) \oplus H^{\mathrm{odd}>1}(L;\Lambda)$ defined by

$$P(y_1, \ldots, y_\ell, b_{>1}) := \sum_{k,\beta} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} m^J_{k,\beta}((b_{>1})^{\otimes k}).$$

Theorem 3.1. *[13, Theorem 1.2] P is convergent in*

$$\mathcal{V}_\delta = \{(y_1, \ldots, y_\ell, b_{>1}) |\ val(y_1), \ldots, val(y_\ell) \in (-\delta, \delta),\ b_{>1} \in H^{odd>1}(L;\Lambda_+)\}$$

where $\delta > 0$ is a positive constant.

The proof is based on the following argument which is what the Fukaya's trick refers to. Consider a Weinstein neighbourhood U of L. Then every small $\alpha = v_1 e_1 + \cdots + v_\ell e_\ell \in H^1(L;\mathbb{R})$, i.e. $v_1, \ldots, v_\ell$ are real numbers close enough to 0, gives rise to a nearby Lagrangian $L(\alpha)$ lying inside U which is expressed as the graph of a closed 1-form representing α. Take a diffeomorphism $\mathcal{F}_\alpha$ of X such that $\mathcal{F}_\alpha(L) = L(\alpha)$ and $(\mathcal{F}_\alpha)_* J := d\mathcal{F}_\alpha \circ J \circ (d\mathcal{F}_\alpha)^{-1}$

[4] In order for $y_i = e^{x_i}$ to have meaning, it is necessary and sufficient that $y_i \in U_\Lambda$. But for the purpose of extending the domain of definition of the function P which will be introduced shortly, we regard each y_i formally as an element of $\Lambda^\times$.

is ω-tame[5]. Write $\beta' := (\mathcal{F}_\alpha)_*\beta$, $e_i' = (\mathcal{F}_\alpha)_* e_i$, etc. Let $E(\beta)$ denote the symplectic area $\int_\beta \omega$.

By the facts that $E(\beta') = E(\beta) + \langle \partial\beta, \alpha\rangle$ and the moduli spaces $\overline{\mathcal{M}}_{k+1}(L, \beta, J)$ are identified with $\overline{\mathcal{M}}_{k+1}(L(\alpha), \beta', (\mathcal{F}_\alpha)_* J)$ as *Kuranishi spaces* (see [4, 13, 16, 17] for the definition), we have

$$\begin{aligned} &P(y_1, \ldots, y_\ell, b_{>1}) \\ &= \sum_{k,\beta'} T^{E(\beta')} \cdot T^{-\langle \partial\beta, v_1 e_1 + \cdots + v_\ell e_\ell\rangle} y_1^{\langle \partial\beta', e_1'\rangle} \cdots y_\ell^{\langle \partial\beta', e_\ell'\rangle} m_{k,\beta'}^{(\mathcal{F}_\alpha)_* J}((b'_{>1})^{\otimes k}) \\ &= \sum_{k,\beta'} T^{E(\beta')} \left(T^{-v_1} y_1\right)^{\langle \partial\beta', e_1'\rangle} \cdots \left(T^{-v_\ell} y_\ell\right)^{\langle \partial\beta', e_\ell'\rangle} m_{k,\beta'}^{(\mathcal{F}_\alpha)_* J}((b'_{>1})^{\otimes k}). \end{aligned}$$

It follows that $P(y_1, \ldots, y_\ell, b_{>1})$ is convergent if $val(y_i) = v_i,\ i = 1, \ldots, \ell$, by Gromov compactness.

Definition 3.2. Define the *local mirror* of L, denoted by $\mathcal{M}_{weak}(L, m^J)$, to be

$$\{\mathbf{b} = (y_1, \ldots, y_\ell, b_{>1}) \in \mathcal{V}_\delta |\ P(\mathbf{b}) = W(\mathbf{b}) \cdot 1 \text{ for some scalar } W(\mathbf{b})\}/_\sim$$

where $\sim$ is the gauge equivalence [17]. Notice that it depends on J.

The proof of Theorem 3.1 suggests that $\mathcal{M}_{weak}(L, m^J)$ contains not only the weak bounding cochains (over Λ_+) on L but also those on all nearby Lagrangians (up to Hamiltonian isotopy). Therefore, for any two Lagrangians L and L' which are close to each other, their local mirrors overlap. More precisely, they contain weak bounding cochains (over Λ_+) on all Lagrangians which are close to L and L' simultaneously. The gluing function defined on this overlapping region can be obtained as follows.

Take a smooth path $\{\mathcal{F}^t_{L,L'}\}_{t\in[0,1]}$ of diffeomorphisms of X such that $\mathcal{F}^0_{L,L'} = \mathrm{id}$, $\mathcal{F}^1_{L,L'}(L) = L'$ and $(\mathcal{F}^t_{L,L'})_*^{-1} J$ remains ω-tame for all t. Put $\mathcal{J} := \{(\mathcal{F}^t_{L,L'})_*^{-1} J\}_{t\in[0,1]}$. Then $\mathcal{J}$ induces an A_∞ quasi-isomorphism $F^{\mathcal{J}}$ satisfying (3.2). Define

$$F_*^{\mathcal{J}} : \mathcal{M}_{weak}(L, m^J) \dashrightarrow \mathcal{M}_{weak}(L, m^{(\mathcal{F}^1_{L,L'})_*^{-1} J})$$

by

$$F_*^{\mathcal{J}}(\mathbf{b}) := \left(y_1 e^{\langle \mathrm{pr}_1(f(\mathbf{b})), e_1^\vee\rangle}, \ldots, y_\ell e^{\langle \mathrm{pr}_1(f(\mathbf{b})), e_\ell^\vee\rangle}, \mathrm{pr}_{\neq 1}(f(\mathbf{b}))\right)$$

[5]The latter condition is satisfied if α is small enough.

where $\{e_1^\vee,\ldots,e_\ell^\vee\}$ is the dual basis of $\{e_1,\ldots,e_\ell\}$, pr_1 (resp. $\mathrm{pr}_{\neq 1}$) is the projection of $H^\bullet(L;\Lambda)$ onto $H^1(L;\Lambda)$ (resp. $\bigoplus_{d\neq 1} H^d(L;\Lambda)$), and $f(\mathbf{b}) := \sum_{k,\beta} T^{E(\beta)} y_1^{\langle\partial\beta,e_1\rangle}\cdots y_\ell^{\langle\partial\beta,e_\ell\rangle} F^{\mathcal{J}}_{k,\beta}((b_{>1})^{\otimes k})$. Here the dash arrow indicates that $F^{\mathcal{J}}_*$ is defined only on an open subset of the domain due to the convergence issue which will be discussed shortly.

On the other hand, the diffeomorphism $\mathcal{F}^1_{L,L'}$ induces a bijection $\psi : \mathcal{M}_{weak}(L, m^{(\mathcal{F}^1_{L,L'})_*^{-1}J}) \to \mathcal{M}_{weak}(L', m^J)$ given by

$$\psi(y_1,\ldots,y_\ell,b_{>1}) := \left(T^{-v_1}(\mathcal{F}^1_{L,L'})_* y_1,\ldots,T^{-v_\ell}(\mathcal{F}^1_{L,L'})_* y_\ell,(\mathcal{F}^1_{L,L'})_*(b_{>1})\right)$$

where we regard L' as the graph of a closed 1-form representing $v_1e_1+\cdots+v_\ell e_\ell$.

Then the desired gluing function $\Psi_{L,L'}$ is defined to be the composite

$$\mathcal{M}_{weak}(L,m^J) \overset{F^{\mathcal{J}}_*}{\dashrightarrow} \mathcal{M}_{weak}(L, m^{(\mathcal{F}^1_{L,L'})_*^{-1}J}) \xrightarrow{\psi} \mathcal{M}_{weak}(L', m^J).$$

It turns out that $F^{\mathcal{J}}_*$ is convergent at any point which corresponds to a third Lagrangian L'' which is close to L and L' simultaneously (i.e. the overlapping region). This is proved by a family version of the Fukaya's trick [31]. The key point is to find a family $\{\mathcal{G}_t\}_{t\in[0,1]}$ of diffeomorphisms such that $\mathcal{G}_t(L) = L''$ for all t and $\{(\mathcal{G}_t)_*\mathcal{J}\}_{t\in[0,1]}$ is a family of ω-tame almost complex structures joining $(\mathcal{F}_\alpha)_*J$ and $(\mathcal{F}_{\alpha'})_*J$ where $L'' = \mathcal{F}_\alpha(L) = \mathcal{F}_{\alpha'}(L')$ and $\mathcal{F}_\alpha$ (resp. $\mathcal{F}_{\alpha'}$) are the diffeomorphisms used in the proof of Theorem 3.1 for L (resp. L').

3.2. Action on $(\mathcal{M}_{weak}(L), W)$

Recall the $\mathbb{Z}_{2n}$-local system $\mathcal{E}_L$ from Section 2.1. Since L is oriented, $\mathcal{E}_L$ is reduced to a $\mathbb{Z}_n$-local system. See Remark 2.5. Hence $\mathcal{E}_L$ is represented by an element $\gamma \in Hom(H_1(L;\mathbb{Z}),\mathbb{C}^\times)$ with the property that

$$\gamma(\partial\beta) = \zeta^{\mu(\beta)}$$

for any $\beta\in\pi_2(X,L)$ where $\zeta = e^{\frac{2\pi i}{2n}}$. See (2.8). Write $\gamma_i := \gamma(e_i^\vee)$, $i=1,\ldots,\ell$.

Apply the operator $(\cdot)^\zeta$ from Section 2.2 to $\mathrm{id}_{H^\bullet(L;\Lambda_+)}$. Recall it means

$$\mathrm{id}^\zeta|_{H^d(L;\Lambda_+)} = \zeta^{-d}\mathrm{id}_{H^d(L;\Lambda_+)}.$$

Definition 3.3. Define $\tau : \mathcal{V}_\delta \to \mathcal{V}_\delta$ by

$$\tau(y_1,\ldots,y_\ell,b_{>1}) := \left(\gamma_1 y_1,\ldots,\gamma_\ell y_\ell,\zeta\mathrm{id}^\zeta(b_{>1})\right).$$

Lemma 3.4. *If* $\mathbf{b} \in \mathcal{M}_{weak}(L, m^J)$, *then* $\tau(\mathbf{b}) \in \mathcal{M}_{weak}(L, m^J)$ *and*

$$W(\tau(\mathbf{b})) = \zeta^2 W(\mathbf{b}).$$

Proof. Write $b_{>1} = \sum b_i$ with $b_i \in H^i(L; \Lambda_+)$. We have

$$\begin{aligned} P(\mathbf{b}) &= \sum_{k,\beta} \sum_{i_1,\ldots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_\ell}) \\ &= W(\mathbf{b}) \cdot 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} &P(\tau(\mathbf{b})) \\ =&\sum_{k,\beta} \sum_{i_1,\ldots,i_k} T^{E(\beta)} (\gamma_1 y_1)^{\langle \partial\beta, e_1\rangle} \cdots (\gamma_\ell y_\ell)^{\langle \partial\beta, e_\ell\rangle} \zeta^{k-(i_1+\cdots+i_k)} m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_\ell}) \\ =&\sum_{k,\beta} \sum_{i_1,\ldots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} \gamma(\beta) \zeta^{k-(i_1+\cdots+i_k)} m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_\ell}) \\ =&\sum_{k,\beta} \sum_{i_1,\ldots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} \zeta^{\mu(\beta)+k-(i_1+\cdots+i_k)} m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_\ell}) \\ =&\ \zeta^2 \sum_{k,\beta} \sum_{i_1,\ldots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} \zeta^{-[i_1+\cdots+i_k+2-k-\mu(\beta)]} m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_\ell}) \end{aligned}$$

Notice that $m^J_{k,\beta}(b_{i_1}, \ldots, b_{i_k})$ has degree $i_1 + \cdots + i_k + 2 - k - \mu(\beta)$, and hence

$$P(\tau(\mathbf{b})) = \zeta^2 \ \mathrm{id}^\zeta(P(\mathbf{b})) = \zeta^2 \ \mathrm{id}^\zeta(W(\mathbf{b}) \cdot 1) = \zeta^2 W(\mathbf{b}).$$

The proof that τ preserves gauge equivalences is similar. □

It is clear that $\tau^n = \mathrm{id}$. We have proved

Proposition 3.5. *(=Proposition 1.2) There is a* $\mathbb{Z}_n$*-action on* $\mathcal{M}_{weak}(L, m^J)$ *with respect to which* W *is equivariant.*

3.3. Commute with wall-crossing

Recall $\Psi_{L,L'}$ is defined to be the composite

$$\mathcal{M}_{weak}(L, m^J) \overset{F_*^{\mathcal{J}}}{\dashrightarrow} \mathcal{M}_{weak}(L, m^{(\mathcal{F}^1_{L,L'})_*^{-1} J}) \xrightarrow{\psi} \mathcal{M}_{weak}(L', m^J).$$

By Proposition 3.5, $\mathbb{Z}_n$ acts on the following local mirrors

$$\mathcal{M}_{weak}(L, m^J), \mathcal{M}_{weak}(L, m^{(\mathcal{F}^1_{L,L'})^{-1}_* J}), \mathcal{M}_{weak}(L', m^J)$$

with respect to which W is equivariant.

Lemma 3.6. *τ commutes with $F^{\mathcal{J}}_*$.*

Proof. Write $b_{>1} = \sum b_i$ as before. We have

$$\begin{aligned}
&f(\tau(\mathbf{b}))\\
&= \sum_{k,\beta} T^{E(\beta)} (\gamma_1 y_1)^{\langle \partial\beta, e_1\rangle} \cdots (\gamma_\ell y_\ell)^{\langle \partial\beta, e_\ell\rangle} F^{\mathcal{J}}_{k,\beta}((\zeta\ \mathrm{id}^\zeta(b_{>1}))^{\otimes k})\\
&= \sum_{k,\beta} \sum_{i_1,\dots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} \zeta^{\mu(\beta)+k-(i_1+\cdots+i_k)} F^{\mathcal{J}}_{k,\beta}(b_{i_1}, \dots, b_{i_k})\\
&= \zeta \sum_{k,\beta} \sum_{i_1,\dots,i_k} T^{E(\beta)} y_1^{\langle \partial\beta, e_1\rangle} \cdots y_\ell^{\langle \partial\beta, e_\ell\rangle} \zeta^{-[i_1+\cdots+i_k+1-k-\mu(\beta)]} F^{\mathcal{J}}_{k,\beta}(b_{i_1}, \dots, b_{i_k}).
\end{aligned}$$

Since $F^{\mathcal{J}}_{k,\beta}(b_{i_1}, \dots, b_{i_k})$ has degree $i_1 + \cdots + i_k + 1 - k - \mu(\beta)$, we have

$$f(\tau(\mathbf{b})) = \zeta\ \mathrm{id}^\zeta(f(\mathbf{b})).$$

Then

$$\begin{aligned}
F^{\mathcal{J}}_* \circ \tau(\mathbf{b}) &= \left(\gamma_1 y_1 e^{\langle \mathrm{pr}_1(f(\tau(\mathbf{b}))), e_1^\vee\rangle}, \dots, \gamma_\ell y_\ell e^{\langle \mathrm{pr}_1(f(\tau(\mathbf{b}))), e_\ell^\vee\rangle}, \mathrm{pr}_{\neq 1}(f(\tau(\mathbf{b})))\right)\\
&= \left(\gamma_1 y_1 e^{\langle \mathrm{pr}_1(f(\mathbf{b})), e_1^\vee\rangle}, \dots, \gamma_\ell y_\ell e^{\langle \mathrm{pr}_1(f(\mathbf{b})), e_\ell^\vee\rangle}, \zeta\ \mathrm{id}^\zeta(\mathrm{pr}_{\neq 1}(f(\mathbf{b})))\right)\\
&= \tau \circ F^{\mathcal{J}}_*(\mathbf{b}).
\end{aligned}$$

(We have used the fact that id^ζ commutes with pr_1 and $\mathrm{pr}_{\neq 1}$.) ☐

Lemma 3.7. *τ commutes with ψ.*

Proof. It follows from the fact that if $\iota_t : L \to X$ is a Lagrangian isotopy, then the local systems $\iota_0^* \mathcal{E}_{\iota_0(L)}$ and $\iota_1^* \mathcal{E}_{\iota_1(L)}$ on L are isomorphic, as they are isomorphic to a local system on $L \times [0, 1]$ restricted to the slices $L \times \{0\}$ and $L \times \{1\}$ respectively. ☐

Therefore, we have proved

Proposition 3.8. *(=Proposition 1.3) τ commutes with $\Psi_{L,L'}$.*

4. Extending the action to the complete mirror

Let X, D and $(\check{X}^\circ, W)$ be given as in the introduction. Recall that X is Kähler, D is an anticanonical divisor of X and $(\check{X}^\circ, W)$ is the uncompactified mirror obtained by gluing the local mirror charts of the smooth torus fibers of an SYZ fibration on $X - D$ following Fukaya's scheme.

Assumptions (B).

- $\check{X}^\circ$ is an analytic variety over $\mathbb{C}$.
- $(\check{X}^\circ, W)$ can be completed to $(\check{X}, W)$ where $\check{X}$ is a normal affine analytic variety and $W : \check{X} \to \mathbb{C}$ is a holomorphic function.
- The complement $\check{X} - \check{X}^\circ$ is contained in a closed analytic subset A which has codimension at least two.
- There is a $\mathbb{Z}_n$-action on $(\check{X}^\circ, W)$, i.e. there is a biholomorphism $\tau : \check{X}^\circ \to \check{X}^\circ$ such that $\tau^n = \mathrm{id}$ and

$$W(\tau \cdot x) = e^{\frac{2\pi i}{n}} W(x),\ \ x \in \check{X}^\circ.$$

Notice that the last assumption is actually the outcome of Corollary 1.5. As for the first three, we emphasize that they are reasonable if X is Fano. For example, consider the complete SYZ mirror of the famous special Lagrangian torus fibration defined on $\mathbb{C}P^2$ minus a line and a conic which is given by

$$\begin{aligned} \check{X} &= \{(u, v) \in \mathbb{C}^2 |\ uv \neq 1\} \\ W &= u + \frac{v^2}{uv - 1}. \end{aligned}$$

The only point that the local mirror charts of the Clifford tori and Chekanov tori do not cover is $(0, 0)$. (It is covered by the local mirror chart of the immersed 2-sphere.) This point has codimension two, and the desired action is given by $(u, v) \mapsto (\zeta u, \zeta^{-1} v)$ where $\zeta = e^{\frac{2\pi i}{3}}$.

Back to the general case.

Proposition 4.1. *Under the assumptions (B), the $\mathbb{Z}_n$-action on $(\check{X}^\circ, W)$ extends to a unique $\mathbb{Z}_n$-action on $(\check{X}, W)$.*

Proof. Let $U := \check{X} - A$. Define

$$V := \bigcap_{i=0}^{n-1} \tau^i(U).$$

Then V is an open analytic subset of $\check{X}$ whose complement has codimension at least two. Moreover, V is contained in $\check{X}^\circ$ and is invariant under the given $\mathbb{Z}_n$-action. It follows that $\mathbb{Z}_n$ acts on the ring $\mathbb{C}[V]$ of holomorphic functions on V. But the inclusion $V \hookrightarrow \check{X}$ induces an isomorphism $\mathbb{C}[\check{X}] \simeq \mathbb{C}[V]$ by

Lemma 4.2. *(The second Riemann extension theorem, see e.g. [20, Chapter 7]) Every holomorphic function on V extends to a unique holomorphic function on $\check{X}$.*

It follows that $\mathbb{Z}_n$ acts on the ring $\mathbb{C}[\check{X}]$ and hence on the space $\check{X}$, since $\check{X}$ is affine. It is clear that this action is the unique extension of the given one on $\check{X}^\circ$ and W is equivariant in the above sense. □

Appendix A. *Fuk*(X)

The objects of $Fuk(X)$ consist of $\mathbb{L} = (L, \mathcal{E})$[6] where L is taken from a fixed finite collection of immersed compact oriented Lagrangians of X with clean self-intersection and $\mathcal{E}$ is a $\mathbb{C}^\times$-local system on L.

For any pair L_0, L_1 of such Lagrangians, let $\iota : L_i \to X,\ i = 0, 1$ denote the immersion. Suppose L_0 and L_1 intersect cleanly. Recall it means the fiber product

$$L_0 \times_\iota L_1 := \{(x, y) \in L_0 \times L_1 |\ \iota(x) = \iota(y)\}$$

is smooth and satisfies

$$T_{(x,y)}(L_0 \times_\iota L_1) = T_x L_0 \times_{d\iota} T_y L_1$$

for any $(x, y) \in L_0 \times_\iota L_1$.

Define the morphism space between two objects $\mathbb{L}_i = (L_i, \mathcal{E}_i),\ i = 0, 1$ by

$$Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_1) := \bigoplus_{C \in \pi_0(L_0 \times_\iota L_1)} \Omega^\bullet(C; Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C)) \tag{A.1}$$

where $\Omega^\bullet(C; \mathcal{E})$ is any of the standard models (de Rham, singular cochain, etc) of the cohomology group $H^\bullet(C; \mathcal{E})$ with local coefficient $\mathcal{E}$[7].

---

[6]A relative spin structure on L is also included as part of the data. But since it is not relevant for our construction, we drop it from the discussion.

[7]Strictly speaking, the local system $Hom(\mathcal{E}_0|_C, \mathcal{E}_1|_C)$ in (A.1) has to be twisted by a $\mathbb{Z}_2$-local system which is used to orient the moduli spaces of holomorphic disks. Since our Φ will not modify it, we drop it from the discussion.

The A_∞ structure on $Fuk(X)$ is defined by making sense of the slogan "counting holomorphic polygons". Let $\overrightarrow{\mathbb{L}} = (\mathbb{L}_0, \dots, \mathbb{L}_k)$ be a sequence of $k+1$ objects of $Fuk(X)$ such that L_{i-1} and L_i intersect cleanly for each $i = 0, \dots, k$. (Here $\mathbb{L}_{-1} = \mathbb{L}_k$.) Let $\overrightarrow{L} = (L_0, \dots, L_k)$. For each $i = 0, \dots, k$, fix a connected component $C_i \in \pi_0(L_{i-1} \times_\iota L_i)$. Let C_0' denote the connected component C_0 regarded as an element of $\pi_0(L_0 \times_\iota L_k)$. Fix a homotopy class $\beta \in \pi_2\left(X, \bigcup_{i=0}^k \iota(L_i)\right)$ and an ω-tame almost complex structure J on X.

The domain Σ of holomorphic polygons are bordered Riemann surfaces of genus zero with boundary marked points $\xi_0, \dots, \xi_k, \xi_{k+1} = \xi_0$ arranged counterclockwise. For each pair of consecutive marked points ξ_i and ξ_{i+1}, we denote by $[\xi_i, \xi_{i+1}]$ the arc in $\partial\Sigma$ drawn from ξ_i to ξ_{i+1} counterclockwise.

Define $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J)$ to be the moduli space of J-holomorphic polygons u which represents β and has continuous lifts $\tilde{u}_{[\xi_i,\xi_{i+1}]}, i = 0, \dots, k$ into L_i along the arc $[\xi_i, \xi_{i+1}]$ such that for each i

$$\begin{aligned}\tilde{u}(\xi_i) &:= (\tilde{u}(\xi_i^-), \tilde{u}(\xi_i^+)) \\ &:= \left(\lim_{\substack{\xi \to \xi_i \\ \xi \in (\xi_{i-1}, \xi_i)}} \tilde{u}_{[\xi_{i-1},\xi_i]}(\xi), \lim_{\substack{\xi \to \xi_i \\ \xi \in (\xi_i, \xi_{i+1})}} \tilde{u}_{[\xi_i,\xi_{i+1}]}(\xi) \right) \in C_i.\end{aligned}$$

The virtual dimension of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J)$ is equal to $\frac{1}{2}\dim(X) + \mu(\beta) + k - 2$ where $\mu(\beta)$ is the Maslov index of β which is defined in the standard way. See [4]. Notice that in the presence of corners, $\mu(\beta)$ depends on an assignment to each marked point ζ_i a path in the Lagrangian Grassmannian $LG(T_{u(\xi_i)}X, \omega_{u(\xi_i)})$ joining the two limiting Lagrangian subspaces at $u(\xi_i)$ (from the left and from the right) determined by a representative u. In our case, we have chosen the canonical short path (2.5) from Section 2.2.

By performing abstract or rigid count of elements of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J)$, weighted by the holonomy of $\mathcal{E}_i$'s along their boundary arcs, one obtains a multilinear map

$$\begin{aligned}\text{(A.2)} \qquad m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}} : \bigotimes_{i=1}^{k} \Omega^\bullet(C_i; Hom(\mathcal{E}_{i-1}|_{C_i}, \mathcal{E}_i|_{C_i})) \\ \to \Omega^\bullet(C_0'; Hom(\mathcal{E}_0|_{C_0'}, \mathcal{E}_k|_{C_0'}))\end{aligned}$$

of degree $\left(\dim(C_0) - \frac{1}{2}\dim(X)\right) + 2 - k - \mu(\beta)$. (The degree can be seen from the dimension of $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J)$ given above.)

The A_∞ product map

$$m_k : \bigotimes_{i=1}^{k} Hom_{Fuk(X)}(\mathbb{L}_{i-1}, \mathbb{L}_i) \to Hom_{Fuk(X)}(\mathbb{L}_0, \mathbb{L}_k)$$

is then defined to be

$$m_k := \sum_{\overrightarrow{C},\beta} T^{E(\beta)} m_{k,\beta,\overrightarrow{\mathbb{L}},\overrightarrow{C}} \tag{A.3}$$

over all $\overrightarrow{C} = (C_0, \ldots, C_k)$ with $C_i \in \pi_0(L_{i-1} \times_\iota L_i)$ and $\beta \in \pi_2\left(X, \bigcup_{i=0}^k \iota(L_i)\right)$ for which $\overline{\mathcal{M}}_{k+1}(\overrightarrow{L}, \overrightarrow{C}, \beta, J) \neq \emptyset$. (Here $E(\beta) = \int_\beta \omega$ is the symplectic area.)

In general, $Fuk(X)$ is defined over the Novikov ring Λ_0.

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