JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 20, Number 4, 911–926, 2022

Reductive subalgebras of semisimple Lie algebras and Poisson commutativity

Dmitri I. Panyushev and Oksana S. Yakimova

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a reductive subalgebra such that the orthogonal complement $\mathfrak{h}^\perp$ is a complementary $\mathfrak{h}$-submodule of $\mathfrak{g}$. In 1983, Bogoyavlenski claimed that one obtains a Poisson commutative subalgebra of the symmetric algebra $\mathcal{S}(\mathfrak{g})$ by taking the subalgebra $\mathcal{Z}$ generated by the bi-homogeneous components of all $H \in \mathcal{S}(\mathfrak{g})^\mathfrak{g}$ taken w.r.t. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. But this is false, and we present a counterexample. We also provide a criterion for the Poisson commutativity of such subalgebras $\mathcal{Z}$. As a by-product, we prove that $\mathcal{Z}$ is Poisson commutative if $\mathfrak{h}$ is abelian and describe $\mathcal{Z}$ in the special case when $\mathfrak{h}$ is a Cartan subalgebra. In this case, $\mathcal{Z}$ appears to be polynomial and has the maximal transcendence degree $\boldsymbol{b}(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} + \mathsf{rk}\, \mathfrak{g})$.

Introduction

0.1.

The ground field $\Bbbk$ is algebraically closed and $\mathrm{char}(\Bbbk) = 0$. For any finite-dimensional Lie algebra $\mathfrak{q}$, the dual space $\mathfrak{q}^*$ is a Poisson variety. The algebra of polynomial functions on $\mathfrak{q}^*$, $\Bbbk[\mathfrak{q}^*]$, is isomorphic to the graded symmetric algebra $\mathcal{S}(\mathfrak{q})$ and the Lie–Poisson bracket $\{\ ,\ \}$ on $\mathcal{S}(\mathfrak{q})$ is defined on the elements of degree one by $\{\xi, \eta\} = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{q}$. There is a method for constructing "large" Poisson commutative subalgebras of $\mathcal{S}(\mathfrak{q})$ that exploits pairs of *compatible Poisson brackets*, see [4, Sect. 10], [9]. To apply this, one needs a *suitable* second Poisson bracket $\{\ ,\ \}_{II}$ beside $\{\ ,\ \} = \{\ ,\ \}_\mathfrak{q}$, here suitable (= compatible) means that the sum $\{\ ,\ \} + \{\ ,\ \}_{II}$, as well as any linear combination of $\{\ ,\ \}$ and $\{\ ,\ \}_{II}$, is again a Poisson bracket. Let us

The first author is partially supported by R.F.B.R. grant № 20-01-00515. The second author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — project number 454900253.

recall some situations, where this "general method" (= method of compatible Poisson brackets) works.

I. The celebrated "argument shift method" goes back to [7] (if $\mathfrak{q}$ is semisimple). It employs an arbitrary $\gamma \in \mathfrak{q}^*$ and the Poisson bracket $\{\ ,\ \}_\gamma$, where $\{x, y\}_\gamma = \gamma([x, y])$ for $x, y \in \mathfrak{q}$. The brackets $\{\ ,\ \}$ and $\{\ ,\ \}_\gamma$ are compatible, and the general method produces the *Mishchenko–Fomenko subalgebra* (= $\mathcal{MF}$*-subalgebra*) $(\mathcal{MF})_\gamma \subset \mathcal{S}(\mathfrak{q})$. Let $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ be the *Poisson centre* of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \})$, i.e.,

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \{H \in \mathcal{S}(\mathfrak{q}) \mid \{H, x\} = 0 \ \ \forall x \in \mathfrak{q}\}.$$

For $F \in \mathcal{S}(\mathfrak{q})$, let $\partial_\gamma F$ be the directional derivative of F with respect to $\gamma \in \mathfrak{q}^*$, i.e.,

$$\partial_\gamma F(x) = \frac{d}{dt} F(x + t\gamma)\Big|_{t=0} \ \text{ for all } \ x \in \mathfrak{q}^*.$$

By the original definition of the $\mathcal{MF}$-subalgebras [7], $(\mathcal{MF})_\gamma$ is generated by all $\partial_\gamma^k F$ with $k \geqslant 0$ and $F \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Since then, the algebras $(\mathcal{MF})_\gamma$ and their quantum counterparts attracted a great deal of attention, see e.g. [3, 8, 15] and references therein. If $\mathfrak{q}$ is reductive and γ is regular in $\mathfrak{q}^*$, then $(\mathcal{MF})_\gamma$ is a maximal Poisson commutative subalgebra in $\mathcal{S}(\mathfrak{q})$ of maximal transcendence degree [12].

II. Let $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1$ be a $\mathbb{Z}_2$-grading, i.e., we have $[\mathfrak{q}_i, \mathfrak{q}_j] \subset \mathfrak{q}_{i+j \ (\mathrm{mod}\ 2)}$. Then $\mathfrak{q}$ admits the Inönu–Wigner contraction to the semi-direct product $\tilde{\mathfrak{q}} = \mathfrak{q}_0 \ltimes \mathfrak{q}_1^{\mathsf{ab}}$, and the second bracket is the Lie–Poisson bracket of $\tilde{\mathfrak{q}}$. (Here $\mathfrak{q}$ and $\tilde{\mathfrak{q}}$ are identified as vector spaces.) The compatibility of $\{\ ,\ \}_{\mathfrak{q}}$ and $\{\ ,\ \}_{\tilde{\mathfrak{q}}}$ stems from the presence of $\mathbb{Z}_2$-grading, cf. Section 1.1. The sum $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1$ determines the bi-homogeneous decomposition $\mathcal{S}(\mathfrak{q}) = \bigoplus_{i,j \geqslant 0} \mathcal{S}^i(\mathfrak{q}_0) \otimes \mathcal{S}^j(\mathfrak{q}_1)$. Here the general method yields the Poisson commutative subalgebra generated by the bi-homogeneous components of all $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. This case has been studied in [6] and recently in our article [14]. For substantial applications, one has to assume, of course, that $\mathfrak{q}$ is semisimple.

0.2.

Soon after [14] has been accepted, we came across an article of Bogoyavlenski [1]. He claims that if $\mathfrak{g}$ is semisimple, $\mathfrak{f} \subset \mathfrak{g}$ is reductive and the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{f}$, then the direct sum $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{f}^\perp$ is the orthogonal complement of $\mathfrak{f}$ w.r.t. the Killing form, allows to construct similarly a Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{g})$. Namely, a special case of [1, Theorem 1] (with $n = k = j = 1$ in the original notation) asserts that the bi-homogeneous components of all $F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ generate a Poisson commutative subalgebra. However, this is false and we provide a counterexample to that claim. An explanation for that error is that here one can also consider the contraction $\tilde{\mathfrak{g}} = \mathfrak{k} \ltimes \mathfrak{m}^{\mathsf{ab}}$ and the Poisson bracket $\{\ ,\ \}_{\tilde{\mathfrak{g}}}$ on the vector space $\mathfrak{g} \simeq \tilde{\mathfrak{g}}$, but the brackets $\{\ ,\ \}_{\mathfrak{g}}$ and $\{\ ,\ \}_{\tilde{\mathfrak{g}}}$ are not necessarily compatible. One can also notice that Bogoyavlenski did not properly distinguish a Lie algebra and its dual, and his usage of differentials of elements of $\mathcal{S}(\mathfrak{g})$ is sloppy.

Our main motivation for writing this note was just to clarify and remedy this situation. However, we also discovered some exciting new phenomena. Let $\mathfrak{g} = \mathsf{Lie}(G)$ be semisimple and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as above. Let $\mathcal{Z}_{(\mathfrak{g},\mathfrak{k})}$ be the subalgebra of $\mathcal{S}(\mathfrak{g})$ generated by the bi-homogeneous components of all F belonging to $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. The results of this note are:

1) we provide a criterion for $\mathcal{Z}_{(\mathfrak{g},\mathfrak{k})}$ to be Poisson commutative;
2) using our criterion we prove that $\mathcal{Z}_{(\mathfrak{sl}_4,\mathfrak{sl}_2)}$ is not Poisson commutative for the standard embedding $\mathfrak{sl}_2 \subset \mathfrak{sl}_4$;
3) a corollary of our criterion is that $\mathcal{Z}_{(\mathfrak{g},\mathfrak{k})}$ is Poisson commutative whenever $\mathfrak{k}$ is abelian (e.g. if $\mathfrak{k}$ is the Lie algebra of a torus in G);
4) it is proved that if $\mathfrak{k} = \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is polynomial, $\operatorname{tr.deg} \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})} = \boldsymbol{b}(\mathfrak{g}) = (\dim \mathfrak{g} + \mathsf{rk}\, \mathfrak{g})/2$, and $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is *complete* on every regular G-orbit in $\mathfrak{g}$.
5) We point out an algebraic extension $\tilde{\mathcal{Z}} \supset \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ such that $\tilde{\mathcal{Z}}$ is a **maximal** Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ (w.r.t. inclusion) and is still polynomial.

Our criterion for the equality $\{\mathcal{Z}_{(\mathfrak{q},\mathfrak{k})}, \mathcal{Z}_{(\mathfrak{q},\mathfrak{k})}\} = 0$ works also for non-reductive Lie algebras $\mathfrak{q}$, see Theorem 2.1.

1. Preliminaries on the coadjoint representation

Let Q be a connected affine algebraic group with $\mathsf{Lie}(Q) = \mathfrak{q}$. The symmetric algebra $\mathcal{S}(\mathfrak{q})$ over $\Bbbk$ is identified with the graded algebra of polynomial functions on $\mathfrak{q}^*$, and we also write $\Bbbk[\mathfrak{q}^*]$ for it.

Let $\mathfrak{q}^\xi$ denote the stabiliser in $\mathfrak{q}$ of $\xi \in \mathfrak{q}^*$. The *index of* $\mathfrak{q}$, $\operatorname{ind} \mathfrak{q}$, is the minimal codimension of Q-orbits in $\mathfrak{q}^*$. Equivalently, $\operatorname{ind} \mathfrak{q} = \min_{\xi \in \mathfrak{q}^*} \dim \mathfrak{q}^\xi$. By Rosenlicht's theorem [2, I.6], one also has $\operatorname{ind} \mathfrak{q} = \operatorname{tr.deg} \Bbbk(\mathfrak{q}^*)^Q$. The Lie–Poisson bracket for $\Bbbk[\mathfrak{q}^*]$ is defined on the elements of degree 1 (i.e., on $\mathfrak{q}$) by $\{x, y\} := [x, y]$. Set further $\hat{\gamma}(x, y) = \gamma([x, y])$ for $\gamma \in \mathfrak{q}^*$. For any

$F_1, F_2 \in \mathcal{S}(\mathfrak{q})$ and $\gamma \in \mathfrak{q}^*$, we have

$$\{F_1, F_2\}(\gamma) = \hat{\gamma}(d_\gamma F_1, d_\gamma F_2), \tag{1·1}$$

where $d_\gamma F \in \mathfrak{q}$ is the differential of $F \in \mathcal{S}(\mathfrak{q})$ at γ. As Q is connected, we have $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})^Q = \Bbbk[\mathfrak{q}^*]^Q$. The set of *$Q$-regular* elements of $\mathfrak{q}^*$ is

$$\mathfrak{q}^*_{\mathsf{reg}} = \{\eta \in \mathfrak{q}^* \mid \dim \mathfrak{q}^\eta = \operatorname{ind} \mathfrak{q}\}. \tag{1·2}$$

Set $\mathfrak{q}^*_{\mathsf{sing}} = \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathsf{reg}}$. We say that $\mathfrak{q}$ has the *codim–n* property if $\operatorname{codim} \mathfrak{q}^*_{\mathsf{sing}} \geqslant n$. By [5], the semisimple algebras $\mathfrak{g}$ have the *codim*–3 property.

Set $\boldsymbol{b}(\mathfrak{q}) = (\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q})/2$. Since the coadjoint orbits are even-dimensional, this number is an integer. If $\mathfrak{q}$ is reductive, then $\operatorname{ind} \mathfrak{q}$ equals the rank $\mathsf{rk}\, \mathfrak{q}$ of $\mathfrak{q}$ and $\boldsymbol{b}(\mathfrak{q})$ equals the dimension of a Borel subalgebra. A subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is said to be *Poisson commutative* if $\{\mathcal{A}, \mathcal{A}\} = 0$. If $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is Poisson commutative, then $\operatorname{tr.deg} \mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q})$, see e.g. [15, 0.2].

Definition 1. A Poisson commutative subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is said to be *complete* on a coadjoint orbit $Q\gamma \subset \mathfrak{q}^*$ if $\operatorname{tr.deg} (\mathcal{A}|_{Q\gamma}) = \frac{1}{2} \dim(Q\gamma)$.

The notion of completeness originates from the theory of integrable systems.

For a subalgebra $A \subset \mathcal{S}(\mathfrak{q})$ and $\gamma \in \mathfrak{q}^*$, set $d_\gamma A = \langle d_\gamma F \mid F \in A \rangle_{\Bbbk}$.

1.1. Decompositions and compatibility

Let $\mathfrak{q} = \mathfrak{f} \oplus V$ be a vector space decomposition, where $\mathfrak{f}$ is a subalgebra. For any $s \in \Bbbk^\times$, define a linear map $\varphi_s\colon \mathfrak{q} \to \mathfrak{q}$ by setting $\varphi_s|_{\mathfrak{f}} = \mathsf{id}$, $\varphi_s|_V = s{\cdot}\mathsf{id}$. Then $\varphi_s \varphi_{s'} = \varphi_{ss'}$ and $\varphi_s^{-1} = \varphi_{s^{-1}}$, i.e., this yields a one-parameter subgroup of $\mathrm{GL}(\mathfrak{q})$. For each s, the formula

$$[x, y]_{(s)} = \varphi_s^{-1}([\varphi_s(x), \varphi_s(y)]) \tag{1·3}$$

defines a modified Lie algebra structure on the vector space $\mathfrak{q}$. All these structures are isomorphic to the initial one. The corresponding Poisson bracket is denoted by $\{\ ,\ \}_{(s)}$. We naturally extend φ_s to an automorphism of $\mathcal{S}(\mathfrak{q})$. Then the centre of the Poisson algebra $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \}_{(s)})$ equals $\varphi_s^{-1}(\mathcal{S}(\mathfrak{q})^{\mathfrak{q}})$. For $x \in \mathfrak{q}$, write $x = x_{\mathfrak{f}} + x_V$ with $x_{\mathfrak{f}} \in \mathfrak{f}$, $x_V \in V$.

If $\mathfrak{q} = \mathfrak{f} \oplus V$ is a $\mathbb{Z}_2$-grading, i.e., $[\mathfrak{f}, V] \subset V$ and $[V, V] \subset \mathfrak{f}$, then $\{\ ,\ \}_{(0)} = \lim_{s \to 0} \{\ ,\ \}_{(s)}$ is a Poisson bracket; furthermore $\{\ ,\ \}_{(s)} = \{\ ,\ \}_{(-s)}$ and $\{\ ,\ \}_{(s)} + \{\ ,\ \}_{(s')} = 2\{\ ,\ \}_{(\tilde{s})}$ with $2\tilde{s}^2 = s^2 + (s')^2$. The brackets $\{\ ,\ \}_{(s)}$ are pairwise compatible and together with the line $\Bbbk(\{\ ,\ \} - \{\ ,\ \}_{(0)})$ build a two-dimensional pencil.

Lemma 1.1. *Suppose that* $\mathfrak{q} = \mathfrak{f} \oplus V$*, where* $\mathfrak{f} \subset \mathfrak{q}$ *is a subalgebra and* $[\mathfrak{f}, V] \subset V$*. For any* $x = x_{\mathfrak{f}} + x_V, y = y_{\mathfrak{f}} + y_V \in \mathfrak{q}$*, we have*

$$(1{\cdot}4) \qquad [x,y]_{(s)} = [x_{\mathfrak{f}}, y_{\mathfrak{f}}] + [x_{\mathfrak{f}}, y_V] + [x_V, y_{\mathfrak{f}}] + s[x_V, y_V]_V + s^2[x_V, y_V]_{\mathfrak{f}}.$$

Proof. The statement is verified by a straightforward computation. □

Assume that $\mathfrak{q} = \mathfrak{f} \oplus V$ is an $\mathfrak{f}$-stable decomposition. One of the crucial properties of $[\ ,\ ]_{(s)}$ is that if $x \in \mathfrak{f}$ and $y \in \mathfrak{q}$, then $[x,y]_{(s)} = [x,y]$ for all $s \in \Bbbk$. Then (1·4) shows also that if $[\mathfrak{f}, \mathfrak{q}] \neq 0$ and $[V,V]$ is not contained in either $\mathfrak{f}$ or V, then the brackets $\{\ ,\ \}_{(s)}$ do not build a two-dimensional pencil.

2. A criterion for commutativity

Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{q}$. Suppose that there is an $\mathfrak{f}$-stable decomposition $\mathfrak{q} = \mathfrak{f} \oplus \mathfrak{m}$, i.e., $[\mathfrak{f}, \mathfrak{m}] \subset \mathfrak{m}$. This yields a bi-homogeneous structure for $\mathcal{S}(\mathfrak{q})$:

$$\mathcal{S}(\mathfrak{q}) = \bigoplus_{i,j \geqslant 0} \mathcal{S}^i(\mathfrak{f}) \otimes \mathcal{S}^j(\mathfrak{m}).$$

For any $H \in \mathcal{S}(\mathfrak{q})$, we have $H = \sum_{i,j\geqslant 0} H_{(i,j)}$, where $H_{(i,j)} \in \mathcal{S}^i(\mathfrak{f}) \otimes \mathcal{S}^j(\mathfrak{m})$ are the *bi-homogeneous components* of H. Let $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ be the subalgebra of $\mathcal{S}(\mathfrak{q})$ generated by the bi-homogeneous components of all $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Since each bi-homogeneous component of $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is $\mathfrak{f}$-invariant, we have $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{f}}$. It is claimed in [1, Theorem 1] that if $\mathfrak{q}$ is semisimple and the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{f}$ (so that an $\mathfrak{f}$-stable decomposition of $\mathfrak{q}$ does exist), then $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ is Poisson commutative. However, this is **false**! Below, we give a criterion for the Poisson commutativity of $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ and provide a counterexample to the assertion of [1]. On the positive side, we deduce from our criterion that $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ is Poisson commutative whenever $\mathfrak{f}$ is an abelian subalgebra.

Given $\gamma \in \mathfrak{q}^*$, we decompose it as $\gamma = \gamma_{\mathfrak{f}} + \gamma_{\mathfrak{m}}$, where $\gamma_{\mathfrak{f}}|_{\mathfrak{m}} = 0$ and $\gamma_{\mathfrak{m}}|_{\mathfrak{f}} = 0$. Let $\varphi_s : \mathfrak{q} \to \mathfrak{q}$ be the same as in Section 1.1 with $V = \mathfrak{m}$. Set $\varphi_s(\gamma) = \gamma_{\mathfrak{f}} + s\gamma_{\mathfrak{m}}$. It is well known and easily verified that, for any $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and $\xi \in \mathfrak{q}^*$, one has $d_\xi H \in \mathfrak{z}(\mathfrak{q}^\xi)$, where $\mathfrak{z}(\mathfrak{q}^\xi)$ is the centre of $\mathfrak{q}^\xi$. A standard calculation with differentials shows that

$$d_\gamma(\varphi_s(F)) = \varphi_s(d_{\varphi_s(\gamma)}F) \tag{2·1}$$

for any $F \in \mathcal{S}(\mathfrak{q})$.

Theorem 2.1. *The subalgebra $\mathcal{Z} = \mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ is Poisson commutative if and only if*

$$\hat{\gamma}_{\mathfrak{f}}\big((d_{\varphi_s(\gamma)}H)_{\mathfrak{f}}, (d_{\varphi_{s'}(\gamma)}H')_{\mathfrak{f}}\big) = 0$$

for each $\gamma \in \mathfrak{q}^*$, all nonzero $s, s' \in \Bbbk$, and all $H, H' \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$.

Proof. It suffices to prove the assertion for homogeneous $H, H' \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Note that if $H \in \mathcal{S}^d(\mathfrak{q})$ and $H = \sum_{j=0}^d H_{(d-j,j)}$, then $\varphi_s(H) = \sum_j s^j H_{(d-j,j)}$. Therefore, employing the standard argument with the Vandermonde determinant, one shows that

$$\mathcal{Z} = \mathsf{alg}\langle \varphi_s(H) \mid H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}, s \in \Bbbk^\times \rangle. \tag{2·2}$$

Hence the algebra $\mathcal{Z}$ is Poisson commutative if and only if for all $H, H' \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$, all nonzero $s, s' \in \Bbbk$, and any $\gamma \in \mathfrak{q}^*$, by (1·1), we have

$$A_{s,s'} = A_{s,s',H,H',\gamma} := \hat{\gamma}(d_\gamma\varphi_s(H), d_\gamma\varphi_{s'}(H')) = \{\varphi_s(H), \varphi_{s'}(H')\}(\gamma) = 0.$$

Suppose that H, H' and γ are fixed. Then there is no ambiguity in the use of $A_{s,s'}$.

Set $\xi = d_{\varphi_s(\gamma)}H$ and $\eta = d_{\varphi_{s'}(\gamma)}H'$. Since $\varphi_s(H)$ belongs to the Poisson centre of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \}_{(s^{-1})})$, we derive from (2·1) that

$$\begin{aligned}\gamma([d_\gamma\varphi_s(H), d_\gamma\varphi_{s'}(H')]_{(s^{-1})}) &= \gamma([\varphi_s(\xi), \varphi_{s'}(\eta)]_{(s^{-1})}) \\ &= \gamma([\xi_{\mathfrak{f}} + s\xi_{\mathfrak{m}}, \eta_{\mathfrak{f}} + s'\eta_{\mathfrak{m}}]_{(s^{-1})}) = 0.\end{aligned}$$

Similarly, $\varphi_{s'}(H')$ belongs to the Poisson centre of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \}_{((s')^{-1})})$ and hence

$$\gamma([\xi_{\mathfrak{f}} + s\xi_{\mathfrak{m}}, \eta_{\mathfrak{f}} + s'\eta_{\mathfrak{m}}]_{((s')^{-1})}) = 0.$$

For all $\tilde{s} \in \Bbbk^\times$ and $\tilde{H} \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$, we have $\hat{\gamma}(\mathfrak{f}, d_\gamma\varphi_{\tilde{s}}(\tilde{H})) = 0$, since $\mathcal{Z} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{f}}$. Therefore, $\hat{\gamma}(\varphi_s(\xi), \eta_{\mathfrak{f}}) = \hat{\gamma}(\xi_{\mathfrak{f}}, \varphi_{s'}(\eta)) = 0$. Thus,

$$\begin{aligned}C &:= \hat{\gamma}_{\mathfrak{f}}((d_{\varphi_s(\gamma)}H)_{\mathfrak{f}}, (d_{\varphi_{s'}(\gamma)}H')_{\mathfrak{f}}) \\ &= \hat{\gamma}(\xi_{\mathfrak{f}}, \eta_{\mathfrak{f}}) = -s'\hat{\gamma}(\xi_{\mathfrak{f}}, \eta_{\mathfrak{m}}) = -s\hat{\gamma}(\xi_{\mathfrak{m}}, \eta_{\mathfrak{f}}).\end{aligned} \tag{2·3}$$

Let us substitute this into the formulas

$$\begin{aligned}\gamma([\varphi_s(\xi),\varphi_{s'}(\eta)]_{(s^{-1})}) &= \hat\gamma(\xi_{\mathfrak f},\eta_{\mathfrak f}) + s\hat\gamma(\xi_{\mathfrak m},\eta_{\mathfrak f}) + s'\hat\gamma(\xi_{\mathfrak f},\eta_{\mathfrak m})\\ &\quad+\frac{s'}{s}\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + s'\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})=0,\\ \gamma([\varphi_s(\xi),\varphi_{s'}(\eta)]_{((s')^{-1})}) &= \hat\gamma(\xi_{\mathfrak f},\eta_{\mathfrak f}) + s\hat\gamma(\xi_{\mathfrak m},\eta_{\mathfrak f}) + s'\hat\gamma(\xi_{\mathfrak f},\eta_{\mathfrak m})\\ &\quad+\frac{s}{s'}\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + s\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})=0,\end{aligned}$$

obtaining the equalities

$$\begin{aligned}&C-C-C+s^{-1}s'\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + s'\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})=0,\\ &C-C-C+s(s')^{-1}\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + s\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})=0.\end{aligned}$$

Furthermore $A_{s,s'}=-C+ss'\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f})+ss'\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})$.

Suppose that $C=0$, then

$$\begin{aligned}s^{-1}\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + \gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})&=0,\\ (s')^{-1}\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f}) + \gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})&=0.\end{aligned}$$

Thereby $(s^{-1}-(s')^{-1})\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f})=0$ and $(s-s')\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})=0$. If $s\neq s'$, then necessarily $A_{s,s'}=0$. Since $A_{s,s'}$ is a polynomial in s and s' with constant coefficients, $A_{s,s'}=0$ for all nonzero s,s'. This settles the 'if' part.

In order to prove the 'only if' implication, suppose that $A_{s,s'}=0$ for all $s,s'\in\Bbbk^\times$. Then $x=\gamma_{\mathfrak f}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak f})$ and $y=\gamma_{\mathfrak m}([\xi_{\mathfrak m},\eta_{\mathfrak m}]_{\mathfrak m})$ satisfy $s^{-1}s'x+s'y=s(s')^{-1}x+sy=ss'(x+y)=C$. Assume that $s\neq s'$ and that $s,s'\neq 1$. Then

$$\begin{cases}\frac{s'+s}{ss'}\cdot x+y=0;\\ \frac{s+1}{s}\cdot x+y=0,\end{cases}$$

and the only solution of this system is $x=y=0$. Hence $C=0$. Since C is a polynomial in s and s' with constant coefficients, the equality

$$\hat\gamma_{\mathfrak f}((d_{\varphi_s(\gamma)}H)_{\mathfrak f},(d_{\varphi_{s'}(\gamma)}H')_{\mathfrak f})=0$$

holds for all $s,s'\in\Bbbk^\times$. □

Corollary 2.2. *If $\mathfrak f$ is an abelian Lie algebra, then $\mathcal Z_{(\mathfrak q,\mathfrak f)}$ is Poisson commutative.*

Proof. Since $[\mathfrak{f},\mathfrak{f}]=0$, we have $[(d_{\varphi_s(\gamma)}H)_{\mathfrak{f}},(d_{\varphi_{s'}(\gamma)}H')_{\mathfrak{f}}]=0$ for each $\gamma\in\mathfrak{q}^*$, all nonzero $s,s'\in\Bbbk$, and all $H,H'\in\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Hence $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})}$ is Poisson-commutative by Theorem 2.1. □

Let $\mathfrak{g}=\mathsf{Lie}(G)$ be a reductive Lie algebra. Then $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ via a G-invariant non-degenerate symmetric bilinear form (,) and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial ring. Let $\{H_1,\dots,H_l\}$ be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $\deg H_j=:d_j$. By the *Kostant regularity criterion* for $\mathfrak{g}$ [5, Theorem 9],

$$(2{\cdot}4)\qquad \langle d_\xi H_j \mid 1\leqslant j\leqslant l\rangle_{\Bbbk}=\mathfrak{g}^\xi \ \text{ if and only if } \ \xi\in\mathfrak{g}^*_{\mathsf{reg}}.$$

Recall that $\mathfrak{g}^\xi=\mathfrak{z}(\mathfrak{g}^\xi)$ if and only if $\xi\in\mathfrak{g}^*_{\mathsf{reg}}$ [10, Theorem 3.3].

Example 2.3. If $\mathfrak{g}=\mathfrak{gl}_n$, then $x^k\in\mathfrak{g}^x$ for any $x\in\mathfrak{g}$ and $k\in\mathbb{N}$. (Here x^k is the usual matrix power.) Moreover, if we identify $\mathfrak{g}$ and $\mathfrak{g}^*$, then $d_x\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}=\langle x^k \mid 0\leqslant k<n\rangle_{\Bbbk}$.

Consider the pair $(\mathfrak{g},\mathfrak{f})=(\mathfrak{gl}_4,\mathfrak{sl}_2)$ with $\mathfrak{sl}_2$ embedded in the right lower corner.

$$\text{Take } \gamma=\begin{pmatrix}1&0&0&1\\0&0&1&0\\1&0&1&0\\0&1&0&-1\end{pmatrix}.\ \text{ Then } \gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}}=\begin{pmatrix}s&0&0&s\\0&0&s&0\\s&0&1&0\\0&s&0&-1\end{pmatrix}.$$

Note that $\gamma_{\mathfrak{f}}\neq 0$. For any $k\geqslant 0$, $(\varphi_s(\gamma))^k=(\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^k$ belongs to $d_{\varphi_s(\gamma)}\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Hence $((\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^k)_{\mathfrak{f}}\in(d_{\varphi_s(\gamma)}\mathcal{S}(\mathfrak{g})^{\mathfrak{g}})_{\mathfrak{f}}$. Let us do calculations for $k=2,3$:

$$(\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^2=\begin{pmatrix}s^2&s^2&0&s^2-s\\s^2&0&s&0\\s^2+s&0&1&s^2\\0&-s&s^2&1\end{pmatrix}\quad\text{and}$$

$$(\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^3=\begin{pmatrix}*&*&*&*\\ *&*&*&*\\ *&*&1&s^3\\ *&*&0&-1\end{pmatrix}.$$

Let $e=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $h=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, and $f=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ form the standard basis of $\mathfrak{sl}_2$. Then

$$\gamma_{\mathfrak{f}}=h,\ \ ((\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^2)_{\mathfrak{f}}=s^2(e+f),\ \text{ and } ((\gamma_{\mathfrak{f}}+s\gamma_{\mathfrak{m}})^3)_{\mathfrak{f}}=s^3e+h.$$

Therefore, if $s\neq 0$, then $\big\langle (d_{\varphi_s(\gamma)}H)_{\mathfrak{f}} \mid H\in\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}\big\rangle_{\Bbbk}=\mathfrak{f}$. Since $(h,[\mathfrak{f},\mathfrak{f}])\neq 0$, we conclude that

$$\hat{\gamma}_{\mathfrak{f}}((d_{\varphi_s(\gamma)}H)_{\mathfrak{f}}, (d_{\varphi_{s'}(\gamma)}H')_{\mathfrak{f}}) \neq 0$$

for all nonzero s, s'. Thus, by Theorem 2.1, $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})}$ is not Poisson commutative.

Remark 2.4. Example 2.3 also implies that $\mathcal{Z}_{(\mathfrak{g},\mathfrak{sl}_2)}$ is not Poisson commutative if $\mathfrak{g} = \mathfrak{gl}_4$ is replaced with $\mathfrak{sl}_4$. For, $\mathfrak{gl}_4 = \mathfrak{z} \oplus \mathfrak{sl}_4$ with $\mathfrak{z} = \Bbbk I_4$, hence $\mathcal{S}(\mathfrak{gl}_4)^{\mathfrak{gl}_4}$ is generated by $\mathcal{S}(\mathfrak{sl}_4)^{\mathfrak{sl}_4}$ and $\mathfrak{z}$. For any reductive $\mathfrak{f} \subset \mathfrak{sl}_4$, the algebra $\mathcal{Z}_{(\mathfrak{gl}_4,\mathfrak{f})} = \mathsf{alg}\langle \mathcal{Z}_{(\mathfrak{sl}_4,\mathfrak{f})}, \mathfrak{z}\rangle$ is Poisson commutative if and only if $\mathcal{Z}_{(\mathfrak{sl}_4,\mathfrak{f})}$ is.

Example 2.3 easily generalises to the pairs $(\mathfrak{gl}_n, \mathfrak{gl}_m)$ with $n \geqslant m+2$. On the other hand, one can prove that the algebra $\mathcal{Z}_{(\mathfrak{gl}_3,\mathfrak{sl}_2)}$ or $\mathcal{Z}_{(\mathfrak{sl}_3,\mathfrak{sl}_2)}$ is still Poisson commutative.

Example 2.5. Let us show that, for a special choice of $\mathfrak{f}$, the algebra $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})}$ is rather close to an $\mathcal{M}F$-subalgebra.

Let $h \in \mathfrak{g}$ be a semisimple element such that $(h, h) \neq 0$. Set $\mathfrak{f} = \langle h\rangle = \langle h\rangle_{\Bbbk}$. Then $\mathfrak{m} \subset \mathfrak{g}$ is the orthogonal complement of h with respect to $(\ , \ )$ and the bi-homogeneous decomposition of $H_j \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is

$$H_j = H_{j,0}h^{d_j} + H_{j,1}h^{d_j-1} + \ldots + H_{j,k}h^{d_j-k} + \ldots + H_{j,d_j},$$

where $H_{j,k} \in \mathcal{S}^k(\mathfrak{m})$. By definition, $\mathcal{Z}_{(\mathfrak{g},\langle h\rangle)}$ is generated by $H_{j,k}h^{d_j-k}$ with $1 \leqslant j \leqslant l$ and $0 \leqslant k \leqslant d_j$. On the one hand, we had $\mathfrak{f} = \langle h\rangle$. On the other hand, let $\gamma \in \mathfrak{g}^*$ be such that $\gamma(\mathfrak{m}) = 0$ and $\gamma(h) = 1$. Actually, $\gamma = \frac{h}{(h,h)}$ under the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$. Then

$$\partial_\gamma^k H_j = \sum_{r=k}^{d_j} r(r-1)\ldots(r-k+1)h^{r-k}H_{j,d_j-r}.$$

If $H \in \mathcal{S}^2(\mathfrak{g})^{\mathfrak{g}}$ is the quadratic form corresponding to $(\ , \ )$, then $\partial_\gamma H = ch$ for some $c \in \Bbbk^\times$. Hence $h \in (\mathcal{MF})_\gamma$. Arguing by induction on k, we obtain $H_{j,k} \in (\mathcal{MF})_\gamma$ for $k \leqslant d_j$. Thus

$$\mathcal{Z}_{(\mathfrak{g},\langle h\rangle)} \subset (\mathcal{MF})_\gamma = (\mathcal{MF})_h \subset \mathsf{alg}\left\langle \mathcal{Z}_{(\mathfrak{g},\langle h\rangle)}, h, h^{-1}\right\rangle = \mathsf{alg}\left\langle \mathcal{Z}_{(\mathfrak{g},\langle h\rangle)}, h^{-1}\right\rangle,$$

where the last equality holds, because $h^2 \in \mathcal{Z}_{(\mathfrak{g},\langle h\rangle)}$.

3. Properties of the algebra $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$

Suppose that $\mathfrak{g}$ is semisimple. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and Δ the root system of $(\mathfrak{g},\mathfrak{t})$. By Corollary 2.2, the algebra $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is Poisson commutative, and our goal is to prove that this algebra has a number of remarkable properties. Let $\mathfrak{g}_\gamma$ be the root space corresponding to $\gamma \in \Delta$ and let $e_\gamma \in \mathfrak{g}_\gamma$ be a nonzero vector. Then $\mathfrak{m} = \mathfrak{t}^\perp = \bigoplus_{\gamma\in\Delta} \mathfrak{g}_\gamma$.

Recall that $\{H_1, \dots, H_l\}$ is a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and $\deg H_j = d_j$. One has $\sum_{j=1}^{l} d_j = \boldsymbol{b}(\mathfrak{g})$. The vector space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ provides the bi-homogeneous decomposition of each H_j:

$$H_j = \sum_{i=0}^{d_j} (H_j)_{(i,d_j-i)},$$

where $(H_j)_{(i,d_j-i)} \in \mathcal{S}^i(\mathfrak{t}) \otimes \mathcal{S}^{d_j-i}(\mathfrak{m}) \subset \mathcal{S}^{d_j}(\mathfrak{g})$. Recall that $\mathcal{Z} := \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is the algebra generated by

$$\{(H_j)_{(i,d_j-i)} \mid j = 1, \dots, l; i = 0, 1, \dots, d_j\}. \tag{3·1}$$

Since each H_j is $\mathfrak{g}$-invariant, all the bi-homogeneous components in (3·1) are $\mathfrak{t}$-invariant. Hence $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$. The total number of these functions is $\sum_{j=1}^{l}(d_j + 1) = \boldsymbol{b}(\mathfrak{g}) + l$, but some of them are identically equal to zero. Indeed, $(H_j)_{(d_j-1,1)} \in \mathcal{S}^{d_j-1}(\mathfrak{t}) \otimes \mathfrak{m}$ and $\mathfrak{m}^{\mathfrak{t}} = \{0\}$, hence $(H_j)_{(d_j-1,1)} \equiv 0$ for $j = 1, \dots, l$. Therefore, the number of nonzero generators of $\mathcal{Z}$ is at most $\boldsymbol{b}(\mathfrak{g})$.

The bi-homogeneous component $(H_j)_{(d_j,0)} \in \mathcal{S}^{d_j}(\mathfrak{t})$ is the restriction of H_j to $\mathfrak{t} \simeq \mathfrak{t}^*$. Therefore, by the Chevalley restriction theorem, the polynomials $(H_j)_{(d_j,0)}$, $j = 1, \dots, l$, are the free generators of $\mathcal{S}(\mathfrak{t})^W$, where W is the Weyl group of $\mathfrak{t}$. This means that having replaced $(H_1)_{(d_1,0)}, \dots, (H_l)_{(d_l,0)}$ with a basis of $\mathfrak{t}$ and keeping intact all other bi-homogeneous components (generators of $\mathcal{Z}$), we obtain a larger subalgebra $\tilde{\mathcal{Z}}$, which is an algebraic extension of $\mathcal{Z}$ (i.e. $\operatorname{tr.deg} \tilde{\mathcal{Z}} = \operatorname{tr.deg} \mathcal{Z}$). Furthermore, since $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$, $\tilde{\mathcal{Z}}$ is still Poisson commutative.

Once again, we use the map φ_s defined in Section 1.1. By (2·4), if $\varphi_s(\gamma) \in \mathfrak{g}^*_{\mathsf{reg}}$, then $d_{\varphi_s(\gamma)}\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathfrak{g}^{\varphi_s(\gamma)}$; and by (2·1), we have

$$d_\gamma \varphi_s(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}) = \varphi_s(\mathfrak{g}^{\varphi_s(\gamma)}). \tag{3·2}$$

As before, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$.

Lemma 3.1. *Let $h \in \mathfrak{t}$ and $x \in \mathfrak{m}$ be such that $(h + \Bbbk x) \cap \mathfrak{g}^*_{\mathsf{reg}} \neq \varnothing$. Then $d_{h+x}\tilde{\mathcal{Z}} = \mathfrak{t} + d_h((\mathcal{MF})_x)$. Moreover, if $h \in \mathfrak{g}^*_{\mathsf{reg}}$, then $d_{h+x}\tilde{\mathcal{Z}} = d_h((\mathcal{MF})_x) = d_x((\mathcal{MF})_h)$.*

Proof. The assumption $(h \oplus \Bbbk x) \cap \mathfrak{g}^*_{\mathsf{reg}} \neq \varnothing$ implies that

$$\Omega := \{s \in \Bbbk^\times \mid h + sx \in \mathfrak{g}^*_{\mathsf{reg}}\}$$

is a nonempty open subset of $\Bbbk^\times$. Since Ω is infinite, we can strengthen (2·2) as

$$\mathcal{Z} = \mathsf{alg}\langle \varphi_s(H) \mid H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}, s \in \Omega \rangle. \tag{3·3}$$

Combining this with (3·2), we obtain

$$d_{h+x}\tilde{\mathcal{Z}} = \mathfrak{t} + \sum_{s\in\Omega} d_{h+x}\varphi_s(\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}) = \mathfrak{t} + \sum_{s\in\Omega} \varphi_s(\mathfrak{g}^{h+sx}) = \mathfrak{t} + \sum_{s\in\Omega} \mathfrak{g}^{h+sx}.$$

Set $\Omega' = \Omega \sqcup \{0\}$ if $h \in \mathfrak{g}^*_{\mathsf{reg}}$ and $\Omega' = \Omega$ otherwise. Then the equality $\sum_{s\in\Omega'} \mathfrak{g}^{h+sx} = d_h((\mathcal{MF})_x)$ follows from [13, Lemma 1.3], see also the proof of Lemma 2.1 in [13, Sect. 2]. If $h \notin \mathfrak{g}^*_{\mathsf{reg}}$, we are done. If $h \in \mathfrak{g}^*_{\mathsf{reg}}$, then $\mathfrak{g}^h = \mathfrak{t}$ and $\mathfrak{t} + \sum_{s\in\Omega} \mathfrak{g}^{h+sx} = \sum_{s\in\Omega'} \mathfrak{g}^{h+sx}$.

Finally, we recall that $d_h((\mathcal{MF})_x) = d_x((\mathcal{MF})_h)$ for any $x, h \in \mathfrak{g}$ by [13, Eq. (2·3)]. □

Theorem 3.2. *For $\mathcal{Z} = \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ and $\tilde{\mathcal{Z}}$ as above, we have*

- (i) tr.deg $\mathcal{Z}$ = tr.deg $\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$ *and both algebras $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are polynomial;*
- (ii) *both $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are complete on each regular orbit;*
- (iii) *$\tilde{\mathcal{Z}}$ is a maximal Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{g})$.*

Proof. (i) Since $\mathcal{Z} \subset \tilde{\mathcal{Z}}$ is an algebraic extension, the first equality follows. Take a principal $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$ such that $h \in \mathfrak{t}$ and $e, f \in \mathfrak{m}$. Note that any nonzero element of $\langle e, h, f\rangle_\Bbbk$ is regular in $\mathfrak{g} \simeq \mathfrak{g}^*$. Pick a nonzero $x \in \langle e, f\rangle_\Bbbk \subset \mathfrak{m}$ and consider the subspace

$$d_{h+x}\tilde{\mathcal{Z}} := \{d_{h+x}F \mid F \in \tilde{\mathcal{Z}}\} \subset \mathfrak{g}.$$

Because $\varphi_s(h + x) = h + sx \in \mathfrak{g}^*_{\mathsf{reg}}$ for all s, we have $d_{h+x}\tilde{\mathcal{Z}} = d_h((\mathcal{MF})_x)$ by Lemma 3.1.

One of the properties of $\mathcal{MF}$-subalgebras is that $\dim d_h((\mathcal{MF})_x) = \boldsymbol{b}(\mathfrak{g})$, if

$$(\Bbbk x \oplus \Bbbk h) \cap \mathfrak{g}_{\mathsf{sing}} = \{0\},$$

see [12], [13, Cor. 1.6 & Lemma 2.1]. Thus $\dim d_{h+x}\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$. It follows that tr.deg $\tilde{\mathcal{Z}} \geqslant \boldsymbol{b}(\mathfrak{g})$, and since tr.deg $\mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{g})$ for any Poisson commutative subalgebra, we actually get the equality. As both $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ have at most $\boldsymbol{b}(\mathfrak{g})$ generators, they are polynomial. Therefore, $\mathcal{Z}$ is freely generated by

$$\{(H_j)_{(i,d_j-i)} \mid 1 \leqslant j \leqslant l;\ i = 0, 1, \ldots, d_j - 2, d_j\},$$

while $\tilde{\mathcal{Z}}$ is freely generated by a basis of $\mathfrak{t}$ and the components $(H_j)_{(i,d_j-i)}$, where $1 \leqslant j \leqslant l$ and $i = 0, 1, \ldots, d_j - 2$.

(ii) In part (i), we proved that $\dim d_{h+x}\tilde{\mathcal{Z}} = \boldsymbol{b}(g)$. Then [13, Lemma 1.2] implies that $\tilde{\mathcal{Z}}$ is complete on the orbit $G(h + x)$. For an appropriate choice of $x \in \langle e, f\rangle$, we obtain a nilpotent element $h + x \in \langle e, h, f\rangle_{\Bbbk} \simeq \mathfrak{sl}_2$. Hence $\tilde{\mathcal{Z}}$ is complete on the regular nilpotent orbit. Then a standard deformation argument, see [13, Cor. 2.6], shows that $\tilde{\mathcal{Z}}$ is complete on every regular orbit. The same line of argument applies to $\mathcal{Z}$, since $d_{h+x}\mathcal{S}(\mathfrak{t})^W = d_h\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathfrak{t}$ and $d_{h+x}\mathcal{Z} = d_{h+x}\tilde{\mathcal{Z}}$.

(iii) The maximality of $\tilde{\mathcal{Z}}$ will follow from the fact that the subvariety $Y = \{\gamma \in \mathfrak{g}^* \mid \dim d_\gamma\tilde{\mathcal{Z}} < \boldsymbol{b}(\mathfrak{g})\}$ is of codimension $\geqslant 2$ in $\mathfrak{g}^*$ (see below). We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form and regard Y as a subvariety of $\mathfrak{g}$. Write $\gamma = h' + x'$ with $h' \in \mathfrak{t}$, $x' \in \mathfrak{m}$. If $\langle h', x'\rangle_{\Bbbk} \cap \mathfrak{g}_{\mathsf{sing}} = \{0\}$, then $\dim d_{h'}((\mathcal{MF})_{x'}) = \boldsymbol{b}(\mathfrak{g})$ [12, Theorem 2.5] and $\dim d_\gamma\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$ by Lemma 3.1.

Consider the map $\psi : \mathfrak{g}_{\mathsf{sing}} \times \Bbbk \to \mathfrak{g}$ defined by $\psi(\xi, s) = \xi_{\mathfrak{t}} + s\xi_{\mathfrak{m}}$ and let $\tilde{Y}$ be the closure of $\mathrm{Im}\,(\psi)$. Set $\mathfrak{t}_{\mathsf{sing}} := \mathfrak{t} \cap \mathfrak{g}_{\mathsf{sing}}$ and $\mathfrak{m}_{\mathsf{sing}} := \mathfrak{m} \cap \mathfrak{g}_{\mathsf{sing}}$. Then

$$Y \subset \tilde{Y} \cup (\mathfrak{t}_{\mathsf{sing}} \times \mathfrak{m}) \cup (\mathfrak{t} \times \mathfrak{m}_{\mathsf{sing}}).$$

- Since $\operatorname{codim} \mathfrak{g}_{\mathsf{sing}} = 3$, we have $\dim \tilde{Y} \leqslant \dim \mathfrak{g} - 2$.
- As $\mathfrak{m}_{\mathsf{sing}}$ is conical and $\langle e, f\rangle_{\Bbbk} \cap \mathfrak{m}_{\mathsf{sing}} = \{0\}$, we have $\dim \mathfrak{m}_{\mathsf{sing}} \leqslant \dim \mathfrak{m} - 2$. Therefore, $\mathfrak{t} \times \mathfrak{m}_{\mathsf{sing}} \subset \mathfrak{g}$ does not contain divisors.
- We prove below that $\dim(Y \cap (\mathfrak{t}_{\mathsf{sing}} \times \mathfrak{m})) \leqslant \dim \mathfrak{g} - 2$, which yields the required estimate of $\operatorname{codim} Y$.

The subset $\mathfrak{t}_{\mathsf{sing}} \subset \mathfrak{t}$ is the union of all reflection hyperplanes in $\mathfrak{t}$. That is, if

$$\mathcal{H}_\gamma = \{x \in \mathfrak{t} \mid (\gamma, x) = 0\},$$

then $\mathfrak{t}_{\mathsf{sing}} = \bigcup_{\gamma\in\Delta} \mathcal{H}_\gamma$. (Of course, $\mathcal{H}_\gamma = \mathcal{H}_{-\gamma}$.) Suppose that $h' \in \mathcal{H}_\nu$ is generic, i.e., $h' \in \mathcal{H}_\nu \setminus \bigcup_{\gamma\ne\pm\nu} \mathcal{H}_\gamma$. Then $h' \in \mathfrak{g}$ is subregular and

$$\mathfrak{g}^{h'} = \mathfrak{t} \oplus \mathfrak{g}_\nu \oplus \mathfrak{g}_{-\nu} = \mathcal{H}_\nu \oplus \langle e_\nu, h_\nu, e_{-\nu}\rangle_{\Bbbk} \simeq \mathcal{H}_\nu \oplus \mathfrak{sl}_2,$$

where $h_\nu = [e_\nu, e_{-\nu}]$ and $\mathcal{H}_\nu$ is the centre of $\mathfrak{g}^{h'}$. Note also that $\mathcal{H}_\nu = d_{h'}\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \subset \mathfrak{t}$, cf. [14, Lemma 4.9]. Without loss of generality, we may assume that ν is a **simple** root with respect to some choice of $\Delta^+ \subset \Delta$. Let $\Pi \subset \Delta^+$ be the corresponding set of simple roots and $\mathfrak{m} = \mathfrak{u} \oplus \mathfrak{u}^-$. We may also assume that $e = \sum_{\alpha\in\Pi} c_\alpha e_\alpha \in \mathfrak{u}$ with $c_\alpha \in \Bbbk^\times$ and $f = \sum_{\alpha\in\Pi} e_{-\alpha} \in \mathfrak{u}^-$ for a principal $\mathfrak{sl}_2$-triple $\{e, h, f\}$ with $h \in \mathfrak{t}$, cf. [5, Theorem 4]. For $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$, we have $f + \mathfrak{b} \subset \mathfrak{g}_{\mathsf{reg}}$ by [5, Lemma 10]. In particular, $h' + sf \in \mathfrak{g}_{\mathsf{reg}}$ for any $s \in \Bbbk^\times$.

Lemma 3.3. *If $h' \in \mathcal{H}_\nu$ is generic, then $\mathfrak{g}^{h'+sf} \subset \mathcal{H}_\nu \oplus \mathfrak{u}^-$ for any $s \ne 0$.*

Proof. As is well known, $(\mathfrak{g}^{h'+sf})^\perp = [\mathfrak{g}, h' + sf]$. Hence it suffices to prove that $[\mathfrak{g}, h' + sf] \supset (\mathcal{H}_\nu \oplus \mathfrak{u}^-)^\perp = \langle h_\nu\rangle_{\Bbbk} \oplus \mathfrak{u}^-$.

Since $h' + sf \in \mathfrak{t} \oplus \mathfrak{u}^- =: \mathfrak{b}^-$ is regular in $\mathfrak{g}$, we have $\mathfrak{g}^{h'+sf} \subset \mathfrak{b}^-$. Hence $[\mathfrak{g}, h' + sf] \supset (\mathfrak{b}^-)^\perp = \mathfrak{u}^-$. Next, $[e_\nu, h' + sf] = [e_\nu, sf] = s[e_\nu, e_{-\nu}] = s{\cdot}h_\nu \in [\mathfrak{g}, h' + sf]$. □

Now, set

$$\mathbb{V} := d_{h'}((\mathcal{MF})_f) = \sum_{s\ne0} \mathfrak{g}^{h'+sf},$$

where the last equality stems from [13, Lemma 1.3]. On the one hand, $\mathbb{V} \subset \mathcal{H}_\nu \oplus \mathfrak{u}^-$ by the above lemma. On the other hand, $\dim \mathbb{V} = \boldsymbol{b}(\mathfrak{g}) - 1$ in view of [13, proof of Theorem 2.4]. Hence $\mathbb{V} = \mathcal{H}_\nu \oplus \mathfrak{u}^-$.

The differentials $d_{h'+f}((H_j)_{(d_j,0)}) = d_{h'}((H_j)_{(d_j,0)}) = d_{h'}H_j$ with $1 \leqslant j \leqslant l$ are linearly dependent, because they are contained in $\mathcal{H}_\nu$, hence $\dim d_{h'+f}\mathcal{Z} \leqslant \boldsymbol{b}(\mathfrak{g}) - 1$. Recall that $h' + sf \in \mathfrak{g}_{\mathsf{reg}}$ for any $s \in \Bbbk^\times$. Combining (3·3) with (3·2), we obtain

$$\mathbb{V}_{\mathcal{Z}} := d_{h'+f}\mathcal{Z} \supset \sum_{s\ne0} \varphi_s(\mathfrak{g}^{h'+sf}) = \varphi_s(\mathbb{V}) = \mathcal{H}_\nu \oplus \mathfrak{u}^-.$$

Thus $\mathbb{V}_{\mathcal{Z}} = \mathcal{H}_\nu + \mathfrak{u}^-$. Next $d_{h'+f}\tilde{\mathcal{Z}} \ne d_{h'+f}\mathcal{Z}$, since $\mathfrak{t} \not\subset \mathbb{V}_{\mathcal{Z}}$. We obtain $\dim d_{h'+f}\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$, which means that $\dim d_\gamma\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$ on a dense open subset of $\mathcal{H}_\nu \times \mathfrak{m}$. Since $\nu \in \Delta$ is arbitrary, this implies that $\dim(Y \cap (\mathfrak{t}_{\mathsf{sing}} \times \mathfrak{m})) \leqslant \dim \mathfrak{g} - 2$.

Thus, we have proved that $\dim Y \leqslant \dim \mathfrak{g} - 2$.

Since $\tilde{\mathcal{Z}}$ is generated by algebraically independent homogeneous polynomials and $\operatorname{codim} Y \geqslant 2$, it follows from [11, Theorem 1.1] that $\tilde{\mathcal{Z}}$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$ (i.e., if $F \in \mathcal{S}(\mathfrak{g})$ is algebraic over the quotient field of $\tilde{\mathcal{Z}}$, then $F \in \tilde{\mathcal{Z}}$). An inclusion $\tilde{\mathcal{Z}} \subset \mathcal{A} \subset \mathcal{S}(\mathfrak{g})$, where $\{\mathcal{A}, \mathcal{A}\} = 0$, is only possible if $\mathcal{A}$ is an algebraic extension of $\tilde{\mathcal{Z}}$, because $\operatorname{tr.deg} \tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$ and $\operatorname{tr.deg} \mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{g})$. Therefore we must have $\tilde{\mathcal{Z}} = \mathcal{A}$. □

Remark 3.4. We know that $\tilde{\mathcal{Z}} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$ and $\operatorname{tr.deg} \tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$. If $h \in \mathfrak{t}^*_{\mathsf{reg}}$, then these two properties are also satisfied for $(\mathcal{MF})_h$ [12]. One may say that $\tilde{\mathcal{Z}}$, as well as $\mathcal{Z}$, resembles all such $\mathcal{MF}$-subalgebras. However, there is no choice of $h \in \mathfrak{t}^*$ involved in the construction of $\tilde{\mathcal{Z}}$ and $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{N_G(\mathfrak{t})}$ unlike any of $(\mathcal{MF})_h$ with $h \in \mathfrak{t}^*_{\mathsf{reg}}$.

Furthermore, by Lemma 3.1, we have $d_{h+x}\tilde{\mathcal{Z}} = d_h((\mathcal{MF})_x) = d_x((\mathcal{MF})_h)$ for any $x \in \mathfrak{m}$ and $h \in \mathfrak{t}^*_{\mathsf{reg}}$. It is tempting to further investigate this relationship.

Another intriguing task is to produce a quantisation of $\tilde{\mathcal{Z}}$, i.e., a commutative subalgebra of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ such that its graded image in $\mathcal{S}(\mathfrak{g})$ is $\tilde{\mathcal{Z}}$.

References

[1] O. I. Bogoyavlenski. Integrable Euler equations associated with filtrations of Lie algebras, *Matem. Sbornik*, **121**, no. 2 (1983), 233–242 (Russian). English transl.: *Math. USSR-Sb.*, **49**:1 (1984), 229–238.

[2] M. Brion. Invariants et covariants des groupes algébriques réductifs, Dans: "*Théorie des invariants et géometrie des variétés quotients*" (Travaux en cours, t. **61**), 83–168, Paris: Hermann, 2000.

[3] B. Feigin, E. Frenkel, and L. Rybnikov. Opers with irregular singularity and spectra of the shift of argument subalgebra, *Duke Math. J.*, **155** (2010), no. 2, 337–363.

[4] I. M. Gelfand and I. S. Zakharevich. Webs, Lenard schemes, and the local geometry of bi-Hamiltonian Toda and Lax structures, *Selecta Math., New Ser.*, **6** (2000), 131–183.

[5] B. Kostant. Lie group representations on polynomial rings, *Amer. J. Math.*, **85** (1963), 327–404.

[6] I. V. Mikityuk. Integrability of Euler equations connected with filtrations of semisimple Lie algebras, *Matem. Sb.*, **125**, no. 4 (1984), 539–546 (Russian). English transl.: *Math. USSR-Sb.*, **53**:2 (1986), 541–549.

[7] A. S. Mishchenko and A. T. Fomenko. Euler equation on finite-dimensional Lie groups, *Izv. AN SSSR. Ser. Matem.* **42**, no. 2 (1978), 396–415 (Russian). English transl.: *Math. USSR-Izv.* **12** (1978), 371–389.

[8] A. Molev and O. Yakimova. Quantisation and nilpotent limits of Mishchenko–Fomenko subalgebras, *Represent. Theory*, **23** (2019), 350–378.

[9] A. Panasyuk. Veronese webs for bi-Hamiltonian structures of higher corank. *Poisson geometry* (Warszawa, 1998), 251–261, Banach Center Publ., **51**, Polish Acad. Sci. Inst. Math., Warszawa, 2000.

[10] D. Panyushev. The index of a Lie algebra, the centraliser of a nilpotent element, and the normaliser of the centraliser, *Math. Proc. Camb. Phil. Soc.*, **134**, Part 1 (2003), 41–59.

[11] D. Panyushev, A. Premet and O. Yakimova. On symmetric invariants of centralisers in reductive Lie algebras, *J. Algebra*, **313** (2007), 343–391.

[12] D. Panyushev and O. Yakimova. The argument shift method and maximal commutative subalgebras of Poisson algebras, *Math. Res. Letters*, **15**, no. 2 (2008), 239–249.

[13] D. Panyushev and O. Yakimova. Poisson commutative subalgebras and complete integrability on non-regular coadjoint orbits and flag varieties, *Math. Zeitschrift*, **295** (2020), 101–127.

[14] D. Panyushev and O. Yakimova. Poisson commutative subalgebras of $\mathcal{S}(\mathfrak{g})$ associated with involutions, *Int. Math. Res. Notices*, **2021**, no. 23 (2021), 18367–18406.

[15] Э.Б. Винберг. О некоторых коммутативных подалгебрах универсальной обертывающей алгебры, *Изв. АН СССР. Сер. Матем.* **54**, № 1 (1990), 3–25 (Russian). English translation: E. B. Vinberg. On certain commutative subalgebras of a universal enveloping algebra, *Math. USSR-Izv.* **36** (1991), 1–22.

[16] O. Yakimova. Symmetric invariants of $\mathbb{Z}_2$-contractions and other semi-direct products, *Int. Math. Res. Notices*, (2017) 2017 (6): 1674–1716.

Institute for Information Transmission Problems of the R.A.S.
Bolshoi Karetnyi per. 19, Moscow 127051, Russia
E-mail address: panyushev@iitp.ru

Institut für Mathematik, Friedrich-Schiller-Universität Jena
Jena, 07737, Germany
E-mail address: oksana.yakimova@uni-jena.de

Received December 18, 2020
Accepted November 19, 2021