

Caustics of Lagrangian homotopy spheres with stably trivial Gauss map

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For each positive integer n , we give a geometric description of the stably trivial elements of the group $\pi_n U_n / O_n$. In particular, we show that all such elements admit representatives whose tangencies with respect to a fixed Lagrangian plane consist only of folds. By the h-principle for the simplification of caustics, this has the following consequence: if a Lagrangian distribution is stably trivial from the viewpoint of a Lagrangian homotopy sphere, then by an ambient Hamiltonian isotopy one may deform the Lagrangian homotopy sphere so that its tangencies with respect to the Lagrangian distribution are only of fold type. Thus the stable triviality of the Lagrangian distribution, which is a necessary condition for the simplification of caustics to be possible, is also sufficient. We give applications of this result to the arborealization program and to the study of nearby Lagrangian homotopy spheres.

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*DA was partially supported by the Simons Foundation.

1. Introduction

1.1. Main result

Let (M, ω) be a $2n$ -dimensional symplectic manifold, $\gamma \subset TM$ a Lagrangian distribution and $L \subset M$ a closed Lagrangian submanifold. In [AG18b] the first author established the following h-principle: if γ is homotopic to a Lagrangian distribution with respect to which L only has fold type tangencies, then L is Hamiltonian isotopic to a Lagrangian submanifold which only has fold type tangencies with respect to γ .

This h-principle reduces the problem of eliminating higher tangencies to the underlying homotopical problem. In the present article we solve this homotopical problem in the case where L has the homotopy type of a sphere. The central notion is that of stable triviality, which we now define.

Definition 1.1. We say that $\gamma|_L$ is *stably trivial* if $\gamma|_L \oplus \mathbf{R}$ is homotopic to $TL \oplus \mathbf{R}$ as Lagrangian distributions in the symplectic vector bundle $\mathbf{C}^{n+1} \rightarrow TM|_L \oplus \mathbf{C} \rightarrow L$.

Our main result is the following, where we assume $n > 1$ (for $n = 1$ the problem is trivial).

Theorem 1.2. *Let $L \subset M$ be a Lagrangian homotopy sphere in a symplectic manifold (M, ω) and $\gamma \subset TM$ a Lagrangian distribution. The tangencies of L with respect to γ can be simplified to consist of only folds via a Hamiltonian isotopy of L if and only if $\gamma|_L$ is stably trivial.*

Remark 1.3. We observe:

- (i) Since any homotopy sphere is stably parallelizable [KM63] and $\pi_{n-1}U_n \rightarrow \pi_{n-1}U_{n+1}$ is an isomorphism, note that $T(T^*L)|_L \simeq L \times \mathbf{C}^n$ as symplectic vector bundles. Hence we may think of the homotopy class of $\gamma|_L$ as an element of $\pi_n\Lambda_n$, where Λ_n the Grassmannian of Lagrangian planes in \mathbf{C}^n . From this viewpoint stable triviality is equivalent to asking that this element is in the kernel of the stabilization map $\pi_n\Lambda_n \rightarrow \pi_n\Lambda_{n+1}$.
- (ii) The hypothesis that $\gamma|_L$ is stably trivial is automatically satisfied if n is congruent to 0, 4, 6 or 7 modulo 8, since $\pi_n\Lambda_{n+1} = 0$ for those values of n . Full tables of the relevant homotopy groups can be found in Section 3.1.1 below.

- (iii) The subgroup $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$ of $\pi_n \Lambda_n$ is always cyclic, in fact it is infinite cyclic for even n and cyclic of order 2 for odd $n > 1$. We will exhibit an explicit generator in each dimension, see Remark 3.7.

The homotopical problem underlying Theorem 1.2, which by the h-principle [AG18b] is equivalent to Theorem 1.2 itself, is to show that each element in the kernel of the stabilization map $\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}$ admits a representative $S^n \rightarrow \Lambda_n$ which only has fold type tangencies with respect to some fixed but arbitrary Lagrangian plane $P \subset \mathbf{C}^n$. This is the problem that is addressed in the present article. We formulate this precisely as Theorem 2.12 below, after introducing the notion of a formal fold, which is a special case of Entov's notion of a chain of corank 1 Lagrangian singularities [En97].

The Lagrangian Grassmannian Λ_n admits a description as the homogeneous quotient U_n/O_n where U_n is the unitary group and O_n the orthogonal group. Thus the homotopy groups of O_n , U_n and Λ_n are related via the long exact sequence in homotopy associated to the Serre fibration $O_n \rightarrow U_n \rightarrow \Lambda_n$. These homotopy groups were computed by Bott in the stable range [B59]. However, while $\pi_n U_n$ lies in the stable range, $\pi_{n-1} O_n$ does not, and neither does $\pi_n \Lambda_n$.

In fact, $\pi_{n-1} O_n$ and $\pi_n \Lambda_n$ are the first nonstable homotopy groups of O_n and Λ_n , i.e. as soon as we stabilize them once we enter the stable range. Moreover, the stabilization maps $\pi_{n-1} O_n \rightarrow \pi_{n-1} O_{n+1}$ and $\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}$ are epimorphisms. These groups lie in the so-called metastable range, which is somewhat more subtle than the stable range, but has also been studied in the literature and exhibits a secondary form of 8-fold periodicity for $n \geq 8$. In particular $\pi_n \Lambda_n$ has been computed [K78], and this computation is essential input for our approach.

When n is even the problem is simpler because $\pi_n U_n = 0$, as was already observed in [AG18b]. The main novelty of the present article is to tackle the case of n odd. The special cases $n = 3, 7$ are particularly subtle due to the parallelizability of S^n and need to be addressed individually. We tackle the special cases $n = 3, 7$ by making explicit use of the geometry of the quaternions and octonions respectively. The key homotopical input is the well-known fact that multiplication by unit quaternions (resp. octonions), thought of as an element of $\pi_3 O_4$ (resp. $\pi_7 O_8$), maps under stabilization to a generator of $\pi_3 O$ (resp. $\pi_7 O$). In the case $n = 3$ we also sketch an alternative argument using Entov's technique of surgery of corank 1 Lagrangian singularities.

We will give some applications of Theorem 1.2 in Section 1.3 below, but first we will briefly discuss what kind of results one might hope for in the general case where the distribution $\gamma|_L$ is not assumed to be stably trivial.

1.2. Homotopically essential caustics of Lagrangian spheres

In order to go beyond the results of the present article and achieve a full classification of the homotopically essential caustics of Lagrangian spheres with respect to an arbitrary Lagrangian distribution it will be necessary to understand the geometry of the elements of $\pi_n \Lambda_n$ coming from the generators of the stable groups $\pi_n U_n \simeq \pi_n U$, since these elements are in general not stably trivial. The group $\pi_n U$ is of course well understood from Bott periodicity: it is isomorphic to \mathbf{Z} for n odd and it is trivial for n even.

While there exist explicit descriptions of the generators of the groups $\pi_{2k-1} U$, for example see [PR03] for simple formulas in $\pi_{2k-1} U_{2k-1}$, these formulas become quite complicated after de-stabilizing down to $\pi_{2k-1} U_{2k-1}$. In particular it is not clear what type of singularities of tangency one obtains, or to what extent they can be simplified.

Problem 1.4. For each odd integer n , exhibit an explicit representative for a generator of $\pi_n U_n$ so that the corresponding element of $\pi_n \Lambda_n$ has the simplest possible tangencies with respect to a fixed but arbitrary Lagrangian plane $P \subset \mathbf{C}^n$.

For example, when $n = 3$ a generator of $\pi_3 \Lambda_3 \simeq \mathbf{Z}/4$, which is the image of a generator of $\pi_3 U_3$ since $\pi_2 \mathcal{O}_3 = 0$, admits a representative which has folds along a torus $T \subset S^3$ and pleats along a $(1, 1)$ curve on T , where we embed the torus in S^3 as the boundary of a standard handlebody. See Figure 1.1, as well as Remark 4.7.

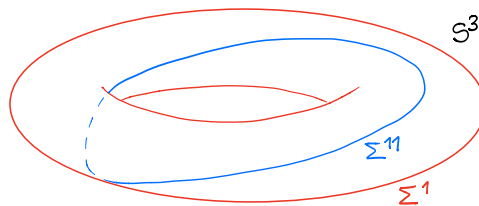


Figure 1.1: The chain of singularities for a generator of $\pi_3 \Lambda_3$.

For $n = 5$ it is not known to us how simple of a tangency locus one can achieve for the image of a generator of $\pi_5 U_5$ in $\pi_5 \Lambda_5 \simeq \mathbf{Z} \oplus \mathbf{Z}/2$.

Moreover, note that understanding the image of $\pi_n U_n \rightarrow \pi_n \Lambda_n$ would not be by itself sufficient to achieve a full classification of the homotopically essential caustics of Lagrangian spheres. As the simplest example consider the case $n = 2$, where we have $\pi_2 U_2 = 0$ but the subgroup of stably trivial elements in $\pi_2 \Lambda_2 \simeq \mathbf{Z}$ has index 2. In this case the situation is not so bad: a generator of $\pi_2 \Lambda_2$ admits a representative with a circle of folds and a single pleat at a point on the circle, see Figure 1.2. However in general it is not clear to us what one should expect.

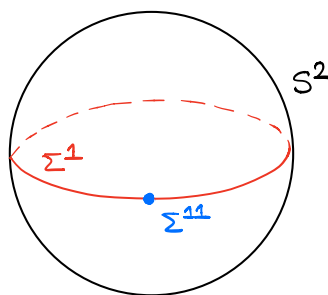


Figure 1.2: The chain of singularities for a generator of $\pi_2 \Lambda_2$.

The most optimistic hope is that it is always possible to find Σ^2 -nonsingular representatives. While this seems unlikely, we do not know of a counterexample. Hence we formulate the following:

Question 1.5. Do all elements of $\pi_n \Lambda_n$ admit representatives whose intersection with a fixed Lagrangian plane $P \subset \mathbf{C}^n$ has dimension ≤ 1 ?

If we set $\Sigma^2 = \{Q \in \Lambda_n : \dim(P \cap Q) \geq 2\} \subset \Lambda_n$ for a fixed but arbitrary Lagrangian plane $P \subset \mathbf{C}^n$ whose choice is immaterial, then the above question is equivalent to asking whether the inclusion $(\Lambda_n \setminus \Sigma^2) \subset \Lambda_n$ induces a surjection $\pi_n(\Lambda_n \setminus \Sigma^2) \rightarrow \pi_n \Lambda_n$. We note that the inclusion $(\Lambda_n \setminus \Sigma^2) \subset \Lambda_n$ is far from being a homotopy equivalence, as can be easily deduced from the cohomological calculations in the literature.

For example if $n = 3$, denote by $D^3 \subset \mathbf{R}^3$ the unit disk and let $D^3 \rightarrow \Lambda_3$ be the Gauss map of a neighborhood of a generic isolated Lagrangian Σ^2 singularity [AGV85]. Then the resulting element in $\pi_3(\Lambda_3, \Lambda_3 \setminus \Sigma^2)$ can be shown to be non-trivial by means of a characteristic class in $H^3(\Lambda_3; \mathbf{Z}/2)$ which is Poincaré dual to the codimension 3 cycle $\Sigma^2 \subset \Lambda_3$, see [A67].

In fact for any positive integer n the integral cohomology ring of Λ_n is generated by characteristic classes dual to similarly defined cycles [F68]. While these classes may be used to prove results establishing the necessity of higher singularities, a different method will most likely be needed to prove results in the opposite direction.

1.3. Applications

We present two applications of our main result Theorem 1.2, one to the arborealization program and another to the study of nearby Lagrangian homotopy spheres.

1.3.1. Arborealization program. As our first application we give a simple proof that polarized Weinstein manifolds which are obtained from the standard Darboux ball by a single handle attachment admit arboreal skeleta. This recovers a special case of the main theorem of [AGEN20b], where it is shown more generally that any polarized Weinstein manifold admits an arboreal skeleton. The argument used in [AGEN20b] is rather involved due to the subtleties arising from the interaction of three or more strata, whereas for the special class of polarized Weinstein manifolds obtained from a single handle attachment one can give a rather simple argument. Namely, the proof consists of a direct application of Theorem 1.2 together with Starkston's local model for the arborealization of a semi-cubical cusp [St18], which was used in that paper to arborealize Weinstein manifolds of dimension four.

In addition to the simplicity of the argument, a novel feature of the result we establish is that the arboreal skeleton we end up with has arboreal singularities of a particularly simple type. This conclusion does not follow directly from [AGEN20b].

Before we state the result, recall that arboreal singularities are modeled on rooted trees equipped with a decoration of a sign ± 1 for each edge not adjacent to a root [St18, AGEN20a]. By the height of a vertex we mean the number of edges between that vertex and the root, by the height of a tree we mean the maximal height among all vertices and by the height of an arboreal singularity we mean the height of the corresponding signed rooted tree.

Corollary 1.6. *Let (W, λ, ϕ) be a Weinstein manifold such that TW admits a global field of Lagrangian planes and such that the Morse Lyapunov function ϕ only has two critical points. Then by a homotopy of the Weinstein structure we can arrange it so that the skeleton of (W, λ) becomes arboreal,*

and moreover so that the arboreal singularities which appear in the skeleton have height ≤ 2 .

We briefly describe the proof, which follows the blueprint of [St18]. First one blows up a Darboux ball around the origin into the cotangent bundle of a Morse-Bott disk. The stable manifold of the other critical point then lands on this Morse-Bott disk along a front projection. The singularities of this front are a priori very complicated, but existence of a polarization is precisely the homotopical input needed for Theorem 1.2 to apply. Hence by a Legendrian isotopy of the attaching Legendrian, which can be realized by a homotopy of the Weinstein structure, we may assume that the front only has semi-cubical cusp singularities. Finally the cusps can be traded for arboreal singularities as shown in [St18].

1.3.2. Nearby Lagrangian homotopy spheres. As our second application we show that any nearby Lagrangian in the cotangent bundle of a homotopy sphere can be deformed via a Hamiltonian isotopy so that it is generated by a framed generalized Morse family on some bundle of tubes. We briefly explain the terminology before formally stating the result.

Following Igusa [I87], a framed generalized Morse family, or *framed function* for short, on the total space of a fibre bundle $W \rightarrow M$ is a function $f : W \rightarrow \mathbf{R}$ such that the restriction of f to each fibre is Morse or generalized Morse (i.e. we allow cubic birth/death of Morse critical points), and moreover such that the negative eigenspaces of the fibrewise Hessian at the fibrewise critical points are equipped with framings which vary continuously over M and are suitably compatible at the birth death points.

Following Waldhausen [W82], tubes are codimension zero submanifolds with boundary $T \subset \mathbf{R}^{n+1}$ which up to a compactly supported isotopy are given by the standard model for a smooth handle attachment on the boundary of the half space $\{x_{n+1} \leq 0\}$. A *tube bundle* is a fibre bundle of tubes $T \rightarrow W \rightarrow M$ where we assume that all tubes are contained in a fixed Euclidean space, i.e. $W \subset M \times \mathbf{R}^{n+1}$ and $W \rightarrow M$ is the restriction of the obvious projection $M \times \mathbf{R}^{n+1} \rightarrow M$.

We can now state:

Corollary 1.7. *Let Σ_0, Σ_1 be homotopy spheres and $\Sigma_1 \subset T^*\Sigma_0$ a Lagrangian embedding. There exists a Hamiltonian isotopy φ_t of Σ_1 such that $\varphi_1(\Sigma_1)$ is generated by a framed function on some tube bundle $T \rightarrow W \rightarrow \Sigma_0$.*

The starting point of the argument is the recent paper of Abouzaid, Courte, Guillermou and Kragh [ACGK20], where they show that if Σ_0, Σ_1

are homotopy spheres and $\Sigma_1 \subset T^*\Sigma_0$ is a Lagrangian embedding, then Σ_1 admits a generating function on some tube bundle over Σ_0 . In particular it follows from their result that the stable Gauss map $\Sigma_1 \rightarrow U/O$ is trivial.

Unwinding the definition, this means that Theorem 1.2 applies to $M = T^*\Sigma_0$, $L = \Sigma_1$ and $\gamma = \ker(d\pi)$ the vertical distribution, where $\pi : T^*\Sigma_0 \rightarrow \Sigma_0$ is the cotangent bundle projection. Therefore, Σ_1 can be deformed by a Hamiltonian isotopy φ_t so that $\varphi_1(\Sigma_1)$ only has fold tangencies with respect to the vertical distribution.

One may then apply the existence theorem for generating functions [ACGK20] once again to the deformed Lagrangian $\varphi_1(\Sigma_1)$ to obtain a generating function $f : W \rightarrow \mathbf{R}$ for $\varphi_1(\Sigma_1)$ on some tube bundle $W \rightarrow \Sigma_0$. Note that the restriction of f to each fibre only has Morse or Morse birth/death critical points. Indeed, Morse critical points correspond to points where $\varphi_1(\Sigma_1)$ and the vertical distribution are transverse and Morse birth/death critical points correspond to fold type tangencies.

Finally, this function may not admit a framing but one can fix this by further replacing W with a twisted stabilization of W using the fact that the projection $\varphi_1(\Sigma_1) \rightarrow \Sigma_0$ is a homotopy equivalence [A12].

Framed functions are the homotopically canonical way of studying fibre bundles via parametrized Morse theory [I02]. It is our hope that Corollary 1.7 may be useful for the study of nearby Lagrangians via parametrized Morse theory.

1.4. Structure of the article

In Section 2 we introduce the notion of a formal fold and translate the geometric problem into a homotopical problem. In Section 3 we perform the homotopical calculation necessary to establish our main theorem in dimensions not equal to 3 or 7. In Section 4 we tackle the special dimensions 3 and 7. In Section 5 we give the proofs of the applications stated above.

1.5. Acknowledgements

This article was the outcome of an undergraduate Research Opportunities Program (UROP) which the second author undertook at MIT under the supervision of the first author. We are grateful to the UROP program for enabling this collaboration. We would also like to thank Sasha Givental for useful conversations. Finally, we are grateful to the referee for useful suggestions and corrections.

2. Formal folds

2.1. Tangencies of fold type

2.1.1. Lagrangian tangencies. Let (M, ω) be a $2n$ -dimensional symplectic manifold, $L \subset M$ a smooth Lagrangian submanifold and $\gamma \subset TM$ a Lagrangian distribution.

Definition 2.1. A *tangency* between L and γ is a point $x \in L$ such that $T_x L \cap \gamma_x \neq 0$.

If $\gamma = \ker(d\pi)$ for a Lagrangian fibration $\pi : M \rightarrow B$, then tangencies of L with respect to γ are the same as singular points of the restriction $\pi|_L : L \rightarrow B$, i.e. points $x \in L$ at which the differential $d\pi_x : T_x L \rightarrow T_{\pi(x)} B$ fails to be an isomorphism. If L is exact then we may lift it to a Legendrian \widehat{L} in the contactization $M \times \mathbf{R}$ and the tangencies of L with respect to γ can also be thought of as the singularities of the front $\widehat{L} \rightarrow B \times \mathbf{R}$, which is known as the caustic in the literature [A90].

A tangency point $x \in L$ is said to be of corank 1, or Σ^2 -nonsingular, if $\dim(T_x L \cap \gamma_x) = 1$. The locus of corank 1 tangencies $\Sigma^1 = \{x \in L : \dim(L \cap \gamma) = 1\}$ is C^∞ -generically a smooth hypersurface in L and $\ell = (TL \cap \gamma)|_{\Sigma^1}$ is a line field inside $TL|_{\Sigma^1}$. We say that γ is Σ^2 -nonsingular if all its tangencies with L are Σ^2 -nonsingular, so the tangency locus of L with γ is equal to Σ^1 , which in this case is C^∞ -generically a smooth, closed hypersurface in L without boundary.

While C^∞ -generic Lagrangian tangencies are non-classifiable, the class of Σ^2 -nonsingular tangencies does admit a finite list of local models, at least in the case where γ is integrable [AGV85]. The simplest type of Σ^2 -nonsingular tangency is called a fold. This is the only type of tangency we will need to consider in the present article.

Definition 2.2. We say that a tangency point $x \in \Sigma^1$ is of *fold type* if Σ^1 is transversely cut out in a neighborhood of x and $\ell_x \pitchfork T_x \Sigma^1$ inside $T_x L$.

When γ is integrable, a fold tangency is locally symplectomorphic to the normal form

$$L = \{q = p^2\} \times \mathbf{R}^{n-1} \subset T^*\mathbf{R} \times T^*\mathbf{R}^{n-1}, \quad \gamma = \ker(T^*\mathbf{R}^n \rightarrow \mathbf{R}^n).$$

Remark 2.3. We note that in the contactization, fold tangencies correspond to semi-cubical cusps of the Legendrian front.

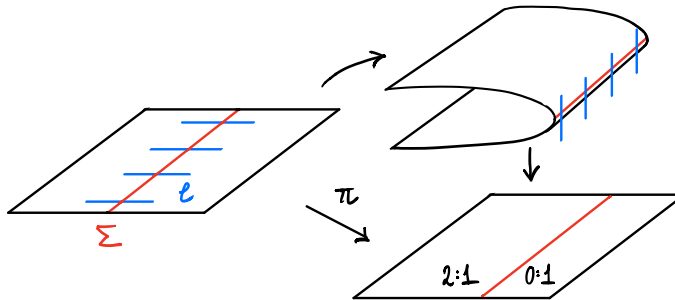


Figure 2.1: A fold type tangency. Note that $\ell = \ker(d\pi|_L)$ is transverse to the singular locus $\Sigma \subset L$.

2.1.2. The h-principle for the simplification of caustics. In order to reduce Theorem 1.2 to a homotopical problem, we use the h-principle for the simplification of caustics established by the first author in [AG18b]. It states the following:

Theorem 2.4 ([AG18b]). *Let (M, ω) be a symplectic manifold, $L \subset M$ a Lagrangian submanifold and $\gamma \subset TM$ a Lagrangian distribution. Suppose that γ is homotopic through Lagrangian distributions to a Lagrangian distribution with respect to which L only has fold tangencies. Then L is Hamiltonian isotopic to a Lagrangian submanifold which only has fold tangencies with respect to γ .*

Hence to prove Theorem 1.2 it suffices to show that under the stated hypotheses γ is homotopic to a Lagrangian distribution which only has fold tangencies with L .

Remark 2.5. The hypothesis in Theorem 2.4 only cares about the restriction of γ to L , since any homotopy of $\gamma|_L$ can be extended to a homotopy of γ . Furthermore, by taking a Weinstein neighborhood of L we may immediately reduce to the case $M = T^*L$, which is therefore the only case we will consider in what follows.

2.2. Formal folds and their stable triviality

2.2.1. Formal folds. The homotopical object underlying a Lagrangian distribution with only fold type tangencies is a formal fold, which is defined as follows:

Definition 2.6. A *formal fold* in a smooth manifold L consists of a pair (Σ, v) , where $\Sigma \subset L$ is a co-orientable smooth closed hypersurface in L and v is a choice of co-orientation of Σ .

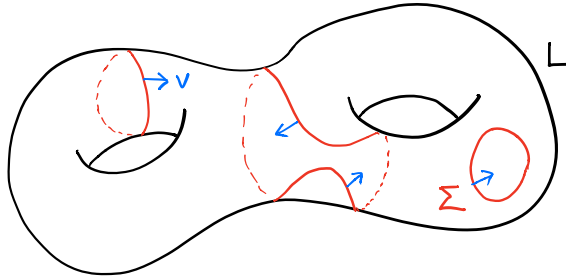


Figure 2.2: A formal fold on a surface consists of a disjoint union of co-oriented simple closed curves.

Remark 2.7. Formal folds are the simplest version of the notion of a chain of Lagrangian singularities as defined by Entov [En97], generalizing the notion of a chain of singularities for smooth maps [E72]. We will not need this more general notion in what follows and hence will not discuss it further, with the exception of the non-essential Remark 4.7.

Let $\gamma \subset T(T^*L)|_L$ be a Lagrangian distribution which has only fold type tangencies with respect to L . That is, the intersection $\gamma_x \cap T_x L \subset T_x(T^*L)$ has dimension ≤ 1 for any $x \in L$, the subset $\Sigma = \{x \in L : \gamma_x \cap T_x L \neq 0\} \subset L$ is a transversely cut out hypersurface and $\ell = \gamma|_\Sigma \cap TL$ is a line field along Σ which is transverse to Σ . To such a γ we associate a formal fold (Σ, v) by specifying v to be the Maslov co-orientation [A67, En97].

Conversely, if (Σ, v) is a formal fold on L , there is a homotopically unique Lagrangian distribution $\gamma(\Sigma, v) \subset T(T^*L)|_L$ which has only fold type tangencies with respect to L and whose associated formal fold is (Σ, v) . For existence, let $\Sigma \times [-1, 1]$ be a tubular neighborhood of Σ in L such that the coordinate $t \in [-1, 1]$ is compatible with the co-orientation, i.e. $v = \partial/\partial t$. On $L \setminus (\Sigma \times [-1, 1])$ we define $\gamma(\Sigma, v)$ to be the vertical distribution. On $\Sigma \times [-1, 1]$ we define it to be the direct sum of the vertical distribution in

$T(T^*\Sigma)|_\Sigma$ and the line field $\ell \subset T(T^*[-1, 1])|_{[-1, 1]}$ defined by

$$\ell_t = \text{span} \left\langle \sin \left(\frac{\pi t}{2} \right) \frac{\partial}{\partial u} + \cos \left(\frac{\pi t}{2} \right) \frac{\partial}{\partial t} \right\rangle \subset T_t(T^*[-1, 1])$$

where u is the momentum coordinate dual to t , see Figure 2.3.

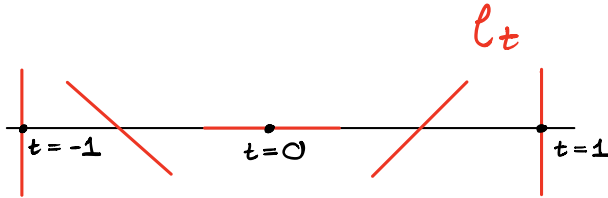


Figure 2.3: The line field ℓ_t .

The fact that $\gamma(\Sigma, v)$ is homotopically unique is straightforward to verify using the well-known fact that the space of Lagrangian planes in \mathbf{C}^n which are transverse to a fixed Lagrangian plane P is contractible; indeed this space can be identified with the (convex) space of quadratic forms on any Lagrangian plane Q which is transverse to P .

Finally, we note that the homotopy class of γ only depends on the formal fold (Σ, v) up to ambient isotopy in L .

2.2.2. Stable triviality of formal folds. Let $\gamma \subset T(T^*L)|_L$ be a Lagrangian distribution defined along L . We say that γ is trivial if it is homotopic through Lagrangian distributions to the vertical distribution, which is defined to be $\nu = \ker(d\pi)$ for $\pi : T^*L \rightarrow L$ the cotangent bundle projection. More generally:

Definition 2.8. We say that γ is *stably trivial* if $\gamma \oplus \mathbf{R}$ and $\nu \oplus \mathbf{R}$ are homotopic as Lagrangian distributions in $T(T^*L)|_L \oplus \mathbf{C}$.

Remark 2.9. This notion of stable triviality is equivalent to the one given in Definition 1.1 since TL and ν are homotopic Lagrangian distributions in $T(T^*L)|_L$. For example, this can be seen by rotating one to the other via a compatible almost complex structure J on T^*L such that $JT_x L = \nu_x$ in $T_x(T^*L)$ for all $x \in L$.

Lemma 2.10. *Let (Σ, v) be a formal fold in L . Assume L is simply connected. Then $\gamma(\Sigma, v)$ is stably trivial.*

Proof. Consider the path $p_1 : [-1, 1] \rightarrow U_2$ given by

$$t \mapsto \begin{pmatrix} e^{\pi i(t+1)/2} & 0 \\ 0 & 1 \end{pmatrix}$$

and the path $p_2 : [-1, 1] \rightarrow U_2$ given by

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i(t+1)/2} \end{pmatrix}$$

Post-composing p_1 and p_2 with the projection $U_2 \rightarrow \Lambda_2$ (i.e. taking the images $p_i(t)(\mathbf{R}^2) \subset \mathbf{C}^2$) we obtain loops $\eta_1, \eta_2 : ([-1, 1], \{\pm 1\}) \rightarrow (\Lambda_2, \mathbf{R}^2)$, i.e. $\eta_i(-1) = \eta_i(1) = \mathbf{R}^2$ for $i = 1, 2$. Since the isomorphism $\pi_1 \Lambda_2 \simeq \mathbf{Z}$ is induced by $\det^2 : \Lambda_2 \rightarrow U_1$ and $\det^2 \circ \eta_1 = \det^2 \circ \eta_2$ (both are equal to the function $e^{\pi i(t+1)}$) it follows that η_1 and η_2 are homotopic relative to $\{\pm 1\}$, as can be verified explicitly.

At a point $x \in \Sigma \times [-1, 1]$ we may split $T_x L \oplus \mathbf{C} \simeq T_x \Sigma \oplus \mathbf{C}^2$. From the above observation it follows that $\gamma(\Sigma, v) \oplus \mathbf{R}$ is homotopic to the distribution $\nu \oplus \widehat{\ell}$, where $\widehat{\ell}$ denotes the line field in $L \times \mathbf{C}$ defined as $i\mathbf{R}$ outside of $\Sigma \times [-1, 1]$ and for $(x, t) \in \Sigma \times [-1, 1]$ given by

$$\widehat{\ell} = \text{span} \left\langle \sin \left(\frac{\pi t}{2} \right) \frac{\partial}{\partial x} + \cos \left(\frac{\pi t}{2} \right) \frac{\partial}{\partial y} \right\rangle \subset \mathbf{C}.$$

But every map $L \rightarrow \Lambda_1$ is null-homotopic when $n > 1$ since $\Lambda_1 \simeq S^1$ and L is simply connected. Hence $\widehat{\ell}$ is homotopic to the trivial distribution $L \times \mathbf{R} \subset S^n \times \mathbf{C}$ and consequently $\gamma(\Sigma, v) \oplus \mathbf{R}$ is homotopic to $\nu \oplus \mathbf{R}$, which was to be proved. \square

Remark 2.11. The statement of Lemma 2.10 is false if L is not assumed to be simply connected. For the simplest example take $L = S^1$ and let Σ consist of a single point. Of course the Lemma can be salvaged by adding the assumption that the class in $H^1(L; \mathbf{Z})$ Poincaré dual to (Σ, v) is trivial, but we won't need this in what follows.

2.3. Reduction to homotopy theory

2.3.1. Formal folds in \mathbf{R}^n . Let (Σ, v) be a formal fold in \mathbf{R}^n . We assume $\Sigma \subset \mathbf{R}^n$ to be compact, hence the corresponding Lagrangian distribution $\gamma(\Sigma, v) \subset T(T^*\mathbf{R}^n)|_{\mathbf{R}^n}$ is vertical at infinity. In other words, $\gamma(\Sigma, v)$ is equal to the vertical distribution $\nu = \ker(d\pi)$ outside of a compact subset, where $\pi : T^*\mathbf{R}^n \rightarrow \mathbf{R}^n$ is the standard projection.

Since $T(T^*\mathbf{R}^n)|_{\mathbf{R}^n} \simeq \mathbf{R}^n \times \mathbf{C}^n$ as symplectic vector bundles, there is a one to one correspondence between homotopy classes of Lagrangian distributions in $T(T^*\mathbf{R}^n)|_{\mathbf{R}^n}$ which are vertical at infinity and elements of $\pi_n\Lambda_n$, where Λ_n is the Grassmannian of linear Lagrangian subspaces of \mathbf{C}^n . Thus to a formal fold (Σ, v) in \mathbf{R}^n is associated an element $\alpha(\Sigma, v) \in \pi_n\Lambda_n$. Here we think of the n -sphere as the one-point compactification of \mathbf{R}^n with the basepoint at infinity and we take the (vertical) imaginary plane $i\mathbf{R}^n$ as the basepoint of Λ_n .

By Lemma 2.10, every element of the form $\alpha(\Sigma, v)$ is in the kernel of the stabilization map $\pi_n\Lambda_n \rightarrow \pi_n\Lambda_{n+1}$ induced by the inclusion $\Lambda_n \hookrightarrow \Lambda_{n+1}$, which we recall is given by

$$P \mapsto P \oplus \mathbf{R} \subset \mathbf{C}^n \oplus \mathbf{C}, \quad P \subset \mathbf{C}^n.$$

In Sections 3 and 4 below we will prove that the converse is also true:

Theorem 2.12. *Every element of $\ker(\pi_n\Lambda_n \rightarrow \pi_n\Lambda_{n+1})$ admits a representative of the form $\alpha(\Sigma, v)$ for some formal fold (Σ, v) in \mathbf{R}^n .*

In the rest of the present section we will show how Theorem 2.12, together with the h-principle 2.4, imply our main result Theorem 1.2.

2.3.2. Formal folds in homotopy spheres. Let L be an n -dimensional homotopy sphere and denote by $\mathbf{C}^n \rightarrow E_L \rightarrow L$ the symplectic vector bundle $T(T^*L)|_L$. Let $\Lambda_n \rightarrow \Lambda_L \rightarrow L$ denote the associated Grassmann bundle, whose fibre over $x \in L$ is the Grassmannian of linear Lagrangian subspaces of $T_x(T^*L)$. Let $f : D^n \rightarrow L$ be a smooth embedding of the closed unit disk $D^n = \{\|x\| \leq 1, x \in \mathbf{R}^n\}$, which is unique up to isotopy (and precomposition with an orientation-reversing diffeomorphism).

After identifying the interior $B^n = \{\|x\| < 1, x \in \mathbf{R}^n\}$ of D^n with \mathbf{R}^n , the embedding f induces a map

$$f_* : \pi_n\Lambda_n \rightarrow \pi_0\Gamma(\Lambda_L),$$

where $\Gamma(\Lambda_L)$ denotes the space of sections of Λ_L . This is induced from a pushforward at the level of spaces, i.e. from the space of maps $(D^n, \partial D^n) \rightarrow (\Lambda_n, i\mathbf{R}^n)$ to the section space $\Gamma(\Lambda_L)$. Explicitly, a Lagrangian distribution in $T(T^*D^n)|_{D^n}$ which is vertical near ∂D^n is extended to $T(T^*L)|_L$ as the vertical distribution outside of $f(D^n)$. Note that at the level of spaces the pushforward takes formal folds to formal folds, see Figure 2.4.

Lemma 2.13. $f_*\pi_n\Lambda_n = \pi_0\Gamma(\Lambda_L)$.

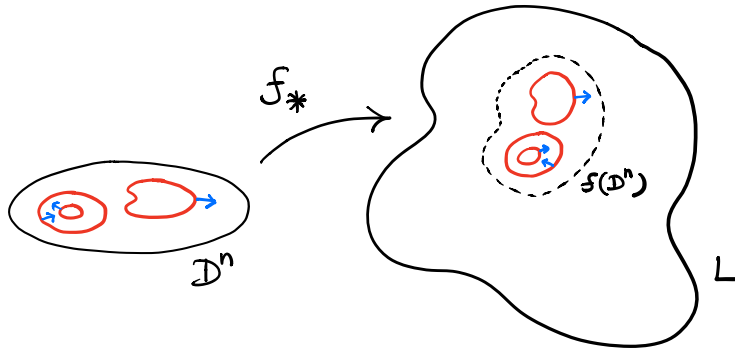


Figure 2.4: Pushing forward a formal fold from D^n to L via f .

Proof. Any Lagrangian distribution $\gamma \in T(T^*L)|_L$ may be deformed so that it is equal to the vertical distribution on a neighborhood U of $L \setminus f(B^n)$ since $L \setminus f(B^n)$ is contractible. \square

Denote by $\Gamma^{\text{st}}(\Lambda_L) \subset \Gamma(\Lambda_L)$ the subspace of stably trivial sections. It is clear that $f_* \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}) \subset \pi_0 \Gamma^{\text{st}}(\Lambda_L)$. Again we have surjectivity:

Lemma 2.14. $f_* \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}) = \pi_0 \Gamma^{\text{st}}(\Lambda_L)$.

Proof. If a Lagrangian distribution $\gamma \in T(T^*L)|_L$ which is vertical in a neighborhood U of $L \setminus f(B)$ is stably trivial, then $\gamma \oplus \mathbf{R}$ and $\nu \oplus \mathbf{R}$ are homotopic in $T(T^*L)|_L \oplus \mathbf{C}$, but the homotopy need not be fixed in U . So we need to fix this.

We may assume that U itself is contractible, for example we can set $U = L \setminus f(\frac{1}{2}B^n)$ for $\frac{1}{2}B^n = \{\|x\| < 1/2, x \in \mathbf{R}^n\}$. Let $x_0 \in U$ be a point outside of $f(D^n)$. The restriction of the homotopy between $\gamma \oplus \mathbf{R}$ and $\nu \oplus \mathbf{R}$ to x_0 determines an element of $\pi_1 \Lambda_{n+1}$. Now, $\pi_1 \Lambda_n \rightarrow \pi_1 \Lambda_{n+1}$ is an isomorphism for any $n \geq 1$, hence after a suitable deformation of γ we may assume that this homotopy is through Lagrangian planes of the form $\gamma_t(x_0) \oplus \mathbf{R}$, where $\gamma_t(x_0) \subset T_{x_0}(T^*L)|_L$.

We may then use the homotopy $\gamma_t(x_0)$ to further deform γ so that it is equal to the vertical distribution ν at the point x_0 and so that $\gamma \oplus \mathbf{R}$ is homotopic to $\nu \oplus \mathbf{R}$ through distributions which are equal to $\nu \oplus \mathbf{R}$ at the point x_0 . Explicitly, trivialize a neighborhood $V \simeq \mathbf{R}^n$ of x_0 contained in U and in this neighborhood replace $\gamma(x)$ with $\gamma_{\phi(x)}(x) \subset \mathbf{C}^n$ where $\phi : \mathbf{R}^n \rightarrow [0, 1]$ is a compactly supported function such that $\phi(x_0) = 1$.

Finally, since U is contractible we may further deform γ so that the same property holds over all of U , i.e. γ is vertical over U and $\gamma \oplus \mathbf{R}$ is homotopic to $\nu \oplus \mathbf{R}$ through distributions which are equal to $\nu \oplus \mathbf{R}$ over U . This proves the lemma. \square

We are now ready to prove our main result.

Proof of Theorem 1.2 assuming Theorem 2.12. The necessity of stable triviality follows from Lemma 2.10. The sufficiency of stable triviality follows from Lemma 2.14 and Theorem 2.12, which show that the hypothesis needed to apply the h-principle Theorem 2.4 is satisfied. \square

It therefore remains for us to establish Theorem 2.12. This will be achieved in Section 3 for those dimensions not equal to 3 or 7 and in Section 4 for the exceptional dimensions 3 and 7.

3. Homotopical computation

3.1. Homotopical background

We begin by reviewing some relevant background in homotopy theory, in particular we review for future reference certain stable and nonstable homotopy groups of the unitary and orthogonal groups and of their homogeneous quotient, the Lagrangian Grassmannian.

3.1.1. The classical groups. Recall that to a Serre fibration $F \rightarrow E \rightarrow B$ is associated a long exact sequence in homotopy groups:

$$\cdots \rightarrow \pi_{n+1}B \rightarrow \pi_n F \rightarrow \pi_n E \rightarrow \pi_n B \rightarrow \pi_{n-1}F \rightarrow \cdots .$$

From the fibration $U_n \rightarrow U_{n+1} \rightarrow S^{2n+1}$ given by the standard action of U_{n+1} on S^{2n+1} one deduces that the stabilization map $U_n \rightarrow U_{n+1}$, which is given by adding a row and a column with zeros everywhere except for a 1 in the diagonal entry, induces isomorphisms on all π_k for $k < 2n$ and an epimorphism on π_{2n} . Indeed, $\pi_k S^{2n+1} = 0$ for $k < 2n + 1$. The homotopy groups $\pi_k U := \pi_k U_n$ in the stable range $k < 2n$ exhibit 2-fold periodicity and were computed by Bott [B59] as follows:

mod(k , 2)	$\pi_k U$
0	0
1	\mathbf{Z}

Similarly, from the fibration $O_n \rightarrow O_{n+1} \rightarrow S^n$ given by the standard action of O_{n+1} on S^n one deduces that the analogous stabilization map $O_n \rightarrow O_{n+1}$ induces isomorphisms on all π_k for $k < n - 1$ and an epimorphism on π_{n-1} . The homotopy groups $\pi_k O := \pi_k O_n$ in the stable range $k < n - 1$ exhibit 8-fold periodicity and were also computed by Bott as follows:

$\text{mod}(k, 8)$	$\pi_k O$
0	$\mathbf{Z}/2$
1	$\mathbf{Z}/2$
2	0
3	\mathbf{Z}
4	0
5	0
6	0
7	\mathbf{Z}

The Lagrangian Grassmannian Λ_n admits a transitive action of U_n with the stabilizer O_n , hence can be described as the homogeneous space U_n/O_n . By considering the long exact sequence in homotopy associated to the resulting fibration $O_n \rightarrow U_n \rightarrow \Lambda_n$, it follows from the above that the stabilization map $\Lambda_n \rightarrow \Lambda_{n+1}$, which is given by taking the direct sum in $\mathbf{C}^{n+1} = \mathbf{C}^n \oplus \mathbf{C}$ of a linear Lagrangian subspace of \mathbf{C}^n and $\mathbf{R} \subset \mathbf{C}$, induces isomorphisms on all π_k for $k < n$ and an epimorphism on π_n .

The homotopy groups $\pi_k \Lambda := \pi_k \Lambda_n$ in the stable range $k < n$ exhibit 8-fold periodicity and were also computed by Bott, in fact they are just a shift of the stable homotopy groups $\pi_k O$ due to the homotopy equivalence $\Omega \Lambda \simeq \mathbf{Z} \times BO$.

$\text{mod}(k, 8)$	$\pi_k \Lambda$
0	0
1	\mathbf{Z}
2	$\mathbf{Z}/2$
3	$\mathbf{Z}/2$
4	0
5	\mathbf{Z}
6	0
7	0

For the purposes of this article we are interested not in the stable homotopy groups of Λ_n but in the unstable group $\pi_n \Lambda_n$. Via the long exact sequence in homotopy of the fibration $O_n \rightarrow U_n \rightarrow \Lambda_n$ we may relate this group to the homotopy groups $\pi_n U_n$ and $\pi_{n-1} O_n$, the first of which is in the

stable range but the second of which is not. The groups $\pi_n \Lambda_n$ and $\pi_{n-1} O_n$ are the first nonstable homotopy groups of Λ_n and O_n respectively.

These homotopy groups, though nonstable, are also understood. Not only do they surject onto the corresponding stable groups, but they exhibit a secondary form of 8-fold periodicity, with three exceptions related to the parallelizability of S^1 , S^3 and S^7 .

The computation of $\pi_{n-1} O_n$ is mostly straightforward, see [S51], but the non-parallelizability of S^n for $n \neq 1, 3, 7$ [BM58, K58, M58] plays an essential role. Here is the table for $\pi_{n-1} O_n$, where we remark that the indexing of π_* is by $n - 1$ instead of n for future convenience when analyzing the sequence $\pi_n U_n \rightarrow \pi_n \Lambda_n \rightarrow \pi_{n-1} O_n$.

$\text{mod}(n, 8), n \geq 8$	$\pi_{n-1} O_n$	(small n)
0	$\mathbf{Z} \oplus \mathbf{Z}$	$\pi_0 O_1 = \mathbf{Z}/2$
1	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\pi_1 O_2 = \mathbf{Z}$
2	$\mathbf{Z}/2 \oplus \mathbf{Z}$	$\pi_2 O_3 = 0$
3	$\mathbf{Z}/2$	$\pi_3 O_4 = \mathbf{Z} \oplus \mathbf{Z}$
4	$\mathbf{Z} \oplus \mathbf{Z}$	$\pi_4 O_5 = \mathbf{Z}/2$
5	$\mathbf{Z}/2$	$\pi_5 O_6 = \mathbf{Z}$
6	\mathbf{Z}	$\pi_6 O_7 = 0$
7	$\mathbf{Z}/2$	$\pi_7 O_8 = \mathbf{Z} \oplus \mathbf{Z}$

Briefly, to relate this table with that of the stable groups $\pi_{n-1} O$ one uses the fact that $\pi_{n-1} O_n \rightarrow \pi_{n-1} O$ is an epimorphism and $\ker(\pi_{n-1} O_n \rightarrow \pi_{n-1} O) = \text{im}(\pi_{n-1} S^{n-1} \rightarrow \pi_{n-1} O_n)$ is generated by the class of the tangent bundle TS^n , which has infinite order if n is even, has order 2 if n is odd and not equal to 1, 3, 7, and is trivial if $n = 1, 3$ or 7.

The groups $\pi_n \Lambda_n$ were computed by Kachi in [K78] and are given as follows:

$\text{mod}(n, 8), n \geq 8$	$\pi_n \Lambda_n$	(small n)
0	\mathbf{Z}	$\pi_0 \Lambda_0 = 0$
1	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\pi_1 \Lambda_1 = \mathbf{Z}$
2	$\mathbf{Z}/2 \oplus \mathbf{Z}$	$\pi_2 \Lambda_2 = \mathbf{Z}$
3	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\pi_3 \Lambda_3 = \mathbf{Z}/4$
4	\mathbf{Z}	$\pi_4 \Lambda_4 = \mathbf{Z}$
5	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\pi_5 \Lambda_5 = \mathbf{Z} \oplus \mathbf{Z}/2$
6	\mathbf{Z}	$\pi_6 \Lambda_6 = \mathbf{Z}$
7	$\mathbf{Z}/2$	$\pi_7 \Lambda_7 = \mathbf{Z}/2$

Remark 3.1. Strictly speaking the computation in [K78] is for $\pi_n SU_n/SO_n$, however this group is isomorphic to $\pi_n U_n/O_n$ whenever $n > 1$.

This follows immediately from the long exact sequences in homotopy associated to the determinant fibrations $SU_n \rightarrow U_n \rightarrow U_1$ and $SO_n \rightarrow O_n \rightarrow O_1$.

Finally, $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$ is given as follows:

$\text{mod}(n, 2), n > 1$	$\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$
0	\mathbf{Z}
1	$\mathbf{Z}/2$

This table follows almost immediately from the previous ones and in any case is a consequence of the computation below. In almost all cases the subgroup $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$ is a direct summand of $\pi_n \Lambda_n$ (with the other direct summand given by $\pi_n \Lambda_{n+1} \simeq \pi_n \Lambda$), but there are some exceptions in which it is given by:

- ($n=1$) The trivial subgroup.
- ($n=2$) The index 2 subgroup $2\mathbf{Z} \subset \mathbf{Z} \simeq \pi_2 \Lambda_2$.
- ($n=3$) The cyclic subgroup of order 2 in $\pi_3 \Lambda_3 \simeq \mathbf{Z}/4$.

Remark 3.2. Note that in all cases $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}) \subset \pi_n \Lambda_n$ is cyclic and we will give an explicit generator.

3.2. A homotopical lemma

The following lemma is the main homotopical input needed to prove our main theorem in the non-exceptional dimensions $n \neq 1, 3, 7$.

Lemma 3.3. *Let $n \neq 1, 3, 7$. Then $\ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n) \cap \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1}) = 0$.*

Proof. Let $\beta \in \ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n) \cap \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$. We proceed by cases to show that $\beta = 0$.

3.2.1. The case $n \equiv 0 \pmod{2}$. If n is even then $\pi_n U_n = 0$, hence the map $\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n$ is a monomorphism and so $\beta \in \ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n)$ is necessarily zero.

3.2.2. The case $n \equiv 7 \pmod{8}$, $n \geq 15$. In this case the map $\pi_n O_n \rightarrow \pi_n U_n$ is an epimorphism by commutativity of the diagram

$$\begin{array}{ccc} \pi_n O_n & \longrightarrow & \pi_n U_n \\ \downarrow & & \downarrow \\ \pi_n O & \longrightarrow & \pi_n U \end{array}$$

Indeed, for $n \equiv 7 \pmod{8}$, $n \geq 15$, we note:

- (i) $\pi_n O_n \rightarrow \pi_n O$ an epimorphism as shown by Kervaire [K60],
 - (ii) $\pi_n O \rightarrow \pi_n U$ is an isomorphism since $\pi_{8k+6}\Lambda = \pi_{8k+7}\Lambda = 0$,
 - (iii) $\pi_n U_n \rightarrow \pi_n U$ is also an isomorphism since $\pi_n U_n$ is in the stable range,
- from which the conclusion follows. Hence $\pi_n U_n \rightarrow \pi_n \Lambda_n$ is the zero map, so $\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n$ is a monomorphism and we can argue as in the previous case.

3.2.3. The case $n \equiv 1, 5 \pmod{8}$. In this case $\pi_n O_{n+1} \rightarrow \pi_n U_{n+1}$ is the zero map since $\pi_n O_{n+1}$ is isomorphic to 0 or $\mathbf{Z}/2$ for n congruent to 1 or 5 respectively while $\pi_n U_{n+1}$ is isomorphic to \mathbf{Z} . Hence the map $\pi_n U_{n+1} \rightarrow \pi_n \Lambda_{n+1}$ is a monomorphism. We can therefore argue as follows.

Let $\beta \in \ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n) \cap \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$. Since $\ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n) = \text{im}(\pi_n U_n \rightarrow \pi_n \Lambda_n)$ we can lift β to an element $\widehat{\beta} \in \pi_n U_n$. By commutativity of the diagram

$$\begin{array}{ccc} \pi_n U_n & \longrightarrow & \pi_n U_{n+1} \\ \downarrow & & \downarrow \\ \pi_n \Lambda_n & \longrightarrow & \pi_n \Lambda_{n+1} \end{array}$$

it follows that the image $s(\widehat{\beta})$ of $\widehat{\beta}$ under the stabilization map $\pi_n U_n \rightarrow \pi_n U_{n+1}$ is in the kernel of the map $\pi_n U_{n+1} \rightarrow \pi_n \Lambda_{n+1}$. Since $\pi_n U_{n+1} \rightarrow \pi_n \Lambda_{n+1}$ is a monomorphism, this implies $s(\widehat{\beta}) = 0$. But $\pi_n U_n \rightarrow \pi_n U_{n+1}$ is an isomorphism, so we must also have $\widehat{\beta} = 0$ and hence we conclude $\beta = 0$.

3.2.4. The case $n \equiv 3 \pmod{8}$, $n \geq 11$. In this case we have

$$\pi_n \Lambda_n = \mathbf{Z}/2 \oplus \mathbf{Z}/2, \quad \pi_n \Lambda_{n+1} = \mathbf{Z}/2, \quad \pi_{n-1} O_n = \mathbf{Z}/2, \quad \pi_{n-1} O_{n+1} = 0.$$

Hence $\pi_n U_{n+1} \rightarrow \pi_n \Lambda_{n+1}$ is the unique nontrivial map $\mathbf{Z} \rightarrow \mathbf{Z}/2$ with the kernel $2\mathbf{Z} \subset \mathbf{Z}$.

Let $\beta \in \ker(\pi_n \Lambda_n \rightarrow \pi_{n-1} O_n) \cap \ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$. As in the previous case, we may choose $\widehat{\beta} \in \pi_n U_n$ a lift of β , and the image $s(\widehat{\beta})$ of $\widehat{\beta}$ under the stabilization map $\pi_n U_n \rightarrow \pi_n U_{n+1}$ is in the kernel of $\pi_n U_{n+1} \rightarrow \pi_n \Lambda_{n+1}$. It follows that $s(\widehat{\beta})$ is divisible by 2 in $\pi_n U_{n+1}$.

Since $\pi_n U_n \rightarrow \pi_n U_{n+1}$ is an isomorphism, we deduce that $\widehat{\beta}$ is also divisible by 2, hence the same is true of β . But $\pi_n \Lambda_n$ is 2-torsion, so we conclude $\beta = 0$.

Having exhausted all cases, the proof is complete. □

3.3. Proof of the main theorem for $n \neq 1, 3, 7$

Assume $n > 1$ in what follows.

3.3.1. An Euler number computation. Recall that a formal fold (Σ, v) in \mathbf{R}^n determines an element $\alpha(\Sigma, v)$ in $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$. The image of $\alpha(\Sigma, v)$ in $\pi_{n-1} O_n$ lies in the kernel of the map $\pi_{n-1} O_n \rightarrow \pi_{n-1} O_{n+1}$ by commutativity of the diagram

$$\begin{array}{ccc} \pi_n \Lambda_n & \longrightarrow & \pi_n \Lambda_{n+1} \\ \downarrow & & \downarrow \\ \pi_{n-1} O_n & \longrightarrow & \pi_{n-1} O_{n+1} \end{array}$$

This is just a diagram chasing way of saying that since $\alpha(\Sigma, v)$ is a stably trivial Lagrangian distribution, in particular the underlying real vector bundle is stably trivial. It turns out that all stably trivial real vector bundles arise in this way:

Lemma 3.4. *The images of the elements $\alpha(\Sigma, v)$ in $\pi_{n-1} O_n$ generate the subgroup $\ker(\pi_{n-1} O_n \rightarrow \pi_{n-1} O_{n+1})$.*

Proof. Consider first the case where n is even. The subgroup $\ker(\pi_{n-1} O_n \rightarrow \pi_{n-1} O_{n+1}) = \text{im}(\pi_n S^n \rightarrow \pi_{n-1} O_n)$ is infinite cyclic and generated by TS^n , so it is enough to show that the Euler numbers of the real vector bundles underlying the distributions $\alpha(\Sigma, v)$ can realize any even integer. Let $\Omega \subset \mathbf{R}^n$ be a compact domain with smooth boundary. Set $\Sigma = \partial\Omega$ and let v be the outward normal to Ω . A straightforward application of the Poincaré-Hopf index theorem will show that the Euler number of (Σ, v) is equal to $\pm 2\chi(\Omega)$.

To this end, first note that as a real vector bundle, $\alpha(\Sigma, v)$ is the result of gluing trivial bundles on Ω and on its complement via the isomorphism

of $\Sigma \times \mathbf{R}^n \simeq T\Sigma \oplus \mathbf{R}v$ given by

$$\text{id} \oplus (-1) : T\Sigma \oplus \mathbf{R}v \rightarrow T\Sigma \oplus \mathbf{R}v.$$

Let us compute the Euler number of $\alpha(\Sigma, v)$, presented in this way, using a section σ which is constant and non-zero outside of a large ball $B \subset \mathbf{R}^n$ containing Ω and which restricts inside Ω to an outwards pointing vector field. Thus the zeros of σ inside Ω yield a total index of $\chi(\Omega)$ by Poincaré-Hopf. Note that in light of the above gluing, σ restricts on the complement of Ω to a vector field which is also outwards pointing along $\partial\Omega$. Therefore, also by Poincaré-Hopf, inside $B \setminus \Omega$ the zeros of σ yield a total of $\chi(B \setminus \Omega) - 1$. The summand -1 comes from the fact that σ does not point outwards pointing along ∂B (it is constant), but can be arranged to do so after the introduction of a source-type zero, which has index 1.

Next, note that $\chi(B \setminus \Omega) - 1 = \chi(B) - \chi(\Omega) - 1 = \chi(\Omega)$. Finally, in the above index computation we have used the canonical orientation of \mathbf{R}^n on each of the regions Ω and $B \setminus \Omega$. To count the total index of σ we must use a global orientation of $\alpha(\Sigma, v)$, which means we must switch one of the orientations on either Ω or $B \setminus \Omega$ to account for the fold along Σ (in other words, the gluing map $\text{id} \oplus (-1)$ is orientation reversing). Thus we obtain an Euler number of $\pm 2\chi(\Omega)$ for $\alpha(\Sigma, v)$. This completes the Euler number calculation.

It now suffices to observe that since $n > 1$, we can arrange for $\chi(\Omega)$ to take any integer value, which completes the proof in the case where n is even.

Consider next the case where n is odd. For $n = 1, 3, 7$ the group $\pi_{n-1}O_n$ is trivial so there is nothing to prove. For $n \neq 1, 3, 7$ the group $\pi_{n-1}O_n$ has a single stably trivial element, which is the class of TS^n . By direct inspection this element is equal to the image of $\alpha(S^{n-1}, n_{D^n})$ in $\pi_{n-1}O_n$, where n_{D^n} is the outward normal to the unit disk $D^n = \{\|x\| \leq 1 : x \in \mathbf{R}^n\}$. \square

Corollary 3.5. *For $n \neq 1, 3, 7$ the map $\ker(\pi_n\Lambda_n \rightarrow \pi_n\Lambda_{n+1}) \rightarrow \ker(\pi_{n-1}O_n \rightarrow \pi_{n-1}O_{n+1})$ is an isomorphism.*

Proof. Injectivity is given by Lemma 3.3 while surjectivity is given by Lemma 3.4. \square

Remark 3.6. When $n = 3$ we have $\ker(\pi_3\Lambda_3 \rightarrow \pi_3\Lambda_4) \simeq 2 \cdot \mathbf{Z}/4 \subset \mathbf{Z}/4 \simeq \pi_3\Lambda_3$ while $\pi_2O_3 = 0$. When $n = 7$ we have $\ker(\pi_7\Lambda_7 \rightarrow \pi_7\Lambda_8) = \pi_7\Lambda_7 \simeq \mathbf{Z}/2$ while $\pi_6O_7 = 0$.

3.3.2. Conclusion of the proof. We are now ready to prove Theorem 2.12 in the non-exceptional dimensions, which we recall states that every element of $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$ admits a representative of the form $\alpha(\Sigma, v)$ for some formal fold (Σ, v) in \mathbf{R}^n .

Proof of Theorem 2.12 for $n \neq 1, 3, 7$. Let $\gamma \in \ker(\pi_n \Lambda_n \rightarrow \Lambda_{n+1})$. By Lemma 3.4 there exists a formal fold (Σ, v) such that the images of γ and $\alpha(\Sigma, v)$ in $\pi_{n-1} O_n$ are equal. But Corollary 3.5 implies that γ and $\alpha(\Sigma, v)$ are in fact equal in $\pi_n \Lambda_n$. \square

Remark 3.7. It follows that for any $n \neq 1, 3, 7$ the subgroup $\ker(\pi_n \Lambda_n \rightarrow \pi_n \Lambda_{n+1})$ is cyclic with a generator given by $\alpha(S^{n-1}, n_{D^n})$, where n_{D^n} is the outward normal the unit disk $D^n = \{\|x\| \leq 1 : x \in \mathbf{R}^n\}$. As we will see below the same is true for the exceptional dimensions $n = 1, 3, 7$. The element $\alpha(S^{n-1}, n_{D^n})$ is 2-torsion for n odd. For n even it is not and we can obtain representatives for its multiples as follows. Given $k > 0$, the element of $\pi_n \Lambda_n$ given by k times $\alpha(S^{n-1}, n_{D^n})$ is equal to $\alpha(\partial\Omega_k, n_{\Omega_k})$, where Ω_k is the disjoint union of k disks in \mathbf{R}^n and n_{Ω_k} is the outward normal. More generally, if $\Omega \subset \mathbf{R}^n$ is any domain of Euler characteristic k then $\alpha(\partial\Omega, n_\Omega)$ is a representative for $k\alpha(S^{n-1}, n_{D^n})$, where n_Ω is the outward normal to Ω . Similarly, one can obtain a representative for $-k\alpha(S^{n-1}, n_{D^n})$ by taking $\alpha(\partial\Omega, n_\Omega)$, where $\Omega \subset \mathbf{R}^n$ is any domain with Euler characteristic $-k$ and n_Ω is the outward normal to Ω .

4. The exceptional cases

4.1. Complex trivializations

To tackle the exceptional dimensions $n = 3, 7$ we will make use of an explicit complex trivialization of $T(T^*S^n)|_{S^n}$ together with a certain property of stable triviality satisfied by the trivialization. This trivialization is defined for all $n \geq 1$, but for $n = 3, 7$ we will further examine its interaction with the trivializations coming from quaternionic and octonionic geometry (which are not stably trivial), leading to a proof of Theorem 2.12 in those dimensions.

4.1.1. The isomorphism $T(T^*S^n)|_{S^n} \simeq S^n \times \mathbf{C}^n$. We construct an explicit isomorphism of symplectic vector bundles between $T(T^*S^n)|_{S^n}$ and $S^n \times \mathbf{C}^n$ as follows. It will be useful to keep in mind that there is a canonical symplectic bundle isomorphism $T(T^*S^n)|_{S^n} \simeq TS^n \oplus T^*S^n$.

Let e be the first unit vector in \mathbf{R}^{n+1} , and let $\theta = \theta(g)$ measure the angle of a vector $g \in S^n$ away from e ; importantly, $\cos \theta = \langle e, g \rangle$. Each level set

$\theta = \theta_0$ (other than $\theta_0 = 0, \pi$) is isometric to the scaled sphere $\sin(\theta) S^{n-1}$ by the mapping $g \mapsto g - \cos(\theta_0)e$. For any point g in the level set $\theta = \theta_0$ and each non-zero vector $v \in \{e\}^\perp \subset \mathbf{R}^{n+1}$, define the coordinate $\theta_v = \theta_v(g)$ to be the angle between $g - \cos(\theta)e$ and v . These coordinates are well-defined except at the poles $\pm e$. In particular, fix an orthonormal basis e_1, \dots, e_n of $\{e\}^\perp$, and define $\theta_i = \theta_{e_i}$.

Now define the (discontinuous) vector fields $X = -\partial_\theta$ and $X_v = -\partial_{\theta_v}$, and write $X_i = -\partial_{\theta_i}$; note that X_v depends only on the normalized vector $v/\|v\|$. Writing J for the standard¹ almost complex structure on $T(T^*S^n)|_{S^n}$, we define a complex trivialization of $T(T^*S^n)|_{S^n}$ by

$$(1) \quad E_i = -\cos(\theta_i)e^{J\theta}X + \sin(\theta_i)X_i.$$

Convenient formulas for X_i and E_i are given as

$$(2) \quad \begin{aligned} X_i &= \csc(\theta_i)e_i - \cot(\theta_i)\partial_R, \\ E_i &= \cos(\theta_i)\cos(\theta)\partial_\theta + J\cos(\theta_i)\sin(\theta)\partial_\theta + e_i - \cos(\theta_i)\partial_R. \end{aligned}$$

It is clear from these formulas that E_i is everywhere continuous—indeed, $\cos(\theta_i) = 0$ when θ is 0 or π , which are the two discontinuities of ∂_θ , and the latter two terms in the above formula are smooth. In fact, the same analysis shows that E_i is smooth away from the two poles $\theta = 0$ and $\theta = \pi$.

Definition 4.1. For any $n \geq 1$, define the bundle map $F : T(T^*S^n)|_{S^n} \rightarrow S^n \times \mathbf{C}^n$ to be the one taking $E_i|_g$ to (g, e_i) at each point $g \in S^n$.

Lemma 4.2. For any $n \geq 1$, the map $F : T(T^*S^n)|_{S^n} \rightarrow S^n \times \mathbf{C}^n$ is a complex vector bundle isomorphism.

Proof. The lemma boils down to showing that the maps $F|_g : T_g(T^*S^n) \rightarrow \{g\} \times \mathbf{C}^n$ are linear isomorphisms and vary continuously with $g \in S^n$; in turn, this follows from showing that $\{E_i\}$ is a continuous complex frame.

It is clear from the expressions (1) that the sections E_i are continuous and well-defined, so it remains to be seen that they are complex-linearly independent.

Let $a, b \in \{e\}^\perp \subset \mathbf{R}^{n+1}$, and set $v = a + \sqrt{-1}b = \sum(a^j + \sqrt{-1}b^j)e_j \in \mathbf{C}^n$. We show that the complex linear combination $\sum(a^i + Jb^i)E_i$ results in

¹That is, J is compatible with the round metric and the canonical symplectic structure.

$E_v := E_a + JE_b$, where we define

$$E_a = \|a\| \left(-\cos(\theta_a)e^{J\theta} X + \sin(\theta_a)X_a \right)$$

and similarly for E_b . Indeed, for any $g \in S^n$,

$$\begin{aligned} \sum (a^i + Jb^i)E_i|_g &= - \left[\sum (a^i + Jb^i) \cos(\theta_i) \right] e^{J\theta} X \\ &\quad + \sum (a^i + Jb^i) \sin(\theta_i)X_i. \\ (3) \qquad &= - [\langle a, g \rangle + J\langle a, g \rangle] e^{J\theta} X \\ &\quad + \sum (a^i + Jb^i) \sin(\theta_i)X_i. \\ &= - [\|a\| \cos(\theta_a) + J\|b\| \cos(\theta_b)] e^{J\theta} X \\ &\quad + \sum (a^i + Jb^i) \sin(\theta_i)X_i. \end{aligned}$$

To deal with the second term above, rewrite

$$X_u|_g = (\sin \theta_u)^{-1} \left(u - \cos(\theta_u) \cdot \frac{g - \cos(\theta)e}{\sin(\theta)} \right).$$

Then we find

$$\begin{aligned} \sum (a^i + Jb^i) \sin(\theta_i)X_i &= \sum (a^i + Jb^i) \left(e_i - \cos(\theta_i) \cdot \frac{g - \cos(\theta)e}{\sin(\theta)} \right) \\ &= a + Jb - (\langle a, g \rangle + J\langle b, g \rangle) \cdot \frac{g - \cos(\theta)e}{\sin(\theta)} \\ &= a + Jb - (\|a\| \cos(\theta_a) + J\|b\| \cos(\theta_b)) \cdot \frac{g - \cos(\theta)e}{\sin(\theta)} \\ &= \|a\| \sin(\theta_a)X_a + J\|b\| \sin(\theta_b)X_b. \end{aligned}$$

Putting these two elements together implies

$$\sum (a^i + Jb^i)E_i = E_a + JE_b = E_v.$$

Since E_v is nonzero for v nonzero, this proves our result. □

Remark 4.3. Consider the Lagrangian distribution $\gamma \subset T(T^*S^n)|_{S^n}$ which is the preimage by F of $S^n \times i\mathbf{R}^n$; that is, the distribution spanned by $JE_i|_g$ at each $g \in S^n$. From the equations 2, it is clear that γ has fold type tangencies with S^n along the equator $S^{n-1} = \{x_0 = 0\} \cap S^n$ and is transverse to S^n everywhere else.

4.1.2. Stable triviality of the frame. We will also need the fact that the frame defined above is *stably trivial*, in the following sense. By stabilizing once, the vector bundle isomorphism $F : T(T^*S^n)|_{S^n} \rightarrow S^n \times \mathbf{C}^n$ extends to an isomorphism $\widehat{F} : T(T^*S^n)|_{S^n} \times \mathbf{C} \rightarrow S^n \times \mathbf{C}^{n+1}$. Identifying the extra factor of \mathbf{C} as the complexification of the normal direction to the sphere and using the trivialization $T(T^*\mathbf{R}^{n+1})|_{S^n} \simeq S^n \times \mathbf{C}^{n+1}$ we may rewrite this as a map

$$\widehat{F} : S^n \times \mathbf{C}^{n+1} \rightarrow S^n \times \mathbf{C}^{n+1}$$

which is a lift of the identity map $S^n \rightarrow S^n$ by fibrewise linear isomorphisms $\mathbf{C}^{n+1} \rightarrow \mathbf{C}^{n+1}$.

In fact, these are unitary transformations, as can be verified using the explicit formulas provided by Definition 4.1.

Lemma 4.4. *The map $\widehat{F} : S^n \times \mathbf{C}^{n+1} \rightarrow S^n \times \mathbf{C}^{n+1}$ is trivial as an element of $\pi_n U_{n+1}$.*

Remark 4.5. We note:

- (i) As a basepoint of S^n we take the point e where the frame $(E_i)_i$ agrees with the frame e_i , and as a basepoint of U_{n+1} we take the identity matrix.
- (ii) It is sufficient to prove the triviality of \widehat{F} as an element of $\pi_n \text{GL}(\mathbf{C}^{n+1})$ since the inclusion $U_{n+1} \subset \text{GL}(\mathbf{C}^{n+1})$ is a homotopy equivalence.

Proof. We continue in the language of the proof of Lemma 4.2.

By stabilizing, we introduce a new vector field E_0 to our frame, everywhere orthogonal to $E_{i \neq 0}$. We can view this as an outward normal field to S^n . In short, $E_0 \equiv \partial_R$, where R is the outward radial coordinate (the norm in \mathbf{R}^{n+1}). In this setting, \widehat{F} takes the form

$$\widehat{F} : S^n \times \mathbf{C}^{n+1} \rightarrow S^n \times \mathbf{C}^{n+1}, \quad E_i|_g \mapsto (g, e_i),$$

with $\{e = e_0, e_1, \dots, e_n\}$ an orthonormal basis of \mathbf{C}^{n+1} . Our lemma thus boils down to the following claim: the frame $\{E_i\}$ is homotopic to the trivial frame $\{e_i\}$ through maps $S^n \rightarrow \text{GL}(\mathbf{C}^{n+1})$. Indeed, if this is the case, then we can pre-compose the map $\widehat{F} : E_i \mapsto e_i$ with this homotopy to perturb \widehat{F} itself continuously to the identity map $e_i \mapsto e_i$.

We prove this by supplying a sequence of homotopies bringing E_i to e_i ; it of course is crucial (and we will prove this along the way) that the images of the frames are complex-linearly independent at each point in time.

To begin, we apply two continuous homotopies from $E_i = E'_i(0)$ to $E'_i(1)$.

$$\begin{aligned} E'_i(t) &= E_i + t \cos(\theta_i) \sin(\theta) \partial_R, \\ E'_0(t) &= e^{-J\theta t} E_0 + t \sin(\theta) X, \end{aligned}$$

where we let t go from 0 to 1 (and we exclude 0 from the index i). We can extend these homotopies to general E_v , where $v = a + \sqrt{-1}b = \sum_{i>0} (a^i e_i + \sqrt{-1}b^i e_i)$, $a, b \in \{e\}^\perp \subset \mathbf{R}^{n+1}$ using the formula

$$E'_v(t) = E_v + t (\|a\| \cos(\theta_a) + J\|b\| \cos(\theta_b)) \sin(\theta) \partial_R.$$

The resulting transformation $E_v \mapsto E'_v$ is a complex-linear isomorphism, as we see from the following calculation:

$$\begin{aligned} \sum_i (a^i E'_i(t) + b^i J E'_j(t)) \Big|_g &= \sum_i (a^i E_i + b^i J E_j) \\ &\quad + t \sum_i (a^i \cos(\theta_i) + b^i J \cos(\theta_i)) \sin(\theta) \partial_R, \\ &= E_v + t \sum_i ((a^i + b^i J) \langle x, e_i \rangle) \sin(\theta) \partial_R, \\ &= E_v + t (\langle g, a \rangle + J \langle g, b \rangle) \sin(\theta) \partial_R, \\ &= E_v + t (\|a\| \cos(\theta_a) + J\|b\| \cos(\theta_b)) \sin(\theta) \partial_R, \end{aligned}$$

In particular, this means that if $E'_0(t)$ were in the span of $\{E'_i(t)\}$, then we would have $E'_0(t) = E'_v(t)$ for some nonzero v . We can see that this would require $t = 1$ and $\sin \theta = \pm 1$; otherwise, the projection of E'_0 to $T(T^*S^n)|_{S^n}$ has norm strictly larger than the projection of E'_0 to $\text{span}(\partial_R)$, while the opposite is true of E'_v .

Next, note that the projection of $E'_0(t)$ to $\text{span}(\partial_R)$ is real; to obtain $E'_0(t)|_g = E'_v(t)|_g$, we thus also require $\|b\| = 0$. Following the same argument as in the preceding paragraph, this implies that $\cos(\theta_a) = \pm 1$. Knowing this, we have

$$E'_0 = -J\partial_R + X, \quad E'_v = \mp JX \pm \partial_R.$$

These are indeed independent; within the complex subspace spanned by X and ∂_R , these two vectors give a complex determinant of $\pm 2 \neq 0$.

Finally, we make the two homotopies

$$\begin{aligned} E''_i(s) &= E'_i(1) + sJ \cos(\theta_i) \sin(\theta) X, \\ E''_0(s) &= E'_0(1) + sJ \sin(\theta) \partial_R, \end{aligned}$$

which extend as before to E'_v for general $v \in \{e\}^\perp \subset \mathbf{C}^{n+1}$. For clarity, here are the closed-form expressions of E''_0 and E''_u , for $u \in \{e\}^\perp \subset \mathbf{R}^{n+1}$.

$$E''_u(s) = \|u\| [\sin(\theta_u)X_u - \cos(\theta_u) \cos(\theta)X + \cos(\theta_u) \sin(\theta)\partial_R + (s - 1)J \cos(\theta_u) \sin(\theta)X],$$

$$E''_0(s) = \cos(\theta)\partial_R + \sin(\theta)X + (s - 1)J \sin(\theta)\partial_R.$$

Since $v \mapsto E''_v(s)$ is complex-linear, the vector fields $\{E''_0(s), E''_i(s)\}_i$ can again only be linearly dependent if $E''_0(s) = E''_v(s)$ for some $v \neq 0$. Suppose $v = a + \sqrt{-1}b$, for $a, b \in \{e\}^\perp \subset \mathbf{R}^{n+1}$. From the above expressions, we can only satisfy $E''_0(s) = E''_a(s) + JE''_b(s)$ at points $g \in S^n$ where the components along X_a and X_b vanish; this requires $\sin(\theta_a) = \sin(\theta_b) = 0$, further implying that $\cos(\theta_a), \cos(\theta_b) = \pm 1$, that a is a multiple of b , and thus that $E''_v(s) = \beta E''_a(s)$ for some nonzero $\beta \in \mathbf{C}$. In this case, the determinant of $E''_0(s), E''_v(s)$ within the subspace $\text{span}(X, \partial_R)$ is

$$\beta \|v\| \det \begin{pmatrix} \sin \theta & \mp \cos \theta \pm (s-1)J \sin \theta \\ \cos \theta + (s-1)J \sin \theta & \pm \sin(\theta) \end{pmatrix}$$

$$= \pm \beta \|v\| (1 + (s - 1)^2 \sin^2 \theta) \neq 0.$$

Thus, the frame remains linearly independent over the full homotopy. Finally, it is clear from the above expressions that $E''_v(1) \equiv v$ and $E''_0(1) \equiv e$, which proves the lemma. □

4.2. Proof of the main theorem in the exceptional cases

It remains to prove Theorem 2.12 in the cases $n = 1, 3, 7$. The case $n = 1$ is trivial and will not be discussed further.

4.2.1. The case $n = 3$. Identify $S^3 \subset \mathbf{R}^4 \simeq \mathbf{H}$ with the set of unit quaternions, giving it the structure of a Lie group. As such, we can recover an orthonormal trivialization E_1, E_2, E_3 of left-invariant vector fields in TS^3 by extending the basis $\{i, j, k\}$ of T_1S^3 .

This gives rise to a complex trivialization of $T(T^*S^3)|_{S^3}$ (equipped with the unique almost complex structure compatible with the symplectic form and the round metric) and thus an isomorphism

$$G : T(T^*S^3)|_{S^3} \xrightarrow{\sim} S^3 \times \mathbf{C}^3$$

distinct from the isomorphism F considered in Lemma 4.2.

Thus we obtain an element $\beta := iF \circ G^{-1} \in \pi_3 U_3$.

Lemma 4.6. *The image α of β in $\pi_3\Lambda_3$ is equal to the element $\alpha(S^2, n_{D^3})$.*

Proof. We argue as follows. On the one hand, G^{-1} maps $S^3 \times \mathbf{R}^3$ to TS^3 and $S^3 \times i\mathbf{R}^3$ to the vertical distribution ν of $T(T^*S^3)|_{S^3}$, by construction. On the other hand, $iF(TS^3)$ is transverse to $S^3 \times \mathbf{R}^3$ everywhere away from the equator $\{\mathbf{1} \in \mathbf{H}\}^\perp \cap S^3$, but has fold tangencies with $S^3 \times \mathbf{R}^3$ along that equator. Thus, α has folds along that same equator, but no other singularities anywhere else. \square

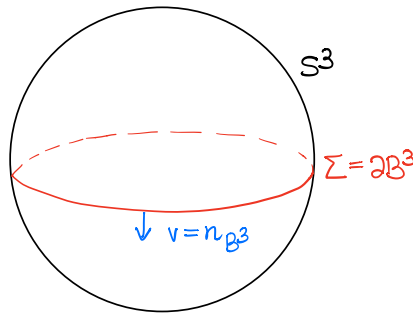


Figure 4.1: The element $\alpha(S^2, n_{B^3}) \in \pi_3\Lambda_3$.

Next, consider the element $\eta \in \pi_3O_4$ given by left-quaternion multiplication—that is, for $g \in S^3 \subset \mathbf{H}$ and $h \in \mathbf{R}^4 \simeq \mathbf{H}$, we have $\eta(g) \cdot h = gh \in \mathbf{R}^4$. We claim that the image of η in $\pi_3U_4 \simeq \mathbf{Z}$ is equal to twice a generator, which can be seen as follows. It is well known that the image of η in π_3O is a generator [AH61]. In particular, its image in the stable group π_3O_5 must be a generator. Consider the map $\pi_3O_5 \rightarrow \pi_3U_5$ in the following diagram:

$$(4) \quad \begin{array}{ccccccc} \pi_3O_5 & \longrightarrow & \pi_3U_5 & \longrightarrow & \pi_3\Lambda_5 & \longrightarrow & \pi_2O_5 \\ \uparrow & & \uparrow & & & & \\ \pi_3O_4 & \longrightarrow & \pi_3U_4 & & & & \end{array}$$

Since $\pi_3\Lambda_5 \simeq \mathbf{Z}/2$ and $\pi_2O_5 \simeq 0$, the map $\pi_3O_5 \rightarrow \pi_3U_5$, which is $\mathbf{Z} \rightarrow \mathbf{Z}$, sends a generator to twice a generator. Since $\pi_3U_4 \rightarrow \pi_3U_5$ is an isomorphism, it follows that the image of η in π_3U_4 is equal to twice a generator, as claimed.

Proof of Theorem 2.12 in the case $n = 3$. Recall that $\pi_3\Lambda_3 \simeq \mathbf{Z}/4$ and $\pi_3\Lambda_4 \simeq \mathbf{Z}/2$, hence the stabilization map $\pi_3\Lambda_3 \rightarrow \pi_3\Lambda_4$ is the unique epimorphism $\mathbf{Z}/4 \rightarrow \mathbf{Z}/2$ whose kernel is the single non-zero element $2 \in \mathbf{Z}/4$ corresponding to twice a generator.

Hence in view of Lemma 4.6 it suffices to prove that $\alpha \in \pi_3\Lambda_3$ is equal to twice a generator. Since the map $\pi_3U_3 \rightarrow \pi_3\Lambda_3$ sends a generator to a generator, it also suffices to prove that $\beta \in \pi_3U_3$ is equal to twice a generator. Finally, since $\pi_3U_3 \rightarrow \pi_3U_4$ is an isomorphism and we know the image of the element η under the map $\pi_3O_4 \rightarrow \pi_3U_4$ to equal twice a generator, it suffices to show that the images of β and η in π_3U_4 are identical.

In stabilizing β to an element $\hat{\beta} \in \pi_3U_4$, consider the additional 4th unit vector field E_0 as the complexification of the outward unit normal to the sphere. Considering $S^3 \times \mathbf{C}^4$ as a restriction of the tangent bundle of the complexified Lie group $\mathbf{H} \otimes \mathbf{C}$, note that E_0 is also left-invariant (similar to E_1, E_2, E_3). Extend F and G to complex vector bundle isomorphisms $\hat{F}, \hat{G} : S^3 \times \mathbf{C}^4 \rightarrow S^3 \times \mathbf{C}^4$ by $\hat{F}(E_0|_g) = \hat{G}(E_0|_g) = (g, e_0)$. Since E_0 is left-invariant in the sense mentioned above, \hat{G}^{-1} is simply the complexified quaternion multiplication map, i.e. the image of η in π_3U_4 .

That $i\hat{F}$ is homotopic to the identity follows from Lemma 4.4; thus, $\hat{\beta} = i\hat{F} \circ \hat{G}^{-1}$ is equal to the image of η in π_3U_4 , and we are done. \square

Remark 4.7. We note that in the case $n = 3$ one may alternatively argue in the following way. By chasing the diagram

$$\begin{array}{ccc} \pi_3U_2 & \longrightarrow & \pi_3U_3 \\ \downarrow & & \downarrow \\ \pi_3\Lambda_2 & \longrightarrow & \pi_3\Lambda_3 \end{array}$$

it follows that a generator of $\pi_3\Lambda_3$ is given by the stabilization of the image under $\pi_3U_2 \rightarrow \pi_3\Lambda_2$ of a generator of π_3U_2 . But from the determinant fibration $SU_2 \rightarrow U_2 \rightarrow U_1$ we see that a generator of π_3U_2 is given by the image of a generator of π_3SU_2 , which we can take to be the identity map under the standard identification $S^3 \simeq SU_2$ given by

$$(z, w) \mapsto \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}, \quad (z, w) \in S^3 \subset \mathbf{C}^2.$$

One may explicitly compute the tangencies of the resulting map $S^3 \rightarrow \Lambda_2$ with a suitable Lagrangian plane to be Σ^2 -nonsingular, and more precisely to consist of a Σ^1 locus on a torus which is the boundary of a standard genus

1 handlebody in \mathbf{R}^3 , with Σ^{110} pleats on a $(1, 1)$ curve on the torus and no Σ^{1110} points, see Figure 1.1.

It is then an entertaining exercise in Entov’s surgery of singularities [En97] to show that the disjoint union of two copies of this chain of singularities can be surgered into a sphere of Σ^{10} folds, i.e. into the element $\alpha(S^2, n_{B^3})$. Hence the generator of $\ker(\pi_3\Lambda_3 \rightarrow \pi_3\Lambda_4)$, which is equal to twice a generator of $\pi_3\Lambda_3$, is represented by $\alpha(S^2, n_{B^3})$. We know of no analogous explicit argument in the case $n = 7$, which we discuss next.

4.2.2. The case $n = 7$. Just as in the quaternionic case earlier, we identify $S^7 \subset \mathbf{R}^8 \simeq \mathbf{O}$ with the set of unit octonions. Let $e = \mathbf{1} \in \mathbf{O}$ be the first unit vector, and—for any unit octonion $g \in S^7$ —define the left-multiplication map

$$L_g : T_e S^7 \rightarrow T_g S^7, \quad (e, v) \mapsto (g, g \cdot v),$$

where we view $v \in T_e S^7 \simeq \{x \in \mathbf{R}^8 \simeq \mathbf{O} \mid \langle x, e \rangle = 0\}$ as an octonion itself. This yields a trivialization E_1, \dots, E_7 of TS^7 , where $E_i|_e = e_i$ is the i^{th} unit vector in $\mathbf{R}^8 \supset T_e S^7$ and $E_i|_g = L_g E_i|_e$.

This gives rise to a complex trivialization of $T(T^*S^7)|_{S^7}$ (equipped with the unique almost complex structure compatible with the symplectic form and the round metric), and thus an isomorphism

$$G : T(T^*S^7)|_{S^7} \xrightarrow{\sim} S^7 \times \mathbf{C}^7$$

distinct from the isomorphism F considered in Lemma 4.2.

Thus we obtain an element $\beta := iF \circ G^{-1} \in \pi_7 U_7$.

Lemma 4.8. *The image α of β in $\pi_7 \Lambda_7$ is equal to the element $\alpha(S^6, n_{D^7})$.*

Proof. On the one hand, G^{-1} maps $S^7 \times \mathbf{R}^7$ to TS^7 and $S^7 \times i\mathbf{R}^7$ to the vertical distribution ν of $T(T^*S^7)|_{S^7}$, by construction. On the other, $iF(TS^7)$ is transverse to $S^7 \times \mathbf{R}^7$ everywhere away from the equator $\{e\}^\perp \cap S^7$, and has fold tangencies with $S^7 \times \mathbf{R}^7$ along that equator. Thus, α has folds along that same equator, but no other singularities anywhere else. \square

Next, consider the element $\eta \in \pi_7 O_8$ given by left-octonion multiplication—that is, for $g \in S^7 \simeq O_1(\mathbf{O})$ and $h \in \mathbf{R}^8 \simeq \mathbf{O}$, we have $\eta(g) \cdot h = gh \in \mathbf{R}^8$. We claim that the image of η in $\pi_7 U_8$ is a generator, which can be seen as follows. It is well known that that the image of

η generates $\pi_7 O$ [AH61]; in particular, it generates $\pi_7 O_9$. Consider the diagram

$$(5) \quad \begin{array}{ccccc} \pi_7 O_9 & \longrightarrow & \pi_7 U_9 & \longrightarrow & \pi_7 \Lambda_9 \\ \uparrow & & \uparrow & & \\ \pi_7 O_8 & \longrightarrow & \pi_7 U_8 & & \end{array}$$

From the fact that $\pi_7 \Lambda_9 = 0$, we see that $\pi_7 O_9 \rightarrow \pi_7 U_9$ is an isomorphism $\mathbf{Z} \rightarrow \mathbf{Z}$. Since $\pi_7 U_8 \rightarrow \pi_7 U_9$ is an isomorphism, it follows that the image of η in $\pi_7 U_8$ must be a generator, which establishes the claim.

Proof of Theorem 2.12 in the case $n = 7$. Recall that $\pi_7 \Lambda_7 \simeq \mathbf{Z}/2$ has only one nonzero element; it is stably trivial because $\pi_7 \Lambda_8 = 0$.

In view of Lemma 4.8 we need to prove that $\alpha \in \pi_7 \Lambda_7$ is this unique non-zero element. Since the map $\pi_7 U_7 \rightarrow \pi_7 \Lambda_7$ is the unique non-zero map $\mathbf{Z} \rightarrow \mathbf{Z}/2$, it suffices to show that $\beta \in \pi_7 U_7$ is a generator. Finally, since the image of η under the map $\pi_7 O_8 \rightarrow \pi_7 U_8$ is a generator and $\pi_7 U_7 \rightarrow \pi_7 U_8$ is an isomorphism, it suffices to show that the images of β and η in $\pi_7 U_8$ are identical.

In stabilizing β to an element $\widehat{\beta} \in \pi_7 U_8$, consider the additional 8th unit vector field E_0 as the outward unit normal to the sphere. The multiplication map L_g is diagonal along the radial coordinate, so we have $E_0|_g = L_g E_0|_e$, just as with E_1, \dots, E_7 .

Extend F and G to (vector bundle) isomorphisms $\widehat{F}, \widehat{G} : S^7 \times \mathbf{C}^8 \rightarrow S^7 \times \mathbf{C}^8$ by $\widehat{F}(E_0|_g) = \widehat{G}(E_0|_g) = (g, e_0)$. We can see that \widehat{G}^{-1} is simply the complexified octonion multiplication map, i.e. the image of η in $\pi_7 U_8$. Indeed, suppose $v = \sum v^i e_i$, and calculate

$$\widehat{G}^{-1}(g, v) = \sum v^i \widehat{G}^{-1}(g, e_i) = \sum v^i E_i|_g = \sum v^i L_g e_i = L_g v.$$

That $i\widehat{F}$ is homotopic to the identity follows from Lemma 4.4; thus, $\widehat{\beta} = i\widehat{F} \circ \widehat{G}^{-1}$ is equal to the image of η in $\pi_7 U_8$, and the theorem is proved. \square

5. Applications

5.1. Arborealization of Weinstein manifolds with a single handle

We begin by briefly recalling some basic definitions of symplectic topology.

Definition 5.1. A *Liouville domain* (W, λ) consists of a compact manifold with boundary equipped with an exact symplectic form $\omega = d\lambda$ together with a choice of primitive λ such that the vector field Z which is ω -dual to λ is outwards pointing along ∂W .

Definition 5.2. A *Weinstein domain* (W, λ, ϕ) consists of a Liouville domain (W, λ) together with a Morse function $\phi : W \rightarrow \mathbf{R}$ which is Lyapunov for Z .

The Lyapunov condition means that Z is gradient-like for ϕ . The skeleton of a Weinstein domain (W, λ, ϕ) is the union of the stable manifolds of the critical points of ϕ , hence is a union of isotropic submanifolds. The skeleton is in general a quite singular object, but there is a particularly simple class of Lagrangian singularities introduced by Nadler in [N13, N15] and further developed in [St18, AGEN20a] called arboreal singularities. That four dimensional Weinstein manifolds admit skeleta with arboreal singularities was proved in [St18].

In arbitrary dimensions, it was proved in [AGEN20b] that if TW admits a global field of Lagrangian planes, then the Weinstein structure of (W, λ, ϕ) can be deformed so that the skeleton has arboreal singularities. The proof relies on the ridgification theorem [AGEN19], which builds on the h-principle for the simplification of caustics [AG18b] but is somewhat more subtle and has a greater range of applicability.

As a corollary of our main result Theorem 1.2 we will now show that for the class of polarized Weinstein domains admitting a Lyapunov function with only two critical points it is possible to apply the h-principle for the simplification of caustics directly, following the approach of [St18], and thus avoiding the more complicated treatment of [AGEN20b], which is only necessary when one needs to control the interaction of three or more strata in the skeleton. Moreover, for this special class of Weinstein domains we show that the skeleton can be arranged to have arboreal singularities of a particularly simple type, which does not directly follow from [AGEN20b].

Remark 5.3. We claim no originality in the arborealization strategy, which simply follows the blueprint of [St18], all we do is verify that the necessary homotopical hypotheses are satisfied, which is a consequence of Theorem 2.12.

We recall that arboreal singularities are classified by finite rooted trees equipped with a decoration of signs ± 1 on each edge not adjacent to the root. The height of a vertex is defined to be the minimal number of edges in

a path between that vertex and the root. The height of a tree is defined to be the maximal height among all vertices. The height of an arboreal singularity is defined to be the height of the corresponding signed rooted tree.

Corollary 5.4. *Let (W, λ, ϕ) be a Weinstein manifold such that TW admits a global field of Lagrangian planes and such that the Morse Lyapunov function ϕ only has two critical points. Then by a homotopy of the Weinstein structure we can arrange it so that the skeleton of (W, λ) is arboreal, and moreover so that the arboreal singularities which appear in the skeleton have height ≤ 2 .*

Proof. One of the critical points of ϕ is a minimum x_0 . Let us assume that the other critical point x_n has the maximal index n , the subcritical case being easier and left as an exercise for the reader. A neighborhood of x_0 is exact symplectomorphic to the standard Darboux ball $(B^{2n}, pdq - qdp)$ and the stable manifold $W^s(x_n)$ of x_n intersects $\partial B^{2n} = S^{2n-1}$ in an $(n-1)$ -dimensional sphere Λ which is Legendrian for the standard contact structure.

Following Starkston [St18] we may deform the Weinstein structure of (W, λ, ϕ) in a neighborhood of x_0 from the standard Darboux model $(B^{2n}, pdq - qdp)$ to the standard cotangent model (U^*D^n, pdq) , where $U^*D^n = \{(q, p) \in T^*D^n, \|p\| \leq 1\}$. Moreover, we may arrange it so that the global Lagrangian distribution $\eta \subset TW$ agrees with the vertical distribution of U^*D^n at its center point x_0 , and hence after a homotopy of η we may assume that it agrees with the vertical distribution on all of U^*D^n .

Next, observe that $\partial D^n \subset \partial(U^*D^n)$ corresponds to a Legendrian unknot in $(\partial B^{2n}, \xi_{\text{std}})$ which by a general position argument may be assumed to be disjoint from Λ . Hence we may now think of Λ as a Legendrian submanifold in $S^*D^n = \{(p, q) \in T^*D^n, \|p\| = 1\} \subset \partial(U^*D^n)$. The singularities of the restriction $\pi|_{\Lambda} : \Lambda \rightarrow D^n$ of the front projection $\pi : S^*D^n \rightarrow D^n$ are the same as the tangencies of Λ with respect to the distribution $\nu \subset T(S^*D^n)$ tangent to the fibres of π . Theorem 1.2 says that it will be possible to deform Λ by a Legendrian isotopy so that these singularities consist only of semi-cubical cusps as soon as we know that $\nu|_{\Lambda}$ is stably trivial as an element of $\pi_{n-1}\Lambda_{n-1}$. We remind the reader that here we are implicitly using the trivialization induced by a Weinstein neighborhood and the isomorphism of symplectic vector bundles $T(T^*\Lambda)|_{\Lambda} \simeq \Lambda \times \mathbf{C}^{n-1}$.

Now, the image of $\nu|_{\Lambda}$ under the stabilization map $\pi_{n-1}\Lambda_{n-1} \rightarrow \pi_{n-1}\Lambda_n$ can be identified with the direct sum of $\nu|_{\Lambda}$ with the Liouville direction, which is the vertical distribution of U^*D^n restricted to Λ . By construction, on U^*D^n this vertical distribution agrees with our globally defined

Lagrangian field $\eta \subset TW$. In particular we see that the stabilization of $\nu|_\Lambda$ extends to the n -disk given by the stable manifold $W^s(x_n)$, which implies that the stabilization of $\nu|_\Lambda$ is trivial as an element of $\pi_{n-1}\Lambda_n$. This is precisely what we needed to show.

Therefore by Theorem 1.2 we may find a Legendrian isotopy Λ_t of $\Lambda = \Lambda_0$ in S^*D^n such that $\pi|_{\Lambda_1} : \Lambda_1 \rightarrow D^n$ has singularities consisting only of semi-cubical cusps, and this Legendrian isotopy can be realized by a homotopy of the ambient Weinstein structure. The new skeleton is arboreal outside of the cusp locus, with arboreal singularities of height ≤ 1 . To conclude the proof it remains to arborealize the semi-cubical cusps. To do this one may directly invoke [St18], hence the proof is complete.

For the benefit of the reader let us briefly explain how this works. First, one introduces an explicit local model near the cusps to replace them with arboreal singularities, which are of height 2. The model propagates new arboreal singularities in the Liouville direction, so this modification is not local near the cusps. To fix this, one can insert a wall along Λ_1 on which the propagated singularities land. After a generic perturbation this results in new arboreal singularities of height 2 where before there were fold tangencies of Λ_1 with respect to ν . \square

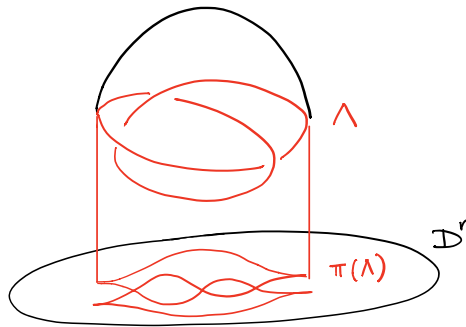


Figure 5.1: The skeleton after the application of the h-principle but before trading the cusps for arboreal singularities of height 2.

5.2. Nearby Lagrangian homotopy spheres admit framed generating functions

Let Σ_0 be an n -dimensional homotopy sphere and let $\Sigma_1 \subset T^*\Sigma_0$ be a Lagrangian embedding of another homotopy sphere Σ_1 . In [ACGK20] it is

proved that the stable Gauss map $\Sigma_1 \rightarrow U/O$ is trivial, which is equivalent to the statement that the vertical distribution of $T^*\Sigma_0$ is stably trivial as a Lagrangian distribution defined along Σ_1 . Therefore, Theorem 1.2 implies the following result.

Corollary 5.5. *There exists a compactly supported Hamiltonian isotopy φ_t of $T^*\Sigma_0$ such that $\varphi_1(\Sigma_1) \subset T^*\Sigma_0$ only has fold tangencies with respect to the vertical distribution.*

This result has the following consequence. In [ACGK20], the triviality of the stable Gauss map $\Sigma_1 \rightarrow U/O$ is deduced as a consequence of an existence theorem for generating functions. This theorem states that Σ_1 can be presented as the Cerf diagram of a function $f : W \rightarrow \mathbf{R}$, where $W \rightarrow \Sigma_0$ is a bundle of tubes in the sense of Waldhausen [W82]. We briefly recall the relevant definitions, and refer the reader to [ACGK20] for further details.

Let E be a k -dimensional linear subspace of \mathbf{R}^N . Consider the codimension zero submanifold $T_E \subset \mathbf{R}^{N+1}$ obtained by attaching to the half-space $\{x_{N+1} \leq 0\}$ a standard $(N + 1)$ -dimensional index k handle along the unit sphere of $E \subset \{x_{N+1} = 0\}$. We call T_E a *rigid tube*. We call a *tube* any codimension zero submanifold $T \subset \mathbf{R}^{N+1}$ which is the image of a rigid tube under a smooth isotopy fixed outside of a compact set.

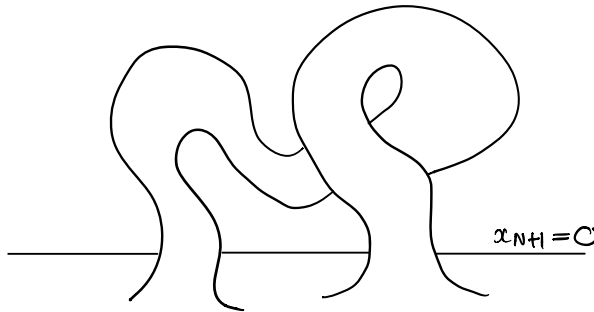


Figure 5.2: A tube.

Definition 5.6. Let M be a closed manifold. A *tube bundle* $W \rightarrow M$ is a smooth fibre bundle of manifolds whose fibres are tubes $T \subset \mathbf{R}^{N+1}$ in a fixed Euclidean space.

Let $T \subset \mathbf{R}^{N+1}$ be a tube. We consider functions $g : T \rightarrow \mathbf{R}$ such that:

- (1) ∂T is a regular level set of g .
- (2) $g = x_{N+1}$ outside of a compact set.

Let $W \rightarrow M$ be a tube bundle. We consider functions $f : W \rightarrow \mathbf{R}$ such that the restriction of f to each fibre is a function satisfying (1) & (2) and such that:

- (3) the fibrewise Euclidean gradient $\nabla_T f : W \rightarrow \mathbf{R}^{N+1}$ has 0 as a regular value.

We denote by $f_m : T_m \rightarrow \mathbf{R}$ the restriction of f to the fibre over $m \in M$.

Definition 5.7. Let $W \rightarrow M$ be a tube bundle. The *Cerf diagram* of a function $f : W \rightarrow \mathbf{R}$ is the subset $\{(m, z) \in M \times \mathbf{R} : z \text{ is a critical value of } f_m\} \subset M \times \mathbf{R}$.

Recall that $J^1M = T^*M \times \mathbf{R}$ is equipped with the canonical contact form $\alpha = dz - pdq$ for pdq the Liouville form on T^*M . A Legendrian submanifold $\Lambda \subset J^1M$ is a smooth submanifold of the same dimension as M such that $\alpha|_\Lambda = 0$. The front projection of a Legendrian $\Lambda \subset J^1M$ is its image under the map $J^1M \rightarrow J^0M$, where we recall $J^0M = M \times \mathbf{R}$.

Definition 5.8. Let $W \rightarrow M$ be a tube bundle, $\Lambda \subset J^1M$ a Legendrian submanifold and $f : W \rightarrow \mathbf{R}$ a function satisfying (1), (2) & (3). We say that f is a *generating function* for Λ if the symplectic reduction defines an embedding $\{\partial_F f = 0\} \rightarrow J^1M$ with image Λ .

Remark 5.9. In particular, note the front projection of Λ is the Cerf diagram of f .

We can now state the existence theorem for generating functions:

Theorem 5.10 ([ACGK20]). *Let $\Sigma_1 \subset T^*\Sigma_0$ be a Lagrangian homotopy sphere. Then there exists a tube bundle $W \rightarrow \Sigma_1$ and a function $f : W \rightarrow \mathbf{R}$ such that f generates Λ_1 .*

Work in progress of the first author with K. Igusa aims to study such Lagrangian homotopy spheres via the parametrized Morse theory of the generating function $f : W \rightarrow \mathbf{R}$, thought of as a family of functions on the fibres $f_x : W_x \rightarrow \mathbf{R}$, $x \in \Sigma_0$. However, theorem [ACGK20] provides no a priori control over the singularities of this family. In particular, there is no guarantee that each f_x is Morse or generalized Morse, nor can this be

arranged by a generic perturbation. Here by generalized Morse we mean cubic, i.e. the normal form for Morse birth/death, so that at the moment of bifurcation f_x takes the normal form given by x^3 direct sum with the Morse normal form in the other coordinates.

While existing h-principles in the literature [I87], [EM12] ensure that the function $f : W \rightarrow \mathbf{R}$ may be deformed by a homotopy $f_t : W \rightarrow \mathbf{R}$ so that the restriction of $f_1 : W \rightarrow \mathbf{R}$ to each fibre is Morse or generalized Morse, in general such a homotopy will generate a Lagrangian cobordism rather than a Lagrangian isotopy, and moreover will introduce self-intersection points so that in particular the end result is an immersed rather than embedded exact Lagrangian submanifold. One may overcome this issue by using Corollary 5.5 instead.

Corollary 5.11. *There exists a compactly supported Hamiltonian isotopy φ_t of $T^*\Sigma_0$ such that $\varphi_1(\Sigma_1) \subset T^*\Sigma_0$ is generated by a function $f : W \rightarrow \mathbf{R}$ on a tube bundle $W \rightarrow \Sigma_0$ with the property that the restriction $f_x : W_x \rightarrow \mathbf{R}$ to each fibre is Morse or generalized Morse.*

Proof. This follows from an application of Corollary 5.5 followed by an application of Theorem 5.10. Indeed, if a function generates a Lagrangian whose tangencies with the vertical distribution consist only of folds, then the function has Morse birth/death singularities along the tangency locus and is Morse elsewhere. □

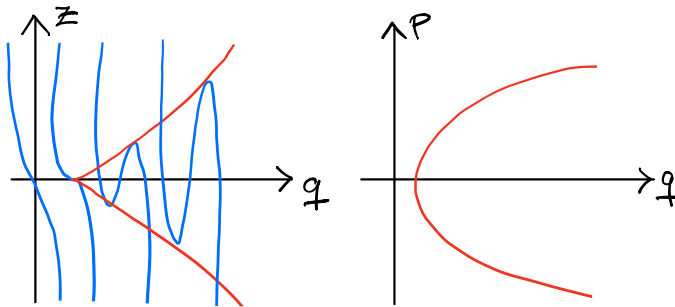


Figure 5.3: A birth/death of Morse critical points corresponds to a semi-cubical cusp in the front projection and to a fold type tangency in the Lagrangian projection.

Finally, we prove that in the situation under consideration it is moreover possible to find a generating function $f : W \rightarrow \mathbf{R}$ which is *framed*, i.e. the

restriction f_x of f to each fibre is Morse or generalized Morse and furthermore the negative eigenspaces to the Hessian of f_x at the critical points are equipped with framings that vary continuously with $x \in \Sigma_0$ and are suitably compatible at the birth/death points. Framed functions are useful because they are homotopically canonical [I87], [EM12], and can be used to compute higher K-theoretic invariants of the bundle they are defined over purely in terms of the associated family of Thom-Smale complexes [I02].

Intuitively, near a birth/death point the negative eigenspaces of the two critical points which come to be born or die differ by the 1-dimensional subspace in which the function is cubic, which is canonically framed by the direction in which the function is increasing. The compatibility requirement is that the framing for the negative eigenspace of the critical point of greater index is obtained from the framing of the negative eigenspace of the critical point of smaller index by adding the canonical framing of the cubic direction.

An equivalent formulation (up to stabilization of f) is the following: for a function $f : W \rightarrow \mathbf{R}$ whose restriction to each fibre $f_x : W \rightarrow \mathbf{R}$ only has Morse or generalized Morse critical points, the negative eigenspaces to the Hessian of f_x can be suitably stabilized depending on the index and assembled into a real vector bundle over the fibrewise critical locus of W , whose class in reduced topological K-theory is called the *stable bundle*. Then the condition that f admits a framing is, up to stabilization, equivalent to the condition that the stable bundle is trivial, see [I87] or [EM12] for details.

Corollary 5.12. *There exists a compactly supported Hamiltonian isotopy φ_t of $T^*\Sigma_0$ such that $\varphi_1(\Sigma_1) \subset T^*\Sigma_0$ is generated by a framed function $f : W \rightarrow \mathbf{R}$ on a tube bundle $W \rightarrow \Sigma_0$.*

Proof. The function $f : W \rightarrow \mathbf{R}$ produced by Corollary 5.12 need not admit a framing, however we may easily correct this. Let $E \rightarrow \varphi_1(\Sigma_1)$ be the stable bundle of f as explained above. It is known that the projection $\Sigma_1 \rightarrow \Sigma_0$ is a homotopy equivalence [A12], hence $\varphi_1(\Sigma_1) \rightarrow \Sigma_0$ is also a homotopy equivalence, hence we may find a real vector bundle $F \rightarrow \Sigma_0$ such that the direct sum of E with the pullback π^*F of F by $\varphi_1(\Sigma_1) \rightarrow \Sigma_0$ is trivial.

We may use F to perform a twisted stabilization of W to obtain a new tube bundle $W_F \rightarrow \Sigma_0$, namely this is the result of ‘folding down’ the extra dimensions on the function $f + Q_F$, where $Q_F : \mathbf{R}^{2m} \rightarrow \mathbf{R}$ is a family of quadratic forms parametrized by Σ_0 whose negative eigenspaces form a real vector bundle isomorphic to F . By construction, the new tube bundle W_F has the property that there exists a function $g : W_F \rightarrow \mathbf{R}$ generating $\varphi_1(\Sigma_1)$

which near its critical points coincides with $f + Q_F$. Hence the new generating function g has stable bundle $E \oplus \pi^*F$, which is trivial. This completes the proof. \square

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RECEIVED MAY 20, 2021

ACCEPTED JANUARY 4, 2022