JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 20, Number 5, 1037–1066, 2022

Contact $(+1)$-surgeries on rational homology 3-spheres

Fan Ding, Youlin Li, and Zhongtao Wu

In this paper, sufficient conditions for contact $(+1)$-surgeries along Legendrian knots in contact rational homology 3-spheres to have vanishing contact invariants or to be overtwisted are given. They can be applied to study contact (± 1)-surgeries along Legendrian links in the standard contact 3-sphere. We also obtain a sufficient condition for contact $(+1)$-surgeries along Legendrian two-component links in the standard contact 3-sphere to be overtwisted via their front projections.

1. Introduction

There is a dichotomy of contact structures on 3-manifolds: tight and overtwisted. Given a contact 3-manifold (Y, ξ), it is a fundamental question to ask whether it is tight or overtwisted. In [33], Ozsváth and Szabó introduced contact invariants $c(\xi) \in \widehat{HF}(-Y)$ and its image $c^+(\xi) \in HF^+(-Y)$, and proved that if (Y, ξ) is overtwisted then $c(\xi)$ vanishes. Moreover, Ghiggini proved that if (Y, ξ) is strongly symplectically fillable then $c^+(\xi)$, and hence $c(\xi)$, are non-trivial [19, Theorem 2.13]. So it is crucial to determine whether the contact invariant is trivial or not. In [10], the first author and Geiges proved that any closed connected contact 3-manifold can be obtained by contact surgery on the standard contact 3-sphere (S^3, ξ_{st}) along a Legendrian link $\mathbb{L}_1 \cup \mathbb{L}_2$ with coefficients $+1$ for each component of $\mathbb{L}_1$ and -1 for each component of $\mathbb{L}_2$. This leads us to study the tightness and contact invariant of (Y, ξ) through its contact (± 1)-surgery diagram along a Legendrian link in (S^3, ξ_{st}). If $\mathbb{L}_1$ is empty, then (Y, ξ) is Stein fillable, and $c(\xi)$ is nontrivial, and hence (Y, ξ) is tight. So we consider the case that $\mathbb{L}_1$ is non-empty, namely there are contact $(+1)$-surgeries involved in the surgery. In [13], the authors studied contact $(+1)$-surgeries along Legendrian two-component links in (S^3, ξ_{st}).

In many situations, a problem related to contact (± 1)-surgery along a Legendrian link in (S^3, ξ_{st}) can be reduced to a problem related to contact $(+1)$-surgery along a Legendrian link in a contact rational homology 3-sphere. This may occur, for example, when contact (-1)-surgery along the sublink $\mathbb{L}_2$ yields a contact rational homology 3-sphere.

For a Legendrian knot L in the standard contact 3-sphere (S^3, ξ_{st}), whether or not the result of contact $\frac{p}{q}$-surgery along L has non-vanishing contact invariant has been well studied by Lisca and Stipsicz [27, 28], Golla [20], and Mark and Tosun [29], etc. Recall that the contact invariant is natural under the cobordism induced by contact $(+1)$-surgery [33]. Thus, if (Y, ξ) is a contact 3-manifold whose contact invariant vanishes, then the result of contact $(+1)$-surgery along any Legendrian knot in (Y, ξ) has vanishing contact invariant as well. By Theorem 1.2 in [39] and Proposition 8 in [9], the result of contact $(+1)$-surgery along any Legendrian knot in an overtwisted closed connected contact 3-manifold (Y, ξ) is overtwisted. In this paper, we are mainly concerned with the contact invariant and overtwistedness of the result of contact $(+1)$-surgery along a Legendrian knot in a contact rational homology 3-sphere.

The last two authors introduced an invariant $\tau^*_{c(\xi)}(Y, K)$ for a rationally null-homologous knot K in a contact 3-manifold (Y, ξ) with non-vanishing contact invariant $c(\xi)$ [25], and proved that this invariant gives an upper bound for the sum of the rational Thurston-Bennequin invariant and the absolute value of the rational rotation number of all Legendrian knots isotopic to K, i.e.

$$tb_{\mathbb{Q}}(L) + |rot_{\mathbb{Q}}(L)| \leq 2\tau^*_{c(\xi)}(Y, K) - 1,$$

where L is a Legendrian knot in (Y, ξ) isotopic to K. This is a generalization of the inequalities appeared in [5], [15], [36], [22], etc. We give a sufficient condition for the result of contact $(+1)$-surgery having vanishing contact invariant. Let $(Y_{+1}(L), \xi_{+1}(L))$ denote the result of contact $(+1)$-surgery on (Y, ξ) along L.

Theorem 1.1. *Suppose K is a knot in a rational homology 3-sphere Y, and ξ is a contact structure on Y with nontrivial contact invariant $c(\xi) \in \widehat{HF}(-Y)$. Let L be a Legendrian knot in (Y, ξ) isotopic to K. Then the contact invariant $c(\xi_{+1}(L))$ vanishes if*

$$tb_{\mathbb{Q}}(L) + |rot_{\mathbb{Q}}(L)| < 2\tau^*_{c(\xi)}(Y, K) - 1.$$

Let $(S^3(\mathbb{L}_1^+ \cup \mathbb{L}_2^-), \xi_{st}(\mathbb{L}_1^+ \cup \mathbb{L}_2^-))$ denote the contact 3-manifold obtained by contact surgery on (S^3, ξ_{st}) along a Legendrian link $\mathbb{L}_1 \cup \mathbb{L}_2$ with coefficients $+1$ for each component of $\mathbb{L}_1$ and -1 for each component of $\mathbb{L}_2$. Özbağci showed in [31] that if some component of $\mathbb{L}_1$ contains an isolated stabilized arc which does not tangle with any other component of $\mathbb{L}_1 \cup \mathbb{L}_2$, then $(S^3(\mathbb{L}_1^+ \cup \mathbb{L}_2^-), \xi_{st}(\mathbb{L}_1^+ \cup \mathbb{L}_2^-))$ is overtwisted. In fact, thanks to Theorem 1.2 in [39], the condition in Özbağci's result can be slightly relaxed to be that some component of $\mathbb{L}_1$ contains an isolated stabilized arc which does not tangle with any component of $\mathbb{L}_2$. Applying Theorem 1.1 we obtain a result similar to that of Özbağci. Here we consider isolated Legendrian connected summands. See Figure 1. We refer the reader to [17] for Legendrian connected sum.

Proposition 1.2. *Let $\mathbb{L}_1 \cup \mathbb{L}_2 \subset (S^3, \xi_{st})$ be an oriented Legendrian link. If the contact 3-manifold $S^3(\mathbb{L}_2^-)$ is a rational homology 3-sphere, and there exists a front projection of $\mathbb{L}_1 \cup \mathbb{L}_2$ such that a component L_1 of $\mathbb{L}_1$ contains an isolated connected summand L_3 which does not tangle with $\mathbb{L}_2$ and satisfies*

$$tb(L_3) + |rot(L_3)| < 2\tau(L_3) - 1,$$

then the contact invariant $c(\xi_{st}(\mathbb{L}_1^+ \cup \mathbb{L}_2^-))$ vanishes.

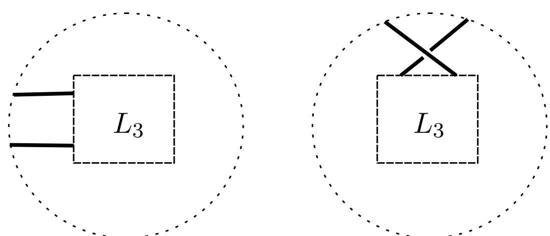

Figure 1: Isolated Legendrian connected sum.

On the other hand, we remark that the second part of [28, Proposition 1.4] can be generalized to Legendrian knots in contact L-spaces.

Proposition 1.3. *Suppose (Y, ξ) is a contact L-space, and L is a Legendrian knot in (Y, ξ). If $tb_{\mathbb{Q}}(L) < -1$, then the contact invariant $c^+(\xi_{+1}(L))$ vanishes.*

As mentioned earlier, this result may be used to study certain contact (± 1)-surgeries along Legendrian links in (S^3, ξ_{st}).

Corollary 1.4. *Let $\mathbb{L} = L_1 \cup L_2$ be a Legendrian two-component link in (S^3, ξ_{st}) whose two components have linking number l. Suppose that L_2 is a Legendrian L-space knot and $l^2 > 2g(L_2)(tb(L_1)+1)$, where $g(L_2)$ denotes the genus of L_2. Then the contact invariant $c^+(\xi_{st}(\mathbb{L}^+))$ vanishes.*

Now we deal with the overtwistedness of contact (+1)-surgeries. Among other things, Conway [7] and Onaran [30] obtained sufficient conditions for the overtwistedness of contact (+1)-surgeries along Legendrian null-homologous knots. Here we generalize Conway's results to Legendrian knots in contact rational homology 3-spheres. Likewise, it is useful for determining the overtwistedness of contact (± 1)-surgery along Legendrian links in (S^3, ξ_{st}).

Theorem 1.5. *Let L be a Legendrian knot in a contact rational homology 3-sphere (Y, ξ). Let q be the order of $[L]$ in $H_1(Y;\mathbb{Z})$, and $\chi(F)$ be the Euler characteristic of a rational Seifert surface F for L.*
(1) If

$$tb_{\mathbb{Q}}(L) < -1 \quad \textit{and} \quad tb_{\mathbb{Q}}(L) - |rot_{\mathbb{Q}}(L)| < \frac{\chi(F)}{q},$$

then $(Y_{+1}(L), \xi_{+1}(L))$ is overtwisted.
(2) If

$$tb_{\mathbb{Q}}(L) + |rot_{\mathbb{Q}}(L)| < \frac{\chi(F)}{q} - 2,$$

then the result of any positive contact surgery along L is overtwisted.

In [13, Theorem 1.6 and Corollary 6.4], the authors obtained sufficient conditions for the result of contact (+1)-surgery along a Legendrian two-component link in (S^3, ξ_{st}) to be overtwisted via some specific configurations in the front projection. The following theorem is an improvement of [13, Corollary 6.4].

Theorem 1.6. *Suppose the front projection of a Legendrian two-component link $\mathbb{L} = L_1 \cup L_2$ in the standard contact 3-sphere (S^3, ξ_{st}) contains a configuration exhibited in Figure 2, then contact $(+1)$-surgery on (S^3, ξ_{st}) along $\mathbb{L}$ yields an overtwisted contact 3-manifold.*

Acknowledgements. The authors would like to thank Roger Casals for sharing us his alternative proof of Theorem 1.6. We are also grateful to the referee for valuable suggestions. The first author was partially supported by

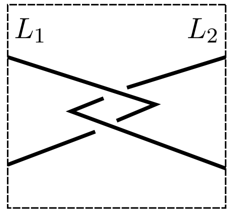

Figure 2: A configuration in the front projection of a Legendrian two-component link $\mathbb{L}$.

National Key R&D Program of China (No. 2020YFA0712800) and Grants No. 12131009 and 11371033 of the National Natural Science Foundation of China. The second author was partially supported by Grant No. 11871332 and 12271349 of the NNSFC. The third author was partially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. 14300018 and 14301819).

2. knots in rational homology 3-spheres

Suppose Y is an oriented rational homology 3-sphere and K is an oriented knot in Y. Suppose the order of $[K]$ in $H_1(Y;\mathbb{Z})$ is q. According to [1], we define a *rational Seifert surface* for K to be a smooth map $j : F \to Y$ from a connected compact oriented surface F to Y that is an embedding from the interior of F into $Y \setminus K$, and a q-fold cover from its boundary ∂F to K. Denote by $N(K)$ a closed tubular neighborhood of K in Y, and by $\mu \subset \partial N(K)$ a meridian of K. We can assume that $j(F) \cap \partial N(K)$ is composed of c parallel oriented simple closed curves, each of which has homology $\nu \in H_1(\partial N(K);\mathbb{Z})$. Then we can choose a canonical longitude λ_{can} such that $\nu = t\lambda_{can} + r\mu$, where homology classes of λ_{can} and μ are also denoted by λ_{can} and μ, respectively, t and r are coprime integers, and $0 \leq r < t$ (cf. [38, Section 2.6]). Certainly we have $ct = q$.

2.1. Filtrations.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a doubly pointed Heegaard diagram of K in Y. Then the set of relative Spin^c-structures determines a filtration of the chain complex $\widehat{CF}(Y)$ via a map

$$\underline{\mathfrak{s}}_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \underline{\mathrm{Spin}}^c(Y, K).$$

Each relative Spin^c structure $\underline{\mathfrak{s}}$ for (Y,K) corresponds to a Spin^c structure $\mathfrak{s}$ on Y via a natural map $G_{Y,K} : \underline{\mathrm{Spin}^c}(Y,K) \to \mathrm{Spin}^c(Y)$ (see [35, Section 2.2]).

Fix a rational Seifert surface F for K. As in [23], the Alexander grading of a relative Spin^c-structure $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}^c}(Y,K)$ is defined by

$$A(\underline{\mathfrak{s}}) = \frac{1}{2q}(\langle c_1(\underline{\mathfrak{s}}), [F]\rangle + q).$$

Moreover, the Alexander grading of an intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is defined by

$$A(x) = \frac{1}{2q}(\langle c_1(\underline{\mathfrak{s}}_{w,z}(x)), [F]\rangle + q).$$

In general, the Alexander grading $A(x)$ is a rational number. Nonetheless, observe that for any two relative Spin^c structures $\underline{\mathfrak{s}}_1, \underline{\mathfrak{s}}_2 \in G^{-1}_{Y,K}(\mathfrak{s})$ of a fixed $\mathfrak{s}$, we have $\underline{\mathfrak{s}}_2 - \underline{\mathfrak{s}}_1 = l\,\mathrm{PD}[\mu]$ for some integer l, and $A(\underline{\mathfrak{s}}_2) - A(\underline{\mathfrak{s}}_1) = l$.

2.2. Rational τ invariants and rational ν invariants

For a fixed $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}^c}(Y,K)$, let $C_{\underline{\mathfrak{s}}}$ be the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex $CFK^\infty(Y,K,\underline{\mathfrak{s}})$ (see [35, Section 3]).

Let

$$\iota_k : C_{\underline{\mathfrak{s}}}\{i=0, j\le k\} \to C_{\underline{\mathfrak{s}}}\{i=0\}$$

be the inclusion map, where $k\in\mathbb{Z}$. It induces a homomorphism between the homologies

$$\iota_{k*} : H_*(C_{\underline{\mathfrak{s}}}\{i=0, j\le k\}) \to \widehat{HF}(Y,\mathfrak{s}),$$

where $\mathfrak{s} = G_{Y,K}(\underline{\mathfrak{s}})$. Let

$$v_k : C_{\underline{\mathfrak{s}}}\{\max(i, j-k)=0\} \to C_{\underline{\mathfrak{s}}}\{i=0\}$$

be the composition of ι_k and the quotient map from $C_{\underline{\mathfrak{s}}}\{\max(i,j-k)=0\}$ to $C_{\underline{\mathfrak{s}}}\{i=0, j\le k\}$. It induces a homomorphism between the homologies

$$v_{k*} : H_*(C_{\underline{\mathfrak{s}}}\{\max(i,j-k)=0\}) \to \widehat{HF}(Y,\mathfrak{s}).$$

Next we recall the definition of rational τ invariants [22].

Definition 2.1. For any $[x]\neq 0 \in \widehat{HF}(Y,\mathfrak{s})$, define

$$\tau_{[x]}(Y,K) = \min\{A(\underline{\mathfrak{s}}) + k \mid [x]\in \mathrm{Im}(\iota_{k*})\}.$$

Then we introduce the definition of rational ν invariants in the same manner as Hom-Levine-Lidman did in the integral homology sphere case ([24]).

Definition 2.2. For any $[x] \neq 0 \in \widehat{HF}(Y, \mathfrak{s})$, define

$$\nu_{[x]}(Y, K) = \min\{A(\underline{\mathfrak{s}}) + k \mid [x] \in \mathrm{Im}(v_{k*})\}.$$

Lemma 2.3. $\nu_{[x]}(Y, K) = \tau_{[x]}(Y, K)$ *or* $\tau_{[x]}(Y, K) + 1$.

Proof. Due to [22, Proposition 24] and [25, Proposition 2.5], the proof is similar to the case where $Y = S^3$ [35, Equation 34], and is straightforward. □

Consider the orientation reversal $-Y$ of Y. We have a pairing

$$\langle -, - \rangle : \widehat{CF}(-Y, \mathfrak{s}) \otimes \widehat{CF}(Y, \mathfrak{s}) \to \mathbb{Z}/2\mathbb{Z},$$

given by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

It descends to a pairing

$$\langle -, - \rangle : \widehat{HF}(-Y, \mathfrak{s}) \otimes \widehat{HF}(Y, \mathfrak{s}) \to \mathbb{Z}/2\mathbb{Z}.$$

Definition 2.4. For any $[y] \neq 0 \in \widehat{HF}(-Y, \mathfrak{s})$, define

$$\tau^*_{[y]}(Y, K) = \min\{A(\underline{\mathfrak{s}}) + k \mid \exists \alpha \in \mathrm{Im}(\iota_{k*}), \text{such that} \langle [y], \alpha \rangle \neq 0\}.$$

Proposition 2.5. *[25, Proposition 2.3] Let* $[y] \neq 0 \in \widehat{HF}(-Y, \mathfrak{s})$. *Then*

$$\tau_{[y]}(-Y, K) = -\tau^*_{[y]}(Y, K).$$

Proposition 2.6. *[25, Proposition 2.4] For any* $[y_i] \neq 0 \in \widehat{HF}(-Y_i, \mathfrak{s}_i)$, $i = 1, 2$, *we have*

$$\tau^*_{[y_1] \otimes [y_2]}(Y_1 \# Y_2, K_1 \# K_2) = \tau^*_{[y_1]}(Y_1, K_1) + \tau^*_{[y_2]}(Y_2, K_2).$$

Suppose (Y_1, ξ_1) and (Y_2, ξ_2) are two contact rational homology 3-spheres with nonvanishing contact invariants $c(\xi_1)$ and $c(\xi_2)$, then the contact invariant of $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ is $c(\xi_1) \otimes c(\xi_2) \in \widehat{HF}(-Y_1) \otimes \widehat{HF}(-Y_2)$. See for example [22, Page 105]. As a corollary of Proposition 2.6, we have the following proposition.

Proposition 2.7. *Let $K_1 \subset (Y_1, \xi_1)$ and $K_2 \subset (Y_2, \xi_2)$ be two smooth knots. Then*

$$\tau^*_{c(\xi_1)\otimes c(\xi_2)}(Y_1 \# Y_2, K_1 \# K_2) = \tau^*_{c(\xi_1)}(Y_1, K_1) + \tau^*_{c(\xi_2)}(Y_2, K_2).$$

2.3. Mapping cone for Morse surgery along knots in rational homology 3-spheres

For $p \in \mathbb{Z}$, let $Y_p(K)$ denote the 3-manifold obtained by performing Dehn surgery along K in Y with coefficient p with respect to the canonical longitude λ_{can}. The meridian μ is isotopic to a core circle of the glued-in solid torus. Let K_p denote this core circle, with the orientation inherited from $-\mu$ (see [23, Section 1.1]). The sets $\underline{\mathrm{Spin}}^c(Y, K)$ and $\underline{\mathrm{Spin}}^c(Y_p(K), K_p)$ are naturally identified and the fibers of the map

$$G_{Y_p(K),K_p} : \underline{\mathrm{Spin}}^c(Y_p(K), K_p) \to \mathrm{Spin}^c(Y_p(K))$$

are the orbits of the action of $PD[\lambda]$, where $\lambda = \lambda_{can} + p\mu$. Let $X_p(K)$ be the 4-manifold obtained by attaching a 4-dimensional 2-handle H to $Y \times I$ along $K \times \{1\}$ with coefficient p with respect to λ_{can}. Then $\partial X_p(K) = (-Y) \sqcup Y_p(K)$. Let C denote the core disk of the attached 2-handle in $X_p(K)$ with $\partial C = K \times \{1\}$. For a rational Seifert surface $j : F \to Y$ for K, $F_{qC} = (j(F) \times \{1\}) \cup (-qC)$ represents a homology class $[F_{qC}]$ in $H_2(X_p(K); \mathbb{Z})$. Given a Spin^c structure $\mathfrak{s}$ on Y, we can extend $\mathfrak{s}$ to a Spin^c structure $\mathfrak{t}$ on $X_p(K)$. All Spin^c structures $\mathfrak{t}$ on $X_p(K)$ with $\mathfrak{t}|_Y = \mathfrak{s}$ are distinguished by $\langle c_1(\mathfrak{t}), [F_{qC}]\rangle$. For each $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(Y, K)$ with $G_{Y,K}(\underline{\mathfrak{s}}) = \mathfrak{s}$, there is a unique Spin^c structure $\mathfrak{t}$ on $X_p(K)$ such that $\mathfrak{t}|_Y = \mathfrak{s}$ and

$$\langle c_1(\mathfrak{t}), [F_{qC}]\rangle + pq - cr = 2qA(\underline{\mathfrak{s}})$$

(see [38, Theorem 4.2]).

For $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(Y, K)$, let $A_{\underline{\mathfrak{s}}} = C_{\underline{\mathfrak{s}}}\{\max(i, j) = 0\}$ and $B_{\underline{\mathfrak{s}}} = C_{\underline{\mathfrak{s}}}\{i = 0\}$. There are two natural projection maps

$$v_{\underline{\mathfrak{s}}} : A_{\underline{\mathfrak{s}}} \to B_{\underline{\mathfrak{s}}}, \quad h_{\underline{\mathfrak{s}}} : A_{\underline{\mathfrak{s}}} \to B_{\underline{\mathfrak{s}}+PD[\lambda]}.$$

Define

$$\Phi : \bigoplus_{\underline{\mathfrak{s}}} A_{\underline{\mathfrak{s}}} \to \bigoplus_{\underline{\mathfrak{s}}} B_{\underline{\mathfrak{s}}}, \quad (\underline{\mathfrak{s}}, a) \mapsto \big(\underline{\mathfrak{s}}, v_{\underline{\mathfrak{s}}}(a)\big) + \big(\underline{\mathfrak{s}} + PD[\lambda], h_{\underline{\mathfrak{s}}}(a)\big),$$

which is often written in the following form

$$\begin{array}{ccccccccccc}
\cdots & & A_{\underline{\mathfrak{s}}-PD[\lambda]} & & & A_{\underline{\mathfrak{s}}} & & & A_{\underline{\mathfrak{s}}+PD[\lambda]} & & \cdots \\
& \searrow & \downarrow{\scriptstyle v_{\underline{\mathfrak{s}}-PD[\lambda]}} & \overset{h_{\underline{\mathfrak{s}}-PD[\lambda]}}{\searrow} & & \downarrow{\scriptstyle v_{\underline{\mathfrak{s}}}} & \overset{h_{\underline{\mathfrak{s}}}}{\searrow} & & \downarrow{\scriptstyle v_{\underline{\mathfrak{s}}+PD[\lambda]}} & \overset{h_{\underline{\mathfrak{s}}+PD[\lambda]}}{\searrow} & \\
\cdots & & B_{\underline{\mathfrak{s}}-PD[\lambda]} & & & B_{\underline{\mathfrak{s}}} & & & B_{\underline{\mathfrak{s}}+PD[\lambda]} & & \cdots
\end{array}$$

Note that the mapping cone of Φ splits over equivalence classes of relative Spin^c structures, where $\underline{\mathfrak{s}}_1$ and $\underline{\mathfrak{s}}_2$ are equivalent if $\underline{\mathfrak{s}}_2 = \underline{\mathfrak{s}}_1 + mPD[\lambda]$ for some integer m.

Theorem 2.8. *[35, Theorem 6.1] Denote $\widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]}$ the summand of the cone of Φ corresponding to the equivalence class of $\underline{\mathfrak{s}}$. Then there exists a quasi-isomorphism of the complexes*

$$\Psi : \widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]} \cong \widehat{CF}(Y_p(K), G_{Y_p(K),K_p}(\underline{\mathfrak{s}})).$$

Theorem 2.9. *[38, Theorem 4.2] Suppose $\mathfrak{s} \in Spin^c(Y)$ and $\mathfrak{t} \in Spin^c(X_p(K))$ extends $\mathfrak{s}$. Then the map*

$$\widehat{CF}(Y, \mathfrak{s}) \to \widehat{CF}(Y_p(K), \mathfrak{t}|_{Y_p(K)})$$

induced by $\mathfrak{t}$ corresponds via Ψ to the inclusion of $B_{\underline{\mathfrak{s}}}$ in $\widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]}$, where $\underline{\mathfrak{s}} \in \underline{Spin}^c(Y, K)$ is determined by $G_{Y,K}(\underline{\mathfrak{s}}) = \mathfrak{s}$ and

$$\langle c_1(\mathfrak{t}), [F_{qC}] \rangle + pq - cr = 2qA(\underline{\mathfrak{s}}).$$

3. Contact (+1)-surgeries on rational homology 3-spheres with vanishing contact invariants

For a rationally null-homologous oriented Legendrian knot L in a contact 3-manifold (Y, ξ), one can define the rational Thurston-Bennequin invariant $tb_{\mathbb{Q}}(L)$ and the rational rotation number $rot_{\mathbb{Q}}(L)$. We refer the reader to [1] for more details.

Suppose Y is an oriented rational homology 3-sphere and ξ is a contact structure on Y. Let K be an oriented knot in Y. Suppose the order of $[K]$ in $H_1(Y; \mathbb{Z})$ is q. Let L be an oriented Legendrian knot in (Y, ξ) isotopic to K. Let F be a rational Seifert surface for L. Suppose the longitude of L determined by the contact framing is $\lambda_c = \lambda_{can} + (p-1)\mu$ for some integer p. We use the notation in Section 2 with K replaced by L. Performing contact

(+1)-surgery along L, we obtain a contact 3-manifold $(Y_{+1}(L), \xi_{+1}(L))$. This contact (+1)-surgery induces a cobordism $X_p(L)$, also denoted by W, from Y to $Y_{+1}(L)$. Notice that $Y_p(L) = Y_{+1}(L)$.

The contact structure ξ (respectively, $\xi_{+1}(L)$) defines an almost complex structure J (on W) along Y (respectively, $Y_{+1}(L)$) by requiring ξ (respectively, $\xi_{+1}(L)$) to be J-invariant and J to map the inward (respectively, outward) normal along Y (respectively, $Y_{+1}(L)$) to a vector in Y (respectively, $Y_{+1}(L)$) positively transverse to ξ (respectively, $\xi_{+1}(L)$). This J can be extended to the closure of the complement of a 4-disk $D_H \subset int(H) \subset W$ such that $d_3(\xi_H) = \frac{1}{2}$, where ξ_H denotes the plane field on ∂D_H induced by J (see [12, Section 3]). The Spin^c structure on W induced by J is denoted by $\mathfrak{t}_1$.

Mimicking the proof of [12, Proposition 5.2], we have

Lemma 3.1. $\langle c_1(\mathfrak{t}_1), [F_{qC}]\rangle = q \cdot rot_{\mathbb{Q}}(L)$. □

Lemma 3.2. $pq - cr = q \cdot (tb_{\mathbb{Q}}(L) + 1)$.

Proof. Recall that the longitude of L determined by the contact framing is $\lambda_c = \lambda_{can} + (p-1)\mu$. By definition in Baker-Etnyre [1], $tb_{\mathbb{Q}}(L)$ is the rational linking number of L and λ_c. So

$$\begin{aligned} tb_{\mathbb{Q}}(L) + 1 = lk_{\mathbb{Q}}(L, \lambda_c) + 1 &= \frac{1}{q}[F] \bullet \lambda_c + 1 \\ &= \frac{1}{q}(q\lambda_{can} + cr\mu) \bullet (\lambda_{can} + (p-1)\mu) + 1 = \frac{1}{q}(pq - cr), \end{aligned}$$

where the second intersection product is on $\partial(\overline{Y \setminus N(L)})$. □

By Baldwin [4, Theorem 1.2] (see also Mark-Tosun [29, Theorem 3.1]), there exists a Spin^c structure $\mathfrak{t}_2$ on $-W$ such that the homomorphism

$$F_{-W,\mathfrak{t}_2} : \widehat{HF}(-Y) \to \widehat{HF}(-Y_{+1}(L))$$

satisfies

$$F_{-W,\mathfrak{t}_2}(c(\xi)) = c(\xi_{+1}(L)).$$

The following lemma is similar to [29, Corollary 3.6].

Lemma 3.3. *The Spin^c structure $\mathfrak{t}_2$ satisfies*

$$\langle c_1(\mathfrak{t}_2), [F_{qC}]\rangle = \pm\langle c_1(\mathfrak{t}_1), [F_{qC}]\rangle = \pm q \cdot rot_{\mathbb{Q}}(L).$$

Proof. First, assume that $pq - cr = 0$, which is equivalent to $r = p = 0$. In this case, the map $H^2(X_p(L);\mathbb{Q}) \to H^2(Y_{+1}(L);\mathbb{Q})$ induced by inclusion is an isomorphism. Combining this with the fact that $\mathfrak{t}_1, \mathfrak{t}_2$ induce the same Spin^c structure on $Y_{+1}(L)$, i.e. the Spin^c structure induced by $\xi_{+1}(L)$, we have

$$\langle c_1(\mathfrak{t}_2), [F_{qC}]\rangle = \langle c_1(\mathfrak{t}_1), [F_{qC}]\rangle = q \cdot rot_{\mathbb{Q}}(L).$$

Now suppose $pq - cr \neq 0$. In this case, $Y_{+1}(L)$ is a rational homology 3-sphere and $c_1(\xi_{+1}(L))$ is torsion. For an oriented plane filed η with torsion first Chern class in an oriented 3-manifold R, Gompf [21] defined the 3-dimensional invariant $d_3(\eta)$ to be the rational number

$$\frac{1}{4}(c_1^2(Z,J) - 3\sigma(Z) - 2\chi(Z)),$$

where (Z,J) is an almost complex 4-manifold having $\partial Z = R$ and $\eta = TR \cap J(TR)$. According to [33] or [37], the absolute grading of the contact invariant $c(\xi)$ for a contact 3-manifold (R,ξ) is related to the 3-dimensional invariant $d_3(\xi)$ by

$$\widetilde{gr}(c(\xi)) = -d_3(\xi) - \frac{1}{2}.$$

So it follows from the degree shift formula [34, Theorem 7.1] in Heegaard Floer homology that

$$\frac{1}{4}(c_1^2(\mathfrak{t}_2)_{-W} - 3\sigma(-W) - 2\chi(-W)) = -d_3(\xi_{+1}(L)) + d_3(\xi). \tag{3.1}$$

Let W' denote $W - int(D_H)$. Since $\partial W' = Y_{+1}(L) \sqcup (-Y) \sqcup (-\partial D_H)$ and the almost complex structure J induces the plane fields $\xi_{+1}(L), \xi$ and ξ_H on $Y_{+1}(L), Y$ and ∂D_H, respectively, we have

$$\begin{aligned} d_3(\xi_{+1}(L)) - d_3(\xi) - d_3(\xi_H) &= \frac{1}{4}(c_1^2(J)_{W'} - 3\sigma(W') - 2\chi(W')) \\ &= \frac{1}{4}(c_1^2(\mathfrak{t}_1)_W - 3\sigma(W) - 2\chi(W)) + \frac{1}{2}. \end{aligned}$$

Note that $\chi(W) = 1$ and $d_3(\xi_H) = \frac{1}{2}$. Thus

$$d_3(\xi_{+1}(L)) - d_3(\xi) - \frac{1}{2} = \frac{1}{4}(c_1^2(\mathfrak{t}_1)_W - 3\sigma(W)).$$

With (3.1), we get $c_1^2(\mathfrak{t}_2)_{-W} = -c_1^2(\mathfrak{t}_1)_W$. Since $H_2(W;\mathbb{Q})$ is generated by $[F_{qC}]$ and $[F_{qC}]^2 = q(pq - cr) \neq 0$, we obtain

$$\langle c_1(\mathfrak{t}_2), [F_{qC}]\rangle = \pm\langle c_1(\mathfrak{t}_1), [F_{qC}]\rangle = \pm q \cdot rot_{\mathbb{Q}}(L).$$

□

For $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(-Y, L)$, by Theorem 2.8, there exists a quasi-isomorphism of the complexes

$$\Psi : \widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]} \cong \widehat{CF}(-Y_p(L), G_{-Y_p(L),-L_p}(\underline{\mathfrak{s}})).$$

The isomorphism

$$H_*(\widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]}) \cong \widehat{HF}(-Y_p(L), G_{-Y_p(L),-L_p}(\underline{\mathfrak{s}}))$$

induced by Ψ is denoted by Ψ_*. Let $\mathfrak{s}_\xi$ (respectively, $\mathfrak{s}_{\xi_{+1}(L)}$) denote the Spin^c structure on Y (respectively, $Y_{+1}(L)$) induced by ξ (respectively, $\xi_{+1}(L)$). By Theorem 2.9, the map

$$\widehat{CF}(-Y, \mathfrak{s}_\xi) \to \widehat{CF}(-Y_p(L), \mathfrak{s}_{\xi_{+1}(L)})$$

induced by $\mathfrak{t}_2$ corresponds via Ψ to the inclusion of $B_{\underline{\mathfrak{s}}}$ in $\widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]}$, where $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(-Y, L)$ satisfies $G_{-Y,L}(\underline{\mathfrak{s}}) = \mathfrak{s}_\xi$ and

$$\langle c_1(\mathfrak{t}_2), [F_{qC}] \rangle - pq + cr = 2qA(\underline{\mathfrak{s}}).$$

Applying Lemmata 3.2 and 3.3, we have

Corollary 3.4. *Via Ψ_*, the contact invariant $c(\xi_{+1}(L)) \in \widehat{HF}(-Y_p(L))$ is the image of $c(\xi)$ under the homomorphism of homologies induced by inclusion*

$$B_{\underline{\mathfrak{s}}} \hookrightarrow \widehat{\mathbb{X}}_{[\underline{\mathfrak{s}}]},$$

where $\underline{\mathfrak{s}} \in \underline{Spin}^c(-Y, L)$ satisfies $G_{-Y,L}(\underline{\mathfrak{s}}) = \mathfrak{s}_\xi$ and

$$2A(\underline{\mathfrak{s}}) = -tb_{\mathbb{Q}}(L) \pm rot_{\mathbb{Q}}(L) - 1.$$

Lemma 3.5. *Suppose $c(\xi) \neq 0$. For $\underline{\mathfrak{s}}$ in the above corollary, $c(\xi) \notin Im(v_{\underline{\mathfrak{s}}*})$ if and only if both*

$$\nu_{c(\xi)}(-Y, K) = -\tau^*_{c(\xi)}(Y, K) + 1$$
$$\text{and} \quad \tau^*_{c(\xi)}(Y, K) = \frac{1}{2}(tb_{\mathbb{Q}}(L) \mp rot_{\mathbb{Q}}(L) + 1).$$

Proof. By the definition of $\nu_{c(\xi)}(-Y, K)$, the contact invariant $c(\xi) \notin \mathrm{Im}(v_{\underline{\mathfrak{s}}*})$ if and only if $A(\underline{\mathfrak{s}}) < \nu_{c(\xi)}(-Y, K)$. By Lemma 2.3 and Proposition 2.5, $\nu_{c(\xi)}(-Y, K)$ equals either $-\tau^*_{c(\xi)}(Y, K)$ or $-\tau^*_{c(\xi)}(Y, K) + 1$. On the other hand, it follows from [25, Theorem 1.1] that $\frac{1}{2}(-tb_{\mathbb{Q}}(L) \pm rot_{\mathbb{Q}}(L) - 1) \geq$

$-\tau^*_{c(\xi)}(Y,K)$. Applying Corollary 3.4, we conclude that $\nu_{c(\xi)}(-Y,K) = -\tau^*_{c(\xi)}(Y,K)+1$ and $-\tau^*_{c(\xi)}(Y,K) = A(\underline{\mathfrak{s}}) = \frac{1}{2}(-tb_{\mathbb{Q}}(L) \pm rot_{\mathbb{Q}}(L) - 1)$. □

Proof of Theorem 1.1. Since

$$\frac{1}{2}(tb_{\mathbb{Q}}(L) \pm rot_{\mathbb{Q}}(L) + 1) < \tau^*_{c(\xi)}(Y,K) = -\tau_{c(\xi)}(-Y,K),$$

Lemma 3.5 implies that $c(\xi)$ lies in the image of $v_{\underline{\mathfrak{s}}*}$. By Corollary 3.4, it suffices to find a cycle $c \in A_{\underline{\mathfrak{s}}}$ such that $v_{\underline{\mathfrak{s}}*}([c]) = c(\xi) \in H_*(B_{\underline{\mathfrak{s}}})$, while $h_{\underline{\mathfrak{s}}*}([c]) = 0$. Recall that $A_{\underline{\mathfrak{s}}}$ is the subquotient complex $C_{\underline{\mathfrak{s}}}\{\max(i,j)=0\}$ of $CFK^\infty(-Y,K,\underline{\mathfrak{s}})$. By the definition of $\tau_{c(\xi)}(-Y,K)$, there is a cycle c in the vertical complex $B_{\underline{\mathfrak{s}}} = C_{\underline{\mathfrak{s}}}\{i=0\}$ that is supported in $C_{\underline{\mathfrak{s}}}\{i=0, j \leq \tau_{c(\xi)}(-Y,K) - A(\underline{\mathfrak{s}})\}$ and $[c] = c(\xi) \in H_*(B_{\underline{\mathfrak{s}}})$. By our assumption, $A(\underline{\mathfrak{s}}) = \frac{1}{2}(\pm rot_{\mathbb{Q}}(L) - tb_{\mathbb{Q}}(L) - 1) > \tau_{c(\xi)}(-Y,K)$, so c can be considered as a cycle in $A_{\underline{\mathfrak{s}}}$, and since it lies in the subcomplex with $j < 0$, it vanishes under $h_{\underline{\mathfrak{s}}}$. □

Remark 3.6. Note that we do not really need the condition on the rational ν invariant in Lemma 3.5 for the proof of Theorem 1.1 or any other main results in this paper. On the other hand, it is natural and convenient to study this rational ν invariant in the context of the paper, which is likely to be useful in future research.

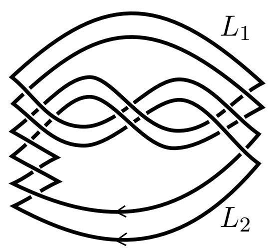

Figure 3: A Legendrian link $L_1 \cup L_2$, where L_2 is a Legendrian push-off of L_1.

Remark 3.7. For a Legendrian knot L in (S^3, ξ_{st}), if $rot(L) \neq 0$, then [20, Theorem 1.1] tells us that $\xi_{+1}(L)$ has vanishing contact invariant. However, this is not true for Legendrian knots in contact rational homology 3-spheres. For example, in Figure 3, L_1 is a Legendrian right handed trefoil with $tb(L_1) = 0$ and $rot(L_1) = -1$, and L_2 is a Legendrian push-off of L_1. The 3-manifold $S^3(L_2^-)$ is an integral homology 3-sphere. By [18, Lemma 3.1], the Thurston-Bennequin invariant of L_1 in $(S^3(L_2^-), \xi_{st}(L_2^-))$ is 0 and the rotation number of L_1 in $(S^3(L_2^-), \xi_{st}(L_2^-))$ is -1. Contact $(+1)$-surgery along L_1 in $(S^3(L_2^-), \xi_{st}(L_2^-))$ yields (S^3, ξ_{st}) (see [9]) which certainly has nonvanishing contact invariant.

Now we turn to some applications of Theorem 1.1. First we recall the following proposition.

Proposition 3.8. *[25, Lemma 3.2] For $i = 1, 2$, suppose that L_i is a Legendrian knot in a contact rational homology 3-sphere (Y_i, ξ_i). Then the rational Thurston-Bennequin invariant and the rational rotation number of the Legendrian knot $L_1 \# L_2$ in the contact 3-manifold $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ satisfy*

$$tb_{\mathbb{Q}}(L_1 \# L_2) = tb_{\mathbb{Q}}(L_1) + tb_{\mathbb{Q}}(L_2) + 1,$$

$$rot_{\mathbb{Q}}(L_1 \# L_2) = rot_{\mathbb{Q}}(L_1) + rot_{\mathbb{Q}}(L_2).$$

Proof of Proposition 1.2. It suffices to prove the case that $\mathbb{L}_1$ contains only one component L_1. Suppose L_1 is the Legendrian connected sum of L_3' and L_3. Then we have

$$(S^3(\mathbb{L}_2^-), L_1) = (S^3(\mathbb{L}_2^-), L_3') \# (S^3, L_3).$$

By [25, Theorem 1.1],

$$tb_{\mathbb{Q}}(L_3') + |rot_{\mathbb{Q}}(L_3')| + 1 \leq 2\tau^*_{c(\xi_{st}(\mathbb{L}_2^-))}(S^3(\mathbb{L}_2^-), L_3').$$

By assumption,

$$tb(L_3) + |rot(L_3)| + 1 < 2\tau(L_3).$$

So by Propositions 2.7 and 3.8, we have

$$tb_{\mathbb{Q}}(L_1) + |rot_{\mathbb{Q}}(L_1)| + 1 < 2\tau^*_{c(\xi_{st}(\mathbb{L}_2^{-1}))}(S^3(\mathbb{L}_2^-), L_1).$$

The proposition now follows from Theorem 1.1. □

Corollary 3.9. *Let $L_1 \cup L_2 \subset (S^3, \xi_{st})$ be an oriented Legendrian link with two components which has a front projection depicted at the top of Figure 4. If $tb(L_2) \neq 1$ and $tb(L_1) + |rot(L_1)| < 2\tau(L_1) - 1$, then the contact invariant $c(\xi_{st}(L_1^+ \cup L_2^-))$ vanishes.*

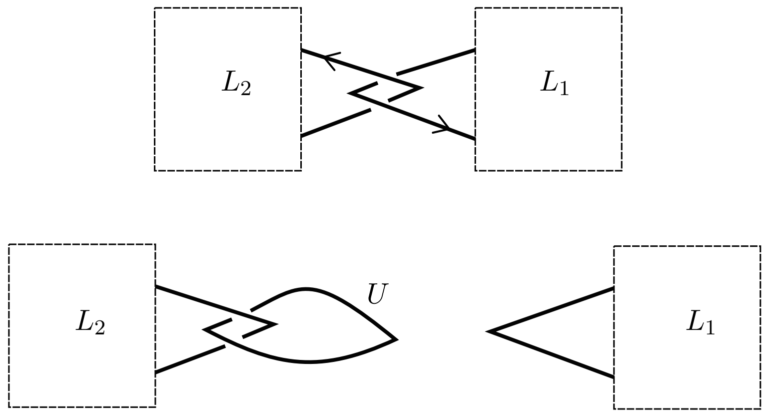

Figure 4: A Legendrian link $L_1 \cup L_2$.

Proof. Clearly, the contact invariant $c(\xi_{st}(L_2^-))$ is non-trivial. Since $tb(L_2) - 1 \neq 0$, $S^3(L_2^-)$ is a rational homology 3-sphere. We have

$$(S^3(L_2^-), L_1) = (S^3(L_2^-), U)\#(S^3, L_1),$$

where U is a Legendrian unknot shown at the bottom left of Figure 4.

Since

$$tb(L_1) + |rot(L_1)| + 1 < 2\tau(L_1),$$

the corollary follows from Proposition 1.2. □

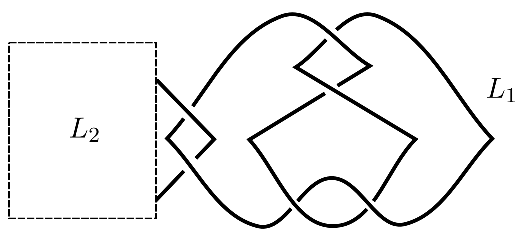

Figure 5: A Legendrian link $L_1 \cup L_2$, where L_1 is a Legendrian figure eight knot.

Example 3.10. Let $L = L_1 \cup L_2$ be a Legendrian link in (S^3, ξ_{st}) depicted in Figure 5. Note that L_1 is a Legendrian figure eight knot with $tb(L_1) = -3$ and $rot(L_1) = 0$. Since $\tau(L_1) = 0$, Corollary 3.9 implies that $(S^3(L_1^+ \cup L_2^-), \xi_{st}(L_1^+ \cup L_2^-))$ is a contact 3-manifold with vanishing contact invariant for any Legendrian knot L_2 with $tb(L_2) \neq 1$.

In the last part of this section, we prove Proposition 1.3 and its application to contact (+1)-surgeries along Legendrian links in (S^3, ξ_{st}), Corollary 1.4. Note that the vanishing result in Proposition 1.3 is only obtained for the plus-version of the contact invariant $c^+(\xi)$ as opposed to $c(\xi)$ in the other parts of the paper. To the best of the authors' knowledge, there is no known example of contact 3-manifold (Y, ξ) with vanishing $c^+(\xi)$ but nonvanishing $c(\xi)$.

Proof of Proposition 1.3. Let W be the cobordism from Y to $Y_{+1}(L)$ induced by contact (+1)-surgery. Then the map $F^+_{-W} : HF^+(-Y) \to HF^+(-Y_{+1}(L))$ send $c^+(\xi)$ to $c^+(\xi_{+1}(L))$. By [29, Lemma 5.1] and Lemma 3.2, the self-intersection of a generator of $H_2(-W;\mathbb{Z}) \cong \mathbb{Z}$ is $-q^2(tb_{\mathbb{Q}}(L) + 1) > 0$. So $-W$ is positive definite. By [34, Lemma 8.2], $F^\infty_{-W} : HF^\infty(-Y) \to HF^\infty(-Y_{+1}(L))$ vanishes. Since Y is an L-space, $HF^\infty(-Y) \to HF^+(-Y)$ is onto. Hence $F^+_{-W} = 0$, and the contact invariant $c^+(\xi_{+1}(L))$ vanishes. □

Proof of Corollary 1.4. If L_2 is an unknot, then the corollary follows from [13, Theorem 1.1]. In the following we assume that L_2 is nontrivial.

Since L_2 is an L-space knot, $g(L_2) = \tau(L_2)$. If $tb(L_2) < 2g(L_2) - 1 = 2\tau(L_2) - 1$, then [20, Theorem 1.1] implies that $c^+(\xi_{st}(L_2^+))$ vanishes. So the contact invariant $c^+(\xi_{st}(\mathbb{L}^+))$ vanishes for any Legendrian knot L_1.

From Bennequin inequality, we can now assume that $tb(L_2) = 2g(L_2) - 1$. (Indeed, Lidman and Sivek conjectured in [26, Conjecture 1.19] that any L-space knot K has maximal Thurston-Bennequin invariant $2g(K) - 1$.) By [20, Theorem 1.1], $(S^3(L_2^+), \xi_{st}(L_2^+))$ is a tight contact L-space. Using [18, Lemma 3.1], we know that the rational Thurston-Bennequin invariant of L_1 in $(S^3(L_2^+), \xi_{st}(L_2^+))$ is

$$tb(L_1) + \frac{\det\begin{pmatrix} 0 & l \\ l & tb(L_2)+1 \end{pmatrix}}{tb(L_2)+1} = tb(L_1) - \frac{l^2}{2g(L_2)},$$

which is less than -1 by assumption. So the corollary follows immediately from Proposition 1.3. □

Example 3.11. Contact (+1)-surgery along the Legendrian link $\mathbb{L} = L_1 \cup L_2$ in (S^3, ξ_{st}) depicted in Figure 6 yields a contact 3-manifold with vanishing contact invariant $c^+(\xi_{st}(\mathbb{L}^+))$.

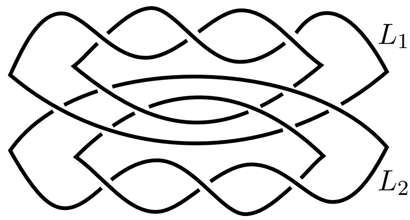

Figure 6: A Legendrian link $L_1 \cup L_2$, where both components are Legendrian right handed trefoil knots with $tb = 1$, and the linking number is 4.

4. Contact (±1)-surgeries on rational homology 3-spheres

Let $\mathbb{L} = L_1 \cup \cdots \cup L_n$ be a Legendrian link in a contact rational homology 3-sphere (Y, ξ). Suppose that contact (± 1)-surgery along the components of $\mathbb{L}$ produces (Y', ξ'). Set $a_i = tb_{\mathbb{Q}}(L_i) \pm 1$, depending on whether we perform contact $(+1)$-surgery or (-1)-surgery along L_i. Let L_0 be a Legendrian knot in (Y, ξ) disjoint from $\mathbb{L}$. For $i \neq j$, denote the rational linking number $lk_{\mathbb{Q}}(L_i, L_j)$ by l_{ij}. Let $M = (m_{ij})_{i,j=1}^n$ be the linking matrix of $\mathbb{L}$, i.e.

$$m_{ij} = \begin{cases} a_i & \text{if } i = j, \\ l_{ij} & \text{if } i \neq j. \end{cases}$$

Let $M_0 = (m_{ij})_{i,j=0}^n$ be the extended matrix given by

$$m_{ij} = \begin{cases} 0 & \text{if } i = j = 0, \\ a_i & \text{if } i = j \geq 1, \\ l_{ij} & \text{if } i \neq j. \end{cases}$$

Y' is still a rational homology 3-sphere if and only if $\det M \neq 0$ (see the proof of the following lemma). Let L be the image of L_0 in (Y', ξ'). The following lemma is a generalization of [18, Lemma 3.1].

Lemma 4.1. *Suppose* $\det M \neq 0$. *Then*

$$tb_{\mathbb{Q}}(L) = tb_{\mathbb{Q}}(L_0) + \frac{\det M_0}{\det M}$$

and

$$rot_{\mathbb{Q}}(L) = rot_{\mathbb{Q}}(L_0) - \left\langle \begin{pmatrix} rot_{\mathbb{Q}}(L_1) \\ \vdots \\ rot_{\mathbb{Q}}(L_n) \end{pmatrix}, M^{-1} \begin{pmatrix} l_{01} \\ \vdots \\ l_{0n} \end{pmatrix} \right\rangle .$$

Proof. For $i = 0, 1, \dots, n$, denote by $N(L_i)$ a closed tubular neighborhood of L_i in Y. Suppose for $i \neq j$, $N(L_i)$ and $N(L_j)$ are disjoint. Denote the knot exterior $\overline{Y \setminus N(L_i)}$ by $X(L_i)$. Denote $\overline{Y \setminus \bigcup_{i=1}^{n} N(L_i)}$ by $X(\mathbb{L})$ and $\overline{Y \setminus \bigcup_{i=0}^{n} N(L_i)}$ by $X(L_0 \cup \mathbb{L})$. Suppose the order of $[L_i]$ in $H_1(Y;\mathbb{Z})$ is q_i. Let F_i be a rational Seifert surface for L_i. We can assume that $F_i \cap \partial N(L_i)$ is composed of c_i parallel oriented simple closed curves, each of which has homology $t_i\lambda_i + r_i\mu_i \in H_1(\partial N(L_i);\mathbb{Z})$, where λ_i is the class of a canonical longitude and μ_i is the class of a meridian, t_i, r_i are coprime and $0 \leq r_i < t_i$. Certainly we have $c_it_i = q_i$.

Note that

$$H_1(X(L_0 \cup \mathbb{L});\mathbb{Q}) \cong \mathbb{Q}\langle\mu_0\rangle \oplus \cdots \oplus \mathbb{Q}\langle\mu_n\rangle,$$

where $\mathbb{Q}\langle\mu_i\rangle$ denotes the vector space over $\mathbb{Q}$ generated by the class μ_i. In $H_1(X(L_0 \cup \mathbb{L});\mathbb{Q})$, we have

$$c_i(t_i\lambda_i + r_i\mu_i) = \sum_{\substack{j=0 \\ j\neq i}}^{n} q_i l_{ij}\mu_j.$$

Since $c_it_i = q_i$, this is equivalent to

$$\lambda_i = -\frac{r_i}{t_i}\mu_i + \sum_{\substack{j=0 \\ j\neq i}}^{n} l_{ij}\mu_j.$$

Suppose the class of a longitude of L_i determined by the contact framing is $\lambda_i + m_i\mu_i$. Then

$$tb_{\mathbb{Q}}(L_i) = \frac{1}{q_i}[F_i] \bullet (\lambda_i + m_i\mu_i) = \frac{1}{q_i}(c_it_im_i - c_ir_i) = m_i - \frac{r_i}{t_i}.$$

Contact (± 1)-surgery along L_i implies that we glue a meridional disc along

$$\lambda_i + (m_i \pm 1)\mu_i = a_i\mu_i + \sum_{\substack{j=0 \\ j\neq i}}^{n} l_{ij}\mu_j,$$

$i = 1, \dots, n$. It follows that

$$H_1(X'(L);\mathbb{Q}) \cong \mathbb{Q}\langle\mu_0\rangle \oplus \cdots \oplus \mathbb{Q}\langle\mu_n\rangle / \langle a_i\mu_i + \sum_{\substack{j=0 \\ j\neq i}}^{n} l_{ij}\mu_j,\ i = 1, \dots, n\rangle,$$

where $X'(L)$ denotes the knot exterior $\overline{Y' \setminus N(L_0)}$. Similarly,

$$H_1(Y';\mathbb{Q}) \cong \mathbb{Q}\langle\mu_1\rangle \oplus \cdots \oplus \mathbb{Q}\langle\mu_n\rangle / \langle a_i\mu_i + \sum_{\substack{j=1\\ j\neq i}}^{n} l_{ij}\mu_j,\ i=1,\ldots,n\rangle.$$

Hence Y' is a rational homology 3-sphere if and only if $\det M \neq 0$.

Since $\det M \neq 0$, Y' is a rational homology 3-sphere and $H_1(X'(L);\mathbb{Q}) \cong \mathbb{Q}\langle\mu_0\rangle$. Thus there is a unique $a_0 \in \mathbb{Q}$ such that $\lambda_0 + a_0\mu_0 = 0$ in $H_1(X'(L);\mathbb{Q})$. The rational Thurston-Bennequin invariant of L in (Y',ξ') can be computed as

$$tb_{\mathbb{Q}}(L) = (\lambda_0 + a_0\mu_0) \bullet (\lambda_0 + m_0\mu_0) = m_0 - a_0,$$

where the intersection product is on $\partial X'(L)$. Since

$$\lambda_0 = -\frac{r_0}{t_0}\mu_0 + \sum_{j=1}^{n} l_{0j}\mu_j \in H_1(X(L_0 \cup \mathbb{L});\mathbb{Q}),$$

and

$$\lambda_0 + a_0\mu_0 = 0 \in H_1(X'(L);\mathbb{Q}),$$

$(a_0 - \frac{r_0}{t_0})\mu_0 + \sum\limits_{j=1}^{n} l_{0j}\mu_j$ is a linear combination of the relations in $H_1(X'(L);\mathbb{Q})$, which gives

$$0 = \begin{vmatrix} a_0 - \frac{r_0}{t_0} & l_{01} & \cdots & l_{0n} \\ l_{10} & a_1 & \cdots & l_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n0} & l_{n1} & \cdots & a_n \end{vmatrix} = (a_0 - \frac{r_0}{t_0}) \det M + \det M_0.$$

Hence $tb_{\mathbb{Q}}(L) = m_0 - (\frac{r_0}{t_0} - \frac{\det M_0}{\det M}) = tb_{\mathbb{Q}}(L_0) + \frac{\det M_0}{\det M}$.

The Poincaré dual of the relative class $e(\xi, L_i)$ over $\mathbb{Q}$, $\mathrm{PD}(e(\xi,L_i))_{\mathbb{Q}} \in H_1(X(L_i);\mathbb{Q}) \cong \mathbb{Q}\langle\mu_i\rangle$, is equal to $rot_{\mathbb{Q}}(L_i)\mu_i$. Since under the map $H_1(X(L_0 \cup \mathbb{L});\mathbb{Q}) \to H_1(X(L_i);\mathbb{Q})$ induced by inclusion, $\mathrm{PD}(e(\xi, \bigcup\limits_{i=0}^{n} L_i))_{\mathbb{Q}}$ maps to $\mathrm{PD}(e(\xi,L_i))_{\mathbb{Q}}$, we have

$$\mathrm{PD}(e(\xi, \bigcup_{i=0}^{n} L_i))_{\mathbb{Q}} = \sum_{i=0}^{n} rot_{\mathbb{Q}}(L_i)\mu_i.$$

Under the map $H_1(X(L_0 \cup \mathbb{L});\mathbb{Q}) \to H_1(X'(L);\mathbb{Q})$ induced by inclusion, $\mathrm{PD}(e(\xi, \bigcup\limits_{i=0}^{n} L_i))_{\mathbb{Q}}$ maps to $\mathrm{PD}(e(\xi',L))_{\mathbb{Q}}$ (see [18, Lemma 3.2]). Therefore, in $H_1(X'(L);\mathbb{Q})$, we have

$$
\begin{aligned}
rot_{\mathbb{Q}}(L)\mu_0 &= \sum_{i=0}^{n} rot_{\mathbb{Q}}(L_i)\mu_i \\
&= \left(rot_{\mathbb{Q}}(L_0) - \left\langle \begin{pmatrix} rot_{\mathbb{Q}}(L_1) \\ \vdots \\ rot_{\mathbb{Q}}(L_n) \end{pmatrix},\ M^{-1} \begin{pmatrix} l_{01} \\ \vdots \\ l_{0n} \end{pmatrix} \right\rangle \right) \mu_0.
\end{aligned}
$$

This proves the second formula in the lemma. □

5. Overtwisted contact surgeries on rational homology 3-spheres

We use the following result as the main tool of the proof of Theorem 1.5.

Theorem 5.1 (Świątowski[14], Etnyre[16], Baker-Onaran[3]). *If $L \subset (Y,\xi)$ is a rationally null-homologous Legendrian knot such that the complement of a regular neighborhood of L is tight, then*

$$-|tb_{\mathbb{Q}}(L)| + |rot_{\mathbb{Q}}(L)| \leq -\frac{\chi(F)}{q},$$

where q is the order of $[L]$ in $H_1(Y;\mathbb{Z})$, and $\chi(F)$ is the Euler characteristic of a rational Seifert surface F for L.

Proof of Theorem 1.5. Consider contact $(+1)$-surgery along L. Let L^* be the surgery dual. Let L_0 be a Legendrian push-off of L. Then L_0^*, the image of L_0 in $(Y_{+1}(L),\xi_{+1}(L))$, is Legendrian isotopic to L^*. We use Lemma 4.1 to compute $tb_{\mathbb{Q}}(L_0^*)$ and $rot_{\mathbb{Q}}(L_0^*)$. Now

$$M = (tb_{\mathbb{Q}}(L)+1) \quad \text{and} \quad M_0 = \begin{pmatrix} 0 & tb_{\mathbb{Q}}(L) \\ tb_{\mathbb{Q}}(L) & tb_{\mathbb{Q}}(L)+1 \end{pmatrix}.$$

Hence

$$
\begin{aligned}
tb_{\mathbb{Q}}(L_0^*) &= tb_{\mathbb{Q}}(L_0) + \frac{\det M_0}{\det M} \\
&= tb_{\mathbb{Q}}(L) - \frac{(tb_{\mathbb{Q}}(L))^2}{tb_{\mathbb{Q}}(L)+1} = \frac{tb_{\mathbb{Q}}(L)}{tb_{\mathbb{Q}}(L)+1}
\end{aligned}
$$

and

$$rot_{\mathbb{Q}}(L_0^*) = rot_{\mathbb{Q}}(L_0) - \frac{rot_{\mathbb{Q}}(L)\cdot tb_{\mathbb{Q}}(L)}{tb_{\mathbb{Q}}(L)+1} = \frac{rot_{\mathbb{Q}}(L)}{tb_{\mathbb{Q}}(L)+1}.$$

Lemma 5.2. *The order of* $[L_0^*]$ *in* $H_1(Y_{+1}(L);\mathbb{Z})$ *is* $q \cdot |tb_{\mathbb{Q}}(L)+1|$.

Proof. We use the notation in the first paragraph of Section 2 with K replaced by L. Let $X(L)$ denote the knot exterior $\overline{Y \setminus N(L)}$. Let $i_*: H_1(\partial N(L);\mathbb{Z}) \to H_1(X(L);\mathbb{Z})$ be the map induced by inclusion. Then $\ker i_*$ is generated by $c(t\lambda_{can}+r\mu) = q\lambda_{can}+cr\mu$. Write μ' and λ' for the classes of a meridian and a longitude, respectively, of the solid torus we glue in to perform the surgery. Suppose the class of a longitude of L determined by the contact framing is $\lambda_c = \lambda_{can}+(p-1)\mu$ for some integer p. Contact (+1)-surgery can be described by the map

$$\mu' \mapsto \mu+\lambda_c = p\mu+\lambda_{can}, \;\; \lambda' \mapsto \lambda_c = (p-1)\mu+\lambda_{can}.$$

Then in terms of μ' and λ', $\ker i_*$ is generated by $(pq-cr)\lambda'+(cr-pq+q)\mu'$. Thus the order of $[L^*]$ in $H_1(Y_{+1}(L);\mathbb{Z})$ is $|pq-cr|$. Since L_0^* is isotopic to L^*, by Lemma 3.2, $[L_0^*]$ is of order $q\cdot|tb_{\mathbb{Q}}(L)+1|$ in $H_1(Y_{+1}(L);\mathbb{Z})$. $\square$

(1) Let L_k^* be the Legendrian knot obtained from L_0^* after k stabilizations. If $rot_{\mathbb{Q}}(L_0^*)$ is nonnegative, then we choose k positive stabilizations. Otherwise, we choose k negative stabilizations. Assume that k is sufficiently large. Since $tb_{\mathbb{Q}}(L)<-1$, $tb_{\mathbb{Q}}(L_0^*)$ is positive. So we have

$$\begin{aligned}-|tb_{\mathbb{Q}}(L_k^*)|+|rot_{\mathbb{Q}}(L_k^*)| &= -|tb_{\mathbb{Q}}(L_0^*)-k|+|rot_{\mathbb{Q}}(L_0^*)|+k\\ &= -(k-tb_{\mathbb{Q}}(L_0^*))+|rot_{\mathbb{Q}}(L_0^*)|+k\\ &= |tb_{\mathbb{Q}}(L_0^*)|+|rot_{\mathbb{Q}}(L_0^*)|\\ &= \frac{|tb_{\mathbb{Q}}(L)|+|rot_{\mathbb{Q}}(L)|}{|tb_{\mathbb{Q}}(L)+1|}.\end{aligned}$$

The order of $[L_k^*]$ in $H_1(Y_{+1}(L);\mathbb{Z})$ is the same as that of $[L_0^*]$, that is, $q|tb_{\mathbb{Q}}(L)+1|$. Denote $F\cap X(L)$ by F^0. We can radially cone $\partial F^0 \subset \partial X(L)$ in the solid torus we glue in to get a rational Seifert surface F^* for L^* in $Y_{+1}(L)$. Since L_k^* is smoothly isotopic to L^*, there is a rational Seifert surface F_k^* for L_k^* in $Y_{+1}(L)$ with $\chi(F_k^*)=\chi(F^*)=\chi(F)$. Since $tb_{\mathbb{Q}}(L)-|rot_{\mathbb{Q}}(L)|<\frac{\chi(F)}{q}$, $|tb_{\mathbb{Q}}(L)|+|rot_{\mathbb{Q}}(L)|>\frac{-\chi(F)}{q}$ and $-|tb_{\mathbb{Q}}(L_k^*)|+|rot_{\mathbb{Q}}(L_k^*)|>\frac{-\chi(F_k^*)}{q|tb_{\mathbb{Q}}(L)+1|}$. By Theorem 5.1, the complement of L_k^* in $(Y_{+1}(L),\xi_{+1}(L))$ is overtwisted.

(2) Since $\chi(F)\le 1$, $tb_{\mathbb{Q}}(L)+|rot_{\mathbb{Q}}(L)|<\frac{\chi(F)}{q}-2$ implies that $tb_{\mathbb{Q}}(L)<-1$. Consider L_+^* and L_-^*, the positive and negative stabilizations of L_0^*. We

have

$$tb_{\mathbb{Q}}(L_+^*) = tb_{\mathbb{Q}}(L_0^*) - 1 = -\frac{1}{tb_{\mathbb{Q}}(L)+1}$$

and

$$rot_{\mathbb{Q}}(L_+^*) = rot_{\mathbb{Q}}(L_0^*) + 1 = \frac{rot_{\mathbb{Q}}(L) + tb_{\mathbb{Q}}(L) + 1}{tb_{\mathbb{Q}}(L)+1}.$$

It follows that

$$-|tb_{\mathbb{Q}}(L_+^*)| + |rot_{\mathbb{Q}}(L_+^*)| = \frac{|rot_{\mathbb{Q}}(L) + tb_{\mathbb{Q}}(L) + 1| - 1}{|tb_{\mathbb{Q}}(L)+1|} > -\frac{\chi(F)}{q|tb_{\mathbb{Q}}(L)+1|}.$$

So L_+^* is loose by Theorem 5.1. Similarly, L_-^* is also loose.

Let $X_+^*(L)$ (respectively, $X_-^*(L)$) denote the complement of a standard neighbourhood of L_+^* (respectively, L_-^*) in $(Y_{+1}(L), \xi_{+1}(L))$. Then $(X_+^*(L), \xi_{+1}(L))$ and $(X_-^*(L), \xi_{+1}(L))$ are overtwisted. Since the result of any positive contact surgery along L in (Y, ξ) contains either $(X_+^*(L), \xi_{+1}(L))$ or $(X_-^*(L), \xi_{+1}(L))$ (see [7, Section 4]), it is overtwisted. □

We give some applications of Theorem 1.5. In practice, the most difficult part is to find a rational Seifert surface.

Corollary 5.3. *Let $L_1 \cup L_2 \subset (S^3, \xi_{st})$ be an oriented Legendrian link with two components which has a front projection depicted at the top of Figure 4. If*

$$tb(L_2) \neq 1,\ tb(L_1) + \frac{1}{1 - tb(L_2)} < -1$$

and

$$|rot(L_1) + \frac{rot(L_2)}{1-tb(L_2)}| > 2g_1 + \frac{2g_2 - 1}{|1 - tb(L_2)|} + tb(L_1) + \frac{1}{1-tb(L_2)},$$

where g_i is the genus of L_i for $i = 1, 2$, then $(S^3(L_1^+ \cup L_2^-), \xi_{st}(L_1^+ \cup L_2^-))$ is overtwisted.

Proof. Let L be the image of L_1 in $(S^3(L_2^-), \xi_{st}(L_2^-))$. By Lemma 4.1, $tb_{\mathbb{Q}}(L) = tb(L_1) + \frac{1}{1-tb(L_2)}$ and $rot_{\mathbb{Q}}(L) = rot(L_1) + \frac{rot(L_2)}{1-tb(L_2)}$. The order q of $[L]$ in $H_1(S^3(L_2^-); \mathbb{Z})$ is $|1 - tb(L_2)|$. The Legendrian knot L in $(S^3(L_2^-), \xi_{st}(L_2^-))$ can be seen as the connected sum of a Legendrian knot U in $(S^3(L_2^-), \xi_{st}(L_2^-))$ and L_1 in (S^3, ξ_{st}) (see the bottom of Figure 4). The order of $[U]$ in $H_1(S^3(L_2^-); \mathbb{Z})$ is also $|1 - tb(L_2)|$. Since U is smoothly isotopic to the core of the solid torus we glue in to get $S^3(L_2^-)$, it has a rational Seifert surface in $S^3(L_2^-)$ with Euler characteristic $1-2g_2$. By [6, (2.3.1)], L has a rational Seifert surface F in $S^3(L_2^-)$ with Euler characteristic

$$\begin{aligned}1-2g_2+|1-tb(L_2)|\cdot(1-2g_1)-|1-tb(L_2)|\\=1-2g_2-2g_1\cdot|1-tb(L_2)|.\end{aligned}$$

So we have $tb_{\mathbb{Q}}(L)<-1$ and $tb_{\mathbb{Q}}(L)-|rot_{\mathbb{Q}}(L)|<\frac{\chi(F)}{q}$. By Theorem 1.5, $(S^3(L_1^+\cup L_2^-),\xi_{st}(L_1^+\cup L_2^-))$ is overtwisted. □

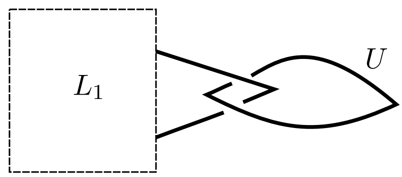

Figure 7: A Legendrian link $L_1\cup U$.

Example 5.4. Let $L_1\cup U$ be the Legendrian link in (S^3,ξ_{st}) shown in Figure 7. By Corollary 5.3, if $tb(L_1)\leq -2$ and $|rot(L_1)|>tb(L_1)+2g_1$, where g_1 is the genus of L_1, then $(S^3(L_1^+\cup U^-),\xi_{st}(L_1^+\cup U^-))$ is overtwisted.

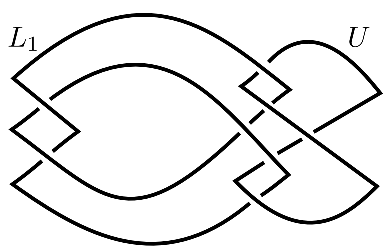

Figure 8: A Legendrian link $L_1\cup U$ with linking number 0.

Example 5.5. Let $L_1\cup U$ be the Legendrian link in (S^3,ξ_{st}) shown in Figure 8. Denote the image of L_1 in $(S^3(U^-),\xi_{st}(U^-))$ by L. Since the linking number of $L_1\cup U$ is 0, L is null-homologous in $S^3(U^-)$. By [18, Lemma 3.1] or Lemma 4.1, $tb_{\mathbb{Q}}(L)=-6$ and $|rot_{\mathbb{Q}}(L)|=1$. There is a genus-one Seifert surface F_0 of L_1 in S^3 which intersects U in exactly two points. Removing the open disk neighborhoods of these two intersection points from the interior of F_0, and gluing a tube which wraps around an arc of U bounded by the two intersection points, one can obtain a Seifert surface F for L in $S^3(U^-)$. Note that $\chi(F) = -3$. By Theorem 1.5 or [7, Theorem 1.1], $(S^3(L_1^+ \cup U^-), \xi_{st}(L_1^+ \cup U^-))$ is overtwisted.

Remark 5.6. In [2, Proposition 3.4], Baker and Grigsby proved that any Legendrian knot L in a universally tight contact lens space $(L(a,b), \xi_{UT})$ has a twisted toroidal front projection. The invariant $tb_{\mathbb{Q}}(L)$ can be computed via such a front projection (see [2, Proposition 6.8] and [8, Corollary 3.3]). So we can apply Proposition 1.3 for Legendrian knots in $(L(a,b), \xi_{UT})$ conveniently. In [8, Proposition 3.6], the invariant $rot_{\mathbb{Q}}(L)$ is also computed via the front projection. Possibly, one can construct a rational Seifert surface for L via the front projection in a similar way as in [32, Section 3.4]. Then we can apply Theorem 1.5 for Legendrian knots in $(L(a,b), \xi_{UT})$.

6. Overtwisted contact (+1)-surgeries along Legendrian two-component links

In this section, we prove Theorem 1.6.

Proof of Theorem 1.6. We shall construct an explicit overtwisted disk in the contact 3-manifold $(S^3(\mathbb{L}^+), \xi_{st}(\mathbb{L}^+))$. First, we construct a Legendrian knot L' in (S^3, ξ_{st}) disjoint from the link $\mathbb{L} = L_1 \cup L_2$. See Figure 9. Outside the dashed box, L' consists of two Legendrian arcs which are downward Legendrian push-offs of the parts of L_1 and L_2 outside the dashed box, respectively. There is a thrice-punctured sphere S shown in Figure 9 whose boundary $\partial S = L_1 \cup L_2 \cup L'$.

We orient L_1, L_2 and L' as the boundary of S. Suppose the linking number of L_1 and L_2 is l.

Lemma 6.1. $tb(L') = tb(L_1) + tb(L_2) + 2(l+1)$.

Proof. The proof is similar to that of [13, Lemma 6.1]. The number of cusps of L' is $c(L') = c(L_1) + c(L_2) - 2$. The writhe of L' is $w(L') = w(L_1) + w(L_2) + 2(l+1) - 1$, where the self-crossings of L' outside the dashed box contribute $w(L_1) + w(L_2) + 2(l+1)$ to $w(L')$, and the self-crossing of L' inside the dashed box contributes -1 to $w(L')$. So $tb(L') = w(L') - \frac{1}{2}c(L') = tb(L_1) + tb(L_2) + 2(l+1)$. □

Lemma 6.2. *(1) For $i = 1, 2$, the framing of L_i induced by S is $tb(L_i) + 1$ with respect to the Seifert surface framing of L_i.*
(2) The framing of L' induced by S is $tb(L_1) + tb(L_2) + 2(l+1)$ with respect

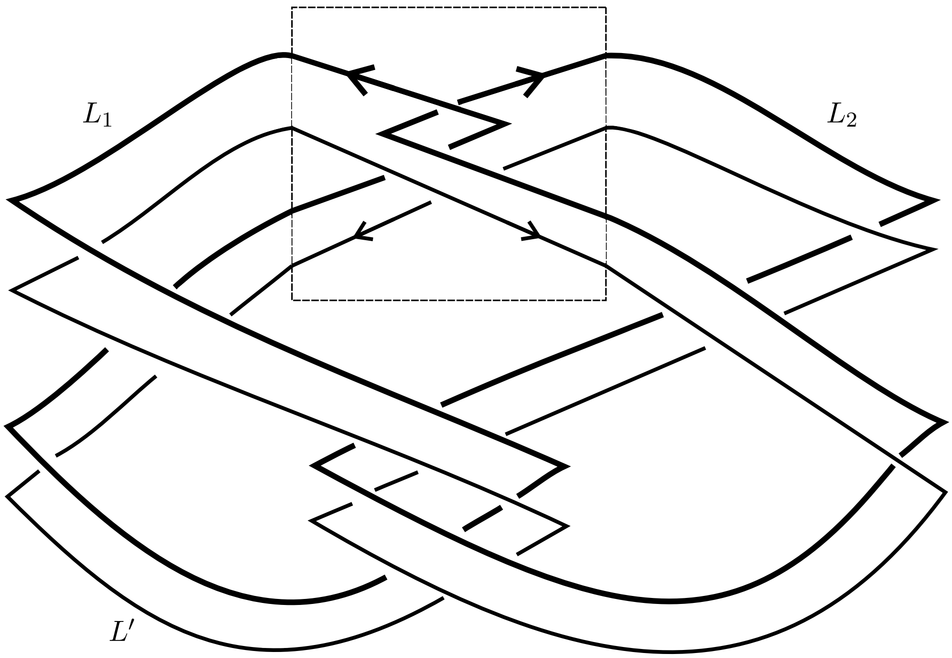

Figure 9: The thin knot is L'. The shaded area is a thrice-punctured sphere.

to the Seifert surface framing of L'; that is, the framing of L' induced by S coincides with the contact framing of L'.

Proof. (1) For $i = 1, 2$, the framing of L_i induced by S, with respect to the Seifert surface framing of L_i, is the linking number of L_i and its push-off in the interior of S. The verification is straightforward.
(2) Let L_0' be the push-off of L' in the interior of S. We compute the linking number $lk(L', L_0')$ as the number of crossings where L_0' crosses under L', counted with sign. The crossings outside the dashed box contribute $tb(L_1) + tb(L_2) + 1 + 2(l+1)$ to $lk(L', L_0')$. The crossing inside the dashed box contributes -1 to $lk(L', L_0')$. So $lk(L', L_0') = tb(L_1) + tb(L_2) + 2(l+1)$. □

By Lemma 6.2(1), after we perform contact (+1)-surgery along $L_1 \cup L_2$, S caps off to a disk with boundary L'. It follows from Lemma 6.2(2) that this disk is an overtwisted disk. □

Roger Casals provided an alternative proof of Theorem 1.6.

Alternative proof of Theorem 1.6. We prove that the Legendrian unknot L_0 with $tb(L_0) = -1$ in $(S^3(\mathbb{L}^+), \xi_{st}(\mathbb{L}^+))$ in the first picture in the sequence of

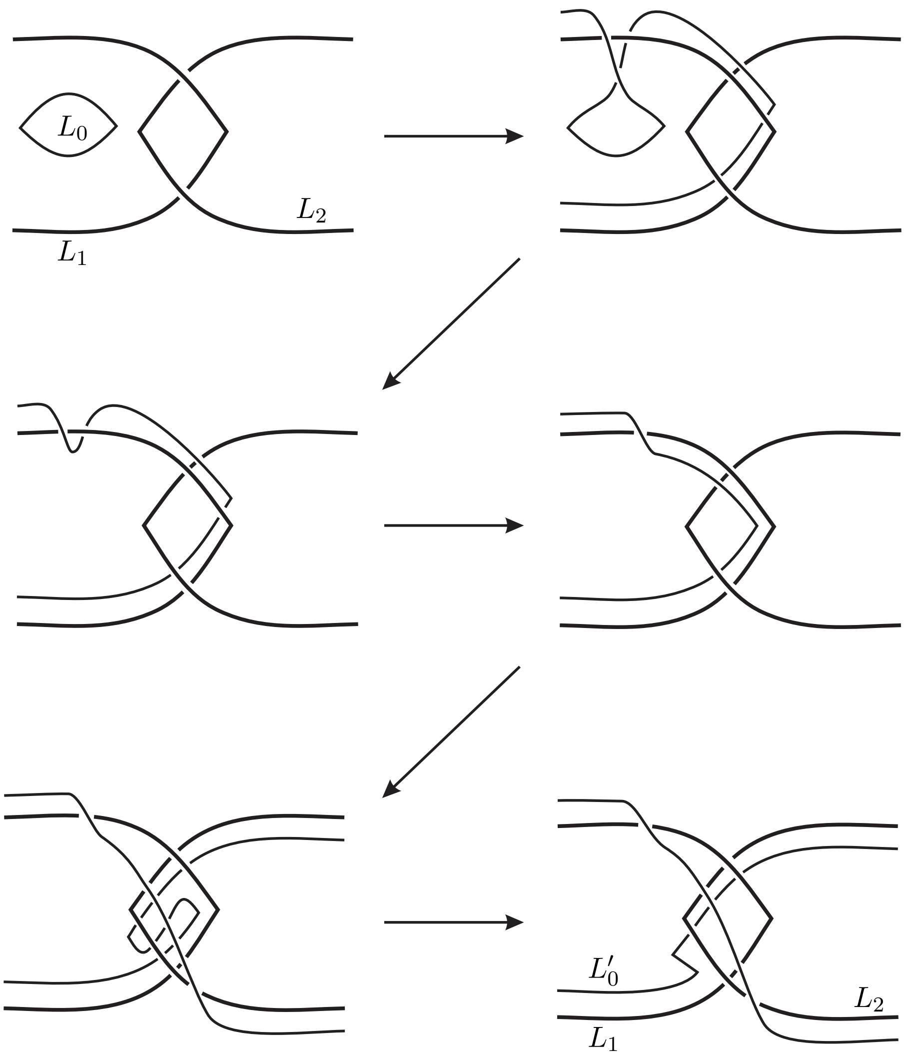

Figure 10: The thick curves present the two components of the link $\mathbb{L} = L_1 \cup L_2$. The thin curves present L_0 and the resulting Legendrian knots after Legendrian Reidemeister moves and Kirby moves.

Figure 10 destabilizes. In fact, L_0 is Legendrian isotopic to the Legendrian knot L'_0 in $(S^3(\mathbb{L}^+), \xi_{st}(\mathbb{L}^+))$ in the final picture which contains an isolated stabilized arc.

In the first and fourth steps in the sequence of Figure 10, we perform Kirby moves of the second kind (see [11, Proposition 1]). The remaining moves are Legendrian Reidemeister moves. □

References

[1] K. L. Baker and J. Etnyre, *Rational linking and contact geometry*, Perspectives in analysis, geometry, and topology, 19–37, Progr. Math., 296, Birkhäuser/Springer, New York, 2012.

[2] K. L. Baker and J. E. Grigsby, *Grid diagrams and Legendrian lens space links,* J. Symplectic Geom. 7 (2009), no. 4, 415–448.

[3] K. L. Baker and S. Onaran, *Nonlooseness of nonloose knots*, Algebr. Geom. Topol. 15 (2015), no. 2, 1031–1066.

[4] J. A. Baldwin, *Capping off open books and the Ozsváth-Szabó contact invariant*, J. Symplectic Geom. 11 (2013), no. 4, 525–561.

[5] D. Bennequin, *Entrelacements et équations de Pfaff*, Asterisque, 107-108:87-161, 1983.

[6] D. Calegari and C. Gordon, *Knots with small rational genus*, Comment. Math. Helv. 88 (2013), no. 1, 85–130.

[7] J. Conway, *Overtwisted positive contact surgeries*, Topology Appl. 261 (2019), 51–62.

[8] C. R. Cornwell, *Bennequin type inequalities in lens spaces,* Int. Math. Res. Not. IMRN 2012, no. 8, 1890–1916.

[9] F. Ding and H. Geiges, *Symplectic fillability of tight contact structures on torus bundles,* Algebr. Geom. Topol. 1 (2001), 153–172.

[10] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 3, 583–598.

[11] F. Ding and H. Geiges, *Handle moves in contact surgery diagrams,* J. Topol. 2 (2009), no. 1, 105–122.

[12] F. Ding, H. Geiges and A. I. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math. 28 (2004), no. 1, 41–74.

[13] F. Ding, Y. Li and Z. Wu, *Contact* (+1)*-surgeries along Legendrian two-component links*, Quantum Topol. 11 (2020), no. 2, 295–321.

[14] K. Dymara, *Legendrian knots in overtwisted contact structures on* S^3, Ann. Global Anal. Geom. 19 (2001), no. 3, 293–305.

[15] Y. Eliashberg, *Legendrian and transversal knots in tight contact 3-manifolds*, Topological methods in modern mathematics (Stony Brook, NY, 1991), 171–193, Publish or Perish, Houston, TX, 1993.

[16] J. B. Etnyre, *On knots in overtwisted contact structures*, Quantum Topol. 4 (2013), no. 3, 229–264.

[17] J. B. Etnyre and K. Honda, *On connected sums and Legendrian knots,* Adv. Math. 179 (2003), no. 1, 59–74.

[18] H. Geiges and S. Onaran, *Legendrian rational unknots in lens spaces,* J. Symplectic Geom. 13 (2015), no. 1, 17–50.

[19] P. Ghiggini, *Ozsváth-Szabó invariants and fillability of contact structures,* Math. Z. 253 (2006), no. 1, 159–175.

[20] M. Golla, *Ozsváth-Szabó invariants of contact surgeries*, Geom. Topol. 19 (2015), no. 1, 171–235.

[21] R. E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998), no. 2, 619–693.

[22] M. Hedden, *An Ozsváth-Szabó Floer homology invariant of knots in a contact manifold*, Adv. Math. 219 (2008), no. 1, 89–117.

[23] M. Hedden and A. Levine, *A surgery formula for knot Floer homology*, `arXiv:1901.02488`.

[24] J. Hom, A. Levine and T. Lidman, *Knot concordance in homology cobordisms*, Duke Math. J. 171 (2022), no. 15, 3089–3131.

[25] Y. Li and Z. Wu, *A bound for rational Thurston-Bennequin invariants*, Geom. Dedicata, 200 (2019), 371–383.

[26] T. Lidman and S. Sivek, *Contact structures and reducible surgeries,* Compos. Math. 152 (2016), no. 1, 152–186.

[27] P. Lisca and A. I. Stipsicz, *Ozsváth-Szabó invariants and tight contact three-manifolds. I,* Geom. Topol. 8 (2004), 925–945.

[28] P. Lisca and A. I. Stipsicz, *Notes on the contact Ozsváth-Szabó invariants,* Pacific J. Math. 228 (2006), no. 2, 277–295.

[29] T. E. Mark and B. Tosun, *Naturality of Heegaard Floer invariants under positive rational contact surgery*, J. Differential Geom. 110 (2018), no. 2, 281–344.

[30] S. Onaran, *On overtwisted contact surgeries*, Bull. Aust. Math. Soc. 98 (2018), 144–148.

[31] B. Özbağci, *A note on contact surgery diagrams*, Internat. J. Math. 16 (2005), no. 1, 87–99.

[32] P. Ozsváth, A. I. Stipsicz and Z. Szabó, *Grid homology for knots and links,* Mathematical Surveys and Monographs, 208. American Mathematical Society, Providence, RI, 2015. x+410 pp.

[33] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005), 39–61.

[34] P. Ozsváth and Z. Szabó, *Holomorphic triangles and invariants for smooth four-manifolds,* Adv. Math. 202 (2006), no. 2, 326–400.

[35] P. Ozsváth and Z. Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. 11 (2011), no. 1, 1–68.

[36] O. Plamenevskaya, *Bounds for the Thurston-Bennequin number from Floer homology*, Algebr. Geom. Topol. 4 (2004), 399–406.

[37] O. Plamenevskaya, *Contact structures with distinct Heegaard Floer invariants*, Math. Res. Lett. 11 (2004), no. 4, 547–561.

[38] K. Raoux, *τ-invariants for knots in rational homology spheres*, Algebr. Geom. Topol. 20 (2020), no. 4, 1601–1640.

[39] A. Wand, *Tightness is preserved by Legendrian surgery*, Ann. of Math. (2) 182 (2015), no. 2, 723–738.

School of Mathematical Sciences and LMAM
Peking University, Beijing 100871, China
E-mail address: dingfan@math.pku.edu.cn

School of Mathematical Sciences
Shanghai Jiao Tong University, Shanghai 200240, China
E-mail address: liyoulin@sjtu.edu.cn

Department of Mathematics
The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: ztwu@math.cuhk.edu.hk

Received November 12, 2020
Accepted February 19, 2022