

Poisson maps between character varieties: gluing and capping

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Let G be a compact Lie group or a complex reductive affine algebraic group. We explore induced mappings between G -character varieties of surface groups by mappings between corresponding surfaces. It is shown that these mappings are generally Poisson. We also give an effective algorithm to compute the Poisson bi-vectors when $G = \mathrm{SL}(2, \mathbb{C})$. We demonstrate this algorithm by explicitly calculating the Poisson bi-vector for the 5-holed sphere, the first example for an Euler characteristic -3 surface.

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1. Introduction

Suppose that $\Sigma_{n,g}$ is a surface of genus g with n boundary circles (or n punctures), and G is either a complex reductive affine algebraic group or compact Lie group. The moduli space of representations of the fundamental group $\pi_1(\Sigma_{n,g})$ into G , the G -character variety of $\pi_1(\Sigma_{n,g})$, has a natural Poisson structure. This structure was given by Goldman [Gol1] in the closed case, and extended in [La3] to the case of surfaces with boundary. The representation space also has an equivalent interpretation as a space of flat connections, and from this point of view one can define the Poisson structure with an approach pioneered by Atiyah and Bott [AB] when the surface is compact, and extended by Jeffrey in [Je] to punctured surfaces (see also [BJ]). In this paper we consider the effect of mappings between surfaces. Given an appropriate mapping between two surfaces $f : \Sigma_1 \rightarrow \Sigma_2$, there is a natural morphism between their character varieties:

$$\Phi : \mathfrak{X}_{\Sigma_2}(G) \rightarrow \mathfrak{X}_{\Sigma_1}(G).$$

We will see that this is a Poisson map, both from the point of view of representations (Theorem 2.8) and from the point of view of flat connections (Theorem 4.2). Precisely, we prove:

Theorem A. Let G be either a compact Lie group, or a complex reductive affine algebraic group. Let $q : \Sigma_1 \rightarrow \Sigma_2$ be a continuous map between compact orientable surfaces that preserves transversality of based loops, and double points. Then the induced map

$$q^* : \mathfrak{X}_{\Sigma_2}(G) \rightarrow \mathfrak{X}_{\Sigma_1}(G)$$

is Poisson whenever q preserves orientation, and is anti-Poisson if q reverses orientation.

Next, we review past computations of Poisson bi-vectors on character varieties and show that the bi-vector determines the underlying surface (Theorem 2.6). In particular, we prove:

Theorem B. There is a homeomorphism between compact, connected orientable surfaces $\Sigma_{n_1,g_1} \cong \Sigma_{n_2,g_2}$ if and only if there is an equivalence of Poisson varieties $\mathfrak{X}_{n_1,g_1}(G) \cong \mathfrak{X}_{n_2,g_2}(G)$.

We then extend (Theorem 2.7) the known examples of explicitly computed bi-vectors on character varieties by determining the case of the 5-holed sphere $\Sigma_{5,0}$ and $G = \text{SL}(2, \mathbb{C})$.

Theorem C. For any $n, g \geq 0$, there is an effective algorithm to compute the Poisson bi-vector of $\mathfrak{X}_{n,g}(\text{SL}(2, \mathbb{C}))$. The bi-vector of $\mathfrak{X}_{5,0}(\text{SL}(2, \mathbb{C}))$ is:

$$\begin{aligned} \mathfrak{a}_{5,0}(\text{SL}(2, \mathbb{C})) = & \mathfrak{a}_{1324} \frac{\partial}{\partial t_{\{1,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} + \Sigma_1 \left(\mathfrak{a}_{1214} \frac{\partial}{\partial t_{\{1,2\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} \right) \\ & + \Sigma_2 \left(\mathfrak{a}_{12314} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} + \mathfrak{a}_{12324} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} \right. \\ & \left. + \mathfrak{a}_{12334} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{3,4\}}} \right) \\ & + \Sigma_2 \left(\mathfrak{a}_{123124} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,2,4\}}} \right) \\ & + \Sigma_3 \left(\mathfrak{a}_{123134} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,3,4\}}} \right), \end{aligned}$$

where Σ_i are symmetry operators defined by the mapping class group of the surface and \mathfrak{a}_x are explicit polynomials (see Section 7).

Remark 1.1. We give only one new example in Theorem C since all cases of less complexity exist in the literature (references in Subsection 2.5) and this is the first (and most tractable) example of a bi-vector for an Euler characteristic -3 surface. Other examples are possible to compute by following the general algorithm we describe, however such examples are significantly more onerous to determine in detail.

In Sections 3 and 4, we provide an analytic point of view on Theorem A. In Section 5 we give simple examples, in which we “cap” some of the boundary circles of a surface Σ_1 (with a disk, a cylinder, a genus one surface, etc.) to obtain a surface Σ_2 , and see what we obtain with Φ . By capping all the boundaries, we obtain symplectic character varieties mapping to Poisson character varieties. In Section 6, we discuss gluing maps via symplectic quotients. The final section (Section 7) is devoted to the computer aided proof of Theorem 2.7.

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2. Poisson structure on character varieties: Betti point-of-view

2.1. Reductive groups

Let G be a connected reductive affine algebraic group over \mathbb{C} . By the central isogeny theorem, $G \cong DG \times_F T$, where $DG = [G, G]$ is the derived subgroup, $T \cong (\mathbb{C}^*)^s$ is the maximal central torus, and $F = T \cap DG$. The group G acts on itself by conjugation and the geometric invariant theoretic quotient $G//G$ is isomorphic to \mathbf{T}/W , where $\mathbf{T} \subset G$ is a maximal torus and W is the Weyl group $N_G(\mathbf{T})/\mathbf{T}$. By potentially enlarging F , we can assume DG is simply connected. With that assumption made, by results of Steinberg [St], we can say more:

$$G//G \cong \mathbf{T}//W \cong \mathbb{C}^r \times_F (\mathbb{C}^*)^s,$$

where r is the rank of DG . Therefore, the coordinate ring $\mathbb{C}[G//G]$ is isomorphic to

$$\mathbb{C}[t_1, \dots, t_r, d_1, \dots, d_{s+1}]^F / (d_1 \cdots d_{s+1} - 1).$$

We denote points in $G//G$ by $(\tau_1, \dots, \tau_r, \delta_1, \dots, \delta_s)$.

Example 2.1. If G is $\mathrm{SL}(n, \mathbb{C})$, then $G//G \cong \mathbb{C}^{n-1}$. Therefore, if $G = \mathrm{GL}(n, \mathbb{C})$, then $G//G \cong \mathbb{C}^{n-1} \times_{\mathbb{Z}/n\mathbb{Z}} \mathbb{C}^*$, where $\mathbb{Z}/n\mathbb{Z}$ arises as the group of scalar matrices of determinant 1. The coordinates describing points are the coefficients of the characteristic polynomial which can take any value freely, except the determinant which can take any value except 0.

2.2. Character varieties of surfaces

Let $\Sigma_{n,g}$ be a compact connected orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components; we assume that $n \geq 2$ if $g = 0$ since otherwise

the surface is simply-connected and the moduli spaces we will consider are trivial. Pick a base point $*$ in the interior of $\Sigma_{n,g}$. The fundamental group of $\Sigma_{n,g}$ admits the presentation:

$$\pi_1(\Sigma_{n,g}, *) \cong \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n \mid \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^n c_j = 1 \rangle,$$

where $[x, y] = xyx^{-1}y^{-1}$ is the commutator.

The set of homomorphisms $\text{Hom}(\pi_1(\Sigma_{n,g}, *), G)$ is naturally an affine algebraic subvariety of G^{2g+n} by evaluating a homomorphisms at generators. The group G acts rationally on $\text{Hom}(\pi_1(\Sigma_{n,g}, *), G)$ by conjugation, that is, $g \cdot \rho = g\rho g^{-1}$. The geometric invariant theoretic quotient of this action is denoted

$$\mathfrak{X}_{n,g}(G) := \text{Hom}(\pi_1(\Sigma_{n,g}, *), G) // G,$$

and is called the G -character variety of $\Sigma_{n,g}$. More generally, if (Σ, x_0) is any (pointed) compact orientable surface we will denote the G -character variety of $\pi_1(\Sigma, x_0)$ by $\mathfrak{X}_{\Sigma, x_0}(G)$.

Note that the conjugation action of the center $Z(G)$ of G is trivial and thus it suffices to consider the conjugation action of $PG := G/Z(G)$; making it an effective action. The following lemma overlaps with [Sik1, Proposition 49].

Lemma 2.2. *Assume G is non-abelian. The (complex) dimension of $\mathfrak{X}_{n,g}(G)$ is*

$$-\chi(\Sigma_{n,g}) \dim G + \zeta_{n,g},$$

where

- $\zeta_{n,g} = s$ if $n > 0$ and $2g + n \geq 3$,
- $\zeta_{n,g} = r + s$ if $n > 0$ and $2g + n = 2$,
- $\zeta_{n,g} = 2s$ if $n = 0$ and $g \geq 2$, and
- $\zeta_{n,g} = 2(r + s)$ if $n = 0$ and $g = 1$.

Proof. Recall our standing assumption that $n \geq 2$ if $g = 0$, which rules out only the 2-sphere and the disk (both simply-connected).

If $n > 0$, then $\Sigma_{n,g}$ deformation retracts to a 1-complex whose fundamental group is free of rank $2g + n - 1$. In this case $\chi(\Sigma_{n,g}) = 1 - (2g + n - 1) = 2 - 2g - n$ and so the rank of $\pi_1(\Sigma_{n,g}, *)$ is $1 - \chi(\Sigma_{n,g})$.

If $n = 0$, then the Euler characteristic is $1 - 2g + 1 = 2 - 2g$. Either way, $\chi(\Sigma_{n,g}) = 2 - 2g - n$.

When $n > 0$ then $\pi_1(\Sigma_{n,g}, *)$ is a free group of rank $2g + n - 1$ which is greater than or equal to 2 if and only if $g \geq 1$ or $n \geq 3$. Thus, $\pi_1(\Sigma_{n,g}, *)$ surjects onto a rank 2 free group F_2 , and so $\text{Hom}(F_2, G)$ injects into $\text{Hom}(\pi_1(\Sigma_{n,g}, *), G)$. The generic dimension of a PG -conjugation stabilizer of $\text{Hom}(F_2, G)$ is 0 since a generic pair of elements in G generates a Zariski dense subgroup; hence the same is true for $\text{Hom}(\pi_1(\Sigma_{n,g}, *), G)$.

Thus, since $\text{Hom}(\pi_1(\Sigma_{n,g}, *), G) \cong G^{2g+n-1}$ the dimension of $\mathfrak{X}_{n,g}(G)$ is $(2g + n - 1) \dim G - \dim PG = (2g + n - 1) \dim G - \dim G + \dim Z(G) = -\chi(\Sigma_{n,g}) \dim(G) + s$.

Now, still assuming $n > 0$, if $\pi_1(\Sigma_{n,g}, *)$ is a free group of rank 1 (only occurring when $g = 0$ and $n = 2$), then the character variety is isomorphic to $G//G$ which we have already seen is of dimension $r + s$ which is also the dimension of a generic G -conjugation stabilizer.

Likewise, if $n = 0$ and $g \geq 2$ then $\pi_1(\Sigma_{n,g}, *)$ surjects onto a free group of rank 2. Thus, we have that generic dimension of a stabilizer is the dimension of $Z(G)$, which is s . Moreover, the commutator map $G^{2g} \rightarrow DG$ (defining the relation in $\pi_1(\Sigma_{0,g}, *)$) is dominant (and hence a generic submersion), and therefore $\dim \text{Hom}(\pi_1(\Sigma_{0,g}, *), G) = 2g \dim G - \dim DG$ which then implies $\dim \mathfrak{X}_{0,g}(G) = 2g \dim G - \dim DG - \dim PG = -\chi(\Sigma_{0,g}) \dim G + 2 \dim Z(G) = -\chi(\Sigma_{0,g}) \dim G + 2s$.

Lastly, when $n = 0$ and $g = 1$, then the identity component of $\mathfrak{X}_{0,1}(G)$ satisfies $\mathfrak{X}_{0,1}^0(G) \cong \mathbf{T}^2/W$ which has dimension $2(r + s)$, see [FL3]. We note that if G is simply connected then $\mathfrak{X}_{0,1}(G)$ is connected but otherwise it has smaller dimensional components. Regardless, in this case $\chi(\Sigma_{0,1}) = 0$, and so we have established the formula. □

Remark 2.3. If G is abelian then the conjugation action is trivial, and so

$$\begin{aligned} \mathfrak{X}_{n,g}(G) &= \text{Hom}(\pi_1(\Sigma_{n,g}, *), G) \\ &= \text{Hom}(\pi_1(\Sigma_{n,g}, *) / [\pi_1(\Sigma_{n,g}, *), \pi_1(\Sigma_{n,g}, *)], G) \\ &= \text{Hom}(\mathbb{Z}^{\epsilon_{n,g}}, G) = G^{\epsilon_{n,g}}, \end{aligned}$$

where $\epsilon_{0,g} = 2g$ and $\epsilon_{n,g} = 2g + n - 1$ if $n > 0$. The dimension in these cases is obvious. In the cases when $g = 0$ and $n = 0$ or 1, then the character variety is a point and so has dimension 0.

The algebraic structure of $\mathfrak{X}_{n,g}(G)$, up to biregular mappings, does not depend on the presentation of the fundamental group of $\Sigma_{n,g}$. In fact, it only depends on the Euler characteristic $\chi(\Sigma_{n,g})$.

Proposition 2.4. *There is a biregular morphism $\mathfrak{X}_{n_1,g_1}(G) \cong \mathfrak{X}_{n_1,g_2}(G)$ if and only if both n_1, n_2 are either positive or 0, and $\chi(\Sigma_{n_1,g_1}) = \chi(\Sigma_{n_2,g_2})$.*

Proof. Assume first that $\mathfrak{X}_{n_1,g_1}(G) \cong \mathfrak{X}_{n_2,g_2}(G)$. Then their dimensions are equal. Suppose $n_1 > 0$ and $n_2 = 0$. Then $\mathfrak{X}_{n_1,g_1}(G)$ strong deformation retracts onto $\mathfrak{X}_{n_1,g_1}(K)$, where K is a maximal compact subgroup of G , while $\mathfrak{X}_{0,g_2}(G)$ is not even homotopy equivalent to $\mathfrak{X}_{0,g_2}(K)$ as long as $g_2 \geq 2$ by [FL1, BF, FL4]. Thus, if $\mathfrak{X}_{n_1,g_1}(G) \cong \mathfrak{X}_{n_1,g_2}(G)$, then both surfaces are open or both are closed. From Lemma 2.2 we conclude that $\chi(\Sigma_{n_1,g_1}) = \chi(\Sigma_{n_2,g_2})$. Note that this deduction holds *without* the assumptions on n_1 and n_2 .

Conversely, assume that $\chi(\Sigma_{n_1,g_1}) = \chi(\Sigma_{n_2,g_2})$. There are three cases to consider: (1) $n_1n_2 > 0$, (2) $n_1 = 0 = n_2$, and (3) $n_1n_2 = 0$ but $n_1 + n_2 \neq 0$.

In Case (1), both the fundamental groups $\pi_1(\Sigma_{n_1,g_1}, *)$ and $\pi_1(\Sigma_{n_2,g_2}, *)$ are free of the same rank, and hence the character varieties are isomorphic (note that the surfaces need not be homeomorphic, but they will be homotopic).

In Case (2), the Euler characteristics being equal implies the surfaces are homeomorphic and hence their fundamental groups are isomorphic. Hence the character varieties too are isomorphic.

Case (3) does occur since the Euler characteristics can be equal (with one surface open and the other closed) since $2 - 2g_1 - n_1 = 2 - 2g_2$ requires only that $n_1 = 2(g_1 - g_2)$. In this situation (without loss of generality assume $n_1 > 0$ and $n_2 = 0$), as noted in the first paragraph of this proof, $\mathfrak{X}_{n_1,g_1}(G)$ is homotopic to $\mathfrak{X}_{n_1,g_1}(K)$, while $\mathfrak{X}_{0,g_2}(G)$ is not homotopic to $\mathfrak{X}_{0,g_2}(K)$ for $g_2 \geq 2$. So the converse (without additional assumptions on n_1 and n_2) does not hold.

To exhaust the possibilities with Case (3), suppose $g_2 = 1$, then $2 - 2g_1 - n_1 = 0$ and hence $2g_1 + n_1 = 2$ and so we have that $\mathfrak{X}_{n_1,g_1}(G) \cong G//G$ of dimension $s + t$ while $\mathfrak{X}_{0,g_2}(G)$ has dimension $2(s + t)$, and so they are not isomorphic. In short, every time Case (3) arises we have simultaneously that the Euler characteristics are equal yet the character varieties are not isomorphic. □

Remark 2.5. Proposition 2.4 is non-trivial in the sense that in general if two character varieties are isomorphic it does not imply the underlying

groups are isomorphic. As a simple example of this observe that

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$$

as each is a set of two points.

2.3. Relative character varieties

When $n > 0$, for every $1 \leq i \leq n$ define the boundary map

$$\mathfrak{b}_i : \mathfrak{X}_{n,g}(G) \longrightarrow G // G$$

by sending a representation class $[\rho]$ to $[\rho|_{c_i}]$. Subsequently, we define

$$\mathfrak{b}_{n,g} := (\mathfrak{b}_1, \dots, \mathfrak{b}_n) : \mathfrak{X}_{n,g}(G) \longrightarrow (G // G)^n.$$

We emphasize that the map $\mathfrak{b}_{n,g}$ depends on the surface, not only its fundamental group.

Let $\tau \in \mathfrak{b}_{n,g}(\mathfrak{X}_{n,g}(G)) \subset (G // G)^n$ be a point in the image of the boundary map and define $\mathfrak{L}_\tau = \mathfrak{b}_{n,g}^{-1}(\tau)$. The singular locus of $\mathfrak{X}_{n,g}(G)$ is a proper closed sub-variety; denote its complement by $\mathcal{X}_{n,g}(G)$. So $\mathcal{X}_{n,g}(G)$ is a complex manifold that is dense in $\mathfrak{X}_{n,g}(G)$. Since $\mathfrak{b}_{n,g}$ is dominant, its regular values are generic. Thus, at such a point, $\mathcal{L}_\tau := \mathfrak{L}_\tau \cap \mathcal{X}_{n,g}(G)$ is a submanifold of dimension

$$\chi(\Sigma_{n,g}) \dim G + \zeta_{n,g} - n(r + s).$$

It is shown in [La3] that $\cup_\tau \mathcal{L}_\tau$ foliate $\mathcal{X}_{n,g}(G)$ by complex symplectic submanifolds, making $\mathcal{X}_{n,g}(G)$ a complex Poisson manifold. This structure continuously extends over all of $\mathfrak{X}_{n,g}(G)$ making it a Poisson variety; a variety whose sheaf of regular functions is a sheaf of Poisson algebras (see [BLR] for details).

We now review the explicit definition of this structure.

2.4. Poisson structure

For an affine variety V defined over \mathbb{C} , a Poisson structure on V is a Lie bracket operation $\{, \}$ on its coordinate ring $\mathbb{C}[V]$ that acts as a formal derivation (satisfies the Leibniz rule).

The smooth stratum of V , denoted \mathcal{V} , is a complex Poisson manifold in the usual sense by the Stone-Weierstrass Theorem. For any holomorphic

function f on \mathcal{V} , there is a Hamiltonian vector field $H_f := \{f, \cdot\}$ on \mathcal{V} defined in terms of the Poisson bracket. There exists an exterior bi-vector field $\mathfrak{a} \in H^0(\mathcal{V}, \wedge^2 T\mathcal{V})$ whose restriction to symplectic leaves (with $(2, 0)$ -form ω) is given by $\{f, g\} = \omega(H_g, H_f)$. Let $f, g \in \mathbb{C}[V]$. Then with respect to interior multiplication $\{f, g\} = \mathfrak{a} \cdot (df \wedge dg) = (df \wedge dg)(\mathfrak{a})$. In local coordinates (z_1, \dots, z_k) it takes the form

$$\mathfrak{a} = \sum_{i,j} \mathfrak{a}_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

and so

$$\begin{aligned} \{f, g\} &= \sum_{i,j} \left(\mathfrak{a}_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \right) \cdot \left(\frac{\partial f}{\partial z_i} dz_i \wedge \frac{\partial g}{\partial z_j} dz_j \right) \\ &= \sum_{i,j} \mathfrak{a}_{i,j} \left(\frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial z_i} \right). \end{aligned}$$

Any reductive G has a symmetric, non-degenerate bilinear form \mathfrak{B} on its Lie algebra \mathfrak{g} that is invariant under the adjoint representation. Fix such an invariant form $\mathfrak{B} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$. If G is semisimple \mathfrak{B} is a multiple of the Killing form.

Returning to our varieties $\mathfrak{X}_{n,g}(G)$, in [GHJW] it is established that ω , in the following commutative diagram, defines a symplectic form on the leaf \mathcal{L}_τ :

$$\begin{array}{ccc} H^1(\Sigma_{n,g}, \partial\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho}) \times H^1(\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho}) & \xrightarrow{\cup} & H^2(\Sigma_{n,g}, \partial\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho} \otimes \mathfrak{g}_{\text{Ad}_\rho}) \\ \uparrow & & \downarrow \mathfrak{B}_* \\ & & H^2(\Sigma_{n,g}, \partial\Sigma_{n,g}; \mathbb{C}) \\ & & \downarrow \cap[Z] \\ H^1_{\text{par}}(\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho}) \times H^1_{\text{par}}(\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho}) & \xrightarrow{\omega} & H_0(\Sigma_{n,g}; \mathbb{C}) \cong \mathbb{C}. \end{array}$$

Note that $H^1_{\text{par}}(\Sigma_{n,g}; \mathfrak{g}_{\text{Ad}_\rho})$ is a model for the tangent space at a class $[\rho]$ in \mathcal{L}_τ .

With respect to this 2-form, in [La3], it is shown that Goldman’s proof [Gol1, Gol2] of the Poisson bracket in the closed surface case generalizes directly to relative and parabolic cohomology and establishes a Poisson bracket on the coordinate ring $\mathbb{C}[\mathfrak{X}_{n,g}(G)]$.

Let $\alpha, \beta \in \pi_1(\Sigma_{n,g}, *)$. Up to homotopy, we can always arrange for α and β to intersect at worst in transverse double points. Let $\alpha \cap \beta$ be the set of (transverse) double point intersections of α and β . Let $\epsilon(p, \alpha, \beta)$ be the oriented intersection number at $p \in \alpha \cap \beta$ and let $\alpha_p \in \pi_1(\Sigma, p)$ be the curve α based at p .

For a given $f \in \mathbb{C}[G//G]$ we obtain $f_\alpha : \mathfrak{X}_{n,g}(G) \rightarrow \mathbb{C}$ defined by $f_\alpha([\rho]) = f(\rho(\alpha))$. Define the variation F of an invariant function f by

$$\mathfrak{B}(F(\mathbf{A}), X) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(tX)\mathbf{A}).$$

In special cases, F can be computed explicitly; see [Gol2] for further details. In these terms the bracket is defined on $\mathbb{C}[\mathfrak{X}_{n,g}(G)]$ by:

$$(2.1) \quad \{f_\alpha([\rho]), g_\beta([\rho])\} = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \mathfrak{B}(F_{\alpha_p}([\rho]), G_{\beta_p}([\rho])).$$

See [La3, Sections 3 and 4] for further details when $n > 0$, [Gol2] when $n = 0$ and $g \geq 2$, and [Sik2] for $n = 0$ and $g = 1$. We will denote the bi-vector associated to this Poisson bracket on $\mathfrak{X}_{n,g}(G)$ by $\mathfrak{a}_{n,g}(G)$.

Note that when α represents one of the boundary curves in $\Sigma_{n,g}$, it can be chosen to not intersect any of the other generators of the fundamental group. Consequently, Formula (2.1) implies that f_α Poisson commutes with all other functions; such functions are called *Casimirs*.

2.5. Bi-vectors on character varieties

In contrast to Proposition 2.4, the Poisson bi-vector completely determines the isomorphism class of the underlying surface.

Theorem 2.6. *There is a homeomorphism $\Sigma_{n_1,g_1} \cong \Sigma_{n_2,g_2}$ if and only if there is an equivalence of Poisson varieties $\mathfrak{X}_{n_1,g_1}(G) \cong \mathfrak{X}_{n_2,g_2}(G)$.*

Proof. The forward direction is obvious. We break the converse direction into three cases.

First, assume $n_1, n_2 > 0$. Then the Casimir subalgebra of $\mathbb{C}[\mathfrak{X}_{n_1,g_1}(G)]$ differs from the Casimir subalgebra of $\mathbb{C}[\mathfrak{X}_{n_1,g_1}(G)]$ unless $n_1 = n_2$. In that case, the Euler characteristic, which is read off the dimension of $\mathfrak{X}_{n,g}(G)$, determines the genus and so $g_1 = g_2$ and we are done.

Second, assume $n_1 = 0$ but $n_2 > 0$ (which implies that the two surfaces are not isomorphic). In that case, $\mathfrak{X}_{n_1,g_1}(G)$ is symplectic but $\mathfrak{X}_{n_2,g_2}(G)$ is

not (non-trivial Casimirs). Thus, their bi-vectors could not be equivalent either.

Lastly, assume that $n_1 = 0 = n_2$. Then Proposition 2.4 tells us that since the dimensions of $\mathfrak{X}_{n_1, g_1}(G)$ and $\mathfrak{X}_{n_2, g_2}(G)$ are the same the Euler characteristics of the surfaces are the same. But since each surface is closed, they are isomorphic. \square

We next consider some examples. Since the character variety is a point when $(n, g) = (0, 0)$ or $(1, 0)$ these cases are trivial for any G . Likewise, for any G the $(2, 0)$ case has 0 dimensional symplectic leaves and so the bi-vector is trivial.

The next simplest example is the 3-holed sphere. For $G = \text{SL}(2, \mathbb{C})$, we have $\mathfrak{X}_{3,0}(G) \cong \mathbb{C}^3$ with coordinates $\text{tr}_{c_1}, \text{tr}_{c_2}$, and $\text{tr}_{c_1 c_2}$ (see [ABL] for a proof). Since the boundary curves are disjoint, they have no intersections and thus the Poisson bracket is trivial. Alternatively, the symplectic leaves are the level sets obtained by fixing the three boundary invariants. But since each point in $\mathfrak{X}_{3,0}(G)$ is uniquely determined by $\text{tr}_{c_1}, \text{tr}_{c_2}$, and $\text{tr}_{c_1 c_2}$, each symplectic leaf is a point.

When $G = \text{SL}(3, \mathbb{C})$ the bi-vector for $\Sigma_{3,0}$ was worked out in [La1, La3]. Unlike the case of $\text{SL}(2, \mathbb{C})$ where the symplectic leaves are 0 dimensional, the symplectic leaves in $\mathfrak{X}_{3,0}(\text{SL}(3, \mathbb{C}))$ are 2 dimensional.

To describe it we need to briefly review the structure of $\text{Hom}(F_2, \text{SL}(3, \mathbb{C}))//\text{SL}(3, \mathbb{C})$ from [La1, La2]. The $\text{SL}(3, \mathbb{C})$ -character variety of a free group F_2 of rank 2 is a hypersurface in \mathbb{C}^9 , which is a branched double cover of \mathbb{C}^8 under projection. The coordinate ring is generated by 9 trace functions of simple closed curves (in the 1-holed torus) denoted $\{t_{(\pm 1)}, \dots, t_{(\pm 4)}, t_{(5)}\}$, and satisfies a single relation of the form $t_{(5)}^2 - Pt_{(5)} + Q$ where $P, Q \in \mathbb{C}[t_{(\pm 1)}, \dots, t_{(\pm 4)}]$. Let $\mathfrak{a}_{i,j} = \{t_{(i)}, t_{(j)}\}$. In these terms, the Poisson bi-vector is:

$$\mathfrak{a}_{3,0}(G) = (P - 2t_{(5)}) \frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(-4)}} + (1 - \mathfrak{i}) \left(\mathfrak{a}_{4,5} \frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(5)}} \right),$$

where $\mathfrak{a}_{4,5} = \frac{\partial}{\partial t_{(-4)}}(Q - t_{(5)}P)$, and $\mathfrak{i} \in \text{Out}(F_2)$ is the outer automorphism of $F_2 = \langle a, b \rangle$ defined by $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$.

The other surface with Euler -1 is the 1-holed torus. In this case, the bi-vector for $\text{SL}(2, \mathbb{C})$ is computed in [Gol3], and for $\text{SL}(3, \mathbb{C})$ it is computed in [La3, La4]. Additionally, in [Gol3] the bi-vector is computed for $\text{SL}(2, \mathbb{C})$ and the Euler characteristic -2 open surfaces: the 4-holed sphere and the 2-holed torus. No other examples have been computed.

In Section 7 we add to the known examples by computing the bi-vector for the 5-holed sphere (one of three Euler characteristic -3 orientable surfaces). This computation uses the computational program in [ABL] that allows one to compute the generators and relations of *any* $SL(2, \mathbb{C})$ -character variety. There are 45 required computations and diagrams. In fact, the algorithm we describe and use is *effective*, that is, the algorithm terminates after a finite number of steps that can in principle be done “by hand,” and always produces a correct answer if the steps are correctly followed. Here is the theorem:

Theorem 2.7. *For any $n, g \geq 0$, there is an effective algorithm to compute the Poisson bi-vector of $\mathfrak{X}_{n,g}(SL(2, \mathbb{C}))$. The bi-vector of $\mathfrak{X}_{5,0}(SL(2, \mathbb{C}))$ is:*

$$\begin{aligned} \mathfrak{a}_{5,0}(SL(2, \mathbb{C})) = & \mathfrak{a}_{1324} \frac{\partial}{\partial t_{\{1,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} + \Sigma_1 \left(\mathfrak{a}_{1214} \frac{\partial}{\partial t_{\{1,2\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} \right) \\ & + \Sigma_2 \left(\mathfrak{a}_{12314} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} + \mathfrak{a}_{12324} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} \right. \\ & \left. + \mathfrak{a}_{12334} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{3,4\}}} \right) \\ & + \Sigma_2 \left(\mathfrak{a}_{123124} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,2,4\}}} \right) \\ & + \Sigma_3 \left(\mathfrak{a}_{123134} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,3,4\}}} \right), \end{aligned}$$

where Σ_i are symmetry operators defined by the mapping class group of the surface and \mathfrak{a}_x are explicit polynomials; both are described in detail in Section 7. Moreover, the polynomial coefficients do not exhibit any further mapping class group symmetry from boundary permutation.

We remark again that this is the first explicit example of the Poisson structure on a character variety of an Euler characteristic -3 surface.

2.6. Mappings between surfaces

Let Σ_1 and Σ_2 be compact orientable surfaces (possibly with boundary), and G , as before, is a reductive affine algebraic group over \mathbb{C} . If $q : \Sigma_1 \rightarrow \Sigma_2$ is a continuous map and $q(x) = y$, then there is an induced homomorphism

$$q\# : \pi_1(\Sigma_1, x) \rightarrow \pi_1(\Sigma_2, y).$$

In turn, we have an induced *continuous* map

$$q^* : \text{Hom}(\pi_1(\Sigma_2, y), G) \longrightarrow \text{Hom}(\pi_1(\Sigma_1, x), G)$$

given by $q^*(\rho) = \rho \circ q_\#$. This function is equivariant with respect to G -conjugation, and thus there is a morphism

$$\mathfrak{q}^* : \text{Hom}(\pi_1(\Sigma_2, y), G) // G \longrightarrow \text{Hom}(\pi_1(\Sigma_1, x), G) // G$$

given by $\mathfrak{q}^*([\rho]) = [\rho \circ q_\#]$. Lastly, we have an algebra morphism between coordinate rings

$$\mathfrak{q}_* : \mathbb{C}[\text{Hom}(\pi_1(\Sigma_1, x), G)]^G \longrightarrow \mathbb{C}[\text{Hom}(\pi_1(\Sigma_2, y), G)]^G$$

given by $\mathfrak{q}_*(f)(\rho) = f(\rho \circ q_\#)$.

Theorem 2.8. *Let $q : \Sigma_1 \longrightarrow \Sigma_2$ be a continuous map between compact orientable surfaces that preserves transversality of based loops, and double points. Then the induced algebra morphism of coordinate rings $\mathfrak{q}_* : \mathbb{C}[\mathfrak{X}_{\Sigma_1, x}(G)] \longrightarrow \mathbb{C}[\mathfrak{X}_{\Sigma_2, y}(G)]$ is a morphism of Poisson algebras if q preserves orientation and is an anti-Poisson morphism if q reverses orientation. Regardless, the image of \mathfrak{q}_* is a Poisson subalgebra.*

Proof. The first part of the theorem implies the second, so we only prove that. Since \mathfrak{q}_* is an algebra morphism and the bracket is a derivation, it is enough to verify the claim on all generators of the algebra.

Since q preserves transversality of based loops, double points, and either preserves or reverses (globally) orientation, it follows that for any two based loops α and β in Σ_1 used in computing the bi-vector $\mathfrak{a}_{\Sigma_1}(G)$ we have from Equation (2.1):

$$\begin{aligned} (2.2) \quad & \mathfrak{q}_* (\{f_\alpha([\rho]), g_\beta([\rho])\}) \\ &= \sum_{q(p) \in q(\alpha) \cap q(\beta)} \epsilon(q(p), q(\alpha), q(\beta)) \mathfrak{B}(F_{q(\alpha)_{q(p)}}([q_\#(\rho)]), G_{q(\beta)_{q(p)}}([q_\#(\rho)]) \\ &= \pm \{f_{q(\alpha)}([q_\#(\rho)]), g_{q(\beta)}([q_\#(\rho)])\} \\ &= \pm \{\mathfrak{q}_*(f_\alpha([\rho])), \mathfrak{q}_*(g_\beta([\rho]))\}. \end{aligned}$$

However, the intersection numbers $\epsilon(q(p), q(\alpha), q(\beta))$ and $\epsilon(p, \alpha, \beta)$ will be reversed if q reverses orientation and will be preserved if q preserves orientation. □

Corollary 2.9. *The map between character varieties induced by q in Theorem 2.8*

$$q^* : \mathfrak{X}_{\Sigma_2, y}(G) \longrightarrow \mathfrak{X}_{\Sigma_1, x}(G)$$

is Poisson whenever q preserves transversality of based loops, double points and orientation.

Proof. Whenever there is a morphism of Poisson algebras, the dual map, between spaces, is always Poisson [LGPV, Chapter 1]. Thus, the corollary follows from Theorem 2.8. □

Remark 2.10. For K a compact Lie group, $\mathfrak{X}_{n, g}(K) = \text{Hom}(\pi_1(\Sigma_{n, g}), K)/K$ is semi-algebraic. As the complexification of K is a reductive algebraic group G , $\mathfrak{X}_{n, g}(K)$ naturally embeds into the \mathbb{R} -locus of $\mathfrak{X}_{n, g}(G)$ by [FL2, Theorem 4.3]. Consequently, Equation (2.1) defines a Poisson bracket on the real coordinate ring of $\mathfrak{X}_{n, g}(K)$ by restriction of scalars. Thus, Theorem 2.8 and its corollary remain valid in this context as well. The proof is exactly the same.

Remark 2.11. More generally, for a real form H of G , the map $\mathfrak{X}_{n, g}(H) \rightarrow \mathfrak{X}_{n, g}(G)$ need not be an embedding (as it is for the compact real form), but it will be a finite map by the paragraph following [CFLO, Proposition 6.1]. It would be interesting to explore if the above theorems remain valid in this context.

Example 2.12. For any two surfaces Σ_{n_1, g_1} and Σ_{n_2, g_2} with $n_1 > n_2 > 0$ and $\chi(\Sigma_{n_1, g_1}) = \chi(\Sigma_{n_2, g_2})$, there is a quotient mapping $q : \Sigma_{n_1, g_1} \rightarrow \Sigma_{n_2, g_2}$ identifying one or more pairs of boundary components. Both $\pi_1(\Sigma_{n_1, g_1}, *)$ and $\pi_1(\Sigma_{n_2, g_2}, q(*))$ are isomorphic to a free group of rank $1 - \chi(\Sigma_{n_1, g_1})$. Since q satisfies the conditions of Theorem 2.8, the induced gluing map

$$q^* : \mathfrak{X}_{n_2, g_2}(G) \longrightarrow \mathfrak{X}_{n_1, g_1}(G)$$

is Poisson. Since $\mathfrak{X}_{n_2, g_2}(G) \cong \mathfrak{X}_{n_1, g_1}(G)$ we have Poisson morphisms between isomorphic varieties with *different* Poisson structures. For a detailed example of this phenomenon see [La4].

Example 2.13. Another natural example that satisfies the conditions of Theorem 2.8 is the inclusion $\iota : \Sigma_1 \hookrightarrow \Sigma_2$ of a subsurface Σ_1 into a surface Σ_2 . In this case, the induced map on character varieties $q^* : \mathfrak{X}_{\Sigma_2, y}(G) \rightarrow \mathfrak{X}_{\Sigma_1, x}(G)$, which is Poisson, is the restriction map. In the case when Σ_2

is closed and Σ_1 has boundary, we have a Poisson map whose domain is symplectic.

3. Poisson structure on character varieties: De Rham point-of-view

Let, almost as before, $\Sigma_{n,g}$ be a connected orientable surface of genus $g \geq 0$ with $n \geq 0$ punctures, the differences being that:

- 1) the surface is C^∞ , not just topological, and
- 2) unlike in the previous case where we had surfaces with boundary, we only consider here their interior, so that we have open surfaces when $n > 0$.

Let now G be connected Lie group such that either:

- G is compact, or
- G is a reductive affine algebraic group defined over the field of complex numbers.

Note that the reductive affine algebraic group defined over \mathbb{C} are complexifications of compact connected Lie groups.

Fix a base point $x_0 \in \Sigma_{n,g}$. When G is a reductive affine algebraic group defined over \mathbb{C} , a homomorphism $\rho : \pi_1(\Sigma_{n,g}, x_0) \rightarrow G$ is called *reductive* if the Zariski closure of $\rho(\pi_1(\Sigma_{n,g}, x_0))$ in G is a reductive subgroup. We note that ρ is reductive if and only if satisfies the following condition: if P is a parabolic subgroup of G such that $\rho(\pi_1(\Sigma_{n,g}, x_0)) \subset P$, then $\rho(\pi_1(\Sigma_{n,g}, x_0))$ is contained in a Levi factor of P (see [Bo, 11.2] and [Hu, p. 184] for parabolic subgroups and their Levi factors).

If G is a compact Lie group, then all homomorphisms $\rho : \pi_1(\Sigma_{n,g}, x_0) \rightarrow G$ are reductive.

Let

$$\mathcal{R}_G^r(\Sigma_{n,g}) \subset \text{Hom}(\pi_1(\Sigma_{n,g}, x_0), G)$$

be the space of all reductive homomorphisms. The adjoint action of G on itself produces an action of G on $\mathcal{R}_G^r(\Sigma_{n,g})$. The corresponding quotient space

$$(3.1) \quad \mathfrak{X}_{n,g}(G) := \mathcal{R}_G^r(\Sigma_{n,g})/G$$

is again the character variety, this time with G either reductive over the complex numbers or compact. Note that $\mathfrak{X}_{n,g}(G)$ is actually independent of the choice of the base point x_0 . When G is a reductive affine algebraic group defined over \mathbb{C} , then, as we have seen, $\text{Hom}(\pi_1(\Sigma_{n,g}, x_0), G)$ is a complex affine variety, because G is a complex affine variety and the group $\pi_1(\Sigma_{n,g}, x_0)$ is finitely generated. In this case, restricting to reductive representations and taking the ordinary quotient coincides with the geometric invariant theoretic quotient $\text{Hom}(\pi_1(\Sigma_{n,g}, x_0), G)//G$ (in the analytic topology) by [FL3, Theorem 2.1], and so inherits a natural algebraic structure. It is also homotopic to the non-Hausdorff quotient $\text{Hom}(\pi_1(\Sigma_{n,g}, x_0), G)/G$ by [FLR, Proposition 3.4].

A G -connection on $\Sigma_{n,g}$ is a C^∞ principal G -bundle on $\Sigma_{n,g}$ equipped with a connection. If the curvature of a connection vanishes identically, then it is called a flat G -connection. A flat G -connection on $\Sigma_{n,g}$ is called reductive if the corresponding monodromy homomorphism is reductive. The character variety is identified with the moduli space of reductive flat G -connections on $\Sigma_{n,g}$. This identification sends a flat G -connection to the monodromy homomorphism corresponding to the flat connection.

Let E_G be a C^∞ principal G -bundle on $\Sigma_{n,g}$, and let ∇ be a reductive flat connection on E_G . So (E_G, ∇) gives a point

$$(3.2) \quad (E_G, \nabla) := z \in \mathfrak{X}_{n,g}(G),$$

where $\mathfrak{X}_{n,g}(G)$ is constructed in (3.1).

As before, the Lie algebra of G will be denoted by \mathfrak{g} . The adjoint bundle

$$\text{ad}(E_G) = E_G \times^G \mathfrak{g}$$

is the vector bundle over $\Sigma_{n,g}$ associated to E_G for the adjoint action of G on \mathfrak{g} . So the fibers of $\text{ad}(E_G)$ are Lie algebras identified with \mathfrak{g} uniquely up to automorphisms of \mathfrak{g} given by conjugations. Fix a nondegenerate G -invariant symmetric bilinear form

$$(3.3) \quad B \in \text{Sym}^2(\mathfrak{g}^*)^G$$

on \mathfrak{g} ; the assumptions on G ensure that such a form B exists. Since B in (3.3) is G -invariant, it produces a C^∞ pairing

$$(3.4) \quad \mathcal{B} : \text{ad}(E_G) \otimes \text{ad}(E_G) \longrightarrow \Sigma_{n,g} \times k,$$

where $k = \mathbb{R}$ (respectively, $k = \mathbb{C}$) when G is compact (respectively, complex reductive). Since B in (3.3) is also nondegenerate, the pairing \mathcal{B} in (3.4)

is fiberwise nondegenerate. Therefore, \mathcal{B} produces an isomorphism of vector bundles

$$(3.5) \quad \text{ad}(E_G) \xrightarrow{\sim} \text{ad}(E_G)^*.$$

The flat connection ∇ on E_G induces a flat connection on $\text{ad}(E_G)$; this induced connection on $\text{ad}(E_G)$ will be denoted by ∇^{ad} . We note that ∇^{ad} is self-dual with respect to the isomorphism in (3.5); this means that the connection on $\text{ad}(E_G)^*$ given by ∇^{ad} using the isomorphism in (3.5) coincides with the connection on $\text{ad}(E_G)^*$ induced by ∇^{ad} using the duality pairing.

Let $\underline{\text{ad}}(E_G)$ be the local system on $\Sigma_{n,g}$ given by the sheaf of flat sections of $\text{ad}(E_G)$ for the connection ∇^{ad} . We have

$$(3.6) \quad T_z \mathfrak{X}_{n,g}(G) = H^1(\Sigma_{n,g}, \underline{\text{ad}}(E_G)),$$

where z is the point in (3.2) (for example [Gol1, Section 1.8]). Therefore, the Poincaré–Verdier duality gives that

$$(3.7) \quad T_z^* \mathfrak{X}_{n,g}(G) = H^1(\Sigma_{n,g}, \underline{\text{ad}}(E_G))^* = H_c^1(\Sigma_{n,g}, \underline{\text{ad}}(E_G)),$$

where H_c^i denotes the compactly supported i -th cohomology. We have a natural homomorphism

$$T_z^* \mathfrak{X}_{n,g}(G) = H_c^1(\Sigma_{n,g}, \underline{\text{ad}}(E_G)) \longrightarrow H^1(\Sigma_{n,g}, \underline{\text{ad}}(E_G)) = T_z \mathfrak{X}_{n,g}(G).$$

As z moves over $\mathfrak{X}_{n,g}(G)$, these point-wise homomorphisms together produce a C^∞ homomorphism

$$(3.8) \quad \Theta : T^* \mathfrak{X}_{n,g}(G) \longrightarrow T \mathfrak{X}_{n,g}(G).$$

A Poisson structure on a C^∞ manifold A is a section $\theta_A \in C^\infty(A, \wedge^2 TA)$ such that the Schouten–Nijenhuis bracket $[\theta_A, \theta_A]$ vanishes identically [Ar]. The condition that $[\theta_A, \theta_A] = 0$ is equivalent to the following condition: given a pair of C^∞ locally defined functions f_1 and f_2 on A , consider the locally defined C^∞ function

$$\{f_1, f_2\}_{\theta_A} := \theta_A((df_1) \wedge (df_2));$$

then

$$\{f_1, \{f_2, f_3\}_{\theta_A}\}_{\theta_A} + \{f_2, \{f_3, f_1\}_{\theta_A}\}_{\theta_A} + \{f_3, \{f_1, f_2\}_{\theta_A}\}_{\theta_A} = 0$$

for all C^∞ locally defined functions f_1, f_2, f_3 .

The C^∞ homomorphism Θ in (3.8) is a Poisson structure on $\mathfrak{X}_{n,g}(G)$ (see [BJ]).

When $\Sigma_{n,g}$ is a compact oriented surface ($n = 0$), then Θ is an isomorphism, so the Poisson structure given by it is actually a symplectic structure. In that case, it coincides with the symplectic structure constructed by Atiyah–Bott [AB] and Goldman [Gol1].

4. Open subsets and Poisson structure

Let $\Sigma_1 = \Sigma_{n_1,g_1}$ be embedded as a connected open subset of $\Sigma_2 = \Sigma_{n_2,g_2}$. Take the base point $x_0 \in \Sigma_2$ such that $x_0 \in \Sigma_1$.

Restricting the flat G -connections on Σ_2 to the open subset Σ_1 we obtain a map

$$(4.1) \quad \Phi : \mathfrak{X}_{\Sigma_2}(G) \longrightarrow \mathfrak{X}_{\Sigma_1}(G).$$

Indeed, as above, the natural homomorphism $\pi_1(\Sigma_1, x_0) \longrightarrow \pi_1(\Sigma_2, x_0)$ produces a map

$$\text{Hom}(\pi_1(\Sigma_2, x_0), G) \longrightarrow \text{Hom}(\pi_1(\Sigma_1, x_0), G);$$

it in turn gives the map Φ in (4.1) by taking geometric invariant theoretic quotient for the actions of G on $\text{Hom}(\pi_1(\Sigma_2, x_0), G)$ and $\text{Hom}(\pi_1(\Sigma_1, x_0), G)$.

As in (3.2), take any $(E_G, \nabla) := z \in \mathfrak{X}_{\Sigma_2}(G)$. Consider the local system $\underline{\text{ad}}(E_G)$ on Σ_2 (see (3.6), (3.7)). Its restriction to Σ_1 will be denoted by $\underline{\text{ad}}(E_G)_{\Sigma_1}$. The inclusion map $\Sigma_1 \hookrightarrow \Sigma_2$ produces homomorphisms

$$(4.2) \quad \beta : H^1(\Sigma_2, \underline{\text{ad}}(E_G)) \longrightarrow H^1(\Sigma_1, \underline{\text{ad}}(E_G)_{\Sigma_1}),$$

and

$$(4.3) \quad \gamma : H_c^1(\Sigma_1, \underline{\text{ad}}(E_G)_{\Sigma_1}) \longrightarrow H_c^1(\Sigma_2, \underline{\text{ad}}(E_G))$$

(see (3.7)). We note that β is the pullback by the inclusion map of Σ_1 in Σ_2 , while γ is the push-forward by the inclusion map.

For the map Φ in (4.1), let

$$d\Phi(z) : T_z\mathfrak{X}_{\Sigma_2}(G) \longrightarrow T_{\Phi(z)}\mathfrak{X}_{\Sigma_1}(G)$$

be its differential at the point z in (3.2). Let

$$(d\Phi)^*(z) : T_{\Phi(z)}^*\mathfrak{X}_{\Sigma_1}(G) \longrightarrow T_z^*\mathfrak{X}_{\Sigma_2}(G)$$

be its dual homomorphism.

Proposition 4.1. *Invoke the isomorphism in (3.6) (respectively, (3.7)) for $T_z\mathfrak{X}_{\Sigma_2}(G)$ and $T_{\Phi(z)}\mathfrak{X}_{\Sigma_1}(G)$ (respectively, $T_z^*\mathfrak{X}_{\Sigma_2}(G)$ and $T_{\Phi(z)}^*\mathfrak{X}_{\Sigma_1}(G)$).*

- 1) *The homomorphism $d\Phi(z)$ coincides with the homomorphism β in (4.2).*
- 2) *The homomorphism $(d\Phi)^*(z)$ coincides with the homomorphism γ in (4.3).*

Proof. The first statement is standard.

To prove the second statement, take any $\text{ad}(E_G)$ -valued one-form

$$\omega \in C^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$$

such that $\nabla^{\text{ad}}(\omega) = 0$, where ∇^{ad} , as before, is the connection on $\text{ad}(E_G)$ induced by the connection ∇ (see (3.2)). Also take a compactly supported $\text{ad}(E_G)$ -valued one-form $\omega^c \in C_c^\infty(\Sigma_1, (\text{ad}(E_G)|_{\Sigma_1}) \otimes T^*\Sigma_1)$ on Σ_1 such that $\nabla^{\text{ad}}(\omega^c) = 0$. Let $\iota : \Sigma_1 \hookrightarrow \Sigma_2$ be the inclusion map. Then we have

$$(4.4) \quad \int_{\Sigma_1} \mathcal{B}(\omega^c \wedge (\iota^*\omega)) = \int_{\Sigma_2} \mathcal{B}((\iota_*\omega^c) \wedge \omega),$$

where \mathcal{B} is the pairing in (3.4), and $\iota_*\omega^c$ is the push-forward of the compactly supported form ω^c using the inclusion map ι ; note that both $\mathcal{B}(\omega^c \wedge (\iota^*\omega))$ and $\mathcal{B}((\iota_*\omega^c) \wedge \omega)$ are compactly supported 2-forms on Σ_1 , and moreover they coincide. The second statement in the proposition follows from (4.4). □

A smooth map $F : A \rightarrow B$ between Poisson manifolds (A, θ_A) and (B, θ_B) is called *Poisson* if

$$F \circ \{f, g\}_{\theta_B} = \{F \circ f, F \circ g\}_{\theta_A}$$

for all locally defined C^∞ functions f and g on B . This is equivalent to our usage in Corollary 2.9. We note that F is Poisson if and only if the following diagram is commutative

$$(4.5) \quad \begin{array}{ccc} T_{F(x)}^* B & \xrightarrow{(dF)^*(x)=dF(x)^*} & T_x^* A \\ \downarrow \theta_B(F(x)) & & \downarrow \theta_A(x) \\ T_{F(x)} B & \xleftarrow{dF(x)} & T_x A \end{array}$$

for every point $x \in A$, where $dF : TA \rightarrow F^*TB$ is the differential of the map F while $(dF)^*$ is its dual.

Theorem 4.2. *The map Φ in (4.1) is Poisson.*

Proof. In view of Proposition 4.1 it is straight-forward to check that the diagram in (4.5) for $F = \Phi$ commutes. To explain this, take any

$$(E_G, \nabla) := z \in \mathfrak{X}_{\Sigma_1}(G)$$

as in (3.2). Take any compactly supported form

$$\omega \in C_c^\infty(\Sigma_1, (\text{ad}(E_G)|_{\Sigma_1}) \otimes T^*\Sigma_1)$$

such that $\nabla^{\text{ad}}(\omega) = 0$, where ∇^{ad} is the connection on $\text{ad}(E_G)$ induced by the connection ∇ (see (3.2)). Let $\iota_*\omega \in C_c^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$ be the push-forward of ω using the inclusion map $\iota : \Sigma_1 \hookrightarrow \Sigma_2$. Now consider $\iota_*\omega$ as an element of $C^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$; finally, restrict this element of $C^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$ to Σ_1 . This restriction is evidently ω itself. Hence the diagram in (4.5) commutes for $F = \Phi$. \square

The theorem was stated for one surface embedded in another; it holds more generally for suitable ramified covers. Let $\overline{\Sigma_1}$ and $\overline{\Sigma_2}$ be compact connected oriented C^∞ surfaces and

$$\varphi_1 : \overline{\Sigma_1} \rightarrow \overline{\Sigma_2}$$

a possibly ramified covering map which is oriented. Let

$$S_2 \subset \overline{\Sigma_2} \quad \text{and} \quad S_1 \subset \overline{\Sigma_1}$$

be finite subsets such that $\varphi_1^{-1}(S_2) \subset S_1$. Define

$$\Sigma_2 := \overline{\Sigma_2} \setminus S_2 \quad \text{and} \quad \Sigma_1 := \overline{\Sigma_1} \setminus S_1;$$

let

$$\varphi := \varphi_1|_{\Sigma_1} : \Sigma_1 \rightarrow \Sigma_2$$

be the restriction of φ_1 to the open subset Σ_1 .

Consider the character varieties $\mathfrak{X}_{\Sigma_1}(G)$ and $\mathfrak{X}_{\Sigma_2}(G)$. Let

$$(4.6) \quad \Psi : \mathfrak{X}_{\Sigma_2}(G) \longrightarrow \mathfrak{X}_{\Sigma_1}(G)$$

be the map that sends any flat principal G -bundle (E_G, ∇) on Σ_2 to the flat principal G -bundle $(\varphi^*E_G, \varphi^*\nabla)$ on Σ_1 . This map Ψ coincides with the map of character varieties given by the homomorphism

$$\varphi_* : \pi_1(\Sigma_1, y_0) \longrightarrow \pi_1(\Sigma_2, \varphi(y_0))$$

induced by φ .

Proposition 4.3. *The map Ψ in (4.6) is Poisson.*

Proof. Take any

$$(E_G, \nabla) := z \in \mathfrak{X}_{\Sigma_2}(G).$$

Let ∇^{ad} be the flat connection on $\text{ad}(E_G)$ induced by ∇ . Let $\underline{\text{ad}}(E_G)$ be the local system on Σ_2 given by the sheaf of flat sections of $\text{ad}(E_G)$ for the connection ∇^{ad} . From (3.6) we know that $T_z\mathfrak{X}_{\Sigma_2}(G) = H^1(\Sigma_2, \underline{\text{ad}}(E_G))$ and

$$T_{\Psi(z)}\mathfrak{X}_{\Sigma_1}(G) = H^1(\Sigma_1, \underline{\text{ad}}(\varphi^*E_G)) = H^1(\Sigma_1, \varphi^*\underline{\text{ad}}(E_G)),$$

where $\underline{\text{ad}}(\varphi^*E_G)$ is the local system on Σ_1 given by the sheaf of flat sections of $\text{ad}(\varphi^*E_G)$ for the flat connection on $\text{ad}(\varphi^*E_G)$ induced by the flat connection $\varphi^*\nabla$ on φ^*E_G . Note that this induced flat connection on $\text{ad}(\varphi^*E_G)$ coincides with the flat connection $\varphi^*\nabla^{\text{ad}}$ on $\text{ad}(\varphi^*E_G) = \varphi^*\text{ad}(E_G)$.

The differential $d\Psi$ at z

$$\begin{aligned} d\Psi(z) : H^1(\Sigma_2, \underline{\text{ad}}(E_G)) &= T_z\mathfrak{X}_{\Sigma_2}(G) \\ &\longrightarrow T_{\Psi(z)}\mathfrak{X}_{\Sigma_1}(G) = H^1(\Sigma_1, \varphi^*\underline{\text{ad}}(E_G)) \end{aligned}$$

coincides with the homomorphism $H^1(\Sigma_2, \underline{\text{ad}}(E_G)) \longrightarrow H^1(\Sigma_1, \varphi^*\underline{\text{ad}}(E_G))$ that sends any cohomology class μ to its pullback $\varphi^*\mu$.

The dual homomorphism

$$\begin{aligned} (d\Psi)^*(z) : T_{\Psi(z)}^*\mathfrak{X}_{\Sigma_1}(G) &= H_c^1(\Sigma_1, \varphi^*\underline{\text{ad}}(E_G)) \\ &\longrightarrow H_c^1(\Sigma_2, \underline{\text{ad}}(E_G)) = T_z^*\mathfrak{X}_{\Sigma_2}(G) \end{aligned}$$

(see (3.7)) coincides with the trace map. To explain the trace map, take any compactly supported $\varphi^*\text{ad}(E_G)$ -valued 1-form

$$(4.7) \quad \omega \in C_c^\infty(\Sigma_1, \varphi^*\text{ad}(E_G) \otimes T^*\Sigma_1) = C_c^\infty(\Sigma_1, \text{ad}(\varphi^*E_G) \otimes T^*\Sigma_1)$$

such that $\varphi^*\nabla^{\text{ad}}(\omega) = 0$. Now construct

$$\widehat{\omega} \in C^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$$

as follows: For any $x \in \Sigma_2$ such that φ is unramified over x , take any $v \in T_x\Sigma_2$. Define

$$\widehat{\omega}(x)(v) = \sum_{y \in \varphi^{-1}(x)} \omega(y)((d\varphi(y))^{-1}(v)) \in \text{ad}(E_G)_x,$$

where $d\varphi(y)$ is the differential of the map φ at y . It is straight-forward to check the following:

- The above form $\widehat{\omega}$ extends to entire Σ_2 as a C^∞ section of $\text{ad}(E_G) \otimes T^*\Sigma_2$; this section of $\text{ad}(E_G) \otimes T^*\Sigma_2$ over Σ_2 will be denoted by $\widehat{\omega}$.
- The form $\widehat{\omega}$ on Σ_2 is compactly supported. In fact, the support of $\widehat{\omega}$ is contained in the image of the support of ω under the map φ . This implies that $\widehat{\omega}$ is compactly supported.
- $\nabla^{\text{ad}}(\widehat{\omega}) = 0$.

The trace map

$$H_c^1(\Sigma_1, \varphi^*\underline{\text{ad}}(E_G)) \longrightarrow H_c^1(\Sigma_2, \underline{\text{ad}}(E_G))$$

mentioned earlier is constructed by sending any ω as in (4.7) to $\widehat{\omega}$ constructed above from it.

From the above construction of $\widehat{\omega}$ we conclude that $\widehat{\omega}$ has the following property: For any $\omega' \in C^\infty(\Sigma_2, \text{ad}(E_G) \otimes T^*\Sigma_2)$,

$$(4.8) \quad \int_{\Sigma_2} \mathcal{B}(\widehat{\omega} \wedge \omega') = \int_{\Sigma_1} \mathcal{B}(\omega \wedge (\varphi^*\omega')),$$

where \mathcal{B} is the pairing in (3.4).

Using (4.8) it is straightforward to check that the diagram in (4.5) commutes for $F = \Psi$. Hence the map Ψ is Poisson. □

5. Capping: symplectic and Poisson extensions of Poisson character varieties

For a given $\Sigma_1 = \Sigma_{n_1, g_1}$ with a non-empty boundary, we consider different Σ_2 we can obtain by gluing onto the boundary components of Σ_1 . This yields

Poisson maps of the character varieties:

$$(5.1) \quad \Phi : \mathfrak{X}_{\Sigma_2}(G) \longrightarrow \mathfrak{X}_{\Sigma_1}(G),$$

and we will examine the images and fibres of these maps. When Σ_2 is closed, this gives us in some sense symplectic completions of the character varieties.

Number the n_1 boundary components of $\Sigma_1 = \Sigma_{n_1, g_1}$. Choose a base point x_1 on the first boundary component γ_1 of Σ_1 . For every $2 \leq i \leq n_1$, fix a path p_i from x_1 to base point x_i on the i -th boundary component γ_i of Σ_1 . With parametrizations μ_i respecting the orientations of the γ_i and the right choice of paths, we get some standard generators $c_1 = \mu_1, c_i = p_i \mu_i p_i^{-1}$ of the fundamental group $\pi_1(\Sigma_1, x_1)$.

5.1. Case 1: Capping with disks

We consider first the case when Σ_2 is obtained from $\Sigma_1 = \Sigma_{n_1, g_1}$ by gluing in $k \leq n_1$ disks, identifying the boundaries $\tilde{\gamma}_i$ of disks D_i with γ_i , so that Σ_2 has k of the holes of Σ_1 filled in. In this case, it is straightforward to see that Φ is an injection, with image the representations ρ with $\rho(c_i) = 1$. This is a union of a family of symplectic leaves, and if $k = n_1$, a single symplectic leaf.

5.2. Case 2: Capping with a cylinder

Now consider the situation where one glues in a cylinder, attaching the boundary circles $\tilde{\gamma}_1, \tilde{\gamma}_2$ of a cylinder to boundary circles γ_1, γ_2 of Σ_1 . The resulting Σ_2 has two less boundary components and genus one more.

One can again ask what the image and fibres of Φ are in this case. For a flat connection on Σ_1 , choose a flat trivialization along the path p_2 . We then, for the circles γ_i , have holonomies C_i . For these to lie in the image of the flat connections on Σ_2 , we need to be able to glue a flat connection on Σ_1 to a flat connection on the cylinder.

On the cylinder, choose base points on the circles $\tilde{\gamma}_1, \tilde{\gamma}_2$, and a path β from $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$. On the cylinder, we have the relation in the fundamental groupoid:

$$\tilde{\gamma}_1 \beta^{-1} \tilde{\gamma}_2^{-1} \beta = 1.$$

Once one trivializes at the two base points, a flat connection determines corresponding holonomies \tilde{C}_i, B , satisfying

$$\tilde{C}_1 B^{-1} \tilde{C}_2^{-1} B = 1.$$

The gluing, on the level of connections matches C_i with \tilde{C}_i . We then must have:

$$C_1 B^{-1} C_2^{-1} B = 1$$

for some matrix B , which is the image of an extra cycle created by the gluing. Thus the image in $\mathfrak{X}_{\Sigma_1}(G)$ of $\mathfrak{X}_{\Sigma_2}(G)$, is the union of symplectic leaves for which the conjugacy class along γ_1 is the inverse of the conjugacy class along γ_2 , while the fibre is isomorphic to the stabilizer of C_2 under conjugation. Note that $\mathfrak{X}_{\Sigma_1}(G)$ and $\mathfrak{X}_{\Sigma_2}(G)$ have the same dimension.

5.3. Case 3: Capping with a k -holed sphere

We now consider the case of gluing a k -holed sphere ($k \geq 3$) with boundary circles $\tilde{\gamma}_i, i = 1, \dots, k$, to the boundary circles $\gamma_i, i = 1, \dots, k$, of Σ_1 . The resulting Σ_2 will have k less boundary components, and genus $k - 1$ more.

On the sphere, choose base points on the circles $\tilde{\gamma}_i, i = 1, \dots, k$, and paths $\beta_i, i = 1, \dots, k$, from $\tilde{\gamma}_1$ to $\tilde{\gamma}_i$. We have the relation in the fundamental group:

$$\tilde{\gamma}_1^{-1} \beta_2^{-1} \tilde{\gamma}_2^{-1} \beta_2 \dots \beta_i^{-1} \tilde{\gamma}_i^{-1} \beta_i \dots \beta_k^{-1} \tilde{\gamma}_k^{-1} \beta_k = 1.$$

A flat connection on the punctured sphere, once one trivializes at the base points, gives corresponding holonomies \tilde{C}_i, B_i , satisfying:

$$\tilde{C}_1^{-1} B_2^{-1} \tilde{C}_2^{-1} B_2 \dots B_i^{-1} \tilde{C}_i^{-1} B_i \dots B_k^{-1} \tilde{C}_k^{-1} B_k = 1.$$

The gluing, on the level of connections matches C_i with \tilde{C}_i^{-1} , and so

$$(5.2) \quad C_1 B_2^{-1} C_2 B_2 \dots B_i^{-1} C_i B_i \dots B_k^{-1} C_k B_k = 1.$$

So one needs to be able to find matrices B_i which make this relation true. In short, given k conjugacy classes, we have to be able to find elements in them whose product is one. There are choices for which this is not the case; for example, if one takes $C_1 = \dots = C_{k-1} = 1, C_k = -1$.

The general question of when Equation (5.2) has a solution is known as the Deligne-Simpson Problem, which is only solved when G is of type A_n [Sim, Ko, Cr].

The map $\Phi : \mathfrak{X}_{\Sigma_2}(G) \rightarrow \mathfrak{X}_{\Sigma_1}(G)$ is not surjective.

If g is at least two, $\mathfrak{X}_{\Sigma_2}(G)$ has dimension $(k - 2) \dim(G)$ greater than $\mathfrak{X}_{\Sigma_1}(G)$. This would then be the dimension of the generic fibres when there is a solution to Equation (5.2).

5.4. Case 4: Capping one circle with a genus 1 curve, $n_1 \geq 2$

Now let us glue a punctured genus 1 curve with one boundary circle $\tilde{\gamma}$ to the boundary circle γ_1 of Σ_1 , when $n_1 \geq 2$. The resulting Σ_2 will have 1 less boundary components, and genus 1 more.

On the (closed) genus 1 curve, take a standard basis α, β of the fundamental group. In the fundamental group of the punctured curve:

$$\tilde{\gamma}\alpha\beta\alpha^{-1}\beta^{-1} = 1.$$

Lemma 5.1. *Take G as before. Every element of the group $[G, G]$ can be expressed as*

$$[A, B] := ABA^{-1}B^{-1}$$

for some $A, B \in [G, G]$. Thus elements of the commutator subgroup can be written as a single commutator.

Proof. First assume that G is compact. Then $[G, G]$ is a connected compact semisimple Lie group. A theorem of Gotô says that for every element $C \in [G, G]$, there are elements $A, B \in [G, G]$ such that $C = [A, B] := ABA^{-1}B^{-1}$ [Got, p. 270, Lemma].

Next assume that G is a reductive affine algebraic group defined over \mathbb{C} . Then $[G, G]$ is a connected semisimple affine algebraic group defined over \mathbb{C} . Then for every element $C \in [G, G]$, there are elements $A, B \in [G, G]$ such that $C = [A, B] := ABA^{-1}B^{-1}$ [PW, p. 908], [Re, p. 457]. \square

We then have:

Proposition 5.2. *The image in $\mathfrak{X}_{\Sigma_1}(G)$ of $\mathfrak{X}_{\Sigma_2}(G)$ consists of the representations with the image of γ_1 in $[G, G]$. In particular, if G is semisimple, meaning $G = [G, G]$, then the map $\mathfrak{X}_{\Sigma_2}(G) \rightarrow \mathfrak{X}_{\Sigma_1}(G)$ is surjective.*

If $G = [G, G]$, then by capping successively the boundaries of Σ_1 we get a symplectic manifold $\mathfrak{X}_{\Sigma_2}(G)$ and a Poisson map $\mathfrak{X}_{\Sigma_2}(G) \rightarrow \mathfrak{X}_{\Sigma_1}(G)$. If g is at least two, each cap increases the genus by one and diminishes the number of punctures by one, and so adds $\dim(G)$ dimensions.

5.5. Case 5: Capping with an n -punctured genus 1 curve

We now glue a genus 1 curve C with n boundary circles to the n boundary circles of Σ_1 so that the resulting curve Σ_2 has no punctures, but the

genus is increased by n . Again, for G semisimple, $g_1 \geq 2$, this increases the dimensions of the representation space by $n \dim G$.

We have the fundamental groups

$$\pi_1(\Sigma_1) = \langle a_1, b_1, \dots, a_{g_1}, b_{g_1}, c_1, \dots, c_n \mid \prod_{i=1}^{g_1} [a_i, b_i] \prod_{j=1}^n c_j \rangle$$

where g_1 is the genus of Σ_1 , and

$$\pi_1(C) = \langle a, b, c_1, \dots, c_n \mid [a, b] \prod_{j=1}^n c_j \rangle.$$

We then have similar relations for their representations into G .

Proposition 5.3. *The map $\Phi : \mathfrak{X}_{\Sigma_2}(G) \longrightarrow \mathfrak{X}_{\Sigma_1}(G)$ is surjective and Poisson.*

Proof. For the surjectivity, for a representation of $\pi_1(\Sigma_1)$ into G , we have images A_i, B_i, C_j of the generators a_i, b_i, c_j giving an element $\prod_{i=1}^{g_1} [A_i, B_i]$ of the commutator subgroup of G . Now Lemma 5.1 tells us that this is equal to a single commutator $[A, B]$, giving a representation A, B, C_j of $\pi_1(C)$. Inverting, this can be glued to the representation of $\pi_1(\Sigma_1)$ to obtain a representation of $\pi_1(\Sigma_2)$. Consequently, the map $\Phi : \mathfrak{X}_{\Sigma_2}(G) \longrightarrow \mathfrak{X}_{\Sigma_1}(G)$ is surjective. We have already seen that Φ is Poisson. \square

Thus we have a symplectic “completion” of $\mathfrak{X}_{\Sigma_1}(G)$.

5.6. Case 6: Capping with a mirror image

We note that we can also glue a copy of Σ_1 to itself. Again, the resulting Σ_2 will give a symplectic character variety mapping surjectively onto that of Σ_1 , albeit with an enormous redundancy; the dimension gets doubled. We note that for the representations on one of the copies of Σ_2 , we must invert the matrices A_i, B_i, C_j before gluing. This involves changing the generating set of the fundamental group somewhat, but it can be done.

6. Gluing via symplectic quotients

The symplectic extensions obtained by capping boundary components are, as we have seen, often somewhat inefficient in terms of the dimensions they

add. We close by recalling two classical constructions which accomplish the same task, essentially by adding a trivialisation to our connections on a base point on each boundary circle. The first, due to Alekseev, Malkin and Meinrenken [AMM], takes us outside of the symplectic domain, into quasi-Hamiltonian territory; the second involves the extended moduli spaces of Jeffrey [Je]. The reduction to the flat connection spaces are then group quotients. We note that there is an “imploded” version of the construction of [AMM], considered in [HJ].

6.1. q -Hamiltonian spaces

q -Hamiltonian spaces were defined by Alekseev, Malkin and Meinrenken in [AMM]. They play a role analogous to the symplectic quotient of the space of all connections by the based gauge group (in other words the extended moduli space [Je]).

We quote the following definitions from [AMM].

Definition 6.1 ([AMM, Definition 2.2]). Let G be a compact Lie group. Assume further that G is connected and simply connected. A quasi-Hamiltonian (or q -Hamiltonian) G -space is a G -manifold M together with an invariant 2-form $\omega \in \Omega^2(M)^G$ and a G -equivariant map $\mu \in C^\infty(M, G)$ (where G acts on itself by conjugation) such that:

(B1) The differential of ω is given by

$$d\omega = -\mu^*\chi,$$

where χ is the closed bi-invariant 3-form on G given by $\chi = \frac{1}{12}(\theta, [\theta, \theta])$. Here $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ is the left invariant Maurer-Cartan form (where \mathfrak{g} is the Lie algebra of G) and $\bar{\theta}$ is the right invariant Maurer-Cartan form. This is often denoted $\theta = g^{-1}dg$ if $g : U \rightarrow G$ is a coordinate on a coordinate chart U for G . Similarly $\bar{\theta}$ is denoted $dg g^{-1}$.

(B2) The map μ satisfies:

$$i(v_\xi)\omega = \frac{1}{2}\mu^*(\theta + \bar{\theta}, \xi),$$

where v_ξ is the fundamental vector field on G associated to an element ξ in \mathfrak{g} .

(B3) At each $x \in M$, the kernel of ω_x is given by:

$$\ker(\omega_x) = v_\xi, \quad \xi \in \text{Ker}(\text{Ad}_{\mu(x)} + 1).$$

We will refer to μ as a moment map.

The q -Hamiltonian space whose q -Hamiltonian quotient is the character variety for a 2-manifold of genus g with no boundary is the space denoted M_g , which is isomorphic to G^{2g} . The moment map for the diagonal action of G on M_g by conjugation is the product of commutators

$$(6.1) \quad \mu(x_1, x_2, \dots, x_{2g}) = \prod_{j=1}^g x_{2j-1} x_{2j} x_{2j-1}^{-1} x_{2j}^{-1}.$$

Here $x_j \in G$.

The q -Hamiltonian space whose q -Hamiltonian quotient is the character variety for a 2-manifold which has genus g and r boundary components is the space

$$(6.2) \quad M_{g,r} := G^{2g+2r}.$$

See Section 6.2 below.

If X_1 and X_2 are two q -Hamiltonian G -spaces with moment maps μ_1 and μ_2 , then the fusion product $X_1 \times X_2$ is also a q -Hamiltonian G -space with moment map $\mu_1 \cdot \mu_2$, where \cdot is multiplication in G . See for example [Me], §3.2. This is analogous to the fact that the product of two Hamiltonian G -spaces Y_1 and Y_1 is also a Hamiltonian G -space and its moment map is the sum of the moment maps for the Hamiltonian G actions on Y_1 and Y_2 .

It follows that if M_g is the q -Hamiltonian space associated to a genus g surface Σ_g and M_h is the q -Hamiltonian space associated with a genus h surface Σ_h , then the q -Hamiltonian quotient of $M_g \times M_h$ by the diagonal action of G is a symplectic manifold. It is the character variety of a surface of genus $g + h$ without boundary.

This quotient construction is defined in Section 6.2 below. For completeness, we include the following material from Section 6 of [AMM]. The 2-form on the double $D(G) := G \times G$ is given in [AMM, Section 3.2]:

$$\omega_D = \frac{1}{2}(a^*\theta, b^*\bar{\theta}) + \frac{1}{2}(a^*\bar{\theta}, b^*\theta),$$

where $a, b : G \times G \rightarrow G$ are projections to the first and second factors respectively. Here $\theta, \bar{\theta}$ were introduced in Definition 6.1.

Now let us consider the construction for a surface of genus g and $r + 1$ boundary components ($r \geq 0$). This construction creates a 2-form on a space called the internal fusion of the double and denoted $\mathbf{D}(G) = G \times G$ ([AMM, Example 6.1]):

$$\omega = \frac{1}{2}(a^*\theta, b^*\bar{\theta}) + \frac{1}{2}(a^*\bar{\theta}, b^*\theta) + \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\bar{\theta}).$$

The space $M_{g,r} = G^{2g+2r}$ with coordinates (a_i, b_i, u_j, v_j) for $i = 1, \dots, g$ and $j = 1, \dots, r$. The action of $(z_0, \dots, z_r) \in G^{r+1}$ is given by (see [AMM], Equation (38))

$$\begin{aligned} (6.3) \quad a_i &\longmapsto \text{Ad}_{z_0} a_i, \\ b_i &\longmapsto \text{Ad}_{z_0} b_i, \\ u_j &\longmapsto z_0 u_j z_j^{-1} \\ v_j &\longmapsto \text{Ad}_{z_j} v_j \end{aligned}$$

(for i, j in the ranges listed above). Here, we often use a to denote the tuple (a_1, \dots, a_g) (similarly for b). We also use u to denote (u_1, \dots, u_r) (similarly for v).

Our earlier notation M_g is an abbreviation for $M_{g,1}$.

The components of the moment map μ are (see [AMM], Equation (39))

$$(6.4) \quad \mu_j(a, b, u, v) = (v_j)^{-1} \quad (j = 1, \dots, r)$$

$$\mu_0(a, b, u, v) = \text{Ad}_{u_1}(v_1) \cdots \text{Ad}_{u_r}(v_r)[a_1, b_1] \cdots [a_g, b_g].$$

We point out to the reader that the capping constructions described in the previous section are special cases of the quotient of q -Hamiltonian spaces described in this section (see Remarks (6.4) and (6.5)).

For example:

- 1) §5.2: Case 2 corresponds to the q -Hamiltonian quotient of $M_{g,2} \times M_{0,2}$ by the diagonal action of $G \times G$.
- 2) §5.3: Case 3 corresponds to the q -Hamiltonian quotient of $M_{g,k} \times M_{0,k}$ by the diagonal action of G^k .
- 3) §5.4: Case 4 corresponds to the q -Hamiltonian quotient of $M_{g,1} \times M_{1,1}$ by the diagonal action of G .

- 4) §5.5: Case 5 corresponds to the q -Hamiltonian quotient of $M_{g,n} \times M_{1,n}$ by the diagonal action of G^n .
- 5) §5.6: Case 6 corresponds to the q -Hamiltonian quotient of $M_{g,n} \times M_{g,n}$ by the diagonal action of G^n .

6.2. Fusion

We continue to assume that G and H are compact, connected, simply connected Lie groups.

Theorem 6.2 ([AMM, Theorem 6.1]). *Let M be a q -Hamiltonian $(G \times G \times H)$ -space with moment map (μ_1, μ_2, μ_3) . Let (M, ω) be a q -Hamiltonian $(G \times G \times H)$ -space with moment map (μ_1, μ_2, μ_3) , so $\mu_1 : M \rightarrow G$ and $\mu_2 : M \rightarrow G$, while $\mu_3 : M \rightarrow H$. Let $G \times H$ act by the diagonal embedding $(y, z) \mapsto (y, y, z)$. Then M with 2-form*

$$\omega + (\mu_1^* \theta, \mu_2^* \bar{\theta})/2$$

and moment map $\tilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3) : M \rightarrow G \times H$ is a q -Hamiltonian $(G \times H)$ -space. Here \cdot denotes multiplication in G .

Internal fusion means replacing the $(G \times G \times H)$ -action on a q -Hamiltonian $G \times G \times H$ -space with a $(G \times H)$ -action. The space remains the same but the group that acts on it is different. The 2-form also changes: see §3.2 of [Me].

Example 6.3 ([AMM, Example 6.1]). Internal fusion turns the q -Hamiltonian $G \times G$ -space $D(G)$ into a q -Hamiltonian G -space denoted $\mathbf{D}(G)$.

The space $D(G) = G \times G$ has coordinates $(u, v) \in G \times G$. There may be r copies of the double which are indexed by variables $u_j, v_j, j = 1, \dots, r$.

The internal fusion $\mathbf{D}(G)$ is also $G \times G$ with coordinates $(a, b) (a, b \in G)$. There may be g copies of the internal fusion, which are denoted $(a_i, b_i) (i = 1, \dots, g)$. The space $D(G)$ is a q -Hamiltonian $(G \times G)$ -space with moment maps $(v_j^{-1}, \text{Ad}_{u_j} v_j)$, while $\mathbf{D}(G)$ is a G -space with moment map $[a_i, b_i]$.

A quasi-Poisson manifold is a special type of q -Hamiltonian space (see [AKM]). This is in fact the type of q -Hamiltonian space that we reduce to construct character varieties (see [AKM]). For relations between q -Hamiltonian spaces and Poisson geometry, we refer the reader to [AMM].

The fusion product of $(D(G))^r$ and $(\mathbf{D}(G))^g$ is G^{2g+2r} with the q -Hamiltonian action of G^{r+1} given by Equation (6.3) and moment maps given

by Equation (6.4). If we take the q -Hamiltonian quotient of this space with respect to the G^{r+1} action, we obtain a symplectic manifold. The 2-form ω on $D(G)^r \times (\mathbf{D}(G))^g$ restricts on the level set of moment maps to a form whose quotient under the action of G^{r+1} is a symplectic form.

Remark 6.4. For example, the q -Hamiltonian quotient of the product of the spaces M_g and M_h is the subset

$$\{(x, y) \in M_g \times M_h \mid \mu_g(x) = \mu_h(y)\}/G,$$

where G acts on $M_g \times M_h$ by conjugation and $\mu_g : M_g \rightarrow G$ and $\mu_h : M_h \rightarrow G$ are the respective q -Hamiltonian moment maps. Recall that M_g and M_h are equipped with q -Hamiltonian actions of G . The q -Hamiltonian quotient of $M_g \times M_h$ with respect to the diagonal action of G is the character variety of a surface of genus $g + h$. The 2-forms on M_g and M_h restrict to the level set

$$A := \{(x, y) \in M_g \times M_h \mid \mu_g(x) = \mu_h(y)\},$$

giving a 2-form which is the pullback of a symplectic form on the quotient of A by the G action. This procedure is the gluing procedure corresponding to gluing together the boundaries of two different surfaces, each with one boundary component.

Remark 6.5. To take the q -Hamiltonian quotient of the diagonal action of G on the j_1 -th and j_2 -th copies of G , we set $\mu_{j_1}(a, b, u, v) = \mu_{j_2}(a, b, u, v)$ and then take the quotient by the diagonal action of G on these copies of G . In other words, we require that $v_{j_1} = v_{j_2}$. This operation corresponds to gluing together the j_1 -th and j_2 -th boundary components of a connected surface with $r + 1$ boundary components. This procedure is the gluing procedure corresponding to gluing together two boundary components of a connected surface. Again, the 2-form on the level set

$$B := \{a, b, u, v \mid \mu_{j_1}(a, b, u, v) = \mu_{j_2}(a, b, u, v)\}$$

is the pullback of a symplectic form on the quotient of B by the diagonal G action.

6.3. Extended moduli spaces

For reference, see [Je] and also [GHJW, H1, H2]).

Let $G = \text{SU}(2)$. The extended moduli space [Je] for a genus g surface Σ_g with one boundary component was developed for the same purpose as the quasi-Hamiltonian G -space from [AMM]. It has a symplectic structure and a Hamiltonian action of G . Its symplectic quotient (at a specific coadjoint orbit of G) is the moduli space of parabolic bundles associated to that orbit. It has real dimension $6g$.

6.3.1. Extended moduli space for general G . Let G be a compact Lie group with maximal torus T .

We define

$$(6.5) \quad \mathcal{N}^{\mathfrak{g}} = \{(a_1, b_1, \dots, a_g, b_g) \in G^{2g}, \Lambda \in \mathfrak{g} \mid \prod_{j=1}^g [a_j, b_j] = \exp(\Lambda)\}.$$

The real dimension of this space is $2g \dim(G)$.

6.3.2. More than one boundary component. The generalization to r boundary components (where $r \geq 2$) is given in (5.6) of [Je]:

$$(6.6) \quad \mathcal{N}^{\mathfrak{g},r} = \{(a_1, \dots, a_g, b_1, \dots, b_g, k_2, \dots, k_r, \lambda_1, \dots, \lambda_r) \mid a_j, b_j, k_j \in G,$$

$$\lambda_i \in \mathfrak{g}, \prod_{i=1}^g [a_i, b_i] = e^{\lambda_1} k_2 e^{\lambda_2} k_2^{-1} \dots k_r e^{\lambda_r} k_r^{-1}\}.$$

There is a two-form defined on this space whose restriction to an open dense subset is symplectic. This subject may also be described in terms of connections on the 2-manifold rather than elements of products of G – see [Je]. The ($r \geq 2$)-boundary component case of the extended moduli space is also treated in [HJ].

6.4. Gauge theory version of extended moduli space

There is a description of the extended moduli space in terms of connections.

The space $\mathcal{A}_F^{\mathfrak{g}}$ parametrizes connections $A \in \Omega^1(\Sigma, \mathfrak{g})$ with curvature $F_A = 0$ and A having the form λds on a neighbourhood of the boundary (where $s \in [0, 2\pi]$ is a coordinate on the boundary and λ is a constant in \mathfrak{g}). Then $\mathcal{M}_g(\Sigma)$ consists of the quotient of $\mathcal{A}_F^{\mathfrak{g}}$ by the group of gauge transformations equal to the identity on a neighbourhood of the boundary. The space $\mathcal{M}_g(\Sigma)$ is homeomorphic to $\mathcal{N}^{\mathfrak{g}}$ (see [Je, Proposition 2.5]).

6.4.1. Symplectic form of extended moduli space. For the $SU(2)$ extended moduli space, the symplectic form is given in [Je], Section 3.1 (see also Definition 3.2). It is

$$\omega_\Sigma(a, b) = \int_\Sigma Tr(a \wedge b),$$

where $a, b \in \Omega^1(\Sigma) \otimes \mathfrak{g}$ are flat G -connections on Σ . For general G , the symplectic form of this space also has this form (see Section 5.2 of [Je]). This is a simple generalization of the symplectic form defined by Atiyah and Bott [AB].

Let $G = SU(2)$. By [Je], Proposition 3.1, the symplectic form on \mathcal{M}^g is nondegenerate provided $\Lambda = 0$ or $\Lambda \notin \pi\mathbb{Z}H$, where H is the diagonal 2×2 matrix with entries $(\sqrt{-1}, -\sqrt{-1})$. (a generator of the weight lattice of G).

6.4.2. Moment map for extended moduli space. The moment map for the extended moduli space \mathcal{N}^g described above (corresponding to a surface of genus g with one boundary component) is the map

$$((a_1, \dots, a_{2g}) \in G^{2g}, \Lambda \in \mathfrak{g}) \mapsto \Lambda.$$

6.5. Relation with Poisson geometry

As above, let G be a compact connected simply connected semisimple Lie group. According to §8 of [AMM], there is a bijective correspondence between q -Hamiltonian G -spaces and Hamiltonian LG -spaces. The key result is Theorem 8.3 in that paper. The following section of [AMM], §9, uses this machinery to exhibit spaces of flat connections on oriented 2-manifold as symplectic quotient of Hamiltonian LG -spaces, or equivalently q -Hamiltonian quotients of q -Hamiltonian spaces.

The extended moduli spaces of [Je] represent a choice of a gauge for the LG variables. Any \mathfrak{g} -valued 1-form α on S^1 is gauge equivalent to a 1-form of a particular type: $\gamma^*\alpha = \lambda ds$ where $\lambda \in \mathfrak{g}$ is a constant and $\gamma : S^1 \rightarrow G$ is a gauge transformation. By means of this construction, one no longer needs to use the infinite-dimensional group LG . The price one pays is the (non-canonical) choice of a gauge.

If one takes the symplectic quotient of such a Hamiltonian LG -space by $(LG)^{r+1}$, one recovers the space of gauge equivalence classes of flat connections on an oriented 2-manifold, with fixed values of the holonomy around each boundary component. This means one fixes a value in $L\mathfrak{g}$ (the Lie algebra of LG) for each of the $r + 1$ boundary components of the surface.

This value represents the holonomy of the connection around that boundary component. Taking the quotient by $(LG)^{r+1}$ gives a symplectic manifold.

If one takes the quotient by $(LG)^{r+1}$ but does not fix the holonomy for the boundary components, the result is a Poisson manifold. If one then fixes the holonomies around the $r + 1$ boundary components, this gives a restriction map from this Poisson manifold to one of its symplectic leaves.

7. Proof of Theorem 2.7

The coordinate ring $\mathbb{C}[\mathfrak{X}_{n,g}(\mathrm{SL}(2, \mathbb{C}))]$ is finitely generated by traces of curves in $\Sigma_{n,g}$. Since the Poisson bracket is a derivation and a Lie bracket, it is determined by the pairings of the generators of the coordinate ring.

When $G = \mathrm{SL}(n, \mathbb{C})$, Equation (2.1) simplifies to:

$$(7.1) \quad \{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\} \\ = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \left(\mathrm{tr}(\rho(\alpha_p \beta_p)) - \frac{1}{n} \mathrm{tr}(\rho(\alpha)) \mathrm{tr}(\rho(\beta)) \right).$$

As is apparent from this formula, to compute the requisite pairings, one need only draw the required curves and compute the traces of the resulting words. The algorithm in [ABL] that computes traces of words in $\mathbb{C}[\mathfrak{X}_{n,g}(\mathrm{SL}(2, \mathbb{C}))]$ is effective. Therefore, the algorithm to compute the Poisson bracket is likewise effective.

We now demonstrate this algorithm with a new non-trivial, but tractable, example. In principle, many other such examples could be computed with a fully automated implementation of our algorithm. However, we do this computation by hand with the aid of a compute program implementing the algorithm in [ABL], and verify the computation is correct with *Mathematica*.

There are three open surfaces with Euler characteristic -3 : the 5-holed sphere, the 3-holed torus, and the 1-holed genus 2 surface. In all three cases, the fundamental group is a free group $F_4 = \langle c_1, c_2, c_3, c_4 \rangle$ of rank 4. Let $t_{\{i\}} = \mathrm{tr}_{c_i}$, $t_{\{i,j\}} = \mathrm{tr}_{c_i c_j}$, and $t_{\{i,j,k\}} = \mathrm{tr}_{c_i c_j c_k}$.

Then, the coordinate ring of $\mathrm{Hom}(F_4, \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$ is generated by the 14 trace functions

$$\{t_{\{1\}}, t_{\{2\}}, t_{\{3\}}, t_{\{4\}}, t_{\{1,2\}}, t_{\{1,3\}}, \\ t_{\{1,4\}}, t_{\{2,3\}}, t_{\{2,4\}}, t_{\{3,4\}}, t_{\{1,2,3\}}, t_{\{1,2,4\}}, t_{\{1,3,4\}}, t_{\{2,3,4\}}\},$$

and its ideal of relations is generated by 14 polynomials in these variables (see [ABL] for details).

To compute the bi-vector for the 5-holed sphere, we need to compute the $14 \times 14 = 196$ pairings of the form $\{t_x, t_y\}$. Since Poisson brackets are anti-symmetric and $t_{\{1\}}, t_{\{2\}}, t_{\{3\}}, t_{\{4\}}$ are Casimirs, we are left with 45 pairings. These come in three types: (a) 15 of $\{t_{\{i,j\}}, t_{\{k,l\}}\}$, (b) 24 of $\{t_{\{i,j\}}, t_{\{k,l,m\}}\}$, and (c) 6 of $\{t_{\{i,j,k\}}, t_{\{l,m,n\}}\}$.

The topological model of $\Sigma_{5,0}$ we will use is in Figure 1, and we will use Equation (7.1) in the following subsections without explicit mention.

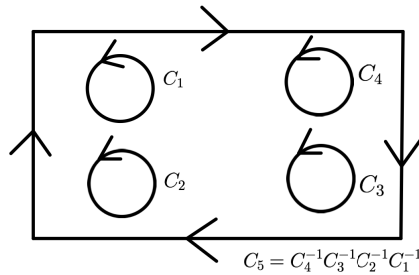


Figure 1: $\Sigma_{5,0}$ with $c_1c_2c_3c_4c_5 = 1$.

7.0.1. Type (a) pairings. There are 15 pairings of type $\{t_{\{i,j\}}, t_{\{k,l\}}\}$.

Since c_1c_2 and c_3c_4 are disjoint, $\{t_{\{1,2\}}, t_{\{3,4\}}\} = 0$. Likewise, $\{t_{\{1,4\}}, t_{\{2,3\}}\} = 0$.

The first non-trivial¹ computation will be for $\{t_{\{1,3\}}, t_{\{2,4\}}\}$. We draw the curves c_1c_3 and c_2c_4 in the 5-holed sphere (see Figure 2). Then we simplify the trace functions in the formula using the algorithm in [ABL] implemented in *Mathematica*.

¹One might be tempted to thinking that these curves can be drawn disjoint by drawing c_2c_4 “going out around” c_3 , but the resulting curve would be homotopic to $c_2c_3c_4c_3^{-1}$ which is *not* homotopic to c_2c_4 .

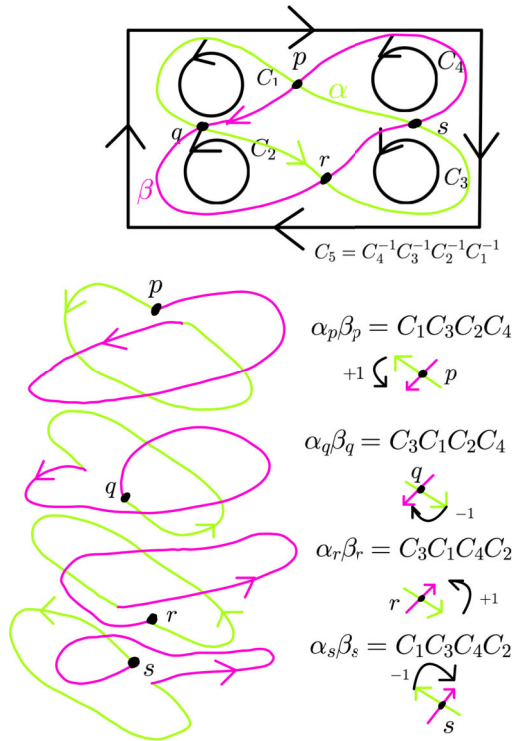


Figure 2: $\{t_{\{1,3\}}, t_{\{2,4\}}\}$.

$$\begin{aligned}
 \{t_{\{1,3\}}, t_{\{2,4\}}\} &= (+1)(\text{tr}(c_1 c_3 c_2 c_4) - \frac{1}{2} \text{tr}(c_1 c_3) \text{tr}(c_2 c_4)) \\
 &\quad + (-1)(\text{tr}(c_3 c_1 c_2 c_4) - \frac{1}{2} \text{tr}(c_1 c_3) \text{tr}(c_2 c_4)) \\
 &\quad + (+1)(\text{tr}(c_3 c_1 c_4 c_2) - \frac{1}{2} \text{tr}(c_1 c_3) \text{tr}(c_2 c_4)) \\
 &\quad + (-1)(\text{tr}(c_1 c_3 c_4 c_2) - \frac{1}{2} \text{tr}(c_1 c_3) \text{tr}(c_2 c_4)) \\
 &= t_{\{3\}} t_{\{4\}} t_{\{1,2\}} - 2t_{\{3,4\}} t_{\{1,2\}} - t_{\{2\}} t_{\{3\}} t_{\{1,4\}} - t_{\{1\}} t_{\{4\}} t_{\{2,3\}} \\
 &\quad + 2t_{\{1,4\}} t_{\{2,3\}} + t_{\{1\}} t_{\{2\}} t_{\{3,4\}}.
 \end{aligned}$$

The next non-trivial computation will be for $\{t_{\{1,2\}}, t_{\{1,4\}}\}$. We draw the curves $c_1 c_2$ and $c_1 c_4$ in Figure 3.

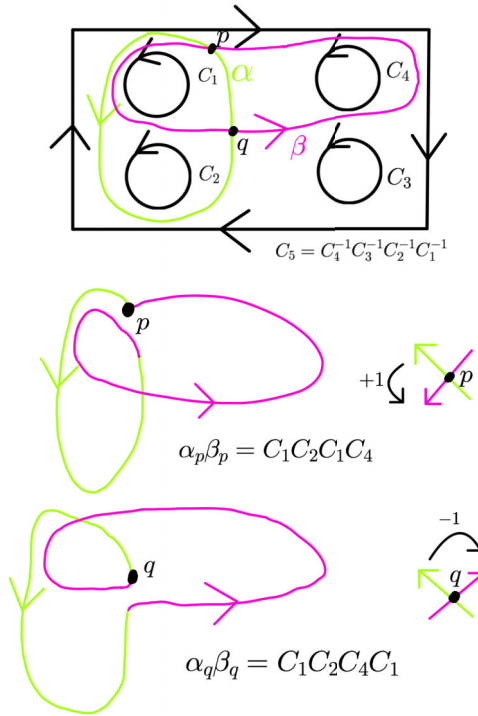


Figure 3: $\{t_{\{1,2\}}, t_{\{1,4\}}\}$.

$$\begin{aligned} \{t_{\{1,2\}}, t_{\{1,4\}}\} &= (+1)(\text{tr}(c_1 c_2 c_1 c_4) - \frac{1}{2} \text{tr}(c_1 c_2) \text{tr}(c_1 c_4)) \\ &\quad + (-1)(\text{tr}(c_1 c_2 c_4 c_1) - \frac{1}{2} \text{tr}(c_1 c_2) \text{tr}(c_1 c_4)) \\ &= t_{\{1,2\}} t_{\{1,4\}} + 2t_{\{2,4\}} - t_{\{1\}} t_{\{1,2,4\}} - t_{\{2\}} t_{\{4\}}. \end{aligned}$$

Likewise, by permuting the indices 3 and 4, we have:

$$\{t_{\{1,2\}}, t_{\{1,3\}}\} = t_{\{1,2\}} t_{\{1,3\}} + 2t_{\{2,3\}} - t_{\{1\}} t_{\{1,2,3\}} - t_{\{2\}} t_{\{3\}},$$

and by permuting 2 and 3, we have:

$$\{t_{\{1,3\}}, t_{\{1,4\}}\} = t_{\{1,3\}} t_{\{1,4\}} + 2t_{\{3,4\}} - t_{\{1\}} t_{\{1,3,4\}} - t_{\{3\}} t_{\{4\}}.$$

We next draw the curves c_1c_2 and c_2c_3 in Figure 4 and compute the resulting bracket.

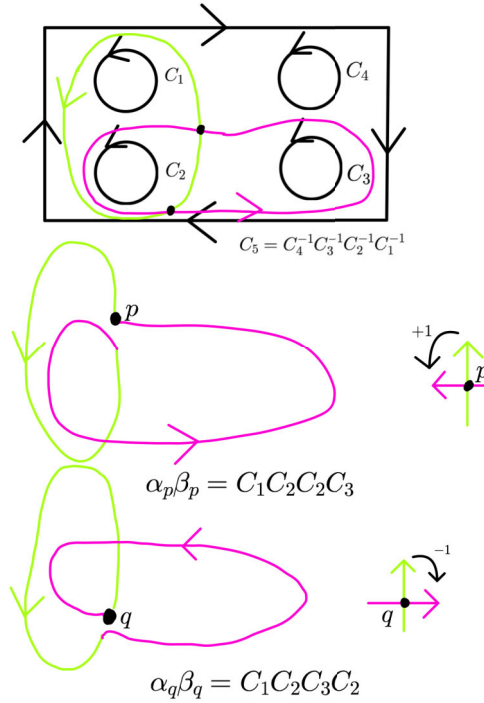


Figure 4: $\{t_{\{1,2\}}, t_{\{2,3\}}\}$.

$$\begin{aligned} \{t_{\{1,2\}}, t_{\{2,3\}}\} &= (+1)(\text{tr}(c_1c_2c_2c_3) - \frac{1}{2}\text{tr}(c_1c_2)\text{tr}(c_2c_3)) \\ &\quad + (-1)(\text{tr}(c_1c_2c_3c_2) - \frac{1}{2}\text{tr}(c_1c_2)\text{tr}(c_2c_3)) \\ &= -2t_{\{1,3\}} - t_{\{1,2\}}t_{\{2,3\}} + t_{\{2\}}t_{\{1,2,3\}} + t_{\{1\}}t_{\{3\}}. \end{aligned}$$

Again, permuting the indices 3 and 4 we obtain

$$\{t_{\{1,2\}}, t_{\{2,4\}}\} = -2t_{\{1,4\}} - t_{\{1,2\}}t_{\{2,4\}} + t_{\{2\}}t_{\{1,2,4\}} + t_{\{1\}}t_{\{4\}},$$

and by permuting 1 and 4 we obtain

$$\{t_{\{2,4\}}, t_{\{2,3\}}\} = -t_{\{2,3\}}t_{\{2,4\}} - 2t_{\{3,4\}} + t_{\{2\}}t_{\{2,3,4\}} + t_{\{3\}}t_{\{4\}}.$$

We next draw the curves c_1c_4 and c_3c_4 in Figure 5 and compute the resulting bracket.

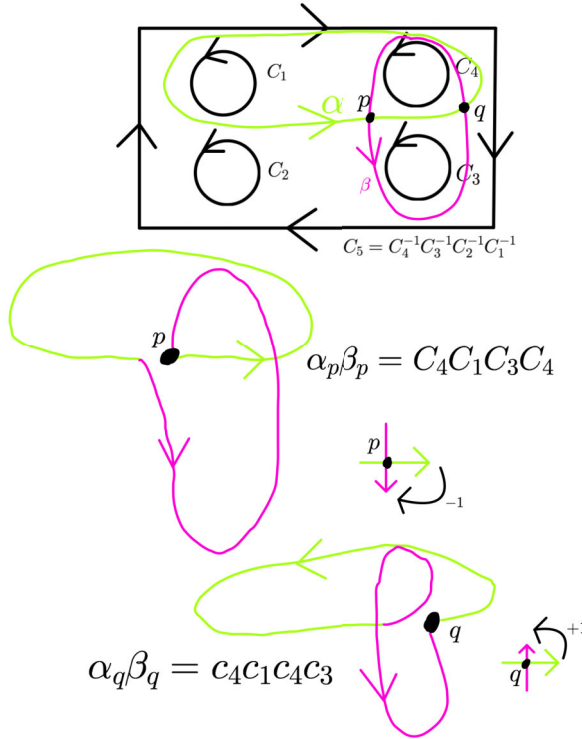


Figure 5: $\{t_{\{1,4\}}, t_{\{3,4\}}\}$.

$$\begin{aligned} \{t_{\{1,4\}}, t_{\{3,4\}}\} &= (-1)(\text{tr}(c_4c_1c_3c_4) - \frac{1}{2}\text{tr}(c_1c_4)\text{tr}(c_3c_4)) \\ &\quad + (+1)(\text{tr}(c_4c_1c_4c_3) - \frac{1}{2}\text{tr}(c_1c_4)\text{tr}(c_3c_4)) \\ &= 2t_{\{1,3\}} + t_{\{1,4\}}t_{\{3,4\}} - t_{\{4\}}t_{\{1,3,4\}} - t_{\{1\}}t_{\{3\}}. \end{aligned}$$

Permuting the indices 1 and 2 we obtain

$$\{t_{\{2,4\}}, t_{\{3,4\}}\} = 2t_{\{2,3\}} + t_{\{2,4\}}t_{\{3,4\}} - t_{\{4\}}t_{\{2,3,4\}} - t_{\{2\}}t_{\{3\}},$$

and by permuting 2 and 3 we obtain

$$\{t_{\{1,4\}}, t_{\{2,4\}}\} = 2t_{\{1,2\}} + t_{\{1,4\}}t_{\{2,4\}} - t_{\{4\}}t_{\{1,2,4\}} - t_{\{1\}}t_{\{2\}}.$$

We next draw the curves c_2c_3 and c_3c_4 in Figure 6 and compute the resulting bracket.

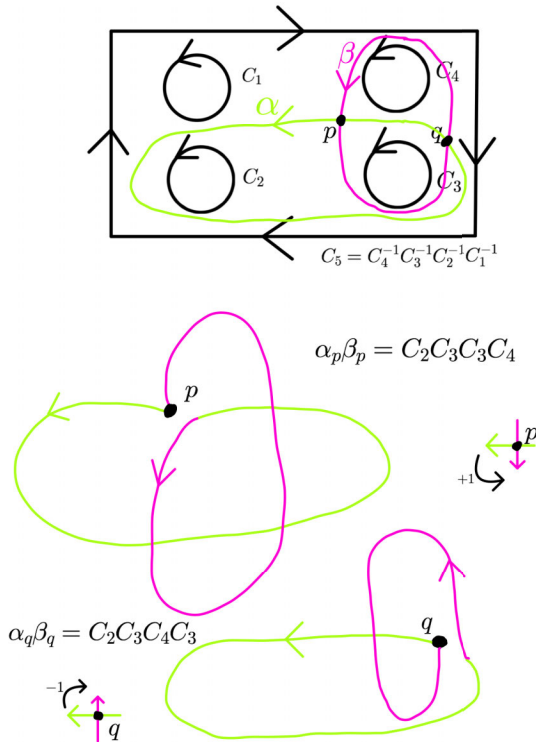


Figure 6: $\{t_{\{2,3\}}, t_{\{3,4\}}\}$.

$$\begin{aligned} \{t_{\{2,3\}}, t_{\{3,4\}}\} &= (+1)(\text{tr}(c_2c_3c_3c_4) - \frac{1}{2}\text{tr}(c_2c_3)\text{tr}(c_3c_4)) \\ &\quad + (-1)(\text{tr}(c_2c_3c_4c_3) - \frac{1}{2}\text{tr}(c_2c_3)\text{tr}(c_3c_4)) \\ &= -2t_{\{2,4\}} - t_{\{2,3\}}t_{\{3,4\}} + t_{\{3\}}t_{\{2,3,4\}} + t_{\{2\}}t_{\{4\}}. \end{aligned}$$

Permuting the indices 1 and 2 we obtain

$$\{t_{\{1,3\}}, t_{\{3,4\}}\} = -2t_{\{1,4\}} - t_{\{1,3\}}t_{\{3,4\}} + t_{\{3\}}t_{\{1,3,4\}} + t_{\{1\}}t_{\{4\}},$$

and by permuting 1 and 4 we obtain

$$\{t_{\{2,3\}}, t_{\{1,3\}}\} = -2t_{\{1,2\}} - t_{\{1,3\}}t_{\{2,3\}} + t_{\{3\}}t_{\{1,2,3\}} + t_{\{1\}}t_{\{2\}}.$$

7.0.2. Type (b) pairings. There are 24 pairings of type $\{t_{\{i,j\}}, t_{\{k,l,m\}}\}$.

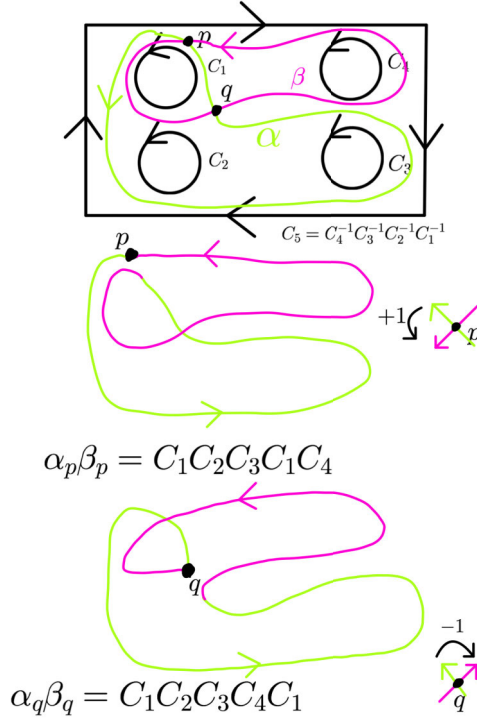


Figure 7: $\{t_{\{1,2,3\}}, t_{\{1,4\}}\}$.

$$\begin{aligned} \{t_{\{1,2,3\}}, t_{\{1,4\}}\} &= (+1)(\text{tr}(c_1 c_2 c_3 c_1 c_4) - \frac{1}{2}\text{tr}(c_1 c_2 c_3)\text{tr}(c_1 c_4)) \\ &\quad + (-1)(\text{tr}(c_1 c_2 c_3 c_4 c_1) - \frac{1}{2}\text{tr}(c_1 c_2 c_3)\text{tr}(c_1 c_4)) \\ &= \frac{1}{2}t_{\{4\}}t_{\{1\}}^2 t_{\{2,3\}} + \frac{1}{2}t_{\{2\}}t_{\{1\}}^2 t_{\{3,4\}} - \frac{1}{2}t_{\{1\}}^2 t_{\{2,3,4\}} \\ &\quad + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1\}}t_{\{1,2\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1\}}t_{\{1,4\}} - \frac{1}{2}t_{\{1\}}t_{\{1,4\}}t_{\{2,3\}} \\ &\quad + \frac{1}{2}t_{\{1\}}t_{\{1,3\}}t_{\{2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{1,2\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{1\}}t_{\{1,2,3\}} \\ &\quad - \frac{1}{2}t_{\{3\}}t_{\{1\}}t_{\{1,2,4\}} - \frac{1}{2}t_{\{2\}}t_{\{1\}}t_{\{1,3,4\}} \\ &\quad - t_{\{4\}}t_{\{2,3\}} + t_{\{1,4\}}t_{\{1,2,3\}} + 2t_{\{2,3,4\}} - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1\}}^2. \end{aligned}$$

One finds similar diagrams for $\{t_{\{1,2,3\}}, t_{\{2,4\}}\}$, and $\{t_{\{1,2,3\}}, t_{\{3,4\}}\}$.
 Indeed, we have:

$$\begin{aligned} \{t_{\{1,2,3\}}, t_{\{2,4\}}\} &= (+1)(\text{tr}(c_1c_2c_3c_2c_4) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_2c_4)) \\ &\quad + (-1)(\text{tr}(c_1c_2c_3c_4c_2) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_2c_4)) \\ &= -t_{\{3\}}t_{\{1,4\}} + t_{\{1\}}t_{\{3,4\}} + t_{\{2,3\}}t_{\{1,2,4\}} - t_{\{1,2\}}t_{\{2,3,4\}}, \end{aligned}$$

and

$$\begin{aligned} \{t_{\{1,2,3\}}, t_{\{3,4\}}\} &= (+1)(\text{tr}(c_1c_2c_3c_3c_4) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_3c_4)) \\ &\quad + (-1)(\text{tr}(c_1c_2c_3c_4c_3) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_3c_4)) \\ &= -\frac{1}{2}t_{\{4\}}t_{\{3\}}^2t_{\{1,2\}} - \frac{1}{2}t_{\{2\}}t_{\{3\}}^2t_{\{1,4\}} + \frac{1}{2}t_{\{3\}}^2t_{\{1,2,4\}} \\ &\quad - \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{3\}}t_{\{2,3\}} + \frac{1}{2}t_{\{3\}}t_{\{1,4\}}t_{\{2,3\}} - \frac{1}{2}t_{\{3\}}t_{\{1,3\}}t_{\{2,4\}} \\ &\quad - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{3,4\}} + \frac{1}{2}t_{\{3\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{4\}}t_{\{3\}}t_{\{1,2,3\}} \\ &\quad + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,3,4\}} + \frac{1}{2}t_{\{1\}}t_{\{3\}}t_{\{2,3,4\}} + t_{\{4\}}t_{\{1,2\}} \\ &\quad - t_{\{3,4\}}t_{\{1,2,3\}} - 2t_{\{1,2,4\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{4\}}t_{\{3\}}^2. \end{aligned}$$

Since the curves are disjoint (Figure 8), we have: $\{t_{\{1,2,3\}}, t_{\{1,2\}}\} = 0$,
 $\{t_{\{1,2,3\}}, t_{\{1,3\}}\} = 0$ and $\{t_{\{1,2,3\}}, t_{\{2,3\}}\} = 0$.

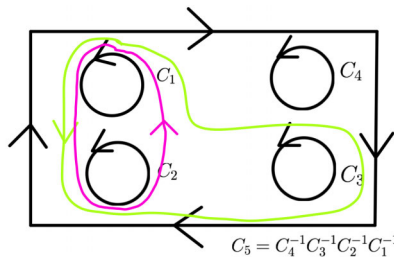


Figure 8: $\{t_{\{1,2,3\}}, t_{\{1,2\}}\} = 0$.

So we see, that for each diagram like Figure 7, which there are 4, we obtain the data for 6 pairings.

Now on to the next diagram (Figure 9).

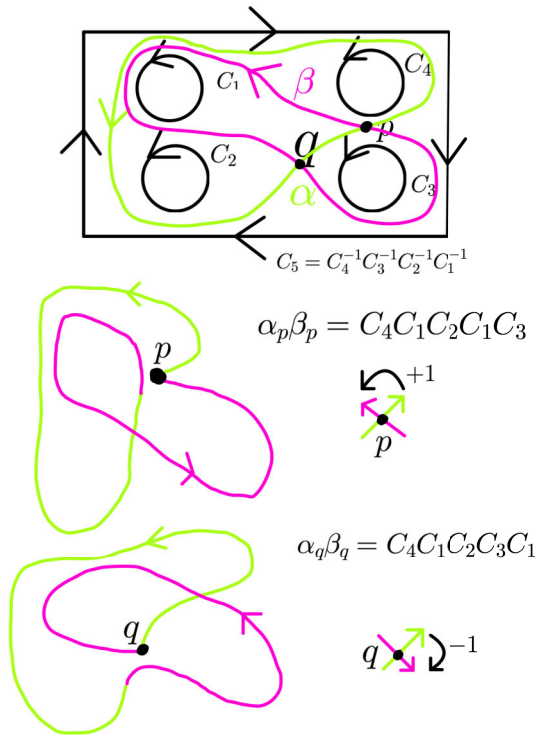


Figure 9: $\{t_{\{1,2,4\}}, t_{\{1,3\}}\}$.

$$\begin{aligned} \{t_{\{1,2,4\}}, t_{\{1,3\}}\} &= (+1)(\text{tr}(c_4 c_1 c_2 c_1 c_3) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_1 c_3)) \\ &\quad + (-1)(\text{tr}(c_4 c_1 c_2 c_3 c_1) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_1 c_3)) \\ &= t_{\{4\}} t_{\{2,3\}} - t_{\{2\}} t_{\{3,4\}} - t_{\{1,4\}} t_{\{1,2,3\}} + t_{\{1,2\}} t_{\{1,3,4\}}. \end{aligned}$$

Again, one finds similar diagrams for $\{t_{\{1,2,4\}}, t_{\{2,3\}}\}$, and $\{t_{\{1,2,4\}}, t_{\{3,4\}}\}$.

Indeed, we have:

$$\begin{aligned} \{t_{\{1,2,4\}}, t_{\{2,3\}}\} &= (+1)(\text{tr}(c_4 c_1 c_2 c_2 c_3) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_2 c_3)) \\ &\quad + (-1)(\text{tr}(c_4 c_1 c_2 c_3 c_2) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_2 c_3)) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}t_{\{3\}}t_{\{2\}}^2t_{\{1,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}^2t_{\{3,4\}} + \frac{1}{2}t_{\{2\}}^2t_{\{1,3,4\}} \\
 &\quad - \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{2\}}t_{\{1,2\}} - \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2\}}t_{\{2,3\}} + \frac{1}{2}t_{\{2\}}t_{\{1,4\}}t_{\{2,3\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{1,3\}}t_{\{2,4\}} + \frac{1}{2}t_{\{2\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{4\}}t_{\{2\}}t_{\{1,2,3\}} \\
 &\quad + \frac{1}{2}t_{\{3\}}t_{\{2\}}t_{\{1,2,4\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{2,3,4\}} + t_{\{3\}}t_{\{1,4\}} \\
 &\quad - t_{\{2,3\}}t_{\{1,2,4\}} - 2t_{\{1,3,4\}} + \frac{1}{2}t_{\{1\}}t_{\{3\}}t_{\{4\}}t_{\{2\}}^2,
 \end{aligned}$$

and

$$\begin{aligned}
 \{t_{\{1,2,4\}}, t_{\{3,4\}}\} &= (+1)(\text{tr}(c_4c_1c_2c_4c_3) - \frac{1}{2}\text{tr}(c_1c_2c_4)\text{tr}(c_3c_4)) \\
 &\quad + (-1)(\text{tr}(c_4c_1c_2c_3c_4) - \frac{1}{2}\text{tr}(c_1c_2c_4)\text{tr}(c_3c_4)) \\
 &= \frac{1}{2}t_{\{3\}}t_{\{4\}}^2t_{\{1,2\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}^2t_{\{2,3\}} - \frac{1}{2}t_{\{4\}}^2t_{\{1,2,3\}} \\
 &\quad + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,4\}} - \frac{1}{2}t_{\{4\}}t_{\{1,4\}}t_{\{2,3\}} + \frac{1}{2}t_{\{4\}}t_{\{1,3\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{4\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2\}}t_{\{3,4\}} - \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2,4\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{4\}}t_{\{1,3,4\}} - \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3,4\}} - t_{\{3\}}t_{\{1,2\}} \\
 &\quad + 2t_{\{1,2,3\}} + t_{\{3,4\}}t_{\{1,2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}^2.
 \end{aligned}$$

Since the curves are disjoint, we have: $\{t_{\{1,2,4\}}, t_{\{1,2\}}\} = 0$, $\{t_{\{1,2,4\}}, t_{\{1,4\}}\} = 0$ and $\{t_{\{1,2,4\}}, t_{\{2,4\}}\} = 0$.

Next we have Figure 10.

$$\begin{aligned}
 \{t_{\{1,3,4\}}, t_{\{2,3\}}\} &= (-1)(\text{tr}(c_3c_4c_1c_2c_3) - \frac{1}{2}\text{tr}(c_1c_3c_4)\text{tr}(c_2c_3)) \\
 &\quad + (+1)(\text{tr}(c_3c_4c_1c_3c_2) - \frac{1}{2}\text{tr}(c_1c_3c_4)\text{tr}(c_2c_3)) \\
 &= \frac{1}{2}t_{\{4\}}t_{\{3\}}^2t_{\{1,2\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}^2t_{\{1,4\}} - \frac{1}{2}t_{\{3\}}^2t_{\{1,2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{3\}}t_{\{2,3\}} - \frac{1}{2}t_{\{3\}}t_{\{1,4\}}t_{\{2,3\}} + \frac{1}{2}t_{\{3\}}t_{\{1,3\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{3,4\}} - \frac{1}{2}t_{\{3\}}t_{\{1,2\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{3\}}t_{\{1,2,3\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,3,4\}} - \frac{1}{2}t_{\{1\}}t_{\{3\}}t_{\{2,3,4\}} - t_{\{2\}}t_{\{1,4\}} \\
 &\quad + 2t_{\{1,2,4\}} + t_{\{2,3\}}t_{\{1,3,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{4\}}t_{\{3\}}^2.
 \end{aligned}$$

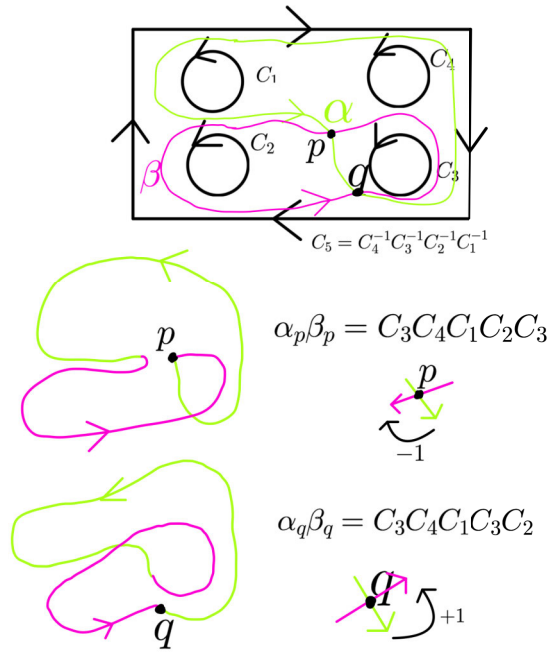


Figure 10: $\{t_{\{1,3,4\}}, t_{\{2,3\}}\}$.

Again, one finds similar diagrams for $\{t_{\{1,3,4\}}, t_{\{2,4\}}\}$ and $\{t_{\{1,3,4\}}, t_{\{1,2\}}\}$.

Indeed, we have:

$$\begin{aligned} \{t_{\{1,3,4\}}, t_{\{2,4\}}\} &= (-1)(\text{tr}(c_3 c_4 c_1 c_2 c_4) - \frac{1}{2} \text{tr}(c_1 c_3 c_4) \text{tr}(c_2 c_4)) \\ &\quad + (+1)(\text{tr}(c_3 c_4 c_1 c_4 c_2) - \frac{1}{2} \text{tr}(c_1 c_3 c_4) \text{tr}(c_2 c_4)) \\ &= t_{\{3\}} t_{\{1,2\}} - t_{\{1\}} t_{\{2,3\}} - t_{\{3,4\}} t_{\{1,2,4\}} + t_{\{1,4\}} t_{\{2,3,4\}}, \end{aligned}$$

and

$$\begin{aligned} \{t_{\{1,3,4\}}, t_{\{1,2\}}\} &= (-1)(\text{tr}(c_3 c_4 c_1 c_2 c_1) - \frac{1}{2} \text{tr}(c_1 c_3 c_4) \text{tr}(c_1 c_2)) \\ &\quad + (+1)(\text{tr}(c_3 c_4 c_1 c_1 c_2) - \frac{1}{2} \text{tr}(c_1 c_3 c_4) \text{tr}(c_1 c_2)) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}t_{\{4\}}t_{\{1\}}^2t_{\{2,3\}} - \frac{1}{2}t_{\{2\}}t_{\{1\}}^2t_{\{3,4\}} + \frac{1}{2}t_{\{1\}}^2t_{\{2,3,4\}} - \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1\}}t_{\{1,2\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1\}}t_{\{1,4\}} + \frac{1}{2}t_{\{1\}}t_{\{1,4\}}t_{\{2,3\}} - \frac{1}{2}t_{\{1\}}t_{\{1,3\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{4\}}t_{\{1\}}t_{\{1,2,3\}} + \frac{1}{2}t_{\{3\}}t_{\{1\}}t_{\{1,2,4\}} \\
 &\quad + \frac{1}{2}t_{\{2\}}t_{\{1\}}t_{\{1,3,4\}} + t_{\{2\}}t_{\{3,4\}} - t_{\{1,2\}}t_{\{1,3,4\}} \\
 &\quad - 2t_{\{2,3,4\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1\}}^2.
 \end{aligned}$$

Since the curves are disjoint, we have: $\{t_{\{1,3,4\}}, t_{\{1,4\}}\} = 0$, $\{t_{\{1,3,4\}}, t_{\{1,3\}}\} = 0$ and $\{t_{\{1,3,4\}}, t_{\{3,4\}}\} = 0$.

Figure 11 depicts the final diagram for this type of pairing.

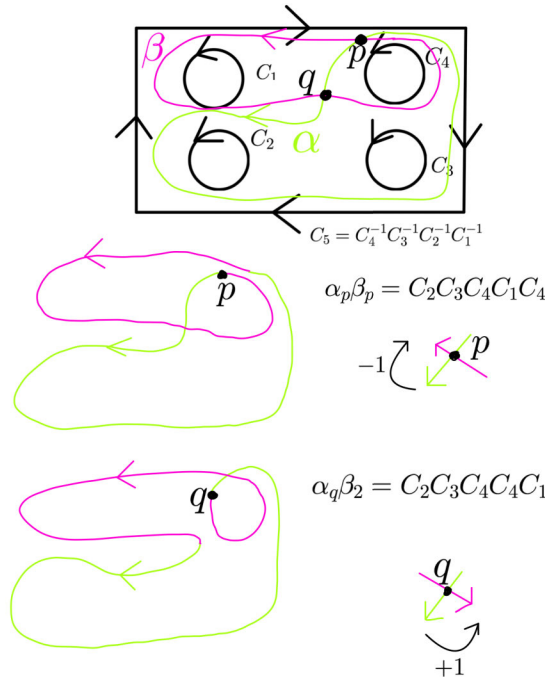


Figure 11: $\{t_{\{2,3,4\}}, t_{\{1,4\}}\}$.

$$\begin{aligned}
 \{t_{\{2,3,4\}}, t_{\{1,4\}}\} &= (-1)(\text{tr}(c_2c_3c_4c_1c_4) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_4)) \\
 &\quad + (+1)(\text{tr}(c_2c_3c_4c_4c_1) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_4)) \\
 &= -\frac{1}{2}t_{\{3\}}t_{\{4\}}^2t_{\{1,2\}} - \frac{1}{2}t_{\{1\}}t_{\{4\}}^2t_{\{2,3\}} + \frac{1}{2}t_{\{4\}}^2t_{\{1,2,3\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,4\}} + \frac{1}{2}t_{\{4\}}t_{\{1,4\}}t_{\{2,3\}} - \frac{1}{2}t_{\{4\}}t_{\{1,3\}}t_{\{2,4\}} \\
 &\quad - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{4\}}t_{\{3,4\}} + \frac{1}{2}t_{\{4\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2,4\}} \\
 &\quad + \frac{1}{2}t_{\{2\}}t_{\{4\}}t_{\{1,3,4\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3,4\}} + t_{\{1\}}t_{\{2,3\}} \\
 &\quad - 2t_{\{1,2,3\}} - t_{\{1,4\}}t_{\{2,3,4\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}^2
 \end{aligned}$$

Again, one finds similar diagrams for $\{t_{\{2,3,4\}}, t_{\{1,3\}}\}$, and $\{t_{\{2,3,4\}}, t_{\{1,2\}}\}$.

Indeed, we have:

$$\begin{aligned}
 \{t_{\{2,3,4\}}, t_{\{1,3\}}\} &= (-1)(\text{tr}(c_2c_3c_4c_1c_3) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_3)) \\
 &\quad + (+1)(\text{tr}(c_2c_3c_4c_3c_1) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_3)) \\
 &= -t_{\{4\}}t_{\{1,2\}} + t_{\{2\}}t_{\{1,4\}} + t_{\{3,4\}}t_{\{1,2,3\}} - t_{\{2,3\}}t_{\{1,3,4\}},
 \end{aligned}$$

and

$$\begin{aligned}
 \{t_{\{2,3,4\}}, t_{\{1,2\}}\} &= (-1)(\text{tr}(c_2c_3c_4c_1c_2) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_2)) \\
 &\quad + (+1)(\text{tr}(c_2c_3c_4c_2c_1) - \frac{1}{2}\text{tr}(c_2c_3c_4)\text{tr}(c_1c_2)) \\
 &= \frac{1}{2}t_{\{3\}}t_{\{2\}}^2t_{\{1,4\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}^2t_{\{3,4\}} - \frac{1}{2}t_{\{2\}}^2t_{\{1,3,4\}} \\
 &\quad + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{2\}}t_{\{1,2\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2\}}t_{\{2,3\}} - \frac{1}{2}t_{\{2\}}t_{\{1,4\}}t_{\{2,3\}} \\
 &\quad + \frac{1}{2}t_{\{2\}}t_{\{1,3\}}t_{\{2,4\}} - \frac{1}{2}t_{\{2\}}t_{\{1,2\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{2\}}t_{\{1,2,3\}} \\
 &\quad - \frac{1}{2}t_{\{3\}}t_{\{2\}}t_{\{1,2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{2,3,4\}} - t_{\{1\}}t_{\{3,4\}} \\
 &\quad + 2t_{\{1,3,4\}} + t_{\{1,2\}}t_{\{2,3,4\}} - \frac{1}{2}t_{\{1\}}t_{\{3\}}t_{\{4\}}t_{\{2\}}^2.
 \end{aligned}$$

Since the curves are disjoint, we have: $\{t_{\{2,3,4\}}, t_{\{3,4\}}\} = 0$, $\{t_{\{2,3,4\}}, t_{\{2,3\}}\} = 0$ and $\{t_{\{2,3,4\}}, t_{\{2,4\}}\} = 0$.

7.0.3. Type (c) pairings. There are 6 pairings of type $\{t_{\{i,j,k\}}, t_{\{l,m,n\}}\}$. Figure 12 is the first one of this type.

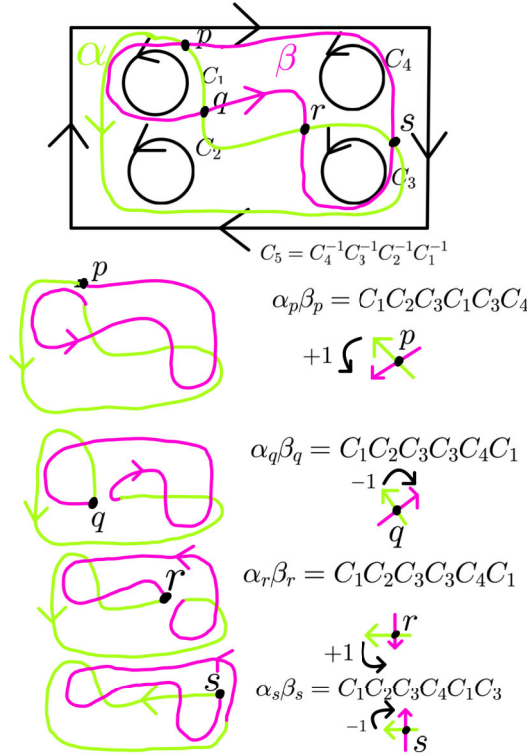


Figure 12: $\{t_{\{1,2,3\}}, t_{\{1,3,4\}}\}$.

$$\begin{aligned}
 \{t_{\{1,2,3\}}, t_{\{1,3,4\}}\} &= (+1)(\text{tr}(c_1 c_2 c_3 c_1 c_3 c_4) - \frac{1}{2} \text{tr}(c_1 c_2 c_3) \text{tr}(c_1 c_3 c_4)) \\
 &+ (-1)(\text{tr}(c_1 c_2 c_3 c_3 c_4 c_1) - \frac{1}{2} \text{tr}(c_1 c_2 c_3) \text{tr}(c_1 c_3 c_4)) \\
 &+ (+1)(\text{tr}(c_1 c_2 c_3 c_3 c_4 c_1) - \frac{1}{2} \text{tr}(c_1 c_2 c_3) \text{tr}(c_1 c_3 c_4)) \\
 &+ (-1)(\text{tr}(c_1 c_2 c_3 c_4 c_1 c_3) - \frac{1}{2} \text{tr}(c_1 c_2 c_3) \text{tr}(c_1 c_3 c_4))
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}t_{\{2,4\}}t_{\{1,3\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,3\}} + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}t_{\{1,3\}} \\
 &+ \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{1,3\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{1,3\}} \\
 &+ \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1,2\}}t_{\{3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{1,3\}} \\
 &- \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{1,3\}} - 2t_{\{2,4\}} \\
 &- 2t_{\{2,3\}}t_{\{3,4\}} + t_{\{1,2,3\}}t_{\{1,3,4\}} + 2t_{\{3\}}t_{\{2,3,4\}} + t_{\{2\}}t_{\{4\}}.
 \end{aligned}$$

Similarly, we have:

$$\begin{aligned}
 \{t_{\{1,2,3\}}, t_{\{1,2,4\}}\} &= (+1)(\text{tr}(c_1c_2c_3c_1c_2c_4) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_1c_2c_4)) \\
 &+ (-1)(\text{tr}(c_1c_2c_3c_4c_1c_2) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_1c_2c_4)) \\
 &= \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}^2 - \frac{1}{2}t_{\{3,4\}}t_{\{1,2\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,2\}} \\
 &+ \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{1,2\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{1,2\}} \\
 &- \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{1,2\}} + \frac{1}{2}t_{\{1,3\}}t_{\{2,4\}}t_{\{1,2\}} \\
 &+ \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{1,2\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{1,2\}} \\
 &- \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{1,2\}} - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{1,2\}} \\
 &- \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{1,2\}} + 2t_{\{3,4\}} + t_{\{1,2,3\}}t_{\{1,2,4\}} - t_{\{3\}}t_{\{4\}},
 \end{aligned}$$

and

$$\begin{aligned}
 \{t_{\{1,2,3\}}, t_{\{2,3,4\}}\} &= (+1)(\text{tr}(c_1c_2c_3c_2c_3c_4) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_2c_3c_4)) \\
 &+ (-1)(\text{tr}(c_1c_2c_3c_4c_2c_3) - \frac{1}{2}\text{tr}(c_1c_2c_3)\text{tr}(c_2c_3c_4)) \\
 &= -\frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}^2 + \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}^2 + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{2,3\}} \\
 &- \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}t_{\{2,3\}} - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{2,3\}} \\
 &- \frac{1}{2}t_{\{1,3\}}t_{\{2,4\}}t_{\{2,3\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{2,3\}} \\
 &+ \frac{1}{2}t_{\{1,2\}}t_{\{3,4\}}t_{\{2,3\}} + \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{2,3\}} + \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{2,3\}} \\
 &+ \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{2,3\}} + \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{2,3\}} - 2t_{\{1,4\}} \\
 &- t_{\{1,2,3\}}t_{\{2,3,4\}} + t_{\{1\}}t_{\{4\}}.
 \end{aligned}$$

Next, and very similarly, we have Figure 13.

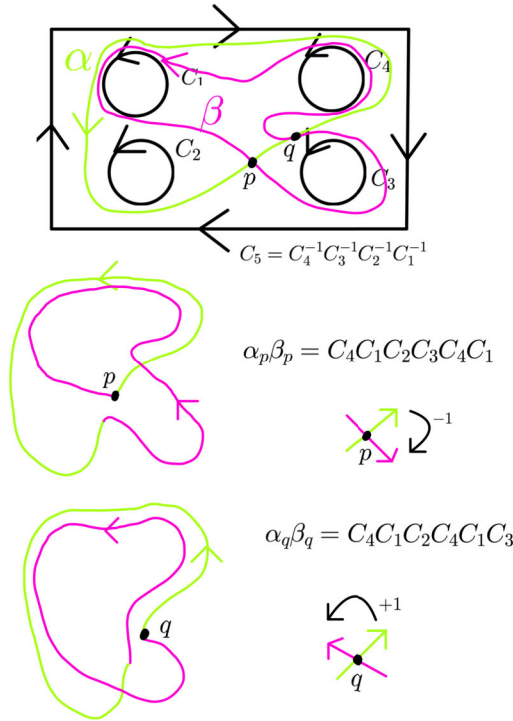


Figure 13: $\{t_{\{1,2,4\}}, t_{\{1,3,4\}}\}$.

$$\begin{aligned}
 \{t_{\{1,2,4\}}, t_{\{1,3,4\}}\} &= (-1)(\text{tr}(c_4 c_1 c_2 c_3 c_4 c_1) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_1 c_3 c_4)) \\
 &\quad + (+1)(\text{tr}(c_4 c_1 c_2 c_4 c_1 c_3) - \frac{1}{2} \text{tr}(c_1 c_2 c_4) \text{tr}(c_1 c_3 c_4)) \\
 &= \frac{1}{2} t_{\{2\}} t_{\{3\}} t_{\{1,4\}}^2 - \frac{1}{2} t_{\{2,3\}} t_{\{1,4\}}^2 - \frac{1}{2} t_{\{1\}} t_{\{2\}} t_{\{3\}} t_{\{4\}} t_{\{1,4\}} \\
 &\quad + \frac{1}{2} t_{\{3\}} t_{\{4\}} t_{\{1,2\}} t_{\{1,4\}} + \frac{1}{2} t_{\{1\}} t_{\{4\}} t_{\{2,3\}} t_{\{1,4\}} \\
 &\quad + \frac{1}{2} t_{\{1,3\}} t_{\{2,4\}} t_{\{1,4\}} + \frac{1}{2} t_{\{1\}} t_{\{2\}} t_{\{3,4\}} t_{\{1,4\}} \\
 &\quad - \frac{1}{2} t_{\{1,2\}} t_{\{3,4\}} t_{\{1,4\}} - \frac{1}{2} t_{\{4\}} t_{\{1,2,3\}} t_{\{1,4\}} - \frac{1}{2} t_{\{3\}} t_{\{1,2,4\}} t_{\{1,4\}} \\
 &\quad - \frac{1}{2} t_{\{2\}} t_{\{1,3,4\}} t_{\{1,4\}} - \frac{1}{2} t_{\{1\}} t_{\{2,3,4\}} t_{\{1,4\}} + 2t_{\{2,3\}} \\
 &\quad + t_{\{1,2,4\}} t_{\{1,3,4\}} - t_{\{2\}} t_{\{3\}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \{t_{\{1,2,4\}}, t_{\{2,3,4\}}\} &= (-1)(\text{tr}(c_4c_1c_2c_3c_4c_2) - \frac{1}{2}\text{tr}(c_1c_2c_4)\text{tr}(c_2c_3c_4)) \\
 &\quad + (+1)(\text{tr}(c_4c_1c_2c_4c_2c_3) - \frac{1}{2}\text{tr}(c_1c_2c_4)\text{tr}(c_2c_3c_4)) \\
 &= \frac{1}{2}t_{\{1,3\}}t_{\{2,4\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}t_{\{2,4\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{2,4\}} - \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{2,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{2,4\}} - \frac{1}{2}t_{\{1,2\}}t_{\{3,4\}}t_{\{2,4\}} \\
 &\quad - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{2,4\}} - \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{2,4\}} \\
 &\quad - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{2,4\}} \\
 &\quad - 2t_{\{1,3\}} - 2t_{\{1,2\}}t_{\{2,3\}} + 2t_{\{2\}}t_{\{1,2,3\}} \\
 &\quad - \frac{1}{2}t_{\{1,2,3\}}t_{\{2,3,4\}} + \frac{3}{2}t_{\{1,2,4\}}t_{\{2,3,4\}} + t_{\{1\}}t_{\{3\}}.
 \end{aligned}$$

Lastly, we consider Figure 14.

$$\begin{aligned}
 \{t_{\{1,3,4\}}, t_{\{2,3,4\}}\} &= (-1)(\text{tr}(c_3c_4c_1c_2c_3c_4) - \frac{1}{2}\text{tr}(c_1c_3c_4)\text{tr}(c_2c_3c_4)) \\
 &\quad + (+1)(\text{tr}(c_3c_4c_1c_3c_4c_2) - \frac{1}{2}\text{tr}(c_1c_3c_4)\text{tr}(c_2c_3c_4)) \\
 &= \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}^2 - \frac{1}{2}t_{\{1,2\}}t_{\{3,4\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{3,4\}} \\
 &\quad + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{3,4\}} \\
 &\quad + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{3,4\}} - \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{3,4\}} \\
 &\quad + \frac{1}{2}t_{\{1,3\}}t_{\{2,4\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{3,4\}} \\
 &\quad - \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{3,4\}} - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{3,4\}} \\
 &\quad - \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{3,4\}} + 2t_{\{1,2\}} \\
 &\quad + t_{\{1,3,4\}}t_{\{2,3,4\}} - t_{\{1\}}t_{\{2\}}.
 \end{aligned}$$

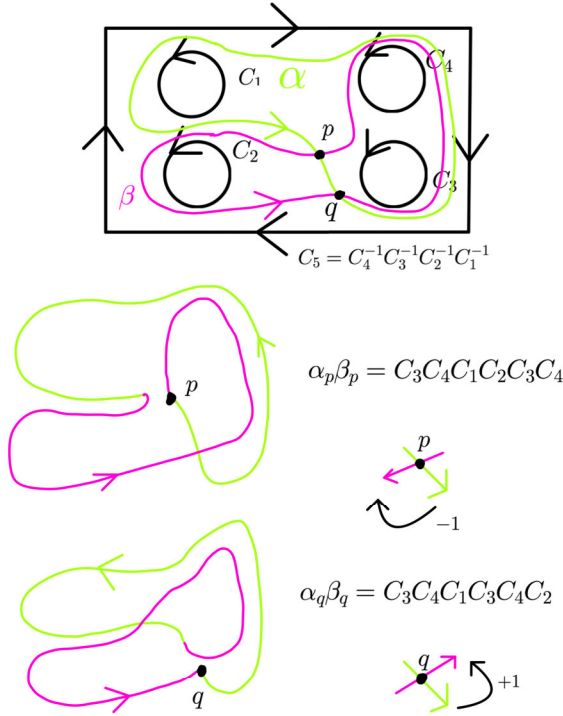


Figure 14: $\{t_{\{1,3,4\}}, t_{\{2,3,4\}}\}$.

7.0.4. The bi-vector and symmetry. Let μ_σ be the mapping class that corresponds to the permutation σ of the boundary components of $\Sigma_{5,0}$; we will use cycle notation for permutations. We consider formal sums of such permutations in the integral group ring associated to the mapping class group of $\Sigma_{5,0}$, and observe that elements in the group ring acts on the coordinate ring of $\mathfrak{X}_{5,0}(\text{SL}(2, \mathbb{C}))$ since the mapping class group acts on $\mathfrak{X}_{5,0}(\text{SL}(2, \mathbb{C}))$.

Let

$$\begin{aligned} \Sigma_1 &= \mu_{(1)} + \mu_{(34)} + \mu_{(23)} + \mu_{(123)} + \mu_{(124)} + \mu_{(142)} + \mu_{(143)} \\ &\quad + \mu_{(13)(24)} + \mu_{(1234)} + \mu_{(1342)} + \mu_{(1324)} + \mu_{(1432)}, \end{aligned}$$

$$\Sigma_2 = \mu_{(1)} + \mu_{(1234)} + \mu_{(13)(24)} + \mu_{(1432)}$$

and $\Sigma_3 = \mu_{(1)} + \mu_{(1432)}$.

The above calculations, after observing symmetry (verified using *Mathematica*), establish the following form for the bi-vector:

$$\begin{aligned} \mathbf{a}_{5,0}(\mathrm{SL}(2, \mathbb{C})) = & \mathbf{a}_{1324} \frac{\partial}{\partial t_{\{1,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} + \Sigma_1 \left(\mathbf{a}_{1214} \frac{\partial}{\partial t_{\{1,2\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} \right) \\ & + \Sigma_2 \left(\mathbf{a}_{12314} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,4\}}} + \mathbf{a}_{12324} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{2,4\}}} \right. \\ & \left. + \mathbf{a}_{12334} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{3,4\}}} \right) \\ & + \Sigma_2 \left(\mathbf{a}_{123124} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,2,4\}}} \right) \\ & + \Sigma_3 \left(\mathbf{a}_{123134} \frac{\partial}{\partial t_{\{1,2,3\}}} \wedge \frac{\partial}{\partial t_{\{1,3,4\}}} \right), \end{aligned}$$

where:

$$\begin{aligned} \mathbf{a}_{1324} = & t_{\{3\}}t_{\{4\}}t_{\{1,2\}} - 2t_{\{3,4\}}t_{\{1,2\}} - t_{\{2\}}t_{\{3\}}t_{\{1,4\}} - t_{\{1\}}t_{\{4\}}t_{\{2,3\}} \\ & + 2t_{\{1,4\}}t_{\{2,3\}} + t_{\{1\}}t_{\{2\}}t_{\{3,4\}}, \\ \mathbf{a}_{1214} = & t_{\{1,2\}}t_{\{1,4\}} + 2t_{\{2,4\}} - t_{\{1\}}t_{\{1,2,4\}} - t_{\{2\}}t_{\{4\}}, \\ \mathbf{a}_{12314} = & \frac{1}{2}t_{\{4\}}t_{\{1\}}^2t_{\{2,3\}} + \frac{1}{2}t_{\{2\}}t_{\{1\}}^2t_{\{3,4\}} - \frac{1}{2}t_{\{1\}}^2t_{\{2,3,4\}} + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1\}}t_{\{1,2\}} \\ & + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1\}}t_{\{1,4\}} - \frac{1}{2}t_{\{1\}}t_{\{1,4\}}t_{\{2,3\}} + \frac{1}{2}t_{\{1\}}t_{\{1,3\}}t_{\{2,4\}} \\ & - \frac{1}{2}t_{\{1\}}t_{\{1,2\}}t_{\{3,4\}} - \frac{1}{2}t_{\{4\}}t_{\{1\}}t_{\{1,2,3\}} - \frac{1}{2}t_{\{3\}}t_{\{1\}}t_{\{1,2,4\}} \\ & - \frac{1}{2}t_{\{2\}}t_{\{1\}}t_{\{1,3,4\}} - t_{\{4\}}t_{\{2,3\}} + t_{\{1,4\}}t_{\{1,2,3\}} \\ & + 2t_{\{2,3,4\}} - \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1\}}^2, \\ \mathbf{a}_{12324} = & -t_{\{3\}}t_{\{1,4\}} + t_{\{1\}}t_{\{3,4\}} + t_{\{2,3\}}t_{\{1,2,4\}} - t_{\{1,2\}}t_{\{2,3,4\}}, \\ \mathbf{a}_{12334} = & -\frac{1}{2}t_{\{4\}}t_{\{3\}}^2t_{\{1,2\}} - \frac{1}{2}t_{\{2\}}t_{\{3\}}^2t_{\{1,4\}} + \frac{1}{2}t_{\{3\}}^2t_{\{1,2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{3\}}t_{\{2,3\}} \\ & + \frac{1}{2}t_{\{3\}}t_{\{1,4\}}t_{\{2,3\}} - \frac{1}{2}t_{\{3\}}t_{\{1,3\}}t_{\{2,4\}} - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{3,4\}} \\ & + \frac{1}{2}t_{\{3\}}t_{\{1,2\}}t_{\{3,4\}} + \frac{1}{2}t_{\{4\}}t_{\{3\}}t_{\{1,2,3\}} + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,3,4\}} \\ & + \frac{1}{2}t_{\{1\}}t_{\{3\}}t_{\{2,3,4\}} + t_{\{4\}}t_{\{1,2\}} - t_{\{3,4\}}t_{\{1,2,3\}} \\ & - 2t_{\{1,2,4\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{4\}}t_{\{3\}}^2, \end{aligned}$$

$$\begin{aligned}
\mathbf{a}_{123124} &= \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}^2 - \frac{1}{2}t_{\{3,4\}}t_{\{1,2\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,2\}} \\
&\quad + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{1,2\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{1,2\}} - \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{1,2\}} \\
&\quad + \frac{1}{2}t_{\{1,3\}}t_{\{2,4\}}t_{\{1,2\}} + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{1,2\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{1,2\}} \\
&\quad - \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{1,2\}} - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{1,2\}} - \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{1,2\}} \\
&\quad + 2t_{\{3,4\}} + t_{\{1,2,3\}}t_{\{1,2,4\}} - t_{\{3\}}t_{\{4\}}, \\
\mathbf{a}_{123134} &= \frac{1}{2}t_{\{2,4\}}t_{\{1,3\}}^2 - \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3\}}t_{\{4\}}t_{\{1,3\}} + \frac{1}{2}t_{\{3\}}t_{\{4\}}t_{\{1,2\}}t_{\{1,3\}} \\
&\quad + \frac{1}{2}t_{\{2\}}t_{\{3\}}t_{\{1,4\}}t_{\{1,3\}} + \frac{1}{2}t_{\{1\}}t_{\{4\}}t_{\{2,3\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1,4\}}t_{\{2,3\}}t_{\{1,3\}} \\
&\quad + \frac{1}{2}t_{\{1\}}t_{\{2\}}t_{\{3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1,2\}}t_{\{3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{4\}}t_{\{1,2,3\}}t_{\{1,3\}} \\
&\quad - \frac{1}{2}t_{\{3\}}t_{\{1,2,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{2\}}t_{\{1,3,4\}}t_{\{1,3\}} - \frac{1}{2}t_{\{1\}}t_{\{2,3,4\}}t_{\{1,3\}} \\
&\quad - 2t_{\{2,4\}} - 2t_{\{2,3\}}t_{\{3,4\}} + t_{\{1,2,3\}}t_{\{1,3,4\}} + 2t_{\{3\}}t_{\{2,3,4\}} + t_{\{2\}}t_{\{4\}}.
\end{aligned}$$

Moreover, using all boundary permutations in the mapping class group (including the permutations of the fifth boundary C_5), we show there do not exist any further symmetries of this type. In other words, the above symmetry is sharp. We used a *Mathematica* notebook for a proof by exhaustion (we checked all 120 induced mappings explicitly).

References

- [AKM] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, Quasi-Poisson manifold, *Canad. J. Math.* **54** (2002), no. 1, 3–29.
- [AMM] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps, *J. Diff. Geom.* **48** (1998) 445–495.
- [Ar] V. I. Arnol'd, *Mathematical methods of classical mechanics*. Translated from the Russian by K. Vogtmann and A. Weinstein, Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.
- [ABL] C. Ashley, J. Burelle and S. Lawton, Rank 1 character varieties of finitely presented groups, *Geom. Dedicata* **192** (2018), 1–19.

- [AB] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London* **308** (1983), 523–615.
- [BF] I. Biswas and C. Florentino, The topology of moduli spaces of group representations: the case of compact surface, *Bull. Sci. Math.* **135** (2011), 395–399.
- [BJ] I. Biswas and L. C. Jeffrey, Poisson structure on character varieties, *Ann. Math. Québec* **45** (2021), 213–219.
- [BLR] I. Biswas, S. Lawton, and D. Ramras, Wonderful compactification of character varieties. With an appendix by Arlo Caine and Samuel Evens, *Pacific J. Math.* **302** (2019), 413–435.
- [Bo] A. Borel, *Linear algebraic groups*, Second edition, Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.
- [CFLO] A. Casimiro, C. Florentino, S. Lawton, A. Oliveira, Topology of moduli spaces of free group representations in real reductive groups, *Forum Math.* **28** (2016), no. 2, 275–294.
- [Cr] W. Crawley-Boevey, Quiver algebras, weighted projective lines, and the Deligne-Simpson problem, *International Congress of Mathematicians*. Vol. II, Eur. Math. Soc., Zürich (2006), 117–129.
- [FL1] C. Florentino and S. Lawton, The topology of moduli spaces of free group representations, *Math. Ann.* **345** (2009), 453–489.
- [FL2] C. Florentino and S. Lawton, Character varieties and moduli of quiver representations. In the tradition of Ahlfors-Bers. VI, 9–38, *Contemp. Math.*, 590, Amer. Math. Soc., Providence, RI, 2013.
- [FL3] C. Florentino and S. Lawton, Topology of character varieties of Abelian groups, *Topology Appl.* **173** (2014), 32–58.
- [FL4] C. Florentino and S. Lawton, Flawed groups and the topology of character varieties, [arXiv:2012.08481](https://arxiv.org/abs/2012.08481).
- [FLR] C. Florentino, S. Lawton, and Daniel Ramras, Homotopy groups of free group character varieties, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **17** (2017), 143–185.
- [GHJW] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, Group systems, groupoids, and moduli spaces of parabolic bundles, *Duke Math. J.* **89** (1997), 377–412.

- [Gol1] W. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. Math.* **54** (1984), 200–225.
- [Gol2] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, *Invent. Math.* **85** (1986), 263–302.
- [Gol3] W. Goldman, Mapping class group dynamics on surface group representations. *Problems on mapping class groups and related topics*, 189–214, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., Providence, RI, 2006.
- [Got] M. Gotô, A theorem on compact semi-simple groups, *J. Math. Soc. Japan* **1** (1949), 270–272.
- [H1] J. Huebschmann, Poisson structure on certain moduli spaces for bundles on a surface. *Annales de l'Institut Fourier* **45** (1995), 65–91,
- [H2] J. Huebschmann, Symplectic and Poisson structures of certain moduli spaces. I. *Duke Math. J.* **80** (1995), no. 3, 737–756.
- [HJ] L. Jeffrey and J. Hurtubise, Representations with weighted frames and framed parabolic bundles, *Canad. J. Math.* **52** (2000), 1235–1268.
- [Hu] J. E. Humphreys, *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York-Heidelberg, 1975.
- [Je] L. C. Jeffrey, Extended moduli spaces of flat connections on Riemann surfaces, *Math. Ann.* **298** (1994), 667–692.
- [JW] L. C. Jeffrey and J. Weitsman, Toric structures on the moduli space of flat connections on a Riemann surface. II. Inductive decomposition of the moduli space, *Math. Ann.* **307** (1997), 93–108.
- [Ko] V. P. Kostov, The Deligne-Simpson problem—a survey, *J. Alg.* **281** (2004), 83–108.
- [La1] S. Lawton, *SL(3, C)-character varieties and $\mathbb{R}P^2$ -structures on a trinion*. Thesis (Ph.D.) University of Maryland, College Park. 2006. 72 pp. ISBN: 978-0542-91017-3, ProQuest LLC
- [La2] S. Lawton, Generators, relations and symmetries in pairs of 3x3 unimodular matrices, *J. Algebra* **313** (2007), 782–801.

- [La3] S. Lawton, Poisson geometry of $SL(3, \mathbb{C})$ -character varieties relative to a surface with boundary, *Trans. Amer. Math. Soc.* **361** (2009), 2397–2429.
- [La4] S. Lawton, Obtaining the one-holed torus from pants: duality in an $SL(3, \mathbb{C})$ -character variety, *Pacific J. Math.* **242** (2009), 131–142.
- [LGPV] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson structures*, volume 347 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2013.
- [Me] E. Meinrenken, Lectures on Group-Valued Moment Maps and Verlinde Formulas, Notre Dame University, May 31–June 4, 2011.
- [PW] S. Pasiencier and H.-C. Wang, Commutators in a semi-simple Lie group, *Proc. Amer. Math. Soc.* **13** (1962), 907–913.
- [Re] R. Ree, Commutators in semi-simple algebraic groups, *Proc. Amer. Math. Soc.* **15** (1964), 457–460.
- [Sik1] A. Sikora, Character varieties, *Trans. Amer. Math. Soc.* **364** (2012), 5173–5208.
- [Sik2] A. Sikora, Character varieties of abelian groups, *Math. Zeit.* **277** (2014), 241–256.
- [Sim] C. T. Simpson, Products of matrices, *Differential geometry, global analysis, and topology*, CMS Conf. Proc., **12**, Amer. Math. Soc. (1991), 157–185.
- [St] R. Steinberg, Regular elements of semisimple algebraic groups, *Inst. Hautes Etudes Sci. Publ. Math.* **25** (1965), 49–80

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