

Every real 3-manifold is real contact

MERVE CENGİZ AND FERİT ÖZTÜRK

A real 3-manifold is a smooth 3-manifold together with an orientation preserving smooth involution, which is called a real structure. A real contact 3-manifold is a real 3-manifold with a contact distribution that is antisymmetric with respect to the real structure. We show that every real 3-manifold can be obtained via surgery along invariant knots starting from the standard real S^3 and that this operation can be performed in the contact setting too. Using this result we prove that any real 3-manifold admits a real contact structure. As a corollary we show that any oriented overtwisted contact structure on an integer homology real 3-sphere can be isotoped to be real. Finally we give construction examples on $S^1 \times S^2$ and lens spaces. For instance on every lens space there exists a unique real structure that acts on each Heegaard torus as hyperelliptic involution. We show that any tight contact structure on any lens space is real with respect to that real structure.

1	Introduction	1314
2	Preliminaries and equivariant contact surgery	1317
3	Going recursively from real S^3 to any real 3-manifold	1324
4	Equivariant Lickorish-Wallace theorem	1329
5	Three-manifolds with real contact structures	1334
6	Examples	1339
	References	1345

1. Introduction

A *real structure* on a smooth oriented $2k$ - (respectively $(2k - 1)$ -) dimensional manifold is a smooth involution which is orientation preserving if k is even and orientation reversing if k is odd, with its fixed point set having dimension k (respectively $k - 1$), if not empty. The idea behind this definition is to mimic the complex conjugation on a complex analytic variety given by functions with real coefficients. For c_M a real structure on M , we call the pair (M, c_M) a *real manifold* and the fixed point set of c_M the *real part*, denoted below by $\text{Fix}(c_M)$.

In this work we will heavily be interested in the 3-dimensional case: a real structure on a smooth, closed, oriented 3-manifold is an orientation preserving smooth involution with the fixed point set being either empty or 1-dimensional. The standard example for a real 3-manifold is the 3-sphere $S^3 \subset \mathbb{C}^2$ with the standard real structure $c_{\text{st}} = \text{conj}|_{S^3}$ where conj denotes the complex conjugation on \mathbb{C}^2 . This involution is known to be the unique real structure with nonempty real part on S^3 up to equivariant isotopy. This fact is a result of the culminated work on masse on the resolution of the Smith conjecture that states originally that a finite-order diffeomorphism of S^3 cannot have a knotted 1-dimensional fixed point set. See [26] for a PL topological solution of that conjecture for even periods and [17] for a detailed exposition of the generalizations and the related work.

Now let ξ be an oriented contact structure on a smooth, compact, oriented real (M, c_M) . If $(c_M)_*(\xi) = -\xi$, then ξ is said to be c_M -real and the triple (M, c_M, ξ) is called a c_M -real contact manifold (see [21], [22] for definitions and discussion). The obvious example is $S^3 \subset \mathbb{C}^2$ with the real structure c_{st} and the unique tight contact structure ξ_{st} on S^3 . More generally real contact 3-manifolds appear naturally as link manifolds of isolated complex analytic singularities defined by analytic functions with real coefficients: the natural tight contact structure induced on the link manifold by the complex tangencies is real with respect to the real structure determined by the complex conjugation. Even more generally, there are various basic conditions for a hypersurface in a real symplectic manifold (i.e. a real smooth manifold with an antisymmetric symplectic form) which makes it naturally a real contact manifold (see e.g. [7], notably for the observation there in Proposition 1.2.4, and for more examples.)

The positive contact structures on a closed, oriented 3-manifold are associated with the open book decompositions on the manifold via the Giroux correspondence; indeed open books and contact structures are in one-to-one correspondence up to positive stabilizations and contact isotopy respectively

[9]. Similarly on a real 3-manifold one can introduce the notion of a real open book decomposition. In [22] we have taken several first steps towards a Giroux correspondence between the real open books and the real contact structures. Namely it has been proven there that every real open book supports a real contact structure, that every real contact structure is supported by a real open book, and that two real contact structures supported by the same real open book are equivariantly isotopic (see [22] for definitions and the exact statements).

Nevertheless it remained an open question whether real open books and real contact structures always exist on a given real 3-manifold, for example whether every real 3-manifold has a contact structure at all that is real with respect to the given real structure. This last question, which was raised in [22] and [7], can be considered as the real version of J. Martinet's result on the existence of contact structures on closed 3-manifolds [16]. The main purpose of the present work is to answer that question affirmatively:

Theorem 1.1. *Every real 3-manifold admits a real contact structure.*

One of the standard ways to prove Martinet's theorem is to recall that every closed 3-manifold can be obtained via a surgery on a link in S^3 and then to argue that this surgery can be performed in the contact setting. A usual expression of the former fact is through the well-known Lickorish-Wallace theorem, which states that every closed orientable 3-manifold may be obtained by a surgery along a link in S^3 where each Dehn surgery coefficient is an integer (see e.g. [23]).

In order to follow this track in the equivariant contact setting, we first define and investigate in Section 2 the notion of Dehn surgery in the equivariant and contact equivariant setup in real contact 3-manifolds. Among others we detect explicitly when equivariant contact $(1/l)$ -surgery ($l \in \mathbb{Z}$) along equivariant Legendrian knots is possible (Theorem 2.4). One of the direct corollaries of that discussion is the following which is proven in Section 2.

Proposition 1.2. *Any overtwisted contact structure on S^3 can be isotoped to be c_{st} -real. More precisely any overtwisted contact structure on S^3 can be obtained by an equivariant contact surgery in $(S^3, c_{\text{st}}, \xi_{\text{st}})$.*

Section 3, which is not directly related to the proof of Theorem 1.1 and may be skipped in the first reading, suggests a method to obtain a given real 3-manifold from the standard real S^3 through a sequence of single surgery operations and intermediate real 3-manifolds, in a way that each

next surgery is equivariant in the previous intermediate real 3-manifold. We call such a link –constituted of an ordered collection of knots– *recursively invariant*. Thus we prove the following theorem in Section 3, which can be considered as a recursively equivariant version of the Lickorish-Wallace theorem. (See Theorem 3.1 for the detailed, precise version.)

Theorem 1.3. *Every closed real 3-manifold can be obtained via a finite number of Dehn surgeries along an ordered, recursively invariant collection of knots starting from the real 3-sphere (S^3, c_{st}) .*

Due to the recursive nature of the construction, this theorem does not lend itself easily to applications, for example in obtaining surgery diagrams in S^3 . Instead, in Section 4 we prove an equivariant version of the Lickorish-Wallace theorem.

Theorem 1.4 (Equivariant Lickorish-Wallace Theorem). *Every closed, oriented real 3-manifold can be obtained via equivariant Dehn surgery along an equivariant link \mathcal{L} in the real 3-sphere (S^3, c_{st}) . The equivariant link can be taken as $\mathcal{L} = L \cup L_S \cup \bar{L}$ where L_S is a c_{st} -equivariant unlink, $c_{\text{st}}(L) = \bar{L}$ and all the surgery coefficients can be taken as ± 1 , (with respect to a framing induced by an invariant Heegaard surface).*

The proofs of Theorems 1.3 and 1.4 respectively follow the proof of the Lickorish-Wallace Theorem where we start with suitable decompositions for S^3 and the given 3-manifold, and look for appropriate factorizations of diffeomorphisms on Heegaard surfaces in the mapping class group (recursively invariant factorization and equivariant factorization respectively).

Employing Theorem 1.4, we prove our main result, Theorem 1.1, in Section 5. For the proof it suffices to show that the equivariant link in Theorem 1.4 can be chosen appropriately so that it is possible to turn the equivariant surgeries into equivariant contact (± 1)-surgeries. The proof is constructive and produces an explicit algorithm that allows explicit equivariant contact surgery descriptions for real contact 3-manifolds.

An immediate consequence of Theorem 1.1 is the following, which we prove in Section 5.

Corollary 1.5. *Any oriented overtwisted contact structure on an integer homology real sphere (Σ, s) of dimension 3 can be isotoped to be s -real.*

In Section 6 we produce examples on $S^1 \times S^2$ and lens spaces. In the first part we show

Theorem 1.6. *The unique tight contact structure on $S^1 \times S^2$ is real with respect to three of the four possible real structures on $S^1 \times S^2$.*

We do not know the answer for the last real structure.

Finally on any lens space there exists a unique real structure that acts on each Heegaard torus as hyperelliptic involution (coined as type A in [12]). We prove in Section 6.2

Theorem 1.7. *For any $p > q > 0$, every tight contact structure on $L(p, q)$ is A-real.*

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2. Preliminaries and equivariant contact surgery

2.1. Basic definitions

In the sequel we always reside in the smooth category, both for spaces and maps, for the sake of keeping the rapport between involutions and contact structures. We note that in the topological category some claims in the equivariant realm may fail, e.g. the Smith conjecture [18].

On a solid torus $S^1 \times D^2$, there are four real structures up to isotopy through real structures [11]. We choose an oriented identification of the boundary T^2 with $\mathbb{R}^2_{(x,y)}/\mathbb{Z}^2$, where x direction corresponds to the meridional direction of T^2 , and fix the coordinates of $S^1 \times D^2$ as (y, t, x) where t is the radial direction of D^2 . In these coordinates, the four real structures on the solid torus are:

- (1) $c_1 : (y, (x, t)) \mapsto (-y, (t, -x))$;
- (2) $c_2 : (y, (t, x)) \mapsto (y, (t, x + \frac{1}{2}))$;
- (3) $c_3 : (y, (t, x)) \mapsto (y + \frac{1}{2}, (t, x))$;
- (4) $c_4 : (y, (t, x)) \mapsto (y + \frac{1}{2}, (t, x + \frac{1}{2}))$.

Any orientation preserving involution on T^2 can be extended to an involution on $S^1 \times D^2$. Such an extension is unique up to isotopy and fixes a core of the solid torus setwise [11]. A c_j -knot is by definition a knot which has a c_j -equivariant neighborhood.

Likewise there are three involutions on the core circle up to isotopy through involutions: reflection, identity and rotation by π (antipodal map).

We also denote them by c_1 , c_2 (or id) and c_3 respectively. Note that $c_4|_{\text{core}} = c_3$ too.

An embedded Heegaard decomposition for a real 3-manifold (M, c_M) is said to be *real* if c_M exchanges the two Heegaard handlebodies. In this case c_M restricts to a real structure on the Heegaard surface H , reversing the orientation. It was proven by T. A. Nagase that every real 3-manifold admits an (embedded) real Heegaard decomposition [19, Proposition 2.4]. As usual one can also construct a real 3-manifold by a pair of handlebodies, both diffeomorphic to a handlebody U , and a gluing map $c : U \rightarrow U$ which is a real structure on U [22]. Such a decomposition of the 3-manifold is called an abstract real Heegaard decomposition, denoted in the sequel by the pair $(\partial U, c)$. The minimal genus among all Heegaard surfaces of all possible real Heegaard decompositions of (M, c_M) is called the real Heegaard genus of (M, c_M) . It is greater than or equal to the Heegaard genus of M . See [22] for a detailed discussion on real Heegaard decompositions.

A particular way to produce a real Heegaard decomposition is through real open books [22]. Let (S, f) be an abstract open book, where S is a compact surface with boundary and $f : S \rightarrow S$ is the monodromy with $f|_{\partial S} = \text{identity}$. For a real structure c on S , the triple (S, f, c) with $f \circ c = c \circ f^{-1}$ is called an (abstract) real open book. The map $c_\pi = c \circ f$ is a real structure on the page π of the open book so that $f = c \circ c_\pi$. An abstract real open book determines a real 3-manifold (M, c_M) uniquely and canonically. The union of the page 0 and the page π is a real Heegaard surface in M and the real structure c_M is the identity map between the two identical handlebodies of the Heegaard splitting. The restrictions of c_M to the page 0 and the page π are respectively c and c_π . As in the usual setting, there is a notion of positive real stabilization developed in [22]. Up to equivariant isotopy there are 9 distinct ways to attach handles to S and to extend the real structure over the new handles (see [22, Figure 3]).

In the sequel, instead of using the term c_M -real, we usually drop the reference to c_M whenever the real structure is understood. The real structures and the real parts will be in red wherever color is possible. We assume that all contact structures are oriented and positive.

2.2. Equivariant surgery

Let (M, c_M) be a closed, oriented real 3-manifold and K be a c_M -invariant knot. Then K has a unique equivariant tubular neighborhood $N(K)$, which is equivariantly isotopic to one of the real solid tori $(S^1 \times D^2, c_i)$. An invariant knot K is called a c_i -knot if it has an equivariant neighborhood of type

$(S^1 \times D^2, c_i)$. The common name used for a c_1 -knot is a strongly invertible knot (under the involution).

A topological (p/q) -surgery $(p, q \in \mathbb{Z})$ along the knot K takes the meridian-longitude pair (μ_0, λ_0) of the new solid torus N_0 to $(p\mu + q\lambda, p'\mu + q'\lambda)$ on the boundary of the excised neighborhood N via a gluing map φ . Note that here $pq' - qp' = -1$ and λ must be chosen. (The following discussion is with respect to a fixed λ .) This surgery can be performed naturally in the real setting along a c_i -knot. The real extension c_j over the surgered solid torus is unique in the following way. Indeed in every possible case it suffices to check that $c_i \circ \varphi = \varphi \circ c_j$.

- If K is a c_1 -knot then the real structure extends as c_1 .
- If K is a c_2 -knot then the real structure extends as

$$\begin{cases} c_2 & \text{if } q \text{ even;} \\ c_3 & \text{if } q \text{ odd, } q' \text{ even;} \\ c_4 & \text{if } q \text{ odd, } q' \text{ odd;} \end{cases}$$

- If K is a c_3 -knot then the real structure extends as

$$\begin{cases} c_2 & \text{if } p \text{ even;} \\ c_3 & \text{if } p \text{ odd, } p' \text{ even;} \\ c_4 & \text{if } p \text{ odd, } p' \text{ odd;} \end{cases}$$

- If K is a c_4 -knot then the real structure extends as

$$\begin{cases} c_2 & \text{if } p + q \text{ even;} \\ c_3 & \text{if } p + q \text{ odd, } p' + q' \text{ even;} \\ c_4 & \text{if } p + q \text{ odd, } p' + q' \text{ odd.} \end{cases}$$

Definition 2.1. If a c_I -solid torus is excised and a c_J -solid torus is glued back, we call such a surgery of type I_J . As a final type of equivariant surgery, consider a knot K satisfying $K \cap \text{Fix}(c_M) = \emptyset$ and its disjoint copy $K' = c_M(K)$. An equivariant pair of surgeries performed along K and K' will be coined as a type-5 surgery (along K and K'). Since c_M is an orientation preserving homeomorphism, the surgery framings along K and K' are equal.

Remark 2.2. Note that a 2_3 -surgery followed by a Dehn twist along the meridian (which alters the parity of q') produces a manifold equivariantly

diffeomorphic to one obtained by a single 2_4 -surgery. Similarly 3_3 - and 3_4 -surgeries are equivalent in that sense since a meridional Dehn twist alters the parity of p' . Similarly for 4_3 - and 4_4 -surgeries. We will use this subtle remark repeatedly in the sequel since we are content with diffeomorphisms, not isotopies. However if the latter is in question then one should a priori distinguish between I_3 - and I_4 -surgeries, $I = 2, 3$ or 4 .

Let us also note that in case of a 1_1 surgery, the number $|\text{Fix}|$ of connected components of the real part may change. In fact, for a c_1 -knot K , if the fixed points of K belong to the same (respectively different) component(s) of the real part, $|\text{Fix}|$ either increases (respectively decreases) by 1 or stays the same. Meanwhile, in case of 2_3 - or 2_4 -surgery, K is a c_2 -knot (a real knot) and $|\text{Fix}|$ decreases by 1; in case of 3_2 - or 4_2 -surgery, $|\text{Fix}|$ increases by 1. The 3_4 - and 4_3 -surgeries do not alter $|\text{Fix}|$.

2.3. Equivariant contact surgery

The possibility of contact surgery for a rational coefficient relies on the existence of a tight contact solid torus with the required contact structure on its convex boundary. It is known that the germ of a contact structure near the convex boundary is determined by a collection of curves on the boundary, called the dividing set. On a convex torus, this picture can be standardized further to obtain linear curves as the dividing set, so that the common slope of these curves determine the contact structure near the torus. Such convex tori are said to be in standard form. The slope on the standard contact neighborhood of a Legendrian curve L is determined by the contact twisting (denoted $tw(L)$) of the curve. The twisting is well-defined after a choice of a longitude for L . (See e.g. [8] or [13] for a thorough discussion for convex surfaces, slopes and twisting.)

Similarly the possibility of equivariant contact surgery for a nonzero rational coefficient relies of course on the existence of an equivariant tight contact solid torus with the required slope on its convex boundary. The equivariant counterpart of the terms above (i.e. the standard equivariant contact neighborhood theorems, equivariant convex surfaces etc.) has been studied in [21] and [20]. Here we put together the previously known existence results for equivariant tight solid tori after [20].

Theorem 2.3. *We have the following listed existence/nonexistence results regarding equivariant tight solid tori with convex boundary. In case of existence, it is unique up to equivariant contact isotopy relative to the equivariant*

convex boundary in standard form such that the dividing set has slope s and has exactly two connected components. Below $k \in \mathbb{Z}$; $1/0$ is considered as ∞ .

- A c_1 -real tight solid torus with $s = 1/k$ exists.
- A c_2 -real tight solid torus exists if and only if $s = 1/k$.
- A c_3 -real tight solid torus exists if and only if $s = 1/(2k + 1)$.
- A c_4 -real tight solid torus exists if and only if $s = 1/(2k)$.

All of the solid tori above are the standard neighborhoods of equivariant Legendrian knots with $tw = 1/s$. Here tw is with respect to a fixed longitude.

Now let K be an equivariant Legendrian knot in a real tight contact manifold (M, ξ, c_M) . Let N be a standard equivariant contact neighborhood of K , the existence of which is warranted by [21] and [20]. A contact (p/q) -surgery on K is with respect to the choice of the longitude λ as the dividing set on the boundary of N . To have a well defined contact structure on the surgered 3-manifold, the dividing sets of ∂N_0 and ∂N must match. By the identification above, the curve $-p'\mu_0 + p\lambda_0$ maps to the longitude λ , so the solid torus N_0 should be a tight contact solid torus with boundary slope $-p/p'$.

We now repeat this discussion in the equivariant setting. We will be interested in the following cases.

Case 1. K is a c_1 -knot. The extension would be c_1 . Since tight c_1 -solid tori of slope $(1/p')$ ($p' \in \mathbb{Z}$) uniquely exist, equivariant contact $(1/q)$ -surgery of type 1_1 is uniquely defined up to equivariant contact isotopy. We fill in the standard c_1 -real tight neighborhood of a c_1 -invariant knot with $tw = -p'$.

Case 2. K is a c_2 -knot. It follows from Theorem 2.3 that equivariant contact (p/q) -surgery is defined if and only if $p = 1$. Similarly for the other cases below.

If q and q' are both odd, then p' must be even and the real contact structure extends uniquely as a c_4 -tight solid torus with slope $-1/p'$ (∞ included here and below whenever $p' = 0$).

If q is odd and q' is even, then p' must be odd and the real contact structure extends uniquely as a c_3 -tight solid torus with slope $-1/p'$.

If q is even, then the real contact structure extends as a c_2 -tight solid torus with slope $-1/p'$.

Note for the first two cases here that the parity of q' is not well-defined: given p and q one can alter the parity of p' and q' by meridional Dehn twists. This is not a problem in the contact setting since a meridional Dehn twist

extends over a solid torus and is smoothly isotopic to the identity. However it is not isotopic to the identity through real structures (See Remark 2.2 and the conclusion in Theorem 2.4).

Case 3. K is a c_3 -knot. Then $p = 1$ as before.

If p' is odd, then the real structure extends as c_4 but no c_4 -real tight solid torus exists with slope $-1/p'$ for odd p' .

If p' is even, then the real structure extends as c_3 but no c_3 -real tight solid torus exists with slope $-1/p'$ for even p' .

Case 4. K is a c_4 -knot. Then $p = 1$.

If $p + q = 1 + q$ is odd (i.e. q is even; so q' is odd) and $p' + q'$ is odd, then p' must be even and the real contact structure extends uniquely as a c_4 -tight solid torus with slope $-1/p'$.

If q is even and $p' + q'$ is even, then p' must be odd and the real contact structure extends uniquely as a c_3 -tight solid torus with slope $-1/p'$.

If q is odd, then the real contact structure extends uniquely as a c_2 -tight solid torus with slope $-1/p'$.

Similar remark as the one following Case 2 above applies here for the parity of p' : it is not well-defined and can be altered. Let us summarize the above discussion.

Theorem 2.4 (Equivariant Contact Surgery along Equivariant Legendrian Knots). *Equivariant contact surgery along a Legendrian c_3 -knot is impossible. Equivariant contact $(1/q)$ -surgery ($q \in \mathbb{Z}$) along a Legendrian c_1 -, c_2 - or c_4 -knot is uniquely defined up to equivariant contactomorphism. For a c_2 - or c_4 -knot the only possible contact surgery coefficient is $1/q$.*

Moreover for the surgery types $1_1, 2_2$ and 4_2 the uniqueness is up to equivariant contact isotopy.

Table 1 details the cases for equivariant contact surgery. The last column describes the type c of the glued back solid torus: a standard c -real tight neighborhood of a c -invariant knot with $tw = -p'$.

Proof of Proposition 1.2. The number of overtwisted contact structures on S^3 up to contact isotopy is countably infinite. They are distinguished by the d_3 invariant, which is a half integer for S^3 . (For d_3 , see [10] where it was first defined or see e.g. [3].) The overtwisted structure ξ_{-2p} with $d_3 = -2p + 1/2$, $p \in \mathbb{Z}$, can be obtained by a contact surgery given explicitly in, for example, [6, Figure 8]. Observing the symmetry here, one can immediately turn this diagram into an equivariant contact one along a pair of c_1 -real knots with surgery coefficients $+1$ and $-1/p$ (see Figure 1), except for $p = 0$ when there is a single knot, the one on the left in Figure 1 with surgery coefficient $+1$.

Knot	Case	Glue back
c_1		$c = c_1$
c_2	q odd, q' odd	$c = c_4$
	q odd, q' even	$c = c_3$
	q even	$c = c_2$
c_3	\emptyset	
c_4	q even, p' even	$c = c_4$
	q even, p' odd	$c = c_3$
	q odd	$c = c_2$

Table 1. Equivariant contact $1/q$ -surgery along c_i -knots. We glue back the standard c -real tight neighborhood of a c -invariant knot with $tw = -p'$

These equivariant contact surgeries are possible by Theorem 2.4. Recall that the equivariant contact connected sum between two real contact 3-manifolds is well-defined thanks to the fact that there is a unique real tight 3-ball [21]. Thus taking connected sums of arbitrarily many (S^3, c_{st}, ξ_k) with $k = 0$ and $d_3 = 1/2$ (respectively $k = -2$ and $d_3 = -3/2$) one obtains every overtwisted 3-sphere with positive (respectively negative) d_3 (cf. [3, Lemma 4.2]). Alternatively one could consider the real contact 3-spheres obtained by taking the connected sum between (S^3, c_{st}, ξ_0) and the 3-sphere in Figure 1 for varying $p \in \mathbb{Z}$. Note that in each case the equivariant contact diagram is simply the disjoint union of the previous ones.

In general one must also check that the final real structure is the desired one. But here that is immediate by the uniqueness of c_{st} . \square

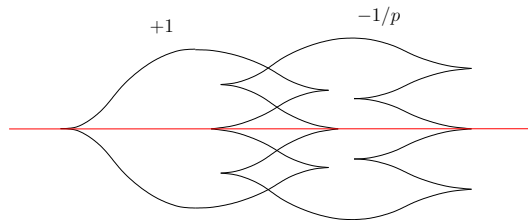


Figure 1. The c_{st} -real overtwisted structure ξ_{-2p} on S^3 , $p \in \mathbb{Z}$. In the case $p = 0$, the ∞ -surgery on the right disappears.

3. Going recursively from real S^3 to any real 3-manifold

In this section we will prove the following theorem of recursive nature.

Theorem 3.1. *Every closed real 3-manifold (M, c_M) can be obtained via a finite number of Dehn surgeries along an ordered, recursively invariant collection of knots starting from the real 3-sphere (S^3, c_{st}) .*

More precisely, there is a sequence $\{(M_j, s_j; K_j)\}_{j=0}^k$ of real 3-manifolds, a common Heegaard surface H and s_j -invariant knots $K_j \subset H \subset M_j$ such that $(M_0, s_0) = (S^3, c_{st})$ and (M_k, s_k) is equivariantly diffeomorphic to (M, c_M) and each (M_{j+1}, s_{j+1}) , $0 \leq j \leq k$, is obtained from (M_j, s_j) via a ± 1 surgery along K_j where the framing along K_j is determined by H ; at each step, the real structure is canonically extended to the surgered region.

First we will prove several lemmata regarding factorizations in the mapping class group of a real Heegaard surface. To start we need some preliminaries. Two real structures (orientation reversing involutions) r and s on an closed, oriented genus g surface Σ_g are said to be equivalent if there is an orientation preserving diffeomorphism h such that $r = h \circ s \circ h^{-1}$. It is well-known that the equivalence class of r is determined by the number of connected components of the real part $\text{Fix}(r)$, and connectedness of $\Sigma_g - \text{Fix}(r)$. (This follows from considering the quotient surface and the classification of 2-manifolds.) In case $\text{Fix}(r)$ is separating (i.e. $\Sigma_g - \text{Fix}(r)$ has exactly 2 connected components), then $1 \leq |\text{Fix}(r)| \leq g + 1$ and g and $|\text{Fix}(r)|$ have opposite parities. We denote by s_{\max} the maximal real structure in the standard form (see Figure 2). If $\text{Fix}(r)$ is non-separating (i.e. $\Sigma_g - \text{Fix}(r)$ is connected), then $0 \leq |\text{Fix}(r)| \leq g$. In this case, the real structure in the standard form is denoted by $\mathfrak{s}_{|\text{Fix}(r)|}$.

Below τ_a always denotes the positive Dehn twist along a curve a on a surface.

Lemma 3.2. *Any real structure s on Σ_g with $|\text{Fix}(s)| = k$ can be expressed in the form*

$$(3.1) \quad s = \tau_\beta^m \cdot \tau_{b_u} \dots \tau_{b_1} \cdot p \cdot s_{\max}$$

where $p = \tau_{a_1}^{\sigma_1} \dots \tau_{a_w}^{\sigma_w} \cdot \tau_{a_w}^{\sigma_w} \dots \tau_{a_1}^{\sigma_1}$ is a palindrome (possibly empty) with even length and with each a_j s_{\max} -invariant, $\sigma_j \in \mathbb{Z}$; each $b_j \subset \text{Fix}(p \cdot s_{\max})$; and $\beta \subset \text{Fix}(s)$ with $m = 1$ if s is separating but not maximal and $m = 0$ otherwise.

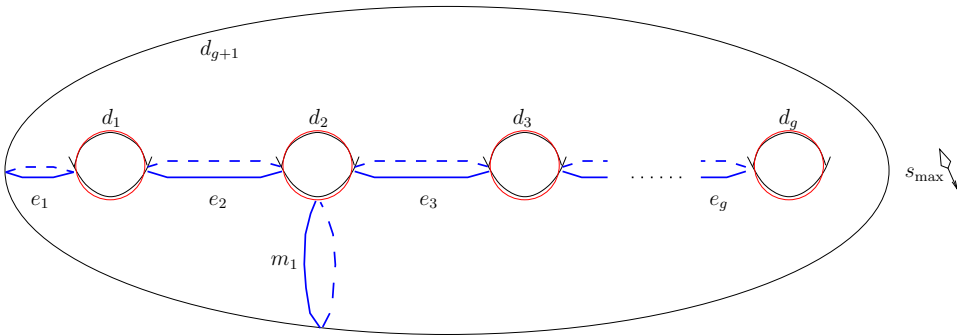


Figure 2. The maximal real structure s_{\max} on Σ_g . The set of fixed points is $d_1 \cup \dots \cup d_g \cup d_{g+1}$; here d_{g+1} is the large outer closed curve. The Humphries generators of the mapping class group are $d_1, \dots, d_g, e_1, \dots, e_g, m_1$; each is an s_{\max} -invariant curve.

Proof. (i) Assume s is non-separating. Set $u = g + 1 - k$ and take s_{\max} -real circles d_1, \dots, d_u on Σ_g . Then $\mathfrak{s}_k = \tau_{d_u} \dots \tau_{d_1} \cdot s_{\max}$ is the standard non-separating real structure with k real circles hence is conjugate to s . Then there is an orientation preserving diffeomorphism f of Σ_g so that

$$\begin{aligned} s &= f \mathfrak{s}_k f^{-1} = f \cdot \tau_{d_u} \dots \tau_{d_1} \cdot s_{\max} \cdot f^{-1} \\ &= f \cdot \tau_{d_u} f^{-1} f \dots f^{-1} f \tau_{d_1} \cdot f^{-1} f s_{\max} \cdot f^{-1} \\ &= \tau_{f(d_u)} \dots \tau_{f(d_1)} \cdot f s_{\max} f^{-1}. \end{aligned}$$

Observe that since each d_j is s_{\max} -invariant, each $b_j = f(d_j)$ is $(f s_{\max} f^{-1})$ -invariant. Furthermore fixing a basis for the mapping class group of Σ_g consisting of twists on s_{\max} -invariant curves (e.g. the s_{\max} -invariant Humphries generators in Figure 2), we can write

$$\begin{aligned} f s_{\max} f^{-1} &= (\tau_{a_1}^{\sigma_1} \dots \tau_{a_w}^{\sigma_w}) \cdot s_{\max} \cdot (\tau_{a_w}^{-\sigma_w} \dots \tau_{a_1}^{-\sigma_1}) \\ &= \tau_{a_1}^{\sigma_1} \dots \tau_{a_w}^{\sigma_w} \tau_{a_w}^{\sigma_w} \dots \tau_{a_1}^{\sigma_1} \cdot s_{\max}. \end{aligned}$$

(ii) If s is separating but not maximal, take any circle $\beta \subset \text{Fix}(s)$. Then $\tau_{\beta}^{-1} \cdot s$ is non-separating and the proof follows from part (i).

(iii) If s is maximal then the equation holds with $m = 0$ and $u = 0$ following part (i). □

On a real surface (Σ_g, s) , we say that the product of Dehn twists $\tau_{r_m}^{\sigma_m} \cdots \tau_{r_1}^{\sigma_1}$ satisfies the recursive s -invariance condition (or is recursively s -invariant) if for each $j = 1, \dots, m$,

- (i) $\tau_{r_{j-1}}^{\sigma_{j-1}} \cdots \tau_{r_1}^{\sigma_1} \cdot s(r_j) = r_j$;
- (ii) or r_j and r_{j+1} are disjoint, and $\tau_{r_{j-1}}^{\sigma_{j-1}} \cdots \tau_{r_1}^{\sigma_1} \cdot s(r_j) = r_{j+1}$.

We will set $s_0 = s$ and, for $j > 0$, $s_j = \tau_{r_j}^{\sigma_j} \cdot s_{j-1}$ in case (i).

Lemma 3.3. *The even palindrome p in the previous lemma has a recursively s_{\max} -invariant factorization in even powers.*

Proof. First, using the fact that p is an even palindrome, observe that it can be written as

$$p = (\tau_{a_1}^{\sigma_1} \cdots \tau_{a_{w-1}}^{\sigma_{w-1}} \tau_{a_w}^{\sigma_w} \tau_{a_{w-1}}^{-\sigma_{w-1}} \cdots \tau_{a_1}^{-\sigma_1})^2 \cdots (\tau_{a_1}^{\sigma_1} \tau_{a_2}^{\sigma_2} \tau_{a_3}^{\sigma_3} \tau_{a_2}^{-\sigma_2} \tau_{a_1}^{-\sigma_1})^2 \cdot (\tau_{a_1}^{\sigma_1} \tau_{a_2}^{\sigma_2} \tau_{a_1}^{-\sigma_1})^2 \cdot \tau_{a_1}^{2\sigma_1}.$$

Now for each $1 \leq j < w$, set $f_j = \tau_{a_1}^{\sigma_1} \cdots \tau_{a_j}^{\sigma_j}$, $\bar{f}_j = \tau_{a_j}^{\sigma_j} \cdots \tau_{a_1}^{\sigma_1}$, and $r_1 = a_1$, $r_{j+1} = f_j(a_{j+1})$. Then the factorization above becomes

$$p = \tau_{r_w}^{2\sigma_w} \cdots \tau_{r_2}^{2\sigma_2} \tau_{r_1}^{2\sigma_1}.$$

We claim that this factorization is recursively s_{\max} -invariant. In fact, $r_1 = a_1$ is invariant under s_{\max} ; furthermore, for each $1 \leq j < w$, (dropping the powers σ_j 's for the sake of clarity)

$$\begin{aligned} (\tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max})(r_{j+1}) &= f_j \bar{f}_j s_{\max}(f_j(a_{j+1})) \\ &= f_j(\bar{f}_j \bar{f}_j^{-1}) s_{\max}(a_{j+1}) \\ &= f_j(a_{j+1}) = r_{j+1}. \end{aligned}$$

To finish the proof, we should also note that $(\tau_{r_{j+1}}^2 \tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max})(r_{j+1}) = r_{j+1}$. □

The proof above shows also that the composition of s_{\max} with τ_{r_j} 's is always a real structure. In fact, for every $1 \leq j < w$, (without writing σ_j 's)

$$\begin{aligned} (\tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max}) \cdot (\tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max}) &= (f_j \bar{f}_j s_{\max}) \cdot (f_j \bar{f}_j s_{\max}) \\ &= f_j \bar{f}_j \bar{f}_j^{-1} f_j^{-1} s_{\max}^2 = \text{id} \end{aligned}$$

and similarly

$$(\tau_{r_{j+1}} \tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max}) \cdot (\tau_{r_{j+1}} \tau_{r_j}^2 \cdots \tau_{r_1}^2 s_{\max}) = \tau_{r_{j+1}} \tau_{r_{j+1}}^{-1} = \text{id}.$$

Of course this fact is valid for every recursively s -invariant product $\tau_{r_w}^{\sigma_w} \dots \tau_{r_1}^{\sigma_1}$. If we set $s_0 = s$ and, for $j > 0$, $s_j = \tau_{r_j}^{\sigma_j} \cdot s_{j-1}$, we see that $\{s_j\}_{j=0}^{j=m}$ is a sequence of real structures on Σ_g .

Proof of Theorem 3.1. We will follow the proof of the Lickorish-Wallace Theorem (see e.g. [23]). Given the real 3-manifold (M, c_M) , consider a real Heegaard splitting of M . Let the Heegaard surface H have genus g . Then we take a genus- g nonseparating real splitting of (S^3, c_{st}) (which exists for any $g > 0$; see [22] or [1]) so that the real map $s = c_{\text{st}}|_H$ equals $\tau_{\delta_1} \dots \tau_{\delta_g} \cdot s_{\max}$ for some s_{\max} -real disjoint curves $\delta_1, \dots, \delta_g$. Note that in these splittings, s and $c = c_M|_H$ are gluing maps. In the usual proof of the Lickorish-Wallace Theorem, the composition $c \cdot s$ is expressed as a product of Dehn twists and then each twist is extended over handlebodies. In our case, we express $c \cdot s$ as a recursively s -invariant product and show that that factorization describes an appropriate recursively equivariant sequence of (± 1) -surgeries. Now, using Lemma 3.2 and Lemma 3.3 we write:

$$\begin{aligned} c \cdot s &= (\tau_{\beta}^m \tau_{b_u} \dots \tau_{b_1} \cdot p \cdot s_{\max}) \cdot (\tau_{\delta_1} \dots \tau_{\delta_g} \cdot s_{\max}) \\ &= \tau_{\beta}^m \tau_{b_u} \dots \tau_{b_1} \cdot p \cdot \tau_{\delta_1}^{-1} \dots \tau_{\delta_g}^{-1} \end{aligned}$$

where $p = \tau_{r_w}^{2\sigma_w} \dots \tau_{r_1}^{2\sigma_1}$ is a (possibly empty) recursively s_{\max} -invariant factorization with even powers, b_j 's are in $\text{Fix}(p \cdot s_{\max})$; δ_j 's are s -invariant and s_{\max} -real; and $\beta \in \text{Fix}(c)$ with $m = 1$ if c is non-separating but not maximal and $m = 0$ otherwise.

We claim that the last factorization for $c \cdot s$ is recursively s -invariant. First since δ_j 's are disjoint s -invariant, following the terms of Lemma 3.3 we have:

$$\begin{aligned} s_0 &= s, \quad r_1 = \delta_g; \\ s_1 &= \tau_{\delta_g}^{-1} \cdot s, \quad r_2 = \delta_{g-1}; \\ &\vdots \\ s_g &= \tau_{\delta_1}^{-1} \dots \tau_{\delta_g}^{-1} \cdot s = s_{\max}, \quad r_{g+1} = \delta_1. \end{aligned}$$

Next, p is already recursively s_{\max} -invariant. Moving on with the remaining terms of the product we set $s_{g+w+1} = p \cdot s_{\max}$ and $r_{g+w+2} = b_1$. Since b_j 's are disjoint and $(p \cdot s_{\max})$ -real and β is c -real the claim of recursive s -invariance follows.

Finally we claim that each of these steps determines an appropriate equivariant (± 1) -surgery with respect to the framing determined by the initial surface H . As usual this is accomplished by *pushing* each curve equivariantly into the Heegaard handlebodies, as the following proposition shows, and thus the proof of Theorem 3.1 follows. \square

Proposition 3.4. *Let X and Y be real 3-manifolds with real Heegaard splittings (H, c) and (H, s) respectively. Suppose $s = \tau_\alpha^{\pm\sigma} \cdot c$ where α is a c -invariant curve on H . Then a $(\mp 1/2\sigma)$ -surgery in X along α , with framing determined by H , followed by uniquely extending the real structure over the surgered region gives a real 3-manifold equivariantly diffeomorphic to (Y, s) .*

Proof. Let H_1 and H_2 be the two handlebodies of the Heegaard splitting of (X, c) and let $X_0 \subset X$ be an equivariant neighborhood of H diffeomorphic to $H \times [-1, 1]$. Set $X_1 = \text{cl}(H_1 - X_0)$ which we consider identical to $X_2 = \text{cl}(H_2 - X_0)$. Then X is equivariantly diffeomorphic to $X_1 \cup_{\text{id}|_{\partial X_1}} X_0 \cup_{c|_{\partial X_0}} X_2$ (here the gluing maps are from the top boundary of the first space to the bottom boundary of the next) with the real structure \tilde{c} defined as

$$\tilde{c} : \begin{cases} X_1 \rightarrow X_2, x \mapsto x \\ X_0 \rightarrow X_0, x \mapsto c(x). \end{cases}$$

Similarly, we consider such a splitting $(Y_1, Y_0, Y_2; \tilde{s})$ for (Y, s) . Here we consider X_1, X_2, Y_1, Y_2 identical and X_0, Y_0 identical.

Let $\nu(\alpha)$ be an annulus neighborhood of α in H and $N = \nu(\alpha) \times [-1, 1]$ be a c -equivariant smooth neighborhood of α in X_0 . Since α is both c - and s -invariant, with a slight abuse, we will also denote by N a small s -equivariant neighborhood of α in Y_0 . We observe that the identity map (again with an abuse) is an equivariant homeomorphism from $X - N$ to $Y - N$, which can be made an equivariant diffeomorphism after smoothing. Now, with $T = S^1 \times D^2$ and φ a $(\mp 1/2\sigma)$ -sloped diffeomorphism on ∂N , $(X - N) \cup_\varphi T$ is diffeomorphic to Y . Here the previous diffeomorphism id_{X-N} extends trivially over T .

What the essence in the claim of the proposition is and what we have to show basically is that the above extension can be performed equivariantly. Indeed the real structure $\tau_\alpha^{\pm 2\sigma} c|_{\partial T}$ (here $\tau_\alpha^{\pm 1}$ is considered to be a twist around a copy of α on $\partial T \cong_\varphi \partial N$) can be extended to a real structure over T uniquely. In fact,

(i) If $c|_\alpha = c_1$ then the unique extension over T up to equivariant isotopy is

- c_1 . This corresponds to a type-1₁ equivariant surgery (see Theorem 2.4).
- (ii) If $c|_\alpha = c_2$, i.e. α is real, then $\tau_\alpha^{\pm 2\sigma} c|_{\partial T}$ is equivariantly diffeomorphic to c_2 since the surgery coefficient $q = \pm 2\sigma$ is even. This corresponds to a type-2₂ equivariant surgery.
- (iii) If $c|_\alpha = c_3$ then $c|_{\partial N} = c_3$ or c_4 . The former is impossible on a real Heegaard surface. Then $\tau_\alpha^{\pm 2\sigma} c|_{\partial T}$ is equivariantly isotopic to c_3 or c_4 . This is an equivariant type-4₄ or 4₃ surgery. Of course the choices here are equivariantly diffeomorphic. □

4. Equivariant Lickorish-Wallace theorem

In this section, we will give an equivariant surgery description for a real 3-manifold and prove the Lickorish-Wallace Theorem in the equivariant setting. In general the surgery link provided by the Lickorish-Wallace Theorem need not accept any kind of symmetry in (S^3, c_{st}) . However we show here that it is always possible to construct an invariant surgery link.

Consider a real three-manifold (M, c_M) and a c_M -equivariant link $L = L_1 \cup \dots \cup L_n$ in M decorated with an integer n -tuple $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{Z}^n$. Let $(M(L), c_M(L), \sigma)$ denote the real 3-manifold resulting from c_M -equivariant σ -surgery along L , i.e. the collection of σ_i -surgeries along L_i 's, $1 \leq i \leq n$. (As usual the surgery coefficients are with respect to a standard or given reference framing.) We will usually write $(M(L), c_M(L))$ instead, when the surgery coefficient σ is clearly understood or is not explicitly required.

Definition 4.1. On a real surface (Σ_g, c) , a product $\tau_{a_t}^{\sigma_t} \dots \tau_{a_1}^{\sigma_1} \cdot \tau_{b_k}^{\pm 1} \dots \tau_{b_1}^{\pm 1} \cdot \tau_{a'_1}^{\sigma_1} \dots \tau_{a'_t}^{\sigma_t}$ of Dehn twists is called an *equivariant product* for c if $c(a_i) = a'_i$ and b_j are disjoint c -invariant curves for all $1 \leq i \leq t$ and $1 \leq j \leq k$.

We start with a technical lemma.

Lemma 4.2. *Let (M, c_M) and $(M', c_{M'})$ be two real manifolds with the associated real Heegaard splittings (H, c) and (H, c') respectively. Assume that $c'c = \tau_{a_t}^{u_t} \dots \tau_{a_1}^{u_1} \cdot \tau_{b_k}^{v_k} \dots \tau_{b_1}^{v_1} \cdot \tau_{a'_1}^{u_1} \dots \tau_{a'_t}^{u_t}$ with all $u_i, v_j \in \{-1, +1\}$ and that this factorization is an equivariant product for c . Then there is an equivariant link L in (M, c_M) decorated with u_i 's and v_j 's such that the real 3-manifold $(M(L), c_M(L))$ (with respect to a framing induced by a real Heegaard surface) is equivariantly diffeomorphic to the manifold $(M', c_{M'})$.*

Proof. Topologically, the manifold M' can be obtained from M following Lickorish and Wallace. Let $M = U_1 \cup_c U_2$ and $M' = U_1 \cup_{c'} U_2$ where U_1 and

U_2 are two genus- g handlebodies bounded by H . Instead, we consider a splitting of (M, c_M) , similar to the proof of Proposition 3.4 but with more intermediate blocks, as follows. Take an equivariant neighborhood $H \times [-1, 1]$ of the Heegaard surface H with c_M fibered in the sense that $c_M|_{H \times \{t\}} = c$. Keeping this model in mind, set $H_i = H \times [i, i + 1]$, $-t - 1 \leq i \leq t$. Then (M, c_M) is equivariantly diffeomorphic to the manifold

$$U_1 \cup_{\text{id}} H_{-t-1} \cup_{\text{id}} \dots \cup_{\text{id}} H_{-2} \cup_{\text{id}} H \times [-1, 1] \cup_{\text{id}} H_1 \cup_{\text{id}} \dots \cup_{\text{id}} H_t \cup_c U_2$$

(as in the proof of Proposition 3.4, here and in every similar notation below the gluing maps are to be understood from the *top* boundary of the previous space to the *lower* boundary of the next) with the real structure defined as

$$\tilde{c}_M = \begin{cases} \text{id} : U_1 \rightarrow U_2, \\ c_M : H_i \rightarrow H_{-i-1}. \end{cases}$$

(See Figure 3, left.) We will use this splitting and real structure for (M, c_M) , and write c_M for \tilde{c}_M , abusing the notation

We push the curves a_i to the knots K_i on the surface h_{-i} (which is the *lower* boundary of H_{-i} ; see Figure 3), the curves a'_i to \bar{K}_i on h_i (lower boundary of H_i) and the curves b_j determine knots C_j on h_0 (which are mutually disjoint by assumption), so that we get the surgery link

$$(L, \sigma) = (\bar{K}_t, u_t) \cup \dots \cup (\bar{K}_1, u_1) \cup (C_k, v_k) \cup \dots \cup (C_1, v_1) \cup \dots \cup (K_t, u_t)$$

for the manifold M' . The surgery coefficients here are with respect to the associated surface h_i . Note that L is a c_M -equivariant link, as the knot pair K_i and \bar{K}_i are swapped by $\tilde{c}_M|_H = c_M|_H = c$ for each $1 \leq i \leq t$ and C_j is a c -invariant knot for all $1 \leq j \leq k$.

We will prove in two steps that the resulting real manifold $(M(L), c_M(L))$ is not only diffeomorphic but also equivariantly diffeomorphic to $(M', c_{M'})$. First we will show that $(M(L), c_M(L))$ is equivariantly diffeomorphic to the intermediate real manifold (E, e) where

$$E = U_1 \cup_{\text{id}} H_{-t-1} \cup_{\tau_{a_t}^{u_t}} \dots \cup_{\tau_{a_1}^{u_1}} H_{-1}, 0] \cup_{\tau_{b_k}^{v_k} \dots \tau_{b_1}^{v_1}} H_0 \cup_{\tau_{a_1}^{u_1}} \dots \cup_{\tau_{a_t}^{u_t}} H_t \cup_c U_2$$

with the real structure

$$e = \begin{cases} \text{id} : U_1 \rightarrow U_2, \\ c_M : H_i \rightarrow H_{-i-1} \text{ (fibered as before)}. \end{cases}$$

(See Figure 3, right.) For the sake of simplicity of the demonstration, let

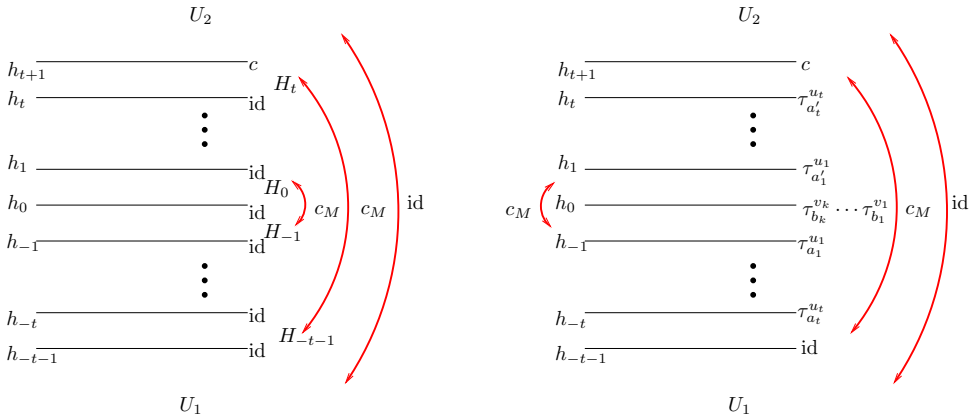


Figure 3. The real splittings associated to (M, c_M) (left) and (E, e) (right). Gluing maps are shown on the right of the surfaces, real structures are shown as –red– thick lines.

us take $k = t = 1$ and drop the indices and powers. Then we have a single invariant knot b lying on the surface h_0 , and a pair of knots (a, a') swapped by the real structure in both manifolds. The general case where there are more curves is handled similarly. Now, take an equivariant neighborhood (a solid torus) V_0 of the curve b , by first taking an equivariant annular neighborhood of b on h_0 , product with a small interval I so that the neighborhood remains equivariant and between but away from the surfaces h_1 and h_{-1} . Similarly take an equivariant pair of neighborhoods in M as follows: neighborhood V' of a' between h_2 and h_0 , and V of a between h_0 and h_{-2} . Maybe the only essential observation here is that since a is invariant under τ_a , $\text{id}(V)$ is a neighborhood of a in E too (similarly for a' and b). Between the complements of the interiors of these solid tori in M and E , there is the equivariant diffeomorphism

$$\text{id} : M - (\overset{\circ}{V}' \cup \overset{\circ}{V}_0 \cup \overset{\circ}{V}) \rightarrow E - (\overset{\circ}{V}' \cup \overset{\circ}{V}_0 \cup \overset{\circ}{V});$$

here $\overset{\circ}{V}$ denotes the interior of V .

For V in M , take a meridian μ and a longitude λ that is a copy of a ; then V in E has meridian $\mu \pm \lambda$. Similarly for V_0 and V' . Then we excise these solid tori from M and glue back identical solid tori T, T_0, T' respectively. The gluing maps are (± 1) -sloped boundary diffeomorphisms ψ, ψ' (i.e. $\mu \mapsto \mu \pm \lambda$) for V and V' , while for T_0 an equivariant diffeomorphism $\psi_0 : (\partial T_0, c_*) \rightarrow (\partial V_0, e|_{V_0})$ can be taken. Here $c_* = c_1, c_2$ or c_4 . Thus one

obtains $(M(L), c(L))$ where the real structure $c(L)$ is given as:

$$c(L) = \begin{cases} c_M : M - \mathring{V}' - \mathring{V}_0 - \mathring{V} \rightarrow M - \mathring{V}' - \mathring{V}_0 - \mathring{V}, \\ \text{id} : T \rightarrow T', \\ c_* : T_0 \rightarrow T_0, \end{cases}$$

Now we define a diffeomorphism F from M_L to E as follows:

$$F = \begin{cases} \text{id} : M - \mathring{V}' - \mathring{V}_0 - \mathring{V} \rightarrow E - \mathring{V}' - \mathring{V}_0 - \mathring{V}, \\ \psi' : T' \rightarrow V', \\ \psi_0 : T_0 \rightarrow V_0, \\ \psi : T \rightarrow V. \end{cases}$$

It is straightforward to check that $F \circ c(L) = e \circ F$ so that F is an equivariant diffeomorphism.

In the second step, we prove that (E, e) is equivariantly diffeomorphic to $(M', c_{M'})$. Consider the following splitting of $(M', c_{M'})$:

$$U_1 \cup_{\text{id}} H_{-t-1} \cup_{\text{id}} \dots \cup_{\text{id}} H_{-1} \cup_{c'} H_0 \cup_{\text{id}} \dots \cup_{\text{id}} H_t \cup_{\text{id}} U_2$$

with the real structure

$$c_{M'} = \begin{cases} \text{id} : U_1 \rightarrow U_2, \\ \text{id} : H_i \rightarrow H_{-i-1}. \end{cases}$$

(See Figure 4.) For each i define the functions f_i and \bar{f}_i as follows:

$$\begin{aligned} f_i &= \left(c \cdot \tau_{a_i}^{-u_t} \dots \tau_{a_{i+1}}^{-u_{i+1}} \right) \times \text{id} : H_i \rightarrow H_i \\ \bar{f}_i &= \left(\tau_{a_t}^{u_t} \dots \tau_{a_{i+1}}^{u_{i+1}} \right) \times \text{id} : H_{-i-1} \rightarrow H_{-i-1} \end{aligned}$$

Now define a map from (E, e) to $(M', c_{M'})$ using the above functions:

$$F = \begin{cases} \text{id} : U_1 \rightarrow U_1, \\ \text{id} : U_2 \rightarrow U_2, \\ f_i : H_i \rightarrow H_i, \\ \bar{f}_i : H_{-i-1} \rightarrow H_{-i-1}, \end{cases}$$

It is straightforward to check that F is a well-defined diffeomorphism from E to M' on the boundaries of the thickened surfaces and handlebodies. We

do that just for h_0 :

$$F^{-1}c'F|_{h_0 \subset \partial H_0} = \bar{f}_0^{-1}c'f_0 = \tau_{a_1}^{-u_1} \dots \tau_{a_t}^{-u_t} c' c \tau_{a'_t}^{-u_t} \dots \tau_{a'_1}^{-u_1} = \tau_{b_k}^{v_k} \dots \tau_{b_1}^{v_1}.$$

Finally F is an equivariant diffeomorphism:

$$FeF^{-1} = \left\{ \begin{array}{l} \text{id} : U_1 \rightarrow U_2 \\ \bar{f}_i^{-1} c_M f_i^{-1} = \text{id} : H_i \rightarrow H_{-i-1} \end{array} \right\} = c_{M'}$$

□

Proof of Theorem 1.4. Let (M, c_M) be a real 3-manifold with a real Heegaard splitting (H, c) and fix a real Heegaard splitting (H, s) of (S^3, c_{st}) with the same surface H (s is to be determined below). The above lemma proves in particular that in case $c \cdot s$ has a factorization which is an equivariant product of Dehn twists, one gets an equivariant surgery description of a real 3-manifold M starting from (S^3, c_{st}) . So to prove the theorem it suffices to show that $c \cdot s$ has a factorization which is an equivariant product. Indeed first assume that the real part of M is non-empty and consists of the circles r_0, \dots, r_k in H . The real structure $c' = \tau_{r_1} \dots \tau_{r_k} \cdot c$ on H has the real part r_0 and is non-separating if and only if either $k \geq 1$ or $k = 0$ and c is non-separating. We know that there is a real Heegaard splitting (H, s) of

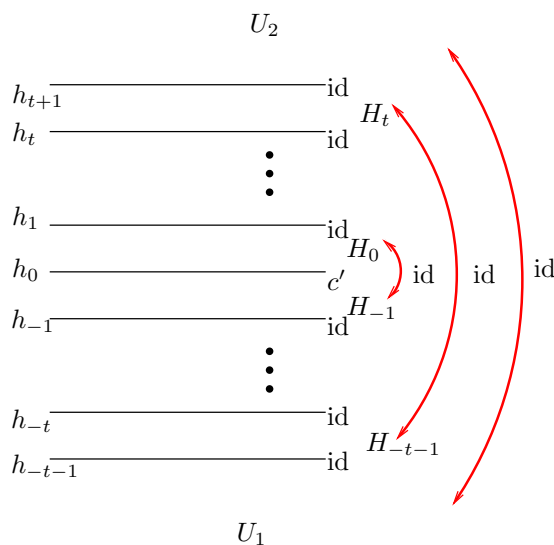


Figure 4. The splitting associated with $(M', c_{M'})$.

(S^3, c_{st}) with a non-separating real part $r \subset H$. Furthermore in case $k = 0$ but c is separating, (S^3, c_{st}) can be assigned a real Heegaard splitting (H, s) with separating s (e.g. see Figure 7). By the classification of real structures on compact surfaces, c' and s are conjugate by an orientation preserving diffeomorphism f . Set $b_i = f(r_i)$ and note that $b_0 = r$. Then we have

$$\begin{aligned} s &= f^{-1} \cdot c' \cdot f \\ &= f^{-1} \cdot \tau_{r_1} \dots \tau_{r_k} \cdot c \cdot f \\ &= (f^{-1} \cdot \tau_{r_1} \cdot f) \cdot f^{-1} \dots f \cdot (f^{-1} \cdot \tau_{r_k} \cdot f) \cdot f^{-1} \cdot c \cdot f \\ &= \tau_{b_1} \dots \tau_{b_k} \cdot f^{-1} \cdot c \cdot f \end{aligned}$$

The curves b_1, \dots, b_k on H are now c_4 -invariant with respect to s . Let $f = \tau_{a_t}^{u_t} \dots \tau_{a_1}^{u_1}$ be an expression for f in terms of generators of the mapping class group of H . Then

$$\begin{aligned} c &= f^{-1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot s \cdot f \\ &= \tau_{a_t}^{u_t} \dots \tau_{a_1}^{u_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot s \cdot \tau_{a_1}^{-u_1} \dots \tau_{a_t}^{-u_t} \\ &= \tau_{a_t}^{u_t} \dots \tau_{a_1}^{u_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot (s \cdot \tau_{a_1}^{-u_1} \cdot s^{-1}) \cdot s \dots s^{-1} \cdot (s \cdot \tau_{a_t}^{-u_t} \cdot s^{-1}) \cdot s \\ &= \tau_{a_t}^{u_t} \dots \tau_{a_1}^{u_1} \cdot \tau_{b_k}^{-1} \dots \tau_{b_1}^{-1} \cdot \tau_{s(a_1)}^{u_1} \dots \tau_{s(a_t)}^{u_t} \cdot s. \end{aligned}$$

The factorization for $c \cdot s$ that appears in the last line above is an equivariant product for s . We complete the proof using the Lemma 4.2 and its proof to get an equivariant link of the form $\mathcal{L} = L \cup L_S \cup \bar{L}$ so that L_S is an equivariant unlink consisting of c_4 -knots and $c_{st}(L) = \bar{L}$.

For the case where the real manifold (M, c_M) has empty real part, by [24, Lemma 4.3.1], $\tau_r \cdot s$ is a real structure with empty real part. We follow exactly the same steps and end up with an equivariant link $\mathcal{L} = L \cup L_S \cup \bar{L}$ so that L_S is a c_2 -knot (the only real knot) in S^3 and $c_{st}(L) = \bar{L}$. □

Let us note the crucial issue that the surgery along the c_4 -knots in L_S is of type 4_2 (purely by the construction of the proof). In other words, the real structure c_* that appears in the expression of the equivariant diffeomorphism ψ_0 in the proof of Lemma 4.2 is c_2 .

5. Three-manifolds with real contact structures

The main result of this section is Theorem 1.1 that asserts that every real 3-manifold admits a real contact structure. In order to prove that, we simply

show that the equivariant surgeries in the previous section can be realized in the contact setting. The proof displays a constructive algorithm. Before that we prove two technical lemmata.

The first one is a Legendrian realization principle in the real setting. Recall that the Legendrian realization principle Theorem 3.7 in [13] gives a criterion for making a collection C of non-isolating curves and arcs on a convex surface Legendrian after a perturbation of the convex surface. (A set C of disjoint curves is said to be non-isolating if C is transverse to Γ_S , every arc in C begins and ends on Γ_S , and every component of $S - C$ has nonempty intersection with Γ_S .) We are going to prove a weaker version in the real setting, for an invariant set of curves on an anti-symmetric convex surface, imposing an additional condition on the set C .

Lemma 5.1. *Let S be an anti-symmetric closed convex surface in (M, ξ, c_M) so that $c = c_M|_S$ is an orientation reversing involution and Γ_S is an anti-symmetric oriented dividing set. Let C be a non-isolating c -invariant set of curves. Assume $S - (\Gamma_S \cup C \cup \text{Fix}(c))$ consists only of disjoint components $S_1, S'_1, \dots, S_k, S'_k$ with $c(S_i) = S'_i$ for $1 \leq i \leq k$. Then there exists an equivariant isotopy ϕ_s ($s \in [0, 1]$) of S :*

- (1) $\phi_0 = \text{id}$;
- (2) $\phi_s(S)$ is convex;
- (3) $\phi_1(\Gamma_S) = \Gamma_{\phi_1(S)}$;
- (4) $\phi_1(C)$ is Legendrian and $(\phi_1 \circ c \circ \phi_1^{-1})$ -invariant.

Proof. In the original proof of Legendrian realization principle, one constructs a singular foliation F containing the set C (or an isotopic copy) which is also divided by Γ_S . Then using the Flexibility Theorem, this foliation can be made a characteristic foliation for Γ and since the leaves of the characteristic foliation are Legendrian, so is the set C . For a detailed proof see [13] or [4].

We will not reproduce the proof here but instead we will point out how one can modify it equivariantly. The singular foliation F is constructed around a boundary collar of each component of $S - (\Gamma_S \cup C)$ and then is extended to the interior. The assumption that C is non-isolating makes it possible to extend C to a singular foliation.

In our case, we consider the set $C \cup \text{Fix}(c)$. Γ_S and $\text{Fix}(c)$ intersect transversally and nontrivially since Γ_S is anti-symmetric, which guarantees that $C \cup \text{Fix}(c)$ is non-isolating provided C is non-isolating. Since $\text{Fix}(c)$ is Legendrian by definition, the components of $S - (\Gamma_S \cup C \cup \text{Fix}(c))$ may be treated as if they are components of $S - (\Gamma_S \cup C)$ with Legendrian boundary

Fix (c) . Now we can construct a singular foliation on each piece S_i in exactly the same way as in the original proof, then extend it to S'_i equivariantly. Then we use the real version of Giroux’s Flexibility Theorem [21, Proposition 3.4] to conclude. \square

Note that if c is separating, then $S - (\Gamma_S \cup C \cup \text{Fix}(c))$ already consists of equivariant pair of components. The assumption is needed when c is non-separating.

The last assumption of the lemma is indispensable. Indeed we are going to show in the next section that there are cases where the assumption does not hold and equivariant Legendrian realization is impossible (see Remark 6.2).

The second lemma is about convex Heegaard splittings of the real tight S^3 , similar to the observations in [22, Section 2.1].

Lemma 5.2. *For any genus $g \geq 1$, $(S^3, \xi_{\text{st}}, c_{\text{st}})$ has a real Heegaard splitting where the Heegaard surface H is convex. Moreover, there are g disjoint c_4 -circles m_1, \dots, m_g on H such that $|m_i \cap \Gamma| = 4$ where Γ is the dividing set of H .*

Proof. We will start with a real open book decomposition of (S^3, c_{st}) with disk pages and by positive real stabilizations we will increase the genus of the induced Heegaard splitting. This proves the first claim as the Heegaard surface induced by an open book decomposition is convex.

Now let $(S_0 = D^2, f_0 = \text{id}, c_0 = \rho_0)$ be the real open book decomposition of (S^3, c_{st}) where ρ_0 is the reflection with respect to the y -axis. After a real positive stabilization of type II, the resulting real pages are annuli with the real structure shown in Figure 5.

Performing real positive stabilizations of type II g times successively, the real pages (S_-, c_-) and (S_+, c_+) become disks with g holes with $c_- = \rho_g \tau_{a_1} \dots \tau_{a_g}$ and $c_+ = \rho_g$ where ρ_g is the reflection with $g + 1$ real arcs; see Figure 6.

We obtain a real Heegaard splitting with convex surface $H = S_- \cup S_+$ of genus g . The dividing set is the binding and the gluing map is $s = c_- \cup c_+$. The blue curves shown in Figure 6 are c_4 -curves with respect to s and satisfy the conditions of the lemma. \square

Proof of Theorem 1.1. Let (M, c_M) be a real 3-manifold with the real Heegaard splitting (H, c) . By Theorem 1.4 there is an equivariant link $\mathcal{L} = L \cup L_S \cup \bar{L}$ and $\sigma \in \{-1, +1\}^n$ in (S^3, c_{st}) such that $(S^3_{\mathcal{L}}, c(\mathcal{L}), \sigma)$ is

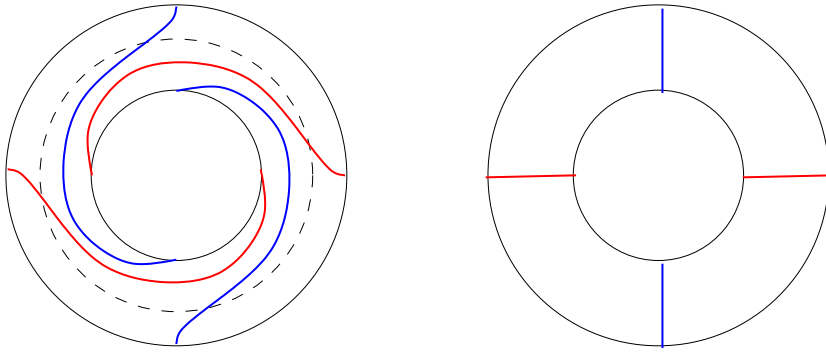


Figure 5. The real pages after a real positive stabilization of type II; left: (S_-, c_-) and right: (S_+, c_+)

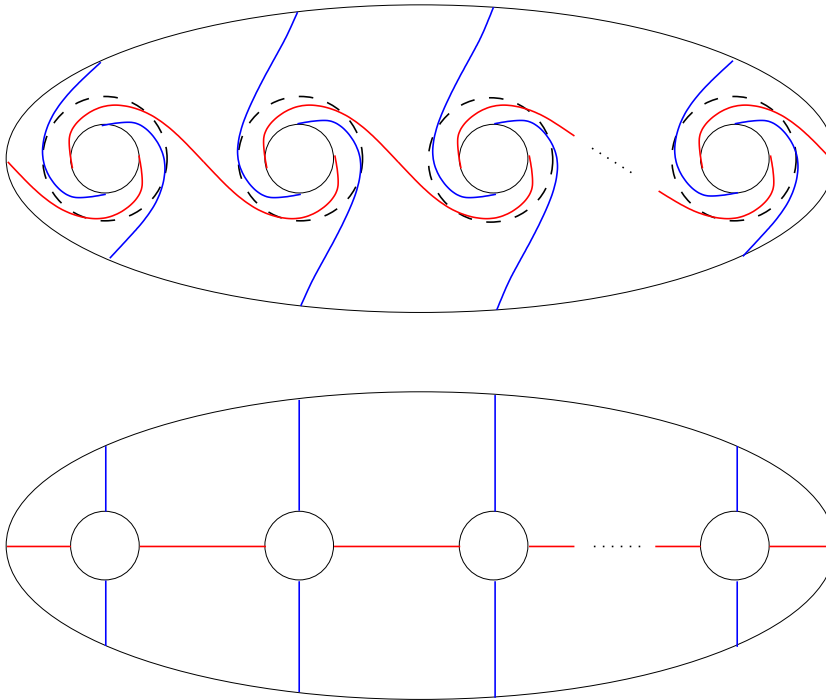


Figure 6. The real pages (S_-, c_-) (top) and (S_+, c_+) (bottom) for (S^3, c_{st}) . The red arcs are real, the blue arcs are c_{st} -invariant.

equivariantly diffeomorphic to (M, c_M) . Here the link L_S consists of unlinked c_{st} -invariant knots and L and \bar{L} are swapped by c_{st} . To turn this

equivariant surgery into an equivariant contact surgery, we must put L_S into an equivariant Legendrian position so that the components of L_S satisfy the hypotheses of Theorem 2.4, summarized in Table 1. For L and \bar{L} , it is enough to make them Legendrian respecting the equivariance.

Using the set up in the proof of Theorem 1.4, let $\text{Fix}(c)$ consist of $k + 1$ real circles r_0, \dots, r_k . Consider (S^3, c_{st}) with the real contact Heegaard splitting (H, s) of genus k induced by the real open book decomposition given in Lemma 5.2. Let $r \subset H$ denote the real circle of s . There are k disjoint c_4 -circles (with respect to s) m_1, \dots, m_k on H by Lemma 5.2. Then $c' = \tau_{m_1}^{-1} \dots \tau_{m_k}^{-1} \cdot s$ is a real structure with $\text{Fix}(c') = \{r, m_1, \dots, m_k\}$. Now that c and c' are conjugate, there is an orientation preserving diffeomorphism f of H given by a product $\tau_{a_1}^{\sigma_1} \dots \tau_{a_t}^{\sigma_t}$ such that $c = f \circ c' \circ f^{-1}$. Following the steps in the proof of Lemma 4.2, we get an equivariant surgery link $L \cup L_S \cup \bar{L}$ where L_S consists of the knots m_1, \dots, m_k on H . On each m_i , we perform a type-4₂, equivariant (-1) -surgery. Note that since $\{m_1, \dots, m_k\}$ is a non-isolating set of curves transverse to the dividing set, we can make each m_i Legendrian with an equivariant perturbation of H by Lemma 5.1. Moreover for each modified m_i , $\text{tw} = -2$ since $|m_i \cap \Gamma| = 4$. Hence, the condition in Theorem 2.4 to perform an equivariant contact surgery on a c_4 -knot is satisfied and we perform an equivariant $(+1)$ -contact surgery on each m_i .

Note that if $k = 0$ and r_0 is separating, we cannot use the real Heegaard splitting given by Lemma 5.2 since r is non-separating there, so that c and s would not be conjugate. In that case we use the real Heegaard splitting induced by the real open book given in Figure 7. (See also [22, Figure 6].) This open book is obtained from the simplest real open book decomposition of S^3 with disk pages and succesively attaching handles of type III k times. The real circle is separating in this case and c and s are conjugate. The rest of the proof follows similarly as above. □

Proof of Corollary 1.5. This follows immediately from Proposition 1.2 and the proof of Theorem 1.1. Given any real integer homology sphere (Σ, s) the proof of Theorem 1.1 shows that one can obtain an s -real contact structure η on Σ via an equivariant contact diagram Γ in $(S^3, c_{\text{st}}, \xi_{\text{st}})$. Then consider the real contact 3-manifold $(\Sigma, s, \eta) \# (S^3, c_{\text{st}}, \xi_p), p \in \mathbb{Z}$, which is equivariantly diffeomorphic to (Σ, s) . Besides the final contact structure is overtwisted, since ξ_p is. In an integer homology sphere, the set of overtwisted contact structures up to contact isotopy is countably infinite and they are distinguished by the d_3 invariant. Again from [3, Lemma 4.2] it follows that varying p one exhausts all overtwisted structures on (Σ, s) . Each is given by the equivariant contact diagram Γ adjoined with the one for $(S^3, c_{\text{st}}, \xi_p)$ obtained in the proof of Proposition 1.2. □

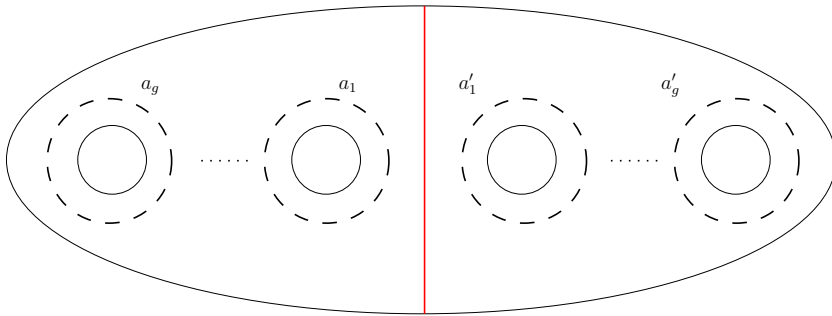


Figure 7. A real open book of S^3 after stabilizations of type III. The real page (S_-, c_-) is depicted.

6. Examples

This section is a display for the applications of the ideas we have harvested in the previous sections. For some basic real 3-manifolds ($S^1 \times S^2$ and lens spaces), we will give equivariant surgery descriptions and provide equivariant contact surgery diagrams in (S^3, c_{st}, ξ_{st}) . Of course, given surgery diagrams for these manifolds we could readily make them equivariant. However then we would not have any control on the real structure obtained at the end. In this section we turn surgery diagrams into equivariant ones with a careful control on the final real structure. For a collection of similar results see [20].

Below we use the genus-1 real contact Heegaard decomposition of (S^3, c_{st}) with the gluing map $c_{st} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with abuse of notation. The convexity of the Heegaard surface is guaranteed by

- Lemma 6.1.** (a) *There is a real contact Heegaard decomposition of (S^3, c_{st}, ξ_{st}) of genus 1, with the Heegaard map c_{st} .*
 (b) *Moreover a parallel copy of the dividing set (which does not satisfy the hypothesis of Lemma 5.1) can be realized in a Legendrian way on the Heegaard surface.*

Proof. We consider the setup in [8, Example 4.6.21] in the real 3-sphere $\{r_1^2 + r_2^2 = 2\} \in \mathbb{R}^4$ with

$$c_{st}(r_1, \varphi_1, r_2, \varphi_2) = (r_2, -\varphi_2, r_1, -\varphi_1).$$

The invariant torus $T : r_1 = r_2 = 1$ of the Hopf vector field is an embedded anti-symmetric Heegaard surface naturally, with the gluing map c_{st} . As noted in [8], such a torus is not convex. However it can be made convex equivariantly using the equivariant perturbation similar to [8, Example 4.8.10]. There the ambient contact space $(T^2 \times \mathbb{R}, d\theta/2 + zd\varphi)$ can be made real by defining $c'(\theta, \varphi, z) = (-\theta, \varphi, -z)$. Now an equivariant neighborhood of the anti-symmetric torus $\{z = 0\}$ is equivariantly contactomorphic to an equivariant neighborhood of T via the map $f : \theta = \varphi_1 + \varphi_2, \varphi = \varphi_1 - \varphi_2, z = r_1^2 - 1$, i.e. f satisfies $f \circ c_{st} = c' \circ f$. Finally observe that the second characteristic foliation in [8, Example 4.8.10] is c' -real when $p = 1$ and $q = 0$. In that (standard) characteristic foliation a knot parallel to the dividing set is Legendrian (see the corresponding figure in [8]). \square

Now we are ready to move on towards our basic examples.

6.1. $S^1 \times S^2$

We refer to the work of J. L. Tollefson [25] for the classification of involutions on $S^1 \times S^2$ up to conjugation by a diffeomorphism of $S^1 \times S^2$. There are 13 classes in total and 4 of them are orientation preserving with one dimensional fixed point set. Hence there are 4 real structures on $S^1 \times S^2$ with nonempty real part, up to equivalence. If we identify $S^1 \times S^2$ as $\{(\theta, (x, y, z)) : x^2 + y^2 + z^2 = 1\} \subset S^1 \times \mathbb{R}^3$ then we can express two of these real structures:

- (1) $s_1 : (\theta, (x, y, z)) \mapsto (\theta, (-x, -y, z))$,
- (2) $s_2 : (\theta, (x, y, z)) \mapsto (-\theta, (x, y, -z))$,

each of which has two real circles.

If we consider $S^1 \times S^2$ as the space $[-1, 1] \times S^2$ with $(-1, (x, y, z))$ and $(1, (-x, -y, z))$ identified, then we can express the remaining two of these real structures as:

- (3) $s_3 : (t, (x, y, z)) \mapsto (t, (x, -y, -z))$,
- (4) $s_4 : (t, (x, y, z)) \mapsto \begin{cases} (1 - t, (-x, -y, -z)) & \text{if } 0 \leq t \leq 1 \\ (-1 - t, (x, y, -z)) & \text{if } -1 \leq t < 1 \end{cases}$,

each of which has one real circle.

Each of the real structures above can be associated with a real genus-1 Heegaard splitting of $S^1 \times S^2$. Consider the real structures on $T = S^1 \times S^1$

given by the matrices

$$(6.1) \quad s_1 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, s_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, s_3 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, s_4 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

The abuse of notation here relies on the fact that the abstract real Heegaard splitting (T, s_i) for any $k \in \mathbb{Z}$ is associated to the real 3-manifold $(S^1 \times S^2, s_i)$. This can be seen by computing the quotients and the number of real components. (Any even -resp. odd- integer at the top right entry of s_1 or s_2 -resp. s_3 or s_4 - would give the same result.)

Now that we have an abstract real Heegaard splitting for each real $S^1 \times S^2$, we will demonstrate how to obtain them via equivariant surgery from the standard real 3-sphere via Theorem 1.4. Below a and b are the curves $S^1 \times \{1\}$ and $\{1\} \times S^1$ on T ; $\tau_a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\tau_b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ are the Dehn twists along a and b respectively.

(1) The gluing involution s_1 satisfies $s_1 = \tau_{a+b}^{-1} c_{st}$ where $a + b$ is a c_4 -knot for c_{st} . Then $(S^1 \times S^2, s_1)$ can be obtained from (S^3, c_{st}) by an equivariant (-1) -surgery of type 4_2 on the unknot represented by $a + b$, with respect to the framing induced from the Heegaard surface H . Since the Heegaard framing minus the Seifert framing along $a + b$ is $+1$, a (-1) -surgery with respect to H corresponds to a topological 0-surgery on the unknot, which gives of course $S^1 \times S^2$.

(2) We have $s_2 = \tau_{a-b} c_{st}$ where $a - b$ is a c_1 -knot for c_{st} . Then $(S^1 \times S^2, s_2)$ can be obtained from (S^3, c_{st}) by an equivariant $(+1)$ -surgery of type 1_1 on the unknot represented by $a - b$, with framing with respect to H .

(3) For $(S^1 \times S^2, s_3)$, we have $s_3 = \tau_b^{-1} \tau_a^{-1} c_{st}$, and $c_{st}(a) = b$. Hence $(S^1 \times S^2, s_3)$ is given by an equivariant (-1) -surgery of type 5 on the Hopf link formed by the knot pair (a, b) . Here Seifert framing and Heegaard framing of a coincide; similarly for b .

(4) For $(S^1 \times S^2, s_4)$, we have $s_3 = \tau_b \tau_a c_{st}$. Hence $(S^1 \times S^2, s_3)$ is given by an equivariant $(+1)$ -surgery of type 5 on the Hopf link formed by the knot pair (a, b) .

The tight contact structure on $S^1 \times S^2$, unique up to isotopy, is given by

$$\xi = \ker(xd\theta + ydz - zdy).$$

Obviously the contact structure ξ is real with respect to s_1 and s_2 . However it is not obvious at all whether ξ is real with respect to s_3 and s_4 . Below

we also answer that question for s_4 affirmatively using equivariant contact surgery.

The unique tight contact structure ξ of $S^1 \times S^2$ is obtained from (S^3, ξ_{st}) through contact $(+1)$ -surgery on the unknot with Thurston-Bennequin number $\mathbf{tb} = -1$ [3] (see Figure 8, left). This contact surgery diagram can be

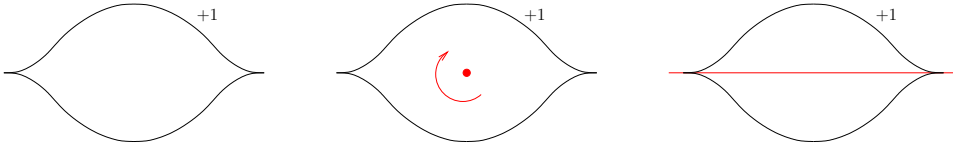


Figure 8. The contact surgery diagram (left) for the unique tight contact structure on $S^1 \times S^2$ can be made s_1 -real (middle) and s_2 -real (right)

readily made equivariant in two ways as depicted in Figure 8, middle and right. Now we argue that these correspond to the equivariant contact surgery descriptions for $(S^1 \times S^2, s_1, \xi)$ and $(S^1 \times S^2, s_2, \xi)$ respectively.

We have already observed that $(S^1 \times S^2, s_1)$ is given by an equivariant (-1) -surgery (with respect to H) of type 4_2 on the unknot represented by $a + b$. We consider a real Heegaard splitting (H, c_{st}) of (S^3, c_{st}, ξ_{st}) ; here, thanks to Lemma 6.1(a), we may assume that H is an equivariant convex torus with two parallel dividing curves, each represented by $a - b$. Here, the Real Legendrian Realization Principle Lemma 5.1 cannot be used to get $a + b$ equivariantly Legendrian on H . Instead we employ Lemma 6.1(b) to accomplish that. Since the contact twisting along $a + b$ with respect to H is $\text{tw}(a + b; H) = -2$, the contact surgery coefficient is $+1$. Moreover, $\mathbf{tb}(a + b) = -1$. Hence we get the corresponding equivariant contact surgery diagram, the one in the middle in Figure 8.

Similarly $(S^1 \times S^2, s_2)$ is given by an equivariant $(+1)$ -surgery of type 1_1 (with respect to H) on the unknot represented by $a - b$. In this the curve $a - b$ can be made equivariantly Legendrian on H according to Lemma 5.1. Then $\text{tw}(a - b; H) = 0$ as $(a - b)$ is parallel to the dividing set. Hence we get the contact surgery coefficient $+1 - 0 = +1$. Since $\mathbf{tb}(a - b) = -1$, we obtain the equivariant contact surgery diagram in Figure 8, right.

$(S^1 \times S^2, s_4)$ is obtained by performing a type 5 $(+1)$ -surgery on the knot pair formed by the curves (a, b) . We make a Legendrian (no need to employ Real Legendrian Realization Principle), and an equivariant copy represents a Legendrian unknot b . If these Legendrian unknots are arranged to have $\text{tw} = \mathbf{tb} = -1$ then we would need an equivariant pair of contact $(+2)$ -surgeries. However such a contact surgery is not well-defined; a priori

it is not unique. However in this case it is: either of the two choices made during the surgery leads to isomorphic structures (see [14, Example 5.3]). Anyhow this topological equivariant surgery must be turned into a well-defined equivariant contact one. For this we follow the idea in [14] to turn the contact (+2)-surgery along the Legendrian knot with $\mathbf{tb} = -1$ into contact (± 1)-surgeries along a pair of Legendrian knots K_1, K_2 and their equivariant copy L_1, L_2 as appears in Figure 9(a). Moreover, this contact diagram gives the unique tight structure on $S^1 \times S^2$.

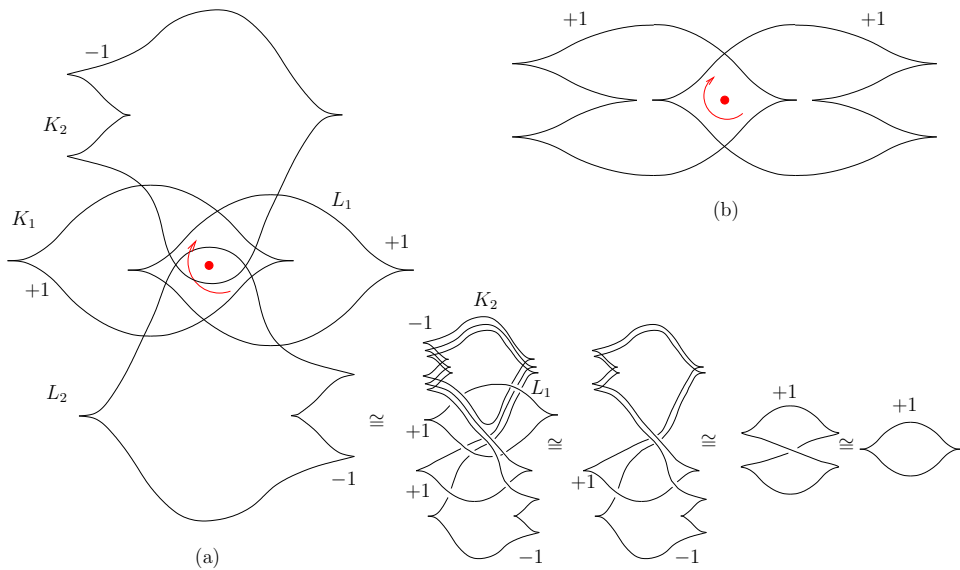


Figure 9. Equivariant contact surgery diagrams (a) for $(S^1 \times S^2, s_4, \xi)$ and (b) for $(S^1 \times S^2, s_3, OT)$.

A proof of tightness is shown in Figure 9. The first isomorphism is by contact sliding of K_1, L_2 over K_2 [2]. In the new diagram L_1 does not link with the slid copies of K_1, L_2 and thus it is a meridian of K_2 . The second isomorphism follows from the fact that a (-1) -contact surgery on some L and a $(+1)$ -contact surgery on its meridian cancels each other. One way to see that is by sliding the meridian over L to get a Legendrian push-off of L . A (-1) -contact surgery on L and a $(+1)$ on its push-off cancel. The third isomorphism is via applying [5, Lemma 2.9] (for two knots and for $n = +1$). This lemma is a generalization of [15, Proposition 2.4]. Finally the last isomorphism is a contact Reidemeister move. The discussion up to now proves Theorem 1.6.

Furthermore $(S^1 \times S^2, s_3)$ is given by a type 5 (-1) -surgery on the same Hopf link as above and a possible corresponding equivariant contact surgery diagram is shown in Figure 9(b). This s_3 -real contact structure is overtwisted (see e.g. [3]). So the attempt to find out whether ξ is s_3 -real or not is inconclusive with this particular surgery. We still do not know whether ξ is s_3 -real or not.

Remark 6.2. Let us start with the genus-1 real contact Heegaard decompositions of (S^3, c_{st}) with the gluing map $f = \tau_b s'_2$ instead of c_{st} , where $s'_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This decomposition is induced from the real open book depicted in Figure 5, thus the Heegaard surface H is equivariantly convex as it comes from a real open book. Since $s'_2 = \tau_b^{-1} f$, $(S^1 \times S^2, s_2)$ can be obtained from (S^3, c_{st}) via an equivariant (-1) -surgery of type 1_1 on the unknot represented by b , with framing with respect to H . We observe that b can be realized in a Legendrian way on H by the Legendrian Realization Principle but that cannot be done equivariantly. Indeed, first note that Lemma 5.1 does not apply here as its last assumption is not satisfied. If equivariant realization were possible, then $\text{tw}(b; H)$ would be zero as b is parallel to the dividing set. Hence we would get $\mathbf{tb}(b) = 0 + 1 = 1$. This is of course impossible for an unknot in the tight S^3 .

6.2. Lens spaces – real structures of type A

As in the previous paragraph let us consider an h -equivariant Heegaard torus T in the real lens space $(L(p, q), h)$, bounding solid tori V and V' . If h is orientation preserving on T and if $h|_V = c_1$ and $h|_{V'} = c_1$, then h is said to be of type A . It is interesting that for any p, q , $L(p, q)$ admits a real structure of type A , unique up to equivariant isotopy [12]. Moreover we claim and prove here Theorem 1.7 which asserts that any tight contact structure on $L(p, q)$ is A -real.

Let $p > q > 0$ and $[r_1, \dots, r_n]$, $r_i \leq -2$ be the continued fraction for $-p/q$. K. Honda proved that on $L(p, q)$, there exists exactly $|(r_1 + 1) \dots (r_k + 1)|$ tight contact structures up to isotopy. Moreover, each tight contact structure on $L(p, q)$ can be obtained from Legendrian $((-1)$ -contact) surgery on a link in the standard tight 3-sphere [13]. Topologically, surgery link is given by a chain of unknots with (ordered) coefficients r_i as in Figure 10. This surgery diagram in fact is an equivariant surgery diagram where all the unknots are c_1 -knots. Performing equivariant surgery produces $(L(p, q), A)$. Indeed, the quotient of the surgered manifold with the

real structure is S^3 . Since the only real structure on $L(p, q)$ having quotient S^3 is A [12], the real structure on the surgered manifold is of type A .

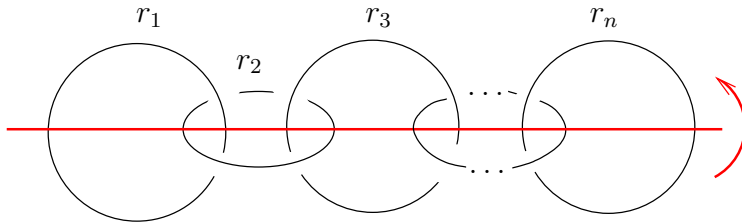


Figure 10. A surgery diagram for $L(p, q)$, which may be taken equivariant.

There are exactly $|(r_i + 1)|$ ways to stabilize an unknot to perform a contact (-1) -surgery on it. Moreover, each choice of stabilization will represent a Legendrian c_1 -unknot. Hence, each contact surgery diagram describing a tight contact structure on $L(p, q)$ is also an equivariant contact surgery diagram (consisting of Legendrian c_1 -knots and contact (-1) -surgeries of type 1_1) describing an A -real tight contact structure.

This argument proves Theorem 1.7.

Finally note that the equivariant contact surgery diagram we mentioned in the previous paragraph cannot be obtained by the construction in the proof of Theorem 1.4 since the latter does not involve any c_1 -knots.

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DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE
CHESTNUT HILL, MA 02467-3806, USA
E-mail address: merveseyhun@gmail.com

DEPARTMENT OF MATHEMATICS, BOGAZICI UNIVERSITY
BEBEK, 34342 ISTANBUL, TURKEY
E-mail address: ferit.ozturk@boun.edu.tr

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