JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 20, Number 6, 1349–1383, 2022

Polyhedral approximation by Lagrangian and isotropic tori

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We prove that every smoothly immersed 2-torus of $\mathbb{R}^4$ can be approximated, in the C^0-sense, by immersed polyhedral Lagrangian tori. In the case of a smoothly immersed (resp. embedded) Lagrangian torus of $\mathbb{R}^4$, the surface can be approximated in the C^1-sense by immersed (resp. embedded) polyhedral Lagrangian tori. Similar statements are proved for isotropic 2-tori of $\mathbb{R}^{2n}$.

1. Introduction

1.1. Statement of results

The main result of this paper is the following approximation theorem, in the context of piecewise linear symplectic geometry.

Theorem A. *Let Σ be a surface diffeomorphic to the 2-dimensional real torus and $\ell : \Sigma \to \mathbb{R}^{2n}$, a smooth isotropic immersion (resp. embedding), for some integer $n \geq 2$. Then, there exist piecewise linear isotropic topological immersions (resp. embeddings) $f : \Sigma \to \mathbb{R}^{2n}$, arbitrarily close to ℓ in the C^1-sense.*

A weak version of Theorem A appeared first in [7]. This improved result now deals with the stronger C^1-estimate and the case of critical dimension

$n = 2$, which is perhaps the most interesting. Indeed this new result can be applied to manifolds, rather than maps, and leads to construction of large families of immersed and embedded polyhedral Lagrangian tori of $\mathbb{R}^4$, obtained by approximation of the smooth ones, according to Corollaries B and C.

Corollary B. *For every integer $n \geq 2$, any smoothly immersed (*resp.. *embedded) isotropic 2-dimensional torus of $\mathbb{R}^{2n}$ can be approximated, in the C^1-sense, by arbitrarily close immersed (*resp.. *embedded) polyhedral isotropic tori.*

Combined with the h-principle, we obtain the following approximation theorem for smooth tori:

Corollary C. *For $n \geq 2$, every smoothly immersed 2-dimensional torus of $\mathbb{R}^{2n}$ can be approximated, in C^0-sense, by arbitrarily close immersed polyhedral isotropic tori.*

If $n \geq 3$, we can replace "immersed" with "embedded" in the above statement.

1.2. Some motivations on piecewise linear symplectic geometry

Flexibility issues have attracted a lot of attention in symplectic geometry. Similarly to the *sphere eversion theorem*, many flexibility questions arise in symplectic geometry and do not have obvious answers. Gromov invented powerful techniques, promoted under the name of *h-principle*, to study certain classes of underdetermined partial differential relations [5]. For example, by a theorem of Gromov, any smoothly immersed surface of $\mathbb{R}^4$ can be approximated by Lagrangian immersed surfaces, in the C^0-sense. Like many flexibility results in the framework of symplectic geometry, the proof of this fact relies on stability properties of symplectic structures and ordinary differential equation techniques. Those properties, specific to smooth manifolds, are not directly available for *piecewise linear* or *piecewise smooth manifolds*. Since these essential tools are missing, very little is known about natural versions of piecewise linear symplectic geometry. We mention now several elementary problems of symplectic geometry, whose piecewise linear counterparts are poorly understood:

1) In spite of the fact that triangulations are quite common in geometry and topology, it is not known whether a smooth symplectic manifold admits a *symplectic triangulation* (cf. the original work of Distexhe

in [2] for more details). Even the case of $\mathbb{CP}^n$ for $n \geq 2$ is not known. Conversely, it is not known if there are suitable smoothing techniques for piecewise linear symplectic manifolds.

2) By Darboux theorem, every smooth symplectic structure is locally standard, which shows that smooth symplectic manifold do not have any local invariants. However, there is no analogue of the Darboux theorem for piecewise linear symplectic manifolds. In fact, there might even be local obstructions for the local triviality of piecewise linear symplectic structures!

3) There are many constructions of Lagrangian manifolds for smooth symplectic manifolds and their deformation theory is very simple. This is natural since Lagrangian manifolds are central object for the symplectic topologists preoccupations. Yet, examples of polyhedral Lagrangian manifolds are scarce. As the main contribution of this paper, we constructs large families of polyhedral Lagrangian tori of $\mathbb{R}^4$ by approximation of the smooth ones in Corollaries B and C.

4) Suitable piecewise linear analogues of the group of symplectic and Hamiltonian diffeomorphisms are not well understood. A first approach was made by Gratza, who introduced some clever (and quite difficult) approximation techniques in [4].

Because piecewise linear or polyhedral geometry is such natural framework, it has attracted a lot of attention in the context of topology, in particular as a tool for studying the relation between smooth and topological manifolds. It seems natural to consider problems of a symplectic nature in the piecewise linear context. For instance, numerical experiments for various geometric flows involving symplectic manifolds rely on the use of polyhedral geometry for discretization purposes. Thus, natural questions about symplectic geometry arise from a numerical perspective as well. The goal of this work, initiated in [7], is an attempt to approximate smooth isotropic or Lagrangian manifolds using piecewise linear geometry. Although this is a first step, it is not clear at this stage what is the relationship between the smooth symplectic and piecewise linear symplectic worlds. Before we get a full picture tremendous efforts have yet to be made.

The proof of Theorem A relies on some techniques introduced in [7]. The main idea is that the problem of finding isotropic surfaces is equivalent to the problem of finding zeroes of a certain *Donaldson moment map*, by the *fixed point principle*. This may seem unusual, but this point of view also provides effective techniques for producing solutions via numerical methods and flows [6, 7]. However, a more straightforward approach, which is expected to work as well, is currently being taken care of by Etourneau in [3].

2. Background material

In this section, we provide some background material and introduce some vocabulary, that have been used in Theorem A and Corollaries B and C, without a proper introduction. Then, an overview of the techniques introduced in [7] is given, since these tools play a central role in the proof of Theorem A.

2.1. Basic definitions of symplectic geometry

The even dimensional Euclidean space $\mathbb{R}^{2n}$, identified to $\mathbb{C}^n$, comes equipped with canonical coordinates $z_j = x_j + iy_j$, for $1 \leq j \leq n$. This Euclidean space is endowed with the *canonical Euclidean inner product* $g = \sum dx_i^2 + dy_j^2$, the corresponding *Euclidean norm* $\|w\| = \sqrt{g(w,w)}$, the *symplectic* and *Liouville forms*

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j, \quad \lambda = \sum_{j=1}^{n} x_j dy_j$$

with the property that $\omega = d\lambda$.

Let Σ be a smooth manifold of real dimension 2, also called a *smooth surface*. A smooth map $\ell : \Sigma \to \mathbb{R}^{2n}$ is called *isotropic* if ω vanishes along ℓ, which is to say

$$\ell^*\omega = 0.$$

If ℓ is an immersion (resp. an embedding), its image $S = \ell(\Sigma)$ is called an *immersed (resp. embedded) isotropic surface* of $\mathbb{R}^{2n}$. In such situation, we have necessarily $n \geq 2$. If $n = 2$, which is the critical dimension, then S is called an immersed (resp. embedded) *Lagrangian* surface of $\mathbb{R}^4$.

2.2. Piecewise smooth and piecewise linear maps

2.2.1. Some basic definitions. We recall some standard definitions of piecewise linear topology.

A *triangulation* of a smooth closed surface Σ is a triple $\mathscr{T} = (\Sigma, \mathscr{K}, \phi)$, where $\mathscr{K}$ is a finite simplicial complex contained in some Euclidean space $\mathbb{R}^d$, together with a homeomorphism $\phi : |\mathscr{K}| \to \Sigma$, where $|\mathscr{K}|$ denotes the topological subspace of $\mathbb{R}^d$ associated to the simplicial complex $\mathscr{K}$. If the restriction of ϕ to every simplex s of the triangulation $\mathscr{K}$ is a *smooth embedding*, we say that the triangulation $\mathscr{T}$ is a *smooth triangulation.* Every surface triangulation in this paper is assumed to be smooth, even if this is not stated.

Given a triangulation $\mathscr{T}$ of Σ, any simplex s of $\mathscr{K}$ is diffeomorphic to its image $\phi(s) \subset \Sigma$, by definition. This identification being understood, s is also referred to as a simplex of Σ, and we drop all references to the map ϕ for simplicity of notations. As the simplicial complex $|\mathscr{K}|$ is homeomorphic to a surface, its simplices can only be 0, 1 and 2-dimensional. They are called respectively *vertices, edges* and *facets* of the triangulation.

A real (or vector) valued *piecewise smooth map* h on Σ is a continuous map, such that there exists a smooth triangulation $\mathscr{T}$ of Σ, with the property that the restriction of h to every facet of the triangulation is smooth. Such a triangulation is said to be *adapted* to the piecewise smooth map h.

Given a smooth triangulation $\mathscr{T}$ of Σ, we can define the class of *piecewise linear maps* on Σ. A real (or vector) valued map h on Σ is piecewise linear with respect to $\mathscr{T}$, if it is continuous and has piecewise linear restriction on each simplex of the triangulation. For a second triangulated surface $\mathscr{T}_2 = (\Sigma_2, \mathscr{K}_2, \phi_2)$, we say that a map $h : \Sigma \to \Sigma_2$ is *piecewise linear with respect to the triangulations* $\mathscr{T}$ and $\mathscr{T}_2$, if the composition $\phi_2^{-1} \circ h : \Sigma \to |\mathscr{K}_2| \subset \mathbb{R}^{d_2}$ is piecewise linear in the previous sense. A *piecewise linear isomorphism* between two triangulated surfaces is a bijective piecewise linear map between the triangulated surfaces. It turns out that the inverse map of a piecewise linear isomorphism is also piecewise linear.

If two triangulations of Σ are piecewise linear isomorphic, they define the same spaces of piecewise linear functions on Σ. A *piecewise linear structure* on a surface Σ is an equivalence class of triangulations, modulo piecewise linear isomorphisms. By definition, a piecewise linear structure on Σ has a well defined class of piecewise linear functions.

Remarks 2.2.2. By Cairns and Whitehead results [1, 8], smooth manifolds admit a unique piecewise linear structure, up to piecewise linear isomorphisms. In other words, the *smooth* triangulations used to define a piecewise linear structures on a smooth manifold exist. Furthermore, they are all piecewise linear isomorphic and locally standard.

According to these results, every linear map on Σ is understood with respect to the essentially unique piecewise linear structure deduced from the smooth structure (like in Theorem A for instance).

Given a piecewise linear map h on Σ with respect to some triangulation, we can always replace the triangulation by a refinement, so that the map h is affine on each simplex of the triangulation. This refinement operation does not change the associated piecewise linear structure. Such a refined triangulation is said to be *adapted* to h.

With the above terminology, a piecewise linear map is piecewise smooth with respect to some adapted triangulation.

Starting at §2.4, the surface Σ is a quotient torus $\mathbb{E}/\Gamma$, where Γ is a lattice of the Euclidean plane $\mathbb{E}$. In this setting, it is convenient use another point of view for triangulations. Suppose we are given a Γ-invariant locally finite *simplicial decomposition* of $\mathbb{E}$. Then there exists a triangulation $\mathscr{T} = (\mathbb{E}/\Gamma, \mathscr{K}, \phi)$, unique up to simplicial isomorphism of simplicial complex and the Γ-action, with the following properties :

1) For each simplex s of $\mathscr{K}$, let $\tilde{\phi} : s \to \mathbb{E}$ be a lift of map ϕ to the universal cover $\mathbb{E}$ of Σ. This map is uniquely defined up to the Γ-action. We require that $\tilde{\phi}$ is an affine map for every simplex s of $\mathscr{K}$.
2) The simplicial decomposition of $\mathbb{E}$ is recovered as the collection of simplices $\tilde{\phi}(s)$, where s is one of the simplices of $\mathscr{K}$ and $\tilde{\phi}$ is one of the lifts of ϕ.

A locally finite Γ-invariant decomposition of $\mathbb{E}$ provides a collection of simplices covering $\mathbb{E}$. This set of simpleces of $\mathbb{E}$ is finite modulo Γ and can be seen as an *abstract simplicial complex.* Then $\mathscr{K}$ is defined as a geometric realization of this abstract simplicial complex, in some Euclidean space $\mathbb{R}^d$. The existence and uniqueness property of a homeomorphism $\phi : |\mathscr{K}| \to \Sigma$ with the above conditions is then essentially trivial.

In conclusion, triangulations of quotient tori can be defined thanks to Γ-invariant locally finite simplicial decomposition of the universal cover $\mathbb{E}$.

2.2.3. Isotropic piecewise smooth maps. Let $h : \Sigma \to \mathbb{R}^{2n}$ be a piecewise smooth map and $\mathscr{T}$ a triangulation of Σ adapted to h. The pull back $h^*\omega$ does not make sense *a priori*, since the tangent map to h is not globally defined on Σ. Yet, the restriction of the map h to any facet $\mathbf{f}$ (i.e. a 2-dimensional simplex of the triangulation identified to its image in Σ) of $\mathscr{T}$ is smooth. In particular, the restriction of the pull-back $h^*\omega|_{\mathbf{f}}$ along a given facet of the triangulation makes sense.

Definition 2.2.4. *We say that a piecewise smooth map $h : \Sigma \to \mathbb{R}^{2n}$, endowed with an adapted triangulation, is isotropic, if*

$$h^*\omega|_{\mathbf{f}} = 0$$

for every facet $\mathbf{f}$ of the triangulation.

Remark 2.2.5. Definition 2.2.4 does not depend on the choice of an adapted triangulation, so that we can talk of piecewise smooth isotropic maps without any reference to an underlying triangulation. In fact, it is easy the see that for a piecewise smooth map $h : \Sigma \to \mathbb{R}^{2n}$, the pullback $h^*\omega$ is defined almost everywhere. Furthermore h is isotropic if, and only if, $h^*\omega = 0$ almost everywhere, which does not involve the choice of an adapted triangulation.

2.2.6. Norms for piecewise smooth maps. The usual C^0-norm for maps $h : \Sigma \to \mathbb{R}^{2n}$ is given by

$$\|h\|_{C^0} = \sup_{x\in\Sigma} \|h(x)\|$$

and more generally, we define the norm on a domain $K \subset \Sigma$ by

$$\|h\|_{C^0(K)} = \sup_{x\in K} \|h(x)\|.$$

If Σ is assumed to be compact, piecewise smooth maps have finite C^0-norm since they are continous. However, C^1-norm of a piecewise smooth map $h : \Sigma \to \mathbb{R}^{2n}$ does not seem to be as standard as the C^0-norm. Lacking some standard classical reference, we ought to give our own definition. For this purpose, let $\mathscr{T}$ be a triangulation of Σ adapted to the piecewise smooth map h. The restriction $h|_{\mathbf{f}}$ of h to every facet $\mathbf{f}$ of $\mathscr{T}$ is smooth and we can make sense of the norm

$$\|dh\|_{C^0(K\cap\mathbf{f})} = \sup_{x\in K\cap\mathbf{f}} \|dh|_{\mathbf{f},x}\|$$

where the norm of *df* is taken with respect to the operator norm induced by the choice of a Riemannian metric g_Σ on Σ and the Euclidean inner product $\|\cdot\|$ on $\mathbb{R}^{2n}$. Using the notation $\mathfrak{C}_2(\mathscr{T})$ for the set of facets of the triangulation, put

$$\|dh\|_{C^0(K)} = \sup_{\mathbf{f}\in\mathfrak{C}_2(\mathscr{T})} \|dh\|_{C^0(K\cap\mathbf{f})}. \tag{2.1}$$

The C^1-norm is eventually defined by

$$\|h\|_{C^1(K)} = \|h\|_{C^0(K)} + \|dh\|_{C^0(K)},$$

and when $K = \Sigma$, we simply write the norm $\|h\|_{C^1}$.

Remark 2.2.7. An alternate definition could be given along the lines of Remark 2.2.5: since h is piecewise smooth, its differential dh makes sense almost everywhere. Thus the C^0-norm of dh makes sense as an *essential supremum.* Therefore, this norm does not depend on the choice of triangulation adapted to a piecewise smooth map h. Furthermore $\|h\|_{C^1}$ is finite for piecewise smooth maps if Σ is compact.

2.3. Embeddings and immersions

Recall that a *topological embedding* $f : \Sigma \to \mathbb{R}^{2n}$ is a continuous map, such that the restriction $f : \Sigma \to f(\Sigma)$ is a homeomorphism, where $f(\Sigma) \subset \mathbb{R}^{2n}$ is endowed with the induced topology. We say that f is a *topological immersion*, if f is locally a topological embedding. If f is not assumed to be smooth, typically piecewise smooth or piecewise linear, we will say that f is an *immersion* (resp. an *embedding*) if it is a *topological immersion* (resp. *embedding*).

A standard application of the implicit function theorem shows that smooth immersions are also topological immersions. There is an easy characterization of piecewise linear maps which are topological immersions thanks to Lemma 2.3.1. Since this lemma is a classical result of piecewise linear geometry and is not used in this paper, we do not give any proof here.

Lemma 2.3.1. *A piecewise linear map $f : \Sigma \to \mathbb{R}^{2n}$ is a topological immersion if, and only if, f is locally injective. If Σ is closed, and f is globally injective, then it is a topological embedding.*

The notion of *polyhedral surface* was used in Corollary B, as well as the condition of a polyhedral surface being an approximation close to a smooth surface, in the C^1-sense. The accurate definitions are given below.

Definition 2.3.2. *A polyhedral surface, $S_0 \subset \mathbb{R}^{2n}$ is the image of a smooth surface Σ by some piecewise linear map $h : \Sigma \to \mathbb{R}^{2n}$.*

If h is a topological immersion (resp. embedding), then S_0 is called an immersed (resp. embedded) polyhedral surface *of $\mathbb{R}^{2n}$. If h is isotropic, in the sense of piecewise smooth isotropic maps, we say that the polyhedral surface S_0 is isotropic. In the case $n = 2$, we say that S_0 is Lagrangian, instead of isotropic.*

For $\varepsilon > 0$, we say that a polyhedral surface S_0 of $\mathbb{R}^{2n}$ is ε-close to a smoothly immersed surface S of $\mathbb{R}^{2n}$, in the C^k-sense (where $k = 0$ or 1), if there exists a smooth surface Σ together with

- *a smooth immersion* $f : \Sigma \to \mathbb{R}^{2n}$ *such that* $f(\Sigma) = S$,
- *a piecewise linear map* $h : \Sigma \to \mathbb{R}^{2n}$ *such that* $h(\Sigma) = S_0$,

with the property that

$$\|f - h\|_{C^k} \leq \varepsilon,$$

where the C^1*-norm is defined using the Riemannian metric* g_Σ *induced on* Σ *by the canonical Euclidean structure of* $\mathbb{R}^{2n}$ *and the immersion* $f : \Sigma \to \mathbb{R}^{2n}$.

A sequence of polyhedral surfaces S_N *of* $\mathbb{R}^{2n}$*, is called an approximation of a smooth surface* S*, in the* C^k*-sense, if there exists a sequence* ε_N *such that* S_N *is* ε_N*-close to* S *in the* C^k*-sense for every* N*, and* $\lim \varepsilon_N = 0$.

Remark 2.3.3. The notion of ε-closeness of Definition 2.3.2 depends on the choice of a Riemannian metric g_Σ on Σ. A choice has to be made for g_Σ. Here, the metric g_Σ is chosen as the one induced by the ambient Euclidean space $\mathbb{R}^{2n}$ and the choice of an immersion $f : \Sigma \to \mathbb{R}^{2n}$. Such a choice is somewhat arbitrary. Nevertheless, this choice is not crucial for our purpose, since another choice of Riemannian metric g_Σ leads to an equivalent C^1-norm. Furthemore, the notion of approximation by a sequence of polyhedral surfaces does not depend on the choice of metric g_Σ.

The C^1-norm of piecewise smooth maps is designed in such a way that various properties, which are classical in the smooth case, extend naturally in the piecewise smooth setting. The next proposition belongs to this category. Although the arguments are quite standard, a proof ought to be given, since we rely on this result for the proof of our main theorem and it seems impossible to find a clear reference for the proof.

Proposition 2.3.4. *Assume* Σ *is a smooth closed surface and* $f : \Sigma \to \mathbb{R}^{2n}$ *is a smooth embedding (resp. immersion). Then, there exists* $\varepsilon > 0$*, such that for every piecewise smooth map* $h : \Sigma \to \mathbb{R}^{2n}$*, the condition*

$$\|f - h\|_{C^1} \leq \varepsilon$$

implies that h *is a topological embedding (resp. immersion).*

Proof. The case of embeddings in the proposition readily follows from the case of immersions exactly as in the smooth setting. Hence, we just prove the proposition in the case of a smooth immersion $f : \Sigma \to \mathbb{R}^{2n}$. Let ε be a positive real number, to be fixed later on, and $h : \Sigma \to \mathbb{R}^{2n}$, be a piecewise smooth map, such that

$$\|f - h\|_{C^1} \leq \varepsilon.$$

We introduce the affine space subspace F of $\mathbb{R}^{2n}$ at a point x of Σ, defined by

$$F = f(x) + \operatorname{Im} df_x,$$

understood as the image of the tangent space $T_x\Sigma$ by the tangent map f_*. The affine map

$$\pi : \mathbb{R}^{2n} \to F$$

denotes the orthogonal projection onto F and the corresponding *projected maps* are given by

$$\hat{f} = \pi \circ f \quad \text{and} \quad \hat{h} = \pi \circ h.$$

Since π is an orthogonal projection, its linear part $\vec{\pi}$ satisfies

$$\|\vec{\pi}\| = 1,$$

which leads to the estimate

$$\|\hat{f} - \hat{h}\|_{C^1} \leq \|f - h\|_{C^1} \leq \varepsilon. \tag{2.2}$$

Since f is a smooth immersion, the map $\hat{f} : \Sigma \to F$ is a local diffeomorphism at x. Let $\mathscr{T}$ be a triangulation of the surface Σ adapted to the piecewise smooth map h. We introduce the closed neighborhood $\overline{U}$ of x corresponding to the *star* of x, which is the union of simplices of $\mathscr{T}$, that contain x. We denote by $\overline{V} \subset F$ its image $\hat{f}(\overline{U})$. Up to passing to a refinement of the triangulation $\mathscr{T}$, we can assume that $\overline{U}$ is an arbitrarily small closed neighborhood of x. In particular, we may assume that the restriction $\hat{f} : \overline{U} \to \overline{V}$, is a diffeomorphism with inverse $\hat{f}^{-1} : \overline{V} \to \overline{U}$.

As the star of x, the domain $\overline{U}$ carries an induced triangulation, obtained as the retriction of $\mathscr{T}$. We deduce a triangulation $\mathscr{T}'$ of $\overline{V}$, obtained as the image triangulation of $\overline{U}$ by $\hat{f}$. By construction, the map

$$\psi : \overline{V} \to F \quad \text{defined by} \quad \psi = \hat{h} \circ \hat{f}^{-1}$$

is piecewise smooth, with $\mathscr{T}'$ as an adapted triangulation.

The next lemma is a local inverse function theorem adapted to the case of the piecewise smooth map ψ.

Lemma 2.3.5. *If $\varepsilon > 0$ is chosen sufficiently small, then for every piecewise smooth map $h : \Sigma \to \mathbb{R}^{2n}$ such that $\|f - h\|_{C^1} \leq \varepsilon$ and every $x \in \Sigma$, the corresponding map $\psi : \overline{V} \to F$, introduced above, is a local homeomorphism near $f(x)$.*

End of the proof of Proposition 2.3.4. Assuming Lemma 2.3.5 holds, we may replace $\overline{V}$ by a smaller open neighborhood V' of $f(x)$ in F, so that the retriction $\psi : V' \to \psi(V')$ is a homeomorphism. Hence, there exists a continuous inverse $\varphi : \psi(V') \to V'$ of $\psi : V' \to \psi(V')$. We define U' as the open neighborhood $U' = \hat{f}^{-1}(V')$ of x. It follows that the map $\pi : h(U') \to \psi(V')$ is a homeomorphism onto its image. Indeed $h \circ \hat{f}^{-1} \circ \varphi : \psi(V') \to h(U')$ is a continuous inverse of $\pi : h(U') \to \psi(V')$. Thus, $h(U')$ is homeomorphic to the open set $\psi(V')$ of the vector space F. This shows that h is a topological immersion, which completes the proof of Proposition 2.3.4, in the case of an immersion. □

Proof of Lemma 2.3.5. By Inequation (2.2), we have the C^0-control

$$\|\psi(z) - z\| = \left\|\hat{h}(\hat{f}^{-1}(z)) - \hat{f}(\hat{f}^{-1}(z))\right\| \leq \varepsilon,$$

for every $z \in \overline{V}$, or, in other words

$$\|\psi - \mathrm{id}\|_{C^0(\overline{V})} \leq \varepsilon. \tag{2.3}$$

The differential of ψ does not make sense everywhere, since the map is only piecewise smooth with respect to the triangulation $\mathscr{T}'$ of $\overline{V}$. For every point $z \in \overline{V}$ contained in the interior of a facet of $\mathscr{T}'$, we have by (2.2)

$$\left\|(d\hat{h} \circ d\hat{f}^{-1})_z - \mathrm{id}\right\| \leq \|\hat{h} - \hat{f}\|_{C^1} \|d\hat{f}_z^{-1}\| \leq \varepsilon \|d\hat{f}_z^{-1}\|. \tag{2.4}$$

The term $\|d\hat{f}_z^{-1}\|$ can be controlled uniformly thanks to the next lemma:

Lemma 2.3.6. *There exists a constant $C > 0$ such that for every $x \in \Sigma$, there exists a refinement of the triangulation $\mathscr{T}$, such that the diffeomorphism $\hat{f} : \overline{U} \to \overline{V}$ defined above satisfies*

$$\|d\hat{f}_z^{-1}\| \leq C$$

for every $z \in \overline{V}$.

Proof. Since f is an immersion and Σ is compact, there exists a constant $C > 0$, such that for every $x \in \Sigma$ and $u \in T_x\Sigma$, we have

$$\frac{2}{C}\|u\| \leq \|df_x \cdot u\|.$$

By definition $d\hat{f} = \vec{\pi} \circ df$, where $\vec{\pi}$ is the linear part of π. Furthermore

$$d\hat{f}_x = \vec{\pi} \circ df_x = df_x,$$

since $\vec{\pi}$ is a projection onto $\operatorname{Im} df_x$. Hence, for every tangent vector $w \in T_{f(x)}\overline{V}$,

$$\left\| d\hat{f}^{-1}_{f(x)} \cdot w \right\| \leq \frac{C}{2}\|w\|,$$

i.e.

$$\left\| d\hat{f}^{-1}_{f(x)} \right\| \leq \frac{C}{2}.$$

Recall that $\overline{U}$ is the star of x with respect to the triangulation $\mathscr{T}$. Up to passing to a refinement of the triangulation $\mathscr{T}$, we may assume that $\overline{U}$ is arbitrarily small, so that by the C^1-continuity of f,

$$\|d\hat{f}_z^{-1}\| \leq C$$

for every $z \in \overline{V} = \hat{f}(\overline{U})$, which proves the lemma. □

In the rest of the proof of Lemma 2.3.5, we assume that a refinement of $\mathscr{T}$ is chosen, so that the conclusion of Lemma 2.3.6 is met. By (2.4) and Lemma 2.3.6, we deduce that for every point z of $\overline{V}$ contained in the interior of a facet of $\mathscr{T}'$, we have

$$\|d\psi_z - \mathrm{id}\| \leq \varepsilon C. \tag{2.5}$$

We choose $\varepsilon = \frac{1}{2}\min\left(\frac{1}{2}, \frac{1}{2}C^{-1}\right)$. Then by (2.3) and (2.5)

$$\|\psi - \mathrm{id}\|_{C^1(\overline{V})} \leq \frac{1}{2}. \tag{2.6}$$

At this stage, it is natural to consider the fixed points of the map

$$\Phi_y(z) = z - \psi(z) + y, \tag{2.7}$$

defined for $y \in F$ and $z \in \overline{V}$.

Lemma 2.3.7. *The map $\Phi_y : \overline{V} \to F$ is a $\frac{1}{2}$-contracting map on a sufficiently small closed ball $\overline{B}_r$ of $\overline{V}$ centered at $f(x)$.*

Proof. This lemma relies on a version of the mean value theorem, in the context of piecewise smooth functions. Obviously

$$\Phi_y(z) - \Phi_y(z') = (\psi(z') - z') - (\psi(z) - z). \tag{2.8}$$

We choose a closed Euclidean ball $\overline{B}_r \subset F$ contained in V and centered at $f(x)$ with radius $r > 0$. For $z, z' \in \overline{B}_r$, the segment $[zz']$ is contained in $\overline{B}_r$ hence in $\overline{V}$. The mean value theorem applies to the function ψ. However some justification is needed since ψ is only piecewise smooth: by transversality, the segment $[zz']$ can be approximated, in the C^1-sense, by a smooth parametric curve $\gamma : [0, 1] \to \overline{V}$, such that $\gamma(0) = z$, $\gamma(1) = z'$ and for every $0 < t < 1$

- $\gamma(t)$ does not go through a vertex of the triangulation $\mathscr{T}'$ of $\overline{V}$.
- Furthermore $\gamma(t)$ intersects transversaly the 1-skeleton of the triangulation.

Such a curve γ intersects the 1-skeleton of $\mathscr{T}'$ only a finite number of times $0 = t_0 < t_1 < \cdots < t_k = 1$. Otherwise $\gamma(t)$ is contained in the interior domain of facets. Then

$$\Phi_y(\gamma(t_{j+1})) - \Phi_y(\gamma(t_j)) = \int_{t_j}^{t_{j+1}} d\Phi_y|_{\gamma(t)} \cdot \gamma'(t)\, dt.$$

Since $d\Phi_y|_{\gamma(t)} = \mathrm{id} - d\psi|_{\gamma(t)}$ for every $t \in (t_j, t_{j+1})$, we deduce by (2.6) that

$$\left\| \Phi_y(\gamma(t_{j+1})) - \Phi_y(\gamma(t_j)) \right\| \le \frac{L_j}{2}.$$

where L_j is the length of the curve $\gamma(t)$ from t_j to t_{j+1}. Adding all the identities together and the triangle inequality give the result

$$\left\| \Phi_y(z) - \Phi_y(z') \right\| \le \frac{L(\gamma)}{2},$$

where $L(\gamma)$ is the length of γ. Passing to the limit with a sequence of curves γ converging in the C^1-sense towards $[zz']$ gives the inequality

$$\left\| \Phi_y(z) - \Phi_y(z') \right\| \le \frac{1}{2} \| z - z' \|$$

for every $z, z' \in \overline{B}_r$, which proves the lemma. □

The end of the proof of Lemma 2.3.5 is based on the fixed point principle, applied to the map Φ_y. Indeed, a solution of $\Phi_y(z) = z$ satisfies $\psi(z) = y$, by definition, which provides a construction for the inverse of ψ. By [7, Proposition 6.3.1], the equation $\Phi_y(z) = z$ admits a unique solution $z = \varphi(y) \in \overline{B}_r$ for every $y \in \overline{B}_{r/2}$. This defines a local inverse $\varphi : \overline{B}_{\frac{r}{2}} \to \overline{B}_r$ of the map ψ.

The fact that φ is continuous is classical: for every $y, y' \in \overline{B}_{r/2}$, we can write

$$\begin{aligned}\|\varphi(y) - \varphi(y')\| &= \|z - z'\| \\ &= \|\Phi_y(z) - \Phi_{y'}(z')\| \\ &= \|\Phi_y(z) - \Phi_y(z') + y - y'\| \\ &\leq \frac{1}{2}\|z - z'\| + \|y - y'\|,\end{aligned}$$

which implies

$$\frac{1}{2}\|\varphi(y) - \varphi(y')\| = \frac{1}{2}\|z - z'\| \leq \|y - y'\|.$$

Therefore φ is a 2-Lipschitz map on the ball $\overline{B}_{r/2}$. In particular φ is continuous. In conclusion ψ is a local homeomorphism near $f(x)$. □

This completes the proof of Proposition 2.3.4. □

Proposition 2.3.4 applies in the special case of piecewise linear maps, which can be stated as the following corollary:

Corollary 2.3.8. *Let Σ be a smooth closed surface and $f : \Sigma \to \mathbb{R}^{2n}$ a smooth immersion (resp. embedding). Any piecewise linear map $h : \Sigma \to \mathbb{R}^{2n}$ sufficiently close to f, in the C^1-sense, is a topological immersion (resp. embedding).*

Equivalently, any polyhedral surface sufficiently close, in the C^1-sense, to a smoothly immersed (resp. embedded) closed surface, is an immersed (resp. embedded) polyhedral surface.

Remark 2.3.9. It is likely that an alternate proof of the above corollary could be given relying on Lemma 2.3.1.

2.4. Overview of the Jauberteau-Rollin-Tapie techniques

A leisurely introduction to the crucial tools of [7] used in the proof of Theorem A is given in this section. The interested reader is advised to keep the article [7] within reach, as a detailed handbook about the constructions that will only be sketched here.

2.4.1. Conformal cover. The Jauberteau-Rollin-Tapie construction starts from a smooth isotropic immersion (or embedding)

$$\ell : \Sigma \to \mathbb{R}^{2n},$$

where Σ is a 2-dimensional smooth torus. The Riemannian metric $g_\Sigma = \ell^* g$, induced on Σ by the canonical Euclidean metric g of $\mathbb{R}^{2n}$, is conformally flat, by the classical uniformization theorem. Hence there exists a covering map

$$p : \mathbb{E} \to \Sigma$$

from a 2-dimensional Euclidean affine space $\mathbb{E}$ with Riemannian metric $g_\mathbb{E}$, such that p is a conformal map with respect to $g_\mathbb{E}$ and g_Σ. The group of deck transformations of p is identified to a lattice Γ of the vector space $\overrightarrow{\mathbb{E}}$ associated to the affine space $\mathbb{E}$, acting by translations. Thus, the flat quotient metric g_σ induced by $g_\mathbb{E}$ on Σ and g_Σ are in the same conformal class. More precicely, there exists a smooth positive function $\theta : \Sigma \to \mathbb{R}$, such that $g_\Sigma = \theta g_\sigma$ and p induces a conformal diffeomorphism

$$\Sigma \simeq \mathbb{E}/\Gamma.$$

An oriented orthonormal frame $(O, \vec{E}_1, \vec{E}_2)$ of the affine space $\mathbb{E}$, induces an affine isometry

$$r : \mathbb{R}^2 \to \mathbb{E},$$

defined by $r(0) = O$, $r(e_1) = O + \vec{E}_1$ and $r(e_2) = O + \vec{E}_2$, where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. A notion of *non degenerate pair* $(p \circ r, \ell)$ is introduced in [7, §5.3]. This technical condition is important for the application of the fixed point principle later on. However, non degeneracy can always be achieved by [7, Proposition 5.3.3], modulo a suitable choice of isometry r. Thus, we could always assume that the non degeneracy condition holds in the rest of our paper.

2.4.2. Standard quadrangulations. For every integer $N > 0$, there is a standard lattice $\Lambda_N \subset \mathbb{R}^2$ given by

$$\Lambda_N = \mathbb{Z}\frac{e_1}{N} \oplus \mathbb{Z}\frac{e_2}{N}.$$

The lattice Λ_N is understood as the set of vertices of the standard *quadrangulation* $\mathcal{Q}_N(\mathbb{R}^2)$ of $\mathbb{R}^2$, tiled by squares of sidelength N^{-1}, according to the Figure 1, where

- the vertices are denoted $\mathbf{v}_{kl} = N^{-1}(ke_1 + le_2) \in \Lambda_N$;
- $\mathbf{f}_{kl}$ are the facets of the quadrangulation.

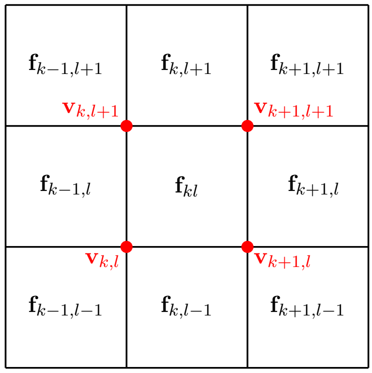

Figure 1: Quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$.

2.4.3. Quotient quadrangulation. The vector spaces $\mathbb{R}^2$ is identified to $\overrightarrow{\mathbb{E}}$ via the linear isometry $\vec{r}$, which is the linear part of the affine isometry $r : \mathbb{R}^2 \to \mathbb{E}$. Hence, the lattices Λ_N and Γ can be understood as lattices of the same 2-dimensional vector space modulo the isomorphism $\vec{r}$. If Γ is a sublattice of Λ_N, we deduce that $\Sigma = \mathbb{E}/\Gamma$ carries a a quotient quadrangulation $\mathcal{Q}_N(\Sigma)$. Furthermore, the quotient quadrangulation is acted on by the residual action of Λ_N. Unfortunately, Γ may not be a sublattice of Λ_N. To get around this technical issue, we can make a suitable choice of a sequence of affine isomorphisms

$$r_N : \mathbb{R}^2 \to \mathbb{E} \quad \text{such that} \quad r_N = r + \mathcal{O}(N^{-1}), \tag{2.9}$$

with the property that the image of Λ_N by $\vec{r}_N$ contains Γ as a sublattice. The construction of the maps r_N is elementary and explained in details in [7, §3.2].

Remark 2.4.4. Although the maps r_N are not isometric, they converge towards an isometry $r : \mathbb{R}^2 \to \mathbb{E}$. In this sense, the maps r_N are *almost isometric*. It follows that the pull-back metrics $r_N^* g_{\mathbb{E}}$ converge towards the canonical Euclidean metric $g_{\mathbb{R}^2}$ and are uniformly commensurate with $g_{\mathbb{R}^2}$. In particular, any of these metrics could be used for uniform estimating purposes.

Given the almost isometric affine maps $r_N : \mathbb{R}^2 \to \mathbb{E}$, the lattices Λ_N are now understood as lattice acting on $\mathbb{E}$, with Γ acting as a sublattice of Λ_N. We deduce that the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ descends as a quotient quadrangulation of $\Sigma \simeq \mathbb{E}/\Gamma$, denoted $\mathcal{Q}_N$ or $\mathcal{Q}_N(\Sigma)$.

2.4.5. Quadrangular meshes. We denote by $\mathfrak{C}_k(\mathcal{Q}_N)$ the set of k-cells of the quadrangulation $\mathcal{Q}_N$ and by $\mathscr{C}^k(\mathcal{Q}_N, X)$ the space of functions on the set $\mathfrak{C}_k(\mathcal{Q}_N)$, with values in some set X. The vector space of *quadrangular meshes*, denoted $\mathscr{M}_N$ or $\mathscr{M}(\mathcal{Q}_N)$, on Σ is defined by

$$\mathscr{M}(\mathcal{Q}_N) = \mathscr{C}^0(\mathcal{Q}_N, \mathbb{R}^{2n}).$$

In other words, a quadrangular mesh $\tau \in \mathscr{M}_N$ is a $\mathbb{R}^{2n}$-valued function on the space of the vertices of the quadrangulation $\mathcal{Q}_N$.

Given a smooth map as $\ell : \Sigma \to \mathbb{R}^{2n}$, there is a natural sequence of *approximations* of ℓ by quadrangular meshes

$$\tau_N \in \mathscr{M}_N$$

called the *samples* of ℓ and defined by

$$\tau_N(\mathbf{v}) = \ell(\mathbf{v}),$$

for every vertex $\mathbf{v}$ of the quadrangulation $\mathcal{Q}_N$.

Remark 2.4.6. If ℓ is isotropic, it is not clear at all whether and, in which sense, the samples τ_N are isotropic. We are going to give a definition of isotropic quadrangular meshes, and show that the samples τ_N are almost isotropic.

2.4.7. Isotropic quadrangular meshes. Let D be a compact oriented surface of $\mathbb{R}^{2n}$ with boundary ∂D. By Stokes theorem

$$\int_D \omega = \int_{\partial D} \lambda,$$

where λ is the Liouville form. If D is isotropic, the integral of the Liouville form along it boundary ∂D vanishes.

Given a quadrangular mesh $\tau \in \mathscr{M}_N$, every facet $\mathbf{f}_{kl}$ of the quadrangulation has an associated quadrilateral of $\mathbb{R}^{2n}$, given by the points $A_j =$

$A_j(\tau, \mathbf{f}_{kl})$ for $0 \leq j \leq 3$, with

$$A_0 = \tau(\mathbf{v}_{kl}), \quad A_1 = \tau(\mathbf{v}_{k+1,l}), \quad A_2 = \tau(\mathbf{v}_{k+1,l+1}), \quad A_3 = \tau(\mathbf{v}_{k,l+1}).$$

The quadrilateral $(A_0A_1A_2A_3)$ is called *the quadrilateral of the mesh* τ, *along the facet* $\mathbf{f}_{kl}$. This motivates the following definition:

Definition 2.4.8. *An oriented closed piecewise smooth curve* γ *of* $\mathbb{R}^{2n}$ *is called isotropic if* $\int_\gamma \lambda = 0$.

In particular if γ *is a parametrization of the four edges of a quadrilateral, we say that the quadrilateral is isotropic.*

A quadrangular mesh $\tau \in \mathscr{M}_N$ *is called isotropic, if every quadrilateral of* $\mathbb{R}^{2n}$ *associated to a facet of the quadrangular mesh* τ *is isotropic.*

Isotropic quadrangular meshes are solutions of an obvious system of quadratic equations, interpreted as the vanishing of a *discrete symplectic density*. Given $\tau \in \mathscr{M}_N$ or a smooth function f on Σ, it is often convenient to work on the cover $p_N = p \circ r_N : \mathbb{R}^2 \to \Sigma$. The pull back $\tilde{f} = f \circ p_N$ of f is understood as a smooth Γ-periodic function of $\mathbb{R}^2$. Similarly, $\tilde{\tau} = \tau \circ p_N$ is understood as a Γ-invariant quadrangular mesh of $\mathscr{M}(\mathcal{Q}_N(\mathbb{R}^2)) = \mathscr{C}^0(\mathcal{Q}_N(\mathbb{R}^2), \mathbb{R}^{2n})$. Most of the time, we will work on the cover $\mathbb{R}^2$ without any warning or special notations.

We then introduce the *vectors fields*

$$\mathscr{U}_\tau \quad \text{and} \quad \mathscr{V}_\tau \in \mathscr{C}^2(\mathcal{Q}_N(\Sigma), \mathbb{R}^{2n})$$

defined on the cover by the formulas

$$\mathscr{U}_\tau(\mathbf{f}_{kl}) = \frac{N}{\sqrt{2}} \left(\tilde{\tau}(\mathbf{v}_{k+1,l+1}) - \tilde{\tau}(\mathbf{v}_{kl})\right)$$

and

$$\mathscr{V}_\tau(\mathbf{f}_{kl}) = \frac{N}{\sqrt{2}} \left(\tilde{\tau}(\mathbf{v}_{k,l+1}) - \tilde{\tau}(\mathbf{v}_{k+1,l})\right)$$

Remark 2.4.9. The above vectors are precisely the diagonals of the quadrilateral of $\mathbb{R}^{2n}$, given by the facet of τ associated to $\mathbf{f}_{kl}$ and renormalized by a factor $\frac{N}{\sqrt{2}}$. Thus $\mathscr{U}_\tau$ and $\mathscr{V}_\tau$ can be understood as finite difference versions of partial derivatives of τ in diagonal directions.

An easy calculation (cf. [7, §4.1]) shows that the integral of the Liouville form along the quadrilateral associated to the quadrangular mesh τ and the facet $\mathbf{f}_{kl}$ is precisely

$$N^{-2}\omega(\mathscr{U}_\tau(\mathbf{f}_{kl}), \mathscr{V}_\tau(\mathbf{f}_{kl})).$$

Notice that N^{-2} is the Euclidean area of the facets of $\mathcal{Q}_N(\mathbb{R}^2)$. This motivates the definition of the symplectic density

$$\mu_N : \mathscr{M}_N \to \mathscr{C}^2(\mathcal{Q}_N, \mathbb{R})$$

given by

$$\mu_N(\tau) = \omega(\mathscr{U}_\tau, \mathscr{V}_\tau). \tag{2.10}$$

By definition $\mu_N^{-1}(0)$ is precisely the set of isotropic quadrangular meshes.

2.4.10. Norms for quadrangular meshes. Certain zeroes of μ_N were constructed in [7], thanks to the fixed point principle. Some adapted norms have to be introduced in order to carry out the construction. There are two special *diagonal* translations T_u and T_v acting on facets or vertices of $\mathcal{Q}_N(\mathbb{R}^2)$ (or $\mathcal{Q}_N(\Sigma)$), given by

$$T_u(\mathbf{v}_{kl}) = \mathbf{v}_{k+1,l+1} \quad \text{and} \quad T_v(\mathbf{v}_{k+1,l}) = \mathbf{v}_{k,l+1}$$

with similar formulas for facets. These operators induce finite difference operators on the space of functions $\mathscr{C}^0$ or $\mathscr{C}^2$. For instance, we define for a quadrangular mesh τ

$$\frac{\partial \tau}{\partial \vec{u}} = \frac{N}{\sqrt{2}}(\tau \circ T_u - \tau) \quad \text{and} \quad \frac{\partial \tau}{\partial \vec{v}} = \frac{N}{\sqrt{2}}(\tau \circ T_v - \tau). \tag{2.11}$$

Such finite differences, are analogues of the derivatives in (u, v)-coordinates obtained by rotating the canonical basis (e_1, e_2) of $\mathbb{R}^2$ by an angle $+\frac{\pi}{4}$. More precisely by setting

$$u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{y-x}{\sqrt{2}}. \tag{2.12}$$

The finite differences induce $\mathcal{C}^k_w$-norms on the space of functions $\mathscr{C}^j(\mathcal{Q}_N)$ on the set of j-cells of $\mathcal{Q}_N$. These norms are discrete versions of the C^k-norms for smooth functions (notice the slightly different typography for the norms in the smooth and discrete cases). The $\mathcal{C}^k_w$-norms are denoted $\|\cdot\|_{\mathcal{C}^k_w}$, where w stands for *weak*. Indeed, the translations T_u and T_v only span a sublattice of index 2 of Λ_N. It follows that these weak norms do not control every finite differences, in particular in the directions

$$k_1 = N^{-1}e_1 \quad \text{and} \quad k_2 = N^{-1}e_2. \tag{2.13}$$

It is also possible to mimic Hölder norms, with Hölder regularity $\alpha \in (0,1)$ in the T_u and T_v directions. The corresponding $\mathcal{C}_w^{k,\alpha}$-norms are denoted $\|\cdot\|_{\mathcal{C}_w^{k,\alpha}}$.

Remark 2.4.11. More details about discrete weak norms and their properties can be found in [7, §3.7]. The reader may still wonder why such weak norms must be used here. Although we do not have a mathematical argument, there is some geometrical evidence, which tends to show why weak norms are natural, and why they are the best one can expect for the discrete analysis of μ_N.

First, the space of solutions of the equation $\mu_N = 0$ has symmetries obtained from the *shear action*, described in [7, §4.2]. The shear action essentially pulls apart two intertwined submeshes of a given quadrangular mesh, by independent translations along two index 2 sublattices of Λ_N. The shear action leaves the equation invariant. This seems to indicate that strong C^1 controls cannot be expected from the mere equations $\mu_N = 0$ (cf. [7, Figure 1]).

The second observation concerns the linearization of the equation $\mu_N = 0$. For the fixed point principle, it is convenient to consider variations of μ_N in some particular directions, given by discrete functions on Σ (cf. [7, Chapter 4]). This idea is motivated by a moment map interpretation of the smooth equation (cf. [7, Chapter 2]), where we are looking for perturbations along complexified orbits of the gauge group. Eventually, such perturbations are given by functions (for the smooth or discrete cases) on Σ, and the linearized operator is essentially a Laplacian (cf. [7, §2.3.3 and §4.6.4]). However, the sequence of discrete Laplacians does not converge towards the operator in the smooth setting, as N goes to infinity. Instead, the sequence of discrete Laplacians converges towards another Fredholm operator, acting on pairs of smooth functions, rather than single functions (see §[7, Theorem 5.1.1]). This phenomenon may be regarded as some infinitesimal reminiscence of the shear action. Hence, natural norms for our problem should allow sequences of solutions of the linearized discrete problem to converge towards pairs of functions for the limiting Fredholm operator on the smooth surface. This feature is the essence of the weak norms (cf. [7, §3.8 and Example 3.8.4]).

However, it is fairly possible that simpler discretization schemes, other than quadrangulations, could be used. Of course, any relevant suggestion would be most welcome.

2.4.12. Existence of isotropic meshes. We can quote the main technical tool for producing isotropic quadrangular meshes:

Theorem 2.4.13 ([7]). *Given a smooth isotropic immersion $\ell : \Sigma \to \mathbb{R}^{2n}$, there exists a sequence of isotropic quadrangular meshes $\rho_N \in \mathscr{M}_N$, such that*

$$\|\rho_N - \tau_N\|_{\mathcal{C}^0} = \mathcal{O}\left(\|\mu_N(\tau_N)\|_{\mathcal{C}^{0,\alpha}_w}\right)$$

where τ_N are the samples of ℓ.

Proof. Although Theorem 2.4.13 is essentially proved in [7, Theorem 6.3.2 and Proposition 6.3.3], the statement is slightly different. In particular, the estimate $\mathcal{O}\left(\|\mu_N(\tau_N)\|_{C^{0,\alpha}_w}\right)$ is replaced with a rough $\mathcal{O}(N^{-1})$ estimate.

Nevertheless, the construction of ρ_N proceeds from the fixed point principle applied to a certain map T_N defined in [7, §6]. The solution $\phi_N = T_N(\phi_N)$ is controlled in $\mathcal{C}^{2,\alpha}_w$-norm by the term $T_N(0) = -G_N(\mu_N(\tau_N))$, where G_N is the Green operator of the discrete Laplacian, defined in [7, §6.1]. By [7, Proposition 6.1.2], the $\mathcal{C}^{0,\alpha}_w$-norm of $\mu_N(\tau_N)$ controls the $\mathcal{C}^{2,\alpha}_w$-norm of $G_N(\mu_N(\tau_N))$. Eventually, the $\mathcal{C}^{2,\alpha}_w$-norm of ϕ_N controls the $\mathcal{C}^0$-norm of $\rho_N - \tau_N$, since $\rho_N = \tau_N - J\delta^\star_N \phi_N$, as defined by [7, Proposition 6.3.3], and the theorem follows. ☐

Remark 2.4.14. The key to obtain a C^1-control, as in Theorem A, is to improve the estimate of the *error term* $\mu_N(\tau_N)$. This is precisely what is done in Proposition 3.2.1, with a $\mathcal{O}(N^{-2})$-estimate, rather than a $\mathcal{O}(N^{-1})$-estimate as in [7].

2.4.15. From quadrangulations to triangulations. Quadrangular meshes do not provide piecewise linear maps in an obvious way. They have to be completed into triangular meshes beforehand. Indeed, a quadrilateral of $\mathbb{R}^{2n}$ is not necessarily contained in a 2-plane. The idea is to fill a quadrilateral $(ABCD)$ with a pyramid as in Figure 2.

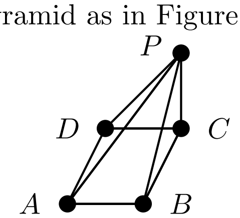

Figure 2: Pyramid with apex P and base $(ABCD)$.

Definition 2.4.16. *A pyramid is called isotropic if its four triangles from the apex P are contained in isotropic planes.*

Given a quadrilateral $(ABCD)$ of $\mathbb{R}^{2n}$, we are looking for a point $P \in \mathbb{R}^{2n}$, such that the corresponding pyramid is isotropic. This equation, bearing on P, is a linear system. The compatibility condition is precisely given by the fact that the quadrilateral is isotropic, in the sense of Definition 2.4.8. The dimension of the affine space spanned by $(ABCD)$ is called the dimension of the quadrilateral. The dimension of the quadrilateral is exactly the codimension of the space of solutions P, such that the pyramid is isotropic. Generic quadrilaterals are 3-dimensional, but they may be flat or more degenerate as well.

Definition 2.4.17. *Let G be the barycenter of an isotropic quadrilateral $(ABCD)$ of $\mathbb{R}^{2n}$. The closest apex P to G, such that the corresponding pyramid is isotropic is called the* optimal apex *of $(ABCD)$.*

Following this idea, we refine the quadrangulations $\mathcal{Q}_N(\mathbb{R}^2)$ into triangulations $\mathscr{T}_N(\mathbb{R}^2)$. In this context, we use the definition of a triangulation as a simplicial decomposition of $\mathbb{R}^2$ invariant by the lattice. To do this, we replace each facet $\mathbf{f}_{kl}$ of the quadrangulation with a vertex $\mathbf{z}_{kl}$ (represented by a black dot) at its barycenter and complete with four edges and four facets, according to Figure 3.

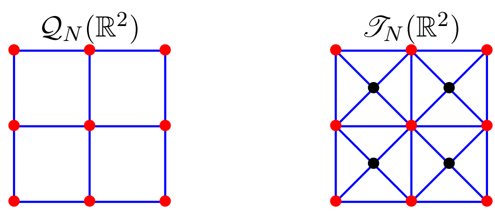

Figure 3: Triangular refinement of a quadrangulation.

Like quadrangulations, the triangulations $\mathscr{T}_N(\mathbb{R}^2)$ descend to the quotient Σ, via the covering map $p \circ r_N : \mathbb{R}^2 \to \Sigma$. The quotient triangulation on Σ is denoted $\mathscr{T}_N(\Sigma)$ or simply $\mathscr{T}_N$. The space of triangular meshes $\mathscr{M}'_N$, analogous to quadrangular meshes $\mathscr{M}_N$, is defined by

$$\mathscr{M}'_N = \mathscr{C}^0(\mathscr{T}_N, \mathbb{R}^{2n}).$$

The samples $\tau_N \in \mathscr{M}_N$ of ℓ can be extended as samples $\tau'_N \in \mathscr{M}'_N$, defined by

$$\tau'_N(\mathbf{v}) = \ell(\mathbf{v})$$

for every vertex $\mathbf{v} \in \mathfrak{C}_0(\mathscr{T}_N)$. Similarly the isotropic quadrangular meshes ρ_N of Theorem 2.4.13 can be extended in two flavors

$$\hat{\rho}_N, \rho'_N \in \mathscr{M}'_N.$$

The former triangular mesh is defined by

$$\hat{\rho}_N(\mathbf{v}) = \rho'_N(\mathbf{v}) = \rho_N(\mathbf{v})$$

for every $\mathbf{v} \in \mathfrak{C}_0(\mathscr{Q}_N)$ (i.e. one of the red vertices in Figure 3). If $\mathbf{z}_{kl}$ is a vertex of $\mathfrak{C}_0(\mathscr{T}_N)$ which does not belong to $\mathfrak{C}_0(\mathscr{Q}_N)$ (i.e. one of the black vertices in Figure 3), we define $\hat{\rho}(\mathbf{z}_{kl})$ as the barycenter of the quadrilateral associated to the facet $\mathbf{f}_{kl}$ of ρ_N. More explicitely

$$\hat{\rho}_N(\mathbf{z}_{kl}) = \frac{1}{4} \sum_{\mathbf{v} \in \mathbf{z}_{kl}} \rho_N(\mathbf{v}).$$

The latter triangular mesh ρ'_N is defined as follows: the quadrilateral associated to the facet $\mathbf{f}_{kl}$ of ρ_N is isotropic, by definition of ρ_N. Let $P_{kl} \in \mathbb{R}^{2n}$ be the optimal apex of this quadrilateral, in the sense of Definition 2.4.17. We then define

$$\rho'_N(\mathbf{z}_{kl}) = P_{kl}.$$

2.4.18. From triangular meshes to piecewise linear maps. A triangular mesh $\rho' \in \mathscr{M}'_N$ defines a natural piecewise linear map. Indeed an affine map along a Euclidean triangle is defined by the values of the map at the three vertices. In particular, the piecewise linear maps ℓ_N associated to ρ'_N are isotropic, by contruction. Furthermore, we will see that the maps ℓ_N provide the solutions to Theorem A. This motivates the following definition.

Definition 2.4.19. *The isotropic piecewise linear map $\ell_N : \Sigma \to \mathbb{R}^{2n}$, defined by the triangular mesh $\rho'_N : \Sigma \to \mathbb{R}^{2n}$ constructed above, is called the N-th optimal piecewise linear isotropic approximation of ℓ.*

3. Proofs

3.1. Proof of Corollaries

Proof of Corollary B. Let S be a smooth immersed (resp. embedded) isotropic surface of $\mathbb{R}^{2n}$ diffeomorphic to a 2-torus Σ. Hence, there exists a smooth isotropic immersion (resp. embedding) $\ell : \Sigma \to \mathbb{R}^{2n}$ such that $\ell(\Sigma) = S$. By Theorem A, given any $\varepsilon > 0$, there exists a piecewise linear isotropic map $f : \Sigma \to \mathbb{R}^{2n}$, with $\|\ell - f\|_{C^1} \leq \varepsilon$, which is a topological immersion (resp. embedding). By definition $S_0 = f(\Sigma)$ is an immersed (resp. embedded) polyhedral isotropic surface, ε-close to S in the C^1-sense. □

Proof of Corollary C. We start with an immersed (resp. embedded) torus S of $\mathbb{R}^{2n}$, where $n \geq 2$. By definition, there exists a smooth torus Σ and a smooth immersion $f : \Sigma \to \mathbb{R}^{2n}$, such that $f(\Sigma) = S$. By a result of Gromov, for every $\varepsilon > 0$, there exists a smooth isotropic immersion $\ell : \Sigma \to \mathbb{R}^{2n}$, such that $\|f - \ell\|_{C^0} \leq \varepsilon$. Furthermore, if $n \geq 3$ and f is an embedding, we may assume that ℓ is an isotropic embedding. As in the proof of Corollary B, we can approximate ℓ by a piecewise linear isotropic topological immersion (or embedding if $n \geq 3$ and S is embedded) $h : \Sigma \to \mathbb{R}^{2n}$, such that $\|\ell - h\|_{C^1} \leq \varepsilon$. We deduce that $\|f - h\|_{C^0} \leq 2\varepsilon$ so that the isotropic immersed polyhedral surface $S_0 = h(\Sigma)$ is 2ε-close to S, in the C^0-sense. This proves the corollary. □

3.2. Symplectic density estimate

The notation $\mathcal{O}(N^{-q})$, for some integer q, is going be used a lot in all the following estimates. Formally, $F(N, \nu_1, \nu_2, \cdots) = \mathcal{O}(N^{-q})$, where F is an application taking values in some Euclidean vector space, means that there exist a constant $C > 0$, such that

$$\|F(N, \nu_1, \cdots)\| \leq CN^{-q}$$

for every integer $N \geq 0$, and every values of the parameters $\nu_1, \cdots$ etc... In our case, the parameters are typically indices (k, l) and points in Σ. Thus the notation $\mathcal{O}(N^{-q})$ is always understood for some uniform constant $C > 0$, which depends only on the choice of the smooth map ℓ, or, more precisely, on its C^k-norm. The $\mathcal{O}(N^{-q})$ will appear in approximations given by the Taylor expansion of ℓ. We use the notation $d^q\ell_z$ for the q-th differential of ℓ at $z \in \Sigma$. The q-th differential is a q-multilinear form, but, for practical reasons, we shall use the notation

$$d^q\ell_z \cdot \zeta = d^q\ell_z(\zeta, \cdots, \zeta)$$

when the same tangent vector ζ is repeated q times.

The key for the proof of Theorem A is a sharp estimate of the error term, essentially the symplectic density, involved in Theorem 2.4.13, as stated in the next proposition.

Proposition 3.2.1. *Given a smooth isotropic map $\ell : \Sigma \to \mathbb{R}^{2n}$ and its sequence of samples $\tau_N \in \mathscr{M}_N$ as defined in §2.4.5, we have the estimates*

$$\|\mu_N(\tau_N)\|_{\mathcal{C}^1_w} = \mathcal{O}(N^{-2}).$$

and

$$\|\mu_N(\tau_N)\|_{\mathcal{C}^{0,\alpha}_w} = \mathcal{O}(N^{-2}).$$

Proof. The second estimate follows from the first according to the following lemma.

Lemma 3.2.2. *The $\mathcal{C}^1_w$-norm controls the $\mathcal{C}^{0,\alpha}_w$-norm. In other words, there exists a constant $C > 0$, such that for every $N > 0$ and every function $\phi \in \mathscr{C}^2(\mathcal{Q}_N(\Sigma))$, we have*

$$\|\phi\|_{\mathcal{C}^{0,\alpha}_w} \leq C\|\phi\|_{\mathcal{C}^1_w}.$$

Proof. This control is obtained in a similar way to the smooth case, where a control on first derivatives induces a control on Hölder regularity. We have not really given the definition of the weak Hölder norms in this paper and the reader should refer to [7, §3.7] for more details. Let ϕ be a function defined on the set of facets of $\mathcal{Q}_N(\mathbb{R}^2)$. If $\mathbf{f}_1$ and $\mathbf{f}_2$ are two facets, their distance is $d(\mathbf{f}_1, \mathbf{f}_2) = \|\mathbf{z}_1 - \mathbf{z}_2\|$ by convention, where $\mathbf{z}_j$ are the barycenters of $\mathbf{f}_j$. If $\mathbf{f}_2$ is obtained by a diagonal translation of $\mathbf{f}_1$, then $\mathbf{f}_2 = T_u^k \circ T_v^l(\mathbf{f}_1)$, where T_u and T_v are the diagonal translations introduced at §2.4.10. By definition, the finite difference satisfy

$$|\phi(\mathbf{f}) - \phi(T_u\mathbf{f})| \leq \frac{\sqrt{2}}{N}\left\|\frac{\partial\phi}{\partial\vec{u}}\right\|_{\mathcal{C}^0},$$

with a similar inequality for the v-finite difference. We deduce

$$|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)| \leq \sqrt{2}\frac{|k| + |l|}{N}\|\phi\|_{\mathcal{C}^1_w}.$$

We deduce that

$$|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)| \leq 2d(\mathbf{f}_1, \mathbf{f}_2)\|\phi\|_{\mathcal{C}^1_w}.$$

If $0 < d(\mathbf{f}_1, \mathbf{f}_2) \leq 1$, we have

$$\frac{|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)|}{d(\mathbf{f}_1, \mathbf{f}_2)^\alpha} \leq 2\|\phi\|_{\mathcal{C}^1_w}.$$

If $d(\mathbf{f}_1, \mathbf{f}_2) > 1$, we have

$$\frac{|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)|}{d(\mathbf{f}_1, \mathbf{f}_2)^\alpha} \leq |\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)| \leq 2\|\phi\|_{\mathcal{C}^0}$$

which proves the estimate of the lemma in the case of $\mathscr{C}^2(\mathcal{Q}_N(\mathbb{R}^2))$. The case of $\mathscr{C}^2(\Sigma)$ endowed with its $\mathcal{C}^1_w$ and $\mathcal{C}^{0,\alpha}_w$-norms follows immediately, since these norms are obtained by passing to the cover $p \circ r_N : \mathbb{R}^2 \to \Sigma$. $\square$

We return to the proof of Proposition 3.2.1 and we use the notation

$$\eta_N = \mu_N(\tau_N)$$

as a shorthand for the rest of the proof. By definition

$$\eta_N(\mathbf{f}) = \omega(\mathscr{U}_{\tau_N}(\mathbf{f}), \mathscr{V}_{\tau_N}(\mathbf{f})).$$

Using the index notations of [7, §3.3.1 and §4.1.4]

$$D^u_{ij} = \frac{\sqrt{2}}{N}\mathscr{U}_{\tau_N}(\mathbf{f}_{ij}) \quad D^v_{ij} = \frac{\sqrt{2}}{N}\mathscr{V}_{\tau_N}(\mathbf{f}_{ij})$$

identifying all the functions with their lift via $p \circ r_N$, we have

$$\frac{2}{N^2}\eta_N(\mathbf{f}_{ij}) = \omega(D^u_{ij}, D^v_{ij})$$

where, by definition

$$D^u_{ij} = \ell(\mathbf{v}_{i+1,j+1}) - \ell(\mathbf{v}_{i,j})$$

and

$$D^v_{ij} = \ell(\mathbf{v}_{i,j+1}) - \ell(\mathbf{v}_{i+1,j})$$

We can use the Taylor formula at the center $\mathbf{z}_{ij}$ of the facet $\mathbf{f}_{ij}$ for the function ℓ. We are going to use the notation

$$\xi = \frac{1}{2N}(e_1 + e_2) = \frac{1}{2}(k_1 + k_2), \quad \zeta = \frac{1}{2N}(e_2 - e_1) = \frac{1}{2}(k_2 - k_1),$$

and the (u, v) coordinates introduced with Formula 2.12, obtained by rotating the canonical basis (e_1, e_2) of $\mathbb{R}^2$ by an angle $+\frac{\pi}{4}$. In particular

$$\xi = \frac{1}{\sqrt{2}N}\frac{\partial}{\partial u}, \quad \zeta = \frac{1}{\sqrt{2}N}\frac{\partial}{\partial v}.$$

Then

$$D^u_{ij} = \ell(\mathbf{z}_{ij} + \xi) - \ell(\mathbf{z}_{ij} - \xi) = 2d\ell|_{\mathbf{z}_{ij}} \cdot \xi + \mathcal{O}(N^{-3})$$

$$D^v_{ij} = \ell(\mathbf{z}_{ij} + \zeta) - \ell(\mathbf{z}_{ij} - \zeta) = 2d\ell|_{\mathbf{z}_{ij}} \cdot \zeta + \mathcal{O}(N^{-3})$$

Hence

$$\omega(D^u_{ij}, D^v_{ij}) = 4\omega(d\ell_{\mathbf{z}_{ij}} \cdot \xi, d\ell_{\mathbf{z}_{ij}} \cdot \zeta) + \mathcal{O}(N^{-4}).$$

Since ℓ is isotropic, the first term of the RHS vanishes, hence

$$\omega(D^u_{ij}, D^v_{ij}) = \mathcal{O}(N^{-4})$$

so that we have

$$\|\eta_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-2}). \tag{3.1}$$

We want to prove a similar estimate of the first order finite differences

$$\frac{\partial \eta_N}{\partial \vec{u}} \quad \text{and} \quad \frac{\partial \eta_N}{\partial \vec{v}}.$$

By definition of finite differences (cf. Formulas (2.11))

$$\frac{\partial \eta_N}{\partial \vec{u}}(\mathbf{f}_{ij}) = \frac{N^3}{\sqrt{2}^3}\big(\omega(D^u_{i+1,j+1}, D^v_{i+1,j+1}) - \omega(D^u_{i,j}, D^v_{i,j})\big)$$

which can be expressed as

$$\begin{aligned}\frac{\sqrt{2}^3}{N^3}\frac{\partial \eta_N}{\partial \vec{u}}(\mathbf{f}_{ij}) = {} & \omega(D^u_{i+1,j+1} - D^u_{ij}, D^v_{i+1,j+1}) \\ & + \omega(D^u_{i,j}, D^v_{i+1,j+1} - D^v_{i,j})\end{aligned} \tag{3.2}$$

The Taylor formula applied at the intersection point $\mathbf{v} = \mathbf{v}_{i+1,j+1}$ between the facets $\mathbf{f}_{ij}$ and $\mathbf{f}_{i+1,j+1}$, gives

$$\begin{aligned} D^u_{ij} &= \ell(\mathbf{v}) - \ell(\mathbf{v} - 2\xi) \\ &= d\ell_{\mathbf{v}} \cdot 2\xi - \frac{1}{2} d^2\ell_{\mathbf{v}} \cdot 2\xi + \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot 2\xi + \mathcal{O}(N^{-4}) \\ &= \sqrt{2} N^{-1} \frac{\partial \ell}{\partial u}(\mathbf{v}) - N^{-2} \frac{\partial^2 \ell}{\partial u^2}(\mathbf{v}) + \frac{\sqrt{2}}{3} N^{-3} \frac{\partial^3 \ell}{\partial u^3}(\mathbf{v}) + \mathcal{O}(N^{-4}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} D^u_{i+1,j+1} &= \ell(\mathbf{v} + 2\xi) - \ell(\mathbf{v}) \\ &= d\ell_{\mathbf{v}} \cdot 2u + \frac{1}{2} d^2\ell_{\mathbf{v}} \cdot 2\xi + \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot 2\xi + \mathcal{O}(N^{-4}) \\ &= \sqrt{2} N^{-1} \frac{\partial \ell}{\partial u}(\mathbf{v}) + N^{-2} \frac{\partial^2 \ell}{\partial u^2}(\mathbf{v}) + \frac{\sqrt{2}}{3} N^{-3} \frac{\partial^3 \ell}{\partial u^3}(\mathbf{v}) + \mathcal{O}(N^{-4}). \end{aligned} \tag{3.4}$$

Hence by Equations (3.3) and (3.4), we have

$$D^u_{i+1,j+1} - D^u_{ij} = 2N^{-2} \frac{\partial^2 \ell}{\partial u^2}(\mathbf{v}) + \mathcal{O}(N^{-4}) \tag{3.5}$$

Using the notations

$$k_1 = \frac{e_1}{N} = N^{-1} \frac{\partial}{\partial x} \quad \text{and} \quad k_2 = \frac{e_2}{N} = N^{-1} \frac{\partial}{\partial y},$$

a similar computation shows that

$$\begin{aligned} D^v_{ij} = \ell(\mathbf{v} - k_1) - \ell(\mathbf{v} - k_2) &= d\ell_{\mathbf{v}} \cdot (k_2 - k_1) \\ &+ \frac{1}{2} d^2\ell_{\mathbf{v}} \cdot k_1 - \frac{1}{2} d^2\ell_{\mathbf{v}} \cdot k_2 - \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot k_1 + \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot k_2 + \mathcal{O}(N^{-4}) \end{aligned}$$

and using the fact that $k_2 + k_1 = 2\xi$, $k_2 - k_1 = 2\zeta$, we have

$$D^v_{ij} = d\ell_{\mathbf{v}} \cdot 2\zeta - \frac{1}{2} d^2\ell_{\mathbf{v}}(2\zeta, 2\xi) - \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot k_1 + \frac{1}{6} d^3\ell_{\mathbf{v}} \cdot k_2 + \mathcal{O}(N^{-4})$$

so that

$$\begin{aligned} D^v_{ij} &= \sqrt{2} N^{-1} \frac{\partial \ell}{\partial u}(\mathbf{v}) - N^{-2} \frac{\partial^2 \ell}{\partial u \partial v}(\mathbf{v}) \\ &\quad + \frac{N^{-3}}{6} \left(\frac{\partial^3 \ell}{\partial y^3}(\mathbf{v}) - \frac{\partial^3 \ell}{\partial x^3}(\mathbf{v}) \right) + \mathcal{O}(N^{-4}) \end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} D^v_{i+1,j+1} &= \ell(\mathbf{v}+k_2) - \ell(\mathbf{v}+k_1) \\ &= d\ell_{\mathbf{v}}\cdot(k_2-k_1) + \frac{1}{2}d^2\ell_{\mathbf{v}}\cdot k_2 - \frac{1}{2}d^2\ell_{\mathbf{v}}\cdot k_1 \\ &\quad + \frac{1}{6}d^3\ell_{\mathbf{v}}\cdot k_2 - \frac{1}{6}d^3\ell_{\mathbf{v}}\cdot k_1 + \mathcal{O}(N^{-4}) \end{aligned}$$

and we have

$$\begin{aligned} D^v_{i+1,j+1} &= \sqrt{2}N^{-1}\frac{\partial\ell}{\partial u}(\mathbf{v}) + N^{-2}\frac{\partial^2\ell}{\partial u\partial v}(\mathbf{v}) \\ &\quad + \frac{N^{-3}}{6}\left(\frac{\partial^3\ell}{\partial y^3}(\mathbf{v}) - \frac{\partial^3\ell}{\partial x^3}(\mathbf{v})\right) + \mathcal{O}(N^{-4}) \end{aligned} \tag{3.7}$$

Then, by Equations (3.6) and (3.7)

$$D^v_{i+1,j+1} - D^v_{ij} = 2N^{-2}\frac{\partial^2\ell}{\partial u\partial v}(\mathbf{v}) + \mathcal{O}(N^{-4}). \tag{3.8}$$

In conclusion, by Equations (3.2),(3.3), (3.5), (3.7) and (3.8)

$$\begin{aligned} N^{-3}\frac{\partial\eta_N}{\partial\vec{u}}(\mathbf{f}_{ij}) &= N^{-3}\omega\left(\frac{\partial^2\ell}{\partial u^2}(\mathbf{v}), \frac{\partial\ell}{\partial v}(\mathbf{v})\right) \\ &\quad + N^{-3}\omega\left(\frac{\partial\ell}{\partial u}(\mathbf{v}), \frac{\partial^2\ell}{\partial u\partial v}(\mathbf{v})\right) + \mathcal{O}(N^{-5}). \end{aligned} \tag{3.9}$$

The lower order term of the above expansion vanishes. Indeed, by isotropy of ℓ, we have the equation

$$0 = \omega\left(\frac{\partial\ell}{\partial u}, \frac{\partial\ell}{\partial v}\right)$$

and differentiating in the u-direction gives

$$0 = \omega\left(\frac{\partial^2\ell}{\partial u^2}, \frac{\partial\ell}{\partial v}\right) + \omega\left(\frac{\partial\ell}{\partial u}, \frac{\partial^2\ell}{\partial u\partial v}\right).$$

Therefore

$$\frac{\partial\eta_N}{\partial\vec{u}}(\mathbf{f}_{ij}) = \mathcal{O}(N^{-2}). \tag{3.10}$$

It is easy to see that the estimate $\mathcal{O}(N^{-2})$ are uniform in $\mathbf{f}_{ij}$, and that the constant involved depend only on the derivatives of ℓ, which are bounded since Σ is closed. By symmetry, we have similar estimates in the v-direction

$$\frac{\partial \eta_N}{\partial \vec{v}}(\mathbf{f}_{ij}) = \mathcal{O}(N^{-2}), \tag{3.11}$$

so that by Equations (3.1), (3.10) and (3.11)

$$\|\eta_N\|_{\mathcal{C}^1_w} = \mathcal{O}(N^{-2})$$

which proves the proposition. □

Corollary 3.2.3. *The isotropic quadrangular meshes ρ_N of Theorem 2.4.13 satisfy*

$$\|\rho_N - \tau_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-2}).$$

Proof. This is an immediate consequence of Theorem 2.4.13 and Proposition 3.2.1. □

3.3. Triangular mesh estimates

The goal of this section is to expand the estimate of Corollary 3.2.3 into an estimate for triangular meshes as follows:

Proposition 3.3.1. *The optimal isotropic triangular mesh ρ'_N and the triangular samples τ'_N of ℓ satisfy the estimate*

$$\|\rho'_N - \tau'_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-2}).$$

Lemma 3.3.2. *Let $\hat{\rho}_N$ and τ'_N be the triangular meshes constructed from ρ_N and τ_N. Then*

$$\|\hat{\rho}_N - \tau'_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-2}).$$

Proof of Lemma 3.3.2. Let $\mathbf{f}_{ij}$ be a facet of $\mathcal{Q}_N$ and $\mathbf{z}_{ij}$ its barycenter. By definition $\hat{\rho}_N(\mathbf{z}_{ij})$ is the barycenter in $\mathbb{R}^{2n}$ of the points $\rho_N(\mathbf{v})$ where $\mathbf{v}$ is a vertex of $\mathbf{f}_{ij}$. Using the Taylor formula we have

$$\sum_{\mathbf{v}\sim\mathbf{z}_{ij}} \ell(\mathbf{v}) = 4\ell(\mathbf{z}_{ij}) + \mathcal{O}(N^{-2}),$$

where $\mathbf{z}_{ij} \sim \mathbf{v}$ means that $\mathbf{v}$ is a vertex of $\mathbf{f}_{ij}$. Since $\ell(\mathbf{v}) = \tau_N(\mathbf{v}) = \rho_N(\mathbf{v}) + \mathcal{O}(N^{-2})$ by Corollary 3.2.3, we deduce that

$$\hat{\rho}_N(\mathbf{z}_{ij}) = \ell(\mathbf{z}_{ij}) + \mathcal{O}(N^{-2})$$

and since $\tau'_N(\mathbf{z}_{ij}) = \ell(\mathbf{z}_{ij})$, by definition, this proves the proposition. □

Proposition 3.3.3.

$$\|\rho'_N - \hat{\rho}_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-2})$$

Proof of Proposition 3.3.3. The optimal isotropic triangular mesh ρ'_N is obtained by choosing the closest point to $\hat{\rho}_N(\mathbf{z}_{ij})$, such that the pyramid constructed from the quadrilateral associated to the facet $\mathbf{f}_{ij}$ of the quadrangular mesh ρ_N is isotropic. We merely need to prove that

$$\hat{\rho}_N(\mathbf{z}_{ij}) - \rho'_N(\mathbf{z}_{ij}) = \mathcal{O}(N^{-2}),$$

since ρ'_N and $\hat{\rho}_N$ agree along vertices of $\mathcal{Q}_N$.

Recall that $\mathbf{v}_{ij}$ is a vertex of $\mathbf{f}_{ij}$. We consider the *parallelogram* $(B_0B_1B_2B_3)$ of $\mathbb{R}^{2n}$ given by $B_0 = \ell(\mathbf{v}_{ij}) = \tau'_N(\mathbf{v}_{ij})$ and

$$\overrightarrow{B_0B_1} = \frac{\partial \ell}{\partial x}(\mathbf{v}_{ij}), \quad \overrightarrow{B_0B_3} = \frac{\partial \ell}{\partial y}(\mathbf{v}_{ij}).$$

Thus, $(B_0B_1B_2B_3)$ is an isotropic parallelogram tangent to the map ℓ at $\ell(\mathbf{v}_{ij})$.

Lemma 3.3.4. *All the parallelograms $(B_0B_1B_2B_3)$ obtained by the above construction are uniformly close to be squares, in the following sense: there exists constants $c_1, c_2 > 0$, independent of N and the choice of facet $\mathbf{f}_{ij}$, such that the side lengths of the parallelogram $(B_0B_1B_2B_3)$, belongs to the interval (c_1, c_2) and*

$$g(\overrightarrow{B_iB_{i+1}}, \overrightarrow{B_{i+1}B_{i+2}}) = \mathcal{O}(N^{-1}),$$

where the index i is understood modulo 4.

Proof of Lemma 3.3.4. The fact that ℓ is a smooth immersion defined on a compact surface implies the existence of the positive constants $c_1, c_2 > 0$. The covering maps $p \circ r_N : \mathbb{R}^2 \to \Sigma$ are *almost* conformal in the sence of Equation (2.9), which implies the almost orthogonality of consecutive sides up to an error term of order $\mathcal{O}(N^{-1})$. □

We return to the proof of Proposition 3.3.3. We consider the quadrilateral $(C_0C_1C_2C_3)$ defined by the facet $\mathbf{f}_{ij}$ and τ_N rescaled by a factor N: we put $C_0 = B_0 = \tau_N(\mathbf{v}_{ij})$ and the other points C_i are given by the vertices of the mesh τ_N around the facet $\mathbf{f}_{ij}$, rescaled by a homothety of center C_0 and scaling factor N.

Similarly, we construct a quadrilateral $(A_0A_1A_2A_3)$ associated to the isotropic quadrangular mesh ρ_N and the facet $\mathbf{f}_{ij}$, rescaled by the same homothety.

It follows from Corollary 3.2.3 that the quadrilaterals $(A_0A_1A_2A_3)$ and $(C_0C_1C_2C_3)$ agree up to a perturbation of order $\mathcal{O}(N^{-1})$ (the loss of one order is due to the rescaling). By the Taylor formula $(B_0B_1B_2B_3)$ agrees with $(C_0C_1C_2C_3)$ up to a perturbation of order $\mathcal{O}(N^{-1})$. In conclusion $(A_0A_1A_2A_3)$ and $(B_0B_1B_2B_3)$ agree up to a perturbation of order $\mathcal{O}(N^{-1})$ and we deduce from Lemma 3.3.4 that the quadrilateral $(A_0A_1A_2A_3)$ is almost a square in the same sense as $(B_0B_1B_2B_3)$.

The optimal apex of $(B_0B_1B_2B_3)$, in the sense of Definition 2.4.17, agrees with its barycenter, since this is a parallelogram contained in an isotropic plane. We expect that a small isotropic deformation of this quadrilateral like $(A_0A_1A_2A_3)$ ought to have an optimal apex very close to its barycenter as well. This is indeed to case: if the isotropic quadrilateral $(A_0A_1A_2A_3)$ is contained in a plane, then this plane must be isotropic and the optimal apex agrees with the barycenter of $(A_0A_1A_2A_3)$, by definition. If $(A_0A_1A_2A_3)$ is not contained in a plane, an explicit apex that completes the quadrilateral $(A_0A_1A_2A_3)$ into an isotropic pyramid can be found by solving the linear system of equations. This work is carried out in details in [7, §7.1], where a particular solution P is given explicitly by [7, Formula (7.16)]. The fact that $(A_0A_1A_2A_3)$ is almost a square in the sense of Lemma 3.3.4 implies that

$$\begin{aligned} &\alpha_0 = \frac{1}{2} + \mathcal{O}(N^{-1}), \quad \alpha_1 = \frac{1}{2} + \mathcal{O}(N^{-1}), \\ &\beta_0 = \mathcal{O}(N^{-1}) \quad \text{and} \quad \beta_1 = \mathcal{O}(N^{-1}). \end{aligned}$$

where the notations of [7, §7.1.5] have been used in the above identities.

Then by [7, Formula (7.15)], we have $\xi(V) = (|\beta_0| + |\beta_1|)\mathcal{O}(N^{-1})$, and we deduce from [7, Formula (7.16)] that the particular solution P satisfies

$$\overrightarrow{GP} = \mathcal{O}(N^{-1})$$

where G is the barycenter of $(A_0A_1A_2A_3)$. In conclusion, the barycenter of the isotropic quadrilateral $(A_0A_1A_2A_3)$ agrees with the optimal apex, up to a perturbation of order $\mathcal{O}(N^{-1})$. After scaling back to the original picture by a factor N^{-1}, this proves the proposition. □

Proof of Proposition 3.3.1. The proposition is an immediate consequence of Lemma 3.3.2 and Proposition 3.3.3. □

3.4. Proof of the main theorem

Proposition 3.4.1. *The optimal piecewise linear isotropic maps $\ell_N : \Sigma \to \mathbb{R}^{2n}$ (cf. Definition 2.4.19) satisfy the estimate*

$$\|\ell - \ell_N\|_{C^0} = \mathcal{O}(N^{-2}), \quad \|\ell - \ell_N\|_{C^1} = \mathcal{O}(N^{-1}).$$

Proof. The piecewise linear isotropic map ℓ_N is defined as an affine map on each facet of the triangulation $\mathscr{T}_N$, which agrees with the isotropic triangular mesh ρ'_N at the vertices (cf. Definition 2.4.19).

Another piecewise linear map $f_N : \Sigma \to \mathbb{R}^{2n}$ can be associated to the mesh τ'_N exactly in the same way. In other words, f_N is an affine map on each facet of the triangulation $\mathscr{T}_N$, which agrees with the map ℓ at the vertices of the triangulation. This type of approximation is classical and by the Taylor formula applied on each facet of the triangulation, we have the following lemma:

Lemma 3.4.2. *The sequence of piecewise linear maps f_N approximates the smooth map ℓ, in the sense that*

$$\|f_N - \ell\|_{C^0} = \mathcal{O}(N^{-2}).$$

Furthermore, f_N piecewise smooth, with $\mathscr{T}_N$ as an adapted triangulation and we have

$$\|df_N - d\ell\|_{C^0} = \mathcal{O}(N^{-1}),$$

where the C^0-norm of differentials of piecewise smooth maps is defined in the sense of Formula (2.1)*.*

The restriction of f_N and ℓ_N along a facet of the triangulation is an affine map. The values of the maps at the vertices are given respectively by τ'_N and ρ'_N. By Proposition 3.3.1, these control values agree up to an error of order $\mathcal{O}(N^{-2})$. Since the facets have size N^{-1}, the estimate extends globally on the entire facet and we obtain the control

$$\|\ell_N - f_N\|_{C^0} = \mathcal{O}(N^{-2}).$$

Together with Lemma 3.4.2, we deduce the estimate

$$\|\ell_N - \ell\|_{C^0} = \mathcal{O}(N^{-2}), \tag{3.12}$$

which proves the first statement of the proposition.

Rescaling the source and target spaces by a factor N for the restriction of maps $u_N = f_N - \ell_N$ on a facet of the triangulation, shows a sequence of affine maps, whose values are of order $\mathcal{O}(N^{-1})$ at the vertices. We deduce that the linear part of the maps u_N along the facet must be of the same order. It follows that

$$\|df_N - d\ell_N\|_{C^0} = \|du_N\|_{C^0} = \mathcal{O}(N^{-1})$$

where the C^0-norm of differential of piecewise smooth maps is taken in the sense of Formula (2.1). Together with Lemma 3.4.2, we deduce the estimate

$$\|d\ell_N - d\ell\|_{C^0} = \mathcal{O}(N^{-1}). \tag{3.13}$$

Formula (3.12) and Formula (3.13) give the control

$$\|\ell_N - \ell\|_{C^1} = \mathcal{O}(N^{-1})$$

which proves the second statement of the proposition. □

Corollary 3.4.3. *For every sufficiently large N, the piecewise linear isotropic map $\ell_N : \Sigma \to \mathbb{R}^{2n}$ is a topological immersion (resp. embedding) if ℓ is an smooth immersion (resp. embedding).*

Proof. The sequence of optimal isotropic piecewise linear maps ℓ_N satisfies

$$\lim_{N\to\infty} \|\ell - \ell_N\|_{C^1} = 0$$

by Proposition 3.4.1. If ℓ is a smooth immersion (resp. embedding), this forces ℓ_N to be a topological immersion (resp. embedding), for every sufficiently large N, by Corollary 2.3.8, which proves the theorem. □

Proof of Theorem A. The theorem is an immediate consequence of Proposition 3.4.1 and Corollary 3.4.3. □

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Received January 25, 2021
Accepted April 19, 2022