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First steps in twisted Rabinowitz–Floer homology

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Rabinowitz–Floer homology is the Morse–Bott homology in the sense of Floer associated with the Rabinowitz action functional introduced by Kai Cieliebak and Urs Frauenfelder in 2009. In our work, we consider a generalisation of this theory to a Rabinowitz–Floer homology of a Liouville automorphism. As an application, we show the existence of noncontractible periodic Reeb orbits on quotients of symmetric star-shaped hypersurfaces. In particular, our theory applies to lens spaces.

1. Introduction

In this paper, we introduce an analogue of the twisted Floer homology [25] in the Rabinowitz–Floer setting. See the excellent survey article [5] for a brief introduction to Rabinowitz–Floer homology and [2] for an overview of common Floer theories. Following [7] and [3], we construct a Morse–Bott homology for a suitable twisted version of the standard Rabinowitz action functional, generalising standard Rabinowitz–Floer homology.

Theorem 1.1 (Twisted Rabinowitz–Floer homology). *Let (M, λ) be the completion of a Liouville domain (W, λ) and let $\varphi \in \operatorname{Diff}(W)$ be of finite order with $\varphi(\partial W) = \partial W$ and $\varphi^*\lambda - \lambda = df_\varphi$ for some smooth compactly supported function $f_\varphi \in C_c^\infty(\operatorname{Int} W)$ in the interior of W.*

(a) *The semi-infinite dimensional Morse–Bott homology $\mathrm{RFH}^\varphi(\partial W, M)$ in the sense of Floer of the twisted Rabinowitz action functional exists and is well-defined. Moreover, twisted Rabinowitz–Floer homology is invariant under twisted homotopies of Liouville domains.*

(b) *If ∂W is simply connected and does not admit any nonconstant twisted Reeb orbits, then $\mathrm{RFH}_*^\varphi(\partial W, M) \cong \mathrm{H}_*(\operatorname{Fix}(\varphi|_{\partial W}); \mathbb{Z}_2)$.*

(c) *If ∂W is displaceable by a compactly supported Hamiltonian symplectomorphism in (M, λ), then $\mathrm{RFH}^\varphi(\partial W, M) \cong 0$.*

Part (a) will be proven in Sections 4 and 5, in particular Theorem 5.2, part (b) is the content of Proposition 4.3 and finally part (c) is the content of Theorem 6.5. First of all, twisted Rabinowitz–Floer homology does indeed generalise standard Rabinowitz–Floer homology as

$$\mathrm{RFH}^{\mathrm{id}_W}(\partial W, M) = \mathrm{RFH}(\partial W, M).$$

Moreover, twisted Rabinowitz–Floer homology can be used to prove existence of noncontractible periodic Reeb orbits on quotients of certain symmetric star-shaped hypersurfaces.

Theorem 1.2. *Let $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, be a compact and connected star-shaped hypersurface invariant under the rotation*

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z^1, \ldots, z^n) := \left(e^{2\pi i k_1/m} z^1, \ldots, e^{2\pi i k_n/m} z^n\right)$$

for some even $m \geq 2$ and $k_1, \ldots, k_n \in \mathbb{Z}$ coprime to m. Then $\Sigma/\mathbb{Z}_m$ admits a noncontractible periodic Reeb orbit generating $\pi_1(\Sigma/\mathbb{Z}_m) \cong \mathbb{Z}_m$.

The proof is straightforward, once we have computed the $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology of the sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$. Indeed, by invariance we may assume that $\Sigma = \mathbb{S}^{2n-1}$, as Σ is star-shaped. Then we use the following elementary topological fact (see Lemma 1.3 below). Let Σ be a simply connected topological manifold and let $\varphi\colon \Sigma \to \Sigma$ be a homeomorphism of finite order m that is not equal to the identity. If the induced discrete action

$$\mathbb{Z}_m \times \Sigma \to \Sigma, \qquad [k] \cdot x := \varphi^k(x)$$

is free, then $\pi\colon \Sigma \to \Sigma/\mathbb{Z}_m$ is a normal covering map [17, Theorem 12.26]. For $x \in \Sigma$ define the ***based twisted loop space of $\boldsymbol{\Sigma}$ and $\boldsymbol{\varphi}$*** by

$$\mathscr{L}_\varphi(\Sigma, x) := \{\gamma \in C(I, \Sigma) : \gamma(0) = x \text{ and } \gamma(1) = \varphi(x)\},$$

where $I := [0, 1]$. Then we have the following result. See Figure 1.

Lemma 1.3. *If $\gamma \in \mathscr{L}_\varphi(\Sigma, x)$ for some $x \in \Sigma$, then $\pi \circ \gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ is not contractible. Conversely, if $\gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ is not contractible, then there exists $1 \leq k < m$ such that $\tilde{\gamma}_x \in \mathscr{L}_{\varphi^k}(\Sigma, x)$ for the unique lift $\tilde{\gamma}_x$ of γ with $\tilde{\gamma}_x(0) = x$.*

For a more detailed study of twisted loop spaces of universal covering manifolds as well as a proof of Lemma 1.3 see Appendix A. To the authors knowledge, there are two similar versions of Theorem 1.2 in the literature.

Theorem 1.4 ([11, Corollary 1.6 (iv)]). *Any contact form on a lens space defining the standard contact structure admits a closed Reeb orbit.*

Using the fact that there is a natural bijection between contact forms on the odd-dimensional sphere equipped with the standard contact structure and star-shaped hypersurfaces, Theorem 1.4 is actually stronger than Theorem 1.2 in that it does not restrict the parity of the lens space. However, Theorem 1.4 does not say anything about the topological nature of the Reeb orbit. The proof of this theorem uses a generalisation of Givental's nonlinear Maslov index to lens spaces.

Theorem 1.5 ([20, Theorem 1.2]). *Let $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, be a dynamically convex star-shaped hypersurface such that $\Sigma = -\Sigma$. Then Σ admits at least two symmetric geometrically distinct closed characteristics.*

Theorem 1.5 has the advantage of being a multiplicity result, but in disadvantage requires the assumption that the hypersurface is dynamically convex and does only work for $\mathbb{Z}_2$-symmetry. The first named author of the paper [20] is currently working on extending Theorem 1.5 to lens spaces. As many multiplicity results, the proof of this theorem makes use of index theory, in particular Ekeland–Hofer theory.

The existence of closed Reeb orbits on lens spaces is important in the study of celestial mechanics. Indeed, by [10, Corollary 5.7.5], the Moser regularised energy hypersurface near the earth or the moon of the planar circular restricted three-body problem for energy values below the first critical value is diffeomorphic to the real projective space $\mathbb{RP}^3$.

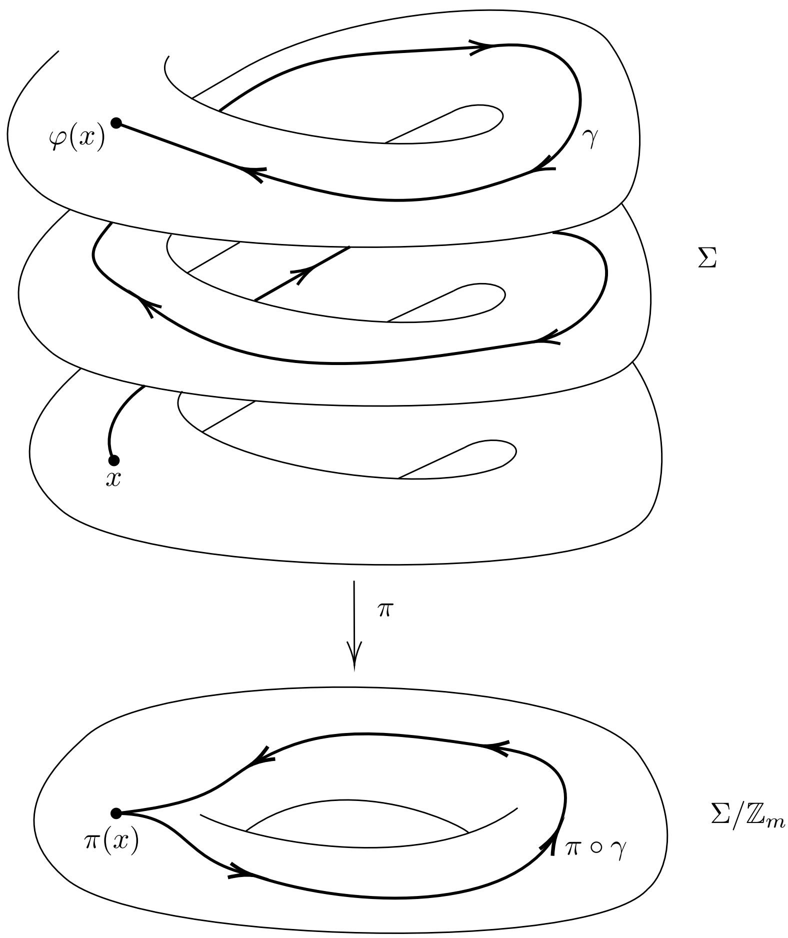

Figure 1. The projection $\pi \circ \gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ of $\gamma \in \mathscr{L}_\varphi(\Sigma, x)$ is not contractible for the deck transformation $\varphi \neq \mathrm{id}_\Sigma$.

2. The twisted Rabinowitz action functional

Definition 2.1 (Free twisted loop space). *Let $\varphi \in \mathrm{Diff}(M)$ be a diffeomorphism of a smooth manifold M. Define the* ***free twisted loop space of M and φ*** *by*

$$\mathscr{L}_\varphi M := \{\gamma \in C^\infty(\mathbb{R}, M) : \gamma(t+1) = \varphi(\gamma(t)) \; \forall t \in \mathbb{R}\}.$$

Let (M, ω) be a symplectic manifold and $\varphi \in \mathrm{Symp}(M, \omega)$. Given a twisted loop $\gamma \in \mathscr{L}_\varphi M$ and $\varepsilon_0 > 0$, we say that a curve

$$(-\varepsilon_0, \varepsilon_0) \to \mathscr{L}_\varphi M, \qquad \varepsilon \mapsto \gamma_\varepsilon$$

starting at γ is ***smooth***, if the induced variation

$$\mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to M, \qquad (t, \varepsilon) \mapsto \gamma_\varepsilon(t)$$

is smooth. Since $\gamma_\varepsilon(t+1) = \varphi(\gamma_\varepsilon(t))$ holds for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $t \in \mathbb{R}$, we call such a variation a ***twisted variation***. Then the infinitesimal variation

$$\delta\gamma := \left.\frac{\partial \gamma_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \in \mathfrak{X}(\gamma),$$

satisfies

$$\delta\gamma(t+1) = D\varphi(\delta\gamma(t)) \qquad \forall t \in \mathbb{R}.$$

Lemma 2.2. *Let (M, ω) be a symplectic manifold and let $\varphi \in \mathrm{Symp}(M, \omega)$ be of finite order. Let $\gamma \in \mathscr{L}_\varphi M$ and let $X \in \mathfrak{X}(\gamma)$ be such that*

$$X(t+1) = D\varphi(X(t)) \qquad \forall t \in \mathbb{R}.$$

Then there exists a twisted variation of γ such that $\delta\gamma = X$.

Proof. As φ is assumed to be of finite order, there exists a φ-invariant ω-compatible almost complex structure J on M by [22, Lemma 5.5.6]. With respect to the induced Riemannian metric

$$m_J := \omega(J\cdot, \cdot),$$

the symplectomorphism φ is an isometry. Define the exponential variation

$$\mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to M, \qquad \gamma_\varepsilon(t) := \exp^{\nabla_J}_{\gamma(t)}(\varepsilon X(t)),$$

for $\varepsilon_0 > 0$ sufficiently small and ∇_J denoting the Levi–Civita connection associated with m_J. Such an $\varepsilon_0 > 0$ does exist by naturality of geodesics [19, Corollary 5.14]. Then we compute

$$\begin{aligned}\gamma_\varepsilon(t+1) &= \exp^{\nabla_J}_{\gamma(t+1)}(\varepsilon X(t+1)) \\ &= \exp^{\nabla_J}_{\varphi(\gamma(t))}(D\varphi(\varepsilon X(t))) \\ &= \varphi\left(\exp^{\nabla_J}_{\gamma(t)}(\varepsilon X(t))\right) \\ &= \varphi(\gamma_\varepsilon(t))\end{aligned}$$

by naturality of the exponential map [19, Proposition 5.20]. □

Remark 2.3. The statement of Lemma 2.2 remains true if $\operatorname{ord}\varphi = \infty$.

This discussion together with Lemma A.3 motivates the following definition of the tangent space to the free twisted loop space.

Definition 2.4 (Tangent space to the free twisted loop space). *Let (M, ω) be a symplectic manifold and $\varphi \in \operatorname{Symp}(M, \omega)$. For $\gamma \in \mathscr{L}_\varphi M$ define the* ***tangent space to the free twisted loop space at γ*** *by*

$$T_\gamma \mathscr{L}_\varphi M := \{X \in \Gamma(\gamma^* TM) : X(t+1) = D\varphi(X(t))\ \forall t \in \mathbb{R}\}.$$

Definition 2.5 (Twisted Hamiltonian function). *Let (M, ω) be a symplectic manifold and $\varphi \in \operatorname{Symp}(M, \omega)$. A function $H \in C^\infty(M \times \mathbb{R})$ is said to be a* ***twisted Hamiltonian function****, if*

$$\varphi^* H_{t+1} = H_t \qquad \forall t \in \mathbb{R}.$$

We denote the space of all twisted Hamiltonian functions by $C^\infty_\varphi(M \times \mathbb{R})$ and the subspace of all autonomous twisted Hamiltonian functions by $C^\infty_\varphi(M)$.

Recall, that an exact symplectic manifold is by definition a pair (M, λ) such that $(M, d\lambda)$ is a symplectic manifold. An exact symplectomorphism of an exact symplectic manifold (M, λ) is a diffeomorphism $\varphi \in \operatorname{Diff}(M)$ such that $\varphi^*\lambda - \lambda$ is exact.

Definition 2.6 (Perturbed twisted Rabinowitz action functional). *Let (M, λ) be an exact symplectic manifold and $\varphi \in \mathrm{Diff}(M)$ an exact symplectomorphism with $\varphi^*\lambda - \lambda = df$. For $H, F \in C^\infty_\varphi(M \times \mathbb{R})$ define the* ***perturbed twisted Rabinowitz action functional***

$$\mathscr{A}_\varphi^{(H,F)} \colon \mathscr{L}_\varphi M \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathscr{A}_\varphi^{(H,F)}(\gamma, \tau) := \int_0^1 \gamma^*\lambda - \tau \int_0^1 H_t(\gamma(t))dt - \int_0^1 F_t(\gamma(t))dt - f(\gamma(0)).$$

If $F = 0$ and $H \in C^\infty_\varphi(M)$, we write $\mathscr{A}_\varphi^H$ for $\mathscr{A}_\varphi^{(H,F)}$ and call $\mathscr{A}_\varphi^H$ the ***twisted Rabinowitz action functional****.*

Remark 2.7. Assume that $m := \operatorname{ord} \varphi < \infty$. Then

$$\mathscr{A}_\varphi^{(H,F)}(\gamma, \tau) = \frac{1}{m}\mathscr{A}^{(H,F)}(\bar{\gamma}, \tau) - \frac{1}{m}\sum_{k=0}^{m-1} f(\gamma(k)),$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M$, where $\bar{\gamma} \in \mathscr{L}M$ is defined by $\bar{\gamma}(t) := \gamma(mt)$ and

$$\mathscr{A}^{(H,F)} \colon \mathscr{L}M \times \mathbb{R} \to \mathbb{R}$$

denotes the standard Rabinowitz action functional.

Definition 2.8 (Differential of the perturbed twisted Rabinowitz action functional). *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ). For $H, F \in C^\infty_\varphi(M \times \mathbb{R})$, define the* ***differential of the perturbed twisted Rabinowitz action functional***

$$d\mathscr{A}_\varphi^{(H,F)}|_{(\gamma,\tau)} \colon T_\gamma \mathscr{L}_\varphi M \times \mathbb{R} \to \mathbb{R}$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$ by

$$d\mathscr{A}_\varphi^{(H,F)}|_{(\gamma,\tau)}(X, \eta) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{A}_\varphi^{(H,F)}(\gamma_\varepsilon, \tau + \varepsilon\eta),$$

where γ_ε is a twisted variation of γ such that $\delta\gamma = X$.

Proposition 2.9 (Differential of the perturbed twisted Rabinowitz action functional). *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of* *an exact symplectic manifold* (M, λ) *and* $H, F \in C^\infty_\varphi(M \times \mathbb{R})$*. Then*

$$d\mathscr{A}^{(H,F)}_\varphi|_{(\gamma,\tau)}(X,\eta) = \int_0^1 d\lambda(X(t), \dot\gamma(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt - \eta \int_0^1 H_t(\gamma(t))dt \tag{2.1}$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$ *and* $(X, \eta) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$*. Moreover, we have that*

$$(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^{(H,F)}_\varphi$$

if and only if

$$\dot\gamma(t) = \tau X_{H_t}(\gamma(t)) + X_{F_t}(\gamma(t)) \qquad \text{and} \qquad \int_0^1 H_t(\gamma(t))dt = 0 \tag{2.2}$$

for all $t \in \mathbb{R}$*.*

Proof. A routine computation shows (2.1). Let $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^{(H,F)}_\varphi$. It follows immediately from (2.1) that

$$\int_0^1 H_t(\gamma(t))dt = 0$$

and

$$\int_0^1 d\lambda(X(t), \dot\gamma(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt = 0$$

for all $X \in T_\gamma \mathscr{L}_\varphi M$. Suppose there exists $t_0 \in \operatorname{Int} I$ such that

$$\dot\gamma(t_0) - \tau X_{H_{t_0}}(\gamma(t_0)) - X_{F_{t_0}}(\gamma(t_0)) \neq 0.$$

By nondegeneracy of the symplectic form $d\lambda$ there exists $v \in T_{\gamma(t_0)}M$ with

$$d\lambda(v, \dot\gamma(t_0) - \tau X_{H_{t_0}}(\gamma(t_0)) - X_{F_{t_0}}(\gamma(t_0))) \neq 0.$$

Fix a Riemannian metric on M and let X_v denote the unique parallel vector field along $\gamma|_I$ such that $X_v(t_0) = v$. As $\operatorname{Int} I$ is open, there exists $\delta > 0$ such that $\bar B_\delta(t_0) \subseteq \operatorname{Int} I$. Fix a smooth bump function $\beta \in C^\infty(I)$ for t_0 supported in $B_\delta(t_0)$. By shrinking δ if necessary, we may assume that

$$\int_{t_0-\delta}^{t_0+\delta} d\lambda(\beta(t)X_v(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \neq 0.$$

Extending

$$(\beta X_v)(t+k) := D\varphi^k(\beta(t)X_v(t)) \qquad \forall t \in I, k \in \mathbb{Z},$$

we have that $\beta X_v \in T_\gamma \mathscr{L}_\varphi M$ and thus we compute

$$\begin{aligned}
0 &= d\mathscr{A}_\varphi^{(H,F)}|_{(\gamma,\tau)}(\beta X_v, 0) \\
&= \int_{t_0-\delta}^{t_0+\delta} d\lambda(\beta(t)X_v(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \\
&\neq 0.
\end{aligned}$$

Hence

$$\dot{\gamma}(t) = \tau X_{H_t}(\gamma(t)) + X_{F_t}(\gamma(t)) \qquad \forall t \in I,$$

implying

$$\begin{aligned}
\dot{\gamma}(t+k) &= D\varphi^k(\dot{\gamma}(t)) \\
&= \tau(D\varphi^k \circ X_{H_t})(\gamma(t)) + (D\varphi^k \circ X_{F_t})(\gamma(t)) \\
&= \tau(D\varphi^k \circ X_{H_t} \circ \varphi^{-k} \circ \varphi^k)(\gamma(t)) + (D\varphi^k \circ X_{F_t} \circ \varphi^{-k} \circ \varphi^k)(\gamma(t)) \\
&= \tau\varphi^k_* X_{H_t}(\gamma(t+k)) + \varphi^k_* X_{F_t}(\gamma(t+k)) \\
&= \tau X_{\varphi^k_* H_t}(\gamma(t+k)) + X_{\varphi^k_* F_t}(\gamma(t+k)) \\
&= \tau X_{H_{t+k}}(\gamma(t+k)) + X_{F_{t+k}}(\gamma(t+k))
\end{aligned}$$

for all $t \in I$ and $k \in \mathbb{Z}$. The other direction is immediate. $\square$

Corollary 2.10. *The differential of the perturbed twisted Rabinowitz action functional is well-defined, that is, independent of the choice of twisted variation, and linear.*

Preservation of energy implies the following corollary.

Corollary 2.11. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. Then $\mathrm{Crit}\,\mathscr{A}_\varphi^H$ consists precisely of all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$ such that $\gamma(\mathbb{R}) \subseteq H^{-1}(0)$ and γ is an integral curve of τX_H.*

There is a natural $\mathbb{R}$-action on the twisted loop space $\mathscr{L}_\varphi M$ given by

$$(s \cdot \gamma)(t) := \gamma(t+s) \qquad \forall t \in \mathbb{R}.$$

If (M, λ) is an exact symplectic manifold and $H \in C^\infty_\varphi(M)$ for an exact symplectomorphism $\varphi \in \mathrm{Diff}(M)$ of finite order such that $\operatorname{supp} f \cap H^{-1}(0) = \varnothing$, then the twisted Rabinowitz action functional $\mathscr{A}^H_\varphi$ is invariant under the induced $\mathbb{S}^1$-action on $\operatorname{Crit} \mathscr{A}^H_\varphi$. In particular, the unperturbed twisted Rabinowitz action functional is never a Morse function.

Definition 2.12 (Hessian of the twisted Rabinowitz action functional). *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. For $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^H_\varphi$, define the* ***Hessian of the twisted Rabinowitz action functional***

$$\operatorname{Hess} \mathscr{A}^H_\varphi|_{(\gamma,\tau)} \colon (T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}) \times (T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}) \to \mathbb{R}$$

by

$$\operatorname{Hess} \mathscr{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) := \left.\frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\right|_{\varepsilon_1=\varepsilon_2=0} \mathscr{A}^H_\varphi(\gamma_{\varepsilon_1,\varepsilon_2}, \tau + \varepsilon_1 \eta + \varepsilon_2 \sigma),$$

for a smooth two-parameter family $\gamma_{\varepsilon_1,\varepsilon_2}$ of twisted loops with

$$\left.\frac{\partial}{\partial \varepsilon_1}\right|_{\varepsilon_1=0} \gamma_{\varepsilon_1,0} = X \qquad \text{and} \qquad \left.\frac{\partial}{\partial \varepsilon_2}\right|_{\varepsilon_2=0} \gamma_{0,\varepsilon_2} = Y.$$

Remark 2.13. Traditionally, the differential and the Hessian of the twisted Rabinowitz action functional are called the first and second variation of the twisted Rabinowitz action functional.

In order to compute the Hessian of the twisted Rabinowitz action functional we need to choose a suitable connection. We will see that this choice is irrelevant in the end.

Definition 2.14 (Symplectic connection). *Let (M, ω) be a symplectic manifold. A* ***symplectic connection on (M, ω)*** *is defined to be a torsion-free connection ∇ in the tangent bundle TM such that $\nabla \omega = 0$.*

Remark 2.15. Every symplectic manifold admits a symplectic connection by [12, p. 308], but in sharp contrast to the Riemannian case, a symplectic connection on a given symplectic manifold is in general not unique.

Lemma 2.16. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ). Fix a symplectic connection ∇ on $(M, d\lambda)$ and a twisted Hamiltonian function $H \in C^\infty_\varphi(M)$. If $(\gamma, \tau) \in \mathrm{Crit}\,\mathscr{A}^H_\varphi$, then*

$$\begin{aligned}(2.3)\quad \mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) &= \int_0^1 d\lambda(Y, \nabla_t X)\\ &- \tau \int_0^1 \mathrm{Hess}^\nabla H(X,Y) - \eta \int_0^1 dH(Y) - \sigma \int_0^1 dH(X)\end{aligned}$$

for all $(X,\eta),(Y,\sigma) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$.

Proof. The proof is a long routine computation. □

Corollary 2.17. *The Hessian of the twisted Rabinowitz action functional is a well-defined, that is, independent of the choice of twisted two-parameter family, symmetric bilinear form.*

Lemma 2.18. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. If $(\gamma, \tau) \in \mathrm{Crit}\,\mathscr{A}^H_\varphi$, then*

$$\begin{aligned}(2.4)\quad \mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) &= \int_0^1 d\lambda(Y, L_{\tau X_H} X - \eta X_H(\gamma))\\ &- \sigma \int_0^1 dH(X)\end{aligned}$$

for all $(X,\eta),(Y,\sigma) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$, where

$$L_{\tau X_H} X(t) = \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau} X(s+t) \qquad \forall t \in I,$$

with ϕ^{X_H} denoting the smooth flow of the Hamiltonian vector field X_H.

Proof. One computes

$$\mathrm{Hess}^\nabla(X,Y) = d\lambda(Y, \nabla_X X_H).$$

Inserting this into (2.3) yields

$$\begin{aligned}\mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) &= \int_0^1 d\lambda(Y, \nabla_t X - \tau \nabla_X X_H)\\ &- \eta \int_0^1 dH(Y) - \sigma \int_0^1 dH(X).\end{aligned}$$

But as ∇ has no torsion by assumption, we compute

$$\nabla_t X - \tau \nabla_X X_H = \nabla_{\dot{\gamma}} X - \tau \nabla_X X_H = \nabla_{\tau X_H} X - \tau \nabla_X X_H = [\tau X_H, X],$$

and

$$\begin{aligned}
[\tau X_H, X](t) &= L_{\tau X_H} X(t) \\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}(X(\phi^{X_H}_{s\tau}(\gamma(t))) \\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}(X(\phi^{X_H}_{s\tau}(\phi^{X_H}_{t\tau}(\gamma(0))))) \\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}(X(\phi^{X_H}_{(s+t)\tau}(\gamma(0)))) \\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau} X(s+t)
\end{aligned}$$

for all $t \in I$. □

Corollary 2.19. *Let $\varphi \in \operatorname{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. The kernel of the Hessian of the twisted Rabinowitz action functional at $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^H_\varphi$ consists precisely of all $(X, \eta) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$ satisfying*

$$L_{\tau X_H} X = \eta X_H(\gamma) \qquad \text{and} \qquad \int_0^1 dH(X) = 0.$$

Lemma 2.20. *Let $\varphi \in \operatorname{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. For every $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^H_\varphi$, there is a canonical isomorphism*

$$\ker \operatorname{Hess} \mathscr{A}^H_\varphi|_{(\gamma,\tau)} \cong \mathfrak{K}(\gamma, \tau), \tag{2.5}$$

where

$$\mathfrak{K}(\gamma, \tau) := \{(v_0, \eta) \in T_{\gamma(0)} M \times \mathbb{R} : \text{solution of } (2.6)\}$$

with

$$D(\phi^{X_H}_{-\tau} \circ \varphi) v_0 = v_0 + \eta X_H(\gamma(0)) \qquad \text{and} \qquad dH(v_0) = 0. \tag{2.6}$$

Proof. We follow [10, p. 99–100]. Let $(X, \eta) \in \ker \operatorname{Hess} \mathscr{A}_\varphi^H|_{(\gamma,\tau)}$ and define

$$v\colon I \to T_{\gamma(0)}M, \qquad v(t) := D\phi^{X_H}_{-\tau t} X(t).$$

We claim that

$$\ker \operatorname{Hess} \mathscr{A}_\varphi^H|_{(\gamma,\tau)} \to \mathfrak{K}(\gamma, \tau), \qquad (X, \eta) \mapsto (v(0), \eta) \tag{2.7}$$

is an isomorphism. First, we show that the above homomorphism is indeed well-defined. The assumption that (X, η) lies in the kernel of the Hessian of the twisted Rabinowitz action functional at the critical point (γ, τ) is by Corollary 2.19 equivalent to

$$\dot{v} = \eta X_H(\gamma(0)) \qquad \text{and} \qquad \int_0^1 dH(v) = 0. \tag{2.8}$$

Integrating the first equation yields

$$v(t) = v_0 + t\eta X_H(\gamma(0)) \qquad \forall t \in I,$$

with $v_0 := v(0)$. Thus $(v_0, \eta) \in \mathfrak{K}(\gamma, \tau)$ follows from

$$\begin{aligned} v(1) &= D\phi^{X_H}_{-\tau} X(1) \\ &= D\phi^{X_H}_{-\tau} D\varphi(X(0)) \\ &= D(\phi^{X_H}_{-\tau} \circ \varphi) X(0) \\ &= D(\phi^{X_H}_{-\tau} \circ \varphi) v_0. \end{aligned} \tag{2.9}$$

That (2.7) is an isomorphism follows by considering the inverse

$$\mathfrak{K}(\gamma, \tau) \to \ker \operatorname{Hess} \mathscr{A}_\varphi^H|_{(\gamma,\tau)}, \qquad (v_0, \eta) \mapsto (X, \eta),$$

where $X \in T_\gamma \mathscr{L}_\varphi M$ is defined by

$$X(t) := D\phi^{X_H}_{\tau t}(v_0 + t\eta X_H(\gamma(0))) \qquad \forall t \in \mathbb{R}.$$

This establishes the canonical isomorphism (2.7). $\square$

In what follows, we assume that the energy hypersurface $H^{-1}(0)$ is a contact manifold. A contact manifold is a pair (Σ, α), where Σ is an odd-dimensional manifold and $\alpha \in \Omega^1(\Sigma)$ is a global contact form. Every contact manifold (Σ, α) admits a unique vector field $R \in \mathfrak{X}(\Sigma)$, called the Reeb vector field, defined implicitly by

$$i_R d\alpha = 0 \qquad \text{and} \qquad i_R \alpha = 1.$$

Recall, that a strict contactomorphism of a contact manifold (Σ, α) is defined to be a diffeomorphism $\varphi \in \mathrm{Diff}(\Sigma)$ such that $\varphi^*\alpha = \alpha$. Note that the Reeb flow always commutes with a strict contactomorphism.

Definition 2.21 (Parametrised twisted Reeb orbit). *For a contact manifold (Σ, α) and a strict contactomorphism $\varphi\colon (\Sigma, \alpha) \to (\Sigma, \alpha)$ define the set of **parametrised twisted Reeb orbits on (Σ, α)** by*

$$\mathscr{P}_\varphi(\Sigma, \alpha) := \{(\gamma, \tau) \in \mathscr{L}_\varphi \Sigma \times \mathbb{R} : \dot{\gamma}(t) = \tau R(\gamma(t)) \; \forall t \in \mathbb{R}\}.$$

Definition 2.22 (Twisted spectrum). *For a contact manifold (Σ, α) and a strict contactomorphism $\varphi\colon (\Sigma, \alpha) \to (\Sigma, \alpha)$ define the **twisted spectrum** by*

$$\mathrm{Spec}(\Sigma, \alpha) := \{\tau \in \mathbb{R} : \exists \gamma \in \mathscr{L}_\varphi \Sigma \textit{ such that } (\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \alpha)\}.$$

Proposition 2.23 (Kernel of the Hessian of the twisted Rabinowitz action functional). *Let $(\Sigma, \lambda|_\Sigma)$ be a regular energy surface of restricted contact type in an exact Hamiltonian system (M, λ, H) with $X_H|_\Sigma = R$. Suppose $\varphi \in \mathrm{Diff}(M)$ is an exact symplectomorphism such that $H \in C^\infty_\varphi(M)$ and $\varphi^*\lambda|_\Sigma = \lambda|_\Sigma$. Then*

$$\mathrm{Crit}\, \mathscr{A}^H_\varphi = \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$$

and

$$\ker \mathrm{Hess}\, \mathscr{A}^H_\varphi|_{(\gamma,\tau)} \cong \ker\left(D(\phi^R_{-\tau} \circ \varphi)|_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}\Sigma}\right)$$

for all $(\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$. Moreover, we have $R(\gamma(0)) \in \ker \mathrm{Hess}\, \mathscr{A}^H_\varphi|_{(\gamma,\tau)}$ and if $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma) \subseteq \Sigma \times \mathbb{R}$ is an embedded submanifold, then $\mathrm{Spec}(\Sigma, \lambda|_\Sigma)$ is discrete.

Remark 2.24. If $(\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$, we have the period-action equality

$$\mathscr{A}^H_\varphi(\gamma, \tau) = \int_0^1 \gamma^*\lambda = \int_0^1 \lambda(\dot{\gamma}) = \tau \int_0^1 \lambda(R(\gamma)) = \tau.$$

Proof. The identity $\operatorname{Crit}\mathscr{A}_\varphi^H = \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ immediately follows from Corollary 2.11 together with [18, Corollary 5.30]. The proof of the formula for the kernel of the Hessian of $\mathscr{A}_\varphi^H$ is inspired by [10, p. 102]. By Lemma 2.20 we have that

$$\ker \operatorname{Hess} \mathscr{A}_\varphi^H|_{(\gamma,\tau)} \cong \mathfrak{K}(\gamma, \tau),$$

where $(v_0, \eta) \in T_{\gamma(0)}M \times \mathbb{R}$ belongs to $\mathfrak{K}(\gamma, \tau)$ if and only if

$$D(\phi_{-\tau}^{X_H} \circ \varphi)v_0 = v_0 + \eta X_H(\gamma(0)) \qquad \text{and} \qquad dH(v_0) = 0.$$

Thus in our setting, the second condition implies $v_0 \in T_{\gamma(0)}\Sigma$. Decompose

$$v_0 = v_0^\xi + aR(\gamma(0)) \qquad v_0^\xi \in \xi_{\gamma(0)}, a \in \mathbb{R},$$

where $\xi := \ker \lambda|_\Sigma$ denotes the contact distribution. Then we compute

$$\begin{aligned} D(\phi_{-\tau}^R \circ \varphi)R(\gamma(0)) &= D(\phi_{-\tau}^R \circ \varphi)\left(\frac{d}{dt}\bigg|_{t=0} \phi_t^R(\gamma(0))\right) \\ &= \frac{d}{dt}\bigg|_{t=0} (\phi_{-\tau}^R \circ \varphi \circ \phi_t^R)(\gamma(0)) \\ &= \frac{d}{dt}\bigg|_{t=0} (\phi_t^R \circ \varphi \circ \phi_{-\tau}^R)(\gamma(0)) \\ &= \frac{d}{dt}\bigg|_{t=0} \phi_t^R(\gamma(0)) \\ &= R(\gamma(0)), \end{aligned}$$

as a strict contactomorphism commutes with the Reeb flow. Hence

$$v_0 + \eta R(\gamma(0)) = D(\phi_{-\tau}^R \circ \varphi)v_0 = D^\xi(\phi_{-\tau}^R \circ \varphi)v_0^\xi + aR(\gamma(0)),$$

where

$$D^\xi(\phi_{-\tau}^R \circ \varphi) := D(\phi_{-\tau}^R \circ \varphi)|_\xi \colon \xi \to \xi,$$

implies

$$\eta = 0 \qquad \text{and} \qquad D^\xi(\phi_{-\tau}^R \circ \varphi)v_0^\xi = v_0^\xi$$

by considering the splitting $T\Sigma = \xi \oplus \langle R \rangle$. Consequently

$$\mathfrak{K}(\gamma, \tau) = \ker\left(D(\phi_{-\tau}^R \circ \varphi)|_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}\Sigma}\right) \times \{0\}.$$

Finally, assume that $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma) \subseteq \Sigma \times \mathbb{R}$ is an embedded submanifold via the obvious identification of $(\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ with $(\gamma(0), \tau) \in \Sigma \times \mathbb{R}$.

Fix a path (γ_s, τ_s) in $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma) = \operatorname{Crit} \mathscr{A}^H_\varphi$ from (γ_0, τ_0) to (γ_1, τ_1). Then using Remark 2.24 we compute

$$\partial_s \tau_s = \partial_s \mathscr{A}^H_\varphi(\gamma_s, \tau_s) = d\mathscr{A}^H_\varphi|_{(\gamma_s, \tau_s)}(\partial_s \gamma_s, \partial_s \tau_s) = 0,$$

implying that τ_s is constant, and in particular $\tau_0 = \tau_1$. Consequently, $\mathscr{A}^H_\varphi$ is constant on each path-connected component of $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$. As $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ is a submanifold of $\Sigma \times \mathbb{R}$, there are only countably many connected components by definition, implying that $\operatorname{Spec}(\Sigma, \lambda|_\Sigma)$ is discrete. $\square$

3. Compactness of the moduli space of twisted negative gradient flow lines

Definition 3.1 (Liouville domain). *A **Liouville domain** is defined to be a compact connected exact symplectic manifold (W, λ) with connected boundary such that the Liouville vector field X defined implicitly by $i_X d\lambda = \lambda$ is outward-pointing along the boundary.*

Definition 3.2 (Liouville automorphism). *Let (W, λ) be a Liouville domain with boundary Σ. A diffeomorphism $\varphi \in \operatorname{Diff}(W)$ is said to be a **Liouville automorphism**, if $\varphi(\Sigma) = \Sigma$, $\varphi^*\lambda - \lambda$ is exact and compactly supported in* $\operatorname{Int} W$, *and* $\operatorname{ord} \varphi < \infty$. *The set of all Liouville automorphisms on the Liouville domain (W, λ) is denoted by* $\operatorname{Aut}(W, \lambda)$.

Remark 3.3. Let $\varphi \in \operatorname{Aut}(W, \lambda)$ be a Liouville automorphism. Then there exists a unique function $f_\varphi \in C^\infty_c(\operatorname{Int} W)$ such that

$$\varphi^*\lambda - \lambda = df_\varphi.$$

Remark 3.4. The set $\operatorname{Aut}(W, \lambda)$ of Liouville automorphisms of a Liouville domain (W, λ) is in general not a group. Indeed, for $\varphi, \psi \in \operatorname{Aut}(W, \lambda)$ it is not necessarily true that $\varphi \circ \psi$ is of finite order unless φ and ψ commute.

Definition 3.5 (Completion of a Liouville domain). *Let (W, λ) be a Liouville domain with boundary Σ. The **completion of (W, λ)** is defined to be the exact symplectic manifold (M, λ), where*

$$M := W \cup_\Sigma [0, +\infty) \times \Sigma \qquad \textit{and} \qquad \lambda|_{[0,+\infty)\times\Sigma} := e^r \lambda|_\Sigma.$$

Definition 3.6 (Twisted defining Hamiltonian function). *Let (W, λ) be a Liouville domain with boundary Σ and $\varphi \in \operatorname{Aut}(W, \lambda)$. A **twisted***

defining Hamiltonian function for $\boldsymbol{\Sigma}$ *is a Hamiltonian function* $H \in C^\infty(M)$ *on the completion* (M, λ) *of* (W, λ)*, satisfying the following conditions:*

(i) $H^{-1}(0) = \Sigma$ *and* $\Sigma \cap \operatorname{Crit} H = \varnothing$.

(ii) $H \in C^\infty_\varphi(M)$.

(iii) dH *is compactly supported.*

(iv) $X_H|_\Sigma = R$ *is the Reeb vector field of the contact form* $\lambda|_\Sigma$.

Denote by $\mathscr{F}_\varphi(\Sigma)$ *the set of twisted defining Hamiltonian functions for* Σ.

Remark 3.7. A necessary condition for $\mathscr{F}_\varphi(\Sigma) \neq \varnothing$ is that $\varphi^* R = R$. This is not true in general if φ does not induce a strict contactomorphism on Σ.

Definition 3.8 (Adapted almost complex structure). *Let* (W, λ) *be a Liouville domain with boundary* Σ. *An* ***adapted almost complex structure*** *is defined to be a* $d\lambda$*-compatible almost complex structure* J *on* (W, λ) *such that* J *restricts to define a* $d\lambda|_\Sigma$*-compatible almost complex structure on the contact distribution* $\ker \lambda|_\Sigma$ *and* $JR = \partial_r$ *holds near the boundary.*

Definition 3.9 (Rabinowitz–Floer data). *Let* (M, λ) *be the completion of a Liouville domain* (W, λ) *with boundary* Σ *and* $\varphi \in \operatorname{Aut}(W, \lambda)$. ***Rabinowitz–Floer data for*** $\boldsymbol{\varphi}$ *is defined to be a pair* (H, J) *consisting of a twisted defining Hamiltonian function* $H \in \mathscr{F}_\varphi(\Sigma)$ *for* Σ *and of a smooth family* $J = (J_t)_{t \in \mathbb{R}}$ *of adapted almost complex structures on* W *such that*

$$\varphi^* J_{t+1} = J_t \qquad \forall t \in \mathbb{R}.$$

Remark 3.10. For simplicity we ignore the fact, that in order to achieve transversality of the moduli spaces in general, the smooth family of almost complex structures does depend on the Lagrange multiplier (see [1]). This technicality does not significantly alter our arguments as explained in [9] and can also be treated abstractly using polyfold theory [14].

Lemma 3.11. *Let* (W, λ) *be a Liouville domain and* $\varphi \in \operatorname{Aut}(W, \lambda)$. *Then there exists Rabinowitz–Floer data for* φ.

Proof. The construction of the twisted defining Hamiltonian H for Σ is inspired by the proof of [8, Proposition 4.1]. Fix $\delta > 0$ such that the exact symplectic embedding

$$\psi\colon \left((-\delta,0]\times\Sigma, e^r\lambda|_\Sigma\right) \hookrightarrow (W,\lambda)$$

defined by

$$\psi(r,x) := \phi_r^X(x)$$

satisfies

$$U_\delta := \psi((-\delta,0]\times\Sigma)\cap \operatorname{supp} f_\varphi = \varnothing. \tag{3.1}$$

Indeed, that ψ is an exact symplectic embedding follows from the computation

$$\begin{aligned}
\frac{d}{dr}\psi_r^*\lambda &= \frac{d}{dr}\left(\phi_r^X\right)^*\lambda\\
&= (\phi_r^X)^* L_X\lambda\\
&= \left(\phi_r^X\right)^*(di_X\lambda + i_X d\lambda)\\
&= \left(\phi_r^X\right)^*(di_X i_X d\lambda + \lambda)\\
&= \left(\phi_r^X\right)^*\lambda\\
&= \psi_r^*\lambda
\end{aligned}$$

implying

$$\psi_r^*\lambda = e^r\lambda|_\Sigma \qquad \forall r\in(-\delta,0],$$

by $\psi_0 = \iota_\Sigma$, where $\iota_\Sigma\colon \Sigma\hookrightarrow W$ denotes the inclusion. Note that $\psi_r^* X = \partial_r$. We claim

$$\varphi(\psi(r,x)) = \psi(r,\varphi(x)) \qquad \forall (r,x)\in(-\delta,0]\times\Sigma, \tag{3.2}$$

that is, φ and ψ commute. Note that (3.2) makes sense because $\varphi(\Sigma)=\Sigma$ by assumption. Indeed, (3.2) follows from the uniqueness of integral curves and the computation

$$\begin{aligned}
\frac{d}{dr}\varphi(\psi(r,x)) &= \frac{d}{dr}\varphi(\phi_r^X(x))\\
&= D\varphi(X(\phi_r^X(x)))\\
&= (D\varphi\circ X|_{U_\delta}\circ\varphi^{-1}\circ\varphi)(\phi_r^X(x))\\
&= (\varphi_* X|_{\varphi(U_\delta)}\circ\varphi)(\phi_r^X(x))\\
&= (X|_{\varphi(U_\delta)}\circ\varphi)(\phi_r^X(x))\\
&= X(\varphi(\psi(r,x)))
\end{aligned}$$

where we used the φ-invariance of the Liouville vector field on U_δ, that is,

$$\varphi_* X|_{\varphi(U_\delta)} = X|_{\varphi(U_\delta)},$$

which in turn follows from

$$\begin{aligned}
i_{\varphi_* X} d\lambda &= d\lambda(\varphi_* X, \cdot) \\
&= d\lambda(D\varphi \circ X \circ \varphi^{-1}, \cdot) \\
&= d\lambda\left(D\varphi \circ X \circ \varphi^{-1}, D\varphi \circ D\varphi^{-1} \cdot\right) \\
&= \varphi^* d\lambda(X \circ \varphi^{-1}, D\varphi^{-1} \cdot) \\
&= d\lambda(X \circ \varphi^{-1}, D\varphi^{-1} \cdot) \\
&= \varphi_*(i_X d\lambda) \\
&= \varphi_* \lambda \\
&= \lambda - d(f_\varphi \circ \varphi^{-1})
\end{aligned}$$

and assumption (3.1).

Next we construct the defining Hamiltonian $H \in C^\infty(M)$. Let $h \in C^\infty(\mathbb{R})$ be a sufficiently small mollification of the piecewise linear function

$$h(r) := \begin{cases} r & r \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right], \\ \frac{\delta}{2} & r \in \left[\frac{\delta}{2}, +\infty\right), \\ -\frac{\delta}{2} & r \in \left(-\infty, -\frac{\delta}{2}\right], \end{cases}$$

as in Figure 2.

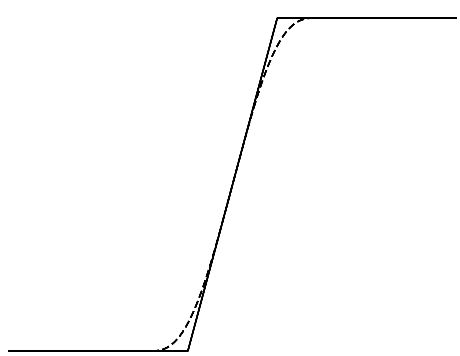

Figure 2. Mollification of the piecewise linear function h.

Define $H \in C^\infty(M)$ by

$$H(p) := \begin{cases} h(r) & p = \psi(r,x) \in U_\delta, \\ h(r) & p = (r,x) \in [0,+\infty) \times \Sigma, \\ -\frac{\delta}{2} & p \in W \setminus U_\delta. \end{cases} \tag{3.3}$$

Then H is a defining Hamiltonian for Σ and dH is compactly supported by construction. Moreover, H is φ-invariant by (3.2). Finally, $X_H|_\Sigma = R$ follows from the observation $X_H = h'(r)e^{-r}R$. Indeed, on U_δ we compute

$$\begin{aligned} i_{h'(r)e^{-r}R}\psi^* d\lambda &= i_{h'(r)e^{-r}R} d(e^r \lambda|_\Sigma) \\ &= i_{h'(r)e^{-r}R}(e^r dr \wedge \lambda|_\Sigma + e^r d\lambda|_\Sigma) \\ &= -h'(r)dr \\ &= -dH. \end{aligned}$$

Next we construct the family $J = (J_t)_{t\in\mathbb{R}}$ of $d\lambda$-compatible almost complex structures on W. Fix a $d\lambda|_\Sigma$-compatible almost complex structure J on the contact distribution $\ker \lambda|_\Sigma$ and choose a path $(J_t^\Sigma)_{t\in I} \subseteq \mathcal{J}(\ker \lambda|_\Sigma, d\lambda|_\Sigma)$ from J to $\varphi_* J$. Extend this smooth family to $(J_t^\Sigma)_{t\in\mathbb{R}}$ satisfying $\varphi^* J_{t+1}^\Sigma = J_t^\Sigma$ for all $t \in \mathbb{R}$. Finally, extend this family to $((-\delta,+\infty) \times \Sigma, d(e^r\lambda|_\Sigma))$ by

$$J_t^\Sigma|_{(a,x)}(b,v) := \left(\lambda_x(v), J_t^\Sigma|_x(\pi(v)) - bR(x)\right), \tag{3.4}$$

where

$$\pi\colon \ker \lambda|_\Sigma \oplus \langle R \rangle \to \ker \lambda|_\Sigma$$

denotes the projection. Choose a smooth family $\left(J_t^{W\setminus\Sigma}\right)_{t\in\mathbb{R}}$ on $W \setminus \Sigma$ twisted by φ, and let $\left\{\beta^\Sigma, \beta^{W\setminus\Sigma}\right\}$ be a partition of unity subordinate to $\{U_\delta, W \setminus \Sigma\}$. Define a smooth family $(m_t)_{t\in\mathbb{R}}$ of Riemannian metrics on W by

$$m_t := \beta^\Sigma m_{\psi_* J_t^\Sigma} + \beta^{W\setminus\Sigma} m_{J_t^{W\setminus\Sigma}}$$

and let $(J_t)_{t\in\mathbb{R}}$ be the corresponding family of $d\lambda$-compatible almost complex structures on W. $\square$

Definition 3.12 ($\boldsymbol{L^2}$-Metric). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. Define an* ***$\boldsymbol{L^2}$-metric*** *on $\mathscr{L}_\varphi M \times \mathbb{R}$*

$$\langle (X, \eta), (Y, \sigma) \rangle_J := \int_0^1 d\lambda(J_t X(t), Y(t)) dt + \eta\sigma \tag{3.5}$$

for all $(X, \eta), (Y, \sigma) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$ and $\gamma \in \mathscr{L}_\varphi M$.

With respect to the L^2-metric (3.5), the gradient of the twisted Rabinowitz action functional $\operatorname{grad}_J \mathscr{A}_\varphi^H \in \mathfrak{X}(\mathscr{L}_\varphi M \times \mathbb{R})$ is given by

$$\operatorname{grad}_J \mathscr{A}_\varphi^H|_{(\gamma,\tau)}(t) = \begin{pmatrix} J_t(\dot\gamma(t) - \tau X_H(\gamma(t))) \\ -\displaystyle\int_0^1 H \circ \gamma \end{pmatrix} \qquad \forall t \in \mathbb{R}.$$

Lemma 3.13 (Fundamental lemma). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. Then there exists a constant $C = C(\lambda, H, J) > 0$ such that*

$$\left\| \operatorname{grad}_J \mathscr{A}_\varphi^H|_{(\gamma,\tau)} \right\|_J < \frac{1}{C} \quad \Rightarrow \quad |\tau| \leq C(|\mathscr{A}_\varphi^H(\gamma, \tau)| + 1)$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$.

Proof. The proof [7, Proposition 3.2] goes through with minor modifications as $\|f_\varphi\|_\infty < +\infty$ by assumption. □

Definition 3.14 (Twisted negative gradient flow line). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. A* ***twisted negative gradient flow line*** *is a tuple $(u, \tau) \in C^\infty(\mathbb{R}, \mathscr{L}_\varphi M \times \mathbb{R})$ such that*

$$\partial_s(u, \tau) = -\operatorname{grad}_J \mathscr{A}_\varphi^H|_{(u(s),\tau(s))} \qquad \forall s \in \mathbb{R}.$$

Definition 3.15 (Energy). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. The* ***energy of a twisted negative gradient flow line $\boldsymbol{(u, \tau)}$*** *is defined by*

$$E_J(u, \tau) := \int_{-\infty}^{+\infty} \|\partial_s(u, \tau)\|_J^2 \, ds = \int_{-\infty}^{+\infty} \left\| \operatorname{grad}_J \mathscr{A}_\varphi^H|_{(u(s),\tau(s))} \right\|_J^2 ds.$$

Theorem 3.16 (Compactness). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. Suppose (u_μ, τ_μ) is a sequence of* *negative gradient flow lines of the twisted Rabinowitz action functional* $\mathscr{A}_\varphi^H$ *such that there exist* $a, b \in \mathbb{R}$ *with*

$$a \leq \mathscr{A}_\varphi^H \big(u_\mu(s), \tau_\mu(s)\big) \leq b \qquad \forall \mu \in \mathbb{N}, s \in \mathbb{R}.$$

Then for every reparametrisation sequence $(s_\mu) \subseteq \mathbb{R}$ *there exists a subsequence* μ_ν *of* μ *and a negative gradient flow line* (u_∞, τ_∞) *of* $\mathscr{A}_\varphi^H$ *such that*

$$\big(u_{\mu_\nu}(\cdot + s_{\mu_\nu}), \tau_{\mu_\nu}(\cdot + s_{\mu_\nu})\big) \xrightarrow{C^\infty_{\mathrm{loc}}} (u_\infty, \tau_\infty) \qquad \textit{as } \nu \to \infty.$$

Proof. The proof [7, p. 268] goes through without any changes as we have a twisted version of the Fundamental Lemma. However, for convenience, we reproduce the main arguments here. We need to establish

- a uniform L^∞-bound on u_μ,
- a uniform L^∞-bound on τ_μ,
- a uniform L^∞-bound on the derivatives of u_μ.

Indeed, by elliptic bootstrapping [21, Theorem B.4.1] the negative gradient flow equation we will obtain C^∞_{loc}-convergence by [21, Theorem B.4.2].

To obtain a uniform L^∞-bound on the sequence of twisted negative gradient flow lines u_μ, observe that by definition of Rabinowitz–Floer data for φ, there exists $r \in (0, +\infty)$ with

$$\operatorname{supp} X_H \cap [r, +\infty) \times \Sigma = \varnothing$$

and J_t is adapted to the boundary of $W \cup_\Sigma [0, r] \times \Sigma$ for all $t \in \mathbb{R}$. Consequently, [21, Corollary 9.2.11] implies that every u_μ remains inside the compact set $W \cup_\Sigma [0, r] \times \Sigma$ as the asymptotics belong to $W \cup_\Sigma [0, r) \times \Sigma$ for all $\mu \in \mathbb{N}$. Indeed, this follows from

$$\begin{aligned}
E_J(u_\mu, \tau_\mu) &= \int_{-\infty}^{+\infty} \|\partial_s(u_\mu, \tau_\mu)\|_J^2 \, ds \\
&= \int_{-\infty}^{+\infty} \langle \partial_s(u_\mu, \tau_\mu), \partial_s(u_\mu, \tau_\mu)\rangle_J ds \\
&= -\int_{-\infty}^{+\infty} \langle \operatorname{grad}_J \mathscr{A}_\varphi^H|_{(u_\mu(s), \tau_\mu(s))}, \partial_s(u_\mu, \tau_\mu)\rangle_J ds \\
&= -\int_{-\infty}^{+\infty} d\mathscr{A}_\varphi^H(\partial_s(u_\mu, \tau_\mu)) ds
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{+\infty} \partial_s \mathscr{A}^H_\varphi(u_\mu, \tau_\mu) ds \\
&= \lim_{s\to-\infty} \mathscr{A}^H_\varphi(u_\mu(s), \tau_\mu(s)) - \lim_{s\to+\infty} \mathscr{A}^H_\varphi(u_\mu(s), \tau_\mu(s)) \\
&\leq b - a,
\end{aligned}
$$

as this implies

$$
\lim_{s\to\pm\infty} \|\partial_s(u_\mu, \tau_\mu)\|_J = \lim_{s\to\pm\infty} \left\| \operatorname{grad}_J \mathscr{A}^H_\varphi|_{(u_\mu(s),\tau_\mu(s))} \right\|_J = 0
$$

by the negative gradient flow equation.

The uniform L^∞-bound on the Lagrange multipliers τ_μ follows from the Fundamental Lemma 3.13 by arguing as in [7, Corollary 3.5].

Lastly, the uniform L^∞-bound on the derivatives of u_μ follows from standard bubbling-off analysis. Indeed, if the derivatives of u_μ are unbounded, then there exists a nonconstant pseudoholomorphic sphere as in [21, Section 4.2]. This is impossible as M is an exact symplectic manifold and thus in particular symplectically aspherical. □

4. Definition of twisted Rabinowitz–Floer homology

In this section we make implicit use of the requirement that a Liouville automorphism has finite order. This is crucial because then the arguments go through as in the case of loops by Remark 2.7.

Definition 4.1 (Transverse Conley–Zehnder index). *Let (W^{2n}, λ) be a Liouville domain with boundary Σ. Let $(\gamma_0, \tau_0), (\gamma_1, \tau_1) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ for some $\varphi \in \operatorname{Aut}(W, \lambda)$ such that there exists a path γ_σ in $\mathscr{L}_\varphi\Sigma$ from γ_0 to γ_1. Define the* ***transverse Conley–Zehnder index*** *by*

$$
\mu((\gamma_0, \tau_0), (\gamma_1, \tau_1)) := \mu_{\mathrm{CZ}}(\Psi^1) - \mu_{\mathrm{CZ}}(\Psi^0) \in \mathbb{Z},
$$

with

$$
\begin{aligned}
\Psi^0 \colon I \to \operatorname{Sp}(n-1), \qquad & \Psi^0_t := \Phi^{-1}_{t,0} \circ D^\xi \phi^R_{\tau_0 t} \circ \Phi_{0,0}, \\
\Psi^1 \colon I \to \operatorname{Sp}(n-1), \qquad & \Psi^1_t := \Phi^{-1}_{t,1} \circ D^\xi \phi^R_{\tau_1 t} \circ \Phi_{0,1},
\end{aligned}
$$

where $\Phi_{t,\sigma} \colon \mathbb{R}^{2n-2} \to \xi_{\gamma_\sigma(t)}$ is a symplectic trivialisation of $F^*\xi$, $\xi := \ker \lambda|_\Sigma$ with $F \in C^\infty(\mathbb{R} \times I, M)$ being defined by $F(t, \sigma) := \gamma_\sigma(t)$, satisfying

$$
\Phi_{t+1,\sigma} = D\varphi \circ \Phi_{t,\sigma} \qquad \forall (t, \sigma) \in \mathbb{R} \times I.
$$

Remark 4.2. The transverse Conley–Zehnder index, or more precisely, the twisted relative transverse Conley–Zehnder index, does not depend on the choice of trivialisation. Denote by

$$\Sigma_\varphi := \frac{\Sigma \times \mathbb{R}}{(\varphi(x), t+1) \sim (x,t)}$$

the mapping torus of φ giving rise to the fibration

$$\pi_\varphi \colon \Sigma_\varphi \to \mathbb{S}^1, \qquad \pi_\varphi([x,t]) := [t].$$

The vertical bundle $\ker D^\xi \pi_\varphi \to \Sigma_\varphi$ is a symplectic vector bundle. One can show, that if $c_1(\ker D^\xi \pi_\varphi) = 0$, then the transverse Conley–Zehnder index is independent of the choice of path in $\mathscr{L}_\varphi \Sigma$ from γ_0 to γ_1.

Let (H, J) be Rabinowitz–Floer data for $\varphi \in \operatorname{Aut}(W, \lambda)$. Set

$$\Sigma := \partial W \qquad \text{and} \qquad M := W \cup_\Sigma [0, +\infty) \times \Sigma.$$

Fix $(\eta, \tau_\eta) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ and denote by $[\eta]$ the corresponding class in $\pi_0 \mathscr{L}_\varphi \Sigma$. Assume that the twisted Rabinowitz action functional $\mathscr{A}^H_\varphi$ is Morse–Bott, that is, $\operatorname{Crit} \mathscr{A}^H_\varphi \subseteq \Sigma \times \mathbb{R}$ is a properly embedded submanifold by Proposition 2.23, and fix a Morse function $h \in C^\infty(\operatorname{Crit} \mathscr{A}^H_\varphi)$. Define the twisted Rabinowitz–Floer chain group $\mathrm{RFC}^\varphi(\Sigma, M)$ to be the $\mathbb{Z}_2$-vector space consisting of all formal linear combinations

$$\zeta = \sum_{\substack{(\gamma,\tau) \in \operatorname{Crit}(h) \\ [\gamma] = [\eta]}} \zeta_{(\gamma,\tau)} (\gamma, \tau)$$

satisfying the Novikov finiteness condition

$$\#\left\{(\gamma,\tau) \in \operatorname{Crit}(h) : \zeta_{(\gamma,\tau)} \neq 0, \mathscr{A}^H_\varphi(\gamma,\tau) \geq \kappa\right\} < \infty \qquad \forall \kappa \in \mathbb{R}.$$

Define a boundary operator

$$\partial \colon \mathrm{RFC}^\varphi(\Sigma, M) \to \mathrm{RFC}^\varphi(\Sigma, M)$$

by

$$\partial(\gamma^-, \tau^-) := \sum_{\substack{(\gamma^+,\tau^+) \in \operatorname{Crit}(h) \\ [\gamma^+] = [\gamma^-]}} n_\varphi(\gamma^\pm, \tau^\pm)(\gamma^+, \tau^+),$$

where

$$n_\varphi(\gamma^\pm, \tau^\pm) := \#_2 \mathscr{M}^0_\varphi(\gamma^\pm, \tau^\pm) \in \mathbb{Z}_2,$$

with $\mathscr{M}^0_\varphi(\gamma^\pm, \tau^\pm)$ denoting the zero-dimensional component of the moduli space of all unparametrised twisted negative gradient flow lines with cascades from (γ^-, τ^-) to (γ^+, τ^+). This is well-defined by Theorem 3.16. Define the ***twisted Rabinowitz–Floer homology of $\boldsymbol{\Sigma}$ and $\boldsymbol{\varphi}$*** by

$$\mathrm{RFH}^\varphi(\Sigma, M) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

Proposition 4.3. *Let (W, λ) be a Liouville domain with simply connected boundary Σ and $\varphi \in \mathrm{Aut}(W, \lambda)$. If there do not exist any nonconstant twisted periodic Reeb orbits on Σ, then*

$$\mathrm{RFH}^\varphi_*(\Sigma, M) \cong \mathrm{H}_*(\mathrm{Fix}(\varphi|_\Sigma); \mathbb{Z}_2).$$

Proof. If there do not exist any nonconstant twisted periodic Reeb orbits,

$$\operatorname{Crit} \mathscr{A}^H_\varphi = \{(c_x, 0) : x \in \mathrm{Fix}(\varphi|_\Sigma)\} \cong \mathrm{Fix}(\varphi|_\Sigma)$$

for any $H \in \mathscr{F}_\varphi(\Sigma)$. Since $\mathrm{Fix}(\varphi|_\Sigma)$ is a properly embedded submanifold of Σ by [19, Problem 8-32] or [22, Lemma 5.5.7], $\mathscr{A}^H_\varphi$ is a Morse–Bott function. Let $x, y \in \mathrm{Fix}(\varphi|_\Sigma)$. As Σ is simply connected by assumption, there exists some path γ from x to y in Σ and a homotopy from γ to $\varphi \circ \gamma$ with fixed endpoints. Extend this homotopy to a path in $\mathscr{L}_\varphi \Sigma$ from c_x to c_y. Choose a Morse function h on $\mathrm{Fix}(\varphi|_\Sigma)$ and any critical point $c_x \in \mathrm{Fix}(\varphi|_\Sigma)$. Then we can define a $\mathbb{Z}$-grading of $\mathrm{RFC}^\varphi(\Sigma, M)$ by

$$\mu((c_y, 0), (c_x, 0)) + \mathrm{ind}_h(c_y) = \mathrm{ind}_h(c_y) \qquad \forall c_y \in \mathrm{Crit}(h),$$

and consequently,

$$\mathrm{RFH}^\varphi_*(\Sigma, M) = \mathrm{HM}_*(\mathrm{Fix}(\varphi|_\Sigma); \mathbb{Z}_2) \cong \mathrm{H}_*(\mathrm{Fix}(\varphi|_\Sigma); \mathbb{Z}_2)$$

as there are only twisted negative gradient flow lines with zero cascades, that is, ordinary Morse gradient flow lines of h. Indeed, suppose that there is a nonconstant twisted negative gradient flow line (u, τ) of $\mathscr{A}^H_\varphi$ such that

$$\lim_{s \to \pm\infty} (u(s), \tau(s)) = (\gamma^\pm, \tau^\pm) \in \operatorname{Crit} \mathscr{A}^H_\varphi.$$

Using the twisted negative gradient flow equation we estimate

$$\tau^- - \tau^+ = \mathscr{A}^H_\varphi(\gamma^-, \tau^-) - \mathscr{A}^H_\varphi(\gamma^+, \tau^+)$$

$$
\begin{aligned}
&= \lim_{s \to -\infty} \mathscr{A}_\varphi^H(u(s), \tau(s)) - \lim_{s \to +\infty} \mathscr{A}_\varphi^H(u(s), \tau(s)) \\
&= \int_{-\infty}^{+\infty} \left\| \operatorname{grad}_J \mathscr{A}_\varphi^H|_{(u(s),\tau(s))} \right\|_J^2 ds \\
&> 0.
\end{aligned}
$$

Hence $\tau^+ < \tau^-$, contradicting $\tau^\pm = 0$. $\square$

5. Invariance of twisted Rabinowitz–Floer homology under twisted homotopies of Liouville domains

Definition 5.1 (Twisted homotopy of Liouville domains). *Given the completion (M, λ) of a Liouville domain (W_0, λ) and $\varphi \in \operatorname{Aut}(W_0, \lambda)$, a **twisted homotopy of Liouville domains in M** is a time-dependent Hamiltonian function $H \in C^\infty(M \times I)$ such that*

(i) *$W_\sigma := H_\sigma^{-1}((-\infty, 0])$ is a Liouville domain with symplectic form $d\lambda|_{W_\sigma}$ and boundary $\Sigma_\sigma := H_\sigma^{-1}(0)$ for all $\sigma \in I$,*

(ii) *$H_\sigma \in \mathscr{F}_\varphi(\Sigma_\sigma)$ for all $\sigma \in I$,*

(iii) *$\Sigma_\sigma \cap \operatorname{supp} f_\varphi = \varnothing$ for all $\sigma \in I$.*

Twisted Rabinowitz–Floer homology is stable under twisted homotopies of Liouville domains. This property is crucial for proving Theorem 1.2.

Theorem 5.2 (Invariance of twisted Rabinowitz–Floer homology). *If $(H_\sigma)_{\sigma \in I}$ is a twisted homotopy of Liouville domains such that both $\mathscr{A}_\varphi^{H_0}$ and $\mathscr{A}_\varphi^{H_1}$ are Morse–Bott, then there is a canonical isomorphism*

$$\mathrm{RFH}^\varphi(\Sigma_0, M) \cong \mathrm{RFH}^\varphi(\Sigma_1, M).$$

Proof. The proof follows from the same adiabatic argument as in [7, p. 275–277]. Crucial is that [7, Theorem 3.6] remains true in our setting as well as the genericity of the Morse–Bott condition. Indeed, if (M, λ) is an exact symplectic manifold and $\varphi \in \operatorname{Diff}(M)$ is of finite order such that $\varphi^*\lambda = \lambda$, then we have the following generalisation of [7, Theorem B.1]. Adapting the proof accordingly, one can show that there exists a subset

$$\mathscr{U} \subseteq \{H \in C_\varphi^\infty(M) : \operatorname{supp} dH \text{ compact}\},$$

of the second category such that for every $H \in \mathscr{U}$, $\mathscr{A}_\varphi^H$ is Morse–Bott with critical manifold being $\mathrm{Fix}(\varphi|_{H^{-1}(0)})$ together with a disjoint union of circles. Again, this works only since the contact condition is an open condition. □

Remark 5.3. Invariance of twisted Rabinowitz–Floer homology allows us to define twisted Rabinowitz–Floer homology also in the case where $\mathscr{A}_\varphi^H$ is not necessarily Morse–Bott. Indeed, as the proof of Theorem 5.2 shows, we can perturb the hypersurface Σ slightly to make it Morse–Bott. Thus we can define the twisted Rabinowitz–Floer homology of such a hypersurface to be the twisted Rabinowitz–Floer homology of any Morse–Bott perturbation. This is well-defined by Theorem 5.2.

Corollary 5.4 (Independence). *Let $\varphi \in \mathrm{Aut}(W, \lambda)$ and $H_0, H_1 \in \mathscr{F}_\varphi(\Sigma)$ be such that either $\mathscr{A}_\varphi^{H_0}$ or $\mathscr{A}_\varphi^{H_1}$ is Morse–Bott. Then the definition of twisted Rabinowitz–Floer homology $\mathrm{RFH}^\varphi(\Sigma, M)$ is independent of the choice of a twisted defining Hamiltonian function for Σ.*

Proof. Note that $\mathscr{F}_\varphi(\Sigma)$ is a convex space. Indeed, set

$$H_\sigma := (1-\sigma)H_0 + \sigma H_1 \qquad \sigma \in I.$$

Then $\varphi^* H_\sigma = H_\sigma$, dH_σ has compact support and $X_{H_\sigma}|_\Sigma = R$ for all $\sigma \in I$. Moreover, for the Liouville vector field $X \in \mathfrak{X}(M)$ we compute

$$\left.\frac{d}{dt}\right|_{t=0} H \circ \phi_t^X|_\Sigma = dH(X)|_\Sigma = d\lambda(X, X_H)|_\Sigma = \lambda(X_H)|_\Sigma = \lambda(R) = 1,$$

for any $H \in \mathscr{F}_\varphi(\Sigma)$, and thus $H < 0$ on $\operatorname{Int} W$ and $H > 0$ on $M \setminus W$. Consequently, $H_\sigma^{-1}(0) = \Sigma$ and so $H_\sigma \in \mathscr{F}_\varphi(\Sigma)$ for all $\sigma \in I$. Hence $(H_\sigma)_{\sigma \in I}$ is a twisted homotopy of Liouville domains in M and Theorem 5.2 implies the claim. □

6. Twisted leaf-wise intersection points

Definition 6.1 (Twisted leaf-wise intersection Point). *Let (M, λ) be the completion of a Liouville domain (W, λ) and let $\varphi \in \mathrm{Aut}(W, \lambda)$ be a Liouville automorphism. A point $x \in \Sigma$ is a **twisted leaf-wise intersection point** for a Hamiltonian symplectomorphism $\varphi_F \in \mathrm{Ham}(M, d\lambda)$, if*

$$\varphi_F(x) \in L_{\varphi(x)} = \{\phi_t^R(\varphi(x)) : t \in \mathbb{R}\}.$$

Definition 6.2 (Twisted Moser pair). *Let $\varphi \in \operatorname{Aut}(W, \lambda)$. A **twisted Moser pair** is defined to be a tuple $\mathfrak{M} := (\chi H, F)$, where*

(i) *$H \in C^\infty_\varphi(M)$, $F \in C^\infty_\varphi(M \times \mathbb{R})$ and $\chi \in C^\infty(\mathbb{S}^1, I)$ such that $\int_0^1 \chi = 1$. Any time-dependent Hamiltonian function χH is said to be **weakly time-dependent**.*

(ii) *$\operatorname{supp} \chi \subseteq \left(0, \frac{1}{2}\right)$ and $F_t = 0$ for all $t \in \left[0, \frac{1}{2}\right]$.*

Lemma 6.3. *Let $\varphi \in \operatorname{Aut}(W, \lambda)$. For all $H \in \mathscr{F}_\varphi(\Sigma)$ and $\varphi_F \in \operatorname{Ham}(M, d\lambda)$ there exists a corresponding twisted Moser pair $\mathfrak{M}$ such that the flow of χX_H is a time-reparametrisation of the flow of X_H.*

Proof. For constructing the Hamiltonian perturbation $\tilde{F}$, fix $\rho \in C^\infty(I, I)$ such that

$$\rho(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right], \\ 1 & t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

See Figure 3a. Then define $\tilde{F} \in C^\infty_\varphi(M \times \mathbb{R})$ by

$$\tilde{F}(x, t) := \dot{\rho}(t - k) F\left(\varphi^{-k}(x), \rho(t - k)\right) \qquad \forall t \in [k, k+1],$$

for $k \in \mathbb{Z}$. See Figure 3b. Then $\tilde{F}_t = 0$ for all $t \in \left[0, \frac{1}{2}\right]$, and

$$\phi_t^{X_{\tilde{F}}} = \phi_{\rho(t)}^{X_F} \qquad \forall t \in I.$$

Indeed, we compute

$$\frac{d}{dt} \phi_{\rho(t)}^{X_F} = \dot{\rho}(t) \frac{d}{d\rho} \phi_{\rho(t)}^{X_F} = \dot{\rho}(t) \left(X_{F_{\rho(t)}} \circ \phi_{\rho(t)}^{X_F}\right) = X_{\tilde{F}_t} \circ \phi_{\rho(t)}^{X_F}.$$

In particular

$$\varphi_{\tilde{F}} = \phi_1^{X_{\tilde{F}}} = \phi_{\rho(1)}^{X_F} = \phi_1^{X_F} = \varphi_F.$$

Finally, we have that

$$\phi_t^{\chi X_H} = \phi_{\tau(t)}^{X_H} \qquad \text{with} \qquad \tau(t) := \int_0^t \chi.$$

Indeed, we compute

$$\frac{d}{dt} \phi_{\tau(t)}^{X_H} = \chi(t) \frac{d}{d\tau} \phi_{\tau(t)}^{X_H} = \chi(t) X_H \circ \phi_{\tau(t)}^{X_H},$$

and thus we conclude by the uniqueness of integral curves. $\square$

(a) The smooth function ρ. (b) The derivative $\dot{\rho}$ of ρ.

Figure 3

Twisted leaf-wise intersection points can be detected variationally by the perturbed twisted Rabinowitz action functional associated with a twisted Moser pair. This is crucial for the proof of Theorem 6.5.

Lemma 6.4. *Let $\varphi \in \mathrm{Aut}(W, \lambda)$ and $\varphi_F \in \mathrm{Ham}(M, d\lambda)$ a Hamiltonian symplectomorphism. If $(\gamma, \tau) \in \mathrm{Crit}\, \mathscr{A}_\varphi^{\mathfrak{M}}$, then $x := \gamma\left(\frac{1}{2}\right)$ is a twisted leaf-wise intersection point for φ_F.*

Proof. Let $\mathfrak{M} = (\chi H, F)$ denote the twisted Moser pair from Lemma 6.3. Using Proposition 2.9 we compute

$$\begin{aligned}
\frac{d}{dt} H(\gamma(t)) &= dH(\dot{\gamma}(t)) \\
&= dH(\tau X_{\chi(t)H}(\gamma(t)) + X_{F_t}(\gamma(t))) \\
&= dH(\tau \chi(t) X_H(\gamma(t))) \\
&= \tau \chi(t) dH(X_H(\gamma(t))) \\
&= 0
\end{aligned}$$

for all $t \in \left[0, \frac{1}{2}\right]$. Thus $H \circ \gamma = c \in \mathbb{R}$ on $\left[0, \frac{1}{2}\right]$ with

$$0 = \int_0^1 \chi H(\gamma) = \int_0^{\frac{1}{2}} \chi H(\gamma) = c \int_0^{\frac{1}{2}} \chi = c \int_0^1 \chi = c.$$

Consequently, $\gamma(0) \in L_x$ and $x \in \Sigma$. Moreover, also $\gamma(1) = \varphi(\gamma(0)) \in \Sigma$ by the φ-invariance of H. For $t \in \left[\frac{1}{2}, 1\right]$, $\dot{\gamma} = X_{F_t}(\gamma)$ and so $\gamma(1) = \varphi_F(x) \in \Sigma$.

We conclude

$$L_{\varphi(x)} = \{\phi_t^R(\varphi(x)) : t \in \mathbb{R}\} = \{\varphi(\phi_t^R(x)) : t \in \mathbb{R}\} = \varphi(L_x)$$

and so $\varphi_F(x) = \gamma(1) = \varphi(\gamma(0)) \in L_{\varphi(x)}$. □

Theorem 6.5. *Let (W, λ) be a Liouville domain with displaceable boundary in the completion (M, λ) and $\varphi \in \operatorname{Aut}(W, \lambda)$. Then $\operatorname{RFH}^\varphi(\Sigma, M) \cong 0$.*

Proof. Suppose that $\Sigma = \partial W$ is displaceable in M via $\varphi_F \in \operatorname{Ham}_c(M, d\lambda)$ and choose Rabinowitz–Floer data (H, J) for φ. Denote by $\mathfrak{M} = (\chi H, F)$ the associated twisted Moser pair from Lemma 6.3. Then $\operatorname{Crit} \mathscr{A}_\varphi^{\mathfrak{M}} = \varnothing$. Indeed, if there exists $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}_\varphi^{\mathfrak{M}}$, then $\gamma\left(\frac{1}{2}\right)$ is a twisted leaf-wise intersection point for φ_F by Lemma 6.4. However, this is impossible as by displaceability we have that $\varphi_F(\Sigma) \cap \Sigma = \varnothing$. Consequently, the perturbed twisted Rabinowitz action functional $\mathscr{A}_\varphi^{\mathfrak{M}}$ is a Morse function. By adapting the Fundamental Lemma to the current setting as in [3, Theorem 2.9], the Floer homology $\operatorname{HF}(\mathscr{A}_\varphi^{\mathfrak{M}})$ is well-defined. By making use of continuation homomorphisms we have that

$$0 = \operatorname{HF}(\mathscr{A}_\varphi^{\mathfrak{M}}) \cong \operatorname{HF}(\mathscr{A}_\varphi^{(\chi H, 0)}) \cong \operatorname{RFH}^\varphi(\Sigma, M),$$

where the last equation is the observation that twisted Rabinowitz–Floer homology in the autonomous case extends to the weakly time-dependent case without any issues. Crucial is, that the period–action equality (see Remark 2.24) is still valid. Indeed, we compute

$$\mathscr{A}_\varphi^{(\chi H, 0)}(\gamma, \tau) = \int_0^1 \gamma^*\lambda = \int_0^1 \lambda(\dot{\gamma}) = \tau \int_0^1 \chi \lambda(R(\gamma)) = \tau \int_0^1 \chi = \tau$$

for all $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}_\varphi^{(\chi H, 0)}$. □

7. Existence of noncontractible periodic Reeb orbits

We define an equivariant version of twisted Rabinowitz–Floer homology following [4, p. 487]. Denote by $(\mathbb{C}^n, \omega)$ the standard symplectic vector space with symplectic form

$$\omega := \sum_{j=1}^n dy^j \wedge dx^j = \frac{i}{2} \sum_{j=1}^n d\bar{z}^j \wedge dz^j,$$

and coordinates $z^j := x^j + iy^j$. Then $\omega = d\lambda$ for

$$\lambda := \frac{1}{2}\sum_{j=1}^{n}\left(y^j dx^j - x^j dy^j\right) = \frac{i}{4}\sum_{j=1}^{n}\left(\bar{z}^j dz^j - z^j d\bar{z}^j\right). \tag{7.1}$$

Consider the free smooth discrete action on the odd-dimensional sphere

$$\mathbb{S}^{2n-1} := \left\{(z^1, \dots, z^n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z^j|^2 = 1\right\}$$

generated by the rotation

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z^1, \dots, z^n) := \left(e^{2\pi i k_1/m} z^1, \dots, e^{2\pi i k_n/m} z^n\right)$$

for $m \geq 1$ and $k_1, \dots, k_n \in \mathbb{Z}$ coprime to m. Define a twisted defining Hamiltonian function $H \in \mathscr{F}_\varphi(\mathbb{S}^{2n-1})$ by

$$H(z) := \frac{1}{2}\left(\beta(|z|^2) - 1\right)$$

for some sufficiently small mollification of the piecewise linear function

$$\beta(r) = \begin{cases} \frac{1}{2} & r \in \left(-\infty, \frac{1}{2}\right], \\ r & r \in \left[\frac{1}{2}, \frac{3}{2}\right], \\ \frac{3}{2} & r \in \left[\frac{3}{2}, +\infty\right). \end{cases}$$

Fix a φ-invariant ω-compatible almost complex structure on $(\mathbb{C}^n, \omega)$. Then the rotation φ induces a free $\mathbb{Z}_m$-action on $\operatorname{Crit} \mathscr{A}_\varphi^H$ and on the moduli space of twisted negative gradient flow lines with cascades of $\mathscr{A}_\varphi^H$. Therefore, we can define ***$\mathbb{Z}_m$-equivariant twisted Rabinowitz-Floer homology***

$$\overline{\mathrm{RFH}}_k^\varphi(\mathbb{S}^{2n-1}/\mathbb{Z}_m) := \frac{\ker \bar{\partial}_k}{\operatorname{im} \bar{\partial}_{k+1}} \qquad \forall k \in \mathbb{Z},$$

as the homology of the $\mathbb{Z}$-graded chain complex (see Remark 4.2)

$$\bar{\partial}_k\colon \mathrm{RFC}_k^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n)/\mathbb{Z}_m \to \mathrm{RFC}_{k-1}^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n)/\mathbb{Z}_m$$

given by

$$\bar{\partial}_k[(\gamma, \tau)] := [\partial_k(\gamma, \tau)] \qquad (\gamma, \tau) \in \operatorname{Crit} h,$$

for some φ-invariant Morse function h on $\operatorname{Crit} \mathscr{A}_\varphi^H$.

Theorem 7.1. *Let $n \geq 2$. For $m \geq 1$ consider the rotation*

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z^1, \dots, z^n) := \left(e^{2\pi i k_1/m} z^1, \dots, e^{2\pi i k_n/m} z^n\right)$$

for $k_1, \dots, k_n \in \mathbb{Z}$ coprime to m. Then

$$\overline{\mathrm{RFH}}_k^{\varphi}(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \begin{cases} \mathbb{Z}_2 & m \text{ even}, \\ 0 & m \text{ odd}, \end{cases} \qquad \forall k \in \mathbb{Z}.$$

If m is even, then $\overline{\mathrm{RFH}}_k^{\varphi}(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ is generated by a noncontractible periodic Reeb orbit in the lens space $\mathbb{S}^{2n-1}/\mathbb{Z}_m$ for all $k \in \mathbb{Z}$.

Proof. First we consider the special case

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \qquad \varphi(z) = e^{2\pi i/m} z.$$

The hypersurface $\mathbb{S}^{2n-1} \subseteq (\mathbb{C}^n, d\lambda)$ is of restricted contact type with contact form $\lambda|_{\mathbb{S}^{2n-1}}$ and associated Reeb vector field

$$R = 2\left(y^j \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial y^j}\right)\Bigg|_{\mathbb{S}^{2n-1}} = 2i\left(\bar{z}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial z}\right)\Bigg|_{\mathbb{S}^{2n-1}}.$$

Suppose $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}_{\varphi}^H$. If $\tau = 0$, then γ is constant. This cannot happen as $\operatorname{Fix}(\varphi|_{\mathbb{S}^{2n-1}}) = \varnothing$. So assume $\tau \neq 0$. Define a reparametrisation

$$\gamma_\tau\colon \mathbb{R} \to \mathbb{S}^{2n-1}, \qquad \gamma_\tau(t) := \gamma(t/\tau).$$

Then γ_τ is the unique integral curve of R starting at $z := \gamma(0)$ and thus

$$\gamma_\tau(t) = e^{-2it} z \qquad \forall t \in \mathbb{R}.$$

From $\gamma(t) = \gamma_\tau(\tau t)$ and the requirement

$$e^{-2i\tau} z = \gamma(1) = \varphi(\gamma(0)) = \varphi(z) = e^{2\pi i/m} z,$$

we conclude $\tau \in \frac{\pi}{m}(m\mathbb{Z} - 1)$. Hence

$$\operatorname{Crit} \mathscr{A}_{\varphi}^H = \left\{\left(\phi^{\tau_k R}(z), \tau_k\right) : k \in \mathbb{Z}, z \in \mathbb{S}^{2n-1}\right\} \cong \mathbb{S}^{2n-1} \times \mathbb{Z},$$

for any $H \in \mathscr{F}_{\varphi}(\mathbb{S}^{2n-1})$, where

$$\tau_k := \frac{\pi}{m}(mk - 1).$$

By Proposition 2.23, $(z_0, \eta) \in T_z\mathbb{S}^{2n-1} \times \mathbb{R}$ belongs to the kernel of the Hessian at $(z, k) \in \operatorname{Crit} \mathscr{A}_\varphi^H$ if and only if $\eta = 0$ and

$$z_0 \in \ker \left(D(\phi^R_{-\tau_k} \circ \varphi)|_z - \mathrm{id}_{T_z\mathbb{S}^{2n-1}} \right).$$

A direct computation yields $D(\phi^R_{-\tau_k} \circ \varphi)|_z = \mathrm{id}_{T_z\mathbb{S}^{2n-1}}$ and thus

$$T_{(z,k)} \operatorname{Crit} \mathscr{A}_\varphi^H = T_z\mathbb{S}^{2n-1} \times \{0\} \cong \ker \operatorname{Hess} \mathscr{A}_\varphi^H|_{(z,k)}.$$

So the twisted Rabinowitz action functional $\mathscr{A}_\varphi^H$ is Morse–Bott with spheres.

The full Conley–Zehnder index [10, Definition 10.4.1] gives rise to a locally constant function

$$\widehat{\mu}_{\mathrm{CZ}} \colon \operatorname{Crit} \mathscr{A}_\varphi^H \to \mathbb{Z}, \qquad \widehat{\mu}(z, k) = (2k-1)n.$$

Note that the definition of the Conley–Zehnder index also applies in this degenerate case, compare [10, Remark 10.4.2]. By the adapted proof of the Hofer–Wysocki–Zehnder Theorem [10, Theorem 12.2.1] to the n-dimensional setting, the full Conley–Zehnder index coincides with the transverse Conley–Zehnder index μ_{CZ}. Indeed, for $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}_\varphi^H$ define a smooth path

$$\Psi \colon I \to \mathrm{Sp}(n), \qquad \Psi_t := D\phi^H_{\tau t}|_{\gamma(0)} \colon \mathbb{C}^n \to \mathbb{C}^n.$$

Adapting the proof of [10, Lemma 12.2.3 (iii)], we get that

$$\Psi_1(R(\gamma(0))) = R(\gamma(1)) \qquad \text{and} \qquad \Psi_1(\gamma(0)) = \gamma(1).$$

Arguing as in [10, p. 235–236] we conclude

$$\mu_{\mathrm{CZ}}(\gamma, \tau) = \widehat{\mu}_{\mathrm{CZ}}(\gamma, \tau).$$

Fix $z_0 \in \mathbb{S}^{2n-1}$ and define $\eta := \phi^{\tau_0 R}(z_0)$. Note that $\phi^{\tau_k R}(z)$ belongs to the same equivalence class in $\pi_0 \mathscr{L}_\varphi \mathbb{S}^{2n-1}$ as η for all $z \in \mathbb{S}^{2n-1}$ and $k \in \mathbb{Z}$ because $\mathbb{S}^{2n-1}$ is simply connected for $n \geq 2$. Let $h \in C^\infty(\mathbb{S}^{2n-1})$ be the standard height function. By Remark 4.2, $\mathrm{RFH}^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n)$ carries the $\mathbb{Z}$-grading

$$\mu((z,k),(z_0,0)) + \mathrm{ind}_h(z) = 2kn + \mathrm{ind}_h(z) \qquad \forall (z,k) \in \mathbb{S}^{2n-1} \times \mathbb{Z}.$$

We claim that the number of twisted negative gradient flow lines between the minimum of $\mathbb{S}^{2n-1} \times \{k+1\}$ and the maximum of $\mathbb{S}^{2n-1} \times \{k\}$ must be odd, so that the critical manifold $\operatorname{Crit} \mathscr{A}^H_\varphi$ looks like a string of pearls, see Figure 4. Indeed, if there is an even number of such negative gradient flow lines, then $\mathrm{RFH}^\varphi_*(\mathbb{S}^{2n-1}, \mathbb{C}^n) \neq 0$, contradicting Theorem 6.5 as $\mathbb{S}^{2n-1}$ is displaceable in the completion $\mathbb{C}^n$.

To compute the $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology, choose the additional $\mathbb{Z}_m$-invariant Morse–Bott function

$$f\colon \mathbb{S}^{2n-1} \to \mathbb{R}, \qquad f(z^1, \dots, z^n) := \sum_{j=1}^n j\,|z^j|^2$$

on each component of $\operatorname{Crit} \mathscr{A}^H_\varphi$. It is easy to check that f is Morse–Bott with circles. Additionaly, choose a $\mathbb{Z}_m$-invariant Morse function on $\operatorname{Crit} f$.

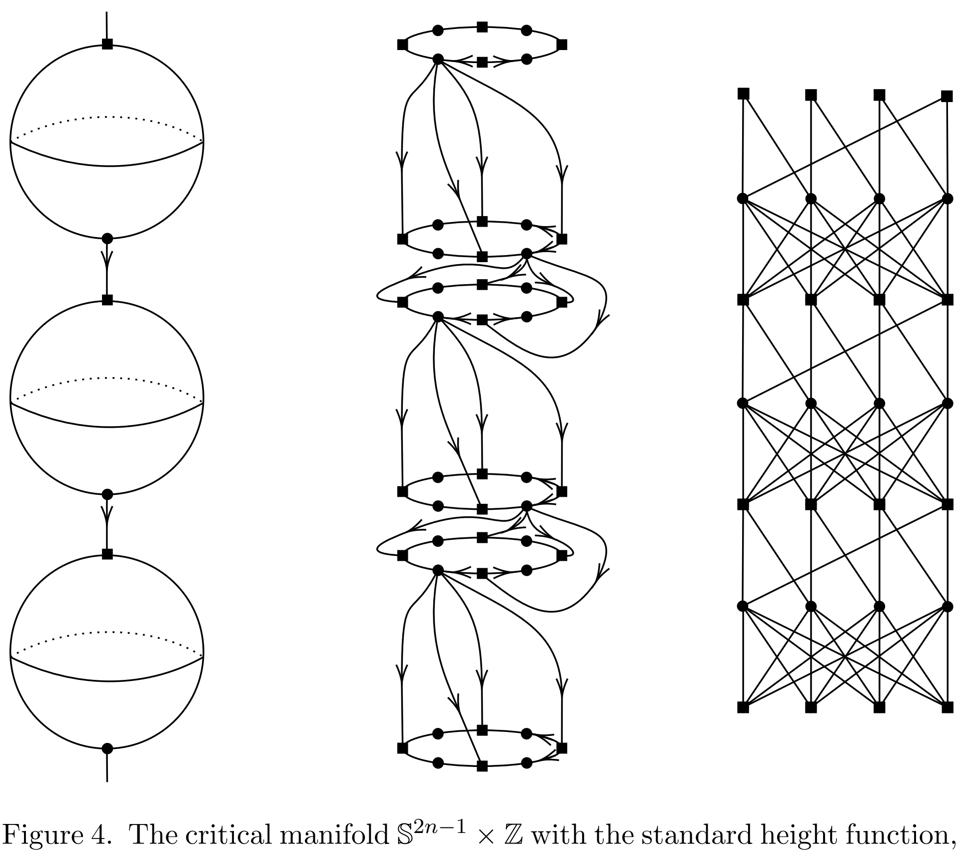

Figure 4. The critical manifold $\mathbb{S}^{2n-1} \times \mathbb{Z}$ with the standard height function, the Morse–Bott function f and the resulting chain complex.

For example, one can take

$$h\colon \mathbb{S}^1 \to \mathbb{R}, \qquad h(t) := \cos(2\pi m t).$$

The resulting chain complex is given by

$$\ldots \longrightarrow \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \xrightarrow{A} \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \xrightarrow{A} \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \longrightarrow \ldots$$

where $\mathbb{1} \in M_{m\times m}(\mathbb{Z}_2)$ has every entry equal to 1 and $A \in M_{m\times m}(\mathbb{Z}_2)$ is defined by

$$A := I_{m\times m} + \sum_{j=1}^{m-1} e_{(j+1)j} + e_{1m},$$

where $e_{ij} \in M_{m\times m}(\mathbb{Z}_2)$ satisfies $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Thus the resulting chain complex looks like a rope ladder. Compare Figure 4.

Passing to the quotient via the free $\mathbb{Z}_m$-action, we get the acyclic chain complex

$$\ldots \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \ldots$$

if m is even and the alternating chain complex

$$\ldots \longrightarrow \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \longrightarrow \ldots$$

if m is odd. From this the statement follows in the special case.

For the general case, we note that

$$\mathbb{C}^n \times [0,1] \to \mathbb{C}^n, \qquad \varphi_s(z^1, \ldots, z^n) := \left(e^{2\pi i s k_1/m} z^1, \ldots, e^{2\pi i s k_n/m} z^n\right)$$

is a smooth path from $\varphi_0 = \mathrm{id}_{\mathbb{C}^n}$ to $\varphi_1 = \varphi$. By adapting the proof of [25, Lemma 2.27], we get an isomorphism of chain complexes

$$\mathrm{RFC}(\mathbb{S}^{2n-1}, \mathbb{C}^n) \cong \mathrm{RFC}^{\varphi}(\mathbb{S}^{2n-1}, \mathbb{C}^n).$$

This isomorphism does not necessarily preserve the grading, but the relative Conley–Zehnder index is preserved. Note that also f is invariant under φ_s for all $s \in [0,1]$. It is no problem to allow twists φ_s of infinite order as the standard Reeb flow on $\mathbb{S}^{2n-1}$ is periodic. Consider the torus action

$$\mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n, \quad (\theta_1, \ldots, \theta_n) \cdot (z^1, \ldots, z^n) := \left(e^{2\pi i \theta_1} z^1, \ldots, e^{2\pi i \theta_n} z^n\right).$$

Since the torus $\mathbb{T}^n$ is abelian, we have that the $\mathbb{Z}_m$-action induced by φ and the different twists along the path $(\varphi_s)_{s\in[0,1]}$ commute. Thus we get an isomorphism of the $\mathbb{Z}_m$-equivariant chain complexes and consequently

$$\overline{\mathrm{RFH}}_*^{\varphi}(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \mathrm{RFH}_*^{\mathbb{Z}_m}(\mathbb{S}^{2n-1}, \mathbb{C}^n),$$

where $\mathrm{RFH}_*^{\mathbb{Z}_m}$ denotes the $\mathbb{Z}_m$-equivariant Rabinowitz–Floer homology constructed in [4, p. 487]. Performing the same computation of the latter homology as before in the special case yields

$$\mathrm{RFH}_k^{\mathbb{Z}_m}(\mathbb{S}^{2n-1}, \mathbb{C}^n) \cong \begin{cases} \mathbb{Z}_2 & m \text{ even}, \\ 0 & m \text{ odd}, \end{cases} \qquad \forall k \in \mathbb{Z}.$$

Finally, $\overline{\mathrm{RFH}}_k^{\varphi}(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ is generated by a noncontractible periodic Reeb orbit in $\mathbb{S}^{2n-1}/\mathbb{Z}_m$ for all $k \in \mathbb{Z}$ by Lemma 1.3. $\square$

For an immediate algebraic corollary recall the definition of Tate cohomology [26, Definition 6.2.4] and Tate homology [6, p. 135]. For a more general result see [24, Theorem 5.6].

Corollary 7.2 (Tate homology). *Let C_m denote the cyclic group of order $m \geq 1$. Then for the trivial left C_m-module $\mathbb{Z}_2$ we have that*

$$\overline{\mathrm{RFH}}_k^{\varphi}(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \widehat{\mathrm{H}}_k(C_m; \mathbb{Z}_2) \qquad \forall k \in \mathbb{Z},$$

where $\widehat{\mathrm{H}}_*(C_m; \mathbb{Z}_2)$ denotes the Tate homology group of C_m with coefficients in the trivial left C_m-module $\mathbb{Z}_2$.

Proof of Theorem 1.2. By assumption, Σ bounds a star-shaped domain D with respect to the origin. Thus $(D \cup \Sigma, \lambda)$ is a Liouville domain with λ given by (7.1). By rescaling we may assume that $\mathbb{S}^{2n-1} \subseteq D$. Define a smooth function

$$\delta \colon \Sigma \to (-\infty, 0)$$

by requiring $\delta(x)$ to be the unique number such that $\phi^X_{\delta(x)}(x) \in \mathbb{S}^{2n-1}$, $x \in \Sigma$, where

$$X = \frac{1}{2}\left(x^j \frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial y^j}\right)$$

denotes the Liouville vector field. We claim that $\delta \circ \varphi = \delta$. Indeed, $\delta(\varphi(x))$ is the unique number such that $\phi^X_{\delta(\varphi(x))}(\varphi(x)) \in \mathbb{S}^{2n-1}$. As the flow of X and φ commute by the proof of Lemma 3.11, we conclude that $\phi^X_{\delta(\varphi(x))}(x) \in \mathbb{S}^{2n-1}$.

Define a smooth family of star-shaped hypersurfaces $(\Sigma_\sigma)_{\sigma \in I}$

$$\Sigma_\sigma := \left\{\phi^X_{\sigma\delta(x)}(x) : x \in \Sigma\right\} \subseteq \mathbb{C}^n.$$

Then we compute

$$\begin{aligned}\varphi(\Sigma_\sigma) &= \left\{\varphi\left(\phi^X_{\sigma\delta(x)}(x)\right) : x \in \Sigma\right\}\\ &= \left\{\phi^X_{\sigma\delta(x)}(\varphi(x)) : x \in \Sigma\right\}\\ &= \left\{\phi^X_{\sigma\delta(\varphi(x))}(\varphi(x)) : x \in \Sigma\right\}\\ &= \left\{\phi^X_{\sigma\delta(y)}(y) : y \in \varphi(\Sigma)\right\}\\ &= \left\{\phi^X_{\sigma\delta(y)}(y) : y \in \Sigma\right\}\\ &= \Sigma_\sigma\end{aligned}$$

for all $\sigma \in I$ and therefore we can find a twisted homotopy $(H_\sigma)_{\sigma \in I}$ of Liouville domains in $\mathbb{C}^n$. By Theorem 5.2 we have that

$$\mathrm{RFH}^\varphi_*(\Sigma, \mathbb{C}^n) \cong \mathrm{RFH}^\varphi_*(\mathbb{S}^{2n-1}, \mathbb{C}^n),$$

giving rise to a canonical isomorphism of the associated $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology

$$\overline{\mathrm{RFH}}^\varphi_*(\Sigma/\mathbb{Z}_m) \cong \overline{\mathrm{RFH}}^\varphi_*(\mathbb{S}^{2n-1}/\mathbb{Z}_m).$$

Indeed, this follows from observing that the standard continuation homomorphism [7, p. 276] is $\mathbb{Z}_m$-invariant and thus descends to the quotient. But by Theorem 7.1 the latter homology does not vanish as $m \geq 2$ is even. □

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Appendix A. Twisted loop spaces

In this Appendix, we will consider the category of topological manifolds rather than the category of smooth manifolds, because smoothness does not add much to the discussion. Free and based loop spaces are fundamental objects in Algebraic Topology, for a vast treatment of the geometry and topology of based as well as free loop spaces see for example [16]. But so-called twisted loop spaces are not considered that much.

Theorem A.1 (Twisted loops in universal covering manifolds). *Let (M, x) be a connected pointed topological manifold and $\pi\colon \tilde{M} \to M$ the universal covering.*

(a) *Fix $[\eta] \in \pi_1(M, x)$ and denote by $U_\eta \subseteq \mathscr{L}(M, x)$ the path component corresponding to $[\eta]$ via the bijection $\pi_0(\mathscr{L}(M, x)) \cong \pi_1(M, x)$. For every $e, e' \in \pi^{-1}(x)$ and $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ such that $\varphi(e) = \tilde{\eta}_e(1)$, where $\tilde{\eta}_e$ denotes the unique lift of η with $\tilde{\eta}_e(0) = e$, we have a commutative diagram of homeomorphisms*

$$\begin{tikzcd} \mathscr{L}_\varphi(\tilde{M}, e) \arrow[rr, "L_\psi"] & & \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M}, e') \\ & U_\eta, \arrow[ul, "\Psi_e"] \arrow[ur, "\Psi_{e'}"'] & \end{tikzcd} \tag{A.1}$$

where $\psi \in \mathrm{Aut}_\pi(\tilde{M})$ is such that $\psi(e) = e'$,

$$L_\psi\colon \mathscr{L}_\varphi(\tilde{M}, e) \to \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M}, e'), \qquad L_\psi(\gamma) := \psi \circ \gamma,$$

and

$$\begin{aligned} \Psi_e\colon U_\eta &\to \mathscr{L}_\varphi(\tilde{M}, e), & \Psi_e(\gamma) &:= \tilde{\gamma}_e, \\ \Psi_{e'}\colon U_\eta &\to \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M}, e'), & \Psi_{e'}(\gamma) &:= \tilde{\gamma}_{e'}. \end{aligned}$$

Moreover, $U_{c_x} \cong \mathscr{L}_\varphi(\tilde{M}, e)$ via Ψ_e if and only if $\varphi = \mathrm{id}_{\tilde{M}}$, where c_x denotes the constant loop at x.

(b) *For every $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ and $e, e' \in \pi^{-1}(x)$ we have a commutative diagram of isomorphisms*

$$\begin{tikzcd} \mathrm{Aut}_\pi(\tilde{M}) \arrow[rr, "C_\psi"] & & \mathrm{Aut}_\pi(\tilde{M}) \\ & \pi_1(M, x), \arrow[ul, "\Phi_e"] \arrow[ur, "\Phi_{e'}"'] & \end{tikzcd}$$

where for $\psi \in \mathrm{Aut}_\pi(\tilde{M})$ *sucht that* $\psi(e) = e'$

$$C_\psi \colon \mathrm{Aut}_\pi(\tilde{M}) \to \mathrm{Aut}_\pi(\tilde{M}), \qquad C_\psi(\varphi) := \psi \circ \varphi \circ \psi^{-1},$$

and

$$\begin{aligned} \Phi_e \colon \pi_1(M, x) &\to \mathrm{Aut}_\pi(\tilde{M}), & \Phi_e([\gamma]) &:= \varphi^e_{[\gamma]}, \\ \Phi_{e'} \colon \pi_1(M, x) &\to \mathrm{Aut}_\pi(\tilde{M}), & \Phi_{e'}([\gamma]) &:= \varphi^{e'}_{[\gamma]}, \end{aligned}$$

with $\varphi^e_{[\gamma]}(e) = \tilde{\gamma}_e(1)$ *and* $\varphi^{e'}_{[\gamma]}(e') = \tilde{\gamma}_{e'}(1)$*.*

(c) *The projection*

$$\tilde{\pi}_x \colon \coprod_{\substack{\varphi \in \mathrm{Aut}_\pi(\tilde{M}) \\ e \in \pi^{-1}(x)}} \mathscr{L}_\varphi(\tilde{M}, e) \to \mathscr{L}(M, x)$$

defined by $\tilde{\pi}_x(\gamma) := \pi \circ \gamma$ *is a covering map with number of sheets coinciding with the cardinality of* $\pi_1(M, x)$*. Moreover,* $\tilde{\pi}_x$ *restricts to define a covering map*

$$\tilde{\pi}_x|_{\mathrm{id}_{\tilde{M}}} \colon \coprod_{e \in \pi^{-1}(x)} \mathscr{L}(\tilde{M}, e) \to U_{c_x},$$

and $\tilde{\pi}_x$ *gives rise to a principal* $\mathrm{Aut}_\pi(\tilde{M})$*-bundle. If* M *admits a smooth structure, then this bundle is additionally a bundle of smooth Banach manifolds.*

Proof. For proving part (a), fix a path class $[\gamma] \in \pi_1(M, x)$. As any topological manifold is Hausdorff, paracompact and locally metrisable by definition, the Smirnov Metrisation Theorem [23, Theorem 42.1] implies that M is metrisable. Let d be a metric on M and $\bar{d}$ be the standard bounded metric corresponding to d, that is,

$$\bar{d}(x, y) = \min\{d(x, y), 1\} \qquad \forall x, y \in M.$$

The metric $\bar{d}$ induces the same topology on M as d by [23, Theorem 20.1]. Topologise the based loop space $\mathscr{L}(M, x) \subseteq \mathscr{L}M$ as a subspace of the free loop space on M, where $\mathscr{L}M$ is equipped with the topology of uniform convergence, that is, with the supremum metric

$$\bar{d}_\infty(\gamma,\gamma') = \sup_{t\in\mathbb{S}^1} \bar{d}\left(\gamma(t),\gamma'(t)\right) \qquad \forall \gamma,\gamma' \in \mathscr{L}M.$$

There is a canonical pseudometric on the universal covering manifold $\tilde{M}$ induced by $\bar{d}$ given by $\bar{d}\circ\pi$. As every pseudometric generates a topology, we topologise the based twisted loop space $\mathscr{L}_\varphi(\tilde{M},e) \subseteq \mathscr{P}\tilde{M}$ as a subspace of the free path space on $\tilde{M}$ for every $e \in \pi^{-1}(x)$ via the supremum metric $\tilde{d}_\infty$ corresponding to $\bar{d}\circ\pi$. In fact, $\tilde{d}_\infty$ is a metric as if $\tilde{d}_\infty(\gamma,\gamma') = 0$, then by definition of $\tilde{d}_\infty$ we have that $\pi(\gamma) = \pi(\gamma')$. But as $\gamma(0) = e = \gamma'(0)$, we conclude $\gamma = \gamma'$ by the unique lifting property of paths [17, Corollary 11.14]. Note that the resulting topology of uniform convergence on $\mathscr{L}_\varphi(\tilde{M},e)$ coincides with the compact-open topology by [23, Theorem 46.8] or [13, Proposition A.13]. In particular, the topology of uniform convergence does not depend on the choice of a metric (see [23, Corollary 46.9]). It follows from [17, Theorem 11.15 (b)], that Ψ_e and $\Psi_{e'}$ are well-defined. Moreover, it is immediate by the fact that the projection $\pi\colon \tilde{M} \to M$ is an isometry with respect to the above metric, that Ψ_e and $\Psi_{e'}$ are continuous with continuous inverse given by the composition with π. It is also immediate that L_ψ is continuous with continuous inverse $L_{\psi^{-1}}$.

Next we show that the diagram (A.1) commutes. Note that

$$\pi \circ L_\psi \circ \Psi_e = \pi \circ \Psi_e = \mathrm{id}_{U_\eta} = \pi \circ \Psi_{e'},$$

thus by

$$(L_\psi \circ \Psi_e(\gamma))(0) = \psi(\tilde{\gamma}_e(0)) = \psi(e) = e' = \tilde{\gamma}_{e'}(0) = \Psi_{e'}(\gamma)(0)$$

and by uniqueness it follows that

$$L_\psi \circ \Psi_e = \Psi_{e'}.$$

In particular

$$\Psi_{e'}(1) = (L_\psi \circ \Psi_e)(1) = \psi(\varphi(e)) = (\psi\circ\varphi\circ\psi^{-1})(e'),$$

and thus $\Psi_{e'}(\gamma) \in \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M},e')$. Consequently, the homeomorphism $\Psi_{e'}$ is well-defined.

Recall, that by the Monodromy Theorem [17, Theorem 11.15 (b)]

$$\gamma \simeq \gamma' \qquad \Leftrightarrow \qquad \Psi_e(\gamma)(1) = \Psi_e(\gamma')(1)$$

for all paths γ and γ' in M starting at x and ending at the same point. Note that the statement of the the Monodromy Theorem is an if-and-only-if statement since $\tilde{M}$ is simply connected.

Suppose $\gamma \in \mathscr{L}(M,x)$ is contractible. Then $\gamma \simeq c_x$, implying $e \in \mathrm{Fix}(\varphi)$. But the only deck transformation of π fixing any point of $\tilde{M}$ is $\mathrm{id}_{\tilde{M}}$ by [17, Proposition 12.1 (a)].

Conversely, assume that $\gamma \in \mathscr{L}(M,x)$ is not contractible. Then we have that $\Psi_e(\gamma)(1) \neq e$. Indeed, if $\Psi_e(\gamma)(1) = e$, then $\gamma \simeq c_x$ and consequently, γ would be contractible. As normal covering maps have transitive automorphism groups by [17, Corollary 12.5], there exists $\psi \in \mathrm{Aut}_\pi(\tilde{M}) \setminus \{\mathrm{id}_{\tilde{M}}\}$ such that $\Psi_e(\gamma)(1) = \psi(e)$.

For proving part (b), observe that Φ_e and $\Phi_{e'}$ are isomorphisms follows from [17, Corollary 12.9]. Moreover, it is also clear that C_ψ is an isomorphism with inverse $C_{\psi^{-1}}$. Let $[\gamma] \in \pi_1(M,x)$. Then using part (a) we compute

$$\begin{aligned}(C_\psi \circ \Phi_e)[\gamma](e') &= (\psi \circ \Phi_e[\gamma] \circ \psi^{-1})(e') \\ &= \psi\left(\varphi^e_{[\gamma]}(e)\right) \\ &= \psi(\tilde{\gamma}_e(1)) \\ &= (L_\psi \circ \Psi_e)(\gamma)(1) \\ &= \Psi_{e'}(\gamma)(1) \\ &= \tilde{\gamma}_{e'}(1) \\ &= \varphi^{e'}_{[\gamma]}(e') \\ &= \Phi_{e'}[\gamma](e').\end{aligned}$$

Thus by uniqueness [17, Proposition 12.1 (a)], we conclude

$$C_\psi \circ \Phi_e = \Phi_{e'}.$$

Finally for proving (c), define a metric $\tilde{d}_\infty$ on

$$E := \coprod_{\substack{\varphi \in \mathrm{Aut}_\pi(\tilde{M}) \\ e \in \pi^{-1}(x)}} \mathscr{L}_\varphi(\tilde{M}, e)$$

by

$$\tilde{d}_\infty(\gamma,\gamma') := \begin{cases} \bar{d}_\infty\left(\pi(\gamma),\pi(\gamma')\right) & \gamma,\gamma' \in \mathscr{L}_\varphi(\tilde{M},e), \\ 1 & \text{else.} \end{cases}$$

Then the induced topology coincides with the disjoint union topology and with respect to this topology, $\tilde{\pi}_x$ is continuous. So left to show is that $\tilde{\pi}_x$ is a covering map. Surjectivity is clear. So let $\gamma \in \mathscr{L}(M,x)$. Then $\gamma \in U_\eta$ for some $[\eta] \in \pi_1(M,x)$. Now note that U_η is open in $\mathscr{L}(M,x)$ and by part (a) we conclude

$$\tilde{\pi}_x^{-1}(U_\eta) = \coprod_{\psi \in \mathrm{Aut}_\pi(\tilde{M})} \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M},\psi(e)) \tag{A.2}$$

for some fixed $e \in \pi^{-1}(x)$ and $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ such that $\varphi(e) = \tilde{\eta}_e(1)$.

As the cardinality of the fibre $\pi^{-1}(x)$ and of $\mathrm{Aut}_\pi(\tilde{M})$ coincides with the cardinality of the fundamental group $\pi_1(M,x)$ by [17, Corollary 11.31] and part (b), we conclude that the number of sheets is equal to the cardinality of the fundamental group $\pi_1(M,x)$.

Equip $\mathrm{Aut}_\pi(\tilde{M})$ with the discrete topology. As the fundamental group of every topological manifold is countable by [17, Theorem 7.21], we have that $\mathrm{Aut}_\pi(\tilde{M})$ is a discrete topological Lie group. Now label the distinct path classes in $\pi_1(M,x)$ by $\beta \in B$ and for fixed $e \in \pi^{-1}(x)$ define local trivialisations

$$(\tilde{\pi}_x,\alpha_\beta)\colon \tilde{\pi}_x^{-1}(U_\beta) \xrightarrow{\cong} U_\beta \times \mathrm{Aut}_\pi(\tilde{M}),$$

making use of (A.2) by

$$\alpha_\beta(\gamma) := \psi^{-1},$$

whenever $\gamma \in \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M},\psi(e))$. Consequently, $\tilde{\pi}_x$ is a fibre bundle with discrete fibre $\mathrm{Aut}_\pi(\tilde{M})$ and bundle atlas $(U_\beta,\alpha_\beta)_{\beta\in B}$. Define a free right action

$$E \times \mathrm{Aut}_\pi(\tilde{M}) \to E, \qquad \gamma\cdot\xi := \xi^{-1}\circ\gamma.$$

Then α_β is $\mathrm{Aut}_\pi(\tilde{M})$-equivariant with respect to this action for all $\beta \in B$. Indeed, using again the commutative diagram (A.1) we compute

$$\alpha_\beta(\gamma\cdot\xi) = \alpha_\beta(\xi^{-1}\circ\gamma) = \left(\xi^{-1}\circ\psi\right)^{-1} = \psi^{-1}\circ\xi = \alpha_\beta(\gamma)\circ\xi$$

for all $\xi \in \mathrm{Aut}_\pi(\tilde{M})$ and $\gamma \in \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\tilde{M},\psi(e))$. Note, that here we use again the fact that $\mathrm{Aut}_\pi(\tilde{M})$ acts transitively on the fibre $\pi^{-1}(x)$.

Suppose that M admits a smooth structure. Then for every compact smooth manifold N we have that the mapping space $C(N,M)$ admits the structure of a smooth Banach manifold by [27]. By [16, Theorem 1.1 p. 24], there is a smooth fibre bundle, called the ***loop-loop fibre bundle***,

$$\mathscr{L}(M,x) \hookrightarrow \mathscr{L}M \xrightarrow{\mathrm{ev}_0} M$$

where

$$\mathrm{ev}_0 \colon \mathscr{L}M \to M, \qquad \mathrm{ev}_0(\gamma) := \gamma(0).$$

Thus the based loop space $\mathscr{L}(M,x) = \mathrm{ev}_0^{-1}(x)$ on M is a smooth Banach manifold by the implicit function theorem [21, Theorem A.3.3] for all $x \in M$. Likewise, by [16, Theorem 1.2 p. 25], there is a smooth fibre bundle, called the ***path-loop fibre bundle***,

$$\mathscr{L}(\tilde{M},e) \hookrightarrow \mathscr{P}(\tilde{M},e) \xrightarrow{\mathrm{ev}_1} \tilde{M},$$

where

$$\mathscr{P}(\tilde{M},e) := \{\gamma \in C(I,\tilde{M}) : \gamma(0) = e\}$$

denotes the based path space and

$$\mathrm{ev}_1 \colon \mathscr{P}(\tilde{M},e) \to \tilde{M}, \qquad \mathrm{ev}_1(\gamma) := \gamma(1).$$

Therefore, the twisted loop space $\mathscr{L}_\varphi(\tilde{M},e) = \mathrm{ev}_1^{-1}(\varphi(e))$ is also a smooth Banach manifold for all $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ and $e \in \pi^{-1}(x)$ by the implicit function theorem [21, Theorem A.3.3]. As the fundamental group $\pi_1(M,x)$ is countable, the topological space E has only countably many connected components being smooth Banach manifolds and thus the total space itself is a smooth Banach manifold. Finally, $\mathrm{Aut}_\pi(\tilde{M})$ is trivially a Banach manifold with $\dim \mathrm{Aut}_\pi(\tilde{M}) = 0$ as a discrete Lie group. $\square$

Corollary A.2. *Let (M,x) be a connected pointed topological manifold and denote by $\pi\colon \tilde{M} \to M$ the universal covering of M. Assume that $\pi_1(M,x)$ is abelian.*

(a) *Fix a path class $[\eta] \in \pi_1(M,x)$. For every $e, e' \in \pi^{-1}(x)$ and deck transformation $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ such that $\varphi(e) = \tilde{\eta}_e(1)$, we have a commutative diagram of homeomorphisms*

$$\begin{tikzcd}
\mathscr{L}_\varphi(\tilde{M},e) \arrow[rr, "L_\psi"] & & \mathscr{L}_\varphi(\tilde{M},e') \\
& U_\eta, \arrow[ul, "\Psi_e"] \arrow[ur, "\Psi_{e'}"'] &
\end{tikzcd}$$

where $\psi \in \mathrm{Aut}_\pi(\tilde{M})$ *is such that* $\psi(e) = e'$.

(b) *For every* $\varphi \in \mathrm{Aut}_\pi(\tilde{M})$ *we have that* $\Phi_e = \Phi_{e'}$ *for all* $e, e' \in \pi^{-1}(x)$.

Lemma 1.3 now follows from part (a) of Theorem A.1. Indeed, by assumption $\varphi \in \mathrm{Aut}_\pi(\Sigma) \setminus \{\mathrm{id}_\Sigma\}$ and using the long exact sequence of homotopy groups of a fibration [13, Theorem 4.41], there is a short exact sequence

$$0 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \pi_1(\Sigma/\mathbb{Z}_m, \pi(x)) \longrightarrow \pi_0(\mathbb{Z}_m) \longrightarrow 0.$$

In particular, by [17, Corollary 12.9] we conclude

$$\mathrm{Aut}_\pi(\Sigma) \cong \pi_1(\Sigma/\mathbb{Z}_m, \pi(x)) \cong \mathbb{Z}_m \cong \{\mathrm{id}_\Sigma, \varphi, \dots, \varphi^{m-1}\}.$$

Finally, we discuss a smooth structure on the continuous free twisted loop space of a smooth manifold.

Lemma A.3. *Let* M *be a smooth manifold and* $\varphi \in \mathrm{Diff}(M)$. *Then the continuous free twisted loop space* $\mathscr{L}_\varphi M$ *is the pullback of*

$$(\mathrm{ev}_0, \mathrm{ev}_1) \colon \mathscr{P}M \to M \times M, \qquad \gamma \mapsto (\gamma(0), \gamma(1)),$$

where we abbreviate $\mathscr{P}M := C(I, M)$, *along the graph of* φ

$$\Gamma_\varphi \colon M \to M \times M, \qquad \Gamma_\varphi(x) := (x, \varphi(x)),$$

in the category of smooth Banach manifolds. Moreover, we have that

$$T_\gamma \mathscr{L}_\varphi M = \{X \in \Gamma^0(\gamma^* TM) : X(1) = D\varphi(X(0))\}$$

for all $\gamma \in \mathscr{L}_\varphi M$.

Proof. Write $f := (\mathrm{ev}_0, \mathrm{ev}_1)$. Then

$$\mathscr{L}_\varphi M = f^{-1}(\Gamma_\varphi(M)).$$

Thus in order to show that the free twisted loop space $\mathscr{L}_\varphi M$ is a smooth Banach manifold, it is enough to show that f is transverse to the properly embedded smooth submanifold $\Gamma_\varphi(M) \subseteq M \times M$. By [15, Proposition 2.4] we need to show that the composition

$$\Phi_\gamma \colon T_\gamma \mathscr{P}M \xrightarrow{Df_\gamma} T_{(x,\varphi(x))}(M \times M) \to T_{(x,\varphi(x))}(M \times M)/T_{(x,\varphi(x))}\Gamma_\varphi(M)$$

is surjective and $\ker \Phi_\gamma$ is complemented for all $\gamma \in f^{-1}(\Gamma_\varphi(M))$, where we abbreviate $x := \gamma(0)$. Note that we have a canonical isomorphism

$$T_{(x,\varphi(x))}(M \times M)/T_{(x,\varphi(x))}\Gamma_\varphi(M) \to T_{\varphi(x)}M, \quad [(v,u)] := u - D\varphi(v).$$

Under this canonical isomorphism, the linear map Φ_γ is given by

$$\Phi_\gamma(X) = X(1) - D\varphi(X(0)), \qquad \forall X \in \Gamma^0(\gamma^*TM).$$

Fix a Riemannian metric on M and let $X_v \in \Gamma(\gamma^*TM)$ be the unique parallel vector field with $X_v(1) = v \in T_{\varphi(x)}M$. Fix a cutoff function $\beta \in C^\infty(I)$ such that $\operatorname{supp}\beta \subseteq \left[\frac{1}{2}, 1\right]$ and $\beta = 1$ in a neighbourhood of 1. Then $\Phi_\gamma(\beta X_v) = v$ and consequently, Φ_γ is surjective. Moreover

$$\ker \Phi_\gamma = \{X \in \Gamma^0(\gamma^*TM) : X(1) = D\varphi(X(0))\}$$

is complemented by the finite-dimensional vector space

$$V := \{\beta X_v \in \Gamma(\gamma^*TM) : v \in T_{\varphi(x)}M\}.$$

Indeed, any $X \in \Gamma^0(\gamma^*TM)$ can be decomposed uniquely as

$$X = X - \beta X_v + \beta X_v, \qquad v := X(1) - D\varphi(X(0)).$$

Abbreviating $Y := X - \beta X_v \in \Gamma^0(\gamma^*TM)$, we have that

$$Y(1) = D\varphi(X(0)) = D\varphi(Y(0)),$$

implying $Y \in \ker \Phi_\gamma$. Thus $\mathscr{L}_\varphi M$ is a smooth Banach manifold.

Now note that $\mathscr{L}_\varphi M$ can be identified with the pullback

$$f^*\mathscr{P}M = \{(x,\gamma) \in M \times \mathscr{P}M : (\gamma(0),\gamma(1)) = (x,\varphi(x))\},$$

making the diagram

$$\begin{tikzcd} f^*\mathscr{P}M \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathscr{P}M \arrow[d, "f"] \\ M \arrow[r, "\Gamma_\varphi"'] & M \times M \end{tikzcd}$$

commute, via the homeomorphism

$$\mathscr{L}_\varphi M \to f^*\mathscr{P}M, \qquad \gamma \mapsto (\gamma(0), \gamma).$$

Finally, one computes

$$T_{(x,\gamma)} f^*\mathscr{P}M = \{(v, X) \in T_x M \times T_\gamma \mathscr{P}M : Df_\gamma X = D\Gamma_\varphi|_x(v)\}$$

for all $(x, \gamma) \in f^*\mathscr{P}M$. □

Remark A.4. Using Lemma A.3 one should be able to prove similar results as in Theorem A.1 in the case of free twisted loop spaces. However, in the non-abelian case the situation gets much more complicated as in general it is not true, that lifts of conjugated elements of the fundamental group lie in the same free twisted loop space by [16, Theorem 1.6 (i)].

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