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Toric generalized Kähler structures. II

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Anti-diagonal toric generalized Kähler (GK) structures of symplectic type on a compact toric symplectic manifold were investigated in [18] . In this article, we consider *general* toric GK structures of symplectic type, without requiring them to be anti-diagonal. Such a structure is characterized by a triple (τ, C, F) where τ is a strictly convex function defined in the interior of the moment polytope Δ and C, F are two constant anti-symmetric matrices. We prove that underlying each such a structure is a *canonical* toric Kähler structure I_0 whose symplectic potential is given by this τ. Conversely, given a toric Kähler structure with symplectic potential τ and two anti-symmetric constant matrices C, F, the triple (τ, C, F) then determines a toric GK structure of symplectic type canonically if F satisfies additionally a certain positive-definiteness condition.

1. Introduction

Generalized Kähler (GK) structures in generalized complex (GC) geometry are a generalization of Kähler structures in complex geometry. M. Gualtieri proved in [10] a remarkable result that such a structure is equivalent to the biHermitian structure first recognized by physicists trying to find the most general 2-dimensional $N = (2, 2)$ supersymmetric σ-models [9].

Compared with Kähler geometry, GK geometry is still not a well-developed discipline and even constructing a nontrivial GK structure needs some effort. Perhaps it helps to study some simple examples first. In Kähler geometry, toric Kähler structures are well-understood mainly through the work of V. Guillemin [12] and M. Abreu [1]. A toric Kähler structure can be efficiently described by a strictly convex function τ (called *symplectic potential*) defined in the interior $\mathring{\Delta}$ of the moment polytope Δ. Often toric Kähler structures provide computable examples to shed some light on abstract ideas in Kähler geometry. The basic goal of [3, 18] and this article as well is to extend the Abreu-Guillemin theory to the context of GK geometry. We hope this study would provide interesting yet simple examples for GK geometry.

In [3] L. Boulanger started to study toric GK structures of symplectic type on a compact toric symplectic manifold $(M, \Omega, \mathbb{T}, \mu)$ ($\mathbb{T}$ is a torus acting on M in an effective and Hamiltonian fashion, μ the moment map and the symplectic form Ω provides one of the two underlying GC structures); in particular, he identified a special class of such structures called *anti-diagonal* ones and found that each such a structure can be characterized by a pair (τ, C), where τ is *again* a strictly convex function on $\mathring{\Delta}$ and C is an anti-symmetric constant matrix.

Anti-diagonal toric GK structures of symplectic type were further explored in [18]. It was found that the above τ is always the symplectic potential of a canonically associated toric Kähler structure and C provides a holomorphic Poisson structure β such that the other GC structure besides the symplectic one is induced from this β up to B-transform. In this article, we continue to study toric GK structures of symplectic type that are *not* necessarily anti-diagonal. Note that a key ingredient in the approach of [18] towards anti-diagonal toric GK structures of symplectic type is to realize that the $\mathbb{T}$-action is strong Hamiltonian in the sense of [17] and thus can be generalized complexified. However, for the most general case, the torus action fails to be strong Hamiltonian and the geometry becomes much more complicated.

It turns out in this article that a general toric GK structure of symplectic type can be characterized by a triple (τ, C, F), where τ is the symplectic potential of a *canonically* associated toric Kähler structure and C, F are two constant anti-symmetric matrices. If $F = 0$, we specialize to the anti-diagonal case, and if $C = F = 0$, this is the classical toric Kähler case. The role of this new matrix F needs to be clarified. Note that $\mu : \mathring{M} \to \mathring{\Delta}$ is a trivial principal $\mathbb{T}$-bundle over $\mathring{\Delta}$ where $\mathring{M} = \mu^{-1}(\mathring{\Delta})$. While for the anti-diagonal case only *one* flat connection on $\mathring{M}$ is involved, in the general case *three* flat connections arise naturally and are related to each other by F. If we interpret F as a deformation of the canonically associated toric Kähler structure, it can be imagined that before deformation, the three connections coincide and as the deformation starts, they become separated: one of them stays unchanged and the other two change in opposite directions.

To understand the different roles of C and F properly, let us resort to a simplified picture. Imagine how one defines a linear complex structure I in a real vector space V. He can choose a basis $\{f_i\}$ of V and a certain matrix A claimed to be the matrix form of I w.r.t. $\{f_i\}$. Now if he is to deform I to obtain new ones, then there are basically two ways to achieve this: on one side he can fix $\{f_i\}$ and deform A, while on the other side, he can also fix A but deform the basis $\{f_i\}$. If we interpret C, F as small deformations of the canonical complex structure, then C corresponds to the first way and F to the second. This explanation will be much clearer in the main body of this article.

The above investigation suggests the possibility of constructing toric GK structures from toric Kähler structures by inputting additionally two constant matrices C and F. In this aspect, C and F again behave very differently. To realize this construction, there is no requirement on the magnitude of C and all feasible C's form a real linear space, but F must satisfy a further positive-definiteness condition and all possible F's only constitute a bounded set.

The article is organized as follows. § 2 is a modest review of the necessary background on GC geometry. § 3.1 is a brief account of Abreu-Guillemin theory and its generalization in [3, 18]. Our study on general toric GK structures of symplectic type actually starts from § 3.2. Basing on some essential remarks on a result in [18], we formulate our main theorem Thm. 3.3. The proof of this theorem is divided into the following two sections § 4, § 5. Besides the proof, the two sections also contain some detailed information towards understanding the underlying geometric structures. § 6 contains an explicit example on $\mathbb{C}P^1 \times \mathbb{C}P^1$ to demonstrate how a toric GK structure of symplectic type can be constructed from a toric Kähler structure and the two additional constant matrices.

2. GK structures of symplectic type

In this section, we collect the most relevant material from GC geometry. Our basic references are [10, 11].

A Courant algebroid E is a real vector bundle E over a smooth manifold M, together with an anchor map π to TM, a non-degenerate inner product $(\cdot,\cdot)$ and a so-called Courant bracket $[\cdot,\cdot]_c$ on $\Gamma(E)$. These data should satisfy some compatibility axioms we won't review here. E is called exact, if the short sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$$

is exact. We only deal with exact Courant algebroids throughout this article. Given E, one can always find an isotropic right splitting $s: TM \to E$, with a curvature form $H \in \Omega^3_{cl}(M)$ defined by $H(X,Y,Z) = ([s(X),s(Y)]_c, s(Z))$, where $X,Y,Z \in \Gamma(TM)$. By the bundle isomorphism $s+\pi^*: TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $TM \oplus T^*M$. Then the inner product $(\cdot,\cdot)$ is the natural pairing, i.e. $(X+\xi, Y+\eta) = \xi(Y)+\eta(X)$. Different splittings are related by B-tranforms: $e^B(X+\xi) = X+\xi+B(X)$, where B is a 2-form.

Definition 2.1. *A GC structure on a Courant algebroid E is a complex structure $\mathbb{J}$ on E orthogonal w.r.t. the inner product and its $\sqrt{-1}$-eigenbundle $L \subset E_{\mathbb{C}}$ is involutive under the Courant bracket. We also say $\mathbb{J}$ is integrable in this case.*

For $H \equiv 0$, ordinary complex and symplectic structures are extreme examples of GC structures. Precisely, for a complex structure I and a symplectic structure Ω, the corresponding GC structures are of the following form:

$$\mathbb{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \quad \mathbb{J}_\Omega = \begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix}.$$

A nontrivial example beyond these is provided by a holomorphic Poisson structure: Let β be a holomorphic Poisson structure on a complex manifold (M,J). Then

$$\mathbb{J}_\beta = \begin{pmatrix} -J & -4\mathrm{Im}\beta \\ 0 & J^* \end{pmatrix}$$

is a GC structure, where $\mathrm{Im}\beta$ is the imaginary part of β.

Definition 2.2. *A generalized metric on a Courant algebroid E is an orthogonal, self-adjoint operator $\mathcal{G}$ such that $(\mathcal{G}\cdot,\cdot)$ is positive-definite on E.*

$\mathcal{G}$ induces a *canonical* isotropic splitting: $E = \mathcal{G}(T^*M) \oplus T^*M$, called *the metric splitting*. Given a generalized metric, we shall always choose its metric splitting to identify E with $TM \oplus T^*M$. Then $\mathcal{G}$ is of the form $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ for a Riemannian metric g. A generalized metric is an ingredient of a GK structure.

Definition 2.3. *A GK structure on E is a pair of commuting GC structures $(\mathbb{J}_1, \mathbb{J}_2)$ such that $\mathcal{G} := -\mathbb{J}_1\mathbb{J}_2$ is a generalized metric.*

A GK structure can be reformulated in many different ways, the basic of which is the biHermitian one: There are two complex structures $J_\pm$ on M compatible with the metric g induced from the generalized metric. Let $\omega_\pm = gJ_\pm$. Then *in the metric splitting* the GC structures and the corresponding biHermitian data are related by

$$\mathbb{J}_1 = \frac{1}{2}\begin{pmatrix} -J_+ - J_- & \omega_+^{-1} - \omega_-^{-1} \\ -\omega_+ + \omega_- & J_+^* + J_-^* \end{pmatrix},$$
$$\mathbb{J}_2 = \frac{1}{2}\begin{pmatrix} -J_+ + J_- & \omega_+^{-1} + \omega_-^{-1} \\ -\omega_+ - \omega_- & J_+^* - J_-^* \end{pmatrix}.$$

Note that $\beta_1 := -\frac{1}{2}(J_+ - J_-)g^{-1}$ and $\beta_2 := -\frac{1}{2}(J_+ + J_-)g^{-1}$ are real Poisson structures associated to $\mathbb{J}_1$ and $\mathbb{J}_2$ respectively. As was noted by N. Hitchin [14], there is a *third* Poisson structure $\beta_3 = \frac{1}{8}[J_+, J_-]g^{-1}$, which is the common imaginary part of a J_+-holomorphic Poisson structure β_+ and a J_--holomorphic Poisson structure β_-.

If $\mathbb{J}_2$ is a B-transform of a GC structure $\mathbb{J}_\Omega$ induced from a symplectic form Ω, the GK manifold $(M, \mathbb{J}_1, \mathbb{J}_2)$ is said to be *of symplectic type*. It is known from [8] that for a given symplectic manifold (M, Ω), compatible GC structures $\mathbb{J}_1$ which, together with a B-transform of $\mathbb{J}_\Omega$, form GK structures on M are in *one-to-one* correspondence with *tamed* integrable complex structures J_+ on M whose *symplectic adjoint* $J^\Omega := -\Omega^{-1}J_+^*\Omega$ is also integrable. This fact greatly facilitates the study of such structures. Precisely, if we set

$$\frac{1}{2}\begin{pmatrix} -J_+ + J_- & \omega_+^{-1} + \omega_-^{-1} \\ -\omega_+ - \omega_- & J_+^* - J_-^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}\begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

then the following basic identities can be easily obtained:

$$(2.1)\quad J_- = J_+^{\Omega} = -\Omega^{-1}J_+^*\Omega, \quad g = -\frac{1}{2}\Omega(J_+ + J_-), \quad b = -\frac{1}{2}\Omega(J_+ - J_-).$$

Recall that tameness of J_+ means that the symmetric part of $-\Omega J_+$ is a Riemannian metric on M. Since

$$(J_+ + J_-)(J_+ - J_-) = -(J_+ - J_-)(J_+ + J_-) = -[J_+, J_-],$$

one can easily derive in this setting that $\beta_3 = -\frac{1}{4}(J_+ - J_-)\Omega^{-1}$.

3. The main theorem

3.1. Abreu-Guillemin theory and Boulanger's generalization

Let us recall briefly the Abreu-Guillemin theory and its generalization in [3, 18] first.

Definition 3.1. *A toric symplectic manifold $(M, \Omega, \mathbb{T}, \mu)$ of dimension $2n$ is a symplectic manifold (M, Ω) with an effective and Hamiltonian action of the n-dimensional torus $\mathbb{T} = \mathbb{T}^n$. Here μ is the moment map.*

Let $(M, \Omega, \mathbb{T}, \mu)$ be a *compact* toric symplectic manifold and $\mathfrak{t} \cong \mathbb{R}^n$ the Lie algebra of $\mathbb{T}$. By the convexity theorem of Atiyah-Guillemin-Sternberg [2, 13], the image Δ of μ is a polytope in $\mathfrak{t}^* = (\mathbb{R}^n)^*$, i.e. the convex hull of the image of fixed points. Δ is called the *moment polytope*. Delzant's famous theorem says that compact toric symplectic manifolds are classified by their moment polytopes up to equivariant symplectomorphism [6]. Polytopes in this classifying scheme are called *Delzant polytopes*.

Given $(M, \Omega, \mathbb{T}, \mu)$ as above, Guillemin in [12] showed that compatible $\mathbb{T}$-invariant Kähler structures are also determined by data specified on Δ. The following is a sketch of the basic ideas.

Let $\mathring{\Delta}$ be the interior of Δ. Then the open dense subset $\mathring{M} := \mu^{-1}(\mathring{\Delta})$ consists of points at which $\mathbb{T}$ acts freely. Topologically, $\mu : \mathring{M} \to \mathring{\Delta}$ is a trivial principal $\mathbb{T}$-bundle over $\mathring{\Delta}$. Denote the set of $\mathbb{T}$-invariant complex structures on M compatible with Ω by $K_{\Omega}^{\mathbb{T}}(M)$, i.e. the set of toric Kähler structures. Let $I \in K_{\Omega}^{\mathbb{T}}(M)$ and $\{X_j\}$ be the fundamental vector fields associated to a fixed integral basis $\{e_j\} \subset \mathfrak{t}$. Then $\{X_j, IX_j\}$ is a global frame of $T\mathring{M}$ and the Lie bracket of any two vector fields in this frame vanishes. Let $\{\zeta_j, \vartheta_j\}$ be the dual frame on $T^*\mathring{M}$. Then $d\zeta_j = d\vartheta_j = 0$ and thus locally $\zeta_j = d\theta_j$ and

$\vartheta_j = du_j$. $\theta_j + \sqrt{-1}u_j$ are then local holomorphic coordinates of $\mathring{M}$. Since $\{\vartheta_j\}$ and $\{d\mu_j\}$ determine the same integrable Lagrangian distribution $\mathcal{D}$ generated by those X_j's, these u_j's are functions depending only on μ, i.e.

$$du_j = -\sum_{k=1}^{n} \phi_{jk}(\mu) d\mu_k, \tag{3.1}$$

or [1]

$$I^* \begin{pmatrix} d\theta \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi \\ -\phi^{-1} & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\mu \end{pmatrix}. \tag{3.2}$$

These θ_j, μ_j are Darboux coordinates, i.e. on $\mathring{M}$, $\Omega = \sum_{j=1}^n d\mu_j \wedge d\theta_j$.

Compatibility of I with Ω forces the matrix $\phi = (\phi_{jk})$ to be symmetric and positive-definite, and integrability of Eq. (3.1) implies that ϕ ought to be the Hessian of a function τ on $\mathring{\Delta}$, or in other words τ is strictly convex. τ is called the *symplectic potential* of the invariant Kähler structure I, providing a useful computational tool in examining geometric ideas in Kähler geometry. The argument can go in the converse direction: a strictly convex function τ on $\mathring{\Delta}$ can be used to construct a toric Kähler structure on $\mathring{M}$. However, to extend the structure smoothly to the whole of M requires τ to satisfy the so-called Guillemin boundary condition. In [7], Donaldson formulated this condition as follows:

- (a) τ is continuous on Δ and smooth in $\mathring{\Delta}$.
- (b) The restriction of the Hessian ϕ of τ to each (open) face is smooth and positive-definite.
- (c) Let p be a boundary point lying on a co-dimension r (open) face of Δ, and w.l.g., assume $p = 0$ and Δ is locally defined by $x_1 > 0, \cdots, x_r > 0$. Then near p, $\tau = \sum_{i=1}^r x_i \ln x_i + v$ where v is smooth.

Let Δ be the Delzant polytope of $(M, \Omega, \mathbb{T}, \mu)$. Associated with Δ is a canonical toric Kähler manifold M_Δ [6], whose symplectic potential is totally determined by the linear data defining Δ [12]. If Δ in $\mathfrak{t}^* = (\mathbb{R}^n)^*$ is defined

[1] As a convention, we have written $d\theta_j$'s or $d\mu_j$'s in a column. Similar notation is used below.

by

$$l_j(x) := (u_j, x) - \lambda_j \geq 0, \quad j = 1, 2, \cdots, d$$

where the linear equations $l_j(x) = 0$ define faces of codimension 1 of Δ and d is the number of such faces, then the symplectic potential of M_Δ is the Guillemin function $\tau_\Delta(x) = \frac{1}{2}\sum_{j=1}^{d} l_j(x) \ln l_j(x)$. In these terms, Abreu formulated the Guillemin boundary condition as follows [1]:

- (d) $\tau - \tau_\Delta$ is smooth on Δ.
- (e) The Hessian ϕ of τ is positive-definite in $\mathring{\Delta}$ and

$$\det(\phi) = [\delta(x) \prod_{i=1}^{d} l(x)]^{-1}$$

  where $\delta(x)$ is smooth and positive-definite on Δ.

Boulanger's generalization went in a similar spirit. Consider $\mathbb{T}$-invariant GK structures $(\mathbb{J}_1, \mathbb{J}_2)$ of symplectic type on $(M, \Omega, \mathbb{T}, \mu)$, where $\mathbb{J}_2$ is a B-transform of $\mathbb{J}_\Omega$. The complex structure I in the above argument is replaced by J_+ underlying the biHermitian description. The weaker condition of tameness no longer in general ensures that θ_j, μ_j be Darboux coordinates. Boulanger thus focused on a special case to reserve this property. Denote the space of $\mathbb{T}$-invariant GK structures of symplectic type by $GK_\Omega^{\mathbb{T}}(M)$. Then an element of $GK_\Omega^{\mathbb{T}}(M)$ is called *anti-diagonal* if $J_+\mathcal{D} = J_-\mathcal{D}$, where $\mathcal{D}$ is again the Lagrangian distribution generated by $\{X_j\}$.

Let us introduce some notation before proceeding further. As in [3], denote the subset of anti-diagonal elements in $GK_\Omega^{\mathbb{T}}(M)$ by $DGK_\Omega^{\mathbb{T}}(M)$. Since an element in $GK_\Omega^{\mathbb{T}}(M)$ is completely parameterized by its complex structure J_+, we usually write $J_+ \in GK_\Omega^{\mathbb{T}}(M)$ to convey this fact. Sometimes we also write $\mathbb{J}_1 \in GK_\Omega^{\mathbb{T}}(M)$ if the GC aspect is emphasized. Similar notation is adopted for elements in $DGK_\Omega^{\mathbb{T}}(M)$.

For $J_+ \in DGK_\Omega^{\mathbb{T}}(M)$, θ_j, μ_j are again Darboux coordinates (called *admissible coordinates associated to* J_+ in [3]) and with such coordinates $J_\pm$ are of a form similar to Abreu-Guillemin's case:

$$J_+^* \begin{pmatrix} d\theta \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi^T \\ -(\phi^{-1})^T & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\mu \end{pmatrix},$$
$$J_-^* \begin{pmatrix} d\theta \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi \\ -\phi^{-1} & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\mu \end{pmatrix}$$

except that ϕ is not necessarily symmetric. Here ϕ^T is the transpose of ϕ. Integrability of $J_\pm$ forces the symmetric part $\phi_s = (\phi + \phi^T)/2$ to be the Hessian of a function τ on $\mathring{\Delta}$ and the anti-symmetric part $C = \phi_a = (\phi - \phi^T)/2$ to be a constant $n \times n$ matrix. Tameness then simply means that τ is strictly convex. A sketch of this argument can be found in the next subsection in a more general setting.

In [18], it was further proved that Boulanger's τ is the symplectic potential of a *genuine* toric Kähler structure J_0 canonically associated to J_+. Conversely, given a toric Kähler structure and an $n \times n$ constant anti-symmetric matrix C, a canonical element in $DGK_\Omega^{\mathbb{T}}(M)$ can be constructed. This is a rather nontrivial statement as it tells us that in this more general setting the potential τ has the same asymptotic behavior near the boundary of Δ as in the toric Kähler case. Moreover, the underlying GC structure $\mathbb{J}_1$ is simply a B-transform of $\mathbb{J}_\beta$ induced from a J_0-holomorphic Poisson structure β characterized by the matrix C.

By abuse of language, we will not distinguish $\mathbb{T}$-invariant smooth functions on M (or $\mathring{M}$) from smooth functions on Δ (or $\mathring{\Delta}$).

3.2. General toric GK structures of symplectic type

Let us begin with recalling a result from [18]. Fix an integral basis $\{e_j\}$ of $\mathfrak{t}$ and let $\{\mu_j\}$ be the corresponding components of μ. Note again that $\mathring{M}$ is a trivial principal $\mathbb{T}$-bundle over $\mathring{\Delta}$. Let $\zeta = \sum_j \zeta_j e_j$ be a flat connection on this bundle. Since the vertical distribution is Lagrangian, there exists a 1-form $\sigma_\zeta = \sum_j h_j d\mu_j$ with h_j depending only on μ such that $\Omega = \sum_j d\mu_j \wedge \zeta_j + d\sigma_\zeta$. We call the matrix $F_\zeta := \frac{1}{2}(h_{k,j} - h_{j,k})$ the associated matrix of the connection ζ. Obviously, F_ζ is determined by ζ. If F_ζ happens to be constant, we say ζ is *admissible*. If furthermore $F_\zeta \equiv 0$, we say ζ is of *Darboux type*.

Lemma 3.2. *([18]) $J_+ \in GK_\Omega^{\mathbb{T}}(\mathring{M})$ is determined by a triple (ζ^+, τ, C) where ζ^+ is an admissible connection on $\mathring{M}$, C is an $n \times n$ constant anti-symmetric real matrix and τ is a strictly convex function on $\mathring{\Delta}$ whose Hessian ϕ_s satisfies the condition*

$$\phi_s + F_{\zeta^+}(\phi_s)^{-1}F_{\zeta^+} \quad \text{is positive-definite on } \mathring{\Delta}. \tag{3.3}$$

Conversely, such a triple (ζ^+, τ, C) also gives rise to an element in $GK_\Omega^{\mathbb{T}}(\mathring{M})$.

Proof. For the reader's convenience, we only sketch the proof. See [18] for the details.

Let $J_+ \in GK^{\mathbb{T}}_{\Omega}(\mathring{M})$ and X_j be the vector fields on $\mathring{M}$ generated by e_j. Tameness of J_+ with Ω assures that $\{X_j, J_+X_j\}$ be a global frame of $T\mathring{M}$. Let $\{\zeta^+_j, \vartheta_j\}$ be the dual frame on $T^*\mathring{M}$. Since J_+ is integrable and the action of $\mathbb{T}$ is abelian, $\zeta^+ := \sum_j \zeta^+_j e_j$ gives rise to a flat connection on $\mathring{M}$. Locally $\zeta^+_j = d\theta^+_j$, $\vartheta_j = du^+_j$ and $\theta^+_j + \sqrt{-1}u^+_j$ are local J_+-holomorphic coordinates on $\mathring{M}$. Since $\{du^+_j\}$ and $\{d\mu_j\}$ determine the same distribution $\mathcal{D}$, $du^+_j = -\sum_k \phi_{jk} d\mu_k$, where ϕ_{jk}'s are functions only of μ; in particular,

$$J^*_+ \begin{pmatrix} \zeta^+ \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi^T \\ -(\phi^{-1})^T & 0 \end{pmatrix} \begin{pmatrix} \zeta^+ \\ d\mu \end{pmatrix}, \tag{3.4}$$

and for a certain matrix-valued function $F = (F_{kj})$,

$$\Omega = \sum_j d\mu_j \wedge \zeta^+_j + \sum_{j,k} F_{kj} d\mu_j \wedge d\mu_k.$$

The same argument applies to J_- as well. There should be a flat connection ζ^- and a matrix-valued function ψ only of μ such that

$$J^*_- \begin{pmatrix} \zeta^- \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \psi^T \\ -(\psi^{-1})^T & 0 \end{pmatrix} \begin{pmatrix} \zeta^- \\ d\mu \end{pmatrix}.$$

However ψ is nothing else but ϕ^T. Indeed, in the coordinates $\{\theta^+_j, \mu_j\}$,

$$\begin{aligned} \zeta^- &= -\psi^T J^*_- d\mu = \psi^T \Omega J_+ \Omega^{-1}(d\mu) \\ &= -\psi^T \Omega J_+(\partial_{\theta^+}) = \psi^T \phi^{-1} \Omega(\partial_\mu) \\ &= \psi^T \phi^{-1}(\zeta^+ + 2F d\mu). \end{aligned}$$

Since $\zeta^\pm$ are both flat connections, we must have $\zeta^-_j = \zeta^+_j + 2df_j$ for some functions f_j depending only on μ. This implies $\psi^T\phi^{-1} = \mathrm{I}$ where I is the identity matrix or equivalently $\psi = \phi^T$. We must also have $F_{kj} = f_{j,k}$. Let $F_{kj,l} = \frac{\partial F_{kj}}{\partial \mu_l}$. Then

$$F_{kj,l} = f_{j,kl} = f_{j,lk} = F_{lj,k},$$

which, together with $F_{kj} = -F_{jk}$, immediately implies that $F_{kj,l} = 0$ and thus that F is constant, i.e. $\zeta^\pm$ are both admissible.

Since $\zeta_j^+ - \sqrt{-1}\sum_k \phi_{jk} d\mu_k$ and $\zeta_j^- - \sqrt{-1}\sum_k \phi_{kj} d\mu_k$ are $J_\pm$-holomorphic 1-forms respectively, integrability of $J_\pm$ implies

$$\phi_{kj,l} = \phi_{lj,k}, \quad \phi_{jk,l} = \phi_{jl,k}. \tag{3.5}$$

Then we can conclude that the anti-symmetric part ϕ_a of ϕ is a constant matrix C and the symmetric part ϕ_s of ϕ is the Hessian of a function τ defined on $\mathring{\Delta}$.

To see what tameness of J_+ with Ω means, we should derive the matrix form of the metric g. In the frame $\{\zeta^+, d\mu\}$,

$$\begin{aligned} J_-^* &\sim \begin{pmatrix} \mathrm{I} & -2F \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} 0 & \phi \\ -\phi^{-1} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 2F \\ 0 & \mathrm{I} \end{pmatrix} \\ &= \begin{pmatrix} 2F\phi^{-1} & 4F\phi^{-1}F + \phi \\ -\phi^{-1} & -2\phi^{-1}F \end{pmatrix}. \end{aligned}$$

Since $g = 1/2(J_+^* + J_-^*)\Omega$, we obtain the matrix form of g relative to $\{\zeta^+, d\mu\}$:

$$g \sim \begin{pmatrix} (\phi^{-1})_s & \phi^{-1}F \\ -F(\phi^T)^{-1} & \phi_s \end{pmatrix}.$$

It's elementary to find that positive-definiteness of g is equivalent to that both ϕ_s and $\phi_s + F(\phi_s)^{-1}F$ are positive-definite. Thus τ should satisfy the properties listed in the theorem. Clearly, the triple (ζ^+, τ, C) determines J_+ uniquely.

Conversely, given the triple (ζ^+, τ, C) satisfying the listed conditions, let ϕ_s be the Hessian of τ and $\phi = \phi_s + C$ and define J_+ via Eq. (3.4). Obviously such a $J_+ \in GK_\Omega^{\mathbb{T}}(\mathring{M})$. □

Before moving on, let us motivate our further steps by giving some remarks on the implication of Lemma 3.2.

- In this lemma, if $F = 0$, then we recover Boulanger's result for anit-diagonal GK structures of symplectic type. In contrast with this more restrictive case, we should emphasize that in general *two* constant anti-symmetric matrices C and F are involved in the characterization of $J_+ \in GK_\Omega^{\mathbb{T}}(M)$. Compared with C, this additional F turns out to play a very different role: In the anti-diagonal case, only one flat connection ζ^+ of Darboux type is involved, and in the *single* frame $\{\zeta^+, d\mu\}$, $J_\pm$ can be anti-diagonalized *simultaneously.* However, in the general case, *three* flat connections are involved: two admissible connections $\zeta^\pm$ associated with $J_\pm$ respectively and a flat connection ζ of Darboux type, i.e., $\zeta := (\zeta^+ + \zeta^-)/2$ such that $\Omega = \sum_j d\mu_j \wedge \zeta_j$. These connections are related by

$$\zeta^\pm = \zeta \mp F d\mu. \tag{3.6}$$

In particular, $J_\pm$ fails to be anti-diagonalized simultaneously in a single frame. To understand the roles played by τ, C and F, it turns out to be very important to distinguish among these flat connections.

- The connection ζ is determined by τ. Given τ, whose Hessian is non-degenerate in $\mathring{\Delta}$, one can define ζ using Eq. (3.2). Then J_+ is completely determined by the triple (τ, C, F) and the two positive-definiteness conditions in the lemma can be reformulated as that the complex matrix $\phi + \sqrt{-1}F$ is positive-definite. This is simply because the positive-definiteness of g is equivalent to that g is a Hermite metric w.r.t. J_+.[2]

We need to generalize Guillemin's boundary condition to our present setting. We formulate it in Donaldson's terms. We replace Donaldson's (b) by the following

- (b') The restriction of the matrix $\phi + \sqrt{-1}F$ ($\phi = Hessian(\tau)$) to each (open) face is smooth and positive-definite.

For completeness, we also include Abreu's version of this Guillemin condition, one simply replaces the condition (e) by the following

- (e') The matrix $\phi + \sqrt{-1}F$ is positive-definite in $\mathring{\Delta}$ and $\det(\phi + \sqrt{-1}F) = [\delta(x) \prod_{i=1}^d l(x)]^{-1}$ where $\delta(x)$ is smooth and positive-definite on Δ.

We won't prove the equivalence of the two versions here since it will be clear in later sections and established in the end of § 5. We are now ready to formulate our main theorem in this paper. For convenience, the triple (τ, C, F) satisfying Guillemin boundary condition (a)(b')(c) will be called a GK triple on Δ and the proof of the theorem will be given in later sections.

Theorem 3.3. *Given $(M, \Omega, \mathbb{T}, \mu)$ with its Delzant polytope $\Delta \subseteq (\mathbb{R}^n)^*$, there is a one-to-one correspondence between the set $GK_\Omega^{\mathbb{T}}(M)$ and the set of GK triples (τ, C, F) on Δ where τ is defined modular the addition of a linear function.*

[2]The author thanks the referee for pointing out this equivalence.

4. The canonical toric Kähler structure

In this section, we prove that the triple (τ, C, F) arising in Lemma 3.2 satisfies Guillemin boundary condition (a)(b')(c). We only need to prove that τ is actually the symplectic potential of a genuine toric Kähler structure on M. (a) and (c) then follow immediately from this claim and (b') can be seen easily by tracing the degeneration of the vector fields X_i on each face of Δ. To prepare for the proof, we investigate the local geometry on $\mathring{M}$ first. To understand the underlying geometry, this investigation is also of its own interest.

In the present setting, for $J_+ \in GK_\Omega^{\mathbb{T}}(M)$, due to Lemma 3.2 we can define two *new* complex structures $I_\pm$ on $\mathring{M}$ by requiring their matrix forms in the frame $\{\zeta, d\mu\}$ be

$$I_+^* \sim \begin{pmatrix} 0 & \phi^T \\ -(\phi^{-1})^T & 0 \end{pmatrix}, \quad I_-^* \sim \begin{pmatrix} 0 & \phi \\ -\phi^{-1} & 0 \end{pmatrix}. \tag{4.1}$$

We emphasize that $I_\pm$ are different from $J_\pm$ for they are defined using the flat connection ζ rather than the admissible ones $\zeta^\pm$; in particular, up to now we only know that $I_\pm$ are defined on $\mathring{M}$ rather than M. By construction, $I_+ \in DGK_\Omega^{\mathbb{T}}(\mathring{M})$.

There is a *fifth* complex structure I_0 whose matrix form w.r.t. $\{\zeta, d\mu\}$ is

$$I_0^* \sim \begin{pmatrix} 0 & \phi_s \\ -(\phi_s)^{-1} & 0 \end{pmatrix}. \tag{4.2}$$

We know from [18, Thm. 4.4, 4.5] that $I_0 \in K_\Omega^{\mathbb{T}}(\mathring{M})$ and τ is the symplectic potential of I_0, and that $\phi_a = C$ determines an I_0-holomorphic Poisson structure β on $\mathring{M}$.

There is a *sixth almost* complex structure J_0 on $\mathring{M}$. Note that $\mathbb{J}_2$ is a B-transform of $\mathbb{J}_\Omega$ by the 2-form b. In this context, the classical infinitesimal action of $\mathfrak{t}$ on M receives a cotangent correction: $X_j \mapsto X_j - b(X_j)$. The latter should be understood as an extended Lie algebra action [17]. Note that

$$-\mathbb{J}_1(X_j - b(X_j) = \mathbb{J}_1\mathbb{J}_2^2(X_j - b(X_j)) = \mathcal{G}\Omega(X_j) = -g^{-1}d\mu_j.$$

Let $Y_j := -g^{-1}d\mu_j$. These Y_j's are orthogonal to X_k's. Indeed $g(Y_j, X_k) = -(d\mu_j, X_k) = 0$ for μ is $\mathbb{T}$-invariant. Thus $\{X_j, Y_j\}$ is a global frame of $T\mathring{M}$ and J_0 could be simply defined by setting $J_0X_j = Y_j$. In the frame $\{\partial_{\theta^+}, \partial_\mu\}$, the matrix form of J_0 is

$$J_0 \sim \begin{pmatrix} -\Xi^{-1}F(\phi_s)^{-1}\phi & -\Xi^{-1} \\ \Xi[\mathrm{I} + (\Xi^{-1}F(\phi_s)^{-1}\phi)^2] & F(\phi_s)^{-1}\phi\Xi^{-1} \end{pmatrix}, \tag{4.3}$$

where $\Xi = \phi_s + F(\phi_s)^{-1}F$.

If $F = 0$ ($\zeta^{\pm}$ and ζ then coincide), i.e. $J_+ \in DGK_\Omega^{\mathbb{T}}(\mathring{M})$, then the above J_0 is integrable, coincides with I_0 and plays a fundamental role in understanding the geometry [18]. In our present setting, J_0 is not integrable and we choose to include it here since it is naturally associated to $\mathbb{J}_1$.

Another use of these X_j, Y_j is that the smooth distribution $\mathcal{D}_1$ (in the sense of Sussmann [15]) generated by them preserves β_1, as was noted in the remark of [17, Prop. 4.6]. This observation implies partially

Proposition 4.1. *For $\mathbb{J}_1 \in GK_\Omega^{\mathbb{T}}(M)$, points in $\mathring{M}$ are all regular, and the common type is the co-rank of the complex matrix $F - \sqrt{-1}\phi_a$.*

Proof. Recall that the *type* of $\mathbb{J}_1$ at a point $p \in \mathring{M}$ is the complex dimension transverse to the symplectic leaf of β_1 through p. p is called *regular* if this number is constant around p. Since the distribution $\mathcal{D}_1$ has full dimension on $\mathring{M}$, $\mathring{M}$ is actually a leaf of $\mathcal{D}_1$ of the highest dimension. Now that β_1 is preserved by $\mathcal{D}_1$, the rank of β_1 on $\mathring{M}$ has to be constant, i.e., points in $\mathring{M}$ are all regular for $\mathbb{J}_1$.

Besides the above intrinsic proof of the first part of Prop. 4.1, we can give an alternative proof by a direct local computation. Note that $\beta_3 = -1/4(J_+ - J_-)\Omega^{-1}$. We can write down the matrix form of β_3 w.r.t. $\{\zeta^+, d\mu\}$:

$$\beta_3 \sim \frac{1}{2}\begin{pmatrix} -\phi_a & F\phi^{-1} \\ (\phi^{-1})^T F & -(\phi^{-1})_a \end{pmatrix},$$

or as a tensor, $2\beta_3$ is

$$\begin{aligned} &\begin{pmatrix} \partial_{\theta^+}^T & (J_+\partial_{\theta^+})^T \end{pmatrix} \\ &\quad \otimes \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\phi^T \end{pmatrix}\begin{pmatrix} -\phi_a & F\phi^{-1} \\ (\phi^{-1})^T F & -(\phi^{-1})_a \end{pmatrix}\begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\phi \end{pmatrix}\begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix} \\ &\quad = \begin{pmatrix} \partial_{\theta^+}^T & (J_+\partial_{\theta^+})^T \end{pmatrix} \otimes \begin{pmatrix} -\phi_a & -F \\ -F & \phi_a \end{pmatrix}\begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix}. \end{aligned}$$

Note that the type of $\mathbb{J}_1$ is half the real dimension of $\ker(J_+ - J_-)$ and that the matrix $\begin{pmatrix} -\phi_a & -F \\ -F & \phi_a \end{pmatrix}$ is constant. We thus know that points in $\mathring{M}$ are all regular for $\mathbb{J}_1$; in particular, if we denote $z_i^+ = \theta_i^+ + \sqrt{-1}u_i^+$, then it can be easily obtained that

$$\beta_+ = \sum_{i,j} [F_{ij} - \sqrt{-1}(\phi_a)_{ij}]\partial_{z_j^+} \wedge \partial_{z_i^+}. \tag{4.4}$$

Consequently, the common type of $\mathbb{J}_1$ in $\mathring{M}$ is $n - \mathrm{rk}(F - \sqrt{-1}\phi_a)$. □

Remark. Similarly, let $z_i^- = \theta_i^- + \sqrt{-1}u_i^-$. Then we have

$$\beta_- = \sum_{i,j} [F_{ij} - \sqrt{-1}(\phi_a)_{ij}]\partial_{z_j^-} \wedge \partial_{z_i^-}.$$

In particular, we find that $\mathbb{J}_1$ at a fixed point of the $\mathbb{T}$-action is of complex type because the vector fields X_j's all vanish there.

Now we address the global smoothness of those structures defined above, i.e. whether they can be extended smoothly on the whole of M.

Lemma 4.2. *$I_+ \in GK_\Omega^{\mathbb{T}}(\mathring{M})$ is the restriction of an element in $GK_\Omega^{\mathbb{T}}(M)$ on $\mathring{M}$ if and only if the tensors I_+ and $(I_+ + I_-)^{-1}$ can both be extended smoothly to M.*

Proof. Obviously, it suffices to prove the sufficiency part.

Since $I_- = -\Omega^{-1}I_+^*\Omega$, global smoothness of I_+ implies that of I_-. A continuity argument makes it clear that $I_\pm$ are integrable complex structures on M. Therefore, $\bar{g} = -\frac{1}{2}\Omega(I_+ + I_-)$ is smooth on M. By continuity, $\bar{g}$ should be nonnegative-definite on $M\backslash\mathring{M}$. Since $\Omega = -2\bar{g}(I_+ + I_-)^{-1}$, smoothness of $(I_+ + I_-)^{-1}$ implies that $\bar{g}$ must be non-degenerate on $M\backslash\mathring{M}$ and thus positive-definite there. □

Let $I_+ \in GK_\Omega^{\mathbb{T}}(\mathring{M})$ be defined in Eq. (4.1). Take J_+ as a reference element in $GK_\Omega^{\mathbb{T}}(M)$. If we can prove I_+ and $(I_+ + I_-)^{-1} - (J_+ + J_-)^{-1}$ both extend smoothly to M, then by Lemma 4.2 I_+ is the restriction of an element in $DGK_\Omega^{\mathbb{T}}(M)$.

Lemma 4.3. *Let $J_+ \in GK_\Omega^{\mathbb{T}}(M)$ and $\Xi := \phi_s + F(\phi_s)^{-1}F$ in the context of Lemma 3.2. Then the inverse Ξ^{-1} admits a smooth extension to M.*

Proof. In the frame $\{\partial_{\theta^+}, \partial_\mu\}$, the invertible map $(J_+ + J_-)/2$ has the matrix form

$$\frac{J_+ + J_-}{2} \sim \begin{pmatrix} -(\phi^T)^{-1}F & -(\phi^{-1})_s \\ \phi_s + 2F(\phi^T)^{-1}F & F(\phi^T)^{-1} \end{pmatrix}.$$

An elementary computation shows that its inverse has the matrix form

$$(\frac{J_+ + J_-}{2})^{-1} \sim \begin{pmatrix} \Xi^{-1}F(\phi_s)^{-1}\phi & \Xi^{-1} \\ (-\phi^T + 2AF\Xi^{-1}F)(\phi_s)^{-1}\phi & 2AF\Xi^{-1} \end{pmatrix}, \tag{4.5}$$

where $A = \frac{\phi^T(\phi_s)^{-1}}{2} - \mathrm{I}$; in particular, we find

$$\Omega((\frac{J_+ + J_-}{2})^{-1}\partial_{\theta_i^+}, \partial_{\theta_j^+}) = \Omega(\sum_k (\Xi^{-1})^{ki}\partial_{\mu_k}, \partial_{\theta_j^+}) = (\Xi^{-1})^{ji}.$$

Since $\Omega((\frac{J_+ + J_-}{2})^{-1}\partial_{\theta_i^+}, \partial_{\theta_j^+})$ is smooth on M, we know that Ξ^{-1} admits a smooth extension to M. □

Theorem 4.4. *For $J_+ \in GK_\Omega^{\mathbb{T}}(M)$, I_+ defined in Eq. (4.1) is the restriction of an element in $DGK_\Omega^{\mathbb{T}}(M)$ on $\overset{\circ}{M}$.*

Proof. Let us prove first that I_+ is smooth. Define $\mathcal{F} : TM \to TM$ by $\mathcal{F} = \mathrm{Id} - \Omega^{-1}\hat{F}$ where $\hat{F} = \sum_{j,k} F_{kj} d\mu_j \wedge d\mu_k$. Then

$$\mathcal{F}^* \begin{pmatrix} \zeta \\ d\mu \end{pmatrix} = \begin{pmatrix} \mathrm{I} & -F \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} \zeta \\ d\mu \end{pmatrix} = \begin{pmatrix} \zeta^+ \\ d\mu \end{pmatrix}$$

and thus $I_+ = \mathcal{F}J_+\mathcal{F}^{-1}$. Since $\mathcal{F}$ is globally well-defined on M, so is I_+.

Next we shall write down the matrix form of $[(I_+ + I_-)/2]^{-1}$ in the frame $\{\zeta^+, d\mu\}$ ($I_\pm$ are defined using $\{\zeta, d\mu\}$). Since ζ^+ has no global meaning on M, we replace it with $-\phi^T J_+^* d\mu$ where $J_+^* d\mu$ is smooth on M.

By Eq. (4.5), in terms of $\{J_+^* d\mu, d\mu\}$ and $\{\partial_{\theta^+}, J_+\partial_{\theta^+}\}$ the tensor $[(J_+ + J_-)/2]^{-1}$ is

$$\begin{pmatrix} (J_+^* d\mu)^T & d\mu^T \end{pmatrix} \otimes \begin{pmatrix} -\phi\Xi^{-1}F(\phi_s)^{-1}\phi & \phi\Xi^{-1}\phi \\ (-\phi^T + 2AF\Xi^{-1}F)(\phi_s)^{-1}\phi & -2AF\Xi^{-1}\phi \end{pmatrix} \begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix}$$

where $A = \phi^T(\phi_s)^{-1}/2 - \mathrm{I}$. Similarly, the tensor $[(I_+ + I_-)/2]^{-1}$ equals to

$$\begin{pmatrix} (J_+^* d\mu)^T & d\mu^T \end{pmatrix} \otimes \begin{pmatrix} -\phi(\phi_s)^{-1}F & \phi(\phi_s)^{-1}\phi \\ -F(\phi_s)^{-1}F - \phi^T(\phi_s)^{-1}\phi & F(\phi_s)^{-1}\phi \end{pmatrix} \begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix}.$$

Thus to prove that $[(I_+ + I_-)/2]^{-1}$ is globally well-defined on M, we have to justify:
i) $\phi\Xi^{-1}F(\phi_s)^{-1}\phi - \phi(\phi_s)^{-1}F$ is smooth on M;
ii) $\phi(\Xi^{-1} - (\phi_s)^{-1})\phi$ is smooth on M;
iii) $AF\Xi^{-1}F(\phi_s)^{-1}\phi + 1/2F(\phi_s)^{-1}F$ is smooth on M;
iv) $1/2F(\phi_s)^{-1}\phi + AF\Xi^{-1}\phi$ is smooth on M.

With Lemma 4.3 in mind, a careful analysis reveals that one only needs to check that the quantities $(\phi_s)^{-1}, \phi_s\Xi^{-1}\phi_s - \phi_s, \Xi^{-1}\phi_s$ are smooth on M. Note that

$$\phi_s\Xi^{-1}\phi_s - \phi_s = [\Xi - F(\phi_s)^{-1}F]\Xi^{-1}\phi_s - \phi_s = -F(\phi_s)^{-1}F\Xi^{-1}\phi_s$$

and

$$\Xi^{-1}\phi_s = \Xi^{-1}[\Xi - F(\phi_s)^{-1}F] = \mathrm{I} - \Xi^{-1}F(\phi_s)^{-1}F.$$

So we only have to prove that $(\phi_s)^{-1}$ is smooth on M.
Claim. $(\phi_s)^{-1}$ admits a smooth extension to M.

Proof. Let us compute the seemingly irrelevant quantity $((J_+ + J_-)^{-1}\partial_{\theta_i^+}, J_+^* d\mu_j)$ first. From Eq. (4.5), we have

$$\begin{aligned} ((J_+ + J_-)^{-1}\partial_{\theta_i^+}, J_+^* d\mu_j) &= -\sum_{k,l}[\Xi^{-1}F(\phi_s)^{-1}\phi]_{ki}(\phi^{-1})^{jl}(\partial_{\theta_k^+}, \zeta_l^+) \\ &= -[\Xi^{-1}F(\phi_s)^{-1}]_{ji}. \end{aligned}$$

Since $((J_+ + J_-)^{-1}\partial_{\theta_i^+}, J_+^* d\mu_j)$ is globally defined on M, we know that $\Xi^{-1}F(\phi_s)^{-1}$ is smooth on M. Additionally, we have

$$\mathrm{I} = \Xi^{-1}(\phi_s + F(\phi_s)^{-1}F) = \Xi^{-1}\phi_s + \Xi^{-1}F(\phi_s)^{-1}F$$

and consequently $\Xi^{-1}\phi_s$ is smoothly defined on M. Note that $\phi_s \geq \Xi$ on $\mathring{M}$ in the sense that their difference $-F(\phi_s)^{-1}F$ is nonnegative-definite. We thus have $(\phi_s)^{-1} \leq \Xi^{-1}$ on $\mathring{M}$ and $\det(\Xi^{-1}\phi_s) = \det\Xi^{-1} \times \det\phi_s \geq 1$ on $\mathring{M}$. By continuity, $\det(\Xi^{-1}\phi_s) \geq 1$ on the whole of M, implying that $\Xi^{-1}\phi_s$ is both smooth and invertible on M. Therefore, $(\phi_s)^{-1}\Xi$ is smooth on M.

The factorization $(\phi_s)^{-1} = [(\phi_s)^{-1}\Xi] \times \Xi^{-1}$ then implies that $(\phi_s)^{-1}$ is also smooth on M. □

By Lemma 4.2, we thus have finally proved that $I_+ \in DGK_\Omega^{\mathbb{T}}(M)$. □

Corollary 4.5. *The complex structure I_0 defined in Eq. (4.2) admits a smooth extension on M. More precisely, $I_0 \in K_\Omega^{\mathbb{T}}(M)$ and τ is its symplectic potential.*

Proof. Since $I_+ \in DGK_\Omega^{\mathbb{T}}(M)$, the result follows immediately from [18, Thm. 4.9]. □

Corollary 4.6. *The almost complex structure J_0 defined by Eq. (4.3) admits a smooth extension to M; in particular, J_0 is compatible with Ω, i.e. J_0 is a toric almost Kähler structure.*

Proof. The proof goes in the same spirit as the proof of Thm. 4.4. J_0 as a tensor is

$$\begin{pmatrix} (J_+^* d\mu)^T & d\mu^T \end{pmatrix} \otimes \begin{pmatrix} \phi\Xi^{-1}F(\phi_s)^{-1}\phi & -\phi\Xi^{-1}\phi \\ \Xi[\mathrm{I} + (\Xi^{-1}F(\phi_s)^{-1}\phi)^2] & -F(\phi_s)^{-1}\phi\Xi^{-1}\phi \end{pmatrix} \begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix}.$$

Similarly, J_+ as a tensor has the form

$$\begin{pmatrix} (J_+^* d\mu)^T & d\mu^T \end{pmatrix} \otimes \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \begin{pmatrix} \partial_{\theta^+} \\ J_+\partial_{\theta^+} \end{pmatrix}.$$

Thus to see J_0 is globally defined on M, we have to prove the functions $\phi\Xi^{-1}F(\phi_s)^{-1}\phi$, $\phi\Xi^{-1}\phi - \phi$, $F(\phi_s)^{-1}\phi\Xi^{-1}\phi$ and $\Xi[\mathrm{I} + (\Xi^{-1}F(\phi_s)^{-1}\phi)^2] - \phi$ can all be extended smoothly to M. In Lemma 4.3 and the proof of Thm. 4.4, we already have the global smoothness of Ξ^{-1}, Ξ $(\phi_s)^{-1}$ and $\Xi^{-1}\phi_s$, which leads to the global smoothness of J_0.

To see J_0 is compatible with Ω, note that there is a natural J_0-Hermitian metric on $\mathring{M}$ defined by the GK structure on M: Identify $T\mathring{M}$ with $\mathbb{K} \oplus \mathbb{J}_1\mathbb{K}$, where $\mathbb{K}$ is the subbundle of $T\mathring{M} \oplus T^*\mathring{M}$ generated by $X_j - b(X_j)$. The restriction of $-\mathbb{J}_1$ and $\mathcal{G} = -\mathbb{J}_1\mathbb{J}_2$ on $\mathbb{K} \oplus \mathbb{J}_1\mathbb{K}$ then gives rise to J_0 and a Hermitian metric $\tilde{g}$. One can easily check that on $\mathring{M}$, $\Omega = \tilde{g}J_0$. The detailed computation involved here is in essence the same as that in the proof of [18, Thm. 4.4] and thus omitted. By continuity, the conclusion can be finally established. □

We have noted that for $\mathbb{J}_1 \in GK^{\mathbb{T}}_{\Omega}(M)$, points in $\mathring{M}$ are all regular and on the other side fixed points are all of complex type. For completeness, let us have a very brief look at those points in $M\backslash\mathring{M}$.

Let P be an open face of codimension k of Δ, defined in $(\mathbb{R}^n)^*$ by $(u_{j_l}, \mu) = \lambda_{j_l}$, $l = 1, 2, \cdots, k$, and $V_P \subset (\mathbb{R}^n)^*$ the linear subspace singled out by $(u_{j_l}, \mu) = 0$, $l = 1, 2, \cdots, k$. These $u_{j_l} \in \mathfrak{t}$ generate a subtorus T_{0P} acting trivially on $M_P = \mu^{-1}(\bar{P})$, where $\bar{P}$ is the closure of P. Let T_P be the quotient of $\mathbb{T}$ by T_{0P}. Note that intrinsically C and F are elements in $\wedge^2\mathfrak{t}$. Let c_P, f_P be the restriction of $c = 1/2\sum_{j,k} C_{kj} e_j \wedge e_k$ and $f = \sum_{j,k} F_{kj} e_j \wedge e_k$ on V_P respectively.

Theorem 4.7. *Let P be an open face of Δ as above. Then M_P is a GK submanifold for $\mathbb{J}_1 \in GK^{\mathbb{T}}_{\Omega}(M)$. More precisely, its GK structure $(\mathbb{J}_{1P}, \mathbb{J}_{2P})$ belongs to $GK^{\mathbb{T}_P}_{\Omega_P}(M_P)$, where $\Omega_P = \Omega|_{M_P}$. M_P inherits a toric Kähler structure from the canonical one on M, which together with c_P and f_P, characterizes the GK structure on M_P.*

Proof. Recall that M_P is a GK submanifold means that the pull-backs of the complex Dirac structures associated with $\mathbb{J}_1, \mathbb{J}_2$ to M_P are themselves GC structures and form a GK structure on M_P. It is a rather standard argument to prove that M_P is a complex submanifold w.r.t. any one of the complex structures $J_\pm$ and I_0. It is known that if a submanifold is both J_+- and J_--invariant, then it is a GK submanifold (see for example [16]). On the other side, the pull-back of $\mathbb{J}_2$ is of course of sympletic type with its symplectic form Ω_P. These structures on M_P are obviously $\mathbb{T}_P$-invariant and consequently the GK structure on M_P lies in $GK^{\mathbb{T}_P}_{\Omega_P}(M_P)$.

Note that the matrices C, F can be equivalently viewed as two canonical I_0-holomorphic Poisson structures on M and M_P is a Poisson submanifold relative to both of them. Obviously, the corresponding restricted holomorphic Poisson structures on M_P are characterized by c_P and f_P. To see these do characterize the toric GK structure on M_P, the most direct way is through the global formula (5.1) in § 5, which shows how $J_+ \in GK^{\mathbb{T}}_{\Omega}(M)$ arises from an element in $DGK^{\mathbb{T}}_{\Omega}(M)$. □

Remark. From the expression of β_+ (Eq (4.4)), we find that on $\mu^{-1}(P)$ the type of $\mathbb{J}_1$ is $n - \mathrm{rk}(f_P - \sqrt{-1}c_P)$ and the type of $\mathbb{J}_{1P}$ is $n - \mathrm{rk}(f_P - \sqrt{-1}c_P) - k$.

To conclude this section, we specialize to the case where $C = 0$ and $F \neq 0$. This is missing in [3]. We begin with an intrinsic characterization of this case. Recall that $\mathcal{D}$ is the distribution on M generated by the $\mathfrak{t}$-action.

Definition 4.8. *If $J_+ \in GK^{\mathbb{T}}_{\Omega}(M)$ satisfies the condition $(J_+ - J_-)\mathcal{D} \subset \mathcal{D}$, we call J_+ a symmetric toric GK structure of symplectic type on M.*

Proposition 4.9. *$J_+ \in GK^{\mathbb{T}}_{\Omega}(M)$ is symmetric if and only if the underlying matrix C in Lemma 3.2 vanishes.*

Proof. Note that in the frame $\{\zeta^+, d\mu\}$,

$$\frac{J_+^* - J_-^*}{2} \sim \begin{pmatrix} -F\phi^{-1} & -2F\phi^{-1}F - \phi_a \\ (\phi^{-1})_a & \phi^{-1}F \end{pmatrix}.$$

Thus $(J_+ - J_-)\mathcal{D} \subset \mathcal{D}$ if and only if $(\phi^{-1})_a \equiv 0$. The latter is equivalent to that ϕ is symmetric, i.e. $C = 0$. □

In the frame $\{\zeta, d\mu\}$, the several geometric structures as linear maps are of the following more compact form:

$$J_\pm^* \sim \begin{pmatrix} \mp F\phi^{-1} & \Xi \\ -\phi^{-1} & \pm\phi^{-1}F \end{pmatrix}, \quad g \sim \begin{pmatrix} \phi^{-1} & 0 \\ 0 & \Xi \end{pmatrix},$$

$$b \sim \begin{pmatrix} 0 & \phi^{-1}F \\ F\phi^{-1} & 0 \end{pmatrix}, \quad \beta_1 \sim \begin{pmatrix} -F & 0 \\ 0 & \phi^{-1}F\Xi^{-1} \end{pmatrix},$$

where $\Xi = \phi + F\phi^{-1}F$. In the present setting, $\beta_\pm = \sum_{j,k} F_{kj}\partial_{z_j^\pm} \wedge \partial_{z_k^\pm}$, where $z_j^\pm := \theta_j^\pm + \sqrt{-1}u_j^\pm$ are $J_\pm$-holomorphic coordinates respectively. Let I_0 be the toric Kähler structure canonically associated to J_+. An interesting thing is the following

Lemma 4.10. *β_3 is also the imaginary part of an I_0-holomorphic Poisson structure and b is the imaginary part of an I_0-holomorphic 2-form.*

Proof. Note that in the admissible coordinates θ_i, μ_i the matrix form of β_3 is of the form $\begin{pmatrix} 0 & F\phi^{-1} \\ \phi^{-1}F & 0 \end{pmatrix}$. Let $z_i = \theta_i + \sqrt{-1}u_i$ be I_0-holomorphic coordinates on $\mathring{M}$. Then

$$\begin{aligned} 2\beta_3 &= \sum_{j,k,l} F_{kj}(\phi^{-1})^{lk}\partial_{\theta_j} \wedge \partial_{\mu_l} = -\sum_{j,k} F_{kj}\partial_{\theta_j} \wedge \partial_{u_k} \\ &= -\sqrt{-1}\sum_{j,k} F_{kj}(\partial_{z_j} + \partial_{\bar{z}_j}) \wedge (\partial_{z_k} - \partial_{\bar{z}_k}) \\ &= -\sqrt{-1}\sum_{j,k} F_{kj}\partial_{z_j} \wedge \partial_{z_k} + \sqrt{-1}\sum_{j,k} F_{kj}\partial_{\bar{z}_j} \wedge \partial_{\bar{z}_k}. \end{aligned}$$

This implies the conclusion for β_3 and that for b can be obtained similarly. $\square$

If $S \in \Gamma(\mathrm{End}(TM))$ is invertible, then S acts naturally on $TM \oplus T^*M$ by acting only on the tangent part: $S \cdot (X + \xi) = S(X) + \xi$. We call this sort of transforms to be *purely tangent*. Generally S won't preserve the natural pairing on $TM \oplus T^*M$.

Proposition 4.11. *If $\mathbb{J}_1 \in GK^{\mathbb{T}}_{\Omega}(M)$ is symmetric, then up to purely tangent transform, $\mathbb{J}_1$ is a B-transform of a GC structure $\mathbb{J}_\beta$ induced from an I_0-holomorphic Poisson structure $\beta = -\frac{1}{2}(I_0\beta_3 + \sqrt{-1}\beta_3)$.*

Proof. Let $S = \frac{J_+ + J_-}{2}$. We rewrite $\mathbb{J}_1$ in terms of S, β_3 and b. Indeed,

$$\beta_1 = -\frac{J_+ - J_-}{2} g^{-1} = (\frac{J_+ + J_-}{2})^{-1}(\frac{J_+ - J_-}{2})(\frac{J_+ + J_-}{2})g^{-1} = S^{-1}\beta_3,$$

where we have used the fact that $(J_+ + J_-)(J_+ - J_-) = -(J_+ - J_-)(J_+ + J_-)$. Similarly,

$$\begin{aligned} -\frac{1}{2}(\omega_+ - \omega_-) &= -g(\frac{J_+ - J_-}{2}) = gS^{-1}(\frac{J_+ - J_-}{2})S \\ &= -\Omega S S^{-1}(\frac{J_+ - J_-}{2})S = bS. \end{aligned}$$

Therefore,

$$\mathbb{J}_1 = \begin{pmatrix} -S & 2S^{-1}\beta_3 \\ bS & S^* \end{pmatrix} = \begin{pmatrix} S^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} -S & 2\beta_3 \\ b & S^* \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Now we shall prove there is a 2-form b_1 on M such that

$$\begin{pmatrix} -S & 2\beta_3 \\ b & S^* \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ -b_1 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} -I_0 & 2\beta_3 \\ 0 & I_0^* \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ b_1 & \mathrm{Id} \end{pmatrix}.$$

b_1 should satisfy the following two equations:

$$S = I_0 - 2\beta_3 b_1, \quad b = b_1 I_0 - 2b_1\beta_3 b_1 + I_0^* b_1.$$

It suffices to set $b_1 = -\frac{1}{2}F_{kj}d\mu_j \wedge d\mu_k$. Since b_1 is global on M, a continuity argument completes the proof. $\square$

The following proposition may have some relevance in understanding the implication of the almost complex structure J_0.

Proposition 4.12. *The almost complex structure J_0 defined in § 3.2, the complex structure I_0 and β_1 are compatible in the sense that $J_0\beta_1 = \beta_1 I_0^*$.*

Proof. It's elementary to find that in the frame $\{\zeta, d\mu\}$ the matrix form of J_0 is $\begin{pmatrix} 0 & -\Xi^{-1} \\ \Xi & 0 \end{pmatrix}$. Then the result can be obtained by using the matrix forms of I_0 and β_1 directly . □

5. Constructing toric GK structures from a GK triple

This section is devoted to proving that a GK triple (τ, C, F) on Δ produces an element in $GK_\Omega^{\mathbb{T}}(M)$ canonically.

The first thing one should bear in mind is that the function τ in a GK triple on Δ is automatically a symplectic potential of a toric Kähler structure. This is simply because Guillemin boundary condition (a)(b')(c) imply (b). Given the symplectic potential τ ($\phi_s = Hessian(\tau)$) of a toric Kähler structure, for technical convenience we introduce the following condition for a constant skew real matrix F:

- (f) $\mathrm{I} + [(\phi_s)^{-1/2}F(\phi_s)^{-1/2}]^2$ is positive-defintie on Δ.

Here $(\phi_s)^{-1/2}$ is the square root of ϕ_s^{-1}, which is continuous on Δ [5].

Lemma 5.1. *If (τ, C, F) is a GK triple on Δ, then the above condition (f) holds.*

Proof. If (τ, C, F) is a GK triple, then τ is the symplectic potential of a toric Kähler structure, and thus ϕ_s has a smooth inverse and is positive-definite on each open face of Δ. This surely implies that

$$(\phi_s)^{-1/2}(\phi_s + \sqrt{-1}F)(\phi_s)^{-1/2} = \mathrm{I} + \sqrt{-1}(\phi_s)^{-1/2}F(\phi_s)^{-1/2}$$

is positive-definite when restricted on each open face of Δ. The same conclusion holds for $\mathrm{I} - \sqrt{-1}(\phi_s)^{-1/2}F(\phi_s)^{-1/2}$. Then

$$\begin{aligned}(\mathrm{I} + \sqrt{-1}(\phi_s)^{-1/2}F(\phi_s)^{-1/2})(\mathrm{I} - \sqrt{-1}(\phi_s)^{-1/2}F(\phi_s)^{-1/2}) \\ = \mathrm{I} + [(\phi_s)^{-1/2}F(\phi_s)^{-1/2}]^2\end{aligned}$$

is also positive-definite on each open face of Δ. Note that at each fixed point $\phi_s^{-1} = 0$ and (f) automatically holds. This completes the proof. □

Before proceeding, we show the condition (f) is *really* a restriction on the magnitude of F. This is another fundamental distinction between the roles of C and F.

Proposition 5.2. *Let $\mathcal{A}_n$ be the linear space of $n \times n$ anti-symmetric real matrices F with norm $\|F\| := \sqrt{-\mathrm{tr}(F^2)}$ and ϕ_s the Hessian of the symplectic potential τ of $I_0 \in K_\Omega^{\mathbb{T}}(M)$. Then the subset $\mathcal{A}_\tau$ of $F \in \mathcal{A}_n$ satisfying the above condition (f) is a symmetric bounded open convex subset of $\mathcal{A}_n$ containing zero.*

Proof. Fix $x \in \mathring{\Delta}$ and let $F_x := (\phi_s)^{-1/2}(x)F(\phi_s)^{-1/2}(x)$. Then $F \in \mathcal{A}_\tau$ implies that $-F_x^2 < \mathrm{I}$ and hence $\|F_x\|^2 = -\mathrm{tr}(F_x^2) < n$. This shows that $\mathcal{A}_\tau$ is bounded.

For $F \in \mathcal{A}_\tau$, since (f) is an open condition, for each $x_0 \in \Delta$, there is a neighbourhood $U_F^{x_0} \subset \mathcal{A}_n$ of F and a neighborhood $V_{x_0} \subset \Delta$ of x_0 such that

$$-F_x^2 < \mathrm{I}, \quad \forall F \in U_F^{x_0}, \quad x \in V_{x_0}.$$

Now that Δ is compact, there is a finite subset $\{x_i\} \subset \Delta$ such that $\Delta = \cup_i V_{x_i}$. Then $\cap_i U_F^{x_i} \subset \mathcal{A}_\tau$ is an open neighbourhood of F in $\mathcal{A}_n$.

To see $\mathcal{A}_\tau$ is convex, let $F_1, F_2 \in \mathcal{A}_\tau$ and $F_\lambda := \lambda F_1 + (1-\lambda)F_2$ for some $\lambda \in (0,1)$. It suffices to prove $-F_{\lambda x}^2 < \mathrm{I}$ for any $x \in \Delta$. Note that $F_{\lambda x} = \lambda F_{1x} + (1-\lambda)F_{2x}$ and let $|\cdot|$ denote the usual Euclidean norm on $\mathbb{R}^n$. For $0 \neq v \in \mathbb{R}^n$, we have

$$\begin{aligned}
&(F_{\lambda x}v, F_{\lambda x}v) \\
&\quad = \lambda^2(F_{1x}v, F_{1x}v) + (1-\lambda)^2(F_{2x}v, F_{2x}v) + 2\lambda(1-\lambda)(F_{1x}v, F_{2x}v) \\
&\quad \le \lambda^2|F_{1x}v|^2 + (1-\lambda)^2|F_{2x}v|^2 + 2\lambda(1-\lambda)|F_{1x}v||F_{2x}v| \\
&\quad = (\lambda|F_{1x}v| + (1-\lambda)|F_{2x}v|)^2 < [\lambda \times |v| + (1-\lambda) \times |v|]^2 \\
&\quad = |v|^2,
\end{aligned}$$

as required. That $\mathcal{A}_\tau$ is symmetric is obvious. □

Example 5.3. Let us analyse the case $n = 2$ in some detail. If

$$\phi_s = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad F = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix},$$

where ϕ_s is the Hessian of the symplectic potential τ of $I_0 \in K_\Omega^{\mathbb{T}}(M)$, then the condition (f) amounts to that $1 - \frac{f^2}{\det \phi_s} > 0$ on M. Let $m := \max_{x \in \Delta} \frac{1}{\det \phi_s}$ on Δ. Then that $1 - \frac{f^2}{\det \phi_s} > 0$ is equivalent to $f \in (-\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}) \cong \mathcal{A}_\tau$.

Given a GK triple (τ, C, F) on Δ, let $I_0 \in K_\Omega^{\mathbb{T}}(M)$ and ζ be the complex structure and the flat connection associated to τ respectively and $\phi_s = Hessian(\tau)$. As what Lemma 3.2 tells us, we can define two one-parameter families of admissible connections $\zeta^{\pm t} = \zeta \mp tFd\mu$ for $t \in [0,1]$ and define $J_+^t \in GK_\Omega^{\mathbb{T}}(\mathring{M})$ by

$$J_+^{t*}\begin{pmatrix} \zeta^{+t} \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi^T \\ -(\phi^{-1})^T & 0 \end{pmatrix}\begin{pmatrix} \zeta^{+t} \\ d\mu \end{pmatrix},$$

where $\phi = \phi_s + C$. In this context,

$$J_-^{t*}\begin{pmatrix} \zeta^{-t} \\ d\mu \end{pmatrix} := -\Omega J_+^t \Omega^{-1}\begin{pmatrix} \zeta^{-t} \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi \\ -\phi^{-1} & 0 \end{pmatrix}\begin{pmatrix} \zeta^{-t} \\ d\mu \end{pmatrix}.$$

Proposition 5.4. *Let $J_+^t \in GK_\Omega^{\mathbb{T}}(\mathring{M})$ be as above. Then $J_+^t \in GK_\Omega^{\mathbb{T}}(M)$ for each $t \in [0,1]$. In particular, setting $t = 1$ we get the canonical GK structure associated with the GK triple (τ, C, F).*

Proof. To see J_+^t is smooth on M, we resort to a global description of $J_\pm^t$. Firstly we can define another complex structure I_+ as follows:

$$I_+^*\begin{pmatrix} \zeta \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & \phi^T \\ -(\phi^T)^{-1} & 0 \end{pmatrix}\begin{pmatrix} \zeta \\ d\mu \end{pmatrix}.$$

Then due to [18, Thm. 4.11], I_+ is globally well-defined on M and $I_+ \in DGK_\Omega^{\mathbb{T}}(M)$.

Secondly, define a map $\mathcal{F}_t$ by $\mathcal{F}_t = \mathrm{Id} - t\Omega^{-1}\hat{F}$, where $\hat{F} = \sum_{j,k} F_{kj} d\mu_j \wedge d\mu_k$. $\mathcal{F}_t$ is smoothly well-defined on M. Then we have

$$J_+^{t*} = \mathcal{F}_t^* I_+^* (\mathcal{F}_t^*)^{-1}, \quad J_-^{t*} = (\mathcal{F}_t^*)^{-1} I_-^* \mathcal{F}_t^* \tag{5.1}$$

where I_- is the symplectic adjoint of I_+. This shows that $J_\pm^t$ are both smooth on M.

To see $J_+^t \in GK_\Omega^{\mathbb{T}}(M)$, by Lemma. 4.2 we only need to prove the global smoothness of $(J_+^t + J_-^t)^{-1}$. We adopt a similar strategy to that of the proof of Thm. 4.4. This time we choose the toric Kähler structure I_0 as the reference.

Let θ_j, μ_j be the admissible coordinates associated to I_0. Then in the frame $\{\partial_\theta, \partial_\mu\}$,

$$\frac{J_+^t + J_-^t}{2} \sim \begin{pmatrix} t(\phi^{-1})_a F & -(\phi^{-1})_s \\ \phi_s + t^2 F(\phi^{-1})_s F & -tF(\phi^{-1})_a \end{pmatrix},$$

and consequently (see Eq. (4.5))

$$
\begin{aligned}
&(\frac{J_+^t+J_-^t}{2})^{-1}\\
&\sim \begin{pmatrix} \mathrm{I} & 0\\ tF & \mathrm{I}\end{pmatrix}\begin{pmatrix} t\Xi_t^{-1}F(\phi_s)^{-1}\phi & \Xi_t^{-1}\\ (-\phi^T+2t^2AF\Xi_t^{-1}F)(\phi_s)^{-1}\phi & 2tAF\Xi_t^{-1}\end{pmatrix}\begin{pmatrix} \mathrm{I} & 0\\ -tF & \mathrm{I}\end{pmatrix}\\
&= \begin{pmatrix} t\Xi_t^{-1}FB & \Xi_t^{-1}\\ -\phi^T(\phi_s)^{-1}\phi+t^2B^TF\Xi_t^{-1}FB & tB^TF\Xi_t^{-1}\end{pmatrix},
\end{aligned}
$$

where $\Xi_t=\phi_s+t^2F(\phi_s)^{-1}F$, $A=\frac{\phi^T(\phi_s)^{-1}}{2}-\mathrm{I}$, $B=(\phi_s)^{-1}\phi_a$ and $\phi_a=C$. Then as a tensor $(\frac{J_+^t+J_-^t}{2})^{-1}$ is of the following form:

$$
\begin{aligned}
&\begin{pmatrix} (I_0^*d\mu)^T & d\mu^T \end{pmatrix}\\
&\quad\otimes \begin{pmatrix} -t\phi_s\Xi_t^{-1}FB & \phi_s\Xi_t^{-1}\phi_s\\ -\phi^T(\phi_s)^{-1}\phi+t^2B^TF\Xi_t^{-1}FB & -tB^TF\Xi_t^{-1}\phi_s\end{pmatrix}\begin{pmatrix} \partial_\theta\\ I_0\partial_\theta\end{pmatrix}.
\end{aligned}
$$

Similarly, the tensor $(\frac{I_0+I_0^\Omega}{2})^{-1}=I_0^{-1}=-I_0$ is of the form:

$$
\begin{pmatrix} (I_0^*d\mu)^T & d\mu^T \end{pmatrix}\otimes\begin{pmatrix} 0 & \phi_s\\ -\phi_s & 0\end{pmatrix}\begin{pmatrix} \partial_\theta\\ I_0\partial_\theta\end{pmatrix}.
$$

Therefore, to prove the global smoothness of $(\frac{J_+^t+J_-^t}{2})^{-1}$, we have to prove:

i) $\phi_s\Xi_t^{-1}F(\phi_s)^{-1}\phi_a$ is smooth on M;

ii) $\phi_s\Xi_t^{-1}\phi_s-\phi_s$ is smooth on M;

iii) $-\phi^T(\phi_s)^{-1}\phi-t^2\phi_a(\phi_s)^{-1}F\Xi_t^{-1}F(\phi_s)^{-1}\phi_a+\phi_s$ is smooth on M.

With the fact that $(\phi_s)^{-1}$ is smooth on M in mind, a careful but elementary analysis shows that it suffices to prove that $\Xi_t^{-1}\phi_s$ is smooth on M. Note that

$$
(\phi_s)^{-1}\Xi_t=\mathrm{I}+t^2[(\phi_s)^{-1}F]^2
$$

is smooth on M. Thus to complete the proof, it suffices to prove that $(\phi_s)^{-1}\Xi_t$ is also non-degenerate on $M\backslash\mathring{M}$. On M we have

$$
\begin{aligned}
\det((\phi_s)^{-1}\Xi_t)&=\det[(\phi_s)^{-1/2}\Xi_t(\phi_s)^{-1/2}]=\det[\mathrm{I}+t^2((\phi_s)^{-1/2}F(\phi_s)^{-1/2})^2]\\
&\geq\det[\mathrm{I}+((\phi_s)^{-1/2}F(\phi_s)^{-1/2})^2]>0,
\end{aligned}
$$

where the condition (f) is used. □

This completes the proof of Thm. 3.3.

To conclude this section, we remark that

Proposition 5.5. *In Thm. 3.3 Donaldson's version of Guillemin boundary condition can be replaced by Abreu's version.*

Proof. If (τ, C, F) is a GK triple, then the condition (d) of course holds. For Abreu's version to hold, it suffices to prove that $\det(\phi + \sqrt{-1}F)$ ($\phi = Hessian(\tau)$) has the correct form. We only need to prove $\det(\mathrm{I} + \sqrt{-1}\phi^{-1}F)$ to be positive-definite on Δ. This is obvious by the proof of Lemma 5.1.

If (τ, C, F) fulfills Abreu's version of Guillemin boundary condition, i.e. (d)(e'), to prove that (τ, C, F) is actually a GK triple, we only need to prove that τ is the symplectic potential of a genuine toric Kähler structure and that the condition (f) holds. Let ϕ_0 be the Hessian of τ_Δ. Then $\det \phi_0 = [\delta_0(x) \prod_{i=1}^{d} l_i(x)]^{-1}$ for a smooth positive-definite function $\delta_0(x)$ on Δ. By the condition (d), we know $\phi\phi_0^{-1}$ is smooth on Δ. Thus by (e') on Δ

$$\det(\phi\phi_0^{-1} + \sqrt{-1}F\phi_0^{-1}) = \delta_0(x)/\delta(x).$$

It's easy to see $\det(\phi\phi_0^{-1}) \geq \det(\phi\phi_0^{-1} + \sqrt{-1}F\phi_0^{-1})$ and therefore $\phi\phi_0^{-1}$ is smooth and invertible on Δ. It follows that (e) holds and hence τ is the symplectic potential of a toric Kähler structure. One immediately obtains condition (f) from this. □

6. An explicit example on $\mathbb{C}\boldsymbol{P}^1 \times \mathbb{C}\boldsymbol{P}^1$

In this section, to demonstrate the general theory developed before, we construct toric GK structures of symplectic type on $M = \mathbb{C}P^1 \times \mathbb{C}P^1$.

Let M be equipped with the symplectic structure

$$\Omega = \frac{\sqrt{-1}}{2}\frac{dz_1 \wedge d\bar{z}_1}{(1+|z_1|^2)^2} + \frac{\sqrt{-1}}{2}\frac{dz_2 \wedge d\bar{z}_2}{(1+|z_2|^2)^2}.$$

The standard $\mathbb{T}^2$-action

$$(e^{\sqrt{-1}\theta_1}, e^{\sqrt{-1}\theta_2}) \cdot ([1 : z_1], [1 : z_2]) = ([1 : e^{\sqrt{-1}\theta_1} z_1], [1 : e^{\sqrt{-1}\theta_2} z_2])$$

on M is Hamiltonian. The infinitesimal action is then given by

$$\partial_{\theta_j} = \sqrt{-1}(z_j\partial_{z_j} - \bar{z}_j\partial_{\bar{z}_j}), \quad j = 1, 2,$$

with a moment map $\mu_j = |z_j|^2/2(1+|z_j|^2)$, $j=1,2$. Then $\Delta = [0,1/2] \times [0,1/2]$. Guillemin function of M_Δ is

$$\tau = \frac{1}{2}\sum_{j=1}^{2}[\mu_j \ln \mu_j + (\frac{1}{2}-\mu_j)\ln(\frac{1}{2}-\mu_j)],$$

whose Hessian ϕ_s is

$$\begin{pmatrix} \frac{1}{4\mu_1(1/2-\mu_1)} & 0 \\ 0 & \frac{1}{4\mu_2(1/2-\mu_2)} \end{pmatrix}.$$

Let $C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$, where $f \neq 0$. By Example 5.3, for the triplet (τ, C, F) to determine a toric GK structure of symplectic type, f must satisfy

$$1 - 16f^2\mu_1\mu_2(1/2-\mu_1)(1/2-\mu_2) > 0, \quad (\mu_1,\mu_2) \in \Delta.$$

The function $1/\det\phi_s = 16\mu_1\mu_2(1/2-\mu_1)(1/2-\mu_2)$ takes its maximum $1/16$ when $\mu_1 = \mu_2 = 1/4$. We thus find that $f \in \mathcal{A}_\tau = (-4,4)$.

Now let $\phi = \phi_s + C$ and consequently

$$\phi^{-1} = \frac{1}{\det\phi}\begin{pmatrix} \frac{1}{4\mu_2(1/2-\mu_2)} & -c \\ c & \frac{1}{4\mu_1(1/2-\mu_1)} \end{pmatrix},$$

where $\det\phi = \frac{1}{16\mu_1(1/2-\mu_1)\mu_2(1/2-\mu_2)} + c^2$. For later convenience, let

$$p := \frac{1}{16\mu_1(1/2-\mu_1)\mu_2(1/2-\mu_2)}, \quad \varrho_j := dz_j/z_j, \quad j=1,2.$$

Note that in the admissible coordinates θ, μ, the matrix form of g is

$$\frac{1}{\det\phi}\begin{pmatrix} \frac{1}{4\mu_2(1/2-\mu_2)} & 0 & cf & 0 \\ 0 & \frac{1}{4\mu_1(1/2-\mu_1)} & 0 & cf \\ cf & 0 & \frac{\det\phi - f^2}{4\mu_1(1/2-\mu_1)} & 0 \\ 0 & cf & 0 & \frac{\det\phi - f^2}{4\mu_2(1/2-\mu_2)} \end{pmatrix}.$$

Similarly, the matrix form of b is

$$\frac{1}{\det\phi}\begin{pmatrix} 0 & -c & 0 & \frac{f}{4\mu_2(1/2-\mu_2)} \\ c & 0 & -\frac{f}{4\mu_1(1/2-\mu_1)} & 0 \\ 0 & \frac{f}{4\mu_1(1/2-\mu_1)} & 0 & c(\det\phi+f^2) \\ -\frac{f}{4\mu_2(1/2-\mu_2)} & 0 & -c(\det\phi+f^2) & 0 \end{pmatrix},$$

or

$$b = \frac{1}{\det\phi}\times[-cd\theta_1 d\theta_2 + \frac{fd\theta_1 d\mu_2}{4\mu_2(1/2-\mu_2)} - \frac{fd\theta_2 d\mu_1}{4\mu_1(1/2-\mu_1)} + c(\det\phi+f^2)d\mu_1 d\mu_2].$$

$\mathbb{J}_1$ on $\mathring{M}$ is as well of symplectic type and in particular its pure spinor[3] is $e^{b'-\sqrt{-1}Q}$, where b' is a real 2-form and Q is a symplectic form (the inverse of β_1). It can be found that

$$Q = -g(\frac{J_+ - J_-}{2})^{-1}, \quad b' = -\frac{1}{2}Q(J_+ + J_-),$$

through which we can obtain the two 2-forms:

$$Q = \frac{1}{c^2+f^2}[fd\theta_1 d\theta_2 + \frac{cd\theta_1 d\mu_2}{4\mu_2(1/2-\mu_2)} - \frac{cd\theta_2 d\mu_1}{4\mu_1(1/2-\mu_1)} + (fc^2+f^3-pf)d\mu_1 d\mu_2],$$

$$b' = -(\frac{f}{c^2+f^2} - \frac{f}{\det\phi})[\frac{d\theta_1 d\mu_2}{4\mu_2(1/2-\mu_2)} - \frac{d\theta_2 d\mu_1}{4\mu_1(1/2-\mu_1)}] + (\frac{c}{c^2+f^2} - \frac{c}{\det\phi})d\theta_1 d\theta_2 + (-\frac{cp}{c^2+f^2} + \frac{cf^2}{\det\phi})d\mu_1 d\mu_2.$$

Finally, let us have a look at the symmetric case, i.e. $c = 0$. Note that

$$d\theta_j = -\frac{\sqrt{-1}}{2}(\varrho_j - \bar{\varrho}_j), \quad d\mu_j = \frac{|z_j|^2}{2(1+|z_j|^2)^2}(\varrho_j + \bar{\varrho}_j).$$

By using these formulae, we can find that in terms of the Euclidean coordinates

$$b' - b - \sqrt{-1}Q = \frac{\sqrt{-1}dz_1\wedge dz_2}{fz_1 z_2} - \frac{\sqrt{-1}fd|z_1|^2\wedge d|z_2|^2}{4[(1+|z_1|^2)(1+|z_2|^2)]^2}.$$

[3]We won't review the spinor description of GC structures here. For this see [11]

Note that the first term of the right hand side corresponds to the I_0-holomorphic Poisson structure $\sqrt{-1} f z_1 z_2 \partial_{z_1} \wedge \partial_{z_2}$ while the second term seems to represent the effect of the purely tangent transform.

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