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Arboreal models and their stability

Daniel Álvarez-Gavela, Yakov Eliashberg, and David Nadler

The main result of this paper is the uniqueness of local arboreal models, defined as the closure of the class of smooth germs of Lagrangian submanifolds under the operation of taking iterated transverse Liouville cones. A parametric version implies that the space of germs of symplectomorphisms that preserve the local model is weakly homotopy equivalent to the space of automorphisms of the corresponding signed rooted tree. Hence the local symplectic topology around a canonical model reduces to combinatorics, even parametrically. This paper can be read independently, but it is part of a series of papers by the authors on the arborealization program.

1. Introduction

1.1. Main results

1.1.1. Brief summary. This is part of a series of papers [AGEN19, AGEN20a, AGEN20b, AGEN20c, AGEN23a, AGEN23b] by the authors on the arborealization program. Besides motivation, this paper can be read independently from the other papers, and we begin here with an account of

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its main results. Its relevance to the arborealization program is discussed in Section 1.2.

The class of arboreal singularities was introduced by the third author in the paper [N13]. The class was defined in [N13] as abstract stratified complexes, and also as stratified singular Lagrangians and Legendrians via concrete embeddings. Subsequently in [St18] and [E18], these constructions were further decorated by signs (one can view the class in [N13] as the "positive definite" version of the "arbitrary index" generalization of [St18] and [E18]).

It is important to point out that the definition in [N13] fixes only the *homeomorphism*, and not *diffeomorphism* type of the singularity. Likewise, the definitions in [St18] lead *a priori* to a class of singularities for each combinatorial type, rather than unique local models. While this is sufficient for many applications, for example for calculating many invariants, the *homeomorphism* type of an arboreal Lagrangian does not determine in general the *symplectomorphism* type of the ambient manifold, even if the Lagrangian is smooth (e.g. see [Ab12]). In [E18] an inductive definition for a concrete representative of each combinatorial type was given, but no explicit formulae were provided, nor was it proved that this concrete representative was diffeomorphic to other possible representatives of the same combinatorial type.

In brief, the main new innovations of the current paper are:

(i) Uniqueness Theorem 1.2: Signed arboreal Lagrangian and Legendrian singularities are determined up to ambient symplectomorphism by their combinatorial type.

(ii) Canonical Model Definition 2.19: Each combinatorial type has a canonical local model, described not only inductively but by simple polynomial equations.

(iii) Automorphism Theorem 1.3: Automorphisms of signed arboreal Lagrangian and Legendrian singularities are encoded by automorphisms of their combinatorial data, even parametrically.

The questions of uniqueness and automorphisms as established in (i) and (iii) were not even considered in prior papers on the subject; the canonical local models of (ii) were also not known prior to this paper. Given a canonical model as in (ii), if we take its Legendrian lift, apply any contactomorphism taking it into generic position, and form its Liouville cone, then (i) implies we once again obtain a canonical model. At its heart, the proof shows any sufficiently small contact deformation of a canonical local model in generic position in a cosphere bundle can be realized by lifting an isotopy of the base. The calculation of automorphisms in (iii) follows from a parametric generalization of this argument.

We find it surprising that canonical models with such good properties exist. Indeed, we do not know of any other sufficiently large class of Lagrangian singularities which admit a discrete classification up to ambient symplectomorphism.

1.1.2. Uniqueness Theorem. To explain the Uniqueness Theorem 1.2 in more detail, we first introduce some auxiliary notions.

A closed subset of a symplectic or contact manifold is called *isotropic* if is stratified by isotropic submanifolds (by stratified, we mean there is a locally finite partition into locally closed submanifolds of the ambient manifold). It is called Lagrangian or Legendrian if it is isotropic and purely of the maximal possible dimension (i.e if any stratum is in the closure of the one of maximal dimension). The germ at the origin of a locally simply-connected isotropic subset $L \subset T^*\mathbb{R}^n$ of the cotangent bundle with its standard Liouville structure $\lambda = pdq$ admits a unique lift to an isotropic germ at the origin $\widehat{L} \subset J^1\mathbb{R}^n = T^*\mathbb{R}^n \times \mathbb{R}$ of the 1-jet bundle. Given an isotropic subset $\Lambda \subset S^*\mathbb{R}^n$ of the cosphere bundle, its Liouville cone $C(\Lambda) \subset T^*\mathbb{R}^n$, i.e. the closure of its saturation by trajectories of the Liouville vector field $Z = p\frac{\partial}{\partial p}$, is an isotropic subset.

We will take the following inductive definition as our starting point; it captures how arboreal singularities typically arise in nature.

Definition 1.1. *Arboreal Lagrangian (resp. Legendrian) singularities* form the smallest class Arb_n^{symp} (resp. Arb_n^{cont}) of germs of closed isotropic subsets in $2n$-dimensional symplectic (resp. $(2n+1)$-dimensional contact) manifolds such that the following properties are satisfied:

(i) (Invariance) Arb_n^{symp} is invariant with respect to symplectomorphisms and Arb_n^{cont} is invariant with respect to contactomorphisms.

(ii) (Base case) Arb_0^{symp} contains $pt = \mathbb{R}^0 \subset T^*\mathbb{R}^0 = pt$.

(iii) (Stabilizations) If $L \subset (X, \omega)$ is in Arb_n^{symp}, then the product $L \times \mathbb{R} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ is in Arb_{n+1}^{symp}.

(iv) (Legendrian lifts) If $L \subset T^*\mathbb{R}^n$ is in Arb_n^{symp}, then its Legendrian lift $\widehat{L} \subset J^1\mathbb{R}^n$ is in Arb_n^{cont}.

(v) (Liouville cones) Let $\Lambda_1, \dots, \Lambda_k \subset S^*\mathbb{R}^n$ be a finite disjoint union of arboreal Legendrian germs from Arb_{n-1}^{cont} centered at points $z_1, \dots, z_k \in S^*\mathbb{R}^n$. Let $\pi : S^*\mathbb{R}^n \to \mathbb{R}^n$ be the front projection. Suppose
  - $\pi(z_1) = \cdots = \pi(z_k)$;
  - For any i, and smooth submanifold $Y \subset \Lambda_i$, the restriction $\pi|_Y : Y \to \mathbb{R}^n$ is an embedding (or equivalently, an immersion, since we only consider germs).
  - For any distinct $i_1, \dots, i_\ell$, and any smooth submanifolds $Y_{i_1} \subset \Lambda_{i_1}, \dots, Y_{i_\ell} \subset \Lambda_{i_\ell}$, the restriction $\pi|_{Y_{i_1} \cup \cdots \cup Y_{i_\ell}} : Y_{i_1} \cup \cdots \cup Y_{i_\ell} \to \mathbb{R}^n$ is self-transverse.

  Then the union $\mathbb{R}^n \cup C(\Lambda_1) \cup \cdots \cup C(\Lambda_k)$ of the Liouville cones with the zero-section form an arboreal Lagrangian germ from Arb_n^{symp}.

With the above classes defined, we can also allow boundary by additionally taking the product $L \times \mathbb{R}_{\geq 0} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ for any arboreal Lagrangian $L \subset (X, \omega)$, and similarly for arboreal Legendrians.

The main technical result of this paper is the Stability Theorem 3.5 for arboreal singularities as inductively characterized by Definition 1.1. We will content ourselves in this introduction with stating its main application, which is the Uniqueness Theorem 1.2. As will be shown, to each member of the class Arb_n^{symp}, one can assign a signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ with at most $n+1$ vertices; here T is a finite acyclic graph, ρ is a distinguished root vertex, and ε is a sign function on the edges of T not adjacent to ρ. The Uniqueness Theorem states that this discrete data completely determines the germ:

Theorem 1.2. *If two arboreal Lagrangian singularities $L \subset (X, \omega)$, $L' \subset (X', \omega')$ of the class Arb_n^{symp} have the same dimension and signed rooted tree $\widehat{\mathscr{T}}$, then there is (the germ of) a symplectomorphism $(X, \omega) \simeq (X', \omega')$ identifying L and L'.*

Similarly, each member of the class Arb_n^{cont} is determined by an associated signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ with at most $n+1$ vertices. Note that the Uniqueness Theorem in particular implies, for fixed dimension n, that Definition 1.1 produces only finitely many local models up to ambient symplectomorphism or contactomorphism.

1.1.3. Canonical Local Models. As a complement to the Uniqueness Theorem 1.2 (and as called upon essentially in its proof), it turns out there is a canonical local model in each arboreal class. This is detailed in Section 2, beginning with explicit iterated quadratic equations and culminating in Definition 2.19 (one can view arboreal singularities as what results from going one step beyond locally linear Lagrangians to allow quadratic behavior.)

As a representative for each signed rooted tree $\widehat{\mathscr{T}}$, we construct in Definition 2.19 a canonical local model $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^n$, where $n = |n(\widehat{\mathscr{T}})|$ is one less than the number of vertices in the tree. The model $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^n$ is presented as the positive conormal to a canonical local front $H_{\widehat{\mathscr{T}}} \subset \mathbb{R}^n$ defined by polynomial equations.

While this material naturally has some overlap with general constructions of [N13], [St18] and [E18], no such canonical local model was known prior to this paper. Indeed, their construction begins in Section 2.1 with explicit equations that had not been written down before. To keep track of their geometry, we use the same combinatorics developed in [N13], [St18] and [E18]. But even so, we have found it necessary to reformulate the signs introduced in [St18] and [E18] from scratch in order to match inductive arguments to come, so we give here a warning that our sign conventions do not agree with prior conventions.

1.1.4. Parametric Stability. In Section 3, we also establish a Parametric Stability Theorem 3.12 extending the scope of the Stability Theorem 3.5. In fact, the proofs of the two are intertwined: we do not know a more elementary proof of the Stability Theorem 3.5 that does not inductively encounter the Parametric Stability Theorem 3.12. Moreover, the parametric version has additional consequences such as the following characterization of the automorphisms of arboreal singularities:

Theorem 1.3. *Fix a signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$, set $n = |n(\widehat{\mathscr{T}})|$ and consider the arboreal $\widehat{\mathscr{T}}$-front $H_{\widehat{\mathscr{T}}} \subset \mathbb{R}^n$. Let $D(\mathbb{R}^n, H_{\widehat{\mathscr{T}}})$ be the group of germs at 0 of diffeomorphisms of $\mathbb{R}^n$ preserving $H_{\widehat{\mathscr{T}}}$ as a front, i.e. as a subset along with its coorientation.*

Then the fibers of the natural map $D(\mathbb{R}^n, H_{\widehat{\mathscr{T}}}) \to \mathrm{Aut}(\widehat{\mathscr{T}})$ are weakly contractible.

Hence, from Theorem 1.2 and Theorem 1.3, we conclude the local symplectic topology of an arboreal singularity is completely encoded by the combinatorics of the underlying signed rooted tree, even parametrically.

1.2. Arborealization program

We conclude this introduction by briefly explaining the role of this paper within the broader arborealization program.

The initial goal of the arborealization program is to determine when a Weinstein manifold can be deformed to have an arboreal skeleton, i.e. a skeleton which is a stratified Lagrangian with arboreal singularities.

It was shown in [N15] that singularities of Whitney stratified Lagrangians can always be locally deformed to arboreal Lagrangians in a non-characteristic fashion, i.e. without changing their microlocal invariants. The question of whether a global theory exists at the level of Weinstein structures is more subtle. In two dimensions the story is classical: generic ribbon graphs provide arboreal skeleta. In four dimensions, Starkston proved in [St18] that arboreal skeleta always exist in the Weinstein homotopy class of any Weinstein domain.

In the article [AGEN20c], we show any polarized Weinstein manifold, i.e. a Weinstein manifold with a global field of Lagrangian planes in its tangent bundle, can be deformed to have an arboreal skeleton. More specifically, the arboreal singularities that arise are positive in the sense that they are indexed by signed rooted trees with all positive signs, and conversely, any Weinstein manifold with a positive arboreal skeleton comes with a canonical (homotopy class of) polarization.

The arguments of [AGEN20c] produce skeleta with singularities satisfying the characterization of Definition 1.1. Without the uniqueness of Theorem 1.2, we would still be faced with the possible moduli of such singularities. It could happen that two arboreal skeleta built from the same smooth pieces with the same combinatorial recipe do not have symplectomorphic, or even diffeomorphic neighborhoods. The uniqueness of Theorem 1.2 guarantees this is not the case: there is no moduli of the singularities arising, and indeed their geometry is unambiguously specified by the combinatorics.

With this in hand, one can still ask: is the symplectic or Weinstein thickening of an arboreal skeleton unique? Using the results of the current paper we prove in [AGEN20b] that a diffeomorphism between arboreal skeleta, preserving some additional discrete orientation data, extends to a symplectomorphism of their symplectic thickenings. The existence of a Weinstein thickening was first explained in [St18]. The uniqueness of a Weinstein thickening is also proved in [AGEN20b]: Weinstein thickenings of an arboreal skeleton that induce equivalent orientation structures, a further combinatorial decoration on the skeleton, are Weinstein homotopic via a homotopy fixing the skeleton. So not only can we unambiguously construct a Weinstein manifold from a combinatorial recipe, but the one we construct is the unique one with those combinatorics. In the present paper we will not consider Weinstein structures, and focus instead on the problem of uniqueness up to symplectomorphism.

Thus pairing the results of the current paper with those of [AGEN20c], one is able to express polarized Weinstein manifolds in purely combinatorial terms. In a forthcoming paper [AGEN23b], we plan to classify all bifurcations (i.e. "Reidemeister moves") relating positive arboreal skeleta of two polarized Weinstein manifolds related by a polarized Weinstein homotopy. This will reduce the classification of (polarized) Weinstein structures, up to deformation equivalence, to the classification of positive arboreal complexes up to diffeomorphism and Reidemeister moves. As it is discussed in [AGEN20c] the arborealization program cannot be extended to *all* Weinstein manifolds, though it is likely can be extended to a larger class of Weinstein manifolds beyond the polarized one.

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2. Arboreal models

2.1. Quadratic fronts

Before we present the local models for arboreal singularities, we introduce the quadratic fronts out of which the models will be built and discuss some of their basic properties.

2.1.1. Basic constructions. For $i \geq 0$, define functions $h_i : \mathbb{R}^i \to \mathbb{R}$ by the inductive formula

$$h_0 = 0 \qquad h_i = h_i(x_1, \dots, x_i) = x_1 - h_{i-1}(x_2, \dots, x_i)^2$$

For example, for small i, we have

$$h_1(x_1) = x_1 \qquad h_2(x_1, x_2) = x_1 - x_2^2 \qquad h_3(x_1, x_2, x_3) = x_1 - (x_2 - x_3^2)^2$$

Fix $n \geq 0$. For $i = 0, \ldots, n$, define smooth graphical hypersurfaces

$$^n\Gamma_i = \{x_0 = h_i^2\} \subset \mathbb{R}^{n+1}$$

equipped with the graphical coorientation, and consider their union

$$^n\Gamma = \bigcup_{i=0}^n {}^n\Gamma_i$$

Note the elementary identities

$$^n\Gamma_i = {}^i\Gamma_i \times \mathbb{R}^{n-i} \qquad i = 0, \ldots, n$$

$$^n\Gamma_i \cap {}^n\Gamma_0 = \{0\} \times {}^{n-1}\Gamma_{i-1} \qquad i = 1, \ldots, n$$

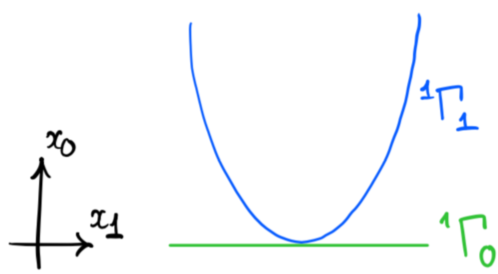

Figure 2.1: The hypersurfaces $^1\Gamma_0$ (green) and $^1\Gamma_1$ (blue).

Let $T^*\mathbb{R}^n$ denote the cotangent bundle with canonical 1-form $pdx = \sum_{i=1}^n p_i dx_i$ where $p = (p_1, \ldots, p_n)$ are dual coordinates to $x = (x_1, \ldots, x_n)$. Let $J^1\mathbb{R}^n = \mathbb{R} \times T^*\mathbb{R}^n$ denote the 1-jet bundle with contact form $dx_0 + pdx = dx_0 + \sum_{i=1}^n p_i dx_i$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with graph $\Gamma_f = \{x_0 = f(x)\} \subset \mathbb{R} \times \mathbb{R}^n$, we have the conormal Lagrangian of the graph $L_{\Gamma_f} = \{x_0 = f(x),\, p_i = -p_0 \partial f / \partial x_i\} \subset T^*\mathbb{R}^{n+1}$, and the conormal Legendrian of the graph $\Lambda_{\Gamma_f} = \{x_0 = f(x),\, p_0 = 1,\, p_i = -\partial f / \partial x_i\} \subset J^1\mathbb{R}^n$.

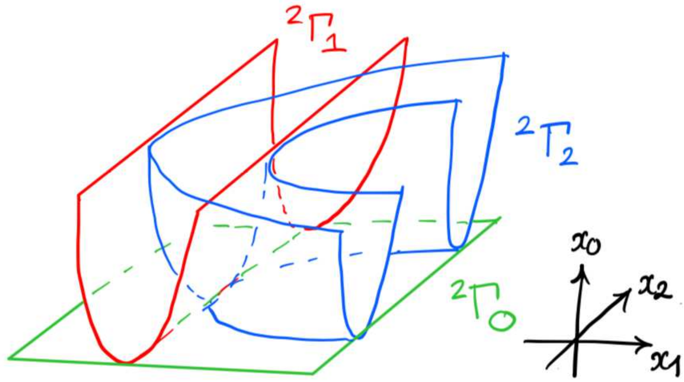

Figure 2.2: The hypersurfaces $^2\Gamma_0$ (green), $^2\Gamma_1$ (red) and $^2\Gamma_2$ (blue).

For $i = 0$, let ${}^nL_0 = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, \ldots, n$, introduce the conormal Lagrangian

$$ {}^nL_i = L_{{}^{n-1}\Gamma_{i-1}} \subset T^*\mathbb{R}^n $$

of the graph ${}^{n-1}\Gamma_{i-1} \subset \mathbb{R}^n$, and consider their union

$$ {}^nL = \bigcup_{i=0}^n {}^nL_i $$

Similarly, for $i = 0, \ldots, n$, introduce the conormal Legendrian

$$ {}^n\Lambda_i = \Lambda_{{}^n\Gamma_i} \subset J^1\mathbb{R}^n $$

of the graph ${}^n\Gamma_i \subset \mathbb{R}^{n+1}$, and consider their union

$$ {}^n\Lambda = \bigcup_{i=0}^n {}^n\Lambda_i $$

Note that the Liouville form vanishes on the conical Lagrangian ${}^nL_i \subset T^*\mathbb{R}^n$, hence its lift to $J^1\mathbb{R}^n = \mathbb{R} \times T^*\mathbb{R}^n$ with zero primitive is a Legendrian. We have the following compatibility:

Lemma 2.1. *The contactomorphism*

$$S : J^1\mathbb{R}^n \longrightarrow J^1\mathbb{R}^n$$

$$S(x_0, x, p) = (x_0 - p_1^2/4, x_1 + p_1/2, x_2, \ldots, x_n, p_1, \ldots, p_n)$$

takes the Legendrian ${}^n\Lambda_i$ *isomorphically to the Legendrian* $\{0\} \times {}^nL_i$*, and thus the union* ${}^n\Lambda$ *isomorphically to the union* $\{0\} \times {}^nL$*.*

Proof. Set $h_{i,1} = h_{i-1}(x_2, \ldots, x_i)$ so that $h_i = x_1 - h_{i,1}^2$. Observe ${}^n\Lambda_i \subset J^1\mathbb{R}^n$ is given by the equations

$$x_0 = h_i^2 \qquad pdx = -dh_i^2 = -2h_i dh_i = -2h_i(dx_1 - 2h_{i,1}dh_{i,1})$$

so in particular $p_1 = -2h_i$ and $\sum_{i=2}^n p_i dx_i = 4h_i h_{i,1} dh_{i,1}$.

If we write $(\hat{x}_0, \hat{x}, p) = S(x_0, x, p)$, for $(x_0, x, p) \in {}^n\Lambda_i$, then we have

$$\hat{x}_0 = x_0 - p_1^2/4 = \pm(x_0 - h_i^2) = 0$$
$$\hat{x}_1 = x_1 + p_1/2 = x_1 - h_i = x_1 - (x_1 - h_{i,1}^2) = h_{i,1}^2$$

Now it remains to observe ${}^nL_i \subset T^*\mathbb{R}^n$ is given by the equations

$$x_1 = h_{i,1}^2 \qquad \sum_{i=2}^n p_i dx_i = -p_1 dh_{i,1}^2 = -2p_1 h_{i,1} dh_{i,1}$$

This completes the proof. □

2.1.2. Distinguished quadrants. We now specify some distinguished quadrants of the ${}^n\Gamma$ which we will use to define our arboreal models. Which of these quadrants are cut out by our sign conventions will become clearer when the arboreal models are introduced.

For $0 \leq j < i \leq n$, set

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_i)$$

so in particular $h_{i,0} = h_i(x_1, \ldots, x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$.

For fixed $0 \leq i \leq n$, consider the collection of functions

$$h_{i,0}, \dots, h_{i,i-1}$$

Note the triangular nature of the linear terms of the collection: for all $0 \leq j \leq i-1$, the subcollection

$$h_{i,j} - x_{j+1}, h_{i,j+1}, \dots, h_{i,i-1}$$

is independent of x_{j+1}. Thus the level sets of the collection are mutually transverse.

Fix once and for all a list of signs $\delta = (\delta_0, \delta_1, \dots, \delta_n)$, $\delta_i \in \{\pm 1\}$. Define the domain quadrant ${}^nQ_i^\delta \subset \mathbb{R}^n$ to be cut out by the inequalities

$$\delta_1 h_{i,0} \leq 0, \dots, \delta_i h_{i,i-1} \leq 0$$

By the transversality noted above, ${}^nQ_i^\delta$ is a submanifold with corners diffeomorphic to $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^{n-i}$. Its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$.

Note ${}^nQ_i^\delta$ only depends on the truncated list $\delta_1, \dots, \delta_i$. In particular, it is independent of δ_0 which will enter the constructions next.

Define the cooriented hypersurface ${}^n\Gamma_i|_\delta \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$${}^n\Gamma_i|_\delta = \{x_0 = \delta_0 h_i^2\}|_{{}^nQ_i^\delta}$$

with the graphical coorientation.

Thus ${}^n\Gamma_i|_\delta$ is cut out by the equations

$$x_0 = \delta_0 h_i^2, \qquad \delta_1 h_{i,0} \leq 0, \dots, \delta_i h_{i,i-1} \leq 0$$

Since ${}^n\Gamma_i|_\delta$ is graphical over ${}^nQ_i^\delta$, it is also a submanifold with corners diffeomorphic to $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^{n-i}$. Likewise, its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$.

Consider as well the union

$${}^n\Gamma|_\delta = \bigcup_{i=0}^n {}^n\Gamma_i|_\delta$$

Remark 2.2. Note that

$$ {}^n\Gamma_i = \bigcup_{\delta,\delta_0=1} {}^n\Gamma_i|_\delta \qquad {}^n\Gamma = \bigcup_{\delta,\delta_0=1} {}^n\Gamma|_\delta $$

since $x \in {}^n\Gamma_i$ implies $x \in {}^n\Gamma_i|_\delta$ where for $1 \leq j \leq i$, we set $\delta_j = -\mathrm{sgn}\,(h_{i,j}(x))$, when $h_{i,j}(x) \neq 0$, and choose it arbitrarily otherwise.

Remark 2.3. Note if we set $\delta' = (\delta_0, \ldots, \delta_{n-1}, -\delta_n)$, then the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(x_0, \ldots, x_{n-1}, x_n) \mapsto (x_0, \ldots, x_{n-1}, -x_n)$, takes ${}^n\Gamma|_\delta$ isomorphically to ${}^n\Gamma|_{\delta'}$ as a cooriented hypersurface. Thus we could always set $\delta_n = 1$ and not miss any new geometry.

Note ${}^n\Gamma_i \cap \{x_0 < 0\}$, hence also ${}^n\Gamma_i|_\delta \cap \{\delta_0 x_0 < 0\}$, is empty since ${}^n\Gamma_i$ is the graph of $h_i^2 \geq 0$.

Lemma 2.4. *Fix* $\delta = (\delta_0, \ldots, \delta_n)$, *and set* $\delta' = (\delta_0\delta_1, \delta_2, \ldots, \delta_n)$. *The homeomorphism*

$$ s : \delta_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \delta_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n $$

$$ s(x_0, x_1, x_2, \ldots, x_n) = (x_0, \delta_0\delta_1(x_1 + \delta_1\sqrt{\delta_0 x_0}), x_2, \ldots, x_n) $$

gives a cooriented identification

$$ s({}^n\Gamma_i|_\delta \cap \{\delta_0 x_0 \geq 0\}) = \delta_0\mathbb{R}_{\geq 0} \times {}^{n-1}\Gamma_{i-1}|_{\delta'} \qquad 0 < i \leq n $$

Proof. Recall ${}^n\Gamma_i|_\delta$ is defined by

$$ x_0 = \delta_0 h_i^2 \qquad \delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0 $$

in particular

$$ x_0 = \delta_0 h_i^2 \qquad \delta_1 h_{i,0} = \delta_1 h_i \leq 0 $$

Note the functions $h_{i,1}, \ldots, h_{i,i-1}$ are independent of the coordinates x_0, x_1.

When $\delta_0 x_0 \geq 0$ and $\delta_1 h_i \leq 0$, the equation $x_0 = \delta_0 h_i^2$ is equivalent to $\sqrt{\delta_0 x_0} = -\delta_1 h_i$. Expanding this in terms of the definitions, we can rewrite this in the form

$$ x_1 + \delta_1\sqrt{\delta_0 x_0} = h_{i-1}(x_2, \ldots, x_i)^2 $$

Thus since $\delta_0' = \delta_0\delta_1$, we see s takes ${}^n\Gamma_i|_\delta \cap \{\delta_0 x_0 \geq 0\}$ into $\delta_0\mathbb{R}_{\geq 0} \times \{x_1 = \delta_0' h_{i-1}^2\}$.

Moreover, the additional functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out ${}^{n-1}\Gamma_{i-1}|_{\delta'} \subset \{x_1 = \delta'_0 h^2_{i-1}\}$ pull back to the same functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out ${}^n\Gamma_i|_\delta$.

Finally, the coorientations of ${}^n\Gamma_i|_\delta, {}^{n-1}\Gamma_{i-1}|_{\delta'}$ are positive on respectively ∂_{x_0}, ∂_{x_1}. Observe the ∂_{x_1}-component of $s_*\partial_{x_0}$ is in the direction of ∂_{x_1}, and hence s gives a cooriented identification. $\square$

2.1.3. Alternative presentation. For compatibility with inductive arguments, it is useful to introduce an alternative sign convention and alternative presentation of the local models.

Fix signs $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$. Consider the involution $\sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\sigma_\varepsilon(x_1, \ldots, x_n) = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Define the domain quadrant ${}^nR^\varepsilon_i \subset \mathbb{R}^n$ cut out by the inequalities

$$\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0$$

Define the cooriented hypersurface ${}^n\Gamma^\varepsilon_i \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$${}^n\Gamma^\varepsilon_i = \{x_0 = \varepsilon_0 h^2_i \circ \sigma_\varepsilon\}|_{{}^nR^\varepsilon_i}$$

with the graphical coorientation. Thus ${}^n\Gamma^\varepsilon_i$ is cut out by the equations

$$x_0 = \varepsilon_0 h^2_i \circ \sigma_\varepsilon \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0$$

Consider as well the union

$${}^n\Gamma^\varepsilon = \textstyle\bigcup_{i=0}^n {}^n\Gamma^\varepsilon_i$$

Remark 2.5. A simple but important observation: ${}^n\Gamma^\varepsilon_i$ in fact only depends on $\varepsilon_0, \ldots, \varepsilon_{i-1}$ and not ε_i. This is because $h_{i,i-1} = x_i$ and so $\varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon = \varepsilon_{i-1} x_i$. In particular, the union ${}^n\Gamma^\varepsilon$ is independent of ε_n.

We have the following adaption of Lemma 2.4.

Lemma 2.6. *Fix $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$, and set $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)$. The homeomorphism*

$$s : \varepsilon_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \varepsilon_0\mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

$$s(x_0, x_1, x_2, \ldots, x_n) = (x_0, x_1 + \varepsilon_0\sqrt{\varepsilon_0 x_0}, x_2, \ldots, x_n)$$

gives a cooriented identification

$$s({}^n\Gamma^\varepsilon_i \cap \{\varepsilon_0 x_0 \geq 0\}) = \varepsilon_0\mathbb{R}_{\geq 0} \times {}^{n-1}\Gamma^{\varepsilon'}_{i-1} \qquad 0 < i \leq n$$

Proof. Recall ${}^n\Gamma_i^\varepsilon$ is defined by

$$x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0$$

in particular

$$x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon = \varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon \leq 0$$

Note the functions $h_{i,1}, \ldots, h_{i,i-1}$ are independent of the coordinates x_0, x_1.

When $\varepsilon_0 x_0 \geq 0$ and $\varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon \leq 0$, the equation $x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon$ is equivalent to $\sqrt{\varepsilon_0 x_0} = -\varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon$. Expanding this in terms of the definitions, we can rewrite this in the form

$$x_1 + \varepsilon_0\sqrt{\varepsilon_0 x_0} = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}$$

Thus we see s takes ${}^n\Gamma_i^\varepsilon \cap \{\varepsilon_0 x_0 \geq 0\}$ into $\varepsilon_0\mathbb{R}_{\geq 0} \times \{x_1 = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}\}$.

Moreover, the additional functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out

$${}^{n-1}\Gamma_{i-1}^{\varepsilon'} \subset \{x_1 = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}\}$$

pull back to the same functions $h_{i,1}, \ldots, h_{i,i-1}$ cutting out ${}^n\Gamma_i^\varepsilon$.

Finally, the coorientations of ${}^n\Gamma_i^\varepsilon, {}^{n-1}\Gamma_{i-1}^{\varepsilon'}$ are positive on respectively ∂_{x_0}, ∂_{x_1}. Observe the ∂_{x_1}-component of $s_*\partial_{x_0}$ is in the direction of ∂_{x_1}, and hence s gives a cooriented identification. $\square$

Here is a useful corollary that "explains" the geometric meaning of the signs ε.

Corollary 2.7. *Fix $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$.*

For $i = 0, \ldots, n-1$, we have $\varepsilon_i = \pm 1$ if and only if ${}^n\Gamma_{i+1}$ is on the $\pm$-side of ${}^n\Gamma_i$ with respect to the graphical dx_0-coorientation.

Moreover, for $i = 1, \ldots, n-1$, we have $\varepsilon_i = \pm 1$ if and only if ${}^n\Gamma_{i+1} \cap {}^n\Gamma_0$ is on the $\pm$-side of ${}^n\Gamma_i \cap {}^n\Gamma_0$ with respect to the graphical dx_1-coorientation.

Proof. For $i = 0$, the first assertion is immediate from the definitions ${}^n\Gamma_0 = \{x_0 = 0\}$ and ${}^n\Gamma_1 = \{x_0 = \varepsilon_0(\varepsilon_1 x_1)^2 = \varepsilon_0 x_1^2, \varepsilon_0\varepsilon_1(\varepsilon_1 x_1) = \varepsilon_0 x_1 \leq 0\}$.

For $i > 0$, both assertions follow by induction from Lemma 2.6. $\square$

Fix signs $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{n-1})$. For $i = 0$, let ${}^nL_0^\varepsilon = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, \dots, n$, introduce the positive conormal bundles

$$ {}^nL_i^\varepsilon = T^+_{{}^{n-1}\Gamma^\varepsilon_{i-1}}\mathbb{R}^n \subset T^*\mathbb{R}^n $$

determined by the graphical coorientation, and consider their union

$$ {}^nL^\varepsilon = \bigcup_{i=0}^n {}^nL_i^\varepsilon $$

Fix signs $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$. For $i = 0, \dots, n$, introduce the Legendrian

$$ {}^n\Lambda_i^\varepsilon \subset J^1\mathbb{R}^n $$

projecting diffeomorphically to the front ${}^n\Gamma_i^\varepsilon \subset \mathbb{R}^{n+1}$, and consider their union

$$ {}^n\Lambda^\varepsilon = \bigcup_{i=0}^n {}^n\Lambda_i^\varepsilon $$

We have the following compatibility of the above Lagrangians and Legendrians analogous to Lemma 2.1.

Lemma 2.8. *Fix signs $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$, and set $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_n)$. The contactomorphism*

$$ S_{\varepsilon_0} : J^1\mathbb{R}^n \longrightarrow J^1\mathbb{R}^n $$

$$ S_{\varepsilon_0}(x_0, x, p) = (x_0 - \varepsilon_0 p_1^2/4, x_1 + \varepsilon_0 p_1/2, x_2, \dots, x_n, p_1, \dots, p_n) $$

takes the Legendrian ${}^n\Lambda_i^\varepsilon$ isomorphically to the Legendrian $\{0\} \times {}^nL_i^{\varepsilon'}$, and thus the union ${}^n\Lambda^\varepsilon$ isomorphically to the union $\{0\} \times {}^nL^{\varepsilon'}$.

Proof. The proof is the same as that of Lemma 2.1 with the following observations. Consider the additional equations

$$ \varepsilon_0\varepsilon_1\delta_1 h_{i,0} \circ \sigma_\delta \leq 0, \dots, \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0 $$

First, over $\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0$, when $p_1 = -2\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon$, we then have $p_1 = -2\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \geq 0$, so we obtain the positive conormal direction. Second, the remaining functions $h_{i,1}, \dots, h_{i,i-1}$ are independent of x_0, x_1. Thus S_{ε_0} indeed takes ${}^n\Lambda_i^\varepsilon$ to $\{0\} \times {}^nL_i^\varepsilon$. $\square$

Remark 2.9. By the lemma, we see the Legendrian ${}^n\Lambda_i^\varepsilon \subset J^1\mathbb{R}^n$ is independent of the initial sign ε_0 so only depends on $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_n)$.

It is also useful to record the following relationship of ${}^n\Gamma^\varepsilon$ with the extended model ${}^n\Gamma$.

Lemma 2.10. *Fix signs $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$.*

Given a contactomorphism $J^1\mathbb{R}^n \to J^1\mathbb{R}^n$ restricting to a closed embedding ${}^n\Lambda^\varepsilon \subset \varepsilon_0 \cdot {}^n\Lambda$ with ${}^n\Lambda_i^\varepsilon \subset \varepsilon_0 \cdot {}^n\Lambda_i$, for all i, consider the front $\Upsilon = \pi({}^n\Lambda^\varepsilon) \subset \varepsilon_0 \cdot {}^n\Gamma$.

Then either the involution σ_ε or its composition with $x_n \mapsto \pm x_n$ takes Υ to ${}^n\Gamma^\varepsilon$.

Proof. Note we have ${}^n\Lambda_0^\varepsilon = \varepsilon_0 \cdot {}^n\Lambda_0 = {}^n\Lambda_0$. Consider the intersection $\Upsilon' = \pi(({}^n\Lambda^\varepsilon \setminus {}^n\Lambda_0) \cap {}^n\Lambda_0)$ as a front inside of $\pi({}^n\Lambda_0) = {}^n\Gamma_0 = \{x_0 = 0\}$. By induction, either the involution σ_ε or its composition with $x_n \mapsto \pm x_n$ takes Υ' to ${}^{n-1}\Gamma^{\varepsilon'}$ where $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_n)$. So we may assume $\Upsilon' = {}^{n-1}\Gamma^{\varepsilon'}$. Now observe ${}^n\Gamma^\varepsilon$ is the unique way to extend ${}^{n-1}\Gamma^{\varepsilon'}$ within $\sigma_\varepsilon(\varepsilon_0 \cdot {}^n\Gamma)$ compatible with coorientations. $\square$

We also have the following observation about signs. See Section 3.1 for notation.

Lemma 2.11. *Let ν_0 be the vertical polarization of $T^*\mathbb{R}^n \to \mathbb{R}^n$.*

Then we have $\varepsilon(\nu_0, {}^nL_1^\varepsilon, {}^nL_2^\varepsilon) = \varepsilon_0$.

Proof. Recall ${}^nL_1^\varepsilon$ is the positive conormal to the graph ${}^{n-1}\Gamma_0^\varepsilon = \{x_0 = 0\}$, and ${}^nL_1^\varepsilon$ is the positive conormal to the graph ${}^{n-1}\Gamma_1^\varepsilon = \{x_0 = \epsilon_0 x_1^2\}$. Since $\varepsilon_0 x_1^2$ is an ε_0-definite quadratic form in x_1, the assertion follows. $\square$

2.2. Arboreal models

We now present the local models for arboreal singularities.

2.2.1. Signed rooted trees.

Definition 2.12. We will use the following terminology throughout:

(i) A *tree* T is a nonempty, finite, connected acyclic graph.

(ii) A *rooted tree* $\mathscr{T} = (T, \rho)$ is a pair of a tree T and a distinguished vertex ρ called the *root*.

(iii) A *signed rooted tree* $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ is a rooted tree (T, ρ) and a decoration ε of a sign ± 1 on each edge of T not adjacent to the root ρ.

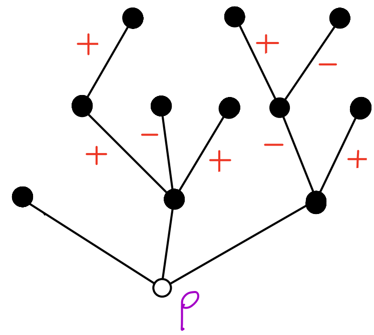

Figure 2.3: A signed rooted tree.

Given a signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$, we write $v(T)$ for the set of vertices, $e(T)$ for the set of edges, and $n(\widehat{\mathscr{T}}) = v(T) \setminus \rho$ for the set of non-root vertices. We regard $v(T)$ as a poset with unique minimum ρ, and in general $\alpha \leq \beta \in v(T)$ when the shortest path connecting β and ρ contains α. We call a non-root vertex β a leaf if exactly one edge of T is adjacent to β, and write $\ell(\widehat{\mathscr{T}}) \subset v(T)$ for the set of leaf vertices.

Remark 2.13. Throughout what follows, for a finite set S, we write $\mathbb{R}^S$ for the Euclidean space of S-tuples of real numbers. One may always fix a bijection $S \simeq \{1, 2, \ldots, n\}$, for some $n \geq 0$, and hence an isomorphism $\mathbb{R}^S \simeq \mathbb{R}^n$, but it will be convenient to avoid choosing such identifications when awkward. We will most often consider $S = n(\widehat{\mathscr{T}})$ the non-root vertices for some rooted tree $\widehat{\mathscr{T}} = (T, \rho)$. Here if one prefers to fix a bijection $b : n(\widehat{\mathscr{T}}) \xrightarrow{\sim} \{1, 2, \ldots, |n(\widehat{\mathscr{T}})|\}$, we recommend choosing b to be order-preserving: if $\alpha \leq \beta$, then one should ensure $b(\alpha) \leq b(\beta)$. This will allow for a clear translation of our constructions.

Definition 2.14. A signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ is called *positive* if the decoration ε consists of signs $+1$.

We will associate to any signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$, a multi-cooriented hypersurface, conic Lagrangian, and Legendrian

$$H_{\widehat{\mathscr{T}}} \subset \mathbb{R}^{n(\widehat{\mathscr{T}})} \qquad L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}})} \qquad \Lambda_{\widehat{\mathscr{T}}} \subset J^1\mathbb{R}^{n(\widehat{\mathscr{T}})}$$

where as usual we write $n(\widehat{\mathscr{T}}) = v(T) \setminus \rho$ for the set of non-root vertices.

By definition, the latter two will be determined by the first as follows:

(i) $L_{\widehat{\mathscr{T}}}$ is the union of the zero-section $\mathbb{R}^{n(\widehat{\mathscr{T}})}$ and the positive conormal to $H_{\widehat{\mathscr{T}}}$.

(ii) $\Lambda_{\widehat{\mathscr{T}}}$ is the Legendrian lift of $L_{\widehat{\mathscr{T}}}$ with zero primitive.

2.2.2. Type *A* trees. Let us first consider the distinguished case of A_{n+1}-trees with extremal root.

Definition 2.15. For $n \geq 0$, a *linear signed A_{n+1}-rooted tree* is a signed rooted tree $\mathcal{A}_{n+1} = (A_{n+1}, \rho, a)$ with vertices $v(A_{n+1}) = \{0, 1, \ldots, n\}$, edges $v(A_{n+1}) = \{[i, i+1] \mid i = 0, \ldots, n-1\}$, and root $\rho = 0$.

By definition, the sign a is a length $n-1$ list of signs $(a_{[1,2]}, \ldots, a_{[n-1,n]})$. Let us set $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1}) = (a_{[1,2]}, \ldots, a_{[n-1,n]}, 1)$ to be the length n list of signs where we pad a by adding a single 1 at the end.

Definition 2.16. The models for A_n-type arboreal singularities are given as follows:

(i) The arboreal $\mathcal{A}_1$-front is the empty set $H_{\mathcal{A}_1} = \emptyset$ inside the point $\mathbb{R}^0$.
For $n \geq 1$, the arboreal $\mathcal{A}_{n+1}$-front is the cooriented hypersurface

$$H_{\mathcal{A}_{n+1}} = {}^{n-1}\Gamma^{\varepsilon} \subset \mathbb{R}^n$$

introduced in Section 2.1.3.

(ii) For $n \geq 0$, the arboreal $\mathcal{A}_{n+1}$-Lagrangian is the union of the zero-section and positive conormal

$$L_{\mathcal{A}_{n+1}} = \mathbb{R}^n \cup T^+_{\mathbb{R}^n} H_{\mathcal{A}_{n+1}} \subset T^*\mathbb{R}^n$$

(iii) For $n \geq 0$, the arboreal $\mathcal{A}_{n+1}$-Legendrian is the lift

$$\Lambda_{\mathcal{A}_{n+1}} = \{0\} \times L_{\mathcal{A}_{n+1}} \subset J^1\mathbb{R}^n$$

Remark 2.17. Following Remark 2.5, the arbitrary choice of the last sign $\varepsilon_{n-1} = 1$ does not affect the arboreal $\mathcal{A}_{n+1}$-models.

Recall the linear signed A_{n+1}-rooted tree $\mathcal{A}_{n+1} = (A_{n+1}, \rho, a)$ has vertices $v(A_{n+1}) = \{0, 1, \ldots, n\}$ with root $\rho = 0$, and so the non-root vertices

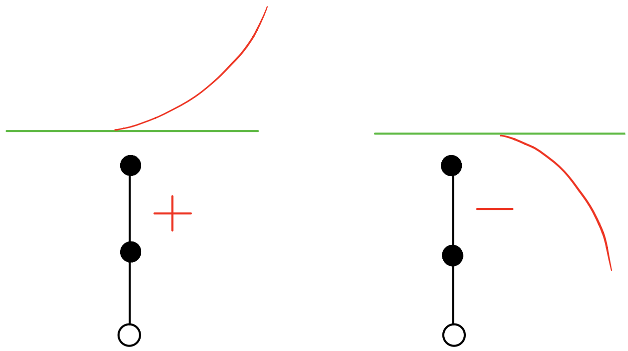

Figure 2.4: The two A_3 fronts with positive and negative sign.

form the set $n(\mathcal{A}_{n+1}) = \{1, \dots, n\}$. In the above definition, we should more invariantly view the ambient Euclidean space $\mathbb{R}^n$ in the form $\mathbb{R}^{n(\mathcal{A}_{n+1})}$ where the ordering of the coordinates matches that of $n(\mathcal{A}_{n+1})$.

With this viewpoint, we rename the smooth pieces of the $\mathcal{A}_{n+1}$-front, indexing them by non-root vertices

$$H_i = {}^{n-1}P^{\varepsilon}_{i-1} \subset H_{\mathcal{A}_{n+1}} \qquad i \in n(\mathcal{A}_{n+1}) = \{1, \dots, n\}$$

Likewise, we rename the smooth pieces of the of the $\mathcal{A}_{n+1}$-Lagrangian, indexing them by vertices

$$L_0 = \mathbb{R}^n \subset L_{\mathcal{A}_{n+1}}$$

$$L_i = T^{+}_{\mathbb{R}^n} H_i \subset L_{\mathcal{A}_{n+1}} \qquad i \in n(\mathcal{A}_{n+1}) = \{1, \dots, n\}$$

and similarly, we rename the smooth pieces of the of the $\mathcal{A}_{n+1}$-Legendrian, indexing them by vertices

$$\Lambda_i = \{0\} \times L_{\mathcal{A}_{n+1},i} \subset \Lambda_{\mathcal{A}_{n+1}} \qquad i \in v(\mathcal{A}_{n+1}) = \{0, 1, \dots, n\}$$

Lemma 2.18. *For $n \geq 1$, and $n \in v(A_{n+1}) = \{0, 1, \dots, n\}$ the unique leaf vertex, and $\mathring{H}_n \subset H_{\mathcal{A}_{n+1}}$ the interior of the corresponding smooth piece, we have*

$$H_{\mathcal{A}_{n+1}} \setminus \mathring{H}_n = H_{\mathcal{A}_n} \times \mathbb{R}$$

inside of $\mathbb{R}^{n(\mathcal{A}_{n+1})} = \mathbb{R}^{n(\mathcal{A}_n)} \times \mathbb{R}$.

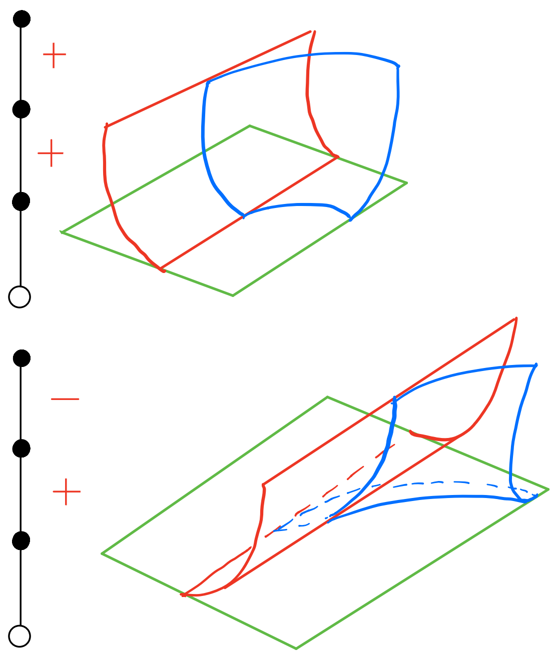

Figure 2.5: Two A_4 fronts with different choices of signs. The other two fronts can be obtained from these two by reflections.

Proof. Recall the other smooth pieces $H_i = {}^{n-1}P^{\varepsilon}_{i-1}$, for $i = 1, \ldots, n-1$, are independent of the last coordinate x_n. $\square$

2.2.3. General trees. Now we consider a general signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$.

To each leaf $\beta \in \ell(\widehat{\mathscr{T}})$, we associate the linear signed A_{n+1}-rooted tree $\mathcal{A}_\beta = (A_\beta, \rho, a)$ where A_β is the full subtree of T on the vertices $v(A_\beta) = \{\alpha \leq \beta \in v(T)\}$, and a is the restricted sign decoration.

Consider the Euclidean space $\mathbb{R}^{n(\widehat{\mathscr{T}})}$. For each $\beta \in \ell(\widehat{\mathscr{T}})$, the inclusion $n(\mathcal{A}_\beta) \subset n(\widehat{\mathscr{T}})$ induces a natural projection

$$\pi_\beta : \mathbb{R}^{n(\widehat{\mathscr{T}})} \longrightarrow \mathbb{R}^{n(\mathcal{A}_\beta)}$$

Definition 2.19. Let $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ be a signed rooted tree.

(i) The arboreal model $\widehat{\mathscr{T}}$-front is the multi-cooriented hypersurface given by the union

$$H_{\widehat{\mathscr{T}}} = \bigcup_{\beta \in \ell(\widehat{\mathscr{T}})} \pi_\beta^{-1}(H_{\mathcal{A}_\beta}) \subset \mathbb{R}^{n(\widehat{\mathscr{T}})}$$

where $H_{\mathcal{A}_\beta} \subset \mathbb{R}^{n(\mathcal{A}_\beta)}$ is the arboreal $\mathcal{A}_\beta$-front.

(ii) The arboreal model $\widehat{\mathscr{T}}$-Lagrangian is the union of the zero-section and positive conormal

$$L_{\widehat{\mathscr{T}}} = \mathbb{R}^{n(\widehat{\mathscr{T}})} \cup T^+_{\mathbb{R}^{n(\widehat{\mathscr{T}})}} H_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$$

(iii) The arboreal model $\widehat{\mathscr{T}}$-Legendrian is the the lift

$$\Lambda_{\widehat{\mathscr{T}}} = \{0\} \times L_{\widehat{\mathscr{T}}} \subset J^1\mathbb{R}^{n(\widehat{\mathscr{T}})}$$

Arboreal models $H_{\widehat{\mathscr{T}}}, L_{\widehat{\mathscr{T}}}$ and $\Lambda_{\widehat{\mathscr{T}}}$ corresponding to positive $\widehat{\mathscr{T}}$ are called positive.

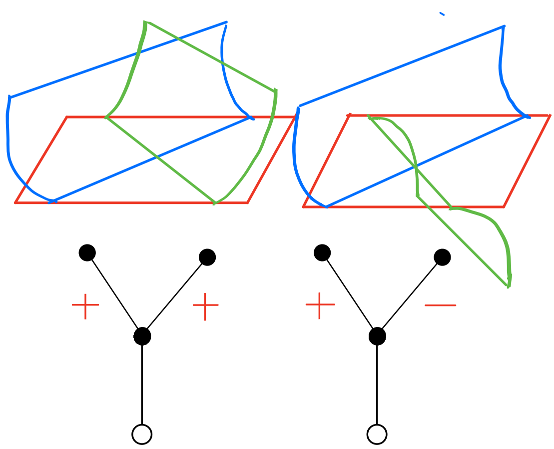

Figure 2.6: Two non A_n-type fronts with different choices of signs.

Remark 2.20. When $\widehat{\mathscr{T}} = \mathcal{A}_{n+1}$, the above definition recovers Definition 2.16 verbatim.

Transporting from the case of $\mathcal{A}_{n+1}$, we may naturally index the smooth pieces of the $\widehat{\mathscr{T}}$-front by non-root vertices

$$H_\alpha = \pi_\beta^{-1}(H_{\mathcal{A}_\beta,\alpha}) \subset H_{\widehat{\mathscr{T}}} \qquad \alpha \in n(\widehat{\mathscr{T}})$$

where $\beta \in \ell(\widehat{\mathscr{T}})$ is any leaf with $\alpha \leq \beta$, and $H_{\mathcal{A}_\beta,\alpha} \subset H_{\mathcal{A}_\beta}$ is the corresponding smooth piece. Likewise, we may index the smooth pieces of the $\widehat{\mathscr{T}}$-Lagrangian by vertices

$$L_\rho = \mathbb{R}^{n(\widehat{\mathscr{T}})} \subset L_{\widehat{\mathscr{T}}}$$

$$L_\alpha = T^+_{\mathbb{R}^{n(\widehat{\mathscr{T}})}} H_\alpha \subset L_{\widehat{\mathscr{T}}} \qquad \alpha \in n(\widehat{\mathscr{T}})$$

and the smooth pieces of the $\widehat{\mathscr{T}}$-Legendrian by vertices

$$\Lambda_\alpha = \{0\} \times L_\alpha \subset \Lambda_{\widehat{\mathscr{T}}} \qquad \alpha \in v(\widehat{\mathscr{T}})$$

Let us record a basic compatibility of the above Lagrangians and Legendrians.

Fix a signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$. Let us first consider the situation when there is a single vertex $\rho' \in \widehat{\mathscr{T}}$ adjacent to ρ. Let $\widehat{\mathscr{T}'} = \widehat{\mathscr{T}} \setminus \rho$ be the signed rooted tree with root ρ' and restricted signs.

Let $\alpha_1, \ldots, \alpha_k \in \widehat{\mathscr{T}'}$ be the vertices adjacent to ρ', and $\varepsilon_1, \ldots, \varepsilon_k$ the signs of $\widehat{\mathscr{T}}$ assigned to the respective edges from ρ' to $\alpha_1, \ldots, \alpha_k$.

Let $L^\infty_{\widehat{\mathscr{T}}} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$ be the ideal Legendrian boundary of $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$. Note that $L^\infty_{\widehat{\mathscr{T}}}$ lies in the open subspace $J^1\mathbb{R}^{n(\widehat{\mathscr{T}'})} \simeq \{p_{\rho'} = 1\} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$.

Lemma 2.21. *The contactomorphism*

$$S : J^1\mathbb{R}^{n(\widehat{\mathscr{T}'})} \longrightarrow J^1\mathbb{R}^{n(\widehat{\mathscr{T}'})}$$

$$S(x_{\rho'}, x, p) = (x_{\rho'} - \textstyle\sum_{i=1}^k \ \varepsilon_i p_{\alpha_i}^2/4, \hat{x}, p)$$

$$\hat{x}_{\alpha_i} = x_{\alpha_i} + \varepsilon_i p_1/2, \ \textit{for } i = 1, \ldots, k, \qquad \hat{x}_\beta = x_\beta \ \textit{else}$$

takes the Legendrian $L^\infty_{\widehat{\mathscr{T}}}$ isomorphically to the Legendrian $\{0\} \times L_{\widehat{\mathscr{T}'}}$.

Thus $L^\infty_{\widehat{\mathscr{T}}}$ itself is a model arboreal Legendrian of type $\widehat{\mathscr{T}'} = \widehat{\mathscr{T}} \setminus \rho$.

Proof. For each leaf vertex of $\widehat{\mathscr{T}}$, we have a linear signed type $\mathcal{A}$ subtree of $\widehat{\mathscr{T}}$ given by the vertices running from ρ to the leaf. By Definition 2.19, $L_{\widehat{\mathscr{T}}}$ is the union of the corresponding linear signed type $\mathcal{A}$ subcomplexes $L_{\mathcal{A}}$. Each such subcomplex is independent of the coordinate x_β indexed by vertices β not in the subtree, hence lies in the zero locus of the dual coordinate p_β. Thus transport of each $L^\infty_{\mathcal{A}}$ under the contactomorphism of the lemma reduces to that of Lemma 2.8. □

More generally, suppose $\rho_1, \dots, \rho_\ell$ are the vertices adjacent to ρ. Observe that $\widehat{\mathscr{T}} \setminus \rho$ is a disjoint union of signed rooted subtrees $\widehat{\mathscr{T}_j} \subset \widehat{\mathscr{T}} \setminus \rho$, for $j = 1, \dots, \ell$, with ρ_j as root and restricted signs. Let $\widehat{\mathscr{T}_j^+} = \widehat{\mathscr{T}_j} \cup \rho \subset \widehat{\mathscr{T}}$ be the signed rooted subtree with ρ readjoined as root and with restricted signs. Set $c_j = n(\widehat{\mathscr{T}}) \setminus n(\widehat{\mathscr{T}_j})$.

Let $L^\infty_{\widehat{\mathscr{T}}} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$ be the ideal Legendrian boundary of $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$. We similarly have $L^\infty_{\widehat{\mathscr{T}_j^+}} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}_j^+})}$ the ideal Legendrian boundary of $L_{\widehat{\mathscr{T}_j^+}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}_j^+})}$.

Since ρ_j is the unique vertex adjacent to ρ within $\widehat{\mathscr{T}_j^+}$, observe that $L_{\widehat{\mathscr{T}_j^+}}$ is connected and in fact lies in

$$J^1\mathbb{R}^{n(\widehat{\mathscr{T}_j})} = \{p_{\rho_j} = 1\} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}_j^+})}.$$

Moreover, observe that $L^\infty_{\widehat{\mathscr{T}}}$ is the disjoint union of the connected components

$$\Lambda_j = L^\infty_{\widehat{\mathscr{T}_j^+}} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^{n(\widehat{\mathscr{T}_j})} \times T^*\mathbb{R}^{c_j} = \{p_{\rho_j} = 1\} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$$

By Lemma 2.21, $L^\infty_{\widehat{\mathscr{T}_j^+}} \subset J^1\mathbb{R}^{n(\widehat{\mathscr{T}_j})}$ is a model arboreal Legendrian of type $\widehat{\mathscr{T}_j}$, so $\Lambda_j = L^\infty_{\widehat{\mathscr{T}_j^+}} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^{n(\widehat{\mathscr{T}_j})} \times T^*\mathbb{R}^{c_j}$ is a stabilized model arboreal Legendrian of type $\widehat{\mathscr{T}_j}$. This proves:

Lemma 2.22. *Fix a signed rooted tree $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$.*

Let $\rho_1, \dots, \rho_k$ be the vertices adjacent to ρ. Let $\widehat{\mathscr{T}_j} \subset \widehat{\mathscr{T}} \setminus \rho$ be the signed rooted subtree with ρ_j as root and restricted signs, and $\widehat{\mathscr{T}_j^+} = \widehat{\mathscr{T}_j} \cup \rho \subset \widehat{\mathscr{T}}$ the signed rooted subtree with ρ readjoined as root and with restricted signs. Set $c_j = n(\widehat{\mathscr{T}}) \setminus n(\widehat{\mathscr{T}_j})$.

Then the ideal Legendrian boundary $L^{\infty}_{\widehat{\mathscr{T}}} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$ of the model arboreal Lagrangian $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\widehat{\mathscr{T}})}$ of type $\widehat{\mathscr{T}}$ is the disjoint union of the Legendrians

$$\Lambda_j = L^{\infty}_{\widehat{\mathscr{T}}_j^+} \times \mathbb{R}^{c_j} \subset S^*\mathbb{R}^{n(\widehat{\mathscr{T}})},$$

which are stabilized model arboreal Legendrians of type $\widehat{\mathscr{T}_j}$.

By Lemma 2.18, we also have the following.

Corollary 2.23. *For $\beta \in \ell(\widehat{\mathscr{T}})$ a leaf vertex, and $\mathring{H}_\beta \subset H_{\widehat{\mathscr{T}}}$ the interior of the corresponding smooth piece, we have*

$$H_{\widehat{\mathscr{T}}} \setminus \mathring{H}_\beta = H_{\widehat{\mathscr{T}}\setminus\beta} \times \mathbb{R}^\beta$$

inside of $\mathbb{R}^{n(\widehat{\mathscr{T}})} = \mathbb{R}^{n(\widehat{\mathscr{T}}\setminus\beta)} \times \mathbb{R}^\beta$.

2.2.4. Extended arboreal models. It will be useful for us also define *extended arboreal models* associated with rooted, but not signed trees $\mathscr{T} = (T, \rho)$.

For the unsigned rooted tree $\mathscr{A}_{n+1} = (A_{n+1}, \rho)$ we define

$$H_{\mathscr{A}_{n+1}} := {}^{n-1}\Gamma \subset \mathbb{R}^n,$$

$$L_{\mathscr{A}_{n+1}} := \mathbb{R}^n \cup T^*_{\mathbb{R}^n} H_{\mathscr{A}_{n+1}} \subset T^*\mathbb{R}^n,$$

$$\Lambda_{\mathscr{A}_{n+1}} := 0 \times L_{\mathscr{A}_{n+1}} \subset J^1\mathbb{R}^n.$$

Similarly, for a general rooted tree $\mathscr{T} = (T, \rho)$ we define

$$H_{\mathscr{T}} = \bigcup\nolimits_{\beta\in\ell(\mathscr{T})} \pi_\beta^{-1}(H_{\mathscr{A}_\beta}) \subset \mathbb{R}^{n(\mathscr{T})}$$

where $H_{\mathscr{A}_\beta} \subset \mathbb{R}^{n(\mathscr{A}_\beta)}$ is the arboreal $\mathscr{A}_\beta$-front. Furthermore, we define

$$L_{\mathscr{T}} = \mathbb{R}^{n(\mathscr{T})} \cup T^+_{\mathbb{R}^{n(\mathscr{T})}} H_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^{n(\mathscr{T})}$$

and

$$\Lambda_{\mathscr{T}} = \{0\} \times \Lambda_{\mathscr{T}} \subset J^1\mathbb{R}^{n(\mathscr{T})}$$

Clearly, for any signed version $\widehat{\mathscr{T}}$ of the tree $\mathscr{T}$ we have $H_{\widehat{\mathscr{T}}} \subset H_{\mathscr{T}}, L_{\widehat{\mathscr{T}}} \subset L_{\mathscr{T}}, \Lambda_{\widehat{\mathscr{T}}} \subset \Lambda_{\mathscr{T}}$.

Lemma 2.24. *Given a closed embedding $\Lambda^\infty_{\widehat{\mathscr{T}}} \subset \Lambda^\infty_{\mathscr{T}}$ with $\Lambda^\infty_{\widehat{\mathscr{T}},\alpha} \subset \Lambda^\infty_{\mathscr{T},\alpha}$, for all α, the front $\pi(\Lambda^\infty_{\widehat{\mathscr{T}}}) \subset H_{\mathscr{T}}$ is an embedding of $H_{\widehat{\mathscr{T}}}$.*

Proof. For each leaf vertex of $\widehat{\mathscr{T}}$, we have a linear signed type $\mathcal{A}$ subtree of $\widehat{\mathscr{T}}$ given by the vertices running from ρ to the leaf. By construction, $\Lambda^\infty_{\widehat{\mathscr{T}}}$ and $\Lambda^\infty_{\mathscr{T}}$ are the union of the corresponding type $\mathcal{A}$ subcomplexes $L^\infty_{\mathcal{A}}$ and $L^\infty_{\mathscr{A}}$. Each such subcomplex is independent of the coordinates x_β indexed by vertices β not in the subtree. Now Lemma 2.10 confirms $\pi(L^\infty_{\mathcal{A}})$ is the standard embedding of $H_{\mathcal{A}}$ after a change of coordinates x_α indexed by vertices α in the subtree. Moreover, the change of coordinates agrees for x_α indexed by vertices α in the intersection of such subtrees. By definition, $H_{\widehat{\mathscr{T}}}$ is the union of the $H_{\mathcal{A}}$. □

3. The stability theorem

In this section we define arboreal Lagrangian and Legendrian subsets and prove their stability under symplectic reduction and Liouville cone operations.

3.1. Arboreal Lagrangians and Legendrians

Definition 3.1. Arboreal Lagrangians and Legendrians are defined as follows:

(a) A closed subset $L \subset X$ of a $2m$-dimensional symplectic manifold (X, ω) is called an *arboreal Lagrangian* if the germ of (X, L) at any point $\lambda \in L$ is symplectomorphic to the germ of the pair $(T^*\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, L_{\widehat{\mathscr{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathscr{T}}$ with $n := n(\widehat{\mathscr{T}}) \leq m$.

(b) A closed subset $\Lambda \subset Y$ of a $(2m+1)$-dimensional contact manifold (Y, ξ) is called am *arboreal Legendrian* if the germ of (Y, Λ) at any point $\lambda \in \Lambda$ is contactomorphic to the germ of $(J^1(\mathbb{R}^n \times \mathbb{R}^{m-n}) = J^1\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, \Lambda_{\widehat{\mathscr{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathscr{T}}$ with $n := n(\widehat{\mathscr{T}}) \leq m$.

(c) A closed subset $H \subset M$ of an $(m+1)$-dimensional manifold M is called an *arboreal front* if the germ of (M, H) at any point $m \in M$ is diffeomorphic to the germ of $(\mathbb{R}^{n+1} \times \mathbb{R}^{m-n}, H_{\widehat{\mathscr{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathscr{T}}$ with $n := n(\widehat{\mathscr{T}}) \leq m$.

The pair $(\widehat{\mathscr{T}}, m)$ is called the *arboreal type* of the germ of L, Λ, or H at the given point. We say L, Λ, or H is *positive* if it is locally modeled on positive arboreal models at all points.

Remark 3.2. Later we will also allow arboreal Lagrangians to have boundary and even corners, but throughout the present discussion we restrict to the above definition for simplicity.

Given an arboreal Lagrangian we call $\sup_{\lambda \in L}\{n(\widehat{\mathscr{T}}(\lambda))\}$ the *maximal order* of L, where $\widehat{\mathscr{T}}(\lambda)$ is a the signed rooted tree describing the germ of L at the point λ. Similarly, we define the maximal order of arboreal Legendrians and fronts.

Every arboreal Lagrangian or Legendrian is naturally stratified by isotropic strata indexed by the corresponding tree type. A Lagrangian distribution η in X is called transverse to an arboreal Lagrangian L if it is transverse to all top-dimensional strata of L. Similarly a Legendrian distribution $\eta \subset \xi$ in a contact (Y, ξ) is called transverse to an arboreal Legendrian Λ if it has trivial intersection with tangent planes to all top-dimensional strata of Λ.

Definition 3.3. A *polarization* of L or Λ is a transverse Lagrangian distribution.

Remark 3.4. We emphasize the transversality to an arboreal Lagrangian means transversality to its *closed* smooth pieces, and not just to open strata.

Before we continue we introduce some auxiliary notions. Let V be a symplectic vector space and $\ell_1, \ell_2, \ell_3 \subset V$ linear Lagrangian subspaces which are pairwise transverse. We write $\ell_1 \prec \ell_2 \prec \ell_3$ if ℓ_3 corresponds to a positive definite quadratic form with respect to the polarization (ℓ_1, ℓ_2) of V. Let $C \subset V$ be a coisotropic subspace. For any linear Lagrangian subspace $\ell \subset V$ we denote by $[\ell]^C$ the symplectic reduction of ℓ with respect to C.

Let L be an arboreal Lagrangian whose germ at a point $\lambda \in L$ has the type $(\widehat{\mathscr{T}} = (T, \rho, \varepsilon), m)$. Let $L_\rho \subset T_\lambda X$ the tangent plane to the root Lagrangian corresponding to the root ρ. For each vertex α connected by an edge with ρ let $L_\alpha \subset T_\lambda X$ denote the Lagrangian plane tangent to the Lagrangian corresponding to the vertex a. We recall that L_ρ and L_α cleanly intersect along a codimension 1 subspace. Consider a coistropic subspace

$C_\alpha := \mathrm{Span}(L_\rho, L_\alpha) \subset T_\lambda X$. Let η be a Lagrangian distribution in X transverse to L. Define the sign

$$\varepsilon(\eta, L, \alpha) = \begin{cases} +1, & \text{if } [L_\rho]^{C_\alpha} \prec [L_\alpha]^{C_\alpha} \prec [\eta]^{C_\alpha}; \\ -1, & \text{if } [L_\rho]^{C_\alpha} \prec [\eta]^{C_\alpha} \prec [L_\alpha]^{C_\alpha}. \end{cases} \tag{1}$$

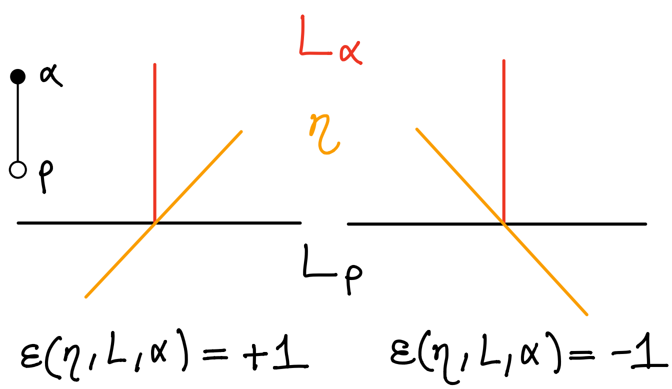

Figure 3.1: The notion of sign for the A_2 singularity.

Similarly, if Λ is an arboreal Legendrian in a contact manifold (Y, ξ), and η a Legendrian distribution transverse to Λ, then for any point $\lambda \in \Lambda$ of type $\widehat{\mathscr{T}} = (T, \rho, \varepsilon)$ we assign a sign $\varepsilon(\eta, \Lambda, \alpha)$ for every vertex α adjacent to the root ρ as equal to ± 1 depending on the $\prec$-order of the triple $[L_\rho]^{C_\alpha}, [L_\alpha]^{C_\alpha}, [\eta]^{C_\alpha}$ in $[\xi_\lambda]^{C_\alpha}$.

3.2. Stability of arboreal Lagrangians and Legendrians

The following is the main result of Section 3. We use below the notation t^*M for the germ of the cotangent bundle T^*M along M.

Theorem 3.5. *Let $\widehat{\mathscr{T}}$ be a signed rooted tree. Let $\rho_1, \dots, \rho_k$ be vertices adjacent to the root ρ and $\widehat{\mathscr{T}_j}$ be subtrees with roots ρ_j (where we removed the decoration of edges $[\rho_j \alpha]$). Let $\phi_j : \mathrm{t}^*\mathbb{R}^m \to J^1\mathbb{R}^m$, $m \geq n = n(\mathscr{T})$, be germs of Weinstein hypersurface embeddings with disjoint images. Denote $z_j := \phi_j(0)$, $\Lambda^j = \phi_j(L_{\widehat{\mathscr{T}_j}} \times \mathbb{R}^{m-n(\widehat{\mathscr{T}_j})})$, $j = 1, \dots, k$. Suppose that*

(i) $\pi(z_j) = 0$;

(ii) *the arboreal Legendrian* $\Lambda := \bigcup_{j=1}^{k} \Lambda^j$ *projects transversely under the front projection* $J^1\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$*;*

(iii) *for each edge* $[\rho_j \alpha]$ *we have* $\varepsilon(\nu, \Lambda^j, \alpha) = \varepsilon_{[\rho_j \alpha]}$.

Then $\mathbb{R}^m \cup C(\Lambda)$*, where* $C(\Lambda)$ *is the Liouville cone of* Λ*, is an arboreal Lagrangian of type* $(\widehat{\mathscr{T}}, m)$ *or equivalently, the germ of the front* $\pi(\Lambda)$ *is diffeomorphic to* $H_{\widehat{\mathscr{T}}} \times \mathbb{R}^{m-n(\widehat{\mathscr{T}})}$.

Theorem 3.5 is a corollary of its unsigned version which is the content of the following proposition.

Proposition 3.6. *Let* $\mathscr{T}$ *be a rooted tree. Let* $\rho_1, \dots, \rho_k$ *be vertices adjacent to the root* ρ *and* $\mathscr{T}_j$ *be subtrees with roots* ρ_j*. Let* $\phi_j : \mathrm{t}^*\mathbb{R}^m \to J^1\mathbb{R}^m$*,* $m \geq n = n(\mathscr{T})$*, be germs of Weinstein hypersurface embeddings. Denote* $z_j := \phi_j(0)$*,* $\Lambda^j = \phi_j(L_{\mathscr{T}_j} \times \mathbb{R}^{m-n(\mathscr{T}_j)})$*,* $j = 1, \dots, k$*. Suppose that*

(i) $\pi(z_j) = 0$*;*

(ii) *the extended arboreal Legendrian* $\Lambda := \bigcup_{j=1}^{k} \Lambda^j$ *projects transversely under the front projection* $J^1\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$*;*

Then $\mathbb{R}^m \cup C(\Lambda)$ *is an extended arboreal Lagrangian of type* $(\mathscr{T}, m)$*, or equivalently, the germ of the front* $\pi(\Lambda)$ *is diffeomorphic to* $H_{\mathscr{T}} \times \mathbb{R}^{m-n(\mathscr{T})}$.

Proof of Theorem 3.5 using Proposition 3.6. Consider the arboreal Legendrian as a closed subcomplex of the extended model. Apply Proposition 3.6 to assume the extended front is in canonical form. Then Lemma 2.24 implies the front of the original arboreal Legendrian is a canonical model. $\square$

Proposition 3.6 will be proven below in this section (see Section 3.6) below, but first we discuss some corollaries of Theorem 3.5.

Corollary 3.7. *Let* $\Lambda \subset \partial_\infty T^*M$ *be an arboreal Legendrian. Suppose that the front projection* $\pi : \Lambda \to M$ *is a transverse immersion. Then* $L := C(\Lambda) \cup M$ *is an arboreal Lagrangian.*

Proof. The intersection $H := M \cap \overline{C(\Lambda)}$ is the front of the Legendrian Λ. Each point $a \in H$ has finitely many pre-images $z_1, \dots, z_k \in \Lambda$. The germs Λ^j of Λ at z_j by our assumption are images of arboreal Lagrangian models under Weinstein embeddings of their symplectic neighborhoods. Hence, by Theorem 3.5 the germ of L at z is of arboreal type. $\square$

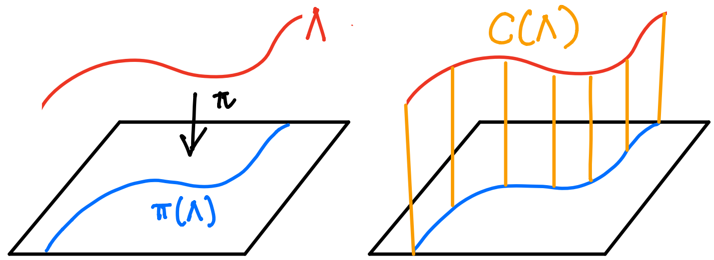

Figure 3.2: In particular, the zero section union the Liouville cone on a regular Legendrian is arboreal with A_2 singularities along its front.

It is not a priori clear that even the standard Lagrangian (resp. Legendrian) arboreal models are arboreal Lagrangians (resp. Legendrians). However, the following corollary shows that they are.

Corollary 3.8. *Consider a model Lagrangian $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^n, n = n(\widehat{\mathscr{T}})$. Then for any point $\lambda \in L_{\widehat{\mathscr{T}}}$ the germ of $L_{\widehat{\mathscr{T}}}$ at λ is a $(\widehat{\mathscr{T}'}, n)$-Lagrangian for a signed rooted tree $\widehat{\mathscr{T}'}$.*

Proof. We argue by induction in n. The base of the induction is trivial. Assuming the claim for $n-1$ we recall that $L_{\widehat{\mathscr{T}}}$ can be presented as $L_\rho \cup C(\Lambda)$, where L_ρ is the smooth piece corresponding to the root ρ of $\widehat{\mathscr{T}}$ and Λ is a union of model Legendrians of dimension $n-1$ in $\partial_\infty T^*(\mathbb{R}^n)$. By the induction hypothesis Λ is an arboreal Legendrian, and hence applying Corollary 3.7 we conclude that $L_{\widehat{\mathscr{T}}}$ is an arboreal Lagrangian. □

Remark 3.9. We will not need it in what follows, so only briefly comment here that it is possible to specify precisely the type $(\widehat{\mathscr{T}'}, n)$ of the germ of $L_{\widehat{\mathscr{T}}}$ at each point $\lambda \in L_{\widehat{\mathscr{T}}}$. Following [N13] the underlying tree T' is a canonically defined subquotient of T, in other words, a diagram $T' \leftarrow S \rightarrow T$, where $S \rightarrow T$ is a full subtree, and $S \rightarrow T'$ contracts some edges; conversely, any such subquotient can occur. Furthermore, if we partially order T with the root $\rho \in T$ as minimum, then the root $\rho' \in T'$ is the unique minimum of the natural induced partial order on T'. Finally, to equip T' with signs, we restrict the signs of T to the subtree S, then push them forward to T' using that each edge of T' is the image of a unique edge of S.

Corollary 3.10. *Let $L_{\widehat{\mathscr{T}}} \subset T^*\mathbb{R}^n$ be a model Lagrangian associated with a signed rooted tree (T, ρ, ε). Let η_0, η_1 be two polarizations transverse to $L_{\widehat{\mathscr{T}}}$. Suppose that for any vertex α of T adjacent to ρ we have*

$$\varepsilon(\eta_0, L, \alpha) = \varepsilon(\eta_1, L, \alpha).$$

Then there is a (germ at the origin of) a symplectomorphism $\psi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ such that $\psi(L) = L$ and $d\psi(\eta_0) = \eta_1$ along L.

Proof. Since η_0 and η_1 are transverse polarizations we may choose embeddings $h_0, h_1 : T^*\mathbb{R}^n \to J^1\mathbb{R}^n$ as Weinstein hypersurfaces such that $h_j(\eta_j) = \nu_0,\ j = 0, 1$, where ν_0 is the canonical Legendrian foliation of $J^1\mathbb{R}^n$ by fibers of the front projection to $\mathbb{R}^n \times \mathbb{R}$. Consider the arboreal Lagrangians $\overline{L}_j := C(h_j(L_{\widehat{\mathscr{T}}})) \cup (\mathbb{R}^n \times \mathbb{R}),\ j = 0, 1$, and note that their arboreal types are described by the same signed rooted tree $\widehat{\widehat{\mathscr{T}}}$ obtained from $\widehat{\mathscr{T}}$ by adding a new root, connecting it by an edge to the old one, and assigning to edges $[\rho\alpha]$ of $\widehat{\mathscr{T}} \subset \widehat{\widehat{\mathscr{T}}}$ adjacent to the old root ρ the sign $\varepsilon(\eta_0, L, \alpha) = \varepsilon(\eta_1, L, \alpha)$. Applying Theorem 3.5 we find the required symplectomorphism ψ. $\square$

Corollary 3.11. *Let $H \subset M$ be an arboreal front. Then for any submanifold $\Sigma \subset M$ transverse to (all strata of) H the intersection $\Sigma \cap H$ is an arboreal front in Σ.*

Proof. We can assume that H is an arboreal front germ at a point $x \in H$, and hence the germ of (M, H) at x is diffeomorphic to the germ of $(\mathbb{R}^{n(\widehat{\mathscr{T}})+1} \times \mathbb{R}^k, H_{\widehat{\mathscr{T}}} \times \mathbb{R}^k)$ for some rooted signed arboreal tree $\widehat{\mathscr{T}}$ and $k = n - n(\widehat{\mathscr{T}})$. Note that the transversality of Σ to H implies that $\operatorname{codim}\Sigma \leq k$ and that the projection of $p : \Sigma \subset \mathbb{R}^{n(\widehat{\mathscr{T}})+1} \times \mathbb{R}^k \to \mathbb{R}^{n(\widehat{\mathscr{T}})+1}$ to the first factor is a submersion, and because we are dealing with germs, it is a trivial fibration. On the other hand, the projection $p|_{\Sigma \cap H} : \Sigma \cap H \to H_{\widehat{\mathscr{T}}}$ is the restriction of this fibration to $H_{\widehat{\mathscr{T}}} \subset \mathbb{R}^{N(\widehat{\mathscr{T}})}$. $\square$

3.3. Parametric version

The following is the parametric version of Theorem 3.5.

Theorem 3.12. *Let $\widehat{\mathscr{T}}$ be a signed rooted tree. Let $\rho_1, \dots, \rho_k$ be vertices adjacent to the root ρ and $\widehat{\mathscr{T}_j}$ be subtrees with roots ρ_j (where we removed the decoration of edges $[\rho_j\alpha]$). Let $\phi_j^y : \mathrm{t}^*\mathbb{R}^m \to J^1\mathbb{R}^m$, $m \geq n = n(\mathscr{T})$, be*

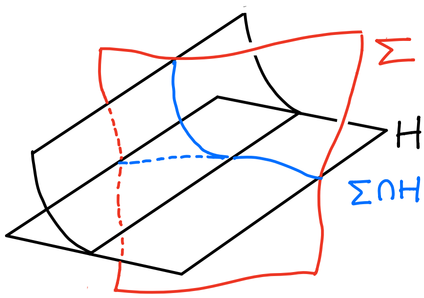

Figure 3.3: Illustration that $\Sigma \cap H$ is an arboreal front in Σ.

families of germs of Weinstein hypersurface embeddings with disjoint images, parametrized by a manifold Y. Denote $z_j^y := \phi_j^y(0)$, $\Lambda_y^j = \phi_j^y(L_{\widehat{\mathcal{T}}_j} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}}_j)})$, $j = 1, \ldots, k$. Suppose that

(i) $\pi(z_j^y) = 0$;

(ii) *the arboreal Legendrian $\Lambda_y := \bigcup_{j=1}^k \Lambda_y^j$ projects transversely under the front projection $J^1\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$;*

(iii) *for each edge $[\rho_j \alpha]$ we have $\varepsilon(\nu, \Lambda_y^j, \alpha) = \varepsilon_{[\rho_j \alpha]}$.*

Then there exists a family of diffeomorphisms ϕ_y between $H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}})}$ and the front $\pi(\Lambda_y)$. If $K \subset Y$ is a closed subset and the ϕ_j^y are the standard embeddings of the local model for $y \in \mathcal{O}p\,(K)$, then we may further assume $\phi_y = Id$ for $y \in \mathcal{O}p\,(K)$.

The parametric version of Proposition 3.6 is formulated similarly. As a consequence of Theorem 3.12 we get the following result:

Corollary 3.13. *Fix a signed rooted tree $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$, set $n = |n(\widehat{\mathcal{T}})|$ and consider the arboreal $\widehat{\mathcal{T}}$-front $H_{\widehat{\mathcal{T}}} \subset \mathbb{R}^n$. Let $D(\mathbb{R}^n, H_{\widehat{\mathcal{T}}})$ be the group of germs at 0 of diffeomorphisms of $\mathbb{R}^n$ preserving $H_{\widehat{\mathcal{T}}}$ as a front, i.e. as a subset along with its coorientation.*

Then the fibers of the natural map $D(\mathbb{R}^n, H_{\widehat{\mathcal{T}}}) \to \mathrm{Aut}(\widehat{\mathcal{T}})$ are weakly contractible.

Proof. We deduce Corollary 3.13 from Theorem 3.12. We will argue for $\widehat{\mathcal{T}} = \mathcal{A}_{n+1}$ when $H_{\mathcal{A}_{n+1}} = {}^{n-1}\Gamma$; the case of general $\widehat{\mathcal{T}}$ is similar.

Since $\mathrm{Aut}(\mathcal{A}_{n+1})$ is trivial, we seek to show $D(\mathbb{R}^n, {}^{n-1}\Gamma)$ is weakly contractible. Note any $\varphi \in D(\mathbb{R}^n, {}^{n-1}\Gamma)$ preserves 0, and moreover, preserves the canonical flag in $T_0\mathbb{R}^n$ given by the tangents to the intersections $\bigcap_{i<i_0} {}^{n-1}\Gamma_i$.

Let $D(\mathbb{R}^n)$ denote the group of germs at 0 of diffeomorphisms of $\mathbb{R}^n$. Consider a k-sphere of maps $f_t \in D(\mathbb{R}^n, {}^{n-1}\Gamma)$, $t \in S^k$. Since all f_t preserve 0 and the canonical flag in $T_0\mathbb{R}^n$, there exists a $k+1$-ball of diffeomorphisms $g_t \in D(\mathbb{R}^n)$, $t \in B^{k+1}$, extending f_t. Applying Theorem 3.12 to the Weinstein hypersurface embeddings induced by g_t, we can find diffeomorphisms h_t such that h_t takes $g_t({}^{n-1}\Gamma)$ back to ${}^{n-1}\Gamma$ and such that h_t is the identity for $t \in S^k$. Then $h_t \circ g_t \in D(\mathbb{R}^n, {}^{n-1}\Gamma)$, $t \in B^{k+1}$, gives an extension of f_t to the $k+1$-ball. $\square$

We also formulate the parametric version of Corollary 3.10.

Corollary 3.14. *Let $L_{\mathscr{T}} \subset T^*\mathbb{R}^n$ be a model Lagrangian associated with a signed rooted tree (T, ρ, ε). Let η_0^y, η_1^y be two families of polarizations transverse to $L_{\mathscr{T}}$ parametrized by a manifold Y. Suppose that for any vertex α of T adjacent to ρ we have*

$$\varepsilon(\eta_0^y, L, \alpha) = \varepsilon(\eta_1^y, L, \alpha).$$

Then there is a family of (germ at the origin of) symplectomorphisms $\psi^y : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ such that $\psi^y(L) = L$ and $d\psi^y(\eta_0^y) = \eta_1^y$ along L. Moreover, if $\eta_0^y = \eta_1^y$ for $y \in \mathcal{O}p\,(K)$ for $K \subset Y$ a closed subset, then we can take $\psi^y = Id$ for $y \in \mathcal{O}p\,(K)$.

The proof is just like in the non-parametric case, but applying Theorem 3.12 instead of Theorem 3.5.

3.4. Tangency loci

Before proving Proposition 3.6 and its parametric analogue we need to analyze more closely the geometry of hypersurfaces forming arboreal fronts.

Definition 3.15. Given smooth hypersurfaces $X_1, X_2 \subset \mathbb{R}^{n+1}$, we denote by $T(X_1, X_2) \subset \mathbb{R}^{n+1}$ their *tangency locus*, i.e. the subset of points $x \in X_1 \cap X_2$ such that $T_xX_1 = T_xX_2$.

Remark 3.16. Given smooth Legendrians $L_1, L_2 \subset J^1\mathbb{R}^n$ whose fronts $X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1}$ are smooth hypersurfaces, note that $T(X_1, X_2) = \pi(L_1 \cap L_2)$.

For $0 \leq j < i \leq n$, recall the notation

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_i)$$

so in particular $h_{i,0} = h_i(x_1, \ldots, x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$. Set

$$T_{i,j} = \{h_{i,j} = 0\} \subset \mathbb{R}^{n+1}$$

Note $h_{i,j}$ is independent of $x_0, \ldots, x_j$, and we have

$$T_{i,j} = \mathbb{R}^{j+1} \times {}^{n-j-1}\Gamma_{i-j-1}$$

Lemma 3.17. *For $0 \leq j < i \leq n$, the tangency locus $T({}^n\Gamma_i, {}^n\Gamma_j) \subset \mathbb{R}^{n+1}$ is the intersection of either ${}^n\Gamma_i$ or ${}^n\Gamma_j$ with the union*

$$\{h_{i,j} = 0\} \cup \bigcup_{k=0}^{j-1} \{h_{i,k} = h_{j,k} = 0\} = T_{i,j} \cup \bigcup_{k=0}^{j-1} (T_{i,k} \cap T_{j,k})$$

Proof. Since ${}^n\Gamma_i, {}^n\Gamma_j$ are the graphs of h_i^2, h_j^2, the projection of $T({}^n\Gamma_i, {}^n\Gamma_j)$ to the domain $\mathbb{R}^n$ is cut out by

$$h_i^2 = h_j^2 \qquad dh_i^2 = dh_j^2$$

Note $h_i = h_{i,0} = x_1 - h_{i,1}^2$, $h_j = h_{j,0} = x_1 - h_{j,1}^2$. By examining the dx_1-component of $dh_i^2 = dh_j^2$, we see it implies $h_i = h_j$. Thus the projection of $T({}^n\Gamma_i, {}^n\Gamma_j)$ is cut out by the single equation $dh_i^2 = dh_j^2$ which in turn implies $h_i = h_j$.

To satisfy $dh_i^2 = dh_j^2$, so in particular $h_i = h_j$, there are two possibilities: (i) $h_i = h_j = 0$; or (ii) $h_i = h_j \neq 0$. In case (i), we find the subset $\{h_{i,0} = h_{j,0} = 0\}$ appearing in the union of the assertion of the lemma. In case (ii), we observe $dh_i^2 = dh_j^2$ is then equivalent to $dh_{i,1}^2 = dh_{j,1}^2$ which in turn implies $h_{i,1} = h_{j,1}$.

Now we repeat the argument. To satisfy $dh_{i,1}^2 = dh_{j,1}^2$, so in particular $h_{i,1} = h_{j,1}$, there are two possibilities: (i) $h_{i,1} = h_{j,1} = 0$; or (ii) $h_{i,1} = h_{j,1} \neq 0$. In case (i), we find the subset $\{h_{i,1} = h_{j,1} = 0\}$ appearing in the union of the assertion of the lemma. In case (ii), we observe $dh_{i,1}^2 = dh_{j,1}^2$ is then equivalent to $dh_{i,2}^2 = dh_{j,2}^2$ which in turn implies $h_{i,2} = h_{j,2}$.

Iterating this argument, we obtain the subset $\bigcup_{k=0}^{j-1}\{h_{i,k} = h_{j,k} = 0\}$, and arrive at the final equation $dh_{i,j}^2 = 0$. By examining the dx_{j+1}-term, we see $dh_{i,j}^2 = 0$ holds if and only if $h_{i,j} = 0$, which gives the remaining subset of the assertion of the lemma. $\square$

Remark 3.18. The only evident redundancy in the description of the lemma is $T_{i,j-1} \cap T_{j,j-1} \subset T_{i,j}$ since $h_{i,j-1} = x_j - h_{i,j}^2$, $h_{j,j-1} = x_j$, so their vanishing implies the vanishing of $h_{i,j}$.

We will be particularly interested in the locus $T_{i,j} \subset T({}^n\Gamma_i, {}^n\Gamma_j)$ and formalize its structure in the following definition.

Definition 3.19. Given smooth hypersurfaces $X_1, X_2 \subset \mathbb{R}^{n+1}$, we denote by $\tau^\circ(X_1, X_2) \subset T(X_1, X_2)$ the subset of points $x \in X_1 \cap X_2$ where in some local coordinates we have $X_1 = \{x_0 = 0\}$, $X_2 = \{x_0 = x_1^2\}$. We write $\tau(X_1, X_2) \subset T(X_1, X_2)$ for the closure of $\tau^\circ(X_1, X_2)$, and refer to it as the *primary tangency* of X_1, X_2.

Remark 3.20. Given smooth Legendrians $L_1, L_2 \subset J^1\mathbb{R}^n$ whose fronts $X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1}$ are smooth hypersurfaces, note that $\tau^\circ(X_1, X_2)$ is the front projection of where L_1, L_2 intersect cleanly in codimension one.

We have the following consequence of Lemma 3.17.

Corollary 3.21. *For $0 \le j < i \le n$, the primary tangency $\tau({}^n\Gamma_i, {}^n\Gamma_j) \subset \mathbb{R}^{n+1}$ is the intersection of either ${}^n\Gamma_i$ or ${}^n\Gamma_j$ with $T_{i,j}$.*

Before continuing, let us record the following for future use.

Lemma 3.22. *Fix $0 \le k < j \le n-1$.*
We have

$$\tau(\tau(\Gamma_n, {}^n\Gamma_k), \tau({}^n\Gamma_j, {}^n\Gamma_k)) = \tau({}^n\Gamma_n, {}^n\Gamma_j) \cap \tau({}^n\Gamma_j, {}^n\Gamma_k)$$

where the primary tangency of $\tau(\Gamma_n, {}^n\Gamma_k)$, $\tau({}^n\Gamma_j, {}^n\Gamma_k)$ of the left hand side is calculated in ${}^n\Gamma_k \simeq \mathbb{R}^n$.

Proof. By the preceding corollary, the left hand side is the intersection ${}^n\Gamma_k \cap \tau(T_{n,k}, T_{j,k})$.

Note ${}^n\Gamma_k \cap T_{j,k} = \tau({}^n\Gamma_j, {}^n\Gamma_k) = {}^n\Gamma_j \cap T_{j,k}$. Hence

$${}^n\Gamma_k \cap \tau(T_{n,k}, T_{j,k}) = {}^n\Gamma_j \cap \tau(T_{n,k}, T_{j,k})$$

since $y \in {}^n\Gamma_k \cap \tau(T_{n,k}, T_{j,k}) \iff y \in {}^n\Gamma_k \cap T_{j,k},\ y \in \tau(T_{n,k}, T_{j,k}) \iff y \in {}^n\Gamma_j \cap T_{j,k},\ y \in \tau(T_{n,k}, T_{j,k}) \iff y \in {}^n\Gamma_j \cap \tau(T_{n,k}, T_{j,k})$.

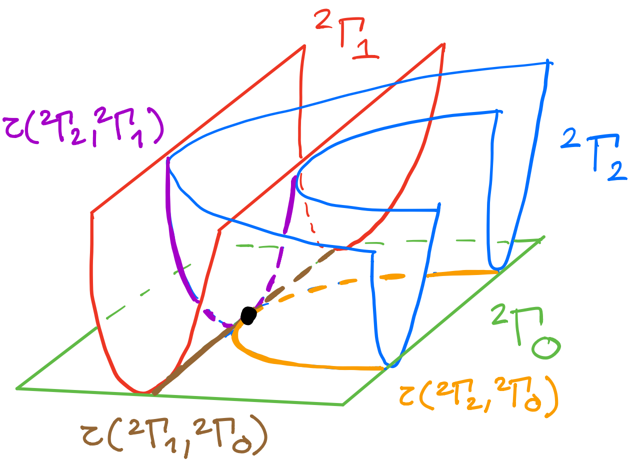

Figure 3.4: Verification of the conclusion of Lemma 3.22 for $n = 2$, in this case both the right and left hand sides of the equality $\tau(\tau(^2\Gamma_2, {}^2\Gamma_0), \tau(^2\Gamma_1, {}^2\Gamma_0)) = \tau(^2\Gamma_2, {}^2\Gamma_1) \cap \tau(^2\Gamma_1, {}^2\Gamma_0)$ consist of the origin.

Next, recall

$$T_{n,k} = \mathbb{R}^{k+1} \times {}^{n-k-1}\Gamma_{n-k-1} \qquad T_{j,k} = \mathbb{R}^{k+1} \times {}^{n-k-1}\Gamma_{j-k-1}$$

Hence by the preceding corollary, we have

$$\tau(T_{n,k}, T_{j,k}) = T_{j,k} \cap \{h_{n,j} = 0\}$$

Thus the left hand side is given by ${}^n\Gamma_j \cap T_{j,k} \cap T_{n,j}$.

On the other hand, by the preceding corollary, the right hand side is also given by ${}^n\Gamma_j \cap T_{n,j} \cap T_{j,k}$. $\square$

3.4.1. More on distinguished quadrants.

Corollary 3.23. *For $0 \leq j < i \leq n$, we have*

$${}^n\Gamma_i^\varepsilon \cap {}^n\Gamma_j^\varepsilon = T({}^n\Gamma_i^\varepsilon, {}^n\Gamma_j^\varepsilon) = \tau({}^n\Gamma_i^\varepsilon, {}^n\Gamma_j^\varepsilon)$$

and they coincide with the closed boundary face of ${}^n\Gamma_i^\varepsilon$ cut out by $h_{i,j} = 0$.

Proof. For $j = 0$, we have ${}^n\Gamma_0^\varepsilon = {}^n\Gamma_0 = \{x_0 = 0\}$. From the definitions, we have

$${}^n\Gamma_i^\varepsilon \cap {}^n\Gamma_0 = T({}^n\Gamma_i^\varepsilon, {}^n\Gamma_0) = \tau({}^n\Gamma_i^\varepsilon, {}^n\Gamma_0)$$

which is cut out of ${}^nP_i^\varepsilon$ by $h_{i,0} = h_i = 0$.

For $j > 0$, the assertions follow from Lemma 2.4 by induction on n. $\square$

Remark 3.24. Note for any $0 \leq j < i \leq n$, we have

$$\tau({}^n\Gamma_i, {}^n\Gamma_j) = \bigcup_\varepsilon \tau({}^n\Gamma_i^\varepsilon, {}^n\Gamma_j^\varepsilon)$$

To see this, consider $x \in \tau({}^n\Gamma_i, {}^n\Gamma_j)$, so that $h_{i,j}(x) = 0$ by Corollary 3.21. Choose ε so that $x \in {}^n\Gamma_i^\varepsilon$. Then by Corollary 3.23, we have $x \in \tau({}^n\Gamma_i^\varepsilon, {}^n\Gamma_j^\varepsilon)$.

For $i = 0$, let ${}^nL_0^\varepsilon = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, \ldots, n$, consider the conormal bundles

$${}^nL_i^\varepsilon = T^*_{{}^{n-1}\Gamma_{i-1}^\varepsilon}\mathbb{R}^n \subset T^*\mathbb{R}^n$$

and their union

$${}^nL^\varepsilon = \bigcup_{i=0}^n {}^nL_i^\varepsilon$$

Similarly, for $i = 0, \ldots, n$, consider the smooth Legendrian

$${}^n\Lambda_i^\varepsilon \subset J^1\mathbb{R}^n$$

that maps diffeomorphically to ${}^n\Gamma_i^\varepsilon \subset \mathbb{R}^{n+1}$ under the front projection $\pi : J^1\mathbb{R}^n \to \mathbb{R}^{n+1}$, and their union

$${}^n\Lambda^\varepsilon = \bigcup_{i=0}^n {}^n\Lambda_i^\varepsilon$$

Note the contactomorphism of Lemma 2.1 takes ${}^n\Lambda_i^\varepsilon \subset J^1\mathbb{R}^n$ isomorphically to $\{0\} \times {}^nL_i^\varepsilon \subset \{0\} \times T^*\mathbb{R}^n$, and thus ${}^n\Lambda^\varepsilon \subset J^1\mathbb{R}^n$ isomorphically to $\{0\} \times {}^nL^\varepsilon \subset \{0\} \times T^*\mathbb{R}^n$.

We have the following topological consequence of Lemma 2.4.

Corollary 3.25. *As a union of smooth manifolds with corners, ${}^n\Gamma^\varepsilon \subset \mathbb{R}^{n+1}$ is given by the gluing*

$${}^n\Gamma^\varepsilon = ({}^{n-1}\Gamma^{\varepsilon'} \times \mathbb{R}_{\geq 0}) \coprod_{({}^{n-1}\Gamma^{\varepsilon'} \times \{0\})} (\mathbb{R}^n \times \{0\})$$

where $\varepsilon' = (\varepsilon_0\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. The front projection takes ${}^nL^\varepsilon \subset J^1\mathbb{R}^n$ homeomorphically to ${}^n\Gamma^\varepsilon \subset \mathbb{R}^{n+1}$.

Before continuing, let us record the following for future use.

Corollary 3.26. *For $0 < j < i \leq n$, the closure of the codimension one clean intersection of ${}^nL_i^\varepsilon, {}^nL_j$ is precisely ${}^nL_i^\varepsilon \cap {}^nL_j^\varepsilon$.*

Proof. The closure of the codimension one clean intersection of ${}^nL_i^\varepsilon, {}^n\Lambda_j$ is conic and projects to the primary tangency of ${}^{n-1}\Gamma_{i-1}^\varepsilon, {}^{n-1}\Gamma_{j-1}$. By Corollary 3.21, the primary tangency of ${}^{n-1}\Gamma_{i-1}, {}^{n-1}\Gamma_{j-1}$ is cut out by $h_{i-1,j-1} = 0$. By Corollary 3.23, this is precisely the tangency $T({}^{n-1}\Gamma_{i-1}^\varepsilon, {}^{n-1}\Gamma_{j-1})$ and hence lifts precisely to the conic intersection ${}^nL_i^\varepsilon \cap {}^nL_j^\varepsilon$. $\square$

3.5. The case of $\mathscr{A}_{n+1}$-tree

The following Theorem 3.27 will play a key role in proving Proposition 3.6.

Theorem 3.27. *Let $\varphi : T^*\mathbb{R}^n \to J^1\mathbb{R}^n$ be an embedding as a Weinstein hypersurface. Assume that the image of ${}^nL$ under φ is transverse to the fibers of the projection $J^1\mathbb{R}^n \to \mathbb{R}^n$. Let $\Upsilon = \pi(\varphi({}^nL)) \subset \mathbb{R} \times \mathbb{R}^n$ be (the germ of) the front at the central point.*

Then there exists a diffeomorphism $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ taking Υ to the germ at the origin of ${}^n\Gamma \subset \mathbb{R} \times \mathbb{R}^n$.

The proof of Theorem 3.27 will proceed by induction on the dimension n. At each stage, we will prove the fully parametric version:

Theorem 3.28. *Let $\varphi^y : T^*\mathbb{R}^n \to J^1\mathbb{R}^n$ be a family of Weinstein hypersurface embeddings parametrized by a manifold Y. Assume that the image of ${}^nL$ under φ^y is transverse to the fibers of the projection $J^1\mathbb{R}^n \to \mathbb{R}^n$. Let $\Upsilon^y = \pi(\varphi^y({}^nL)) \subset \mathbb{R} \times \mathbb{R}^n$ be (the germs of) the fronts at the central points.*

Then there exists a family of diffeomorphisms $\psi^y : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ taking Υ^y to the germ at the origin of ${}^n\Gamma \subset \mathbb{R} \times \mathbb{R}^n$. If $\varphi^y = Id$ for $y \in \mathcal{O}p\,(K)$, where $K \subset Y$ is a closed subset, then we may assume $\psi^y = Id$ for $y \in \mathcal{O}p\,(K)$.

As usual the case of general pairs (Y, K) follows from the case $Y = D^k$ and $K = S^{k-1}$. To simplify notation we set ${}^n\Phi = \varphi({}^nL)$, so that $\Upsilon = \pi({}^n\Phi)$ and similarly with parameter superscripts. We also denote ${}^n\Phi_n = \varphi({}^nL_n)$ and $\Upsilon_n = \pi({}^n\Phi_n)$.

3.5.1. Base case $n = 0$. The k-parametric version states: the germ of any graphical hypersurface $\Upsilon \subset \mathbb{R} \times \mathbb{R}^k$ is diffeomorphic to the germ of the zero-graph ${}^0\Gamma \times \mathbb{R}^k = \{0\} \times \mathbb{R}^k$. This can be achieved by an isotopy generated by a time-dependent vector field of the form $h_t\partial_{x_0}$. This vector field is zero at infinity if Υ is standard at infinity.

3.5.2. Case $n = 1$. The next case of the induction $n = 1$ is elementary but slightly different from the others, so it is more convenient to treat separately.

With the setup of the theorem, consider the front $\Upsilon = \pi({}^1\Phi) \subset \mathbb{R}^2$, and assume without loss of generality that the origin is the central point. By induction, we may assume, the front takes the form $\Upsilon = \Gamma_0 \cup \Upsilon_1 \subset \mathbb{R}^2$ where $\Gamma_0 = \{x_0 = 0\}$. Near the origin, the intersection $\Gamma_0 \cap \Upsilon_1$ and tangency locus $T(\Gamma_0, \Upsilon_1)$ coincide and consist of the origin alone. Moreover, by construction, the origin is a simple tangency, and so $\Upsilon_1 = \{x_0 = \alpha x_1^2\}$ with $\alpha(0) \neq 0$. Now it is elementary to find a time-dependent vector field of the form $h_t x_0 \partial_{x_0}$, hence vanishing on Γ_0, generating an isotopy taking Υ_1 to either $\Gamma_1 = \{x_0 = x_1^2\}$ or $-\Gamma_1 = \{x_0 = -x_1^2\}$. In the former case, we are done; in the latter case, we may apply the diffeomorphism $(x_0, x_1) \mapsto (-x_0, x_1)$ to arrive at the configuration $\Gamma_0 \cup \Gamma_1$. Finally, it is evident the prior constructions can be performed parametrically, with the vector field zero at infinity if Υ is standard at infinity.

3.5.3. Inductive step. The inductive step takes the following form. Suppose the fully parametric assertion has been established for dimension $n-1$. Starting from ${}^n\Phi \subset T^*\mathbb{R}^n$, remove the last smooth piece to obtain ${}^n\Phi' = {}^n\Phi \setminus {}^n\Phi_n$, and consider the corresponding front $\Upsilon' = \pi({}^n\Phi')$. Note that ${}^n\Phi' = {}^{n-1}\Phi \times \mathbb{R} \subset T^*(\mathbb{R}^{n-1} \times \mathbb{R})$, and so by an inductive application of the 1-parametric version of the theorem, we may assume

$$\Upsilon' = {}^{n-1}\Gamma \times \mathbb{R}$$

We will find a diffeomorphism $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that preserves Υ' (as a subset, not pointwise), and takes Υ_n to ${}^n\Gamma_n$. Moreover, it will be evident the diffeomorphism can be constructed in parametric form, including the relative parametric form. This will complete the inductive step and prove the theorem.

3.5.4. Two propositions. The proof of the inductive step is based on the following 2 propositions.

Proposition 3.29. *Fix $n \geq 2$.*

With the setup of Theorem 3.27, suppose $\Upsilon = \bigcup_{i=0}^{n-1} {}^n\Gamma_i \cup \Upsilon_n$ where we recall $\Upsilon_n = \pi({}^n\Phi_n)$. Suppose in addition Υ_n has primary tangency loci satisfying

$$\tau(\Upsilon_n, {}^n\Gamma_i) \supset \tau({}^n\Gamma_n, {}^n\Gamma_i) \qquad i = 0, \ldots, n-1$$

Then $\Upsilon_n = \{x_0 = \alpha h_n^2\}$ *where*

$$\alpha = 1 + \beta \prod_{j=1}^{n-1} h_{n,j}^2 = 1 + \beta h_{n,1}^2 \cdots h_{n,n-1}^2$$

Moreover, the same holds in parametric form.

Proof. We have $\Upsilon_n = \{x_0 = g\}$ for some g. Since $\tau(\Upsilon_n, {}^n\Gamma_0) \supset \tau({}^n\Gamma_n, {}^n\Gamma_0) = \{h_n = 0\}$, we must have g is divisible by h_n^2, hence $g = \alpha h_n^2$, for some α. Next, for any $j \neq 0, n$, by Lemma 3.17, $\tau({}^n\Gamma_n, {}^n\Gamma_j)$ is cut out by $h_{n,j} = 0$. Since $\tau(\Upsilon_n, {}^n\Gamma_j) \supset \tau({}^n\Gamma_n, {}^n\Gamma_j)$, and $h_n \neq 0$ along a dense subset of $\{h_{n,j} = 0\}$, taking the ratio g/h_n^2 shows that we must have $\alpha = 1 + \delta$, where δ is divisible by $h_{n,j}^2$. Repeating this argument, and using the transversality of the level-sets of the collection $h_{n,j}$, we conclude that $\delta = \beta h_{n,1}^2 \cdots h_{n,n-1}^2$. □

Proposition 3.30. *Fix* $n \geq 2$.

With the setup of Theorem 3.27, suppose $\Upsilon = \bigcup_{i=0}^{n-1} {}^n\Gamma_i \cup \Upsilon_n$ *where we recall* $\Upsilon_n = \pi({}^n\Phi_n)$. *Suppose in addition* $\Upsilon_n = \{x_0 = \alpha h_n^2\}$ *where*

$$\alpha = 1 + \beta \prod_{j=1}^{n-1} h_{n,j}^2 = 1 + \beta h_{n,1}^2 \cdots h_{n,n-1}^2$$

Consider the family $\Upsilon_{n,t} = \{x_0 = (1 - t + t\alpha)h_n^2\}$ *so that* $\Upsilon_{n,0} = {}^n\Gamma_n$, $\Upsilon_{n,1} = \Upsilon_n$.

Then there exist functions $g_t : \mathbb{R}^{n+1} \to \mathbb{R}$ *such that the vector fields*

$$g_t v_{n-1} = g_t \textstyle\sum_{i=0}^{n-1} x_i \frac{1}{2^i} \partial_{x_i} = g_t x_0 \partial_{x_0} + \frac{1}{2} g_t x_1 \partial_{x_1} + \cdots + \frac{1}{2^{n-1}} g_t x_{n-1} \partial_{x_{n-1}}$$

generate an isotopy $\varphi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *such that* $\varphi_t(\Upsilon_{n,0}) = \Upsilon_{n,t}$.

In addition, the functions h_t, *hence vector fields* $h_t v_{n-1}$, *are divisible by the product* $\prod_{j=1}^{n-1} h_{n,j}$.

Moreover, all of the above holds in parametric form.

The following lemmas are needed for the proof of Proposition 3.30.

Lemma 3.31. *For all* $0 \leq i \leq n$, *the vector field*

$$v_i = \textstyle\sum_{j=0}^{n} x_j \frac{1}{2^j} \partial_{x_j} = x_0 \partial_{x_0} + \frac{1}{2} x_1 \partial_{x_1} + \cdots + \frac{1}{2^i} x_i \partial_{x_i}$$

preserves each ${}^n\Gamma_j \subset \mathbb{R}^{n+1}$, *for* $j = 0, \ldots, i$.

Proof. Since ${}^n\Gamma_j \subset \mathbb{R}^{n+1}$ is independent of $x_{j+1}, \dots, x_n$, it suffices to prove the case $i = j = n$. Recall ${}^n\Gamma_n$ is the zero-locus of $f = x_0 - h_n^2$. We will show $v(h_n) = \frac{1}{2}h_n$ and so $v(f) = f$. Recall $h_n = h_{n,0} = x_1 - h_{n,1}^2$, and in general $h_{n,j} = x_{j+1} - h_{n,j+1}^2$ with $h_{n,n-1} = x_n$. Thus $v_n(h_{n,n-1}) = \frac{1}{2^n}h_{n,n-1}$, and by induction, $v(h_{n,j}) = \frac{1}{2^{j+1}}h_{n,j}$, so in particular $v(h_{n,0}) = v(h_n) = \frac{1}{2}h_n$. □

Remark 3.32. In the context of the inductive step outlined above, we will use Lemma 3.31 in particular the vector field

$$v_{n-1} = \textstyle\sum_{i=0}^{n-1} x_i \frac{1}{2^i}\partial_{x_i} = x_0\partial_{x_0} + \frac{1}{2}x_1\partial_{x_1} + \cdots + \frac{1}{2^{n-1}}x_{n-1}\partial_{x_{n-1}}$$

to move Υ_n to ${}^n\Gamma_n$. The lemma confirms we will preserve $\Upsilon' = {}^{n-1}\Gamma \times \mathbb{R} = \bigcup_{i=0}^{n-1} {}^n\Gamma_i$.

Lemma 3.33. *For any $0 \le j < i \le n$, and $1 \le k \le i$, we have*

$$\frac{\partial h_i^2}{\partial x_k} = -(-2)^k \prod_{j=0}^{k-1} h_{i,j} = -(-2)^k h_{i,0}h_{i,1}\cdots h_{i,k-1}$$

Proof. Recall $h_i = h_{i,0}$ and the inductive formulas $h_{i,j} = x_{j+1} - h_{i,j+1}^2$ with $h_{i,i-1} = x_i$. Thus we have

$$\frac{\partial h_{i,j}^2}{\partial x_{j+1}} = 2h_{i,j} \qquad \frac{\partial h_{i,j}^2}{\partial x_k} = -2h_{i,j}\frac{\partial h_{i,j+1}^2}{\partial x_k} \quad k > j+1$$

and the assertion follows. □

Proof of Proposition 3.30. Suppose $\Upsilon = \bigcup_{i=0}^{n-1} {}^n\Gamma_i \cup \Upsilon_n$ where Υ_n is the graph of

$$H_\beta = (1 + \beta \prod_{j=1}^{n-1} h_{n,j}^2)h_n^2 = (1 + \beta h_{n,1}^2 \cdots h_{n,n-1}^2)h_n^2$$

Our aim is to find a normalizing isotopy, generated by a time-dependent vector field v_t, taking the graph $\Upsilon_n = \{x_0 = H_\beta\}$ to the standard graph ${}^n\Gamma_n = \{x_0 = h_n^2\}$, i.e. to the graph where $\beta = 0$, while preserving $\bigcup_{i=0}^{n-1} {}^n\Gamma_i$. Thus for any infinitesimal deformation in the class of functions H_β, we seek a vector field v realizing the deformation and preserving the functions

$h_0, \dots, h_{n-1}$, i.e. we seek to solve the system

$$\begin{aligned} &\dot{h}_i = 0, \quad i = 0, \dots, n-1 \\ (2) \qquad &\dot{H}_\beta = \gamma \prod_{j=0}^{n-1} h_{n,j}^2 = \gamma h_{n,0}^2 \cdots h_{n,n-1}^2 \end{aligned}$$

where $\dot{H}_\beta$ denotes the derivative of H_β with respect to v, and γ is any given smooth function.

Let $\Phi_\beta \subset T^*\mathbb{R}^{n+1}$ denote the conormal to the graph of H_β. Any vector field $v = \sum_{j=0}^n v_j \partial/\partial_{x_j}$ on $\mathbb{R}^{n+1}$ extends to a Hamiltonian vector field v_H on $T^*\mathbb{R}^{n+1}$ with Hamiltonian $H = \sum_{j=0}^n p_j v_j$. We will find v deforming the graph of H_β by finding H so that v_H deforms the conormal to the graph Φ_β.

In general, for a function $f : \mathbb{R}^n \to \mathbb{R}$, with graph $\Gamma_f = \{x_0 = f\} \subset \mathbb{R}^{n+1}$, denote the conormal to the graph by $T^*_{\Gamma_f} \subset T^*\mathbb{R}^{n+1}$. With respect to the contact form $p_1 dx_1 + \dots p_n dx_n - x_0 dp_0$, the conormal $T^*_{\Gamma_f}$ is given by the generating function $F(x_1, \dots, x_n) = -p_0 f(x_1, \dots, x_n)$, i.e. it is cut out by the equations

$$\begin{aligned} p_i &= -p_0 \frac{\partial f}{\partial x_i}, \quad i = 1, \dots, n \\ x_0 &= f(x_1, \dots, x_n) \end{aligned}$$

Hence given a Hamiltonian $H = \sum_{j=0}^n p_j v_j$, its restriction to the conormal $T^*_{\Gamma_f}$ is given by

$$H|_{T^*_{\Gamma_f}} = p_0 v_0|_{x_0=f} - p_0 \sum_{j=1}^n \frac{\partial f}{\partial x_j} v_j|_{x_0=f}$$

and so further restricting to $p_0 = 1$, we find the Hamilton-Jacobi equation

$$H|_{T^*_{\Gamma_f} \cap \{p_0=1\}} = v_0|_{x_0=f} - \sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i|_{x_0=f} = v_0|_{x_0=f} - \dot{f}$$

Let us apply the above to H_β and h_i, for $i = 0, \dots, n-1$. It allows us to transform system (2) into the system

$$
\begin{aligned}
&v_0(x_1, \dots, x_n, h_i) - \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} v_j = 0, \quad i = 0, \dots, n-1 \\
(3) \qquad &v_0(x_1, \dots, x_n, H_\beta) - \sum_{j=1}^{n} \frac{\partial H_\beta}{\partial x_j} v_j = \gamma \prod_{j=0}^{n-1} h_{n,j}^2
\end{aligned}
$$

Note we can reformulate Lemma 3.31 from this viewpoint: when $\beta = \gamma = 0$, given any function $h = h(x_1, \dots, x_n)$, the functions

$$
(4) \qquad v_0 = x_0 h, v_1 = \frac{x_1}{2} h, v_2 = \frac{x_2}{4} h, \dots, v_n = \frac{x_n}{2^n} h
$$

satisfy system (3).

Now let us choose $v_0, v_1, \dots v_{n-1}$ as in (4) but set $v_n = 0$. This will satisfy the first n equations of system (3), independently of β, γ. From hereon, we will restrict to this class of vector fields and focus on the last equation of system (3).

Let us first set $\beta = 0$, so that $H_\beta = h_n^2$, and solve system (3) in this case. Using Lemma 3.33, we can then rewrite the left-hand side of the last equation of system (3) in the form

$$
\begin{aligned}
v_0(x_1, \dots, x_n, h_n^2) - \sum_{j=1}^{n-1} \frac{\partial h_n^2}{\partial x_j} v_j &= h \left(h_n^2 - \sum_{j=1}^{n-1} \frac{\partial h_n^2}{\partial x_j} \frac{x_j}{2^j} \right) \\
&= h \left(h_n^2 + \sum_{j=1}^{n-1} (-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} \right)
\end{aligned}
$$

Here we recall the notation $h_{i,j} = h_{i-j}(x_{j+1}, \dots, x_i)$, so that using the relations $h_n = h_{n,0}$, $h_{n,k} - x_{k+1} = -h_{n,k+1}^2$ we have

$$
\frac{\partial h_n}{\partial x_j} = 2^{j-1} (-1)^{j-1} \prod_{k=1}^{j-1} h_{n,k}.
$$

Further, we can inductively simplify the term in parentheses

$$\begin{aligned}
h_n^2 + \sum_{j=1}^{n-1}(-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} &= h_n \left(h_n - x_1 + \sum_{j=2}^{n-1}(-1)^j x_j \prod_{k=1}^{j-1} h_{n,k} \right) \\
&= h_n \left(-h_{n,1}^2 + \sum_{j=2}^{n-1}(-1)^j x_j \prod_{k=1}^{j-1} h_{n,k} \right) \\
&= h_n h_{n,1} \left(-h_{n,1} + x_2 + \sum_{j=3}^{n-1}(-1)^j x_j \prod_{k=2}^{j-1} h_{n,k} \right) \\
&\dots \\
&= (-1)^{n-1} h_n h_{n,1} h_{n,2} \cdots h_{n,n-1} = (-1)^{n-1} \prod_{j=0}^{n-1} h_{n,j}
\end{aligned}$$

Thus for $\beta = 0$, the last equation of system (3) reduces to

$$(-1)^{n-1} h \prod_{j=0}^{n-1} h_{n,j} = \gamma \prod_{j=0}^{n-1} h_{n,j}^2$$

and hence can be solved by

$$h = (-1)^{n-1} \gamma \prod_{j=0}^{n-1} h_{n,j}$$

Now for general β, we will similarly calculate the left-hand side of the last equation of system (3). To simplify the formulas, set

$$F = \prod_{j=0}^{n-1} h_{n,j} \qquad \theta = \beta F^2$$

Thus we have $H_\beta = (1+\theta)h_n^2$, and our prior calculation showed when $\beta = 0$, the last equation of system (3) took the form

$$(-1)^{n-1} hF = \gamma F^2$$

so was solved by $h = (-1)^{n-1}\gamma F$.

For general β, we just need to consider the extra term obtained from the θ part of the factor $(1+\theta)$ which multiplies h_n^2. It therefore follows formally from the previous equation that, after factoring out the function h to be solved for, the left-hand side of the last equation of system (3) takes the form

$$(-1)^{n-1}(1+\theta)F - h_n^2 \sum_{j=1}^{n-1} \frac{1}{2^j}\frac{\partial\theta}{\partial x_j}x_j$$

Thus the equation itself takes the form

$$\left((-1)^{n-1}(1+\theta)F - h_n^2 \sum_{j=1}^{n-1} \frac{1}{2^j}\frac{\partial\theta}{\partial x_j}x_j\right) h = \gamma F^2 \tag{5}$$

Since $\theta = \beta F^2$, we have

$$\frac{\partial\theta}{\partial x_j} = F^2\frac{\partial\beta}{\partial q_j} + \beta\frac{\partial F^2}{\partial q_j} = F^2\frac{\partial\beta}{\partial q_j} + 2F\beta\frac{\partial F}{\partial q_j}$$

and hence $\frac{\partial\theta}{\partial x_j}$ is divisible by F. Thus we can divide equation (5) by F, and after renaming γ, write equation (5) in the form

$$(1+O(x))h = \gamma F$$

where $O(x)$ vanishes at the origin. We conclude we can solve the equation by $h = (1+O(x))^{-1}\gamma F$.

This completes the proof of Proposition 3.30. □

3.5.5. Proof of Theorem 3.27. In this section, we use Propositions 3.29 and Proposition 3.30 to complete the inductive step outlined in 3.5.3, and thus, complete the proof of Theorem 3.27. Let us assume $n \geq 2$. Recall the notation ${}^n\Phi = \varphi({}^nL)$, ${}^n\Phi_n = \varphi({}^nL_n)$, $\Upsilon = \pi({}^n\Phi)$ and $\Upsilon_n = \pi({}^n\Phi_n)$.

Then $\Upsilon = \Upsilon' \cup \Upsilon_n$ where $\Upsilon' = \bigcup_{i=0}^{n-1} {}^n\Gamma_i$ is already standard. We will implement the following strategy. Suppose for some $0 < k \leq n-1$, we have moved Υ_n, while preserving Υ', so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) \supset \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j > k$$

Then using Proposition 3.29 and Proposition 3.30, or alternatively, the cases $n = 0, 1$ when respectively $k = n-1, n-2$, we will move Υ_n, while preserving Υ', so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) \supset \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j \geq k$$

Proceeding in this way, we will arrive at $k = 0$, where all primary tangencies have been normalized. Then a final application of Proposition 3.29 and Proposition 3.30 will complete the proof.

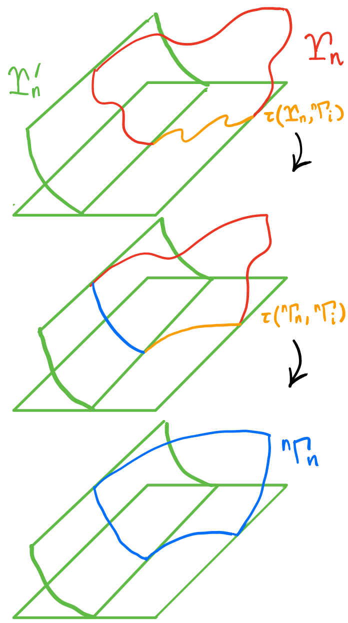

Figure 3.5: The strategy of the proof: inductively normalize tangencies.

To pursue this argument, we need the following control over primary tangencies.

Lemma 3.34. *Fix* $0 \leq k < j \leq n-1$.
We have

$$\tau(\tau(\Upsilon_n, {}^n\Gamma_k), \tau({}^n\Gamma_j, {}^n\Gamma_k)) \supset \tau(\Upsilon_n, {}^n\Gamma_j) \cap \tau({}^n\Gamma_j, {}^n\Gamma_k)$$

Moreover, when $k = n-2$, *the tangency of* $\tau(\Upsilon_n, {}^n\Gamma_{n-2})$ *and* $\tau({}^n\Gamma_{n-1}, {}^n\Gamma_{n-2})$ *is nondegenerate.*

Proof. We will assume $k > 0$ and leave the case $k = 0$ as an exercise.

Fix a point

$$y \in \tau(\Upsilon_n, {}^n\Gamma_j) \cap \tau({}^n\Gamma_j, {}^n\Gamma_k)$$

In particular $y \in \Upsilon_n$ and so $y = \pi(\tilde{y})$ for some $\tilde{y} \in {}^n\Phi_n$. Recall ${}^n\Lambda_n = \bigcup_\varepsilon {}^n\Lambda_n^\varepsilon$. Hence after applying φ we may also write ${}^n\Phi_n = \bigcup_\varepsilon {}^n\Phi_n^\varepsilon$ and so $\tilde{y} \in {}^n\Phi_n^\varepsilon$, for some ε.

Note $y \in \tau(\Upsilon_n, {}^n\Gamma_j)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of ${}^n\Phi_n, {}^n\Lambda_j$.

By applying φ to Corollary 3.26, this locus intersects ${}^n\Phi_n^\varepsilon$ precisely along ${}^n\Phi_n^\varepsilon \cap {}^n\Lambda_j^\varepsilon$ and so $\tilde{y} \in {}^n\Lambda_j^\varepsilon$.

Similarly, note $y \in \tau({}^n\Gamma_j, {}^n\Gamma_k)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of ${}^n\Lambda_j, {}^n\Lambda_k$. By Corollary 3.26, this locus intersects ${}^n\Lambda_j^\varepsilon$ precisely along ${}^n\Lambda_j^\varepsilon \cap {}^n\Lambda_k^\varepsilon$ and so $\tilde{y} \in {}^n\Lambda_k^\varepsilon$.

Thus altogether $\tilde{y} \in {}^n\Phi_n^\varepsilon \cap {}^n\Lambda_j^\varepsilon \cap {}^n\Lambda_k^\varepsilon = ({}^n\Phi_n^\varepsilon \cap {}^n\Lambda_k^\varepsilon) \cap ({}^n\Lambda_j^\varepsilon \cap {}^n\Lambda_k^\varepsilon)$.

By Corollary 3.26, the intersections ${}^n\Phi_n^\varepsilon \cap {}^n\Lambda_k^\varepsilon$ and ${}^n\Lambda_j^\varepsilon \cap {}^n\Lambda_k^\varepsilon$ are closures of clean codimension one intersections, hence their projections lie in the primary tangencies $\tau(\Upsilon_n, {}^n\Gamma_k)$ and $\tau({}^n\Gamma_j, {}^n\Gamma_k)$ (for the first intersection one applies φ to the conclusion of Corollary 3.26). Moreover, ${}^n\Phi_n^\varepsilon \cap {}^n\Lambda_k^\varepsilon$ and ${}^n\Lambda_j^\varepsilon \cap {}^n\Lambda_k^\varepsilon$ intersect along their primary tangency. Since π restricted to ${}^n\Lambda_k$ has no critical points, the projection of this primary tangency is again a primary tangency. Hence $y \in \tau(\tau(\Upsilon_n, {}^n\Gamma_k), \tau({}^n\Gamma_j, {}^n\Gamma_k))$, proving the asserted containment.

We leave the nondegeneracy of the case $k = n - 2$ to the reader. □

Now we are ready to inductively normalize the primary tangencies.

Lemma 3.35. *Fix* $0 \leq k < n - 1$.

Suppose

$$\tau(\Upsilon_n, {}^n\Gamma_j) = \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j > k$$

Then there exists a diffeomorphism $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *preserving* $\Upsilon' = \bigcup_{i=0}^{n-1} {}^n\Gamma_i$ *such that*

$$\tau(\psi(\Upsilon_n), {}^n\Gamma_j) = \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j \geq k$$

Moreover, when $k \neq n - 2$*, the diffeomorphism is an isotopy.*

Proof. We will assume $k < n - 3$. We leave the elementary cases $k = n - 2, n - 3$ to the reader. They can be deduced from the parametric versions of the cases $n = 0, 1$ presented in 3.5.1, 3.5.2 respectively.

Throughout what follows, we use the projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ to identify ${}^n\Gamma_k = \mathbb{R}^n$.

On the one hand, we have

$$\tau({}^n\Gamma_j, {}^n\Gamma_k) = \mathbb{R}^k \times {}^{n-k-1}\Gamma_{j-k-1} \qquad k < j < n$$

On the other hand, by Lemma 3.34 and assumption, we have

$$\begin{aligned}\tau(\tau(\Upsilon_n, {}^n\Gamma_k), \tau({}^n\Gamma_j, {}^n\Gamma_k)) &= \tau(\Upsilon_n, {}^n\Gamma_j) \cap {}^n\Gamma_k \\ &= \tau({}^n\Gamma_n, {}^n\Gamma_j) \cap {}^n\Gamma_k \qquad k < j < n\end{aligned}$$

Hence within ${}^n\Gamma_k = \mathbb{R}^n$, the loci $\tau(\Upsilon_n, {}^n\Gamma_k)$ and $\tau({}^n\Gamma_n, {}^n\Gamma_k)$ have the same tangencies with

$$\tau({}^n\Gamma_j, {}^n\Gamma_k) = \mathbb{R}^k \times {}^{n-k-1}\Gamma_{j-k-1} \qquad k < j < n$$

Thus Proposition 3.29 and Proposition 3.30 provide a time-dependent vector field of the form

$$v_t = h_t \sum_{i=k+1}^{n-1} \tfrac{1}{2^i} x_i \partial_{x_i}$$

generating an isotopy $\varphi : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ satisfying

$$\varphi(\tau(\Upsilon_n, {}^n\Gamma_k)) = \tau({}^n\Gamma_n, {}^n\Gamma_k)$$

In addition, the function h_t, hence vector field v_t, is divisible by the product $\prod_{j=k+1}^{n-1} h_{n,j}$, and thus φ preserves its zero-locus.

Let us complete v_t to the vector field

$$V_t = h_t \sum_{i=0}^{n-1} \tfrac{1}{2^i} x_i \partial_{x_i}$$

and consider the isotopy $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ generated by V_t.

Then ψ satisfies

$$\psi(\tau(\Upsilon_n, {}^n\Gamma_k)) = \tau({}^n\Gamma_n, {}^n\Gamma_k)$$

It also preserves ${}^n\Gamma_i$, for $0 \le i \le n-1$, as well as $\tau(\Upsilon_n, {}^n\Gamma_j) = \tau({}^n\Gamma_n, {}^n\Gamma_j)$, for $j > k$. In addition, it preserves

$$\tau({}^n\Gamma_j, {}^n\Gamma_k) = \mathbb{R}^k \times {}^{n-k-1}\Gamma_{j-k-1} \qquad k < j < n$$

since this is the zero-locus of $h_{n,j}$. $\square$

Finally, let us use the lemma to complete the inductive step of the proof of Theorem 3.27 as outlined above. Suppose for some $0 < k \leq n-1$, we have moved Υ_n, while preserving Υ', so that we have the sought-after primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) = \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j > k$$

Then using Lemma 3.35, we can move Υ_n, while preserving Υ', so that we have the sought-after primary tangencies

$$\tau(\Upsilon_n, {}^n\Gamma_j) = \tau({}^n\Gamma_n, {}^n\Gamma_j) \qquad j \geq k$$

Proceeding in this way, we arrive at $k = 0$, where all primary tangencies have been normalized. Now a final application of Proposition 3.29 and Proposition 3.30 move Υ_n to ${}^n\Gamma_n$, while preserving Υ', and thus complete the proof of Theorem 3.27.

3.6. Conclusion of the proof

We are now ready to prove Proposition 3.6. As a consequence we establish Theorem 3.5, and since all the above also holds parametrically this also establishes the parametric version Theorem 3.12.

Proof of Proposition 3.6. Take any point λ in the front $H := \pi(\Lambda)$ and let $\pi^{-1}(\lambda) = \{\lambda_1, \ldots, \lambda_k\}$. Let $\Lambda_1, \ldots, \Lambda_k$ be germs of Λ at these points of arboreal types $(\mathscr{T}_j, n)$, $n(\mathscr{T}_j) = n_j$. We need to show that the germ of the front H at λ is diffeomorphic to the germ of a model front $H_{\mathscr{T}}$, where $\mathscr{T}$ is a signed rooted tree obtained from $\bigsqcup T_j$ by adding the root ρ and adjoining it to the roots ρ_j of the trees T_j by edges $[\rho\rho_j]$. The signs of all edges of the trees T_j are preserved, while previously unsigned edges $\rho_j\alpha$ get a sign $\varepsilon(\nu, L, \alpha)$, see (1).

We proceed by induction on the number of vertices in the signed rooted tree $\mathscr{T} = (T, \rho, \varepsilon)$.

The base case of a $(\mathscr{A}_1, m)$-front $H \subset \mathbb{R}^m$ is the same geometry as appearing in 3.5.1: any graphical hypersurface $H \subset \mathbb{R} \times \mathbb{R}^{m-1}$ is isotopic to the germ of the zero-graph $\{0\} \times \mathbb{R}^{m-1}$.

For the inductive step, fix a rooted tree $\mathscr{T} = (T, \rho, \varepsilon)$, and as usual set $n = |n(\mathscr{T})|$. Consider a $(\mathscr{T}, m)$-front $H \subset \mathbb{R}^m$, with by necessity $m \geq n$.

Fix a leaf vertex $\beta \in \ell(\mathscr{T})$, which always exists as long as $\mathscr{T} \neq \mathscr{A}_1$. Consider the smaller signed rooted tree $\mathscr{T}' = \mathscr{T} \setminus \beta$, and the corresponding $(\mathscr{T}', m)$-front $H' = H \setminus \mathring{H}[\beta] \subset \mathbb{R}^m$, where $\mathring{H}[\beta] \subset H$ is the interior of the smooth piece indexed by β. By induction, we may assume

$$H' = H_{\mathscr{T}'} \times \mathbb{R}^{m-n+1} \subset \mathbb{R}^m$$

Thus it remains to normalize the smooth piece $H[\beta]$.

Let $\mathscr{A}_\beta = (A_\beta, \rho, \varepsilon_\beta)$ be the linear signed rooted subtree of $\mathscr{T} = (T, \rho, \varepsilon)$ with vertices $v(A_\beta) = \{\alpha \in v(T) \,|\, \alpha \leq \beta\}$. Set $d = v(\mathscr{T}) \setminus v(\mathscr{A}_\beta) = n(\mathscr{T}) \setminus n(\mathscr{A}_\beta)$ to be the complementary vertices.

Consider the $(\mathscr{A}_\beta, m)$-front $K \subset H$ given by the union $K = \bigcup_{\alpha \in n(\mathscr{A}_\beta)} K[\alpha]$ of the smooth pieces of $H \subset \mathbb{R}^m$ indexed by $\alpha \in n(\mathscr{A}_\beta)$. Note for $\mathscr{A}'_\beta = \mathscr{A}_\beta \cap \mathscr{T}'$, and $K' = K \cap H'$, we already have

$$K' = H_{\mathscr{A}'_\beta} \times \mathbb{R}^{m-n+1+d} \subset \mathbb{R}^m$$

and seek to normalize the smooth piece $K[\beta] = H[\beta]$.

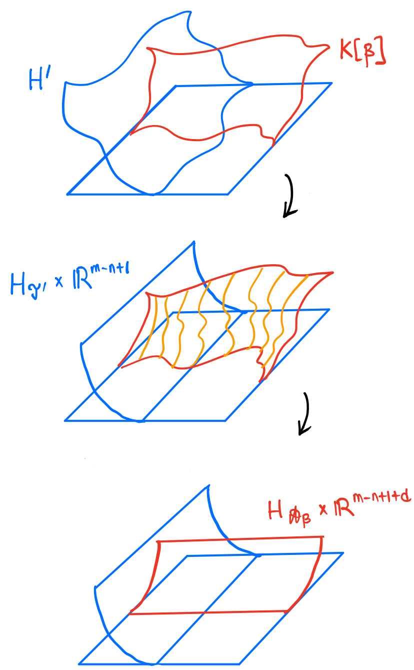

Figure 3.6: Treating the complementary directions as parameters.

Now we can apply Theorem 3.27 to normalize $K[\beta]$ viewed as the final smooth piece of K. More specifically, we can apply Theorem 3.27 to normalize $K[\beta]$ while preserving K' and viewing the complementary directions

$\mathbb{R}^{m-n+1+d}$ as parameters, see Figure 3.6. This insures we preserve H' and hence do not disturb its already arranged normalization.

This concludes the proof of Proposition 3.6. □

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Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, 02139, USA
E-mail address: dgavela@mit.edu

Department of Mathematics, Stanford University
Stanford, CA 94305, USA
E-mail address: eliash@stanford.edu

Department of Mathematics, University of California, Berkeley
Berkeley, CA 94720-3840, USA
E-mail address: nadler@math.berkeley.edu

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